Unified approach for computing sum of sources over CQ-MAC

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***Abstract*—We consider the task of communicating a generic bivariate function of two classical sources over a Classical-Quantum Multiple Access Channel (CQ-MAC). The two sources are observed at the encoders of the CQ-MAC, and the decoder aims at reconstructing a bivariate function from the received quantum state. Inspired by the techniques developed for the classical setting, and employing the technique of simultaneous (joint) decoding developed for the CQ setup, we propose and analyze a coding scheme based on a classical superposition of algebraic structured codes and unstructured codes, and the idea of embedding functions on a prime field. We derive a new set of sufficient conditions that strictly enlarge the largest known set of sources (capable of communicating the bivariate function) for any given CQ-MAC. We provide these conditions in terms of single-letter quantum information-theoretic quantities.**

I. Introduction

In this work, we consider the problem of computing functions of information sources transmitted over a classical-quantum multiple access channel (CQ-MAC) $\mathcal{N}$. Computing functions of sources has been used in a wide range of practical applications. In the classical framework, these include, compute-and-forward strategy for wireless networks [1], interference management in the cellular uplink channel [2], and network coding [3]. As for the CQ setup, recent works have explored compute-and-forward (CAF) relaying technique in quantum one-hop relay network and symmetric private information retrieval (SPIR) over a quantum internet network [4].

The problem can be described as follows: consider a scenario where two distributed parties observe two classical information streams $S_{jt} \in \mathcal{S}_j : t \geqslant 1$, with the pair $(S_{1t}, S_{2t}) : t \geqslant 1$ being independent and identically distributed (IID) according to the distribution $\mathbb{W}_{S_1 S_2}$. These parties intend to send a bivariate function $f(S_1, S_2)$ to a centralized receiver using the CQ-MAC $\mathcal{N}$, and the receiver then aims to reconstruct the bivariate function f from the received quantum state. In this work, we aim to characterize the sufficient conditions, on the distribution of sources $\mathbb{W}_{S_1 S_2}$, such that for a given CQ-MAC, the centralized decoder reconstructs the bivariate function with an arbitrary low probability of error. The conventional approach to characterize the sufficient conditions for this problem rely on enabling the receiver to reconstruct the pair of classical sources, and then computing the function. This would be a direct consequence of the result derived in [5]. The authors in [4] addressed this CQ-MAC problem where the sum of sources is computed directly without explicitly reconstructing the individual sources. However, they restrict their attention to uniform input distribution. The authors in [6] instead employed a different technique, using asymptotically good random nested coset codes that directly reconstruct arbitrary function f of sources of arbitrary distributions. Their work was built on the earlier ideas, developed in the classical setup, of recovering the sum of sources without recovering either of the sources at the receiver [7]–[10]. These classical techniques are part of a broader framework for the multi-terminal problems and are characterized by codes with asymptotically large block-length and algebraic structure. These algebraic structured codes can achieve performance limits that are not achievable with conventional techniques using unstructured random codes [11].

However, even in the classical multi-terminal setup, the coding techniques relying on the algebraic structure may show gains for certain class of problems and in certain rate regimes. Therefore, a unified technique that captures the gain of both unstructured coding techniques and the algebraic structured coding techniques is needed to approach the performance limits for multi-terminal problems. In this regard, considering a multi-terminal problem of classical lossless distributed compression, Alhswede-Han [10] obtained the best known inner bound by combining the Slepian-Wolf [12] coding scheme with the algebraic structured based scheme of Kórner-Marton [7]. Motivated by this, we provide a unified approach for the problem of computing a bivariate function of two sources over CQ-MAC, capitalizing on the gains of the algebraic structured techniques developed in [6], while making the most of the unstructured coding approach developed for this problem [5].

We propose an approach where each transmitter intends to send two pieces of information, about its corresponding source, to the receiver. The first piece of information from both the sources needs to be reconstructed individually at the receiver. Then, conditioned on this reconstruction, we let the receiver reconstruct the necessary function f of the second piece. At the ith transmitter, the two pieces are constructed on auxiliary variables U_i and V_i, and then fused to form the channel input X_i. To model this transmission, as an intermediary step, we construct a $4-$input CQ-MAC with inputs (U_1, U_2, V_1, V_2), and the objective is to decode the triple $(U_1, U_2, V_1 \oplus_q V_2)$, where $\oplus_q$ represents addition with respect to a prime finite field $\mathbb{F}_q$. For this, the decoder needs a CQ simultaneous decoding technique. The ideas of joint typicality using tilting, smoothing, and augmentation introduced by Sen [13], [14] solved the problem of simultaneous decoding of individual messages on CQ-MAC, however, it is based on unstructured

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coding techniques. We develop a unified coding framework that combines unstructured and structured coding techniques while using this jointly typicality approach that enables the decoder to reconstruct $(U_1, U_2, V_1 \oplus_q V_2)$ simultaneously.

In light of this, the main contribution of the current work is in providing a new set of sufficient conditions (see Theorem 1 and 2), while strictly subsuming the current known conditions, for reconstructing arbitrary function of sources over a generic CQ-MAC. Furthermore, we have provided examples where the gains by our unified approach are demonstrated (see [15]). This work opens up an opportunity to investigate a generic approach encompassing both the conventional unstructured and algebraic structured techniques for other multi-terminal problems in the classical-quantum regime [16]–[18].

II. Preliminaries and Notation

Notation: We supplement the notation in [19] with the following. For positive integer n, $[n] \triangleq \{1, \cdots, n\}$. We employ an underline notation to aggregate objects of similar type. For example, $\underline{s}$ denotes (s_1, s_2), $\underline{x}^n$ denotes (x_1^n, x_2^n), $\underline{\mathcal{S}}$ denotes the Cartesian product $\mathcal{S}_1 \times \mathcal{S}_2$. Let c.c.$(\mathcal{S})$ denote the convex closure of the set $\mathcal{S}$.

Consider a (generic) 2-user *CQ-MAC* $\mathcal{N}_2$, which is specified through (i) finite sets $\mathcal{X}_j : j \in [2]$, (ii) Hilbert space $\mathcal{H}_Y$, and (iii) a collection $(\rho_{x_1,x_2} \in \mathcal{D}(\mathcal{H}_Y) : (x_1, x_2) \in \mathcal{X}_1 \times \mathcal{X}_2)$ of density operators. This CQ-MAC is employed to transmit a pair of sources such that the centralized receiver is capable of reconstructing a bivariate function of the classical information streams observed by the senders. Let $\mathcal{S}_1, \mathcal{S}_2$ be finite sets, and $(S_1, S_2) \in \mathcal{S}_1 \times \mathcal{S}_2$, distributed with PMF $\mathbb{W}_{S_1S_2}$, model the pair of information sources observed at the encoders. Specifically, sender j observes the sequence $S_{jt} \in \mathcal{S}_j : t \geqslant 1$. The sequence $(S_{1t}, S_{2t}) : t \geqslant 1$ is assumed to be IID with single-letter PMF $\mathbb{W}_{S_1S_2}$. The receiver aims to recover the sequence $f(S_{1t}, S_{2t}) : t \geqslant 1$ losslessly, where $f : \mathcal{S}_1 \times \mathcal{S}_2 \to \mathcal{R}$ is a specified bivariate function, and $\mathcal{R}$ is some finite set.

Definition 1. A CQ-MAC code $c_f = (n, e_1, e_2, \boldsymbol{\lambda})$ of block-length n for recovering f consists of two encoding maps $e_j : \mathcal{S}^n \to \mathcal{X}_j^n : j \in [2]$, and a POVM $\boldsymbol{\lambda} = \{\lambda_{r^n} \in \mathcal{P}(\mathcal{H}_Y) : r^n \in \mathcal{R}^n\}$. The average error probability of the c_f is

$$\bar{\xi}(c_f) = \quad 1 - \sum_{\underline{s}^n : f(\underline{s}^n) = r^n} \mathbb{W}^n_{S_1S_2}(s_1^n, s_2^n) \operatorname{Tr}\Big(\lambda_{r^n} \rho^{\otimes n}_{c,\underline{s}^n}\Big)$$

where $\rho^{\otimes n}_{c,\underline{s}^n} = \otimes_{i=1}^n \rho_{x_{1i}(s_1^n)x_{2i}(s_2^n)}$, $e_j(s_j^n) = (x_{j1}(s_j^n), x_{j2}(s_j^n), \cdots, x_{jn}(s_j^n))$ for $j \in [2]$.

Definition 2. A function f of the sources $\mathbb{W}_{S_1S_2}$ is said to be reconstructable over a CQ-MAC $\mathcal{N}_2$ if for $\epsilon > 0$, $\exists$ a sequence $c_f^{(n)} = (n, e_1^{(n)}, e_2^{(n)}, \boldsymbol{\lambda})$ such that $\lim_{n\to\infty} \bar{\xi}(c_f^{(n)}, \mathcal{N}_2) = 0$. Restricting f to a sum, we say the sum of sources $\mathbb{W}_{S_1S_2}$ over field $\mathbb{F}_q$ is reconstructable over a CQ-MAC if $\mathcal{S}_1 = \mathcal{S}_2 = \mathbb{F}_q$ and the function $f(S_1, S_2) = S_1 \oplus S_2$ is reconstructable over the CQ-MAC.

III. Main Results

Our objective is to develop a single-letter sufficient conditions for a given pair of sources $\mathbb{W}_{S_1,S_2}$ and function f that is reconstructible over a given CQ-MAC $\mathcal{N}_2$. As an intermediate step toward providing the main result, we present an intermediary result that will be useful in obtaining the main result and can also be of independent interest.

A. 4-to-3 decoding over CQ-MAC

In this subsection, we consider the problem of 4-to-3 decoding over a 4-user CQ-MAC, where the receiver aims to compute functions of messages of user 1 and 2, and the individual message of users 3 and 4. Consider a (generic) 4-user CQ-MAC $\mathcal{N}_4$, which is specified through (i) finite (input) sets $\mathcal{V}_j : j \in [2]$ and $\mathcal{U}_j : j \in [2]$, (ii) a (output) Hilbert space $\mathcal{H}_Z$, and (iii) a collection of density operators $(\rho_{v_1v_2u_1u_2} \in \mathcal{D}(\mathcal{H}_Z) : (v_1, v_2, u_1, u_2) \in \mathcal{V}_1 \times \mathcal{V}_2 \times \mathcal{U}_1 \times \mathcal{U}_2)$.

Definition 3. A code $c = \big(n, \mathbb{F}_q, e_{V_j} : j \in [2], e_{U_j} : j \in [2], \boldsymbol{\lambda}\big)$ of block-length n, for 4-to-3 decoding over $\mathcal{N}_4$ consists of four encoding maps $e_{V_j} : \mathbb{F}_q^l \to \mathcal{V}_j^n : j \in [2]$, $e_{U_j} : [q^{l_j}] \to \mathcal{U}_j^n : j \in [2]$, and a POVM $\boldsymbol{\lambda} = \{\lambda_{\{m^\oplus, m_3, m_4\}} \in \mathcal{P}(\mathcal{H}_Z) : (m^\oplus, m_3, m_4) \in \mathbb{F}_q^l \times [q^{l_1}] \times [q^{l_2}]\}$, where $m^\oplus \triangleq m_1 \oplus m_2$, l, l_1 and l_2 are positive integers, and q is a prime number.

Definition 4. Given a CQ-MAC $\mathcal{N}_4$, and a prime q, a rate triple $(R, R_1, R_2) > \underline{0}$ is said to be achievable for 4-to-3 decoding over the CQ-MAC if given any sequence of triples $(l(n), l_1(n), l_2(n))$, such that $\limsup_{n\to\infty} \frac{l(n)}{n} \log q < R$, $\limsup_{n\to\infty} \frac{l_i(n)}{n} \log q < R_i : i \in [2]$, and any sequence $p^{(n)}_{M_1M_2M_3M_4}$ of PMFs on $\mathbb{F}_q^l \times \mathbb{F}_q^l \times [q^{l_1}] \times [q^{l_2}]$, there exists a code $c^{(n)} = (n, \mathbb{F}_q, e_{V_j} : j \in [2], e_{U_j} : j \in [2], \boldsymbol{\lambda})$ for 4-to-3 decoding over CQ-MAC $\mathcal{N}_4$ of block-length n such that

$$\limsup_{n\to\infty} \bar{\xi}(c^{(n)}, \mathcal{N}_4) = \limsup_{n\to\infty} 1 - \sum_{\underline{m}} p_{\underline{M}}(\underline{m}) \operatorname{Tr}\Big(\lambda_{\{m^\oplus, m_3, m_4\}} \rho^{\otimes n}_{\underline{m}}\Big) = 0,$$

where $\rho^{\otimes n}_{\underline{m}} \triangleq \rho_{v_1^n(m_1)v_2^n(m_2)u_1^n(m_3)u_2^n(m_4)} = \otimes_{i=1}^n \rho_{v_{1i}(m_1)v_{2i}(m_2)u_{1i}(m_3)u_{2i}(m_4)}$ (assuming n-independent uses of $\mathcal{N}_4$). The convex hull of the union of the set of all achievable rate triples (R, R_1, R_2) is the capacity region of the 4-to-3 decoding over CQ-MAC $\mathcal{N}_4$ and prime number q.

Definition 5. Given a CQ-MAC $\mathcal{N}_4$ and a prime q, let $\mathscr{P}(\mathcal{N}_4,\ q)$ be defined as collection of PMF $\{p_{\underline{VU}} : p_{\underline{VU}} = p_{V_1}p_{V_2}p_{U_1}p_{U_2}$ is a PMF on $\underline{\mathcal{V}} \times \underline{\mathcal{U}}\}$. For $p_{\underline{VU}} \in \mathscr{P}(\mathcal{N}_4, q)$, let $\mathscr{R}(p_{\underline{VU}})$ be the set of rate triple (R, R_1, R_2) such that the following inequalities holds:

$$\begin{aligned}
R &\leqslant I(V; Z|U_1, U_2)_\sigma - I_{\max}(V_1, V_2, V)_\sigma,\\
R_1 &\leqslant I(U_1; Z|V, U_2)_\sigma,\\
R_2 &\leqslant I(U_2; Z|V, U_1)_\sigma,\\
R + R_1 &\leqslant I(V, U_1; Z|U_2)_\sigma - I_{\max}(V_1, V_2, V)_\sigma,\\
R + R_2 &\leqslant I(V, U_2; Z|U_1)_\sigma - I_{\max}(V_1, V_2, V)_\sigma,\\
R_1 + R_2 &\leqslant I(U_1, U_2; Z|V)_\sigma
\end{aligned}$$

$$R + R_1 + R_2 \leqslant I(V, U_1, U_2; Z) - I_{\max}(V_1, V_2, V)_\sigma,$$

where $I_{\max}(V_1, V_2, V)_\sigma = \max\{I(V_1; V)_\sigma, I(V_2; V)_\sigma\}$, $V = V_1 \oplus V_2$ and the mutual information quantities are taken with respect to the classical-quantum state $\sigma^{\underline{V}\underline{U}VZ} \triangleq$

$$\sum_{\underline{v},\underline{u},v} p_{V_1}(v_1)p_{V_2}(v_2)p_{U_1}(u_1)p_{U_2}(u_2)\mathbb{1}_{\{v=v_1\oplus v_2\}}|v\rangle\langle v|_V \otimes$$
$$|v_1\rangle\langle v_1|_{V_1} \otimes |v_2\rangle\langle v_2|_{V_2} \otimes |u_1\rangle\langle u_1|_{U_1} \otimes |u_2\rangle\langle u_2|_{U_2} \otimes \rho_{\underline{v}\underline{u}}.$$

Let $\mathscr{R}(\mathcal{N}_4, q) \triangleq \text{c.c.} \bigcup_{p_{\underline{V}\underline{U}} \in \mathscr{P}(\mathcal{N}_4,q)} \mathscr{R}(p_{\underline{V}\underline{U}})$.

Theorem 1. *If the rate triple $(R, R_1, R_2) \in \mathscr{R}(\mathcal{N}_4, q)$, then (R, R_1, R_2) is achievable for 4-to-3 decoding over a CQ-MAC $\mathcal{N}_4$ and prime q.*

Proof. The proof is provided in Section IV. □

B. Decoding Sum of Sources over CQ-MAC $\mathcal{N}_2$

Here we provide our main result characterizing the sufficient conditions on the sources, for the reconstruction of the bivariate function f at the centralized decoder of the given CQ-MAC.

Definition 6. The function $f : \underline{\mathcal{S}} \to \mathcal{R}$ of sources $\mathbb{W}_{S_1S_2}$ is said to be embeddable in a finite field $\mathbb{F}_q$ if there exists (i) a pair of functions $h_j : \mathcal{S}_j \to \mathbb{F}_q$ for $j : \{1, 2\}$, and (ii) a function $g : \mathbb{F}_q \to \mathcal{S}$, such that $f(s_1, s_2) = g(h_1(s_1) \oplus h_2(s_2))$[1].

Theorem 2. *Given the sources $(S_1, S_2, \mathbb{W}_{S_1S_2}, f)$, consider a prime q such that f is embeddable (according to Definition 6) in $\mathbb{F}_q$. Let $\mathcal{P}$ be the set of PMFs $P_{QW_1W_2|S_1S_2}$ defined on $\mathcal{Q} \times \mathcal{W}_1 \times \mathcal{W}_2$ such that (a) Q and (S_1, S_2) are independent, (b) $W_1 - S_1Q - S_2Q - W_2$ forms a Markov chain, and (c) $\mathcal{Q}, \mathcal{W}_1, \mathcal{W}_2$ are finite sets. For $P_{QW_1W_2|S_1S_2} \in \mathcal{P}$, let us define,*

$$\mathscr{R}_S(P_{QW_1W_2|S_1S_2}) \triangleq \Big\{(R, R_1, R_2) : R \geqslant H(S|W_1W_2Q),$$
$$R_1 \geqslant I(S_1; W_1|QW_2), R_2 \geqslant I(S_2; W_2|QW_1),$$
$$R_1 + R_2 \geqslant I(S_1S_2; W_1W_2|Q), \Big\}$$

where $S = h_1(S_1) \oplus h_2(S_2)$.

$$\mathscr{R}_s(\mathbb{W}_{S_1S_2}, f, q) \triangleq \text{c.c.} \bigcup_{p\in\mathcal{P}} \mathscr{R}_S(p).$$

Given a CQ-MAC $\mathcal{N}_2$, and prime q, let $\mathscr{P}$ be the set of PMFs $P_{X_1|U_1V_1}$ and $P_{X_2|U_2V_2}$ with the input alphabets $(\mathcal{U}_1, \mathcal{V}_1)$ and $(\mathcal{U}_2, \mathcal{V}_2)$, and output alphabets $\mathcal{X}_1$ and $\mathcal{X}_2$, respectively. Define,

$$\mathscr{R}_C(P_{X_1|U_1V_1}, P_{X_2|U_2V_2}, q) = \mathscr{R}(\mathcal{N}_4, q),$$

where the corresponding 4-user CQ-MAC $\mathcal{N}_4$ is characterized as: $\rho_{\underline{vu}} = \sum_{x_1x_2} P_{X_1|U_1V_1}(x_1|u_1v_1)P_{X_2|U_2V_2}(x_2|u_2v_2)\rho_{x_1x_2}$. Define,

$$\mathscr{R}_c(\mathcal{N}_2, q) \triangleq \text{c.c} \bigcup_{\{P_{X_j|U_jV_j} : j\in[2]\}\in\mathscr{P}} \mathscr{R}_C(P_{X_j|U_jV_j} : j \in [2]).$$

Then, if $\mathscr{R}_s(\mathbb{W}_{S_1S_2}, f, q) \subset \mathscr{R}_c(\mathcal{N}_2, f, q)$. for some prime q, then the bivariate function f of the sources $\mathbb{W}_{S_1S_2}$ is reconstructible over the CQ-MAC $\mathcal{N}_2$.

Proof. We use the approach of source channel separation with two modules. Consider a source given by $(\mathbb{W}_{S_1,S_2}, f)$. For the source part, the theorem requires showing the above source can be compressed to rates (R, R_1, R_2) that belongs to $\mathscr{R}_s(\mathbb{W}_{S_1,S_2}, f, q)$, which by using Ahlswede-Han [10] source coding scheme is achievable. This forms the source coding module. This module produces messages M_{j2}, M_{j1} at encoder $j \in [2]$, at rates R, R_j, respectively. As for the channel part, the task is to recover $(M_{12} \oplus M_{22}, M_{11}, M_{21})$ reliably and provide it to the source decoder. For this, we employ the result from Theorem 1, which shows that if the triple (R, R_1, R_2) belongs to $\mathscr{R}_c(\mathcal{N}_2, q)$, then for any arbitrary distribution of $p_{M_{11}M_{12}M_{21}M_{22}}$, such a recovery is guaranteed. This completes the proof of the theorem. □

IV. Proof of Theorem 1

Let $p_{\underline{V}\underline{U}} \in \mathscr{P}(\mathcal{N}_4, q)$ be a PMF on $\mathcal{V}_1 \times \mathcal{V}_2 \times \mathcal{U}_1 \times \mathcal{U}_2$ where $\mathcal{V}_1 = \mathcal{V}_2 = \mathbb{F}_q$. We begin by describing the coding scheme in terms of a specific class of codes. In order to choose codewords of a desired empirical distribution p_{V_j}, we employ Nested Coset Code (NCC), as described below.

Definition 7. An $(n, k, l, g_I, g_{O/I}, b^n, e)$ NCC built over a finite field $\mathcal{V} = \mathbb{F}_q$ comprises of (i) generator matrices $g_I \in \mathcal{V}^{k\times n}$, $g_{O/I} \in \mathcal{V}^{l\times n}$ (ii) a bias vector b^n, and (iii) an encoding map $e : \mathcal{V}^l \to \mathcal{V}^k$. We let $v^n(a, m) = ag_I \oplus_q mg_{O/I} \oplus_q b^n : (a, m) \in \mathcal{V}^k \times \mathcal{V}^l$, for $a = e(m)$, and denoted as a_m.

Now, we intend to use the above mapping $v^n(a, m)$ from $\mathcal{V}^l \to \mathcal{V}^n$, as a part of the encoder in relation to Definition 1. Both the encoders $e_{V_j} : j \in [2]$ employ cosets of the *same* linear code. We then consider a 4-to-3 decoding over a 'perturbed' variant of CQ-MAC, which we denote as $\mathcal{N}_4'$. The decoder wants to decode three messages simultaneously and hence we use the framework of CQ joint typicality developed using the ideas of tilting, smoothing and augmentation [14]. This allows us to perform *intersection of non-commuting POVM elements* to construct a set of POVMs for $\mathcal{N}_4'$. Finally, towards bounding the average error probability for $\mathcal{N}_4$, we use an argument, similar to [14, Equation 5], which shows that the outputs of the channel $\mathcal{N}_4'$ and $\mathcal{N}_4$ are indistinguishable in trace norm. Thus, the POVMs constructed for $\mathcal{N}_4'$ can be used for $\mathcal{N}_4$ with an additional boundable error term.

We now define a 4-to-3 decoding over 'perturbed' CQ-MAC $\mathcal{N}_4'$ that consists of the following: (i) Finite (augmented input) sets $(\mathcal{V}_j \times \mathcal{W}_{V_j})$, $(\mathcal{U}_j \times \mathcal{W}_{U_j}) : j \in [2]$. (ii) An (extended output) Hilbert space

$$\mathcal{H}_Z' = \bar{\mathcal{H}}_Z \bigoplus_{j\in[2]} (\bar{\mathcal{H}}_Z \otimes \mathcal{W}_{V_j}) \bigoplus_{j\in[2]} (\bar{\mathcal{H}}_Z \otimes \mathcal{W}_{U_j}),$$

where $\bar{\mathcal{H}}_Z = (\mathcal{H}_Z \otimes \mathbb{C}^2)$, and $\mathcal{W}_{V_j}$ denotes both a finite alphabet as well as a Hilbert space with dimension given by $|\mathcal{W}_{V_j}|$. The states in this Hilbert space are used as quantum

[1]Note that for any given function f, the set of prime q for which f is embeddable with respect to $\mathbb{F}_q$ is always non-empty [11, Def.3.7].

registers to store classical values. Similarly $\mathcal{W}_{U_j}$ is defined. (iii) A collection of density operators

$$\{\rho'_{\underline{vuw}} \in \mathcal{D}(\mathcal{H}'_Z) : (\underline{v}, \underline{u}, \underline{w}_V, \underline{w}_U) \in \underline{\mathcal{V}} \times \underline{\mathcal{U}} \times \underline{\mathcal{W}}_V \times \underline{\mathcal{W}}_U\},$$

where $\underline{w} = (w_{V_1}, w_{V_2}, w_{U_1}, w_{U_2})$, $\underline{w}_V = (w_{V_1}, w_{V_2})$, and $\underline{\mathcal{W}}_V = \mathcal{W}_{V_1} \times \mathcal{W}_{V_2}$. Similarly $\underline{w}_U$ and $\underline{\mathcal{W}}_U$ are defined.

Define $\rho'_{\underline{vuw}} \triangleq \mathcal{T}^{\mathrm{V\underline{U}}}_{\underline{w};\varepsilon}(\rho_{\underline{vu}} \otimes |0\rangle\langle 0|^{\mathbb{C}^2})$, where $\mathcal{T}^{\mathrm{V\underline{U}}}_{\underline{w};\varepsilon}$ is a *tilting map* [14, Section 4] from $\bar{\mathcal{H}}_Z$ to $\mathcal{H}'_Z$ defined as:

$$\mathcal{T}^{\mathrm{V\underline{U}}}_{\underline{w};\varepsilon}(|z\rangle) \triangleq \frac{1}{\sqrt{1+4\varepsilon^2}}(|z\rangle \bigoplus_j \varepsilon\, |z, w_{V_j}\rangle \bigoplus_j \varepsilon\, |z, w_{U_j}\rangle),$$

and ε will be chosen appropriately in the sequel.

Encoding: Consider two NCCs $(n, k, l, g_I, g_{O/I}, b_j^n, e_j)$ having the same parameters except with different bias vectors b_js and encoding maps e_js. For each $j \in [2]$ and $m_j \in \mathbb{F}_q^l$, let

$$\mathcal{A}_j(m_j) \triangleq \begin{cases} \{a_{m_j} : v_j^n(a_{m_j}, m_j) \in T_\delta^n(p_{V_j})\} & \text{if } \theta(m_j) \geqslant 1 \\ \{0^k\} & \text{otherwise,} \end{cases}$$

where $\theta(m_j) \triangleq \sum_{a \in \mathbb{F}_q^k} \mathbb{1}_{\{v_j^n(a, m_j) \in T_\delta^n(p_V)\}}$. For $m_j \in \mathbb{F}_q^l : j \in [2]$, a pre-determined element $a_{m_j} \in \mathcal{A}_j(m_j)$ is chosen and let $v_j^n(a_{m_j}, m_j) \triangleq a_{m_j} g_I \oplus m_j g_{O/I} \oplus b_j^n$ for $(a_{m_j}, m_j) \in \mathbb{F}_q^{k+l}$ for $j \in [2]$. Moreover, for each $j \in [2]$ and $m_{j+2} \in [q^{l_j}]$, construct a codeword $u_j^n(m_{j+2}) \in \mathcal{U}_j^n$. Similarly, for each $j \in [2]$, $m_j \in \mathbb{F}_q^l$ and $m_{j+2} \in [q_j^l]$, construct the codewords $w_{V_j}^n(m_j) \in \mathcal{W}_{V_j}^n$ and $w_{U_j}^n(m_{j+2}) \in \mathcal{W}_{U_j}^n$. For later convenience, we define an additional identical map $w_V^n(m) = w_{V_1}^n(m)$ for all $m \in \mathbb{F}_q^l$. On receiving the message $\underline{m} \in \mathbb{F}_q^l \times \mathbb{F}_q^l \times [q^{l_1}] \times [q^{l_2}]$, the quantum state $\rho'^{\otimes n}_{\underline{m}} \triangleq$

$$\rho'_{v_1^n(a_{m_1}, m_1) w_{V_1}^n(m_1) v_2^n(a_{m_2}, m_2) w_{V_2}^n(m_2)(u_1^n, w_{U_1}^n)(m_3)(u_2^n, w_{U_2}^n)(m_4)}$$

is (distributively) prepared. Towards specifying a decoding POVM's, we define the following associated density operators.

$$\rho \triangleq \sum_{\underline{v}^n, \underline{u}^n} p_{\underline{V}}^n(\underline{v}^n) p_{\underline{U}}^n(\underline{u}^n) \rho_{\underline{v}^n \underline{u}^n}, \quad (1)$$

$$\rho_{v^n} \triangleq \sum_{\underline{v}^n, \underline{u}^n} p_{\underline{V}|V}^n(\underline{v}^n | v^n) p_{\underline{U}}^n(\underline{u}^n) \rho_{\underline{v}^n \underline{u}^n},$$

$$\rho_{u_i^n} \triangleq \sum_{\underline{v}^n u_j^n} p_{U_j}^n(u_j^n) \rho_{\underline{v}^n \underline{u}^n} : i \neq j, i, j \in [2],$$

$$\rho_{v^n u_i^n} \triangleq \sum_{\underline{v}^n, u_j^n} p_{\underline{V}|V}^n(\underline{v}^n | v^n) p_{U_j}^n(u_j^n) \rho_{\underline{v}^n \underline{u}^n} : i \neq j, i, j \in [2],$$

where $p_{\underline{V}|V}^n(\underline{v}^n | v^n) \triangleq p_{\underline{V}}^n(\underline{v}^n) / p_V^n(v^n) \mathbb{1}_{\{v_1^n \oplus v_2^n = v^n\}}$.

Decoding: The decoder is designed to decode the sum of the messages $m^{\oplus}$ along with the individual messages m_3 and m_4 transmitted over the 'perturbed' 4-to-3 CQ-MAC $\mathcal{N}'_4$. To decode m_3 and m_4, we use the codebook used by the encoder, but to decode $m^{\oplus}$, we use the NCC $(n, k, l, g_I, g_{O/I}, b^n, e)$, with all the parameters same as the NCCs used in the encoding, except that $b^n = b_1^n \oplus b_2^n$, and e to be specified later. Define $v^n(a, m) \triangleq a g_I + m g_{O/I} + b^n$. representing a generic codeword and a generic coset, respectively.

POVM construction We start by defining the sub-POVMs for channel $\mathcal{N}$, subsequently we will construct the sub-POVMs for $\mathcal{N}'_4$ using the process of *tilting* [14]. Let π_ρ be the typical projector for the state ρ. Furthermore, for $j \in [2]$ and for all jointly typical vectors $(v^n, \underline{u}^n) \in \mathcal{T}_\delta^{(n)}(p_{V\underline{U}})$, let $\pi_{v^n}, \pi_{u_j^n}, \pi_{v^n u_j^n}, \pi_{\underline{u}^n}$ and $\pi_{v^n \underline{u}^n}$ be the conditional typical projector [19, Def. 15.2.4] with respect to the states $\rho_{v^n}, \rho_{u_i^n}, \rho_{v^n u_i^n}, \rho_{\underline{u}^n}$ and $\rho_{v^n \underline{u}^n}$, respectively. Now, we define the following sub-POVMs in the Hilbert space $\mathcal{H}_Z^{\otimes n}$:

$$\Pi^{\mathrm{V}}_{v^n \underline{u}^n} \triangleq \pi_\rho \pi_{v^n} \pi_{v^n \underline{u}^n} \pi_{v^n} \pi_\rho, \; \Pi^{\mathrm{U_j}}_{v^n \underline{u}^n} \triangleq \pi_\rho \pi_{u_j^n} \pi_{v^n \underline{u}^n} \pi_{u_j^n} \pi_\rho,$$
$$\Pi^{\mathrm{VU_j}}_{v^n \underline{u}^n} \triangleq \pi_\rho \pi_{v^n u_j^n} \pi_{v^n \underline{u}^n} \pi_{v^n u_j^n} \pi_\rho, \; \Pi^{\underline{\mathrm{U}}}_{v^n \underline{u}^n} \triangleq \pi_\rho \pi_{\underline{u}^n} \pi_{v^n \underline{u}^n} \pi_{\underline{u}^n} \pi_\rho,$$
$$\Pi^{\mathrm{V\underline{U}}}_{v^n \underline{u}^n} \triangleq \pi_\rho \pi_{v^n \underline{u}^n} \pi_\rho : i \neq j, i, j \in [2]. \quad (2)$$

The following is a well-known result regarding typical projectors and typical vectors $(\underline{v}^n, \underline{u}^n) \in \mathcal{T}_\delta^{(n)}(p_{\underline{V}\underline{U}})$.

Proposition 1. For all $\epsilon > 0$, and $\delta \in (0, 1)$ sufficiently small and n sufficiently large, and $i, j \in [2]$ with $i \neq j$ the following inequality holds for the sub-POVMs defined in (2).

$$\mathrm{Tr}\left(\Pi^{\Phi}_{v^n \underline{u}^n} \rho_{\underline{v}^n \underline{u}^n}\right) \geqslant 1 - \epsilon,$$
$$\text{for all } \Phi \in \{\mathrm{V}, \mathrm{U_j}, \mathrm{VU_j}, \underline{\mathrm{U}}, \mathrm{V\underline{U}}\},$$
$$\mathrm{Tr}\left(\Pi^{\mathrm{V\underline{U}}}_{v^n \underline{u}^n} \rho\right) \leqslant 2^{-n(I(V, U_1, U_2; Z)_\sigma - \epsilon)},$$
$$\sum_{\underline{u}^n} p_{\underline{U}}^n(\underline{u}^n)\, \mathrm{Tr}\left(\Pi^{\mathrm{V}}_{v^n \underline{u}^n} \rho_{v^n}\right) \leqslant 2^{-n(I(U_1, U_2; Z|V)_\sigma - \epsilon)},$$
$$\sum_{v^n} p_V^n(v^n)\, \mathrm{Tr}\left(\Pi^{\underline{\mathrm{U}}}_{v^n \underline{u}^n} \rho_{\underline{u}^n}\right) \leqslant 2^{-n(I(V; Z|U_1, U_2)_\sigma - \epsilon)},$$
$$\sum_{u_i^n} p_{U_i}^n(u_i^n)\, \mathrm{Tr}\left(\Pi^{\mathrm{VU_j}}_{v^n \underline{u}^n} \rho_{v^n u_j^n}\right) \leqslant 2^{-n(I(U_i; Z|U_j, V)_\sigma - \epsilon)},$$
$$\sum_{v^n u_i^n} p_V^n(v^n) p_{U_i}^n(u_i^n)\, \mathrm{Tr}\left(\Pi^{\mathrm{U_j}}_{v^n \underline{u}^n} \rho_{u_j^n}\right) \leqslant 2^{-n(I(V, U_i; Z|U_j)_\sigma - \epsilon)}.$$

After constructing the sub-POVMs, we now construct the projectors. It is worth to observe that by the Gelfand-Naimark theorem [20], there exists orthogonal projectors $\bar{\Pi}^{\mathrm{V}}_{v^n \underline{u}^n}$, $\bar{\Pi}^{\mathrm{U_j}}_{v^n \underline{u}^n}, \bar{\Pi}^{\mathrm{VU_j}}_{v^n \underline{u}^n}, \bar{\Pi}^{\underline{\mathrm{U}}}_{v^n \underline{u}^n}$ and $\bar{\Pi}^{\mathrm{V\underline{U}}}_{v^n \underline{u}^n}$ in $\bar{\mathcal{H}}_Z^{\otimes n}$ that gives the same measurements statistics on the states $\left(\sigma \otimes |0\rangle\langle 0|^{\mathbb{C}^{2n}}\right) \in \mathcal{D}(\bar{\mathcal{H}}_Z^{\otimes n})$ that sub-POVMs defined in (2) give on the states $\sigma \in \mathcal{D}(\mathcal{H}_Z^{\otimes n})$. To summarize upto this point we have constructed the projectors in $\bar{\mathcal{H}}_Z^{\otimes n}$ for the channel $\mathcal{N}_4$ using the sub-POVMs defined in (2), and we are now equipped to construct the sub-POVMs for $\mathcal{N}'_4$. Let us define $\bar{\Omega}^{\mathrm{V}}_{v^n u^n}$ as the orthogonal complement of the support of $\bar{\Pi}^{\mathrm{V}}_{v^n \underline{u}^n}$. Analogously, we define $\bar{\Omega}^{\mathrm{U_j}}_{v^n \underline{u}^n}, \bar{\Omega}^{\mathrm{VU_j}}_{v^n \underline{u}^n}, \bar{\Omega}^{\underline{\mathrm{U}}}_{v^n \underline{u}^n}$, and $\bar{\Omega}^{\mathrm{V\underline{U}}}_{v^n \underline{u}^n}$. Then we define the corresponding tilted subspace in $\mathcal{H}'^{\otimes n}_Z$ as: $\Omega^{\mathrm{V}}_{v^n \underline{u}^n w_V^n \underline{w}_U^n} \triangleq \mathcal{T}^{\mathrm{V}}_{w_V^n;\varepsilon}(\bar{\Omega}^{\mathrm{V}}_{v^n \underline{u}^n})$, for all $w_V^n \in \mathcal{W}_{V_1}^n$. Likewise, define $\Omega^{\mathrm{U_j}}_{v^n \underline{u}^n w_V^n \underline{w}_U^n}, \Omega^{\mathrm{VU_j}}_{v^n \underline{u}^n w_V^n \underline{w}_U^n}$ and $\Omega^{\underline{\mathrm{U}}}_{v^n \underline{u}^n w_V^n \underline{w}_U^n}$. Also, let us define a new subspace $\widehat{\Omega}_{v^n \underline{u}^n w_V^n \underline{w}_U^n}$, which is analogous to the 'union' of 'complement' of orthogonal projectors corresponding to the sub-POVMs defined in (2).

$$\widehat{\Omega}_{v^n \underline{u}^n w_V^n \underline{w}_U^n} \triangleq \bar{\Omega}^{\mathrm{V\underline{U}}}_{v^n \underline{u}^n} \bigoplus \Omega^{\mathrm{V}}_{v^n \underline{u}^n w_V^n \underline{w}_U^n} \bigoplus_{j \in [2]} \Omega^{\mathrm{U_j}}_{v^n \underline{u}^n w_V^n \underline{w}_U^n}$$
$$\bigoplus_{j \in [2]} \Omega^{\mathrm{VU_j}}_{v^n \underline{u}^n w_V^n \underline{w}_U^n} \bigoplus \Omega^{\underline{\mathrm{U}}}_{v^n \underline{u}^n w_V^n \underline{w}_U^n}. \quad (3)$$

Consider a collection of orthogonal projectors $\widehat{\Pi}'_{v^n\underline{u}^n w_V^n \underline{w}_U^n}$ in $\mathcal{H}_Z'^{\otimes n}$ projecting onto $\widehat{\Omega}_{v^n\underline{u}^n w_V^n \underline{w}_U^n}$, and the orthogonal projector $\widetilde{\Pi}'$ projecting onto $\bar{\mathcal{H}}_Z^{\otimes n}$. Subsequently, define the sub-POVMs in $\mathcal{H}_Z'^{\otimes n}$ for channel $\mathcal{N}_4'$ as follows:

$$\gamma_{v^n\underline{u}^n w_V^n \underline{w}_U^n} \triangleq \left(I - \widehat{\Pi}'_{v^n\underline{u}^n w_V^n \underline{w}_U^n}\right)\ \widetilde{\Pi}'\ \left(I - \widehat{\Pi}'_{v^n\underline{u}^n w_V^n \underline{w}_U^n}\right),$$

The decoder now uses the sub-POVMs $\gamma_{v^n\underline{u}^n w_V^n \underline{w}_U^n}$ as defined above, to construct a *square root measurement* [19], [20] to decode the messages, we define following operators,

$$\lambda_{(a,m),m_3,m_4} \triangleq \bar{\gamma}^{-1/2}\ \gamma_{(a,m),m_3,m_4} \bar{\gamma}^{-1/2}, \tag{4}$$

where $\bar{\gamma} = \left(\sum_{\hat{a},\hat{m}} \sum_{\hat{m}_3,\hat{m}_4} \gamma_{(\hat{a},\hat{m}),\hat{m}_3,\hat{m}_4}\right)$[2].

Distribution of Random Code: The distribution of the random code is completely specified through the distribution $\mathcal{P}(\cdot)$ of $G_I, G_{O/I}, B_j^n, A_{m_j},, W_{V_j}^n(m_j), U_j^n(m_{j+2})$, $W_{U_j}^n(m_{j+2})\colon j \in [2]$. We let

$$\mathcal{P}\begin{pmatrix} G_I = g_I, G_{O/I} = g_{O/I}, B_j^n = b_j^n, A_{m_j} = a_{m_j}, \\ U_j^n(m_{j+2}) = u_j^n(m_{j+2}), \\ W_{V_j}^n(m_j) = w_{v_j}^n(m_j), W_{U_j}^n(m_{j+2}) = w_{u_j}^n(m_{j+2}) \\ : j \in [2]\ ,\ \underline{m} \in \mathbb{F}_q^l \times \mathbb{F}_q^l \times [q^{l_1}] \times [q^{l_2}] \end{pmatrix}$$

$$= \prod_{j\in[2]} \frac{\mathbb{1}_{\{a_{m_j} \in \mathcal{A}_j(m_j)\}}}{\theta(m_j)|\mathcal{W}_{V_j}||\mathcal{W}_{U_j}|} p_{U_j}^n(u_j^n(m_{j+2})) \frac{1}{q^{kn+ln+2n}}. \tag{5}$$

Error Analysis: We derive an upper bound on $\bar{\xi}(c^{(n)}, \mathcal{N}_4')$, by averaging over the above ensemble. Our key insight for the error analysis will be similar to the those adopted in proof of [6, Theorem 2] and [14, Section 4]. Using the encoding and decoding rule stated above, the average probability of error of the code is given as,

$$\bar{\xi}(c^{(n)}, \mathcal{N}_4') = \sum_{\underline{m}} p_{\underline{M}}(\underline{m}) \mathrm{Tr}\left\{\left(I - \sum_a \lambda_{(a,m^\oplus),m_3,m_4}\right) \rho_{\underline{m}}'^{\otimes n}\right\}$$
$$\leqslant \sum_{\underline{m}} p_{\underline{M}}(\underline{m}) \mathrm{Tr}\left\{\left(I - \lambda_{(a^\oplus,m^\oplus),m_3,m_4}\right) \rho_{\underline{m}}'^{\otimes n}\right\}$$

where $a^\oplus \triangleq a_{m_1} \oplus a_{m_2}$ and $\rho'_{\underline{m}}$ is as defined in the Encoding section (Sec. IV). Now consider the event,

$$\mathscr{E} \triangleq \left\{\begin{pmatrix} V_1^n(A_{m_1}, m_1), V_2^n(A_{m_2}, m_2), \\ U_1^n(m_3), U_2^n(m_4), V^n(A^\oplus, m^\oplus) \end{pmatrix} \in T_{8\delta}^{(n)}(p_{\underline{VUV}})\right\}$$

where $V^n(A^\oplus, m^\oplus) \triangleq V_1^n(A_{m_1}, m_1) \oplus V_2^n(A_{m_2}, m_2)$. Then,

$$\mathbb{E}_{\mathcal{P}}\left\{\bar{\xi}(c^{(n)}, \mathcal{N}_4')\right\} = \mathbb{E}_{\mathcal{P}}\left\{\bar{\xi}(c^{(n)}, \mathcal{N}_4')\mathbb{1}_{\mathscr{E}^c} + \bar{\xi}(c^{(n)}, \mathcal{N}_4')\mathbb{1}_{\mathscr{E}}\right\},$$
$$\leqslant \underbrace{\mathbb{E}_{\mathcal{P}}\left\{\mathbb{1}_{\mathscr{E}^c}\right\}}_{T_1} + \underbrace{\mathbb{E}_{\mathcal{P}}\left\{\bar{\xi}(c^{(n)}, \mathcal{N}_4')\mathbb{1}_{\mathscr{E}}\right\}}_{T_2}.$$

To bound the error T_1, we provide the following proposition.

[2] $\gamma_{(a,m),m_3,m_4}$ is an abbreviation for $\gamma_{v^n(a,m)w_V^n(m)u_1^n(m_3)u_2^n(m_4)w_{U_1}^n(m_3)w_{U_2}^n(m_4)}$, and the perturbation w_V used by the decoder is, without loss of generality, identical to w_{V_1}.

Proposition 2. For all $\epsilon \in (0,1)$, and for all sufficiently large n and sufficiently small δ, we have $\mathbb{E}_{\mathcal{P}}\left\{\mathbb{1}_{\mathscr{E}^c}\right\} \leqslant \epsilon$, if $\frac{k}{n}\log q \geqslant \log q - \min\{H(V_1), H(V_2)\} + \delta$.

Proof. Refer to [9, Appendix B] for the proof. □

To upper bound the error probability T_2, we apply the Hayashi-Nagaoka inequality [21]. Then we get,

$$T_2 \leqslant \mathbb{E}_{\mathcal{P}}\left[2T_{20} + 4\left\{T_{2V} + \sum_j T_{2U_j} + \sum_j T_{2VU_j} + T_{2\underline{U}} + T_{2V\underline{U}}\right\}\right],$$

where, $T_{20} \triangleq 1 - \sum_{\underline{m}} p_{\underline{M}}(\underline{m}) \mathrm{Tr}\left(\Gamma_{(A\oplus,m\oplus),m_3,m_4} \rho_{\underline{m}}'^{\otimes n}\right)\mathbb{1}_{\mathscr{E}}$

$$T_{2U_1} \triangleq 4\sum_{\underline{m}} \sum_{\hat{m}_3 \neq m_3} p_{\underline{M}}(\underline{m}) \mathrm{Tr}\left(\Gamma_{(A\oplus,m\oplus),\hat{m}_3,m_4} \rho_{\underline{m}}'^{\otimes n}\right)\mathbb{1}_{\mathscr{E}},$$

$$T_{2V} \triangleq \sum_{\underline{m}} \sum_{\hat{a}} \sum_{\hat{m} \neq m\oplus} p_{\underline{M}}(\underline{m}) \mathrm{Tr}\left(\Gamma_{(\hat{a},\hat{m}),m_3,m_4} \rho_{\underline{m}}'^{\otimes n}\right)\mathbb{1}_{\mathscr{E}},$$

$$T_{2VU_1} \triangleq \sum_{\underline{m}} \sum_{\hat{a}} \sum_{\substack{\hat{m} \neq m\oplus \\ \hat{m}_3 \neq m_3}} p_{\underline{M}}(\underline{m}) \mathrm{Tr}\left(\Gamma_{(\hat{a},\hat{m}),\hat{m}_3,m_4} \rho_{\underline{m}}'^{\otimes n}\right)\mathbb{1}_{\mathscr{E}},$$

$$T_{2\underline{U}} \triangleq \sum_{\underline{m}} \sum_{\substack{\hat{m}_3 \neq m_3 \\ \hat{m}_4 \neq m_4}} p_{\underline{M}}(\underline{m}) \mathrm{Tr}\left(\Gamma_{(A\oplus,m\oplus),\hat{m}_3,\hat{m}_4} \rho_{\underline{m}}'^{\otimes n}\right)\mathbb{1}_{\mathscr{E}},$$

$$T_{2V\underline{U}} \triangleq \sum_{\underline{m}} \sum_{\hat{a}} \sum_{\substack{\hat{m} \neq m\oplus \\ \hat{m}_3 \neq m_3 \\ \hat{m}_4 \neq m_4}} p_{\underline{M}}(\underline{m}) \mathrm{Tr}\left(\Gamma_{(\hat{a},\hat{m}),\hat{m}_3,\hat{m}_4} \rho_{\underline{m}}'^{\otimes n}\right)\mathbb{1}_{\mathscr{E}},$$

and $\Gamma_{(A\oplus,m\oplus),\hat{m}_3,\hat{m}_4}$ is a randomized version of $\gamma_{(a\oplus,m\oplus),\hat{m}_3,\hat{m}_4}$. The analysis of all the above error terms are provided in [15]. Below, we summarize all the rate constraints obtained from bounding these error terms. For any $\epsilon \in (0,1)$ and for all sufficiently large n, we have $\mathbb{E}_{\mathcal{P}}\left\{\bar{\xi}(c^{(n)}, \mathcal{N}_4')\mathbb{1}_{\mathscr{E}}\right\} \leqslant \epsilon$, if the following inequalities holds:

$$\frac{2k + l_j}{n}\log q \leqslant 2\log q + I(U_j; Z|V, U_i) - H(V_1, V_2) - \epsilon,$$
$$\frac{2k + l_1 + l_2}{n}\log q \leqslant 2\log q + I(U_1, U_2; Z|V) - H(V_1, V_2) - \epsilon,$$
$$\frac{3k + l}{n}\log q \leqslant 3\log q + I(V; Z|U_1, U_2) - H_{V_1,V_2} - \epsilon,$$
$$\frac{3k + l + l_j}{n}\log q \leqslant 3\log q + I(V, U_j; Z|U_i) - H_{V_1,V_2} - \epsilon,$$
$$\frac{3k + l + l_1 + l_2}{n}\log q \leqslant 3\log q + I(V, U_1, U_2; Z) - H_{V_1,V_2} - \epsilon,$$

where $i, j \in [2], i \neq j$, and $H_{V_1,V_2} = H(V_1, V_2) + H(V)$.

Now, we need to bound average error probability for $\mathcal{N}_4$. For any $\epsilon \in (0,1)$, if we let $\tau = \epsilon^{1/4}$, and use the following argument, $\left\|\rho'_{\underline{v}^n\underline{u}^n w^n} - \tilde{\rho}_{\underline{v}^n\underline{u}^n}\right\|_1 \leqslant 4\tau$ (similar to the provided in [14, Equation 5]) and the trace inequality $\mathrm{Tr}\{\Delta\rho\} \leqslant \mathrm{Tr}\{\Delta\sigma\} + \frac{1}{2}\|\rho - \sigma\|_1$, where $0 \leqslant \Delta, \rho, \sigma \leqslant I$, then for all sufficiently large n, we have $\bar{\xi}(c^{(n)}, \mathcal{N}_4) \leqslant \bar{\xi}(c^{(n)}, \mathcal{N}_4') + 2\epsilon^{1/4}$. In other words, the average decoding error for CQ-MAC $\mathcal{N}_4$ is bounded from above by the average decoding error for CQ-MAC $\mathcal{N}_4'$ with an additional error of $2\epsilon^{1/4}$ for the same rate constraints and decoding strategy used for $\mathcal{N}_4'$. This concludes the proof of Theorem 1.

References

[1] B. Nazer and M. Gastpar, "Compute-and-forward: Harnessing interference through structured codes," *IEEE Transactions on Information Theory*, vol. 57, no. 10, pp. 6463–6486, 2011.

[2] B. Nazer, A. Sanderovich, M. Gastpar, and S. Shamai, "Structured superposition for backhaul constrained cellular uplink," in *2009 IEEE International Symposium on Information Theory*, 2009, pp. 1530–1534.

[3] N. B. Shah, K. Rashmi, P. V. Kumar, and K. Ramchandran, "Interference alignment in regenerating codes for distributed storage: Necessity and code constructions," *IEEE Transactions on Information Theory*, vol. 58, no. 4, pp. 2134–2158, 2011.

[4] M. Hayashi and Á. Vázquez-Castro, "Computation-aided classical-quantum multiple access to boost network communication speeds," *Physical Review Applied*, vol. 16, no. 5, p. 054021, 2021.

[5] A. Winter, "The capacity of the quantum multiple-access channel," *IEEE Transactions on Information Theory*, vol. 47, no. 7, pp. 3059–3065, 2001.

[6] T. A. Atif, A. Padakandla, and S. S. Pradhan, "Computing sum of sources over a classical-quantum mac," in *2021 IEEE International Symposium on Information Theory (ISIT)*, 2021, pp. 414–419.

[7] J. Korner and K. Marton, "How to encode the modulo-two sum of binary sources (corresp.)," *IEEE Transactions on Information Theory*, vol. 25, no. 2, pp. 219–221, 1979.

[8] B. Nazer and M. Gastpar, "Computation over multiple-access channels," *IEEE Transactions on information theory*, vol. 53, no. 10, pp. 3498–3516, 2007.

[9] A. Padakandla and S. S. Pradhan, "Computing sum of sources over an arbitrary multiple access channel," in *2013 IEEE International Symposium on Information Theory*. IEEE, 2013, pp. 2144–2148.

[10] R. Ahlswede and T. Han, "On source coding with side information via a multiple-access channel and related problems in multi-user information theory," *IEEE Transactions on Information Theory*, vol. 29, no. 3, pp. 396–412, 1983.

[11] S. S. Pradhan, A. Padakandla, and F. Shirani, *An Algebraic and Probabilistic Framework for Network Information Theory*. Foundations and Trends in Communications and Information Theory, 2021, vol. 18, no. 2.

[12] D. Slepian and J. Wolf, "Noiseless coding of correlated information sources," *IEEE Transactions on information Theory*, vol. 19, no. 4, pp. 471–480, 1973.

[13] P. Sen, "A one-shot quantum joint typicality lemma. arxiv e-prints, page," *arXiv preprint arXiv:1806.07278*, 2018.

[14] ——, "Unions, intersections and a one-shot quantum joint typicality lemma," *Sādhanā*, vol. 46, no. 1, pp. 1–44, 2021.

[15] M. A. Sohail, T. A. Atif, S. S. Pradhan, and A. Padakandla, "Unified approach for computing sum of sources over cq-mac," *arXiv preprint arXiv: 2202.10403*, 2022.

[16] O. Fawzi, P. Hayden, I. Savov, P. Sen, and M. M. Wilde, "Classical communication over a quantum interference channel," *IEEE Transactions on Information Theory*, vol. 58, no. 6, pp. 3670–3691, 2012.

[17] I. Savov, "Network information theory for classical-quantum channels," *arXiv preprint arXiv:1208.4188*, 2012.

[18] T. A. Atif, A. Padakandla, and S. S. Pradhan, "Achievable rate-region for 3—user classical-quantum interference channel using structured codes," in *2021 IEEE International Symposium on Information Theory (ISIT)*, 2021, pp. 760–765.

[19] M. M. Wilde, *Quantum Information Theory*, 1st ed. USA: Cambridge University Press, 2013.

[20] A. S. Holevo, *Quantum systems, channels, information: a mathematical introduction*. Walter de Gruyter, 2012, vol. 16.

[21] M. Hayashi and H. Nagaoka, "General formulas for capacity of classical-quantum channels," *IEEE Transactions on Information Theory*, vol. 49, no. 7, pp. 1753–1768, 2003.