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Analysis of a Time-Fractional Substantial Diffusion Equation of Variable Order

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Abstract: A time-fractional substantial diffusion equation of variable order is investigated, in which the variable-order fractional substantial derivative accommodates the memory effects and the structure change of the surroundings of the physical processes with respect to time. The existence and uniqueness of the solutions to the proposed model are proved, based on which the weighted high-order regularity of the solutions, in which the weight function characterizes the singularity of the solutions, are analyzed.

Keywords: time-fractional diffusion equation; fractional substantial derivative; variable-order; existence and uniqueness; regularity



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1. Introduction

Fractional problems with substantial derivatives have attracted increasing amounts of attention due to their applications in, e.g., characterizing the functionals of anomalous diffusion and non-Brownian motions [1,2]. There are several investigations in mathematical analysis and numerical methods to such kinds of problems [3–9], while the corresponding studies for the variable-order fractional substantial derivative models, in which the variable fractional order accommodates, e.g., the structure change of the surroundings with respect to time, are rarely found in the literature.

In some recent works, the following Caputo variable-order time-fractional diffusion equations are investigated mathematically and numerically [10–16]

$$(\partial_t + q \partial_t^{\alpha(\cdot, t)} - \Delta)u(x, t) = f(x, t).$$

Here, the Caputo variable-order fractional derivative operator is defined as [17–19]

$$\partial_t^{\alpha(\cdot, t)} g(t) := \int_0^t \frac{(t-s)^{-\alpha(s, t)}}{\Gamma(1-\alpha(s, t))} \partial_s g(s) ds.$$

The Caputo (variable-order) fractional derivative is widely used in various applications that exhibit the power law type memory effects due to the power function integral kernel. The Caputo variable-order fractional substantial derivative operator ${}^\sigma \partial_t^{\alpha(\cdot, t)}$, which is defined in the following for some $\sigma \geq 0$ and $0 \leq \alpha(s, t) < 1$ [1–3]

$${}^\sigma \partial_t^{\alpha(\cdot, t)} g(t) := \int_0^t \frac{(t-s)^{-\alpha(s, t)} e^{-\sigma(t-s)}}{\Gamma(1-\alpha(s, t))} (\partial_s + \sigma) g(s) ds, \quad (1)$$

tempers the memory effects of the Caputo fractional derivative operator by introducing the exponential factor, such that the impacts located at s are negligible if $t-s$ is large [1,3]. Due

to the complicated form of the variable-order fractional substantial derivative, existing analysis techniques for investigating the Caputo variable-order fractional derivative problems may not apply directly for problems involving (1), which motivates the current work.

In this paper, we analyze the following time-fractional diffusion equation [20–28] equipped with variable-order fractional substantial derivative

$$\begin{aligned} (\partial_t + q {}^\sigma \partial_t^{\alpha(\cdot,t)} - \Delta)u(x, t) &= f(x, t), \quad (x, t) \in \Omega \times (0, T]; \\ u(x, 0) &= u_0(x), \quad x \in \Omega; \quad u(x, t) = 0, \quad (x, t) \in \partial\Omega \times [0, T], \end{aligned} \quad (2)$$

where $q \geq 0$, $T > 0$, Ω is a d -dimensional domain for $1 \leq d \leq 3$ with a smooth boundary $\partial\Omega$. In the following two sections, we rigorously prove the well-posedness and high-order regularity of the solutions to problem (2), respectively, which provide a theoretical basis for this model. In particular, a weight function is introduced in the estimate of high-order smoothing properties to characterize the initial singularity of the solutions.

In the rest of the paper, let $H^1(0, T)$ be the Hilbert space of functions with weak derivatives up to order 1 in $L^2(0, T)$, the space of square-integrable functions on $[0, T]$. The corresponding norms are defined by

$$\begin{aligned} \|g\|_{L^2(0,T)} &:= \left(\int_0^T g^2(t) dt \right)^{1/2}, \\ \|g\|_{H^1(0,T)} &:= (\|g\|_{L^2(0,T)}^2 + \|\partial_t g\|_{L^2(0,T)}^2)^{1/2}. \end{aligned}$$

$H^1(0, T; \mathcal{D})$ for some Banach space $\mathcal{D}$ stands for the space of the functions in $H^1(0, T)$ with respect to the norm of $\mathcal{D}$. Q denotes a generic constant that may assume different values at different cases.

2. Existence and Uniqueness

We prove the existence and uniqueness of the solutions to problem (2) in this section. Throughout the paper, we assume that there exists a constant $0 < \alpha^* < 1$, such that

$$0 \leq \alpha(s, t) \leq \alpha^* \text{ for } 0 \leq s \leq t \leq T.$$

Theorem 1. If $f \in L^2(0, T; L^2(\Omega))$ and $u_0 \in \dot{H}^1(\Omega)$, problem (2) has a unique solution $u \in H^1(0, T; L^2(\Omega)) \cap L^2(0, T; \dot{H}^2(\Omega))$ and

$$\|u\|_{H^1(0,T;L^2(\Omega))} + \|u\|_{L^2(0,T;\dot{H}^2(\Omega))} \leq Q(\|f\|_{L^2(0,T;L^2(\Omega))} + \|u_0\|_{\dot{H}^1(\Omega)})$$

where Q is independent from the data and the solutions and the norm $\|\cdot\|_{\dot{H}^p(\Omega)}$ for $p \geq 0$ is defined in terms of the eigenpairs $\{\lambda_i, \phi_i(x)\}_{i=1}^\infty$ of $-\Delta$ by

$$\|v\|_{\dot{H}^p(\Omega)}^2 := \sum_{i=1}^\infty \lambda_i^p (v, \phi_i)^2.$$

Proof. If we expand u in terms of $\{\phi_i(x)\}_{i=1}^\infty$ as

$$u(x, t) = \sum_{i=1}^\infty u_i(t) \phi_i(x), \quad u_i(t) = (u, \phi_i),$$

then $\{u_i(t)\}_{i=1}^\infty$ satisfy the following ordinary differential equation

$$\partial_t w(t) + q {}^\sigma \partial_t^{\alpha(\cdot,t)} w(t) + \lambda w(t) = h(t), \quad t \in (0, T]; \quad w(0) = w_0 \quad (3)$$

where w, λ, h and w_0 refer to $u_i, \lambda_i, (f, \phi_i)$ and (u_0, ϕ_i) , respectively, for $i \geq 1$. We consider the fractional term as part of the right-hand side term in (3) to obtain

$$\partial_t w(t) + \lambda w(t) = -q {}^\sigma \partial_t^{\alpha(\cdot, t)} w(t) + h(t).$$

An application of the variation-of-constants formula yields

$$w = e^{-\lambda t} w_0 + e^{-\lambda t} * (-q {}^\sigma \partial_t^{\alpha(\cdot, t)} w + h), \quad (4)$$

where $*$ stands for the convolution on $[0, t]$ as follows

$$g_1(t) * g_2(t) := \int_0^t g_1(s) g_2(t-s) ds.$$

Then, we differentiate (4) with respect to t , multiply $e^{-\kappa t}$ for some $\kappa \geq 0$ on the resulting equation and reformulate $e^{-\kappa t} \cdot {}^\sigma \partial_t^{\alpha(\cdot, t)}$ as a sum of a linear operator $\mathcal{S}$ and a function $\mathcal{H}(t)$

$$\begin{aligned} e^{-\kappa t} \cdot {}^\sigma \partial_t^{\alpha(\cdot, t)} w &:= \int_0^t \frac{(t-s)^{-\alpha(s, t)} e^{-(\sigma+\kappa)(t-s)}}{\Gamma(1-\alpha(s, t))} \\ &\times \left(e^{-\kappa s} \partial_s w(s) + \sigma \int_0^s e^{-\kappa(s-y)} e^{-\kappa y} \partial_y w(y) dy + \sigma e^{-\kappa s} w_0 \right) ds \\ &=: \mathcal{S}(e^{-\kappa t} \partial_t w) + \mathcal{H}(t), \\ \mathcal{H}(t) &:= \sigma w_0 e^{-\kappa t} \int_0^t \frac{(t-s)^{-\alpha(s, t)} e^{-\sigma(t-s)}}{\Gamma(1-\alpha(s, t))} ds \end{aligned}$$

to obtain an integral equation in terms of $\tilde{w} := e^{-\kappa t} \partial_t w$ as follows

$$\begin{aligned} \tilde{w} &= -\lambda e^{-(\lambda+\kappa)t} w_0 + (-q \mathcal{S}(\tilde{w}) - q \mathcal{H}(t) + e^{-\kappa t} h) \\ &\quad - \lambda e^{-(\lambda+\kappa)t} * (-q \mathcal{S}(\tilde{w}) - q \mathcal{H}(t) + e^{-\kappa t} h). \end{aligned} \quad (5)$$

In order to prove the well-posedness of this integral equation in $L^2(0, T)$, it suffices to show that the linear operator $-q \mathcal{S} + \lambda q e^{-(\lambda+\kappa)t} * \mathcal{S}$ is a contraction in $L^2(0, T)$. For $v \in L^2(0, T)$, we apply

$$|(t-s)^{-\alpha(s, t)}| = (t-s)^{-\alpha^*} (t-s)^{\alpha^*-\alpha(s, t)} \leq \max\{1, T\} (t-s)^{-\alpha^*}$$

and Young's convolution inequality to bound $\mathcal{S}v$ by

$$\begin{aligned} \|\mathcal{S}v\|_{L^2(0, T)} &\leq Q \left\| \int_0^t \frac{e^{-(\sigma+\kappa)(t-s)}}{(t-s)^{\alpha^*}} \left(|v(s)| + \sigma \int_0^s e^{-\kappa(s-y)} |v(y)| dy \right) ds \right\|_{L^2(0, T)} \\ &\leq Q \|e^{-(\sigma+\kappa)t} t^{-\alpha^*}\|_{L^1(0, T)} \|v\|_{L^2(0, T)} \\ &\leq Q(\sigma + \kappa)^{-(1-\alpha^*)} \|v\|_{L^2(0, T)}. \end{aligned}$$

The operator $\lambda e^{-(\lambda+\kappa)t} * \mathcal{S}$ is then accordingly bounded by

$$\|\lambda e^{-(\lambda+\kappa)t} * \mathcal{S}v\|_{L^2(0, T)} \leq \|\mathcal{S}v\|_{L^2(0, T)} \leq Q(\sigma + \kappa)^{-(1-\alpha^*)} \|v\|_{L^2(0, T)}.$$

We incorporate the above two equations to conclude that for κ large enough, $-qS + \lambda q e^{-(\lambda+\kappa)t} * \mathcal{S}$ is a contraction in $L^2(0, T)$ and, thus, the integral Equation (5) has a unique solution $\tilde{w} \in L^2(0, T)$. We further apply this contractability on (5) to obtain a stability estimate

$$\begin{aligned}\|\tilde{w}\|_{L^2(0,T)} &\leq Q(\lambda^{1/2}|w_0| + \|\mathcal{H}\|_{L^2(0,T)} + \|h\|_{L^2(0,T)}) \\ &\leq Q(\lambda^{1/2}|w_0| + \|h\|_{L^2(0,T)}).\end{aligned}$$

Then, the w defined by

$$w := \int_0^t e^{\kappa s} \tilde{w}(s) ds + w_0 \in H^1(0, T)$$

with the estimate

$$\|w\|_{H^1(0,T)} \leq Q\|\tilde{w}\|_{L^2(0,T)} \leq Q(\lambda^{1/2}|w_0| + \|h\|_{L^2(0,T)})$$

satisfies the differential Equation (3). The unique H^1 solution to (3) follows from that of (5).

Based on the above derivations, we finally estimate the solution u to problem (2) by

$$\begin{aligned}\|u\|_{H^1(0,T;L^2(\Omega))}^2 &\leq Q \sum_{i=1}^{\infty} \|u_i\|_{H^1(0,T)}^2 \leq Q \sum_{i=1}^{\infty} (\lambda_i |(u_0, \phi_i)|^2 \\ &\quad + \|(f, \phi_i)\|_{L^2(0,T)}^2) = Q(\|f\|_{L^2(0,T;L^2(\Omega))}^2 + \|u_0\|_{\check{H}^1(\Omega)}^2).\end{aligned}$$

The estimate of the solutions in $\|\cdot\|_{L^2(0,T;\check{H}^2(\Omega))}$ follows from

$$\begin{aligned}\|u\|_{L^2(0,T;\check{H}^2(\Omega))} &= \|\Delta u\|_{L^2(0,T;L^2(\Omega))} \\ &= \|\partial_t u + q^\sigma \partial_t^{\alpha(\cdot,t)} u - f\|_{L^2(0,T;L^2(\Omega))} \\ &\leq Q(\|f\|_{L^2(0,T;L^2(\Omega))} + \|u_0\|_{\check{H}^1(\Omega)}).\end{aligned}$$

The uniqueness of the solutions to problem (2) follows from that of the differential Equation (3). $\square$

3. Weighted Regularity

We prove the weighted regularity of the solutions to problem (2) in the following theorem.

Theorem 2. Suppose α has bounded first-order derivatives for each coordinate, $u_0 \in \check{H}^{s+3}(\Omega)$, $f \in L^2(0, T; \check{H}^{s+2}(\Omega))$ and $\partial_t f \in L^2(0, T; \check{H}^s(\Omega))$ for some $s \geq 0$. Then, the following estimate holds

$$\begin{aligned}\|t^\delta \partial_t^2 u\|_{L^2(0,T;\check{H}^s(\Omega))} \\ \leq Q(\|u_0\|_{\check{H}^{s+3}(\Omega)} + \|\partial_t f\|_{L^2(0,T;\check{H}^s(\Omega))} + \|f\|_{L^2(0,T;\check{H}^{s+2}(\Omega))})\end{aligned}$$

where

$$\delta := \max\{\alpha(0,0) - 1/2 + \theta, 0\} \text{ with } 0 < \theta \ll 1$$

and Q is independent from the data and the solutions.

Proof. We differentiate (4) twice in time to obtain

$$\begin{aligned}\partial_t^2 w &= \lambda^2 e^{-\lambda t} w_0 + \partial_t(-q^\sigma \partial_t^{\alpha(\cdot,t)} w + h) \\ &\quad - \lambda(-q^\sigma \partial_t^{\alpha(\cdot,t)} w + h) + \lambda^2 e^{-\lambda t} * (-q^\sigma \partial_t^{\alpha(\cdot,t)} w + h).\end{aligned}\tag{6}$$

The $\partial_t^\sigma \partial_t^{\alpha(\cdot,t)} w$ could be simplified as

$$\begin{aligned} |\partial_t^\sigma \partial_t^{\alpha(\cdot,t)} w(t)| &= \left| \partial_t \int_0^t \frac{s^{-\alpha(t-s,t)} e^{-\sigma s}}{\Gamma(1-\alpha(t-s,t))} (\partial_y + \sigma) w(y) \Big|_{y=t-s} ds \right| \\ &= \left| \frac{t^{-\alpha(0,t)} e^{-\sigma t}}{\Gamma(1-\alpha(0,t))} (\partial_t w(0) + \sigma w(0)) \right. \\ &\quad + \int_0^t \partial_t \left(\frac{s^{-\alpha(t-s,t)} e^{-\sigma s}}{\Gamma(1-\alpha(t-s,t))} \right) (\partial_y + \sigma) w(y) \Big|_{y=t-s} ds \\ &\quad \left. + \int_0^t \frac{s^{-\alpha(t-s,t)} e^{-\sigma s}}{\Gamma(1-\alpha(t-s,t))} (\partial_y^2 + \sigma \partial_y) w(y) \Big|_{y=t-s} ds \right| \\ &\leq Q t^{-\alpha(0,t)} (|\partial_t w(0)| + |w(0)|) + Q \int_0^t \frac{|\partial_s^2 u(s)|}{(t-s)^{\alpha(s,t)}} ds. \end{aligned} \quad (7)$$

As

$$t^{-\alpha(0,t)} = t^{-\alpha(0,0)} t^{\alpha(0,0)-\alpha(0,t)} \leq Q t^{-\alpha(0,0)},$$

we multiply t^δ on both sides of (7) and apply

$$(t-s)^{\alpha(s,t)} \geq Q(t-s)^{\alpha(t,t)}$$

to find

$$\begin{aligned} |t^\delta \partial_t^\sigma \partial_t^{\alpha(\cdot,t)} w(t)| &\leq Q t^{\delta-\alpha(0,0)} (|\partial_t w(0)| + |w(0)|) \\ &\quad + Q \int_0^t \frac{s^\delta |\partial_s^2 u(s)|}{(t-s)^{\alpha^*}} ds + Q \int_0^t \frac{(t-s)^\delta |\partial_s^2 u(s)|}{(t-s)^{\alpha(t,t)}} ds. \end{aligned} \quad (8)$$

Since

$$\begin{aligned} &\int_0^t \frac{(t-s)^\delta |\partial_s^2 u(s)|}{(t-s)^{\alpha(t,t)}} ds \\ &= \int_0^t \frac{(t-s)^{\delta-\alpha(t,t)/2}}{s^\delta} \frac{s^\delta |\partial_s^2 u(s)|}{(t-s)^{\alpha(t,t)/2}} ds \\ &\leq \left(\int_0^t \frac{(t-s)^{2\delta-\alpha(t,t)}}{s^{2\delta}} ds \right)^{1/2} \left(\int_0^t \frac{s^{2\delta} |\partial_s^2 u(s)|^2}{(t-s)^{\alpha(t,t)}} ds \right)^{1/2} \\ &\leq Q (t^{1-\alpha(t,t)})^{1/2} \left(\int_0^t \frac{s^{2\delta} |\partial_s^2 u(s)|^2}{(t-s)^{\alpha(t,t)}} ds \right)^{1/2} \\ &\leq Q \left(\int_0^t \frac{s^{2\delta} |\partial_s^2 u(s)|^2}{(t-s)^{\alpha^*}} ds \right)^{1/2}, \end{aligned}$$

we apply $\|e^{-\kappa t} \cdot\|_{L^2(0,T)}$ for some $\kappa > 0$ on (8) to find

$$\begin{aligned} \|e^{-\kappa t} t^\delta \partial_t^\sigma \partial_t^{\alpha(\cdot,t)} w(t)\|_{L^2(0,T)} &\leq Q (|\partial_t w(0)| + |w(0)|) \\ &\quad + Q \|t^{-\alpha^*} e^{-\kappa t}\|_{L^1(0,T)} \|e^{-\kappa t} t^\delta \partial_t^2 u\|_{L^2(0,T)} \\ &\quad + Q \left(\int_0^T \int_0^t \frac{e^{-2\kappa(t-s)}}{(t-s)^{\alpha^*}} (e^{-\kappa s} s^\delta \partial_s^2 u(s))^2 ds dt \right)^{1/2} \\ &\leq Q (|\partial_t w(0)| + |w(0)|) + Q \kappa^{-(1-\alpha^*)/2} \|e^{-\kappa t} t^\delta \partial_t^2 u\|_{L^2(0,T)}. \end{aligned} \quad (9)$$

We, therefore, multiply t^δ on both sides of (6), apply $\|e^{-\kappa t} \cdot\|_{L^2(0,T)}$ on the resulting equation, choosing κ large enough and employ (9) and $(t\lambda)^\delta e^{-\lambda t/2} \leq Q$, as well as similar (and simpler) proofs as above to obtain

$$\|e^{-\kappa t} t^\delta \partial_t^2 w\|_{L^2(0,T)} \leq Q (\lambda^{3/2} |w_0| + \|\partial_t h\|_{L^2(0,T)} + \lambda \|h\|_{L^2(0,T)}).$$

We employ this result to obtain the estimate of $\partial_t^2 u$ as

$$\begin{aligned} \|t^\delta \partial_t^2 u\|_{L^2(0,T;\dot{H}^s(\Omega))}^2 &= \sum_{i=1}^{\infty} \lambda_i^s \|t^\delta \partial_t^2 u_i\|_{L^2(0,T)}^2 \\ &\leq Q \sum_{i=1}^{\infty} (\lambda_i^{s+3} |(u_0, \phi_i)|^2 + \|(\partial_t f, \phi_i)\|_{L^2(0,T)}^2 + \lambda_i^2 \|(f, \phi_i)\|_{L^2(0,T)}^2) \\ &= Q(\|u_0\|_{\dot{H}^{s+3}(\Omega)}^2 + \|\partial_t f\|_{L^2(0,T;L^2(\Omega))}^2 + \|f\|_{L^2(0,T;\dot{H}^2(\Omega))}^2), \end{aligned}$$

which completes the proof. $\square$

4. Conclusions

In this paper we rigorously prove the well-posedness and high-order regularity of the solutions to the time-fractional diffusion equation equipped with variable-order fractional substantial derivative. In particular, a weight function is introduced in the estimate of high-order smoothing properties to characterize the initial singularity of the solutions. Based on these theoretical estimates, we will carry out numerical analysis for the proposed model in the near future in order to provide supports for practical implementations.

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