Lattices from Linear Codes: Source and Channel Networks

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***Abstract*—The paper addresses the fundamental information theoretic limits — in terms of achievable rates and distortions — in a broad class of multiterminal communication scenarios with general continuous-valued sources and channels. A general framework is presented which involves fine discretization of the source and channel variables followed by communication over the resulting discretized network. In order to evaluate fundamental performance limits under the proposed discretization process, convergence results for information measures are provided. The framework is used to study the distributed source coding in source coding, as well as the computation over multiple access channels in channel coding. In each case, a communication scheme is presented, the resulting achievable region is derived, and the region is evaluated for Gaussian sources and channels.**

I. Introduction

Information theory has provided a framework for the study of the fundamental limits of communication — such as achievable rates and distortions — and design of source and channel coding strategies in a wide range of communication scenarios. This has led to the characterization of the optimal rate-distortion function in point-to-point (PtP) source coding and channel capacity in PtP channel coding. These techniques were extended to various multiterminal communication scenarios, where inner and outer bounds for the set of achievable rates and distortions for storage of discrete sources and transmission over discrete channels were derived [1], and the optimal achievable region was derived for special cases of interest such as communication over discrete multiple access channel (MAC) [2], discrete deterministic and semi-deterministic broadcast channels [3], and multiple-descriptions source coding in the no-excess rate regime [4]. However, there is still a lack of a unifying framework for the study of data compression for *general continuous sources* and data transmission over *general continuous channels*.

Many of the derivations in the discrete case rely on the concept of strong typicality which is based on the frequency of occurrence of symbols in sequences of discrete random variables [5]. The notion of strong typicality does not extend naturally to sequences of continuous variables. Prior works address the issue in an ad-hoc fashion. As a result, the optimal performance in terms of achievable rates and distortions is usually known only for special cases when the underlying distributions of all variables are restricted to be Gaussian variables, e.g. distributed storage of Gaussian sources [6], and communication over the Gaussian broadcast channel [7]. For certain problems, such as PtP source coding, PtP channel coding, and communication over MAC, performance limits have been derived using techniques using weak typicality [8] instead of strong typicality. However, weak typicality is not applicable in many multiterminal communication problems such as distributed source coding, and communication over broadcast channels, since for instance, the Markov lemma [9] (a crucial step in the derivation of achievable regions) is not valid for weakly typical sequences. To address this, Wyner [10] proposed a method for the study of PtP source coding with side-information, which can also be used for distributed source coding. Wyner's method involves fine quantization of the source, the side-information, and the auxiliary variables to create a finite-alphabet problem, and then using the achievable results for the finite-alphabet problem to derive performance limits for the original problem using convergence properties of mutual information. This idea of 'discretizing' the continuous communication system, and then applying discrete coding strategies and analytical techniques has also been recently used in the study of the compute-and-forward communication scenario [11], as well as in investigating the correspondence between a set of useful inequalities in terms of entropies of discrete variables and their analogs in terms of differential entropies of continuous variables [12]. Another method to study continuous networks is to modify the notion of weak typicality, and use weak-* typicality instead. In [13], the Markov lemma has been shown to hold for weak-* typical sequences. The results were applied to source compression in the presence of side-information. The derivations in [10], [13] are based on unstructured random code ensembles. An additional technique which has been considered for compression of linear quadratic Gaussian (LQG) sources and channels is to use subtractive dithered lattice codes [14], [15]. The drawback of these lattice codes is that (a) they are very specific to the LQG nature of the problem, and hence not amenable to non-Gaussian and nonlinear problems, and (b) they are based on point-to-point communication perspective, and hence not general enough to be extended to the multiuser techniques such as joint quantization as seen in multiple-description coding, and joint source-channel mapping as seen in transmission of correlated sources over multiple-access channels.

This work develops a unified framework to study the performance limits of communication for general continuous-valued sources and channels in network communication scenarios.

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This builds upon techniques introduced in [16] which studied the problem under restrictive assumptions on the source and channel distributions such as boundedness of the individual support, non-zero lower bound for the joint probability density function (PDF) over the bounded support, and Riemann integrability. The contributions are summarized as follows:

- We build upon Wyner's fine quantization technique, and derive covering bounds, packing bounds, and prove the Markov lemma when unstructured random code ensembles are used as well as for structured code ensembles.
- We apply these results to distributed source coding — where we extend the achievable region in [16] to source distributions which are not Riemann integrable — and to the computation over multiple access channel scenario.
- In each case, we evaluate the achievable region when Gaussian test channels are used. In the case of distributed source coding, we show that the region is strictly larger when the underlying discrete coding techniques use structured codes (e.g. linear codes) instead of unstructured codes.

Notation: Collections of sets are denoted by sans-serif letters such as B, C. The set $\{n, n+1, \cdots, m\}, n, m \in \mathbb{N}$ is denoted by $[n, m]$. For the interval $[1, m]$, we sometimes use the shorthand notation $[m]$. We write x^n to represent the vector $(x_1, x_2, \ldots, x_n)$. The Borel sigma algebra is denoted by $\mathscr{B}$.

II. Framework for Continuous to Discrete Source and Channel Transformation

This section introduces the components of the discretization framework which is considered in subsequent sections.

A. Source and Channel Models

We consider continuous memoryless source and channel networks with real-valued inputs and outputs, and without feedback. Such channel networks (source networks) are completely characterized by their associated channel transition probability (source distribution) and input cost functions (output distortion functions).

Definition 1 (**Transition Probability**). *A transition probability is a function $P : \mathbb{R} \times \mathscr{B} \to \mathbb{R}$ such that:*

- *For $x \in \mathbb{R}$, $P(\cdot|x) : \mathsf{A} \mapsto P(\mathsf{A}|x)$ is a probability measure.*
- *For $\mathsf{A} \in \mathscr{B}$, $P(\mathsf{A}|\cdot) : x \mapsto P(\mathsf{A}|x)$ is a measurable function.*

Remark 1. *We assume that for any variable X the PDF exists and it approaches infinity in at most a finite number of points, and that the set of points of discontinuity has (Lebesgue) measure zero. Furthermore, we assume that the variable does not have discrete points, i.e. $\nexists x \in \mathbb{R}, P(X = x) > 0$).*

Definition 2 (**Memoryless Channel without Feedback**). *A channel is characterized by i) a transition probability $P_{Y|X} : \mathbb{R} \times \mathscr{B} \to \mathbb{R}$, and ii) a continuous cost function $\kappa : \mathbb{R} \to \mathbb{R}^+$, where X and Y are the channel input and output, respectively.*

Since the channel is memoryless and used without feedback, the joint probability measure on $(\mathbb{R}^n, \mathscr{B}^n)$ for n channel-uses with input $x^n \in \mathbb{R}^n$ is given by the unique product measure

$$P(Y_i \in \mathsf{A}_i, i \in [n] | X^n = x^n) = \prod_{i=1}^{n} P_{Y|X}(\mathsf{A}_i | x_i),$$

for all $\mathsf{A}_1, \mathsf{A}_2, \cdots, \mathsf{A}_n \in \mathscr{B}$.

Definition 3 (**Joint Channel Probability Measure**). *For a channel $(P_{Y|X}, \kappa)$, and given probability measure P_X on $(\mathbb{R}, \mathscr{B})$, the joint probability measure P_{XY} on $(\mathbb{R}^2, \mathscr{B}^2)$ is the unique extension of the measure on product sets*

$$P_{XY}(\mathsf{A} \times \mathsf{B}) = \int_{\mathsf{A}} P_X(dx) \int_{\mathsf{B}} P_{Y|X}(dy|x), \qquad \mathsf{A}, \mathsf{B} \in \mathscr{B}.$$

Definition 4 (**Memoryless Source**). *A source is characterized by i) a probability measure $P_X : \mathscr{B} \to \mathbb{R}$, and ii) a jointly continuous distortion function $d : \mathbb{R} \times \mathbb{R} \to \mathbb{R}^+$.*

B. Source and Channel Discretization and Clipping

We will obtain achievable rate-distortion functions for source coding and achievable rate-cost functions for channel coding by first discretizing and clipping the associated random variables. This approach is described in the following.

Definition 5 (**Discretization function**). *Let $n \in \mathbb{N}$, The discretization function $Q_n : \mathbb{R} \to \mathbb{Z}_n$ is defined as*

$$Q_n(s) = \arg\min_{a \in \mathbb{Z}_n} |s - a|, s \in \mathbb{R}, \text{ where } \mathbb{Z}_n := \frac{1}{2^n}\mathbb{Z}.$$

Definition 6 (**Clipping Function**). *Given a random variable U defined on the probability space $(\mathbb{R}, \mathbb{B}, P_U)$, and given $\ell > 0$, the clipping function $C_\ell : \mathbb{R} \to \mathbb{R}$ is defined as*

$$C_\ell(U) := \begin{cases} U & \text{if } U \in [-\ell, \ell], \\ U' & \text{Otherwise.} \end{cases},$$

where U' is independent of U and $f_{U'}(\cdot) := f_{U|U \in [-\ell, \ell]}(\cdot)$.

Remark 2. *We denote $\widehat{S}_{\ell,n} = Q_n(C_\ell(S))$ as $\widehat{S}$, and $\widetilde{S}_\ell = C_\ell(S)$ by $\widetilde{S}$ when the subscript is clear from the context.*

In our derivations, we sometimes need to *smoothen* a given variable U using additive noise N_ϵ, which is uniformly distributed over $[-\epsilon, \epsilon]$ for some small $\epsilon > 0$. The following lemma shows that the smoothing noise becomes independent of the smoothened variable as $\epsilon \to 0$.

Lemma 1 (**Smoothing Noise**). *Consider a bounded continuous random variable U defined on the probability space $([-M, M], \mathscr{B}[-M, M], P_U)$, such that $h(U) < \infty$ and $M > 0$, and let N_ϵ be uniformly distributed over $[-\epsilon, \epsilon], \epsilon > 0$. Assume that U and N_ϵ are independent. Then,*

$$\lim_{\epsilon \to 0} I(N_\epsilon; U + N_\epsilon) = 0.$$

Proof. Please refer to [17]. □

The following lemma is used in proving convergence of information measures under the proposed discretization process.

Lemma 2. *For any quintuple of random variables A, B, C, D, E with joint distribution that satisfies the Markov chain $(A, B) - C - (DE)$, consider a pair of random variables*

$\hat{A}, \hat{E}$ that are correlated with (B, D) such that $P_{BA} = P_{B\hat{A}}$, $P_{DE} = P_{D\hat{E}}$, and $\hat{A} - B - C - D - \hat{E}$, then

$$I(A;C|B) + I(E;C|D) \geq \frac{1}{2\ln 2} V^2(P_{CAE}, P_{C\hat{A}\hat{E}}),$$

where $V(P,Q) := \sup_{\mathsf{A}} |P(\mathsf{A}) - Q(\mathsf{A})|$ is the total variation between P and Q.

III. Discretization of Random Variables and their Sums

In this section, we develop the framework that addresses both structured and unstructured code ensembles for multi-terminal communication. Consider jointly continuous random variables X, Y, U, V with a joint PDF $f_{XY} f_{U|X} f_{V|Y}$. That is the variables satisfy the Markov chain $U \leftrightarrow X \leftrightarrow Y \leftrightarrow V$. We denote the joint probability measure as P_{XYUV}.

A. Discretization of Auxiliary Random Variables U, V

Fix $\ell, \ell', \epsilon > 0$, and $n \in \mathbb{N}$. Define the clipped variables $\widetilde{U}_\ell := C_\ell(U), \widetilde{V}_{\ell'} = C_{\ell'}(V)$ as follows: We take ℓ, ℓ' sufficiently large. Next, define the smoothed random variables $\widetilde{U}_{\ell,\epsilon}, \widetilde{V}_{\ell',\epsilon}$, where

$$\widetilde{U}_{\ell,\epsilon} := \widetilde{U}_\ell + \widetilde{N}_\epsilon, \qquad \widetilde{V}_{\ell',\epsilon} := \widetilde{V}_\ell + \widetilde{N}'_\epsilon,$$
$$f_{\widetilde{N}_\epsilon}(\tilde{n}) = \frac{1}{2\epsilon}, \quad f_{\widetilde{N}'_\epsilon}(\tilde{n}') = \frac{1}{2\epsilon}, \quad \tilde{n}, \tilde{n}' \in (-\epsilon, \epsilon),$$

and the variables $\widetilde{N}_\epsilon$ and $\widetilde{N}'_\epsilon$ are mutually independent of each other and of $X, Y, V, U, U_\ell, V_{\ell'}$. Consider discretizing $\widetilde{U}_{\ell,\epsilon}$ and $\widetilde{V}_{\ell',\epsilon}$ to $\widehat{U}_{n,\ell,\epsilon} := Q_n(\widetilde{U}_{\ell,\epsilon})$ and $\widehat{V}_{n,\ell,\epsilon} := Q_n(\widetilde{V}_{\ell',\epsilon})$, respectively. Note that the by construction the Markov chain $\widehat{U}_{n,\ell,\epsilon} \leftrightarrow X \leftrightarrow Y \leftrightarrow \widehat{V}_{n,\ell,\epsilon}$ holds.

Theorem 1. *Given $\xi > 0$ there exists $n, \ell, \ell', \epsilon > 0$ such that*

$$|I(X; \widehat{U}_{n,\ell,\epsilon}) - I(X;U)| \leq \xi \tag{1}$$
$$|I(Y; \widehat{V}_{n,\ell',\epsilon}) - I(Y;V)| \leq \xi \tag{2}$$
$$|I(\widehat{U}_{n,\ell,\epsilon} + \widehat{V}_{n,\ell',\epsilon}; \widehat{U}_{n,\ell,\epsilon}) - I(U+V;U)| \leq \xi \tag{3}$$
$$|I(\widehat{U}_{n,\ell,\epsilon} + \widehat{V}_{n,\ell',\epsilon}; \widehat{V}_{n,\ell',\epsilon}) - I(U+V;V)| \leq \xi, \tag{4}$$
$$|I(\widehat{U}_{n,\ell,\epsilon}; \widehat{V}_{n,\ell',\epsilon}) - I(U;V)| \leq \xi, \tag{5}$$

Furthermore, given a pair of jointly continuous distortion functions $d_i : \mathbb{R}^2 \to \mathbb{R}^+, i \in \{1,2\}$ and continuous reconstruction functions $g_i : \mathbb{R}^2 \to \mathbb{R}$, we have:

$$|\mathbb{E}(d_1(X, g_1(\widehat{U}_{n,\ell,\epsilon}, \widehat{V}_{n,\ell',\epsilon}))) - \mathbb{E}(d_1(X, g_1(U,V)))| \leq \xi \tag{6}$$
$$|\mathbb{E}(d_2(Y, g_2(\widehat{U}_{n,\ell,\epsilon}, \widehat{V}_{n,\ell',\epsilon}))) - \mathbb{E}(d_2(Y, g_2(U,V)))| \leq \xi \tag{7}$$

Proof. Please refer to [17]. □

B. Discretization of the Source Variables X, Y

In the following, we describe the procedure for discretizing the source variables while ensuring that the long Markov chain holds. Let $\ell, \ell' > 0$, and Z and W be variables that are independent of (X,Y) with distribution $P_Z P_W$ given by

$$P_Z P_W(\mathsf{A} \times \mathsf{B}) = \frac{P_X(\mathsf{A} \cap [-\ell, \ell]) P_Y(\mathsf{B} \cap [-\ell', \ell'])}{P_X([-\ell,\ell]) P_Y([-\ell', \ell])}$$

for all events A and B in Borel sigma algebra. Define the clipped source variables as $\widetilde{X}_\ell = C_\ell(X)$ and $\widetilde{Y}_{\ell'} = C_{\ell'}(Y)$. Furthermore, let $n \in \mathbb{N}$, and define the quantized and clipped source variables $\widehat{X}_{n,\ell} := Q_n(\widetilde{X}_\ell)$, and $\widehat{Y}_{n,\ell'} := Q_n(\widetilde{Y}_{\ell'})$.

Theorem 2. *Given a quadruple of random variables (X,Y,U,V), where i) (X,Y) are jointly continuous with joint PDF $f_{X,Y}$, and ii) U, V are discrete random variables defined on finite sets U and V, respectively, and iii) the long Markov chain $U - X - Y - V$ holds. Then, For any $\xi > 0$ there exists $n, \ell, \ell' > 0$ and variables $\overline{U}_{n,\ell}$ and $\overline{V}_{n,\ell'}$ defined on $\mathsf{U} \times \mathsf{V}$ such that the long Markov chain $\overline{U}_{n,\ell} - \widehat{X}_{n,\ell} - \widehat{Y}_{n,\ell'} - \overline{V}_{n,\ell'}$ holds, and the following conditions are satisfied*

$$|I(\widehat{X}_{n,\ell}; \overline{U}_{n,\ell}) - I(X;U)| \leq \xi \tag{8}$$
$$|I(\widehat{Y}_{n,\ell'}; \overline{V}_{n,\ell'}) - I(Y;V)| \leq \xi \tag{9}$$
$$|I(\overline{U}_{n,\ell} + \overline{V}_{n,\ell'}; \overline{U}_{n,\ell}) - I(U+V;U)| \leq \xi \tag{10}$$
$$|I(\overline{U}_{n,\ell} + \overline{V}_{n,\ell'}; \overline{V}_{n,\ell'}) - I(U+V;V)| \leq \xi, \tag{11}$$
$$|I(\overline{U}_{n,\ell}; \overline{V}_{n,\ell'}) - I(U;V)| \leq \xi, \tag{12}$$

Furthermore, given a pair of jointly continuous distortion functions $d_i : \mathbb{R}^2 \to \mathbb{R}^+, i \in \{1,2\}$ and continuous reconstruction functions $g_i : \mathbb{R}^2 \to \mathbb{R}$, we have:

$$|\mathbb{E}(d_1(\widehat{X}_{n,\ell}, g_1(\overline{U}_{n,\ell}, \overline{V}_{n,\ell'}))) - \mathbb{E}(d_1(X, g_1(U,V)))| \leq \xi,$$
$$|\mathbb{E}(d_2(\widehat{Y}_{n,\ell}, g_2(\overline{U}_{n,\ell}, \overline{V}_{n,\ell'}))) - \mathbb{E}(d_2(Y, g_2(U,V)))| \leq \xi.$$

Proof. Please refer to [17]. □

Corollary 1. *Given a triple of random variables (Y,U,V), For any $\xi > 0$ there exists $\epsilon, n, n', \ell, \ell', \ell'' > 0$ such that the following conditions are satisfied*

$$|I(\widehat{U}_{n,\ell,\epsilon} + \widehat{V}_{n,\ell',\epsilon}, \widehat{Y}_{n',\ell''}; \widehat{U}_{n,\ell,\epsilon}) - I(U+V, Y; U)| \leq \xi,$$

where $\widehat{Y}_{n',\ell''} = Q_{n'}(\widetilde{Y}_{\ell''})$.

IV. Distributed Source Coding Problem

Consider a triple of memoryless continuous-valued sources (X,Y,Z) characterized by a probability measure P_{XYZ}. Let $d : \mathbb{R}^2 \to \mathbb{R}^+$ be a jointly continuous distortion function. The sources X and Y act as helpers for the third source Z. The sources need to be compressed distributively with rates R_1, R_2 and R_3, respectively, into bits to be sent to a joint decoder. For simplicity we let $R_3 = 0$. The joint decoder wishes to reconstruct the source Z with respect to distortion function d.

Definition 7. *An (n, Θ_1, Θ_2) transmission system consists of mappings $e_i : \mathbb{R}^n \to \{1, 2, \ldots, \Theta_i\}$, for $i = 1, 2$, and $f : \{1, 2, \ldots, \Theta_1\} \times \{1, 2, \ldots, \Theta_2\} \to \mathbb{R}^n$. A triple (R_1, R_2, D) is said to be achievable if there exists a sequence of $(n, \Theta_{1n}, \Theta_{2n})$ transmission systems such that for $i = 1, 2$,*

$$\lim_{n\to\infty} \frac{\log \Theta_i}{n} \leq R_i, \quad \lim_{n\to\infty} \mathbb{E} d_n(Z^n, f(e_1(X^n), e_2(Y^n))) \leq D.$$

Let $\mathsf{R}(D)$ denote the set of rates (R_1, R_2) such that (R_1, R_2, D) is achievable.

We provide a coding theorem for the continuous sources.

Theorem 3. *Let $\mathscr{P}(D)$ denote the collection of transition probabilities $P_{QU_1V_1UV\widehat{Z}|XY}$ such that (i) $(UU_1)-(XQ)-(YQ)-(VV_1)$ form a Markov chain, (ii) Q is independent of (X,Y), (iii) $\widehat{Z}=g(U_1,V_1,U+V)$ for some function g, and (iv) $\mathbb{E}d(Z,\widehat{Z})\leq D_1$, where the expectations are evaluated with distribution $P_{XYZ}P_{QU_1V_1UV\widehat{Z}|XY}$. For a $P_{QU_1V_1UV\widehat{Z}|XY}\in\mathscr{P}(D)$, let $\alpha(P_{QU_1V_1UV\widehat{Z}|XY})$ denote the set of rate pairs $(R_1,R_2)\in[0,\infty)^2$ that satisfy*

$$R_1\geq I(X;UU_1|QV_1)+I(U+V;V|QU_1V_1)-I(U;V|QU_1V_1),$$
$$R_2\geq I(Y;VV_1|QU_1)+I(U+V;U|QU_1V_1)-I(U;V|QU_1V_1),$$
$$R_1+R_2\geq I(XY;UVU_1V_1|Q)+I(U+V;V|QU_1V_1)$$
$$+I(U+V;U|QU_1V_1)-I(U;V|QU_1V_1)$$

where the mutual information terms are evaluated with $P_{XY}P_{QU_1V_1UV\widehat{Z}|XY}$. Let the information rate region be defined as

$$\alpha(D)=cl\left(\bigcup_{P_{QU_1V_1UV\widehat{Z}|XY}\in\mathscr{P}(D)}\alpha(P_{QU_1V_1UV\widehat{Z}|XY})\right)$$

Then, we have $\alpha(D)\subseteq\mathsf{R}(D)$, i.e. $\alpha(D)$ is achievable.

To achieve the above rate-distortions, we propose a coding scheme involving two layers to prove the theorem. The first is the Berger-Tung unstructured coding layer. The second is the structured coding layer that uses nested linear codes. Although we do not have space to give a complete proof, the key steps in the proof are as follows. First we quantize the sources and the auxiliary variables, and apply the technique developed in source coding with side-information to come up with a discrete version of the problem at hand. The Berger-Tung unstructured coding is accomplished in a straightforward way. The structured coding is accomplished using nested linear codes. The rates associated with this layer can be understood as follows, for example, assuming U_1,V_1 and Q to be trivial. $R_1\geq I(X;U)+H(U+V)-H(U)=I(X;U)-I(U;V)+I(V;U+V)$, and similarly $R_2\geq I(Y;V)-I(U;V)+I(U;U+V)$. These rates can be achieved using nested linear codes (over arbitrarily large prime fields) along with joint-typical encoding and decoding. Finally, we use the properties of mutual information to show convergence. The complete proof is given in [17].

A. Gaussian Lossy Two-Help-One Example

Consider a pair of zero-mean jointly Gaussian unit variance correlated sources X and Y with correlation coefficient $\rho>0$. Let $Z=X-cY$ for some c, and let $d(z,\hat{z})=(z-\hat{z})^2$. We evaluate a subset of the inner bound to the achievable rate-distortion region using specific Gaussian test channels. Let us denote $\sigma_Z^2=1+c^2-2\rho c$, and let D denote the target distortion. Consider the achievable region in Theorem 3, and let us choose $\mathsf{Q}=\phi$, and $U_1=V_1=0$. Moreover consider

$$U=X+Q_1,\quad\text{and}\quad V=cY+Q_2,$$

where Q_1 and Q_2 are independent zero-mean Gaussian random variables that are independent of the pair (X,Y). We take their variances to be q_1 and $\frac{D\sigma_Z^2}{\sigma_Z^2-D}-q_1$. With this choice we see that $U+V=Z+Q_1+Q_2$, and we take $\widehat{Z}=\mathbb{E}(Z|U+V)=\frac{\sigma_Z^2-D}{\sigma_Z^2}(U+V)$, which results in $\mathbb{E}d(Z,\widehat{Z})=D$. The achievable region is:

$$R_1\geq\frac{1}{2}\log\frac{\sigma_Z^4}{q_1(\sigma_Z^2-D)},\ \text{and}\ R_2\geq\frac{1}{2}\log\frac{\sigma_Z^4}{D\sigma_Z^2-q_1(\sigma_Z^2-D)}.$$

Eliminating q_1 we see that the rate distortion tuple (R_1,R_2,D) satisfying the following equations is achievable:

$$2^{-2R_1}+2^{-2R_2}\leq\left(\frac{\sigma_Z^2}{D}\right)^{-1}.$$

Next, we evaluate rates and distortions which are achievable with unstructured codes, i.e., using just the first layer of the scheme. Let $(q_1,q_2)\in\mathbb{R}_+^2$, and define $U_1=X+Q_{11}$ and $V_1=Y+Q_{12}$, where Q_{11} and Q_{12} are zero-mean independent Gaussian random variables with variances q_1 and q_2, respectively, and independent of the pair (X,Y). Then, by Theorem 3, the convex closure of the region $\mathscr{RD}_{\text{in}}$ is achievable, where $\mathscr{RD}_{\text{in}}$ is defined as the union over all of rate-distortion triples (R_1,R_2,D) satisfying:

$$R_1\geq\frac{1}{2}\log\frac{(1+q_1)(1+q_2)-\rho^2}{q_1(1+q_2)},$$
$$R_2\geq\frac{1}{2}\log\frac{(1+q_1)(1+q_2)-\rho^2}{q_2(1+q_1)},$$
$$R_1+R_2\geq\frac{1}{2}\log\frac{(1+q_1)(1+q_2)-\rho^2}{q_1q_2},$$
$$D\geq\frac{q_1\alpha+q_2c^2\alpha+q_1q_2\sigma_Z^2}{(1+q_1)(1+q_2)-\rho^2}.$$

where $\alpha:=1-\rho^2$.

For a given distortion D, the minimum sum rate $R_{\text{sum}}:=R_1+R_2$ that lies in the convex closure o $\mathscr{RD}_{\text{in}}$ can be evaluated by direct optimization as in [18]. Fig. 1 is a contour plot that illustrates the resulting rate-distortions in detail. We observe that the structured coding scheme performs better than the Berger-Tung based scheme for small distortions, provided ρ is sufficiently high and c lies in a certain interval. The contour labeled R encloses that region in which the pair (ρ,c) should lie for the lattice binning scheme to achieve a sum rate that is at least R units less than the sum rate of the Berger-Tung scheme for some distortion D. We observe that we get improvements only for $c>0$.

V. Computation over MAC

Consider a discrete memoryless two-transmitter multiple-access channel $(P_{Y|X_1,X_2},\kappa_1,\kappa_2)$. The decoder is interested in recovering only single-letter bivariate function $g(\cdot,\cdot)$ of the channel inputs sent by the transmitters reliably. We demonstrate that structured codes can better facilitate the interaction between the two transmitters to ensure that the decoder re-

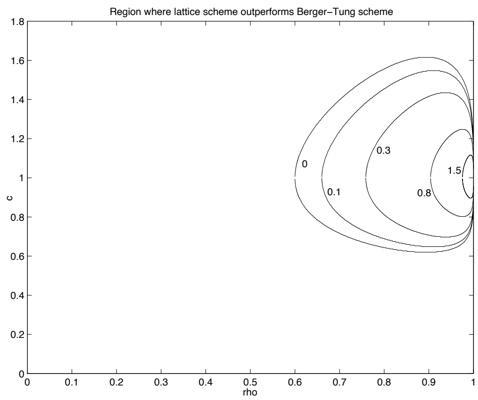

Fig. 1: Range of (ρ, c) where the lattice scheme performs better than the Berger Tung scheme for $D \to 0$.

ceives the desired information while transmitting information at a larger rate than can be sustained by unstructured codes.

Formally, a discrete memoryless two-transmitter multiple-access channel is given by a tuple $(P_{Y|X_1,X_2}, \hat{\mathsf{X}}, g, \kappa_1, \kappa_2)$, consisting of channel input alphabets $\mathsf{X}_1, \mathsf{X}_2$, the channel output alphabet Y, a bivariate function $g : \mathsf{X}_1 \times \mathsf{X}_2 \to \hat{\mathsf{X}}$, and two cost functions κ_1 and κ_2.

Definition 8. *Consider a multiple-access channel $(P_{Y|X_1,X_2}, \hat{\mathsf{X}}, g, \kappa_1, \kappa_2)$. A transmission system with parameters (n, Θ_1, Θ_2) consists of a pair of encoder mappings $e_i : \{1, 2, \ldots, \Theta_i\} \to \mathsf{X}_i^n, i = 1, 2$ and decoder mapping $f : \mathsf{Y}^n \to \hat{\mathsf{X}}^n$. A quadruple of rates and costs $(R_1, R_2, \tau_1, \tau_2)$ is said to be achievable if $\forall \epsilon > 0$, and all sufficiently large n, there exists a transmission system with parameters (n, Θ_1, Θ_2) such that for $i = 1, 2$,*

$$\frac{1}{n} \log \Theta_i \geq R_i - \epsilon, \quad \frac{1}{\Theta_i} \sum_{j=1}^{\Theta_i} \kappa_i(e_i(j)) \leq \tau_i + \epsilon, \quad i = 1, 2$$

$$\sum_{j=1}^{\Theta_1} \sum_{k=1}^{\Theta_2} \frac{1}{\Theta_1 \Theta_2} P^n_{Y_1,Y_2|X_1,X_2} \left[f(Y^n) \neq g^n(X_1^n, X_2^n) \right.$$
$$\left. |X_1^n = e_1(j), X_2^n = e_2(k)) \right] \leq \epsilon, \quad i = 1, 2.$$

The operational capacity region $\mathbb{C}(\tau_1, \tau_2)$ is the set of all rate pairs (R_1, R_2) such that $(R_1, R_2, \tau_1, \tau_2)$ is achievable.

We provide an achievable rate region using the discretization techniques in Section II, and the structured coding strategy for computation over MAC with discrete alphabets introduced in [19, Theorem 4.2]. We focus on continuous-valued channels, i.e., $\mathsf{X}_1 = \mathsf{X}_2 = \mathsf{Y} = \mathbb{R}$, and g to be the addition operation on the real field, i.e, $g(x_1, x_2) = x_1 + x_2$ for all $x_1, x_2 \in \mathbb{R}$.

Theorem 4. *Let $\mathscr{P}(\tau_1, \tau_2)$ denote the collection of pairs formed by a distribution $(P_{QU_1U_2X_1X_2}$ defined on $\mathsf{Q} \times \mathbb{R}^4$ such that (i) $(U_1X_1) - Q - (U_2X_2)$ form a Markov chain, and (ii) $\mathbb{E}(\kappa_i(X_i)) \leq \tau_i$. For a $(P_{QU_1U_2X_1X_2}) \in \mathscr{P}$, let $\alpha_F(P_{QU_1U_2X_1X_2})$ denote the set of rate pairs $(R_1, R_2) \in [0, \infty)^2$ that satisfy*

$$R_1 \leq I(U_1; Y|U_2Q) + I(X; Y|U_1U_2Q) - I(X; X_2|U_1U_2Q)$$
$$R_2 \leq I(U_2; Y|U_1Q) + I(X; Y|U_1U_2Q) - I(X; X_1|U_1U_2Q)$$
$$R_1 + R_2 \leq I(U_1U_2; Y)|Q) + 2I(X; Y|U_1U_2Q)$$
$$- I(X; X_1|U_1U_2Q) - I(X; X_2|U_1U_2Q)$$

where the mutual information terms are evaluated with $P_{QU_1U_2X_1X_2}P_{Y|X_1X_2}$, and $X = X_1 + X_2$. Let the information rate region be defined as

$$\alpha_F = cl \left(\bigcup_{(P_{QU_1U_2X_1X_2}) \in \mathscr{P}} \alpha_F(P_{QU_1U_2X_1X_2}) \right)$$

Then, the operational capacity cost region $\mathbb{C}(\tau_1, \tau_2)$ contains the information capacity region $\alpha(\tau_1, \tau_2)$, i.e., $\alpha(\tau_1, \tau_2) \subseteq \mathbb{C}(\tau_1, \tau_2)$.

Gaussian MAC: Consider the MAC given by $Y = X_1 + X_2 + Z$, where Z is zero-mean Gaussian with variance N. We have power constraints on X_1 and X_2: $\kappa_1(x_1) = x_1^2$ and $\kappa_2(x_2) = x_2^2$, for all $x_1, x_2 \in \mathbb{R}$.

The rates achievable using unstructured code ensembles is given by the standard MAC capacity region given by

$$\left\{ (R_1, R_2) : R_1 \leq \frac{1}{2} \log \left(1 + \frac{P_1}{N} \right), \right.$$
$$\left. R_2 \leq \frac{1}{2} \log \left(1 + \frac{P_2}{N} \right), R_1 + R_2 \leq \frac{1}{2} \log \left(1 + \frac{P_1 + P_2}{N} \right) \right\}.$$

This is achieved using independent Gaussian inputs X_1 and X_2 of variances P_1 and P_2, respectively. Using the same distribution, one can achieve the following rates while employing structured code ensembles.

$$\left\{ (R_1, R_2) : R_1 \leq \frac{1}{2} \log \left(\frac{P_1(P_1 + P_2 + N)}{(P_1 + P_2)N} \right), \right.$$
$$\left. R_2 \leq \frac{1}{2} \log \left(\frac{P_2(P_1 + P_2 + N)}{(P_1 + P_2)N} \right) \right\}.$$

Comparing the sum-rate we see that the structured coding scheme performs better than the unstructured coding scheme when

$$\left(1 + \frac{P_1}{P_2} \right) \left(1 + \frac{P_2}{P_1} \right) \leq 1 + \frac{P_1}{N} + \frac{P_2}{N}.$$

For the case when $P_1 = P_2 = P$ is boils down to the condition that $\frac{P}{N} \geq 1.5$.

VI. Conclusion

The fundamental information theoretic limits in multiterminal communications with general continuous-valued sources and channels were studied. A general framework was presented which involves discretization of the source and channel variables followed by communication over the resulting discretized network. The framework was applied in the distributed source coding and computation over multiple access channels scenarios, where general resulting achievable regions were derived, and they were evaluated for Gaussian sources and channels.

REFERENCES

[1] A. E. Gamal and Y. H. Kim. *Network Information Theory*. Cambridge University Press, 2011.
[2] H Liao. *Multiple Access Channels Ph. D.* PhD thesis, Thesis, Department of Electrical Engineering, University of Hawaii, Honolulu, 1972.
[3] S. I. Gel'fand and M. S. Pinsker. Capacity of a broadcast channel with one deterministic component. *Problemy Peredachi Informatsii*, 16(1):24–34, 1980.
[4] R. Ahlswede. The rate-distortion region for multiple descriptions without excess rate. *IEEE Transactions on Information Theory*, 31(6):721–726, 1985.
[5] I. Csiszár and J. Korner. *Information Theory: Coding Theorems for Discrete Memoryless Systems*. Academic Press Inc. Ltd., 1981.
[6] A. B Wagner, S. Tavildar, and P. Viswanath. Rate region of the quadratic gaussian two-encoder source-coding problem. *IEEE Transactions on Information Theory*, 54(5):1938–1961, 2008.
[7] H. Weingarten, Y. Steinberg, and S. S. Shamai. The capacity region of the gaussian multiple-input multiple-output broadcast channel. *IEEE transactions on information theory*, 52(9):3936–3964, 2006.
[8] T. M. Cover and J. A. Thomas. *Elements of Information Theory*. John Wiley & Sons, Inc., Hoboken, NJ, USA, 2006.
[9] S. Y. Tung. *Multiterminal Source Coding*. PhD thesis, Cornell University, Ithaca, NY, 1978.
[10] A. D. Wyner. The rate-distortion function for source coding with side information at the decoder\3-ii: General sources. *Information and control*, 38(1):60–80, 1978.
[11] A. Pastore, S. H. Lim, C. Feng, B. Nazer, and M. Gastpar. A discretization approach to compute–forward. In *2021 IEEE International Symposium on Information Theory (ISIT)*, pages 2208–2213. IEEE, 2021.
[12] A. V. Makkuva and Y. Wu. Equivalence of additive-combinatorial linear inequalities for shannon entropy and differential entropy. *IEEE Transactions on Information Theory*, 64(5):3579–3589, 2018.
[13] P. Mitran. On a markov lemma and typical sequences for polish alphabets. *IEEE Transactions on Information Theory*, 61(10):5342–5356, 2015.
[14] U. Erez and R. Zamir. Achieving 1/2 log (1+ snr) on the awgn channel with lattice encoding and decoding. *IEEE Transactions on Information Theory*, 50(10):2293–2314, 2004.
[15] R. Zamir, S. Shamai, and U. Erez. Nested linear/lattice codes for structured multiterminal binning. *IEEE Transactions on Information Theory*, 48(6):1250–1276, 2002.
[16] F. Shirani and S. S. Pradhan. Lattices from linear codes and fine quantization: General continuous sources and channels. In *2018 IEEE International Symposium on Information Theory (ISIT)*, pages 2356–2360. IEEE, 2018.
[17] S. Shirani and S. Pradhan. Lattices from linear codes: Source and channel networks. *arXiv preprint*, 2022.
[18] D. Krithivasan and S. S. Pradhan. Lattices for distributed source coding: Jointly gaussian sources and reconstruction of a linear function. *IEEE Transactions on Information Theory*, 55(12):5628–5651, 2009.
[19] S. S. Pradhan, A. Padakandla, and F. Shirani. *An Algebraic and Probabilistic Framework for Network Information Theory*, volume 18-2. Foundations and Trends in Communications and Information Theory, 2021.