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Xiangcheng Zheng,[1] Yiqun Li,[2] and Hong Wang[2,a)]

AFFILIATIONS

[1] School of Mathematical Sciences, Peking University, Beijing 100871, China
[2] Department of Mathematics, University of South Carolina, Columbia, South Carolina 29208, USA

[a)] Author to whom correspondence should be addressed: hwang@math.sc.edu

ABSTRACT

Vibrations are ubiquitous in mechanical or biological systems, and they are ruinous in numerous circumstances. We develop a viscoelastic Timoshenko beam model, which naturally captures distinctive power-law responses arising from a broad distribution of time-scales presented in the complex internal structures of viscoelastic materials and so provides a very competitive description of the mechanical responses of viscoelastic beams, thick beams, and beams subject to high-frequency excitations. We, then, prove the well-posedness and regularity of the viscoelastic Timoshenko beam model. We finally investigate the performance of the model, in comparison with the widely used Euler–Bernoulli and Timoshenko beam models, which shows the utility of the new model.

I. INTRODUCTION

Vibrations are prevalent in mechanical or biological systems, and they are unwanted and even destructive in numerous circumstances. Their precise characterization and modeling are crucial in the design and determination of the dynamic durability of the systems such as turbine blades in jet and helicopter engines[1] and can provide insight into disease evolution[2] and are the key to develop effective ways to mitigate the impact of unwanted vibrations to optimize the performance, to extend the durability of the systems in engineering applications, and to develop replacements that restore the structures and functionality of damaged organs in the medical industry.

A. Modeling issues in beam vibrations

Resonance is one of the key issues in system and structural vibrations, which occurs as a system absorbs more energy when the driving frequency of the external harmonic excitations, e.g., the rotating blades of engines, equals the natural frequency of the system. It causes significant deflections, which may result in structural damage and even system failure.

Consider the dynamic response to a harmonic excitation (e.g., by the cyclic motion of an engine) of a spring-mass-dashpot system,[3]

$$m\ddot{x}(t) + c\dot{x}(t) + kx(t) = q_0 \cos \omega t. \tag{1}$$

Here, m is the mass of the object, c is the viscous damping coefficient, and k is the stiffness of the spring. For an undamped (i.e., $c = 0$) vibration, resonance occurs when the driving frequency ω equals the natural frequency $\omega_{nat} := \sqrt{k/m}$ of the system. The displacement $x(t) = q_0 t \sin \omega t/(2m\omega)$ increases linearly with time, and the system may break. A common practice to mitigate the resonance is to change the natural frequency of the system (e.g., the mass of the object or the stiffness of the spring) or the driving frequency of the harmonic excitation. However, the stiffness of the part may not be allowed to be lower than a certain value in a system design due to load requirements (static deflection) or other design constraints. The change of mass too may have other constraints on it.

Alternatively, a physical damping may be introduced to model (1), yielding a displacement of the form[3,4]

$$x(t) = \frac{q_0 \cos(\omega t - \theta)}{m\sqrt{(\omega_{nat}^2 - \omega^2)^2 + (2\zeta\omega_{nat}\omega)^2}}.$$

Here, the damping ratio $\zeta := c/(2m\omega_{nat})$, the damped natural frequency $\omega_d := \omega_{nat}\sqrt{1-\zeta^2}$, and the phase $\theta := \tan^{-1}\big(2\zeta\omega_{nat}\omega/(\omega_{nat}^2 - \omega^2)\big)$. The displacement does not increase with time although it may have a large amplitude when the driving frequency ω equals the natural frequency ω_{nat} of the system.

Beam vibrations are much more complex and involved. Take the dynamic response of a clamped Euler–Bernoulli beam as an example,[3,4]

$$\begin{aligned} &\rho A(x)\partial_t^2 w + \zeta\rho A(x)\partial_t w + \partial_x^2\big(EI(x)\partial_x^2 w\big) = q(x,t), \\ &w(0,t) = \partial_x w(0,t) = w(l,t) = \partial_x w(l,t) = 0, \quad t \in [0,T], \\ &w(x,0) = \partial_t w(x,0) = 0, \quad x \in [0,l]. \end{aligned} \tag{2}$$

Here, w is the deflection of the beam, ρ is the mass density, $A(x)$ is the cross-sectional area, $I(x)$ is the rotational inertia, $q(x,t)$ is the transverse load per unit length, and E is the elastic modulus. Major complications of beam vibrations include the following: (i) Beams have infinitely many natural frequencies and model shapes, which are of the following form for an undamped vibration:[3]

$$\begin{aligned} &X_n(x) = \cosh\gamma_n x - \cos\gamma_n x - \iota_n(\sinh\gamma_n x - \sin\gamma_n x), \\ &\omega_n = \left(\frac{\gamma_n}{l}\right)^2\sqrt{\frac{EI}{\rho A}}\ \text{rad/s}, \quad \cos\gamma_n l\,\cosh\gamma_n l = 1, \quad n = 1,2,\ldots, \end{aligned} \tag{3}$$

where the weighted natural frequencies γ_n are related to the natural frequencies by the second equations in (3) and $\iota_n = 1$ for $n \geq 4$ and around the unity for $n \leq 3$. Equations (3) show that the natural frequencies may vary significantly (e.g., from full to almost empty in a fuel tank during a flight). In addition, a beam may have a variable cross-sectional area. Hence, the natural frequencies cannot be expressed in a closed form. It is not an easy task to choose the driving frequency not equal to the natural frequency of the system, in general. (ii) A conventional viscous damping (i.e., $\zeta > 0$) does not eliminate the resonance (cf. Sec. VI), which is in sharp contrast to the case of the single-degree-of-freedom spring-mass-damper system (1).

B. Modeling of viscoelastic materials

Many modern structures, e.g., turbine blades in jet engines, are subjected to challenging conditions, such as high temperatures and high tensile stresses. In these circumstances, conventional metals may creep significantly, which is one of the main causes of system failure. This has led to the development of creep-resistant superalloys and ceramic matrix composite materials, which exhibit viscoelastic behaviors. Viscoelastic materials, such as natural and synthetic biomaterials, smart materials, polymers, and elastomers, exhibit both the elastic characteristic of solids and the viscous behavior of fluids. They have widely been used in many applications.[5–13] Consequently, modeling of the mechanical behavior of viscoelastic materials is not a straightforward task.

Classical rheological models consist of combinations of springs and dashpots to describe the elastic and viscous behaviors of viscoelastic materials through simultaneous storage and dissipation of their mechanical energy. The Maxwell model comprises a serial connection of a spring and a dashpot and has a constitutive law $\sigma + (\eta/E)\dot{\sigma} = \eta\dot{\varepsilon}$, where η is the viscosity of the material, σ and ε denote the stress and strain of the material, respectively, and $\dot{\sigma}$ and $\dot{\varepsilon}$ denote their time derivatives.[5,7,11,14] Its stress relaxation in the case of zero initial strain $[\varepsilon(0) = 0]$ reads

$$\sigma(t) = \int_0^t G(t-s)\dot{\varepsilon}(s)ds, \quad G(t) := Ee^{-t/\tau}, \tag{4}$$

with $\tau := \eta/E$ being the retardation time of the material. The Maxwell model describes the relaxation but not the creep behavior of viscoelastic materials. Sophisticated models better describe viscoelastic materials, in which the relaxation modulus may be expressed as a combination of $G(t - t_j)$.

Various experiments show that viscoelastic materials exhibit both restorative elastic mechanism and viscous internal dissipation mechanism simultaneously and demonstrate power-law behaviors.[5,7,11,15–18] Integer-order rheological models, which are expressed as a combination of exponentially decaying modulus, provide satisfactory approximations to the power-law behaviors of viscoelastic materials for short observation times.[10] Fundamentally, this reduces to an approximation of a power-law function by a combination of exponentially decaying functions. A Scott-Blair element with a power-law relaxation modulus and the assumptions that the material is quiescent for $t < 0$ and $\varepsilon(0) = 0$,

$$\sigma(t) = \int_0^t \frac{E_\alpha \dot{\varepsilon}(s) ds}{\Gamma(1-\alpha)(t-s)^\alpha} = E_\alpha \partial_t^\alpha \varepsilon, \quad 0 < \alpha < 1, \tag{5}$$

provides an accurate description as it correctly catches the power-law behavior of viscoelastic materials.[5,9,11,17,18] Here, $\partial_t^\alpha \varepsilon$ is the Caputo fractional differential operator defined by[14]

$$\partial_t^\alpha \varepsilon(t) := {}_0I_t^{1-\alpha} \dot{\varepsilon}(t), \qquad {}_0I_t^\alpha \varepsilon(t) := \frac{1}{\Gamma(\alpha)} \int_0^t \frac{\varepsilon(s)}{(t-s)^{1-\alpha}} ds. \tag{6}$$

E_α is the material dependent parameter and can be viewed as "a fractionalized interpolation" or a weighted average between the modulus of elasticity E and the viscosity η. In particular, E_α with $\alpha = 0$ corresponds to the classical modulus of elasticity E of an elastic material, which also implies that the pseudo-unit $[\mathrm{Pa} \cdot s^\alpha]$ of E_α will be recovered to its equilibrium counterpart when $\alpha \to 0$.[18]

A classical Euler–Bernoulli beam is valid only for long-slender elastic beams and so does not apply to thick beams, composite beams, beams subject to high-frequency excitations, and viscoelastic beams. In this paper, we develop and analyze a viscoelastic Timoshenko beam model to accurately describe the vibrations of viscoelastic beams by properly taking into account for the effect of shear deformation and rotary inertia that are ignored in the Euler–Bernoulli beam.[19,20] We, then, carry out rigorous mathematical analysis to prove the well-posedness of the model and the regularities of its solutions. The rest of this paper is organized as follows: In Sec. II, we develop a viscoelastic Timoshenko–Ehrenfest beam model. In Sec. III, we present preliminaries to be used in the Secs. II–V. In Sec. IV, we prove the well-posedness of the mathematical model. In Sec. V, we prove the regularity of the solutions to the model. In Sec. VI, we investigate the performance of the proposed model in comparison to the integer-order Euler–Bernoulli and Timoshenko models and the fractional Euler–Bernoulli model, which shows the utility of the model.[21]

II. A VISCOELASTIC TIMOSHENKO BEAM MODEL

In this section, we develop a viscoelastic Timoshenko–Ehrenfest beam model involving fractional operators.

A. Force and momentum equilibrium and kinematics

Consider transverse vibrations of a viscoelastic beam under the following hypotheses: (i) The beam has a straight centroidal (labeled x) axis with length l and cross-sectional area $A(x)$. (ii) The loadings are applied in the transverse direction (labeled z axis) to the beam. The beam has a longitudinal $(x - z)$ plane of symmetry. The x, y, and z axes form a right-handed coordinate system. (iii) Cross-sectional planes that are perpendicular to the centroidal axis remain planar (but not necessarily orthogonal to the deformed beam axis as assumed in Euler–Bernoulli beams) after deformation. (iv) The beam is isotropic and homogeneous. Hence, the strains acting in the cross section are only due to bending kinematics and the beam undergoes purely planar flexural vibrations that are small in magnitude. When the beam is deformed, some parts of the beam are compressed and others are stretched. Somewhere between the top and the bottom of the beam, there is a neutral surface of the beam, which retains its original length. The intersection of the neutral surface and the longitudinal plane of symmetry defines the neutral axis of the beam. This reduces the transverse vibration of the beam to the deflection of the one-dimensional neutral axis based on which the deflection of the beam at other locations can be evaluated.

Let the load $q(x,t)$ be positive upward, $M(x,t)$ be the bending moment, and $V(x,t)$ be the shear force. Apply Newton's second law to the dynamic equilibrium of vertical forces in the beam element $(x, x+h)$ to obtain

$$\int_x^{x+h} \rho A(\zeta) \partial_t^2 w(\zeta,t) d\zeta = V(x+h,t) - V(x,t) + \int_x^{x+h} q(\zeta,t) d\zeta.$$

Taking the limit as $h \to 0^+$ yields the vertical force equilibrium equation,

$$\rho A(x) \partial_t^2 w = \partial_x V + q(x,t). \tag{7}$$

Let $\theta(x,t)$ be the angle of rotation of the cross section at x. Use Euler's second law to the beam element $(x, x+h)$ to obtain

$$\int_x^{x+h} \rho I(\zeta) \partial_t^2 \theta(\zeta,t) d\zeta = M(x+h,t) - M(x,t) + V(x+h,t)h + \int_x^{x+h} (\zeta - x) q(\zeta,t) d\zeta,$$

where $I(x) := \int_{A(x)} z^2 dA$ is the rotational inertial. Letting $h \to 0^+$ gives

$$\rho I(x) \partial_t^2 \theta(x,t) = \partial_x M(x,t) + V(x,t). \tag{8}$$

The displacement field $\boldsymbol{w}(x,y,z,t) = (w_x(x,y,z,t), w_y(x,y,z,t), w_z(x,y,z,t))$ is reduced by the Timoshenko beam hypothesis to

$$w_x(x,y,z,t) = -z\theta(x,t), \quad w_y(x,y,z,t) = 0, \quad w_z(x,y,z,t) = w(x,t). \tag{9}$$

The infinitesimal strains are as follows:

$$\begin{aligned}\varepsilon_{xx}(x,y,z,t) &= \partial_x w_x(x,y,z,t) = -z\partial_x\theta(x,t),\\ 2\varepsilon_{xz}(x,y,z,t) &= \partial_z w_x(x,y,z,t) + \partial_x w_z(x,y,z,t) = \partial_x w(x,t) - \theta(x,t),\end{aligned}$$

and all other strains vanish.

B. A fractional Timoshenko beam model

We combine the first equation in (9) with Eq. (5) to get the stress–strain relations in the viscoelastic Timoshenko beam,

$$\begin{aligned}\sigma_{xx}(x,y,z,t) &= E_\alpha \partial_t^\alpha \varepsilon_{xx}(x,y,z,t) = -E_\alpha\, z\, \partial_t^\alpha \partial_x \theta(x,t),\\ \sigma_{xz}(x,y,z,t) &= 2G\, \partial_t^\alpha \varepsilon_{xz}(x,y,z,t) = G\partial_t^\alpha(\partial_x w(x,t) - \theta(x,t)).\end{aligned}$$

Here, G is the shear modulus. We evaluate the net bending moment $M(x,t)$ by

$$M(x,t) = -\int_{A(x)} z\sigma_{xx}(x,y,z,t)dA = E_\alpha\, I(x)\partial_t^\alpha \partial_x\theta(x,t). \tag{10}$$

Let κ be the shear correction coefficient. Evaluate the shear force $V(x,t)$ by

$$V(x,t) = \int_{A(x)} \kappa\sigma_{xz}(x,y,z,t)dA = \kappa G A(x)\partial_t^\alpha(\partial_x w(x,t) - \theta(x,t)). \tag{11}$$

We incorporate Eqs. (10) and (11) into Eqs. (7) and (8) to obtain the system of fractional PDEs for the vibrations of the viscoelastic Timoshenko beam,[22]

$$\begin{aligned}\rho A(x)\partial_t^2 w &= \partial_x\big(\kappa G A(x)\partial_t^\alpha(\partial_x w - \theta)\big) + q(x,t), \quad (x,t) \in (0,l)\times(0,T],\\ \rho I(x)\partial_t^2\theta &= \partial_x\big(E_\alpha I(x)\partial_x\partial_t^\alpha\theta\big) + \kappa G A(x)\partial_t^\alpha(\partial_x w - \theta).\end{aligned} \tag{12}$$

In this paper, we assume that the beam is clamped at both ends, so the displacement w and the angle of rotation θ vanish at both ends, leading to

$$w(0,t) = w(l,t) = 0, \quad \theta(0,t) = \theta(l,t) = 0, \quad t \in [0,T]. \tag{13}$$

Finally, we close system (12) by the initial conditions

$$\begin{aligned}w(x,0) = w_0(x), \quad \partial_t w(x,0) = \check{w}_0(x), \quad x \in [0,l],\\ \theta(x,0) = \theta_0(x), \quad \partial_t\theta(x,0) = \check{\theta}_0(x), \quad x \in [0,l].\end{aligned} \tag{14}$$

III. PRELIMINARIES

Let $C[0,T]$ be the space of continuous functions on $[0,T]$ and $C^m[0,T]$, with $m \in \mathbb{N}$, be the space of m-times continuously differentiable functions on $[0,T]$. For $\mathcal{I} := (0,l)$ or $[0,T]$, let $L_{loc}(\mathcal{I})$ be the space of locally Lesbegue integrable functions on $\mathcal{I}$, $L^p(\mathcal{I})$ $(1 \le p \le \infty)$ be the space of pth Lebesgue integrable functions on $\mathcal{I}$, and $W^{m,p}(\mathcal{I})$ be the subspace of $L^p(\mathcal{I})$ with weak derivatives up to order m being in $L^p(\mathcal{I})$. Denote $H^m(\mathcal{I}) := W^{m,2}(\mathcal{I})$, and $H_0^m(\mathcal{I}) \subset H^m(\mathcal{I})$ is subject to the homogeneous boundary conditions up to order $m-1$. $H^s(\mathcal{I})$ with a non-integer $s \ge 0$ is defined via interpolation. All the spaces are equipped with standard norms.[23,24]

The eigenfunctions $\{\phi_i\}_{i=1}^\infty$ of the Dirichlet Laplacian[24,25]

$$-\partial_x^2\phi_i(x) = \lambda_i\phi_i(x), \quad x \in (0,l), \quad \phi_i(0) = \phi_i(l) = 0$$

form an orthonormal basis in $L^2(0,l)$.

The space-time spaces $C^m([0,T];\mathcal{X})$ and $W^{m,p}(0,T;\mathcal{X})$ with a Banach space $\mathcal{X}$ are defined by[23,24]

$$C^m([0,T];\mathcal{X}) := \left\{g : \left\|\partial_t^k g(\cdot,t)\right\|_{\mathcal{X}} \in C[0,T], \quad 0 \le k \le m\right\},$$
$$W^{m,p}(0,T;\mathcal{X}) := \left\{g : \left\|\partial_t^k g(\cdot,t)\right\|_{\mathcal{X}} \in L^p(0,T), \quad 0 \le k \le m, \ 1 \le p \le \infty\right\},$$

equipped the norms

$$\|g\|_{C^m([0,T];\mathcal{X})} := \max_{0\le k\le m}\max_{t\in[0,T]}\left\|\partial_t^k g\right\|_{\mathcal{X}},$$
$$\|g\|_{W^{m,p}(0,T;\mathcal{X})} := \begin{cases} \left(\sum_{k=0}^m \int_0^T \left\|\partial_t^k g(\cdot,t)\right\|_{\mathcal{X}}^p dt\right)^{1/p}, & 1 \le p < \infty, \\ \max_{0\le k\le m} \operatorname*{esssup}_{t\in(0,T)} \left\|\partial_t^k g(\cdot,t)\right\|_{\mathcal{X}}, & p = \infty. \end{cases}$$

Throughout this paper, we shall use Q and Q_i to denote generic positive constants in which Q may assume different values at different occurrences. We set $\|\cdot\| := \|\cdot\|_{L^2(\mathcal{I})}$ and may write $W^{m,p}(\mathcal{X})$ for $W^{m,p}(0,T;\mathcal{X})$ for simplicity. We, then, refer to the properties of the fractional integral operator.[14,26,27]

Lemma III.1. The left fractional integral operator ${}_0I_t^\beta$ in (6) and the right fractional integral operator ${}_tI_T^\beta$ defined by

$${}_tI_T^\beta v := \frac{1}{\Gamma(\beta)}\int_t^T \frac{v(s)}{(s-t)^{1-\beta}}ds$$

are bounded linear operators from $L^2(0,T)$ to $L^2(0,T)$. For any $v \in L^1(0,T)$,

$${}_0I_t^{\beta_1}\,{}_0I_t^{\beta_2}v = {}_0I_t^{\beta_1}\,{}_0I_t^{\beta_1+\beta_2}v, \quad {}_tI_T^{\beta_1}v, \quad {}_tI_T^{\beta_2}v = v, \quad {}_tI_T^{\beta_1+\beta_2}v \quad \forall t\in[0,T], \ \beta_1,\beta_2 > 0. \tag{15}$$

Furthermore, they are adjoints in the L^2 sense, i.e., for all $\beta > 0$,

$$\int_0^T ({}_0I_t^\beta w)v(t)dt = \int_0^T w(t)({}_tI_T^\beta v)dt, \quad \forall w,v \in L^2(0,T),$$

$$\partial_t^{1-\beta}v = {}_0I_t^\beta \partial_t v = \partial_t\, {}_0I_t^\beta v, \quad \forall v \in W^{1,1}(0,T) \text{ with } v(0) = 0. \tag{16}$$

Lemma III.2. If $g \in L^{2/(1+2\alpha)+\varepsilon}(0,\bar t)$ for any $\bar t \in (0,T)$, $0 < \alpha < 1/2$, and $0 < \varepsilon \ll 1$, then the coercivity estimate holds,

$$\int_0^{\bar t} {}_0I_t^\alpha g \cdot {}_tI_{\bar t}^\alpha g\, dt \ge \cos(\alpha\pi)\left\|{}_0I_t^\alpha g\right\|_{L^2(0,\bar t)}^2, \quad \bar t \in [0,T].$$

Furthermore, if $v \in W^{1,2/(1+2\alpha)+\varepsilon}(0,\bar t)$ for any $\bar t \in (0,T)$, $0 < \alpha < 1/2$, and $0 < \varepsilon \ll 1$, then the coercivity estimate holds,

$$\int_0^{\bar t} {}_0I_t^\alpha \partial_t v \cdot {}_tI_{\bar t}^\alpha \partial_t v\, dt \ge \cos(\alpha\pi)\left\|\partial_t^{1-\alpha} v\right\|_{L^2(0,\bar t)}^2, \quad \bar t \in [0,T].$$

Proof. The proof follows from Theorem 2.23 of the literature[28] and is, thus, omitted. □

We, then, refer to the Gronwall inequality for future use.[29]

Lemma III.3. Let $a \ge 0$, $b > 0$, and $\vartheta, \iota \ge 0$ with $\vartheta + \iota < 1$. Suppose that $v \ge 0$ satisfies the inequality

$$v(t) \le a + b\int_0^t (t-s)^{-\vartheta}s^{-\iota}v(s)ds \text{ for a.e. } t \in [0,T].$$

We write $B_0 := B(1-\vartheta, 1-\iota)$. For $r > 0$, let $t_r := \left(\frac{r}{bB_0}\right)^{\frac{1}{1-\vartheta-\iota}}$ and let $r_0 := bB_0T^{1-\vartheta-\iota}$ so that $t_r \le T$ for $r \le r_0$. Then, if $r \le r_0$ and also $r < 1$, we have

$$v(t) \le \frac{a}{1-r}\exp\left(\frac{bt_r^{-\vartheta}}{(1-r)(1-\iota)}t^{1-\iota}\right) \text{ for a.e. } t \in [0,T].$$

IV. WELL-POSEDNESS OF VISCOELASTIC TIMOSHENKO BEAM MODEL

We prove the well-posedness of the mathematical model (12)–(14). For the convenience of the analysis, we use the substitution

$$u(x,t) := w(x,t) - w_0(x) - t\,\check{w}_0(x), \quad \psi(x,t) := \theta(x,t) - \theta_0(x) - t\,\check{\theta}_0(x) \tag{17}$$

to reformulate system (12) in terms of u and ψ as follows:

$$\begin{aligned} \rho A(x)\partial_t^2 u &= \partial_x\big(\kappa GA(x)\partial_t^\alpha(\partial_x u - \psi)\big) + \tilde{q}(x,t), \quad (x,t) \in (0,l)\times(0,T], \\ \rho I(x)\partial_t^2 \psi &= \partial_x\big(E_\alpha I(x)\partial_x\partial_t^\alpha\psi\big) + \kappa GA(x)\partial_t^\alpha(\partial_x u - \psi) + \tilde{p}(x,t), \end{aligned} \tag{18}$$

along with the homogenous initial and boundary conditions,

$$u(0,t) = \psi(0,t) = u(l,t) = \psi(l,t) = 0, \quad t \in [0,T], \tag{19}$$

$$u(x,0) = \partial_t u(x,0) = \psi(x,0) = \partial_t\psi(x,0) = 0, \quad x \in [0,l]. \tag{20}$$

Here, $\tilde{q}$ and $\tilde{p}$ are defined by

$$\begin{aligned} \tilde{q} &:= q + \kappa G\partial_x\big(A(x)I_t^{1-\alpha}(\partial_x\check{w}_0 - \check{\theta}_0)\big) = q + \frac{t^{1-\alpha}\kappa G\partial_x\big(A(x)(\partial_x\check{w}_0 - \check{\theta}_0)\big)}{\Gamma(2-\alpha)}, \\ \tilde{p} &:= E\partial_x\big(I(x)\partial_x\,{}_0I_t^{1-\alpha}\check{\theta}_0\big) + \kappa GA(x)\,{}_0I_t^{1-\alpha}(\partial_x\check{w}_0 - \check{\theta}_0) \\ &= \frac{t^{1-\alpha}\big[E_\alpha\partial_x\big(I(x)\partial_x\check{\theta}_0\big) + \kappa GA(x)(\partial_x\check{w}_0 - \check{\theta}_0)\big]}{\Gamma(2-\alpha)}. \end{aligned} \tag{21}$$

Integrate Eqs. (18) from 0 to t, apply the homogeneous initial conditions in (20), and use (16) to arrive at a reduced-order system of fractional PDEs,

$$\begin{aligned} \rho A(x)\partial_t u &= \kappa G\partial_x\big(A(x)\,{}_0I_t^{1-\alpha}(\partial_x u - \psi)\big) + f(x,t), \\ \rho I(x)\partial_t\psi &= E_\alpha\partial_x\big(I(x)\,{}_0I_t^{1-\alpha}\partial_x\psi\big) + \kappa GA(x)\,{}_0I_t^{1-\alpha}(\partial_x u - \psi) + g(x,t). \end{aligned} \tag{22}$$

Here, f and g are defined follows with $\tilde{p}$ and $\tilde{q}$ given by (21):

$$\begin{aligned} f(x,t) &:= {}_0I_t^1\tilde{q} = {}_0I_t^1 q + \frac{t^{2-\alpha}\kappa G\partial_x\big(A(x)(\partial_x\check{w}_0 - \check{\theta}_0)\big)}{\Gamma(3-\alpha)}, \\ g(x,t) &:= {}_0I_t^1\tilde{p} = \frac{t^{2-\alpha}\big[E_\alpha\partial_x\big(I(x)\partial_x\check{\theta}_0\big) + \kappa GA(x)(\partial_x\check{w}_0 - \check{\theta}_0)\big]}{\Gamma(3-\alpha)}. \end{aligned} \tag{23}$$

System (22) is closed with the boundary condition (19) and the initial condition

$$u(x,0) = 0, \quad \psi(x,0) = 0, \quad x \in [0,l]. \tag{24}$$

Note that the two homogeneous initial conditions on $\partial_t u$ and $\partial_t\psi$ in (20) can be deduced naturally from (22) in the limit $t \to 0$ since the right-hand side terms on the equations in (22) vanish as $t \to 0$ by (23).

We are now in the position to prove the well-posedness of the model.

Theorem IV.1. *Suppose that $\check{w}_0, \check{\theta}_0 \in H^2$, $q \in L^2(0,T;\,L^2)$, and $A, I \in H^1$ with $0 < A_* \le A(x), I(x) \le A^* < \infty$. Then, problems (22), (19), and (24) have the unique solution $u, \psi \in W^{1,\infty}(0,T;\,L^2) \cap L^2(0,T;\,H^1)$ with the stability estimate*

$$\begin{aligned} &\|u\|_{W^{1,\infty}(0,T;L^2)} + \|u\|_{L^2(0,T;H^1)} + \|\psi\|_{W^{1,\infty}(0,T;L^2)} + \|\psi\|_{L^2(0,T;H^1)} \\ &\quad \le Q\big(\|f\|_{H^1(0,T;L^2)} + \|g\|_{H^1(0,T;L^2)}\big) \le Q\big(\|q\|_{L^2(0,T;L^2)} + \|\check{w}_0\|_{H^2} + \|\check{\theta}_0\|_{H^2}\big), \end{aligned} \tag{25}$$

with $Q = Q(\rho, A_*, A^*, \|A\|_{H^1}, \|I\|_{H^1}, \kappa, G, E_\alpha, \alpha, T)$. Furthermore, if $w_0, \theta_0 \in H^1$, then problems (12)–(14) have the unique solution w, $\theta \in W^{1,\infty}(0,T;\,L^2) \cap L^2(0,T;\,H^1)$ with the stability estimate

$$\|w\|_{W^{1,\infty}(0,T;L^2)} + \|w\|_{L^2(0,T;H^1)} + \|\theta\|_{W^{1,\infty}(0,T;L^2)} + \|\theta\|_{L^2(0,T;H^1)} \\ \leq Q\big(\|q\|_{L^2(0,T;L^2)} + \|w_0\|_{H^1} + \|\theta_0\|_{H^1} + \|\check{w}_0\|_{H^2} + \|\check{\theta}_0\|_{H^2}\big). \tag{26}$$

Proof. We carry out the proofs in four steps.

Step 1. Existence and uniqueness of Galerkin approximations

Let $\{\phi_i\}_{i=1}^{\infty} \subset H_0^1(0,l)$ be an orthonormal basis of $L^2(0,l)$, which is also orthogonal in $H_0^1(0,l)$ (e.g., the eigenfunctions of the Dirichlet Laplacian).[24] For any fixed $m \in \mathbb{N}$, we denote a family of finite-dimensional subspaces,

$$S_m := \mathrm{span}\{\phi_j(x)\}_{j=1}^m \subset H_0^1(0,l). \tag{27}$$

We seek Galerkin approximations $u^{(m)}, \psi^{(m)} : [0,T] \to S_m$,

$$u^{(m)}(x,t) := \sum_{j=1}^m u_j^{(m)}(t)\phi_j(x), \qquad \psi^{(m)}(x,t) := \sum_{j=1}^m \psi_j^{(m)}(t)\phi_j(x), \tag{28}$$

to satisfy the Galerkin weak formulation: For any $\phi \in S_m$,

$$\begin{aligned} &\rho\Big(A\partial_t u^{(m)}(\cdot,t),\phi\Big) + \kappa G\Big(A\,{}_0I_t^{1-\alpha}\partial_x u^{(m)},\partial_x\phi\Big) = \kappa G\Big(A\,{}_0I_t^{1-\alpha}\psi^{(m)},\partial_x\phi\Big) + (f(\cdot,t),\phi),\\ &\rho\Big(I\partial_t\psi^{(m)}(\cdot,t),\phi\Big) + E_\alpha\Big(I\,{}_0I_t^{1-\alpha}\partial_x\psi^{(m)},\partial_x\phi\Big) + \kappa G\Big(A\,{}_0I_t^{1-\alpha}\psi^{(m)},\phi\Big)\\ &\quad = \kappa G\Big(A\,{}_0I_t^{1-\alpha}\partial_x u^{(m)},\phi\Big) + (g(\cdot,t),\phi),\\ &\quad u^{(m)}(x,0) = \psi^{(m)}(x,0) = 0, \quad \forall x \in [0,l]. \end{aligned} \tag{29}$$

This is reformulated as a system of fractional integro-differential equations,

$$\begin{aligned} &\rho \boldsymbol{M}_A \dot{\boldsymbol{u}}^{(m)}(t) + \kappa G \boldsymbol{B}_A\,{}_0I_t^{1-\alpha}\boldsymbol{u}^{(m)} = \kappa G \boldsymbol{C}_A\,{}_0I_t^{1-\alpha}\boldsymbol{\psi}^{(m)} + \boldsymbol{f}(t),\\ &\rho \boldsymbol{M}_I \dot{\boldsymbol{\psi}}^{(m)}(t) + E_\alpha \boldsymbol{B}_I\,{}_0I_t^{1-\alpha}\boldsymbol{\psi}^{(m)} + \kappa G \boldsymbol{M}_A\,{}_0I_t^{1-\alpha}\boldsymbol{\psi}^{(m)} = \kappa G \boldsymbol{D}_A\,{}_0I_t^{1-\alpha}\boldsymbol{u}^{(m)} + \boldsymbol{g}(t),\\ &\boldsymbol{u}^{(m)}(0) = \boldsymbol{0}, \quad \boldsymbol{\psi}^{(m)}(0) = \boldsymbol{0}. \end{aligned} \tag{30}$$

Here, the solution vectors $\boldsymbol{u}^{(m)}(t)$ and $\boldsymbol{\psi}^{(m)}(t)$, the right-hand source vectors $\boldsymbol{f}(t)$ and $\boldsymbol{g}(t)$, the symmetric and positive-definite mass matrices $\boldsymbol{M}_A$ and $\boldsymbol{M}_I$, stiffness matrices $\boldsymbol{B}_A$ and $\boldsymbol{B}_I$, and the nonsymmetric matrices $\boldsymbol{C}_A$ and $\boldsymbol{D}_A$ are defined by

$$\begin{aligned} &\boldsymbol{u}^{(m)}(t) := \Big[u_1^{(m)}(t),\ldots,u_m^{(m)}(t)\Big]^\top, \quad \boldsymbol{f}(t) := [(f(\cdot,t),\phi_1),\ldots,(f(\cdot,t),\phi_m)]^\top,\\ &\boldsymbol{\psi}^{(m)}(t) = \Big[\psi_1^{(m)}(t),\ldots,\psi_m^{(m)}(t)\Big]^\top, \quad \boldsymbol{g}(t) = [(g(\cdot,t),\phi_1),\ldots,(g(\cdot,t),\phi_m)]^\top,\\ &\boldsymbol{M}_A := [(A\phi_i,\phi_j)]_{i,\,j=1}^m, \quad \boldsymbol{M}_I := [(I\phi_i,\phi_j)]_{i,\,j=1}^m, \quad \boldsymbol{C}_A = [(A\phi_i,\partial_x\phi_j)]_{i,\,j=1}^m,\\ &\boldsymbol{D}_A = [(A\partial_x\phi_i,\phi_j)]_{i,\,j=1}^m, \quad \boldsymbol{B}_A := [(A\partial_x\phi_i,\partial_x\phi_j)]_{i,\,j=1}^m, \quad \boldsymbol{B}_I := [(I\partial_x\phi_i,\partial_x\phi_j)]_{i,\,j=1}^m. \end{aligned}$$

We multiply the first and second equations in (30) by $(\rho\boldsymbol{M}_A)^{-1}$ and $(\rho\boldsymbol{M}_I)^{-1}$, respectively, to rewrite system (30) as

$$\begin{aligned} &\dot{\boldsymbol{u}}^{(m)}(t) + \kappa G\rho^{-1}\boldsymbol{M}_A^{-1}\boldsymbol{B}_A\,{}_0I_t^{1-\alpha}\boldsymbol{u}^{(m)} = \kappa G\rho^{-1}\boldsymbol{M}_A^{-1}\boldsymbol{C}_A\,{}_0I_t^{1-\alpha}\boldsymbol{\psi}^{(m)} + \rho^{-1}\boldsymbol{M}_A^{-1}\boldsymbol{f}(t),\\ &\dot{\boldsymbol{\psi}}^{(m)}(t) + E_\alpha\rho^{-1}\boldsymbol{M}_I^{-1}\boldsymbol{B}_I\,{}_0I_t^{1-\alpha}\boldsymbol{\psi}^{(m)} + \kappa G\rho^{-1}\boldsymbol{M}_I^{-1}\boldsymbol{M}_A\,{}_0I_t^{1-\alpha}\boldsymbol{\psi}^{(m)}\\ &\quad = \kappa G\rho^{-1}\boldsymbol{M}_I^{-1}\boldsymbol{D}_A\,{}_0I_t^{1-\alpha}\boldsymbol{u}^{(m)} + \rho^{-1}\boldsymbol{M}_I^{-1}\boldsymbol{g}(t), \quad \boldsymbol{u}^{(m)}(0) = \boldsymbol{0}, \quad \boldsymbol{\psi}^{(m)}(0) = \boldsymbol{0}. \end{aligned} \tag{31}$$

Apply the integral operator $\partial_t\,{}_0I_t^\alpha$ on the two equations in (31) and utilize the semigroup property (15) of the fractional integral operator ${}_0I_t^\alpha$, the commutativity (16) of fractional integral operator ${}_0I_t^\alpha$, and differential operator ∂_t with the homogeneous initial conditions (20) to obtain

$$\partial_t\,{}_0I_t^\alpha\,{}_0I_t^{1-\alpha}v = \partial_t\,I_t^1 v = v, \qquad \partial_t\,{}_0I_t^\alpha\partial_t v = \partial_t^2\,{}_0I_t^\alpha v = {}_0I_t^\alpha\,\partial_t^2 v =: \partial_t^{2-\alpha}v. \tag{32}$$

We, consequently, reformulate system (31) as the initial-boundary value problem of a system of linear fractional ordinary differential equations (fODEs) of order $2-\alpha$ with the unknowns $\{\boldsymbol{u}^{(m)}, \boldsymbol{\psi}^{(m)}\}$,

$$\begin{aligned}
&\partial_t^{2-\alpha}\boldsymbol{u}^{(m)}(t) + \kappa G\rho^{-1}\boldsymbol{M}_A^{-1}\boldsymbol{B}_A\boldsymbol{u}^{(m)}(t) = \kappa G\rho^{-1}\boldsymbol{M}_A^{-1}\boldsymbol{C}_A\boldsymbol{\psi}^{(m)}(t) + \rho^{-1}\boldsymbol{M}_A^{-1}\partial_t^{1-\alpha}\boldsymbol{f}(t),\\
&\partial_t^{2-\alpha}\boldsymbol{\psi}^{(m)}(t) + E_\alpha\rho^{-1}\boldsymbol{M}_I^{-1}\boldsymbol{B}_I\boldsymbol{\psi}^{(m)}(t) + \kappa G\rho^{-1}\boldsymbol{M}_I^{-1}\boldsymbol{M}_A\boldsymbol{\psi}^{(m)}(t)\\
&\quad = \kappa G\rho^{-1}\boldsymbol{M}_I^{-1}\boldsymbol{D}_A\boldsymbol{u}^{(m)}(t) + \rho^{-1}\boldsymbol{M}_I^{-1}\partial_t^{1-\alpha}\boldsymbol{g}(t),\\
&\quad \boldsymbol{u}^{(m)}(0) = \dot{\boldsymbol{u}}^{(m)}(0) = \boldsymbol{0}, \quad \boldsymbol{\psi}^{(m)}(0) = \dot{\boldsymbol{\psi}}^{(m)}(0) = \boldsymbol{0}.
\end{aligned} \tag{33}$$

The theory of linear fODE systems[30] ensures that system (33) admits a unique continuous solution $\{\boldsymbol{u}^{(m)}(t), \boldsymbol{\psi}^{(m)}(t)\}$. Hence, the Galerkin formulation (29) admits the unique Galerkin approximation $u^{(m)}, \psi^{(m)} \in C([0,T]; H_0^1)$ in the form of (28).

Step 2. A stability estimate of the Galerkin approximation

We set $\phi = u^{(m)}$ in the first equation and $\phi = \psi^{(m)}$ in the second equation in (29) to obtain

$$\begin{aligned}
&\rho\Big(A\partial_t u^{(m)}(\cdot,t), u^{(m)}(\cdot,t)\Big) + \kappa G\Big(A\,{}_0I_t^{1-\alpha}\partial_x u^{(m)}(\cdot,t), \partial_x u^{(m)}(\cdot,t)\Big)\\
&\quad = \kappa G\Big(A\,{}_0I_t^{1-\alpha}\psi^{(m)}(\cdot,t), \partial_x u^{(m)}(\cdot,t)\Big) + \Big(f(\cdot,t), u^{(m)}(\cdot,t)\Big),\\
&\rho\Big(I\partial_t\psi^{(m)}(\cdot,t), \psi^{(m)}(\cdot,t)\Big) + E_\alpha\Big(I\,{}_0I_t^{1-\alpha}\partial_x\psi^{(m)}(\cdot,t), \partial_x\psi^{(m)}(\cdot,t)\Big) + \kappa G\Big(A\,{}_0I_t^{1-\alpha}\psi^{(m)}(\cdot,t), \psi^{(m)}(\cdot,t)\Big)\\
&\quad = \kappa G\Big(A\,{}_0I_t^{1-\alpha}\partial_x u^{(m)}(\cdot,t), \psi^{(m)}(\cdot,t)\Big) + \Big(g(\cdot,t), \psi^{(m)}(\cdot,t)\Big).
\end{aligned} \tag{34}$$

We integrate the two equations from 0 to t and sum the resulting equations and use Lemmas III.1 and III.2 to obtain

$$\begin{aligned}
&\frac{\rho A_*}{2}\left\|u^{(m)}(\cdot,t)\right\|^2 + \kappa G A_*\cos\left(\frac{(1-\alpha)\pi}{2}\right)\left\|{}_0I_t^{\frac{1-\alpha}{2}}\partial_x u^{(m)}\right\|^2_{L^2(0,t;L^2)}\\
&\le \frac{\rho}{2}\Big(Au^{(m)}(\cdot,t), u^{(m)}(\cdot,t)\Big) + \kappa G\int_0^t\Big(A\,{}_0I_s^{\frac{1-\alpha}{2}}\partial_x u^{(m)}, {}_sI_t^{\frac{1-\alpha}{2}}\partial_x u^{(m)}\Big)ds\\
&\le \frac{\kappa G A_*}{4}\cos\left(\frac{(1-\alpha)\pi}{2}\right)\left\|{}_0I_t^{\frac{1-\alpha}{2}}\partial_x u^{(m)}\right\|^2_{L^2(0,t;L^2)}\\
&\quad + Q\left\|{}_0I_t^{\frac{1-\alpha}{2}}\psi^{(m)}\right\|^2_{L^2(0,t;L^2)} + \frac{1}{2}\left(\left\|u^{(m)}\right\|^2_{L^2(0,t;L^2)} + \|f\|^2_{L^2(0,t;L^2)}\right),\\
&\frac{\rho A_*}{2}\left\|\psi^{(m)}(\cdot,t)\right\|^2 + E_\alpha A_*\cos\left(\frac{(1-\alpha)\pi}{2}\right)\left\|{}_0I_t^{\frac{1-\alpha}{2}}\partial_x\psi^{(m)}\right\|^2_{L^2(0,t;L^2)}\\
&\quad + \kappa G A_*\cos\left(\frac{(1-\alpha)\pi}{2}\right)\left\|{}_0I_t^{\frac{1-\alpha}{2}}\psi^{(m)}\right\|^2_{L^2(0,t;L^2)}\\
&\le \frac{\rho}{2}\Big(I\psi^{(m)}(\cdot,t), \psi^{(m)}(\cdot,t)\Big) + E_\alpha\int_0^t\Big(I\,{}_0I_t^{\frac{1-\alpha}{2}}\partial_x\psi^{(m)}, {}_sI_t^{\frac{1-\alpha}{2}}\partial_x\psi^{(m)}\Big)ds\\
&\quad + \kappa G\int_0^t\Big(A\,{}_0I_t^{\frac{1-\alpha}{2}}\psi^{(m)}, {}_sI_t^{\frac{1-\alpha}{2}}\psi^{(m)}\Big)ds\\
&\le \frac{\kappa G A_*}{4}\cos\left(\frac{(1-\alpha)\pi}{2}\right)\left\|{}_0I_t^{\frac{1-\alpha}{2}}\partial_x u^{(m)}\right\|^2_{L^2(0,t;L^2)}\\
&\quad + Q\left\|{}_0I_t^{\frac{1-\alpha}{2}}\psi^{(m)}\right\|^2_{L^2(0,t;L^2)} + \frac{1}{2}\left(\|\psi^{(m)}\|^2_{L^2(0,t;L^2)} + \|g\|^2_{L^2(0,t;L^2)}\right).
\end{aligned} \tag{35}$$

We cancel the like terms in inequalities (35), add the resulting inequalities, and use the mapping property of ${}_0I_t^{\frac{1-\alpha}{2}}$ in Lemma III.1 to find

$$\begin{aligned}
&\rho A_*\left(\left\|u^{(m)}(\cdot,t)\right\|^2 + \left\|\psi^{(m)}(\cdot,t)\right\|^2\right) + A_*\cos\left(\frac{(1-\alpha)\pi}{2}\right)\left(\kappa G\left\|{}_0I_t^{\frac{1-\alpha}{2}}\partial_x u^{(m)}\right\|^2_{L^2(0,t;L^2)}\right.\\
&\quad\left. + E_\alpha\left\|{}_0I_t^{\frac{1-\alpha}{2}}\partial_x\psi^{(m)}\right\|^2_{L^2(0,t;L^2)} + \kappa G\left\|{}_0I_t^{\frac{1-\alpha}{2}}\psi^{(m)}\right\|^2_{L^2(0,t;L^2)}\right)\\
&\le Q\int_0^t\left(\left\|u^{(m)}(\cdot,s)\right\|^2 + \left\|\psi^{(m)}(\cdot,s)\right\|^2\right)ds + Q\left(\|f\|^2_{L^2(0,t;L^2)} + \|g\|^2_{L^2(0,t;L^2)}\right).
\end{aligned} \tag{36}$$

We drop the last three terms on the left-hand side of (36) and apply Gronwall's inequality to obtain

$$\|u^{(m)}\|_{C([0,T];L^2)} + \|\psi^{(m)}\|_{C([0,T];L^2)} \le Q\big(\|f\|_{L^2(0,T;L^2)} + \|g\|_{L^2(0,T;L^2)}\big). \tag{37}$$

We incorporate estimate (37) into the right-hand side of (36) and drop the first two terms on the left-hand side of (36) to arrive at

$$\begin{aligned}&\Big\|{}_0I_t^{\frac{1-\alpha}{2}}\partial_x u^{(m)}\Big\|_{L^2(0,T;L^2)} + \Big\|{}_0I_t^{\frac{1-\alpha}{2}}\partial_x \psi^{(m)}\Big\|_{L^2(0,T;L^2)}\\ &\quad\le Q\big(\|f\|_{L^2(0,t;L^2)} + \|g\|_{L^2(0,t;L^2)}\big).\end{aligned} \tag{38}$$

Step 3. A stability estimate of the Galerkin approximation $u^{(m)}$ and $\psi^{(m)}$ in the energy norm

The proof of estimate (25) requires a stability estimate of $u^{(m)}$ and $\psi^{(m)}$ in the energy norm. Following the derivation of a corresponding estimate of the integer-order analog of problem (29), one would set the test function $\phi = \partial_t u^{(m)}$ in the first equation and $\phi = \partial_t\psi^{(m)}$ in the second equation in (29) and add the two equations so that the left-hand side of the resulting equation becomes

$$\begin{aligned}&\rho\Big(A\partial_t u^{(m)}(\cdot,t),\partial_t u^{(m)}(\cdot,t)\Big) + \kappa G\Big(A\,{}_0I_t^{1-\alpha}\partial_x u^{(m)},\partial_x\partial_t u^{(m)}(\cdot,t)\Big)\\ &+\rho\Big(I\partial_t\psi^{(m)}(\cdot,t),\partial_t\psi^{(m)}(\cdot,t)\Big) + E\Big(I\,{}_0I_t^{1-\alpha}\partial_x\psi^{(m)},\partial_x\partial_t\psi^{(m)}(\cdot,t)\Big)\\ &+\kappa G\Big(A\,{}_0I_t^{1-\alpha}\psi^{(m)},\partial_t\psi^{(m)}(\cdot,t)\Big).\end{aligned} \tag{39}$$

Due to the presence of the fractional integral operator ${}_0I_t^{1-\alpha}$, the second and fourth terms in (39) cannot be expressed as the derivatives of the corresponding energy norm squares of $u^{(m)}$ and $\psi^{(m)}$ to ensure the coercivity in the energy norms as in its integer-order analog. We adopt an alternative approach by differentiating Eq. (29) in time. We, then, apply (16), (32), and the homogeneous initial conditions (20) to assert

$$\begin{aligned}&\rho\Big(A\partial_t^2 u^{(m)},\phi\Big) + \kappa G\Big(A\,{}_0I_t^{1-\alpha}\partial_x\partial_t u^{(m)},\partial_x\phi\Big) = \kappa G\Big(A\,{}_0I_t^{1-\alpha}\partial_t\psi^{(m)},\partial_x\phi\Big) + (\partial_t f,\phi),\\ &\rho\Big(I\partial_t^2\psi^{(m)},\phi\Big) + E_\alpha\Big(I\,{}_0I_t^{1-\alpha}\partial_x\partial_t\psi^{(m)},\partial_x\phi\Big) + \kappa G\Big(A\,{}_0I_t^{1-\alpha}\partial_t\psi^{(m)},\phi\Big) = \kappa G\Big(A\,{}_0I_t^{1-\alpha}\partial_x\partial_t u^{(m)},\phi\Big) + (\partial_t g,\phi),\\ &u^{(m)}(x,0) = \partial_t u^{(m)}(x,0) = \psi^{(m)}(x,0) = \partial_t\psi^{(m)}(x,0) = 0.\end{aligned} \tag{40}$$

Equations (40) are in the same form as (29) with $u^{(m)}$, $\psi^{(m)}$, f, and g replaced by $\partial_t u^{(m)}$, $\partial_t\psi^{(m)}$, $\partial_t f$, and $\partial_t g$, respectively. Then, similar estimates such as (37) and (38) yield

$$\begin{aligned}&\Big\|\partial_t u^{(m)}\Big\|_{L^\infty(0,T;L^2)} + \Big\|\partial_t\psi^{(m)}\Big\|_{L^\infty(0,T;L^2)} + \Big\|{}_0I_t^{\frac{1-\alpha}{2}}\partial_t\partial_x u^{(m)}\Big\|_{L^2(0,T;L^2)} + \Big\|{}_0I_t^{\frac{1-\alpha}{2}}\partial_t\partial_x\psi^{(m)}\Big\|_{L^2(0,T;L^2)}\\ &\le Q\Big(\big\|\partial_t f\big\|_{L^2(0,T;L^2)} + \big\|\partial_t g\big\|_{L^2(0,T;L^2)}\Big).\end{aligned} \tag{41}$$

Apply $\partial_t\,{}_0I_t^\alpha$ to Eq. (29) and use (16) with $f(x,0) = g(x,0) = 0$ to find

$$\begin{aligned}&\kappa G\Big(A\partial_x u^{(m)},\partial_x\phi\Big) = -\rho\Big(A\partial_t^{2-\alpha}u^{(m)},\phi\Big) + \kappa G\Big(A\psi^{(m)},\partial_x\phi\Big) + \Big(\partial_t^{1-\alpha}f,\phi\Big),\\ &E_\alpha\Big(I\partial_x\psi^{(m)},\partial_x\phi\Big) + \kappa G\Big(A\psi^{(m)},\phi\Big) = -\rho\Big(I\partial_t^{2-\alpha}\psi^{(m)},\phi\Big) + \kappa G\Big(A\partial_x u^{(m)},\phi\Big) + \Big(\partial_t^{1-\alpha}g,\phi\Big).\end{aligned} \tag{42}$$

We use Lemma III.1 to integrate the first term [excluding $A(x)$] on the right-hand side of the first equation in (42) with $\phi = u^{(m)}$ to obtain

$$\begin{aligned}&\bigg|\int_0^T {}_0I_t^\alpha\Big(\partial_s^2 u^{(m)}(x,s)\Big)u^{(m)}(x,t)dt\bigg| = \bigg|\int_0^T {}_0I_t^{\frac{\alpha}{2}}\Big(\partial_s^2 u^{(m)}(x,s)\Big){}_tI_T^{\frac{\alpha}{2}}\Big(u^{(m)}(x,s)\Big)dt\bigg|\\ &= \bigg|\int_0^T \partial_t\,{}_0I_t^{\frac{\alpha}{2}}\Big(\partial_s u^{(m)}(x,s)\Big){}_tI_T^{\frac{\alpha}{2}}\Big(u^{(m)}(x,s)\Big)dt\bigg| = \bigg|\int_0^T {}_0I_t^{\frac{\alpha}{2}}\Big(\partial_s u^{(m)}(x,s)\Big)\partial_t\Big[{}_tI_T^{\frac{\alpha}{2}}\Big(u^{(m)}(x,s)\Big)\Big]dt\bigg|.\end{aligned} \tag{43}$$

Since $u^{(m)}(x,T)$ might not vanish, the operators ∂_t and ${}_tI_T^{\frac{\alpha}{2}}$ do not commute. We use (41) to bound the integrands in (43) as follows:

$$\begin{aligned}\bigg|\partial_t\Big[{}_tI_T^{\frac{\alpha}{2}}\Big(u^{(m)}(x,s)\Big)\Big]\bigg| &= \bigg|\partial_t\int_t^T \frac{u^{(m)}(x,s)}{\Gamma(\frac{\alpha}{2})(s-t)^{1-\frac{\alpha}{2}}}ds\bigg|\\ &= \bigg|\partial_t\bigg(\frac{(T-t)^{\frac{\alpha}{2}}u^{(m)}(x,T)}{\Gamma(\frac{\alpha}{2}+1)} - \int_t^T \frac{(s-t)^{\frac{\alpha}{2}}}{\Gamma(\frac{\alpha}{2}+1)}\partial_s u^{(m)}(x,s)ds\bigg)\bigg|\\ &\le Q\bigg[(T-t)^{\frac{\alpha}{2}-1}\Big|u^{(m)}(x,T)\Big| + \int_t^T (s-t)^{\frac{\alpha}{2}-1}\Big|\partial_s u^{(m)}(x,s)\Big|ds\bigg],\end{aligned}$$

$$\left\|{}_0I_t^{\frac{\alpha}{2}}\left(\partial_s u^{(m)}(\cdot,s)\right)\right\|^2 = \int_0^l\left[\int_0^t \frac{(t-s)^{(\frac{\alpha}{4}-\frac{1}{2})}(t-s)^{(\frac{\alpha}{4}-\frac{1}{2})}}{\Gamma(\frac{\alpha}{2})}\partial_s u^{(m)}(\cdot,s)ds\right]^2 dx$$
$$\le Q\int_0^l\int_0^t (t-s)^{(\frac{\alpha}{2}-1)}\left(\partial_s u^{(m)}(x,s)\right)^2 ds dx \le Q\int_0^t (t-s)^{(\frac{\alpha}{2}-1)}\left\|\partial_s u^{(m)}(\cdot,s)\right\|^2 ds \le Q\left\|\partial_t u^{(m)}\right\|^2_{L^\infty(0,T;L^2)}.$$

We use the above two estimates, (41), and Cauchy inequality to bound (43) by

$$\left|\int_0^l\int_0^T {}_0I_t^\alpha\left(\partial_s^2 u^{(m)}(x,s)\right)u^{(m)}(x,t)dtdx\right| \le \int_0^T\left\|{}_0I_t^{\frac{\alpha}{2}}\partial_s u^{(m)}(\cdot,s)\right\|\,\left\|\partial_t\left[{}_tI_T^{\frac{\alpha}{2}}\left(u^{(m)}(x,s)\right)\right]\right\|ds$$
$$\le Q\left(\|f\|_{H^1(0,T;L^2)} + \|g\|_{H^1(0,T;L^2)}\right)^2\int_0^T 1 + (T-s)^{\frac{\alpha}{2}-1}ds \le Q\left(\|f\|_{H^1(0,T;L^2)} + \|g\|_{H^1(0,T;L^2)}\right)^2.$$

Similar estimates hold with $u^{(m)}$ replaced by $\psi^{(m)}$. We set $\phi = u^{(m)}$ or $\psi^{(m)}$ in the first and second equations in (42), respectively, and integrate the resulting equations from 0 to T and invoke these estimates and (41) to conclude that

$$\kappa GA_*\left\|\partial_x u^{(m)}\right\|^2_{L^2(0,T;L^2)} + E_\alpha A_*\left\|\partial_x\psi^{(m)}\right\|^2_{L^2(0,T;L^2)}$$
$$\le \frac{\kappa GA_*}{2}\left\|\partial_x u^{(m)}\right\|^2_{L^2(0,T;L^2)} + \frac{E_\alpha A_*}{2}\left\|\partial_x\psi^{(m)}\right\|^2_{L^2(0,T;L^2)} + Q\left(\|f\|_{H^1(0,T;L^2)} + \|g\|_{H^1(0,T;L^2)}\right)^2.$$

Canceling the like terms and adding the two estimates together yield

$$\left\|u^{(m)}\right\|_{L^2(0,T;H^1)} + \left\|\psi^{(m)}\right\|_{L^2(0,T;H^1)} \le Q\left(\|f\|_{H^1(0,T;L^2)} + \|g\|_{H^1(0,T;L^2)}\right). \tag{44}$$

Step 4. Well-posedness of (22).

By estimates (41) and (44) and the fact that fractional integral operators are bounded linear operators from $L^2 \hookrightarrow L^2$, there exist subsequences $\{u^{(m_l)}\}_{l=1}^\infty$ and $\{\psi^{(m_l)}\}_{l=1}^\infty$ and functions $u,\psi \in L^2(0,T;H_0^1)$ with $\partial_t u, \partial_t\psi \in L^2(0,T;\,H^{-1})$ such that $u^{(m_l)}$ and $\psi^{(m_l)}$ converge weakly to u and ψ, ${}_0I_t^{1-\alpha}u^{(m_l)}$ and ${}_0I_t^{1-\alpha}\psi^{(m_l)}$ converge weakly to ${}_0I_t^{1-\alpha}u$ and ${}_0I_t^{1-\alpha}\psi$, respectively, in $L^2(0,T;H_0^1)$, and $\partial_t u^{(m_l)}$ and $\partial_t\psi^{(m_l)}$ converge weakly to $\partial_t u$ and $\partial_t\psi$, respectively, in $L^2(0,T;\,H^{-1})$. We note that Eqs. (29) hold for any

$$v^{(n)}(x,t) = \sum_{j=1}^n v_j^{(n)}(t)\phi_j(x) \in C^1([0,T];S_n), \tag{45}$$

with $\{\phi_j(x)\}_{j=1}^n \subset S_n$. We integrate Eqs. (29) with respect to time t from 0 to T to find that for $m_l \ge n$,

$$\begin{aligned}&\int_0^T \rho\left(A\partial_t u^{(m_l)}, v^{(n)}\right) + \kappa G\left(A\,{}_0I_t^{1-\alpha}\partial_x u^{(m_l)}, \partial_x v^{(n)}\right)dt\\&\quad= \int_0^T \kappa G\left(A\,{}_0I_t^{1-\alpha}\psi^{(m_l)}, \partial_x v^{(n)}\right) + \left(f, v^{(n)}\right)dt,\\&\int_0^T \rho\left(I\partial_t\psi^{(m_l)}, v^{(n)}\right) + E_\alpha\left(I\,{}_0I_t^{1-\alpha}\partial_x\psi^{(m_l)}, \partial_x v^{(n)}\right) + \kappa G\left(A\,{}_0I_t^{1-\alpha}\psi^{(m_l)}, v^{(n)}\right)dt\\&\quad= \int_0^T \kappa G\left(A\,{}_0I_t^{1-\alpha}\partial_x u^{(m_l)}, v^{(n)}\right) + \left(g, v^{(n)}\right)dt.\end{aligned} \tag{46}$$

We take the limit of Eqs. (46) as $l \to \infty$ to deduce

$$\begin{aligned}&\int_0^T \rho\left(A\partial_t u, v^{(n)}\right) + \kappa G\left(A\,{}_0I_t^{1-\alpha}\partial_x u, \partial_x v^{(n)}\right)dt = \int_0^T \kappa G\left(A\,{}_0I_t^{1-\alpha}\psi, \partial_x v^{(n)}\right) + \left(f, v^{(n)}\right)dt,\\&\int_0^T \rho\left(I\partial_t\psi, v^{(n)}\right) + E_\alpha\left(I\,{}_0I_t^{1-\alpha}\partial_x\psi, \partial_x v^{(n)}\right) + \kappa G\left(A\,{}_0I_t^{1-\alpha}\psi, v^{(n)}\right)dt\\&\quad= \int_0^T \kappa G\left(A\,{}_0I_t^{1-\alpha}\partial_x u, v^{(n)}\right) + \left(g, v^{(n)}\right)dt.\end{aligned} \tag{47}$$

Note that the functions $v^{(n)}$ of the form (45) are dense in $L^2(0,T;H_0^1)$. Hence, Eqs. (47) yield that for any $v \in L^2(0,T;H_0^1)$,

$$\begin{aligned}&\int_0^T \rho(A\partial_t u, v) + \kappa G\left(A\,{}_0I_t^{1-\alpha}\partial_x u, \partial_x v\right)dt = \int_0^T \kappa G\left(A\,{}_0I_t^{1-\alpha}\psi, \partial_x v\right) + (f,v)dt,\\&\int_0^T \rho(I\partial_t\psi, v) + E_\alpha\left(I\,{}_0I_t^{1-\alpha}\partial_x\psi, \partial_x v\right) + \kappa G\left(A\,{}_0I_t^{1-\alpha}\psi, v\right)dt = \int_0^T \kappa G\left(A\,{}_0I_t^{1-\alpha}\partial_x u, v\right) + (g,v)dt.\end{aligned} \tag{48}$$

Equations (48) reduce to

$$\rho(A\partial_t u,\phi) + \kappa G\big(A\,{}_0I_t^{1-\alpha}\partial_x u,\partial_x\phi\big) = \kappa G\big(A\,{}_0I_t^{1-\alpha}\psi,\partial_x\phi\big) + (f,\phi),$$
$$\rho(I\partial_t\psi,\phi)+E_\alpha\big(I\,{}_0I_t^{1-\alpha}\partial_x\psi,\partial_x\phi\big)+\kappa G\big(A\,{}_0I_t^{1-\alpha}\psi,\phi\big) = \kappa G\big(A\,{}_0I_t^{1-\alpha}\partial_x u,\phi\big) + (g,\phi),$$

for any $\phi\in H_0^1(0,l)$ and a.e. $t\in(0,T]$.

To show $u(x,0)=\psi(x,0)=0$, we integrate Eqs. (48) with any $v\in C^1([0,T];H_0^1)$ with $v(x,T)=0$ by parts to assert that

$$\begin{aligned}&-\rho(Au(\cdot,0),v(\cdot,0)) - \int_0^T\rho(Au,\partial_t v)dt + \int_0^T\kappa G\big(A\,{}_0I_t^{1-\alpha}\partial_x u,\partial_x v\big)dt\\&\quad= \int_0^T\kappa G\big(A\,{}_0I_t^{1-\alpha}\psi,\partial_x v\big) + (f,v)dt,\\&-\rho(I\psi(\cdot,0),v(\cdot,0)) - \int_0^T\rho(I\psi,\partial_t v)dt + \int_0^T E_\alpha\big(I\,{}_0I_t^{1-\alpha}\partial_x\psi,\partial_x v\big)dt\\&\qquad+\int_0^T\kappa G\big(A\,{}_0I_t^{1-\alpha}\psi,v\big)dt = \int_0^T\kappa G\big(A\,{}_0I_t^{1-\alpha}\partial_x u,v\big) + (g,v)dt.\end{aligned}\tag{49}$$

We integrate Eqs. (46) by parts and enforce the homogeneous initial conditions $u^{(m_l)}(x,0)=\psi^{(m_l)}(x,0)=0$ to obtain

$$\begin{aligned}&-\int_0^T\rho\Big(Au^{(m_l)},\partial_t v^{(n)}\Big)dt + \int_0^T\kappa G\Big(A\,{}_0I_t^{1-\alpha}\partial_x u^{(m_l)},\partial_x v^{(n)}\Big)dt\\&\quad= \int_0^T\kappa G\Big(A\,{}_0I_t^{1-\alpha}\psi^{(m_l)},\partial_x v^{(n)}\Big) + \Big(f,v^{(n)}\Big)dt,\\&-\int_0^T\rho\Big(I\psi^{(m_l)},\partial_t v^{(n)}\Big)dt + \int_0^T E_\alpha\Big(I\,{}_0I_t^{1-\alpha}\partial_x\psi^{(m_l)},\partial_x v^{(n)}\Big)dt\\&\qquad+\int_0^T\kappa G\Big(A\,{}_0I_t^{1-\alpha}\psi^{(m_l)},v^{(n)}\Big)dt = \int_0^T\kappa G\Big(A\,{}_0I_t^{1-\alpha}\partial_x u^{(m_l)},v^{(n)}\Big) + \Big(g,v^{(n)}\Big)dt.\end{aligned}\tag{50}$$

With the help of the same procedure leading to (48), we pass the limit of Eqs. (50) as $l\to\infty$ to obtain

$$\begin{aligned}&-\int_0^T\rho(Au,\partial_t v)dt + \int_0^T\kappa G\big(A\,{}_0I_t^{1-\alpha}\partial_x u,\partial_x v\big)dt = \int_0^T\kappa G\big(A\,{}_0I_t^{1-\alpha}\psi,\partial_x v\big) + (f,v)dt,\\&-\int_0^T\rho(I\psi,\partial_t v)dt + \int_0^T E_\alpha\big(I\,{}_0I_t^{1-\alpha}\partial_x\psi,\partial_x v\big) + \kappa G\big(A\,{}_0I_t^{1-\alpha}\psi,v\big)dt\\&\quad= \int_0^T\kappa G\big(A\,{}_0I_t^{1-\alpha}\partial_x u,v\big) + (g,v)dt.\end{aligned}\tag{51}$$

Subtracting Eqs. (51) from Eqs. (49) gives

$$(A(\cdot)u(\cdot,0),v(\cdot,0)) = 0,\quad (I(\cdot)\psi(\cdot,0),v(\cdot,0)) = 0.\tag{52}$$

Since $v(\cdot,0)\in H_0^1(0,l)$ is arbitrary, Eqs. (52) yield $u(x,0)=\psi(x,0)=0$. Thus, u and ψ are weak solutions of problems (22), (19), and (24).

By (41) and (44), $\{u^{(m_l)}\}_{l=1}^\infty$ and $\{\psi^{(m_l)}\}_{l=1}^\infty$ are bounded in $L^2(0,T;H_0^1)$, and $\{\partial_t u^{(m_l)}\}_{l=1}^\infty$ and $\{\partial_t\psi^{(m_l)}\}_{l=1}^\infty$ are bounded in $L^\infty(0,T;\,L^2)$. Consequently, passing the limit as in Ref. 24 leads to

$$\begin{aligned}&\|\partial_t u\|_{L^\infty(0,T;L^2)} + \|\partial_t\psi\|_{L^\infty(0,T;L^2)} \le Q\Big(\|\partial_t f\|_{L^2(0,T;L^2)} + \|\partial_t g\|_{L^2(0,T;L^2)}\Big),\\&\|u\|_{L^2(0,T;H^1)} + \|\psi\|_{L^2(0,T;H^1)} \le Q\Big(\|\partial_t f\|_{L^2(0,T;L^2)} + \|\partial_t g\|_{L^2(0,T;L^2)}\Big).\end{aligned}\tag{53}$$

We combine estimates (53) with the estimates of f and g,

$$\|f\|_{H^1(0,T;L^2)} \le \|q\|_{L^2(0,T;L^2)} + Q\|\breve{w}_0\|_{H^2} + Q\|\breve{\theta}_0\|_{H^1},\quad \|g\|_{H^1(0,T;L^2)} \le Q\|\breve{\theta}_0\|_{H^2} + Q\|\breve{w}_0\|_{H^1},$$

to complete the proof of estimate (25). We, then, combine estimate (25) with decomposition (17) to finish the proof of estimate (26). □

V. AN ENHANCED REGULARITY ESTIMATE

In this section, we prove the following theorem.

Theorem V.1. *If $\breve{w}_0,\breve{\theta}_0\in H^2$, $q\in H^1(0,T;\,L^2)$, and $A,I\in H^1$ with $A_*\le A(x),I(x)\le A^*$, then the solutions u,ψ of problems (22), (19), and (24) belong to $W^{2,\infty}(0,T;\,L^2)\cap L^2(0,T;\,H^2)$ with the stability estimate*

$$\|u\|_{W^{2,\infty}(0,T;L^2)} + \|\psi\|_{W^{2,\infty}(0,T;L^2)} + \|u\|_{L^2(0,T;H^2)} + \|\psi\|_{L^2(0,T;H^2)} \le Q\big(\|\check{w}_0\|_{H^2} + \|\check{\theta}_0\|_{H^2} + \|q\|_{H^1(0,T;L^2)}\big). \tag{54}$$

Here, $Q = Q(\rho, A_*, A^*, \|A\|_{H^1}, \|I\|_{H^1}, \kappa, G, E_\alpha, \alpha, T)$. *Furthermore, if* $w_0, \theta_0 \in H^2$, *then the solutions* w, θ *of problems (12)–(14) belong to* $\in W^{2,\infty}(0,T;\, L^2) \cap L^2(0,T;\, H^2)$ *and*

$$\begin{aligned}&\|w\|_{W^{2,\infty}(0,T;L^2)} + \|\theta\|_{W^{2,\infty}(0,T;L^2)} + \|w\|_{L^2(0,T;H^2)} + \|\theta\|_{L^2(0,T;H^2)}\\ &\quad\le Q\big(\|w_0\|_{H^2} + \|\theta_0\|_{H^2} + \|\check{w}_0\|_{H^2} + \|\check{\theta}_0\|_{H^2} + \|q\|_{H^1(0,T;L^2)}\big).\end{aligned} \tag{55}$$

Proof. We differentiate Eqs. (29) in time to obtain

$$\begin{aligned}&\rho\Big(A\partial_t^2 u^{(m)}, \phi\Big) + \kappa G\Big(A\,\partial_t^\alpha \partial_x u^{(m)}, \partial_x\phi\Big) = \kappa G\Big(A\,\partial_t^\alpha \psi^{(m)}, \partial_x\phi\Big) + (\partial_t f, \phi),\\ &\rho\Big(I\partial_t^2 \psi^{(m)}, \phi\Big) + E_\alpha\Big(I\,\partial_t^\alpha \partial_x \psi^{(m)}, \partial_x\phi\Big) + \kappa G\Big(A\,\partial_t^\alpha \psi^{(m)}, \phi\Big) = \kappa G\Big(A\,\partial_t^\alpha \partial_x u^{(m)}, \phi\Big) + (\partial_t g, \phi), \quad \forall \phi \in S_m,\end{aligned} \tag{56}$$

with S_m defined in (27), and we differentiate (56) again in time to obtain

$$\begin{aligned}&\rho\Big(A\partial_t^3 u^{(m)}, \phi\Big) + \kappa G\Big(A\,\partial_t^{1+\alpha} \partial_x u^{(m)}, \partial_x\phi\Big) = \kappa G\Big(A\,\partial_t^{1+\alpha} \psi^{(m)}, \partial_x\phi\Big) + (\partial_t^2 f, \phi),\\ &\rho\Big(I\partial_t^3 \psi^{(m)}, \phi\Big) + E_\alpha\Big(I\,\partial_t^{1+\alpha} \partial_x \psi^{(m)}, \partial_x\phi\Big) + \kappa G\Big(A\,\partial_t^{1+\alpha} \psi^{(m)}, \phi\Big)\\ &\quad= \kappa G\Big(A\,\partial_t^{1+\alpha} \partial_x u^{(m)}, \phi\Big) + (\partial_t^2 g, \phi), \quad \forall \phi \in S_m.\end{aligned} \tag{57}$$

Let $t \to 0$ in (56) and use (23) to arrive at the initial conditions

$$\partial_t^2 u^{(m)}(x,0) = \frac{q(x,0)}{\rho A}, \qquad \partial_t^2 \psi^{(m)}(x,0) = 0. \tag{58}$$

Note that Eqs. (57) with the initial conditions (58) are of the same form as (29) with $u^{(m)}$, $\psi^{(m)}$, f, and g replaced by $\partial_t^2 u^{(m)}$, $\partial_t^2 \psi^{(m)}$, $\partial_t^2 f$, and $\partial_t^2 g$, respectively. We adopt a similar approach to the proofs of Theorem IV.1 by choosing $\phi = \partial_t^2 u^{(m)}$ and $\phi = \partial_t^2 \psi^{(m)}$ in the first and second equations of Eqs. (57), respectively, and integrate them in time from 0 to t to derive similar estimates to (34)–(37) in terms of $\partial_t^2 u^{(m)}$ and $\partial_t^2 \psi^{(m)}$. To achieve this goal, we bound the terms on the right-hand side of (57) as follows:

$$\begin{aligned}&\|\partial_t^2 f\| \le \|\partial_t q\| + Qt^{-\alpha}\big(\|\check{w}_0\|_{H^2} + \|\check{\theta}_0\|_{H^1}\big), \quad \|\partial_t^2 g\| \le Qt^{-\alpha}\big(\|\check{w}_0\|_{H^1} + \|\check{\theta}_0\|_{H^2}\big),\\ &\left|\int_0^t \Big(\partial_s^2 f(\cdot,s), \partial_s^2 u^{(m)}(\cdot,s)\Big)ds + \int_0^t \Big(\partial_s^2 g(\cdot,s), \partial_s^2 \psi^{(m)}(\cdot,s)\Big)ds\right|\\ &\quad\le \int_0^t Q\big(\|\check{w}_0\|_{H^2} + \|\check{\theta}_0\|_{H^2}\big)s^{-\alpha}\Big(\Big\|\partial_s^2 u^{(m)}(\cdot,s)\Big\| + \Big\|\partial_s^2 \psi^{(m)}(\cdot,s)\Big\|\Big)ds\\ &\qquad+\Big(\|q\|^2_{H^1(0,T;L^2)} + \Big\|\partial_s^2 u^{(m)}\Big\|^2_{L^2(0,t;L^2)}\Big)\\ &\quad\le Q\Big(\|\check{w}_0\|^2_{H^2} + \|\check{\theta}_0\|^2_{H^2} + \|q\|^2_{H^1(0,T;L^2)} + \Big\|\partial_s^2 u^{(m)}\Big\|^2_{L^2(0,t;L^2)}\Big)\\ &\qquad+\int_0^t s^{-\alpha}\Big(\Big\|\partial_s^2 u^{(m)}(\cdot,s)\Big\|^2 + \Big\|\partial_s^2 \psi^{(m)}(\cdot,s)\Big\|^2\Big)ds.\end{aligned}$$

We can similarly estimate (57) by

$$\begin{aligned}&\rho A_*\Big(\Big\|\partial_t^2 u^{(m)}(\cdot,t)\Big\|^2 + \Big\|\partial_t^2 \psi^{(m)}(\cdot,t)\Big\|^2\Big)\\ &\quad\le Q\Big(\Big\|\partial_s^2 u^{(m)}\Big\|^2_{L^2(0,t;L^2)} + \Big\|\partial_s^2 \psi^{(m)}\Big\|^2_{L^2(0,t;L^2)}\Big) + Q\big(\|\check{w}_0\|^2_{H^2} + \|\check{\theta}_0\|^2_{H^2}\\ &\qquad+ \|q\|^2_{H^1(0,T;L^2)}\big) + \int_0^t s^{-\alpha}\Big(\Big\|\partial_s^2 u^{(m)}(\cdot,s)\Big\|^2 + \Big\|\partial_s^2 \psi^{(m)}(\cdot,s)\Big\|^2\Big)ds.\end{aligned} \tag{59}$$

We further bound the last right-hand side of (59) by

$$\begin{aligned}&\int_0^t s^{-\alpha}\Big(\Big\|\partial_s^2 u^{(m)}(\cdot,s)\Big\|^2 + \Big\|\partial_s^2 \psi^{(m)}(\cdot,s)\Big\|^2\Big)ds\\ &\quad= \int_0^t s^{-\alpha}(t-s)^{-\varepsilon}(t-s)^{\varepsilon}\Big(\Big\|\partial_s^2 u^{(m)}(\cdot,s)\Big\|^2 + \Big\|\partial_s^2 \psi^{(m)}(\cdot,s)\Big\|^2\Big)ds\\ &\quad\le \max\{1,T\}\int_0^t s^{-\alpha}(t-s)^{-\varepsilon}\Big(\Big\|\partial_s^2 u^{(m)}(\cdot,s)\Big\|^2 + \Big\|\partial_s^2 \psi^{(m)}(\cdot,s)\Big\|^2\Big)ds,\end{aligned}$$

for $\varepsilon > 0$ with $\alpha + \varepsilon < 1$ and utilize Lemma III.3 with $a = Q\big(\|\check{w}_0\|_{H^2} + \|\check{\theta}_0\|_{H^2} + \|q\|_{H^1(0,T;L^2)}\big)$, $b = \max\{1, T\}$, $\vartheta = \varepsilon$, and $\iota = \alpha$ to conclude

$$\|\partial_t^2 u^{(m)}\|_{L^\infty(L^2)} + \|\partial_t^2 \psi^{(m)}\|_{L^\infty(L^2)} \le Q\big(\|\check{w}_0\|_{H^2} + \|\check{\theta}_0\|_{H^2} + \|q\|_{H^1(0,T;L^2)}\big). \tag{60}$$

We use the same procedure in step 4 of the proofs of Theorem IV.1 to pass the limit of Eqs. (56) to a subsequence $m = m_l \to \infty$ to find for any $\phi \in H_0^1(0,l)$,

$$\begin{aligned}\rho\big(A\partial_t^2 u, \phi\big) + \kappa G\big(A\,\partial_t^\alpha \partial_x u, \partial_x\phi\big) &= \kappa G\big(A\,\partial_t^\alpha \psi, \partial_x\phi\big) + (\partial_t f, \phi),\\ \rho\big(I\partial_t^2 \psi, \phi\big) + E_\alpha\big(I\,\partial_t^\alpha \partial_x \psi, \partial_x\phi\big) + \kappa G\big(A\,\partial_t^\alpha \psi, \phi\big) &= \kappa G\big(A\,\partial_t^\alpha \partial_x u, \phi\big) + (\partial_t g, \phi).\end{aligned} \tag{61}$$

We, then, pass the limit in (60) to get

$$\|\partial_t^2 u\|_{L^\infty(L^2)} + \|\partial_t^2 \psi\|_{L^\infty(L^2)} \le Q\big(\|\check{w}_0\|_{H^2} + \|\check{\theta}_0\|_{H^2} + \|q\|_{H^1(0,T;L^2)}\big). \tag{62}$$

Similarly to the derivation of (42), apply ${}_0I_t^\alpha$ to Eqs. (61) to find

$$\begin{aligned}\kappa G(A\partial_x u, \partial_x\phi) &= -\rho\big(A\,\partial_t^{2-\alpha} u, \phi\big) - \kappa G(\partial_x(A\psi), \phi) + \big(\partial_t^{1-\alpha} f, \phi\big),\\ E_\alpha(I\partial_x\psi, \partial_x\phi) + \kappa G(A\psi, \phi) &= -\rho\big(I\,\partial_t^{2-\alpha}\psi, \phi\big) + \kappa G(A\partial_x u, \phi) + \big(\partial_t^{1-\alpha} g, \phi\big),\end{aligned} \tag{63}$$

for any $\phi \in H_0^1(0,l)$ and for a.e. $0 \le t \le T$. The application of elliptic regularity theory[24] to Eqs. (63) concludes

$$\begin{aligned}\|u(\cdot,t)\|_{H^2} &\le Q\big(\|\partial_t^{2-\alpha} u(\cdot,t)\| + \|\psi(\cdot,t)\| + \|\partial_x\psi(\cdot,t)\| + \|\partial_t^{1-\alpha} f(\cdot,t)\|\big),\\ \|\psi(\cdot,t)\|_{H^2} &\le Q\big(\|\partial_t^{2-\alpha}\psi(\cdot,t)\| + \|\partial_x u(\cdot,t)\|\big) + \|\partial_t^{1-\alpha} g(\cdot,t)\|\big).\end{aligned} \tag{64}$$

We combine estimate (64) with estimate (62) and the mapping property of fractional integral operator ${}_0I_t^\alpha$ in Lemma III.1 to obtain

$$\|u\|_{L^2(0,T;H^2)} + \|\psi\|_{L^2(0,T;H^2)} \le Q\big(\|\check{w}_0\|_{H^2} + \|\check{\theta}_0\|_{H^2} + \|q\|_{H^1(0,T;L^2)}\big). \tag{65}$$

We combine estimates (62) and (65) to complete the proof of (54). We combine (54) with decomposition (17) to finish the proof of (55). □

VI. MODEL INVESTIGATION

We investigate the behavior and performance of the viscoelastic Timoshenko beam model (12) in the context of real isotropic material, in comparison with the Euler–Bernoulli beam model (2), the fractional Euler–Bernoulli beam model

$$\rho A(x)\partial_t^2 w + \partial_x^2\big(E_\alpha I(x)\partial_t^\alpha \partial_x^2 w\big) = q(x,t), \tag{66}$$

with the initial and boundary conditions in (2), and the integer-order Timoshenko model

$$\begin{aligned}\rho A(x)\partial_t^2 w &= \partial_x(\kappa G A(x)(\partial_x w - \theta)) + q(x,t),\\ \rho I(x)\partial_t^2\theta &= \partial_x(EI(x)\partial_x\theta) + \kappa G A(x)(\partial_x w - \theta),\end{aligned} \tag{67}$$

with boundary condition (13) and initial condition (14).

A central issue in the design and application of beam structures is their durability, while a key factor affecting it is resonance. Therefore, the development of a physical model that accurately predicts and describes the resonance behavior of mechanical or biological systems is of fundamental importance. Since the transient response of a beam system due to the nonzero initial deflection and velocity dies off eventually and we are interested in the dynamic durability of system structures, we focus on the study of the steady state response of the system by assuming homogeneous initial conditions.

We investigate the performance of the preceding beams that are made of a widely used superalloy, i.e., nickel chromium alloy 718 material with a density of $\rho = 8192$ kg/m^3, an elasticity modulus $E = 200$ GPa, a shear modulus of $G = 80$ GPa, and a Timoshenko shear coefficient $\kappa = 5/6$.[31] The material has superior tensile, fatigue, creep, and rupture strength and works in a wide range of temperature environments.

The beams are set to have length $l = 1$ m, width 0.1 m, and thickness $h = 0.025$ and 0.1 m, respectively. The primary natural frequencies (3) for Euler–Bernoulli beams are $\omega_1^{Euler,s} = 798$ rad/s and $\omega_1^{Euler,t} = 3191$ rad/s and for Timoshenko beams are $\omega_1^{timo,s} = 794$ rad/s and $\omega_1^{timo,t} = 2965$ rad/s, where s and t refer to slender and thick beams, respectively.[32] Note that $h_s/l = 0.025$ implies that $(\omega_1^{Euler,s} - \omega_1^{timo,s})/\omega_1^{Euler,s} = 0.5\%$, which increases to $(\omega_1^{Euler,t} - \omega_1^{timo,t})/\omega_1^{Euler,t} = 7\%$ for $h_t/l = 0.1$, where the shear deformation and rotary inertial effects are much more significant. A harmonic load $q = \cos(\omega t)\delta(x - \frac{l}{2})$ with a driving frequency $\omega = \omega_1^{Euler,s}$, $\omega_1^{Euler,t}$, $\omega_1^{timo,s}$, and $\omega_1^{timo,t}$, respectively, is imposed at the middle of the beams over a time period $T = 30$ s.

Because of the high frequency in these processes, a simulation with an insufficient temporal resolution often yields a spurious "beat" phenomenon, i.e., a motion having a rapid oscillation with slowly varying magnitude.[3] Hence, a very fine time step size $\Delta t = \frac{\pi}{1024\omega}$ is used to ensure the temporal resolution to produce physically relevant results. The Euler–Bernoulli models were simulated via cubic Hermite finite elements with the number of elements $N_{Euler} = 64$, and the Timoshenko models were simulated via linear finite elements with $N_{timo} = 2048$, with a fractional order $\alpha = 0.5$ assumed.[33,34] Deflections of the slender and thick integer-order and fractional Euler–Bernoulli beams (2) and (66) and Timoshenko beams (67) and (12) at the final time $T = 30$ s are shown in Fig. 1.

In Figs. 2 and 3, we present the time evolution of the deflection of the center of the neural beam axis. As a complete plot of the time evolution with the fine time resolution will not be presented properly, we, instead, sample the maximum and minimum alternatively every $40\frac{1}{2}$ time period. In Fig. 2, we present the deflection of the center of the slender and think integer-order and fractional Euler–Bernoulli and Timoshenko beams with the same data as in Fig. 1, and in Fig. 3, we present that for the damped thick integer-order Euler–Bernoulli beam. From Figs. 1–3, when the excitation frequency of the external harmonic loads equals their natural frequencies, (i) the slender and thick integer-order Euler–Bernoulli and Timoshenko beams predict vibrations that grow linearly in time as the single-degree mass-spring

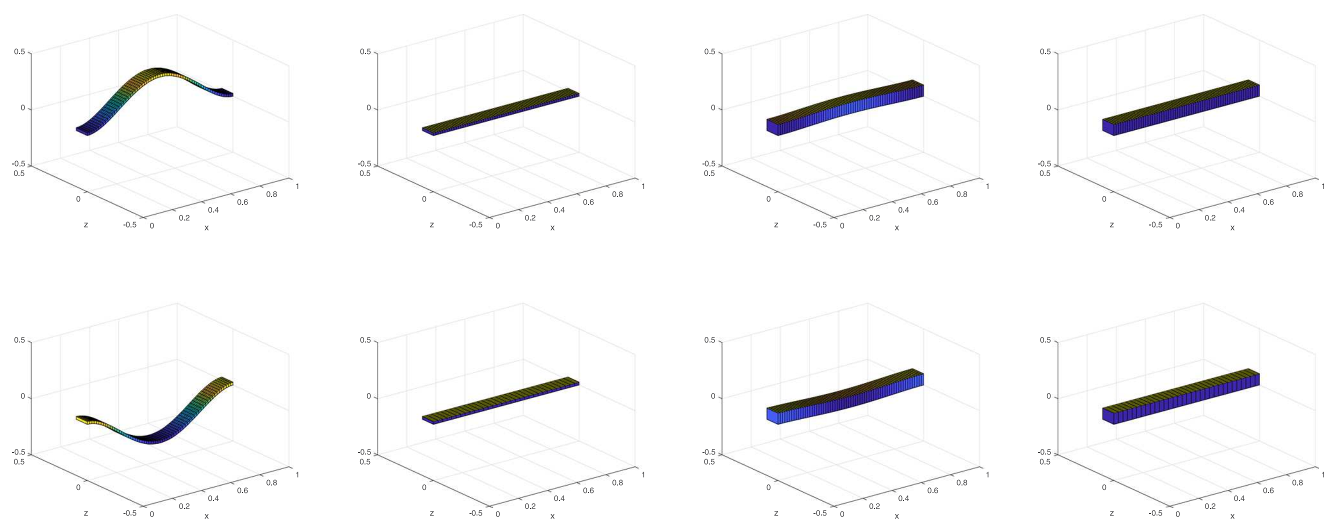

FIG. 1. Left to right: Deflection of the slender integer-order and fractional beams and the thick integer-order and fractional beams at $T = 30$ s with a vertical amplification factor $r = 150$. Row 1: Euler–Bernoulli beams. Row 2: Timoshenko beams.

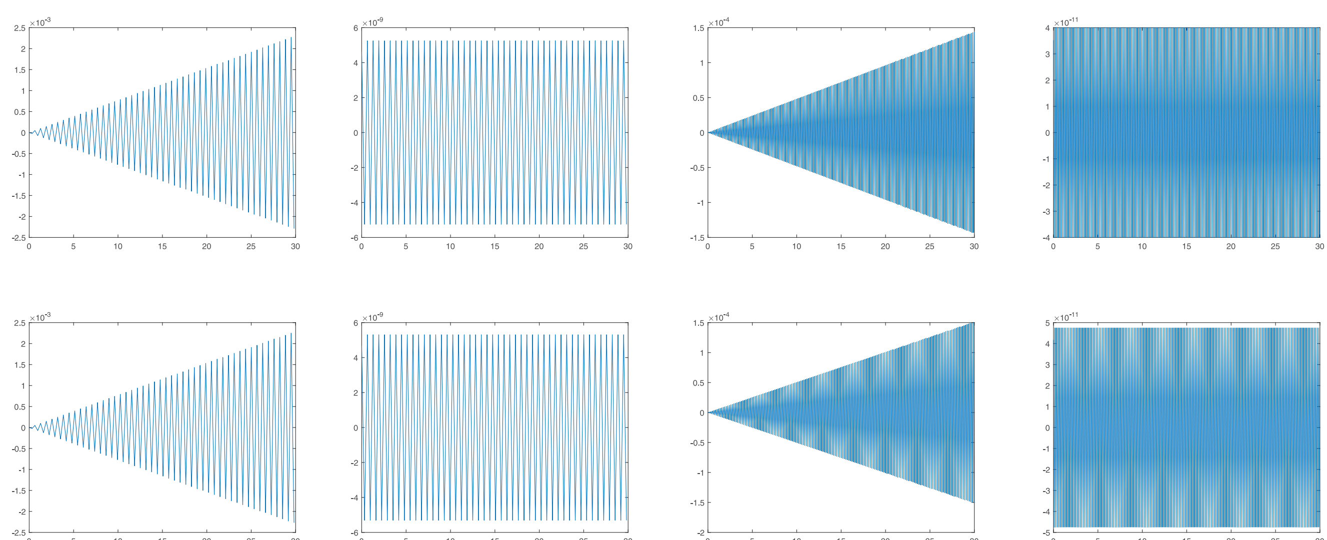

FIG. 2. Left to right: Deflection of the slender integer-order and fractional beams and the thick integer-order and fractional beams on $[0, T]$ with $T = 30$ s. Row 1: Euler–Bernoulli beams. Row 2: Timoshenko beams.

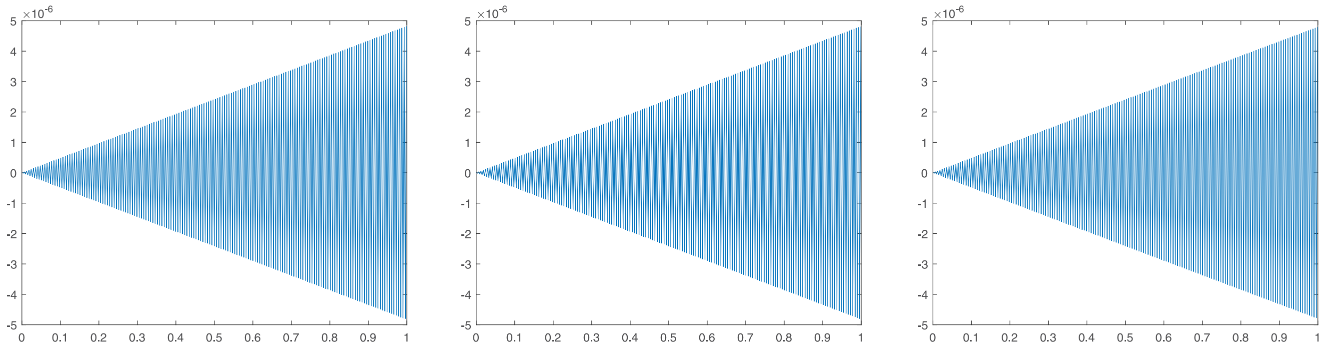

FIG. 3. Left to right: Deflection of the damped thick integer-order Euler–Bernoulli beams (2) on $[0, T]$ with $T = 1$ s, $\zeta = 0.001$, 0.005, and 0.01, respectively.

system does and so are nonphysical because they do not properly take into account for the ubiquitous damping effect; (ii) surprisingly, the damped integer-order Euler–Bernoulli beam also predicts vibrations that grow linearly in time that is in contrast to the single-degree mass-spring-dashpot system does;[3] and (iii) fractional Euler–Bernoulli and Timoshenko beams generate stable predictions of vibrations that do not grow unboundedly and are consistent with physical observations.[3,6,18,35–37] The intrinsic reason why fractional beam models generate better predictions is because they naturally incorporate a viscoelastic damping mechanism and accurately capture the power-law behavior of viscoelastic materials.

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AUTHOR DECLARATIONS

Conflict of Interest

The authors have no conflicts to disclose.

Author Contributions

Xiangcheng Zheng: Formal analysis (equal); Funding acquisition (equal); Investigation (equal); Methodology (equal); Writing – original draft (equal). **Yiqun Li:** Formal analysis (equal); Investigation (equal); Methodology (equal); Writing – original draft (equal). **Hong Wang:** Conceptualization (equal); Funding acquisition (equal); Investigation (equal); Methodology (equal); Writing – review & editing (equal).

DATA AVAILABILITY

Data sharing is not applicable to this article as no new data were created or analyzed in this study.

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