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WELL-POSEDNESS AND REGULARITY OF CAPUTO–HADAMARD TIME-FRACTIONAL DIFFUSION EQUATIONS

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Abstract

Ultraslow diffusion describes the long-time diffusion of particles whose mean square displacement (MSD) grows logarithmically in time. We prove the well-posedness of a Caputo–Hadamard time-fractional diffusion model in multiple space dimensions, in which the MSD in time grows

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logarithmically and thus provides adequate descriptions for the ultraslow diffusion processes, as well as the smoothing properties of the solutions.

Keywords: Ultraslow Diffusion; Mean Square Displacement; Caputo–Hadamard; Time-Fractional Diffusion Equation.

1. INTRODUCTION

The classical Fickian diffusion partial differential equation (PDE) governs the scaling limit of a random walk where the underlying particle jumps have a finite variance, which leads to a normal diffusion that is characterized by a linear growth of the mean square displacement (MSD) in time $\langle \boldsymbol{x}(t)^2 \rangle \simeq t$.[1]

In many scenarios, e.g. the transport of solutes in heterogeneous porous media, the diffusion is anomalous characterized by a power-law growth of the MSD in time $\langle \boldsymbol{x}(t)^2 \rangle \simeq t^\beta$, where $\beta < 1$ and $\beta > 1$ correspond to the subdiffusion and superdiffusion, respectively, and $\beta = 1$ reduces to the normal diffusion.[1,2] This explains why integer-order diffusion PDEs do not accurately describe the diffusive transport of solutes in heterogeneous media, which are instead modeled by the time-fractional PDE (TFPDE) $\partial_t^\beta u - \Delta u = f(\boldsymbol{x}, t)$ with $0 < \beta < 1$ where ∂_t^β is the Caputo fractional differential operator defined by $\partial_t^\beta g := I_t^{1-\beta} g'$ with the convolution $I_t^{1-\beta} g := (t^{-\beta}/\Gamma(1-\beta)) * g$.[3]

The two time-scale mobile–immobile TFPDE model

$$u_t + k(t)\partial_t^\alpha u - \Delta u = f(\boldsymbol{x}, t), \quad 0 < \alpha < 1 \tag{1.1}$$

was derived in Ref. 4 to improve the modeling of subdiffusive transport, in which $k(t)\partial_t^\alpha u$ describes the subdiffusive transport consisting of $k(t)/(1 + k(t))$ portion of the total solute mass and u_t represents the Brownian motion consisting of $1/(1 + k(t))$ portion of the total solute mass. Fractional differential equations have been applied in modeling phenomena in many fields.[5–10] Many diffusive processes are strongly anomalous in that their mean waiting time has a super-heavy tail, which decays slower than any power-law decaying tail does. Their MSD grows logarithmically in time $\langle \boldsymbol{x}(t)^2 \rangle \simeq \log^\mu t$ for some $\mu > 0$.[11–13] In Refs. 14–16, the Caputo–Hadamard fractional calculus with logarithmic kernel is introduced to describe the ultraslow kinetics. Inspired by the above considerations, we consider the two time-scale mobile–immobile Caputo–Hadamard TFPDE of $0 < \alpha < 1$

$$\begin{aligned} &u_t + k(t)\,{}_a\mathcal{D}_t^\alpha u - \Delta u = f(\boldsymbol{x}, t), \\ &\qquad\qquad (\boldsymbol{x}, t) \in \Omega \times (a, T]; \\ &u(\boldsymbol{x}, a) = u_a(\boldsymbol{x}), \quad \boldsymbol{x} \in \Omega; \\ &u(\boldsymbol{x}, t) = 0, \quad (\boldsymbol{x}, t) \in \partial\Omega \times [a, T]. \end{aligned} \tag{1.2}$$

Here, $\Omega \subset \mathbb{R}^d$ $(d = 1, 2, 3)$ is a simply connected bounded domain with a smooth boundary $\partial\Omega$ and convex corners, $\boldsymbol{x} := (x_1, \ldots, x_d)^\top$, $a > 0$ and $|k(t)| \le K_0$. The Caputo–Hadamard fractional derivative is defined by[3,17]

$${}_a\mathcal{D}_t^\alpha f(t) = \frac{1}{\Gamma(1-\alpha)} \int_a^t \left(\log \frac{t}{s}\right)^{-\alpha} f'(s)ds, \qquad \log(\cdot) = \log_e(\cdot).$$

We present several solution curves and MSDs to illustrate the motivations of model (1.2). Let $\alpha = 0.5$, $f = 0$, $a = 1$ and $u_a(x) = e^{-x^2/(2\times 0.01^2)}/(\sqrt{2\pi} \times 0.01)$ in all models. In Fig. 1, we plot short-term solutions to the Caputo–Hadamard TFPDE (1.2) without u_t term under $k(t) = 1$ and (1.2) with $k(t) = 100$ on $\Omega = [-0.1, 0.1]$ during a short time period $[1, 1.01]$. The left plot shows the initial singularity of the solutions, which could be eliminated by adding the u_t term as shown in the right plot, even though the coefficient of the fractional term is large ($k(t) = 100$). In Fig. 2, we plot long-term solutions to the TFPDE (1.1) and the Caputo–Hadamard TFPDE (1.2) with $k(t) = 100$ on the space-time domain $(x, t) \in (-0.1, 0.1) \times [1, 100]$, which shows that the Caputo–Hadamard TFPDE (1.2) exhibits weaker initial singularities and slower decay properties compared with the Caputo TFPDE (1.1). We also explore the MSDs of TFPDEs with different $k(t)$ on $(x, t) \in (-10, 10) \times [1, 100]$ in Fig. 3, which indicates that the model (1.2) with $k(t) \to 0$ models the classical Fickian diffusive transport, while $k(t) \to \infty$ models the ultraslow diffusion. Equation (1.2) with $k(t) = 1$ switches smoothly from the initial Fickian diffusion behavior to the long-term ultraslow diffusion behavior. Therefore,

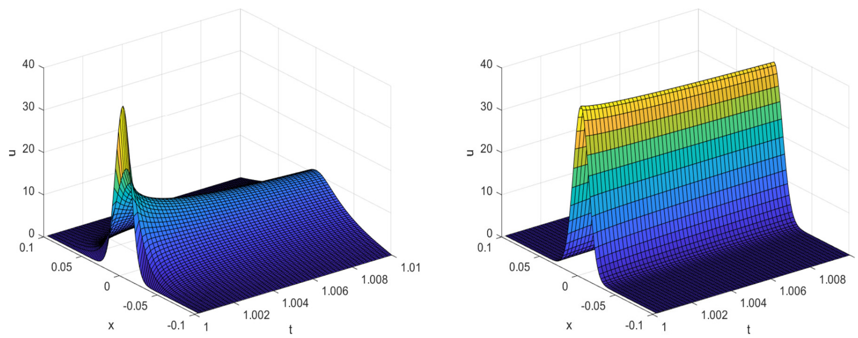

Fig. 1 Short-term solutions to (left) TFPDE (1.2) without u_t term under $\alpha = 0.5$ and $k(t) = 1$ and (right) (1.2) with $\alpha = 0.5$ and $k(t) = 100$.

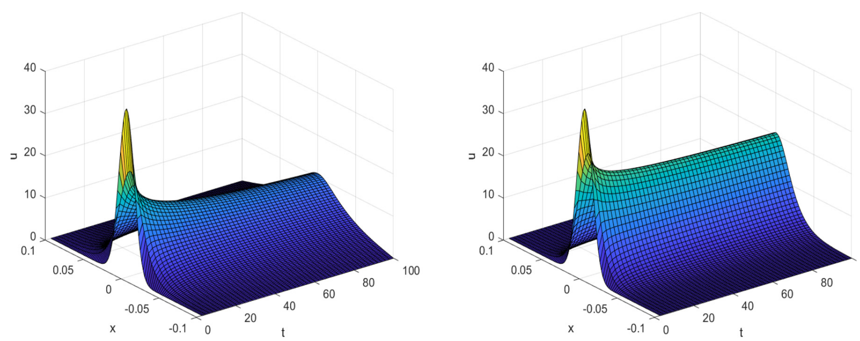

Fig. 2 Long-term solutions to (left) TFPDE (1.1) and (right) the Caputo–Hadamard TFPDE (1.2) under $\alpha = 0.5$ and $k(t) = 100$.

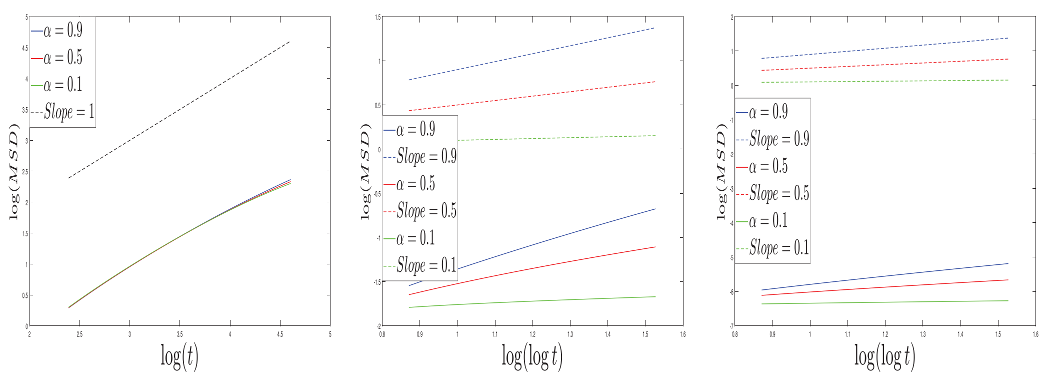

Fig. 3 MSDs for the Caputo–Hadamard TFPDE (1.2) with (left) $k = 0.01$, (middle) $k = 1$ and (right) $k = 100$.

the two time-scale TFPDE (1.2) captures the long-term ultraslow diffusion behavior while eliminating its non-physical initial weak singularity of the Caputo TFPDE, and thus provides a physically relevant extension of TFPDE models.

We prove the existence and uniqueness of the initial-boundary value problem of the Caputo–Hadamard TFPDE (1.2) in multiple space dimensions, as well as the regularity of their solutions that depends on the fractional order α. The rest of the paper is organized as follows: In Sec. 2, we present and prove notations, norms and useful lemmas. In Sec. 3, we prove the mapping properties of integral operator with log-kernel. In Sec. 4, we prove the well-posedness of a Caputo–Hadamard time-fractional ordinary differential equation, based on which we prove the well-posedness and regularity of model (1.2) in Sec. 5.

2. PRELIMINARIES

Let $m \in \mathbb{R}$, $0 \le \mu < 1 \le p \le \infty$ and $\mathrm{I} \subset \mathbb{R}$ be a bounded interval. Let $C^m(\mathrm{I})$ be the space of continuous functions with continuous derivatives up to order m equipped with

$$\|g\|_{C(\mathrm{I})} := \sup_{t\in\mathrm{I}} |g(t)|,$$

$$\|g\|_{C^m(\mathrm{I})} := \max_{0\le n\le m} \|D^n g\|_{C(\mathrm{I})}.$$

We also introduce the space $C_{\gamma,\log}(\mathrm{I})$ of log-Hölder continuous functions on I equipped with the norm[3,18]

$$\|g\|_{C_{\mu,\log}(\mathrm{I})} := \|g\|_{C(\mathrm{I})} + \sup_{t_1,t_2\in\mathrm{I},t_1\neq t_2} \frac{|g(t_2)-g(t_1)|}{|\log t_2 - \log t_1|^{\mu}}.$$

Let $L^2(\Omega)$ be the space of Lebesgue square integrable functions on Ω and $H^m(\Omega)$ be the space of functions with derivatives of order m in $L^2(\Omega)$. Let $H_0^m(\Omega)$ be the completion of $C_0^\infty(\Omega)$, the space of infinitely differentiable functions with compact support in Ω, in $H^m(\Omega)$. For non-integer $r \ge 0$, the fractional Sobolev space $H^r(\Omega)$ is defined by interpolation.[19,20] All the spaces are equipped with the standard norms.

It is well known that the eigenfunctions $\{\phi_i\}_{i=1}^\infty$ of problem

$$-\Delta\phi_i(\boldsymbol{x}) = \lambda_i\phi_i(\boldsymbol{x}),\ \boldsymbol{x}\in\Omega;\quad \phi_i(\boldsymbol{x}) = 0,\ \boldsymbol{x}\in\partial\Omega \tag{2.1}$$

form an orthonormal basis in $L^2(\Omega)$.[21] The eigenvalues $\{\lambda_i\}_{i=1}^\infty$ are positive and form a non-decreasing sequence that tend to ∞ with i. We use the theory of sectorial operators to define the fractional Sobolev spaces[22,23]

$$\check{H}^r(\Omega) := \Big\{ w\in L^2(\Omega) : |v|^2_{\check{H}^r(\Omega)} := ((-\Delta)^r w, w) = \sum_{i=1}^\infty \lambda_i^r (w,\phi_i)^2 < \infty \Big\}.$$

In this paper, we use Q to denote a generic positive constant that may assume different values at different situations, and C_i, M_i and Q_i to denote fixed positive constants.

Lemma 2.1 (Generalized Grönwall inequality (Theorem 3 of Ref. 24)). *Let $0 \le C_0(t) \in L_{\mathrm{loc}}(a,b)$ and C_1 be a non-negative constant. Suppose $0 \le g(t) \in L_{\mathrm{loc}}(a,b)$ satisfies*

$$g(t) \le C_0(t) + C_1\int_a^t g(s)(\log t - \log s)^{\gamma-1}\frac{ds}{s},\quad \forall\, t\in(a,b),\quad 0<\gamma<1.$$

Then $g(t)$ can be bounded by

$$g(t) \le C_0(t) + C_1\int_a^t \sum_{n=1}^\infty \frac{(C_1\Gamma(\gamma))^n}{\Gamma(n\gamma)} \times (\log t - \log s)^{n\gamma-1} C_0(s)\frac{ds}{s},\quad \forall\, t\in(a,b).$$

In particular, if $C_0(t)$ is non-decreasing, then

$$g(t) \le C_0(t) E_{\gamma,1}(C_1\Gamma(\gamma)(\log(t/a))^\gamma),\ \ \forall\, t\in(a,b),$$

where $E_{p,q}(t)$ represents the Mittag-Leffler function defined by

$$E_{p,q}(t) := \sum_{k=0}^\infty \frac{t^k}{\Gamma(pk+q)},\quad t\in\mathbb{R},\ \ p\in\mathbb{R}^+,\ \ q\in\mathbb{R}.$$

We finally give a useful result to be frequently used subsequently.

Lemma 2.2. *The following relation holds for $0 < \zeta < \eta$ and $p, q > 0$:*

$$\int_\zeta^\eta \left(\log\frac{\eta}{t}\right)^{p-1}\left(\log\frac{t}{\zeta}\right)^{q-1}\frac{dt}{t} = \frac{\Gamma(p)\Gamma(q)}{\Gamma(p+q)}\left(\log\frac{\eta}{\zeta}\right)^{p+q-1}.$$

Proof. We omit the proof since it is obvious. □

3. MAPPING PROPERTIES OF THE LOG-KERNEL INTEGRAL OPERATOR

We prove the mapping properties of the following integral operator ${}_b\mathcal{J}_t^\gamma$[3] for $0<\gamma<1$ and $0<b\le t\le c<\infty$:

$${}_b\mathcal{J}_t^\gamma f(t) = \frac{1}{\Gamma(\gamma)}\int_b^t \left(\log\frac{t}{s}\right)^{\gamma-1} f(s)\frac{ds}{s}$$

to facilitate the subsequent analysis.

Theorem 3.1. *For $g\in L^\infty(b,c)$, ${}_b\mathcal{J}_t^\gamma g \in C_{\gamma,\log}[b,c]$ and*

$$\|{}_b\mathcal{J}_t^\gamma g\|_{C_{\gamma,\log}[b,c]} \le Q\|g\|_{L^\infty(b,c)},\quad Q := Q(b,c,\gamma).$$

Proof. For $b\le t_1\le t_2\le c$, direct calculations yield

$$\left[{}_b\mathcal{J}_t^\gamma g(t)\right]\Big|_{t=t_1}^{t=t_2} = \frac{1}{\Gamma(\gamma)}\left[\int_b^{t_2}\left(\log\frac{t_2}{s}\right)^{\gamma-1} g(s)\frac{ds}{s} - \int_b^{t_1}\left(\log\frac{t_1}{s}\right)^{\gamma-1} g(s)\frac{ds}{s}\right]$$

$$= \frac{1}{\Gamma(\gamma)} \int_b^{t_1} \left[\left(\log \frac{t_2}{s} \right)^{\gamma-1} - \left(\log \frac{t_1}{s} \right)^{\gamma-1} \right] g(s) \frac{ds}{s} + \frac{1}{\Gamma(\gamma)} \left[\int_{t_1}^{t_2} \left(\log \frac{t_2}{s} \right)^{\gamma-1} g(s) \frac{ds}{s} \right] =: L_1 + L_2. \tag{3.1}$$

Since $(\log \frac{t_2}{s})^{\gamma-1} - (\log \frac{t_1}{s})^{\gamma-1} \le 0$ for any $b \le s \le t_1$, we bound L_1 by the following:

$$\begin{aligned}|L_1| &\le \frac{\|g\|_{L^\infty(b,c)}}{\Gamma(\gamma)} \int_b^{t_1} \left[\left(\log \frac{t_1}{s} \right)^{\gamma-1} - \left(\log \frac{t_2}{s} \right)^{\gamma-1} \right] \frac{ds}{s} \\ &= \frac{\|g\|_{L^\infty(b,c)}}{\Gamma(1+\gamma)} \left[\left(\log \frac{t_1}{b} \right)^{\gamma} + \left(\log \frac{t_2}{t_1} \right)^{\gamma} - \left(\log \frac{t_2}{b} \right)^{\gamma} \right] \\ &\le \frac{2\|g\|_{L^\infty(b,c)}}{\Gamma(1+\gamma)} \left(\log \frac{t_2}{t_1} \right)^{\gamma} \\ &\le Q \left(\log \frac{t_2}{t_1} \right)^{\gamma} \|g\|_{L^\infty(b,c)}.\end{aligned}$$

Next, we bound L_2 by

$$\begin{aligned}|L_2| &\le \frac{\|g\|_{L^\infty(b,c)}}{\Gamma(\gamma)} \int_{t_1}^{t_2} \left(\log \frac{t_2}{s} \right)^{\gamma-1} \frac{ds}{s} \\ &= \frac{\|g\|_{L^\infty(b,c)}}{\Gamma(1+\gamma)} \left(\log \frac{t_2}{t_1} \right)^{\gamma} \\ &\le Q \left(\log \frac{t_2}{t_1} \right)^{\gamma} \|g\|_{L^\infty(b,c)},\end{aligned}$$

which completes the proof. □

Theorem 3.2. *For $g \in C_{\beta,\log}[b,c]$ with $0 < \gamma + \beta < 1$, ${}_b\mathcal{J}_t^\gamma (g(t) - g(b)) \in C_{\gamma+\beta,\log}[b,c]$ and*

$$\|{}_b\mathcal{J}_t^\gamma (g(t) - g(b))\|_{C_{\gamma+\beta,\log}[b,c]} \le Q\|g\|_{C_{\beta,\log}[b,c]}, \quad Q := Q(b,c,\gamma,\beta). \tag{3.2}$$

Proof. Since $g \in C_{\beta,\log}[b,c]$, $\bar{g} := g(t) - g(b)$ satisfies $|\bar{g}| \le \|g\|_{C_{\beta,\log}[b,c]} (\log \frac{t}{b})^\beta$. For $b \le t_1 \le t_2 \le c$, we rewrite (3.1) by the splitting $g(s) = (g(s) - g(t_1)) + g(t_1)$

$$\begin{aligned}&\left[{}_b\mathcal{J}_t^\gamma \bar{g}\right]\Big|_{t=t_1}^{t=t_2} g \left(\log \frac{t_2}{s} \right)^{\gamma-1} \frac{ds}{s} \\ &= \frac{1}{\Gamma(\gamma)} \int_b^{t_1} \left[\left(\log \frac{t_2}{s} \right)^{\gamma-1} - \left(\log \frac{t_1}{s} \right)^{\gamma-1} \right] \times (\bar{g}(s) - \bar{g}(t_1)) \frac{ds}{s} \\ &\quad + \frac{1}{\Gamma(\gamma)} \int_{t_1}^{t_2} \left(\log \frac{t_2}{s} \right)^{\gamma-1} (\bar{g}(s) - \bar{g}(t_1)) \frac{ds}{s} \\ &\quad + \frac{\bar{g}(t_1)}{\Gamma(\gamma)} \left[\int_b^{t_2} \left(\log \frac{t_2}{s} \right)^{\gamma-1} \frac{ds}{s} - \int_b^{t_1} \left(\log \frac{t_1}{s} \right)^{\gamma-1} \frac{ds}{s} \right] \\ &=: H_1 + H_2 + H_3.\end{aligned}$$

If $\frac{t_1}{b} \le \frac{t_2}{t_1}$, we use the substitution $y = \log t_1 - \log s$ to bound H_1 by the following:

$$\begin{aligned}|H_1| &= \left| \int_0^{\log \frac{t_1}{b}} \frac{\bar{g}(t_1 e^{-y}) - \bar{g}(t_1)}{\Gamma(\gamma)} \times (y^{\gamma-1} - (y + \log t_2 - \log t_1)^{\gamma-1}) dy \right| \\ &\le \left| \int_0^{\log \frac{t_1}{b}} \frac{\|g\|_{C_{\beta,\log}[b,c]}}{\Gamma(\gamma)} \left(\log \left(\frac{t_1}{t_1 e^{-y}} \right) \right)^\beta \times (y^{\gamma-1} - (y + \log t_2 - \log t_1)^{\gamma-1}) dy \right| \\ &= \left| \int_0^{\log \frac{t_1}{b}} \frac{\|g\|_{C_{\beta,\log}[b,c]}}{\Gamma(\gamma)} y^\beta \times (y^{\gamma-1} - (y + \log t_2 - \log t_1)^{\gamma-1}) dy \right| \\ &\le \frac{\|g\|_{C_{\beta,\log}[b,c]}}{\Gamma(\gamma)} \left| \int_0^{\log \frac{t_1}{b}} 2y^{\gamma-1} y^\beta dy \right| \\ &= \frac{2\|g\|_{C_{\beta,\log}[b,c]}}{\Gamma(\gamma)} \left(\log \frac{t_1}{b} \right)^{\gamma+\beta} \\ &\le QM\|g\|_{C_{\beta,\log}[b,c]} \left(\log \frac{t_2}{t_1} \right)^{\gamma+\beta}.\end{aligned} \tag{3.3}$$

Otherwise, we split the first integral in (3.3) on $[0, \frac{t_2}{t_1}]$ and $[\frac{t_2}{t_1}, \frac{t_1}{b}]$ to obtain

$$\begin{aligned}|H_1'| &= \bigg| \int_0^{\log \frac{t_2}{t_1}} \frac{\bar g(t_1 e^{-y}) - \bar g(t_1)}{\Gamma(\gamma)} \\ &\quad \times (y^{\gamma-1} - (y + \log t_2 - \log t_1)^{\gamma-1}) dy \\ &\quad + \int_{\log \frac{t_2}{t_1}}^{\log \frac{t_1}{b}} \frac{\bar g(t_1 e^{-y}) - \bar g(t_1)}{\Gamma(\gamma)} \\ &\quad \times (y^{\gamma-1} - (y + \log t_2 - \log t_1)^{\gamma-1}) dy \bigg| \\ &=: H_{11}' + H_{12}'.\end{aligned}$$

H_{11}' can be bounded by the right-hand side of (3.3) by a similar technique. We bound H_{12}' by the substitution $y = r \log \frac{t_2}{t_1}$

$$\begin{aligned}|H_{12}'| &\le Q \|g\|_{C_{\beta,\log}[b,c]} \left(\log \frac{t_2}{t_1}\right)^{\gamma+\beta} \\ &\quad \times \int_1^{+\infty} (r^{\gamma-1} - (1+r)^{\gamma-1}) r^\beta dr \\ &\le Q \|g\|_{C_{\beta,\log}[b,c]} \left(\log \frac{t_2}{t_1}\right)^{\gamma+\beta} \\ &\quad \times \int_1^{+\infty} \left(1 - \left(1 + \frac{\gamma - 1}{r} + \mathcal{O}\left(\frac{1}{r^2}\right)\right)\right) \\ &\quad \times r^{\alpha+\beta-1} dr \\ &\le Q \|g\|_{C_{\beta,\log}[b,c]} \left(\log \frac{t_2}{t_1}\right)^{\gamma+\beta} \int_1^{+\infty} r^{\gamma+\beta-2} dr \\ &\le Q \|g\|_{C_{\beta,\log}[b,c]} \left(\log \frac{t_2}{t_1}\right)^{\gamma+\beta}.\end{aligned}$$

We then bound H_2 and H_3 for $b \le t_1 < t_2 \le c$ by

$$\begin{aligned}|H_2| &= \frac{1}{\Gamma(\gamma)} \int_{t_1}^{t_2} \left(\log \frac{t_2}{s}\right)^{\gamma-1} (\bar g(s) - \bar g(t_1)) \frac{ds}{s} \\ &\le \frac{\|g\|_{C_{\beta,\log}[b,c]}}{\Gamma(\gamma)} \int_{t_1}^{t_2} \left(\log \frac{t_2}{s}\right)^{\gamma-1} \left(\log \frac{t_2}{s}\right)^{\beta} \frac{ds}{s} \\ &= \frac{\|g\|_{C_{\beta,\log}[b,c]}}{\Gamma(\gamma)} \left(\log \frac{t_2}{t_1}\right)^{\gamma+\beta}\end{aligned}$$

and

$$\begin{aligned}|H_3| &= \frac{|\bar g(t_1)|}{\Gamma(\gamma)} \left| \int_b^{t_2} \left(\log \frac{t_2}{s}\right)^{\gamma-1} \frac{ds}{s} - \int_b^{t_1} \left(\log \frac{t_1}{s}\right)^{\gamma-1} \frac{ds}{s} \right| \\ &= \frac{|\bar g(t_1)|}{\Gamma(1+\gamma)} \left[\left(\log \frac{t_2}{b}\right)^\gamma - \left(\log \frac{t_1}{b}\right)^\gamma\right] \\ &\le Q \|g\|_{C_{\beta,\log}[b,c]} \left(\log \frac{t_1}{b}\right)^\beta \\ &\quad \times \left[\left(\log \frac{t_2}{b}\right)^\gamma - \left(\log \frac{t_1}{b}\right)^\gamma\right]. \end{aligned} \tag{3.4}$$

For $\frac{t_1}{b} \le \frac{t_2}{t_1}$, we use $x_2^\gamma - x_1^\gamma \le (x_2 - x_1)^\gamma$ for $0 < \gamma \le 1$ and $0 \le x_1 < x_2$ to bound the right-hand side of (3.4) by

$$\begin{aligned}&\left(\log \frac{t_1}{b}\right)^\beta \left[\left(\log \frac{t_2}{b}\right)^\gamma - \left(\log \frac{t_1}{b}\right)^\gamma\right] \\ &\quad \le \left(\log \frac{t_2}{t_1}\right)^{\gamma+\beta}.\end{aligned}$$

Otherwise, for $\frac{t_1}{b} > \frac{t_2}{t_1}$, we can use the mean value theorem to obtain for some $\zeta \in (\log \frac{t_1}{b}, \log \frac{t_2}{b})$

$$\begin{aligned}&\left(\log \frac{t_1}{b}\right)^\beta \left[\left(\log \frac{t_2}{b}\right)^\gamma - \left(\log \frac{t_1}{b}\right)^\gamma\right] \\ &\quad = \left(\log \frac{t_1}{b}\right)^\beta \left[\gamma \left(\log \frac{\zeta}{b}\right)^{\gamma-1} \log \frac{t_2}{t_1}\right] \\ &\quad \le Q \left(\log \frac{t_1}{b}\right)^{\gamma+\beta-1} \log \frac{t_2}{t_1} \\ &\quad = Q \left(\log \frac{t_2}{t_1}\right)^{\gamma+\beta}.\end{aligned}$$

Thus, the proof is completed. □

Theorem 3.3. *For $g \in C_{\beta,\log}[b,c]$ with $\gamma + \beta > 1$, ${}_b\mathcal{J}_t^\gamma (g(t) - g(b)) \in C^1[b,c]$ and*

$$\begin{aligned}&\|{}_b\mathcal{J}_t^\gamma (g(t) - g(b))\|_{C^1[b,c]} \\ &\quad \le Q \|g\|_{C_{\beta,\log}[b,c]}, \quad Q := Q(b,c,\gamma,\beta). \end{aligned} \tag{3.5}$$

Proof. For $0 < \sigma \ll 1$ and $g \in C_{\beta,\log}[b,c]$,

$$\begin{aligned}g_\sigma(t) &:= \int_b^{t-\sigma} \bar g(s) \left(\log \frac{t}{s}\right)^{\alpha-1} \frac{ds}{s} \\ &\in C^1[b+\sigma, c] \end{aligned} \tag{3.6}$$

is differentiable and

$$\begin{aligned} g'_\sigma(t) &= \frac{\gamma-1}{t}\int_b^{t-\sigma} \bar{g}(s)\left(\log\frac{t}{s}\right)^{\gamma-2}\frac{ds}{s} \\ &\quad + \frac{\bar{g}(t-\sigma)}{t-\sigma}\left(\log\frac{t}{t-\sigma}\right)^{\gamma-1} \\ &= \frac{\gamma-1}{t}\int_b^{t-\sigma} (\bar{g}(s)-\bar{g}(t))\left(\log\frac{t}{s}\right)^{\gamma-2}\frac{ds}{s} \\ &\quad + \left(\frac{\bar{g}(t-\sigma)}{t-\sigma}-\frac{\bar{g}(t)}{t}\right)\left(\log\frac{t}{t-\sigma}\right)^{\gamma-1} \\ &\quad + \frac{\bar{g}(t)}{t}\left(\log\frac{t}{b}\right)^{\gamma-1} \\ &=: G_1+G_2+G_3. \end{aligned}$$

We use $g \in C_{\beta,\log}[b,c]$ and $\beta+\gamma>1$ to bound $G_1 - G_3$ for $t\in[b+\sigma,c]$ by

$$\begin{aligned} |G_1| &\le \left|\frac{\gamma-1}{t}\int_a^{t-\sigma} (\bar{g}(s)-\bar{g}(t))\left(\log\frac{t}{s}\right)^{\gamma-2}\frac{ds}{s}\right| \\ &\le Q\|g\|_{C_{\beta,\log}[b,c]}\int_b^{t-\sigma}\left(\log\frac{t}{s}\right)^{\beta} \\ &\quad \times\left(\log\frac{t}{s}\right)^{\gamma-2}\frac{ds}{s} \\ &\le Q\|g\|_{C_{\beta,\log}[b,c]}, \quad t\in[b+\sigma,c], \\ |G_2| &= \left|\left(\frac{\bar{g}(t-\sigma)}{t-\sigma}-\frac{\bar{g}(t)}{t}\right)\left(\log\frac{t}{t-\sigma}\right)^{\gamma-1}\right| \\ &= \left|\left(\frac{\bar{g}(t-\sigma)-\bar{g}(t)}{t-\sigma}+\frac{\bar{g}(t)}{t-\sigma}-\frac{\bar{g}(t)}{t}\right)\right. \\ &\quad \left.\times\left(\log\frac{t}{t-\sigma}\right)^{\gamma-1}\right| \\ &\le Q\|g\|_{C_{\beta,\log}[b,c]}\left(\log\frac{t}{t-\sigma}\right)^{\beta+\gamma-1} \\ &\quad + \frac{\|g\|_{C[b,c]}\sigma}{t(t-\sigma)}\left(\log\frac{t}{t-\sigma}\right)^{\gamma-1}. \end{aligned}$$

We apply the variable substitution $x=\frac{t}{t-\sigma}$ and use the L'Hospital's rule to calculate

$$\begin{aligned} \lim_{\sigma\to0^+}\sigma\left(\log\frac{t}{t-\sigma}\right)^{\gamma-1} &= \lim_{x\to1^+}\frac{t-t/x}{(\log x)^{1-\alpha}} \\ &= \lim_{x\to1^+}\frac{1/x^2}{(1-\gamma)(\log x)^{-\gamma}/x} \\ &= \lim_{x\to1^+}\frac{1/x}{(1-\gamma)(\log x)^{-\gamma}} \\ &= 0. \end{aligned}$$

Hence, we can conclude that $G_2\to0$ as $\sigma\to0^+$. Next, we bound G_3 by

$$\begin{aligned} |G_3| &= \frac{\bar{g}(t)}{t}\left(\log\frac{t}{b}\right)^{\gamma-1} \\ &\le Q\|g\|_{C_{\beta,\log}[b,c]}\frac{1}{b}\left(\log\frac{t}{b}\right)^{\gamma+\beta-1} \\ &\le Q\|g\|_{C_{\beta,\log}[b,c]}. \end{aligned}$$

Thus, $g'_\sigma(t)$ is integrable on $[b,c]$ for $0<\sigma\le\epsilon_0\ll1$ and is bounded by

$$\begin{aligned} |g'_\sigma(t)| &\le Q\|g\|_{C_{\beta,\log}[b,c]}\left[\left(\log\frac{t}{b}\right)^{\beta+\gamma-1}\right. \\ &\quad \left.+\left(\log\frac{t}{t-\sigma}\right)^{\beta+\gamma-1}\right] \\ &\le Q\|g\|_{C_{\beta,\log}[b,c]}. \end{aligned}$$

Therefore, the limit function of $g'_\sigma(t)$

$$\begin{aligned} y(t) &:= \lim_{\sigma\to0^+} g'_\sigma(t) \\ &= \frac{\gamma-1}{t}\int_b^t (\bar{g}(s)-\bar{g}(t))\left(\log\frac{t}{s}\right)^{\gamma-2}\frac{ds}{s} \\ &\quad + \frac{\bar{g}(t)}{t}\left(\log\frac{t}{b}\right)^{\gamma-1} \end{aligned}$$

is continuous on $[b,c]$ and can be bounded in terms of $\|g\|_{C_{\beta,\log}[b,c]}$. Use the Lebesgue bounded convergence theorem to obtain

$$\begin{aligned} &\lim_{\sigma\to0^+}[g_\sigma(t)-g_\sigma(b+\sigma)] \\ &\quad = \lim_{\sigma\to0^+}\int_{b+\sigma}^t g'_\sigma(s)ds = \int_b^t y(s)ds. \end{aligned} \tag{3.7}$$

Finally, we combine (3.6) with (3.7) to conclude that

$$\begin{aligned} {}_b\mathcal{J}_t^\gamma\bar{g}(t) &= \frac{1}{\Gamma(\gamma)}\lim_{\sigma\to0^+}[g_\sigma(t)-g_\sigma(b+\sigma)] \\ &= \frac{1}{\Gamma(\gamma)}\int_b^t y(s)ds \end{aligned}$$

is continuous differentiable on $[b,c]$ with the estimate (3.5). □

Theorem 3.4. *For $0<\gamma<1$ and $tg\in C_{\gamma,\log}[b,c]$, it holds $g\in C_{\gamma,\log}[b,c]$.*

Proof. For $b \le t_1 \le t_2 \le c$, we have

$$
\begin{aligned}
|g(t_2) - g(t_1)| &= \left| \frac{t_2 g(t_2)}{t_2} - \frac{t_1 g(t_1)}{t_1} \right| \\
&= \left| \left(\frac{t_2 g(t_2)}{t_2} - \frac{t_2 g(t_2)}{t_1} \right) \right. \\
&\quad \left. + \left(\frac{t_2 g(t_2)}{t_1} - \frac{t_1 g(t_1)}{t_1} \right) \right| \\
&\le Q|t_2 - t_1| \|tg\|_{C[b,c]} + Q \left(\log \frac{t_2}{t_1} \right)^{\gamma} \\
&\quad \times \|tg\|_{C_{\gamma,\log}[b,c]} \\
&\le Q \left(\log \frac{t_2}{t_1} \right)^{\gamma} \|g\|_{C_{\gamma,\log}[b,c]}.
\end{aligned}
$$

Thus, we finish the proof. □

4. ANALYSIS OF A CAPUTO–HADAMARD FRACTIONAL ODE

We prove the well-posedness of the initial value problem of the Caputo–Hadamard fractional ODE for $\alpha \in (0,1)$ and $0 < \lambda < \infty$

$$
\xi'(t) + k(t)\, {}_a\mathcal{D}_t^{\alpha} \xi(t) + \lambda \xi(t) = g(t), \quad t \in (a,T]; \quad \xi(a) = \xi_a. \tag{4.1}
$$

We multiply (4.1) by $e^{\lambda t}$ and integrate the equation from a to t to get

$$
\begin{aligned}
\xi(t) &= \xi_a\, e^{\lambda a - \lambda t} - \frac{1}{\Gamma(1-\alpha)} \int_a^t k(\theta) e^{\lambda(\theta - t)} \\
&\quad \times \int_a^{\theta} \left(\log \frac{\theta}{s} \right)^{-\alpha} \xi'(s)\, ds\, d\theta \\
&\quad + \int_a^t g(\theta) e^{\lambda(\theta - t)} d\theta.
\end{aligned} \tag{4.2}
$$

We differentiate (4.2) with respect to t to arrive at the integral equation in terms of $\eta = \xi'$

$$
\begin{aligned}
\eta(t) &= -\lambda e^{\lambda a - \lambda t} \xi_a + \frac{1}{\Gamma(1-\alpha)} \int_a^t \lambda e^{\lambda(\theta - t)} k(\theta) \\
&\quad \times \int_a^{\theta} \left(\log \frac{\theta}{s} \right)^{-\alpha} \eta(s)\, ds\, d\theta \\
&\quad - \frac{k(t)}{\Gamma(1-\alpha)} \int_a^t \left(\log \frac{t}{s} \right)^{-\alpha} \eta(s)\, ds \\
&\quad - \int_a^t \lambda e^{\lambda(\theta - t)} g(\theta)\, d\theta + g(t).
\end{aligned} \tag{4.3}
$$

Here, ξ can be recovered in terms of η by $\xi(t) = \xi_a + \int_a^t \eta(s) ds$.

Theorem 4.1. *If $g \in C[a,T]$ and $k(t) \in C[a,T]$ holds, problem (4.1) has a unique solution $\xi(t) \in C^1[a,T]$ such that*

$$
\|\xi(t)\|_{C^1[a,T]} \le Q(\lambda|\xi_a| + \|g\|_{C[a,T]}), \quad Q = Q(\alpha, \|k\|_{C[a,T]}, T). \tag{4.4}
$$

Proof. Define a approximation sequence $\{\eta_n\}_{n=0}^{\infty}$ on $[a,T]$ by

$$
\eta_0(t) := -\lambda e^{\lambda a - \lambda t} \xi_a - \int_a^t \lambda e^{\lambda(\theta - t)} g(\theta)\, d\theta + g(t),
$$

$$
\begin{aligned}
\eta_n(t) &:= \eta_0(t) + \frac{1}{\Gamma(1-\alpha)} \int_a^t \lambda e^{\lambda(\theta - t)} k(\theta) \\
&\quad \times \int_a^{\theta} \left(\log \frac{\theta}{s} \right)^{-\alpha} \eta_{n-1}(s)\, ds\, d\theta \\
&\quad - \frac{k(t)}{\Gamma(1-\alpha)} \int_a^t \left(\log \frac{t}{s} \right)^{-\alpha} \eta_{n-1}(s)\, ds.
\end{aligned}
$$

We bound η_0 by

$$
\|\eta_0\|_{C[0,T]} \le M_0 := \lambda|\xi_a| + 2\|g\|_{C[a,T]}. \tag{4.5}
$$

We subtract $\eta_n(t)$ from $\eta_{n+1}(t)$ to obtain the following equation for $n \ge 0$:

$$
\begin{aligned}
&|\eta_{n+1}(t) - \eta_n(t)| \\
&\quad = \left| \int_a^t \frac{\lambda e^{\lambda(\theta - t)} k(\theta)}{\Gamma(1-\alpha)} \int_a^{\theta} \frac{(\eta_n(s) - \eta_{n-1}(s))}{[\log(\theta/s)]^{\alpha}} ds\, d\theta \right. \\
&\quad\quad \left. - \frac{k(t)}{\Gamma(1-\alpha)} \int_a^t \frac{(\eta_n(s) - \eta_{n-1}(s))}{[\log(t/s)]^{\alpha}} ds \right|, \\
&\quad\quad t \in [a,T].
\end{aligned} \tag{4.6}
$$

Here, $\eta_{-1}(s) := 0$. We plug (4.5) into (4.6) with $n = 1$ and use $\int_a^{\theta} (\log \frac{\theta}{s})^{-\alpha} \frac{ds}{s} = \frac{1}{1-\alpha} (\log \frac{\theta}{a})^{1-\alpha}$ to obtain

$$
\begin{aligned}
|\eta_1(t) - \eta_0(t)| &\le \int_a^t \frac{t \lambda e^{\lambda(\theta - t)} |k(\theta)|}{\Gamma(1-\alpha)} \\
&\quad \times \int_a^{\theta} \frac{|\eta_0(s)|}{[\log(\theta/s)]^{\alpha}} \frac{ds}{s} d\theta \\
&\quad + \frac{t|k(t)|}{\Gamma(1-\alpha)} \int_a^t \frac{|\eta_0(s)|}{[\log(t/s)]^{\alpha}} \frac{ds}{s}
\end{aligned}
$$

$$\leq t\int_a^t \frac{\lambda e^{\lambda(\theta-t)}}{\Gamma(1-\alpha)}\frac{K_0M_0}{1-\alpha}\left(\log\frac{t}{a}\right)^{1-\alpha}d\theta + \frac{tK_0}{\Gamma(1-\alpha)}\int_a^t \frac{M_0}{[\log(t/s)]^\alpha}\frac{ds}{s}$$
$$\leq \frac{2K_0M_0t(\log\frac{t}{a})^{1-\alpha}}{\Gamma(2-\alpha)} \leq \frac{2K_0M_0t(\log\frac{T}{a})^{1-\alpha}}{\Gamma(2-\alpha)}, \quad (4.7)$$

where we have used $\int_a^t \lambda e^{\lambda(\theta-t)}d\theta = 1-e^{-\lambda(t-a)} \leq 1$. Assume that for some $n \geq 1$

$$|\eta_n(t)-\eta_{n-1}(t)| \leq \frac{(2K_0)^nM_0t^n\left(\log\frac{t}{a}\right)^{n(1-\alpha)}}{\Gamma(1+n(1-\alpha))}, \quad t\in[a,T]. \quad (4.8)$$

We combine (4.6)–(4.8) and use Lemma 2.2 to obtain

$$|\eta_{n+1}(t)-\eta_n(t)| \leq \frac{2K_0(2K_0)^nM_0t}{\Gamma(1+n(1-\alpha))\Gamma(1-\alpha)} \times \int_a^t\left(\log\frac{t}{s}\right)^{-\alpha}s^n\left(\log\frac{s}{a}\right)^{n(1-\alpha)}\frac{ds}{s}$$
$$\leq \frac{2K_0(2K_0)^nM_0t^{n+1}}{\Gamma(1+n(1-\alpha))\Gamma(1-\alpha)} \times \int_a^t\left(\log\frac{t}{s}\right)^{-\alpha}\left(\log\frac{s}{a}\right)^{n(1-\alpha)}\frac{ds}{s}$$
$$= \frac{(2K_0)^{n+1}M_0t^{n+1}(\log\frac{t}{a})^{(n+1)(1-\alpha)}}{\Gamma(1+n(1-\alpha))\Gamma(1-\alpha)} \times B(1-\alpha,1+n(1-\alpha))$$
$$\leq \frac{(2K_0T)^{n+1}M_0(\log\frac{t}{a})^{(n+1)(1-\alpha)}}{\Gamma(1+(n+1)(1-\alpha))}.$$

By mathematical induction, (4.8) holds for any $n\in\mathbb{N}$. Using the boundedness of the Mittag-Leffler function

$$\sum_{j=0}^{\infty}\frac{M_0(2K_0T)^j(\log\frac{t}{a})^{j(1-\alpha)}}{\Gamma(1+j(1-\alpha))} = M_0E_{1-\alpha,1}\left[2K_0T\left(\log\frac{t}{a}\right)^{1-\alpha}\right] < \infty, \quad t\in[a,T]$$

we conclude that the uniformly convergent limit η given by

$$\eta(t) := \lim_{n\to+\infty}\eta_n(t) = \sum_{n=1}^{\infty}(\eta_n(t)-\eta_{n-1}(t)) + \eta_0(t) \in C[a,T],$$

satisfies Eq. (4.3) with the estimate $\|\eta\|_{C[a,T]} \leq Q(\lambda|\xi_a| + \|g\|_{C[a,T]})$, which further leads to (4.4).

If there exists another C^1 solution $\tilde{\xi}$ to (4.1), then $\zeta = \xi - \tilde{\xi} \in C^1[a,T]$ satisfies

$$|\zeta'| = \left|\frac{1}{\Gamma(1-\alpha)}\int_a^t \lambda e^{\lambda(\theta-t)}k(\theta) \times \int_a^\theta\left(\log\frac{\theta}{s}\right)^{-\alpha}\zeta'(s)dsd\theta - \frac{k(t)}{\Gamma(1-\alpha)}\int_a^t\left(\log\frac{t}{s}\right)^{-\alpha}\zeta'(s)ds\right|$$
$$\leq Q(\lambda)\left[\int_a^\theta|\zeta'(s)|ds + \int_a^t\left(\log\frac{t}{s}\right)^{-\alpha}|\zeta'(s)|ds\right] \leq Q(\lambda)\int_a^t\left(\log\frac{t}{s}\right)^{-\alpha}|\zeta'(s)|ds.$$

By the Grönwall inequality in Lemma 2.1 we conclude that $\zeta' \equiv 0$ and hence $\zeta \equiv 0$ on $[a,T]$ by the homogeneous initial condition, which proves the uniqueness. □

5. WELL-POSEDNESS AND REGULARITY OF CAPUTO–HADAMARD TFPDE

In this section, we prove the well-posedness of model (1.2) and the regularity of its solutions.

Theorem 5.1. *If $u_a \in \check{H}^{r+2}$, $f \in H^\nu(\check{H}^r)$ with $r > d/2$ and $\nu > 1/2$ and $k(t) \in C[a,T]$ holds, then problem (1.2) has a unique solution $u \in C^1([a,T];\check{H}^r)$ and the stability estimate holds for $Q = Q(\alpha, \|k\|_{C[a,T]}, T)$*

$$\|u\|_{C^1([a,T];\check{H}^s)} \leq Q(\|u_a\|_{\check{H}^{2+s}} + \|f\|_{H^\nu(\check{H}^s)}), \quad 0\leq s\leq r.$$

Proof. We express the solution u and f in (1.2) in terms of $\{\phi_i\}_{i=1}^{\infty}$ as

$$u(\boldsymbol{x},t)=\sum_{i=1}^{\infty}u_i(t)\phi_i(\boldsymbol{x}),\quad u_i(t):=(u(\cdot,t),\phi_i),\quad t\in[a,T],$$
$$f(\boldsymbol{x},t)=\sum_{i=1}^{\infty}f_i(t)\phi_i(\boldsymbol{x}),\quad f_i(t):=(f(\cdot,t),\phi_i),\quad t\in[a,T], \tag{5.1}$$

where ϕ_i is the eigenfunction of problem defined in (2.1). We plug these expansions into (1.2) to conclude that for $(\boldsymbol{x},t)\in\Omega\times(a,T]$

$$\sum_{i=1}^{\infty}(u_i'(t)+k(t)_a\mathcal{D}_t^{\alpha}u_i(t))\phi_i(\boldsymbol{x}) = \sum_{i=1}^{\infty}(-\lambda_i u_i(t)+f_i(t))\phi_i(\boldsymbol{x}). \tag{5.2}$$

Hence, u is a solution to problem (1.2) if and only if $\{u_i\}_{i=1}^{\infty}$ satisfy

$$u_i'(t)+k(t)\,{}_a\mathcal{D}_t^{\alpha}u_i(t)=-\lambda_i u_i(t)+f_i(t),\quad t\in(a,T], \tag{5.3}$$
$$u_i(a)=u_{a,i}:=(u_a,\phi_i),\quad i=1,2,\ldots.$$

Note that the above equation has exactly the same form as (4.1) with ξ, λ, g replaced by u_i, λ_i, f_i, respectively. Then by Theorem 4.1, problem (5.3) has a unique solution $u_i\in C^1[a,T]$ and the estimate (4.4) holds by similarly modifying the data. For any $k\in\mathbb{N}$, we use Sobolev embedding and estimate (4.4) to conclude that $S_n(\boldsymbol{x},t):=\sum_{i=1}^{n}u_i(t)\phi_i(\boldsymbol{x})$ satisfies for $n\to\infty$

$$\begin{aligned}\|S'_{n+k}-S'_n\|^2_{C([a,T];C(\bar\Omega))} &\le Q\left\|\sum_{i=n+1}^{n+k}u_i'(t)\phi_i(\boldsymbol{x})\right\|^2_{C([a,T];H^{\gamma}(\Omega))}\\ &\le Q\sum_{i=n+1}^{n+k}\lambda_i^{\gamma}\|u_i\|^2_{C^1[a,T]}\\ &\le Q\sum_{i=n+1}^{n+k}\lambda_i^{\gamma}(\lambda_i^2|u_{a,i}|^2+\|f_i\|^2_{C[a,T]})\to 0.\end{aligned}$$

Hence, the interchange of the differentiation with the summation in (5.2) is satisfied, from which we conclude that u defined in (5.1) belongs to $C^1(H^{\gamma})$ and satisfies problem (1.2). Moreover,

$$\begin{aligned}\|u\|^2_{C^1(\check H^s)} &\le Q\sum_{i=1}^{\infty}\lambda_i^s\|u_i\|^2_{C^1[a,T]}\\ &\le Q\sum_{i=1}^{\infty}\lambda_i^s(\lambda_i^2u_{a,i}^2+\|f_i\|^2_{C[a,T]})\\ &\le Q(\|u_a\|^2_{\check H^{2+s}}+\|f\|^2_{H^{\nu}(\check H^s)}).\end{aligned}$$

The uniqueness of the solution to problem (1.2) follows from that of (5.3). □

Theorem 5.2. *If $u_a\in\check H^{4+s}$, $f\in H^{\nu}(\check H^{2+s})\cap H^{1+\nu}(\check H^s)$ for some $s\ge 0$ and $\nu>1/2$ and $k(t)\in C[a,T]$ holds, $k\in C^1[a,T]$, $u\in C^2((a,T];\check H^s)$ and the following interior estimate holds for any $t\in(a,T]$:*

$$\|u\|_{C^2((a,T];\check H^s)} \le Q\left(\log\frac{t}{a}\right)^{-\alpha}(\|u_a\|_{\check H^{4+s}}+\|f\|_{H^{\nu}(\check H^{2+s})} +\|f\|_{H^{1+\nu}(\check H^s)}), \tag{5.4}$$

with $Q=Q(\alpha,\|k\|_{C^1[a,T]},T)$.

Proof. Similar to (4.1)–(4.3), (5.3) can be rewritten in terms of $v=u_i'(t)$

$$\begin{aligned}v(t)&=-\lambda_i e^{\lambda_i a-\lambda_i t}u_{a,i}\\ &\quad+\frac{1}{\Gamma(1-\alpha)}\int_a^t\lambda_i e^{\lambda_i(\theta-t)}k(\theta)\\ &\quad\times\int_a^{\theta}\left(\log\frac{\theta}{s}\right)^{-\alpha}v(s)dsd\theta\\ &\quad-\frac{k(t)}{\Gamma(1-\alpha)}\int_a^t\left(\log\frac{t}{s}\right)^{-\alpha}v(s)ds\\ &\quad-\int_a^t\lambda e^{\lambda(\theta-t)}f_i(\theta)d\theta+f_i(t).\end{aligned} \tag{5.5}$$

We first prove that v is differentiable. By Theorem 4.1, Eq. (5.5) has s unique solution $v\in C[a,T]$ and (4.4) holds. We multiply (5.5) by $\log\frac{t}{a}$ and use $\log\frac{t}{a}=\log\frac{s}{a}+\log\frac{t}{s}$ to split the third term on the right-hand side of (5.5) to reformulate (5.5) in terms of $v(t)\log\frac{t}{a}$

$$v(t)\log\frac{t}{a}=-k(t)_a\mathcal{J}_t^{1-\alpha}\left(tv(t)\log\frac{t}{a}\right)$$

$$
\begin{aligned}
&-\frac{k(t)}{\Gamma(1-\alpha)}\int_a^t\left(\log\frac{t}{s}\right)^{1-\alpha}v(s)ds\\
&+\frac{\log\frac{t}{a}}{\Gamma(1-\alpha)}\int_a^t\lambda_i e^{\lambda_i(\theta-t)}k(\theta)\\
&\times\int_a^\theta\left(\log\frac{\theta}{s}\right)^{-\alpha}v(s)dsd\theta-\log\frac{t}{a}\\
&\times\lambda_i e^{-\lambda_i t}u_{a,i}-\log\frac{t}{a}\\
&\times\int_a^t\lambda e^{\lambda(\theta-t)}f_i(\theta)d\theta\\
&+\log\frac{t}{a}f_i(t).
\end{aligned}
$$

As $v\in C[a,T]$, all but the first terms on the right-hand side are in $C^1[a,T]$. Without loss of generality, let $N\in\mathbb{N}^+$ be such that $N(1-\alpha)<1$ and $(N+1)(1-\alpha)>1$. Then we apply Theorem 3.2 to conclude that $v(t)\log\frac{t}{a}\in C_{1-\alpha,\log}[a,T]$, and we then employ Theorem 3.4 to find $v(t)\log\frac{t}{a}\in C_{1-\alpha,\log}[a,T]$. We repeat this procedure N times to conclude that $v(t)\log\frac{t}{a}\in C_{N(1-\alpha),\log}[a,T]$. As $N(1-\alpha)+1-\alpha>1$, Theorem 3.3 leads to $v(t)\log\frac{t}{a}\in C^1[a,T]$, which means v is differentiable for $t\in(a,T]$.

To differentiate (5.5), we first differentiate the following integral:

$$
\begin{aligned}
&\left[\int_a^t\left(\log\frac{t}{s}\right)^{-\alpha}v(s)ds\right]'\\
&=\frac{d}{dt}\left[\int_a^t\left(\log\frac{t}{s}\right)^{-\alpha}sv(s)\frac{ds}{s}\right]\\
&=\frac{d}{dt}\left[\int_a^t sv(s)d\left(-\frac{(\log\frac{t}{s})^{1-\alpha}}{1-\alpha}\right)\right]\\
&=\frac{1}{t}\left[av(a)\left(\log\frac{t}{a}\right)^{-\alpha}\right.\\
&\quad\left.+\int_a^t\left(\log\frac{t}{s}\right)^{-\alpha}(v(s)+sv'(s))ds\right].
\end{aligned}
$$

We incorporate this to differentiate (5.5) as $v'(t)=\sum_{j=1}^4 F_j$ where

$$
\begin{aligned}
F_1(t)&=\frac{-1}{\Gamma(1-\alpha)}\int_a^t\lambda_i^2 e^{\lambda_i(\theta-t)}k(\theta)\\
&\quad\times\int_a^\theta\left(\log\frac{\theta}{s}\right)^{-\alpha}v(s)dsd\theta,\\
F_2(t)&=\frac{\lambda_i k(t)-k'(t)}{\Gamma(1-\alpha)}\int_a^t\left(\log\frac{t}{s}\right)^{-\alpha}v(s)ds,\\
F_3(t)&=\frac{-k(t)}{t\Gamma(1-\alpha)}\left[av(a)\left(\log\frac{t}{a}\right)^{-\alpha}\right.\\
&\quad\left.+\int_a^t\left(\log\frac{t}{s}\right)^{-\alpha}(v(s)+sv'(s))ds\right],\\
F_4(t)&=\lambda_i^2 e^{-\lambda_i t}u_{a,i}+f_i'(t)-\lambda_i f_i(t)\\
&\quad+\int_a^t\lambda_i^2 e^{\lambda_i(\theta-t)}f_i(\theta)d\theta.
\end{aligned}
$$

Let M_0 be defined in (4.5). We use (4.4) to bound F_1 by

$$
\begin{aligned}
|F_1|&\le\frac{M_0}{\Gamma(1-\alpha)}\int_a^t\lambda_i^2 e^{\lambda_i(\theta-t)}k(\theta)\\
&\quad\times\int_a^\theta\left(\log\frac{\theta}{a}\right)^{-\alpha}dsd\theta\\
&\le\frac{M_0 t}{\Gamma(1-\alpha)}\int_a^t\lambda_i^2 e^{\lambda_i(\theta-t)}k(\theta)\\
&\quad\times\int_a^\theta\left(\log\frac{\theta}{a}\right)^{-\alpha}\frac{ds}{s}d\theta\\
&=\frac{M_0 t}{\Gamma(2-\alpha)}\int_a^t\lambda_i^2 e^{\lambda_i(\theta-t)}k(\theta)\left(\log\frac{\theta}{a}\right)^{1-\alpha}d\theta\\
&\le\frac{M_0K_0T}{\Gamma(2-\alpha)}\left(\log\frac{t}{a}\right)^{1-\alpha}\lambda_i\int_a^t\lambda_i e^{\lambda_i(\theta-t)}d\theta\\
&\le\frac{\lambda_i M_0K_0T}{\Gamma(2-\alpha)}\left(\log\frac{t}{a}\right)^{1-\alpha}\\
&\le\lambda_i QM_0.
\end{aligned}
$$

We can similarly bound F_2 by

$$
\begin{aligned}
|F_2|&=\left|\frac{\lambda_i k(t)-k'(t)}{\Gamma(1-\alpha)}\int_a^t\left(\log\frac{t}{s}\right)^{-\alpha}v(s)ds\right|\\
&\le\frac{2\lambda_i M_0K_0T}{\Gamma(2-\alpha)}\left(\log\frac{t}{a}\right)^{1-\alpha}\\
&\le\lambda_i QM_0.
\end{aligned}
$$

We bound F_3 by

$$
|F_3|\le\left|\frac{-k(t)}{t\Gamma(1-\alpha)}\left[av(a)\left(\log\frac{t}{a}\right)^{-\alpha}\right]\right|
$$

$$
\begin{aligned}
&+\left|\frac{M_0K_0}{\Gamma(2-\alpha)}\left(\log\frac{t}{a}\right)^{1-\alpha}\right| \\
&+\left|\frac{K_0T}{\Gamma(1-\alpha)}\int_a^t\left(\log\frac{t}{s}\right)^{-\alpha}v'(s)\frac{ds}{s}\right| \\
\leq&\left|\frac{K_0M_0}{\Gamma(1-\alpha)}\left(\log\frac{t}{a}\right)^{-\alpha}\right| \\
&+\frac{M_0K_0}{\Gamma(2-\alpha)}\left(\log\frac{t}{a}\right)^{1-\alpha} \\
&+\left|\frac{K_0T}{\Gamma(1-\alpha)}\int_a^t\left(\log\frac{t}{s}\right)^{-\alpha}v'(s)\frac{ds}{s}\right| \\
\leq& QM_0\left(\log\frac{t}{a}\right)^{-\alpha}+QM_0 \\
&+\left|\frac{K_0T}{\Gamma(1-\alpha)}\int_a^t\left(\log\frac{t}{s}\right)^{-\alpha}v'(s)\frac{ds}{s}\right|.
\end{aligned}
$$

We bound F_4 by

$$
\begin{aligned}
|F_4| \leq\ & M_1 \\
&:= Q(\lambda_i\|f_i\|_{C[a,T]}+\|f_i'\|_{C[a,T]}+\lambda_i^2|u_{a,i}|).
\end{aligned}
$$

We incorporate the preceding estimates to bound $v'(t)$

$$
\begin{aligned}
|v'(t)| \leq Q\int_a^t\left(\log\frac{t}{s}\right)^{-\alpha}v'(s)\frac{ds}{s} \\
+QM_1\left(\log\frac{t}{a}\right)^{-\alpha}, \quad t\in(a,T].
\end{aligned}
$$

We apply Grönwall inequality (Lemma 2.1) to conclude that for $t\in(a,T]$

$$
\begin{aligned}
|v'(t)| \leq& QM_1\left(\log\frac{t}{a}\right)^{-\alpha}+QM_1\sum_{n=1}^{\infty}\frac{(Q\Gamma(1-\alpha))^n}{\Gamma(n(1-\alpha))} \\
&\times\int_a^t\left(\log\frac{t}{s}\right)^{-\alpha}\left(\log\frac{s}{a}\right)^{n(1-\alpha)-1}\frac{ds}{s} \\
\leq& QM_1\left(\log\frac{t}{a}\right)^{-\alpha}+QM_1\left(\log\frac{t}{a}\right)^{-\alpha} \\
&\times\sum_{n=1}^{\infty}\frac{[Q\Gamma(1-\alpha)(\log\frac{t}{a})^{1-\alpha}]^n}{\Gamma((n+1)(1-\alpha))} \\
\leq& QM_1\left(\log\frac{t}{a}\right)^{-\alpha}, \quad t\in(a,T].
\end{aligned}
$$

We then prove estimate (5.4). □

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