Equilibrium forces on nonreciprocal materials

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We discuss and analyze the properties of Casimir forces acting between nonreciprocal objects in thermal equilibrium. By starting from the fluctuation-dissipation theorem and splitting the force into those arising from individual sources, we show that if all temperatures are equal, the resulting force is reciprocal and is derivable as the gradient of a Casimir (free) energy. While the expression for the free energy is identical to the one for reciprocal objects, there are several distinct features: To leading order in reflections, the free energy can be decomposed as the sum of two terms, the first corresponding to two reciprocal objects, and the second corresponding to two antireciprocal objects. The first term is negative and typically yields attraction, while the second can have either sign. For the case of two objects that are each other's mirror images, the second term is positive and yields repulsion. The sum of terms can lead to overall repulsive forces, in agreement with previous observations. Stable configurations, ruled out for reciprocal cases, appear possible for nonreciprocal objects. We show that for three objects, a three-body free energy exists, indicating that previously found persistent heat currents in situations of three objects cannot be used to produce persistent torques.

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I. INTRODUCTION

Steady currents are commonly associated with the lack of thermal equilibrium. The existence of steady currents indicates the absence of detailed balance [1], which can have important consequences, such as violation of the fluctuation-dissipation theorem [2], the presence of activity [3–5], lasing without inversion, [6] and the increase of power extraction in photocells [7].

Recently it was shown that nonreciprocal materials can support steady heat currents in a system that is at thermal equilibrium [8,9] by breaking detailed balance. Examples of nonreciprocal materials include ferrimagnetic compounds, magnetized plasmas, and space-time modulated media, among others [10]. Besides heat currents, fluctuation of charges and electric currents inside nonreciprocal bodies produce Casimir forces that differ from those produced by reciprocal systems [11]. Nonreciprocal systems have been shown to produce new features absent in their reciprocal counterparts, such as novel lateral forces [12–14], repulsive forces [15,16], recoil forces [17], nontrivial optical torques [18,19], lateral thermal-fluctuations-induced forces [9,20–22], and heat transfer with unique properties [8,23–25]. Furthermore, some theorems restricting the properties of forces are based on the assumption that the systems are reciprocal [26,27], indicating the possibility of finding other novel effects in nonreciprocal systems.

In this paper, we study the nature of Casimir forces produced by systems that break detailed balance at thermal equilibrium. We achieve this by analyzing the properties of forces produced by nonreciprocal materials and test whether they are also restricted by standard force theorems. We show that these forces are conservative, forbidding the steady production of mechanical work at equilibrium and therefore preventing them from being used as a way to increase power extraction in photocells or other heat-engine like devices. Moreover, the conservative property of the equilibrium force puts into question the results presented in Ref. [22] on a lateral force in translationally invariant setups, as well as the experimental proposal for using forces to measure equilibrium persistent heat currents in translation invariant geometries proposed in Ref. [9]. As was recently shown, steady work extraction in nonreciprocal heat engines requires at least two different temperatures [12,28]. Furthermore, we show that nonreciprocity may in principle allow for stable configurations, as the Laplacian of the free energy can be positive, in violation of Earnshaw's theorem, which reciprocal systems must fulfill [26]. This comes together with the possibility of repulsion for nonreciprocal objects [11]. Specifically, to leading order in reflections, the free energy is the sum of two terms, corresponding to two reciprocal and two antireciprocal objects, respectively. For two objects that are each other's mirror images, the latter can be shown to yield a positive free energy and a repulsive force. Repulsive Casimir forces are known to exist for bodies immersed in a dielectric [29], or for systems out of equilibrium [30–33].

II. CASIMIR FREE ENERGY

A. System and setup

We consider the situation depicted in Fig. 1, which comprises a collection of bodies immersed in vacuum. The system

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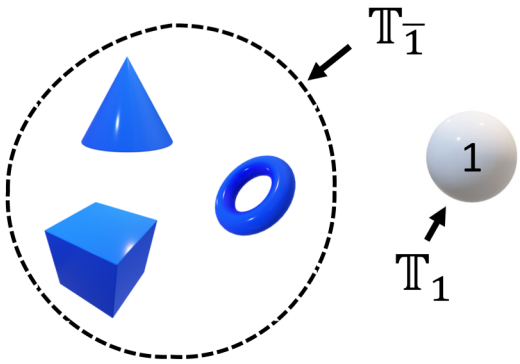

FIG. 1. We consider multiple objects surrounded by vacuum, where all the objects and their surroundings are assumed to be at the same temperature T. We study the properties of the total force on object 1.

is at thermal equilibrium, i.e., all bodies as well as the surroundings are at the same temperature T.

The objects are described by their classical scattering operators $\mathbb{T} = T_{ij}(\mathbf{r}, \mathbf{r}')$ [34,35], which are 3×3 matrices depending on two spatial arguments $\mathbf{r}$ and $\mathbf{r}'$. $\mathbb{T}_1$ denotes the operator for object 1, and $\mathbb{T}_{\bar{1}}$ is the operator for the remaining objects. This notation allows us to treat the case of two objects as well as that of several objects.

For reciprocal bodies, microreversibility constrains $\mathbb{T} = \mathbb{T}^T$, where $[T_{ij}(\mathbf{r}, \mathbf{r}')]^T = T_{ji}(\mathbf{r}', \mathbf{r})$ is the transpose, corresponding to switching spatial arguments and matrix indices. Here we consider objects that are allowed to be nonreciprocal [10,36], so that in general, $\mathbb{T} \neq \mathbb{T}^T$.

B. Derivation of Casimir free energy from fluctuation dissipation theorem

As mentioned above, the systems depicted in Fig. 1, may break time-reversal symmetry and also detailed balance [8]. It is thus not obvious how equilibrium Casimir forces can be computed [11] and what their properties are. We thus start by showing that Casimir forces among nonreciprocal or reciprocal bodies at thermal equilibrium are gradients of a potential energy, whose form is identical to the known case of reciprocal media [34]. This is in stark contrast with nonreciprocal forces [37] and forces on active nonreciprocal mechanical systems [38] where the lack of thermal equilibrium allows the presence of nonconservative active forces.

We therefore start from the corresponding expressions found for *nonequilibrium* forces [12,35], but then take all temperatures equal. This derivation is thus based on the assuming validity of the fluctuation dissipation theorem locally in each nonreciprocal system, which in turn is based on the existence of a Boltzmann distribution [39].

The force on 1 is thus written as a sum of three forces, corresponding to the sources in the different objects and the environment [35],

$$\mathbf{F}^{(1)}_{\rm eq} = \mathbf{F}^{(1)}_{\bar{1}} + \mathbf{F}^{(1)}_{1} + \mathbf{F}^{(1)}_{\rm env}. \tag{1}$$

$\mathbf{F}^{(1)}_{\bar{1}}$ is the force due to fluctuations sourced in objects $\bar{1}$, and reads

$$\mathbf{F}^{(1)}_{\bar{1}} = \frac{\hbar}{\pi}\int_0^\infty d\omega \left[\frac{1}{e^{\frac{\hbar\omega}{k_BT}}-1}+\frac{1}{2}\right] \times \mathrm{Re\,Tr}\left\{\boldsymbol{\nabla}(1+\mathbb{G}_0\mathbb{T}_1)\mathbb{G}_0\mathbb{W}_{\bar{1},1}\mathbb{G}_0 \times \left[\frac{\mathbb{T}_{\bar{1}}-\mathbb{T}^\dagger_{\bar{1}}}{2i} - \mathbb{T}_{\bar{1}}\mathrm{Im}\,[\mathbb{G}_0]\mathbb{T}^\dagger_{\bar{1}}\right]\mathbb{G}^*_0\mathbb{W}^\dagger_{\bar{1},1}\mathbb{G}^*_0\mathbb{T}^\dagger_1\right\}, \tag{2}$$

where $\mathbb{W}_{\bar{1},1} = \mathbb{G}_0^{-1}\frac{1}{1-\mathbb{G}_0\mathbb{T}_{\bar{1}}\mathbb{G}_0\mathbb{T}_1}$ is the multiple scattering operator and $\mathbb{G}_0$ is the free Green's function. Note that, in contrast to the formulas for nonequilibrium Casimir forces [35], we include here the zero-point term: $\mathbf{F}^{(1)}_1$ is the "self-force", which is produced by the fluctuating charges in object 1 itself,

$$\mathbf{F}^{(1)}_{1} = \frac{\hbar}{\pi}\int_0^\infty d\omega \left[\frac{1}{e^{\frac{\hbar\omega}{k_BT}}-1}+\frac{1}{2}\right] \times \mathrm{Re\,Tr}\left\{\boldsymbol{\nabla}(1+\mathbb{G}_0\mathbb{T}_{\bar{1}})\mathbb{G}_0\mathbb{W}_{1,\bar{1}}\mathbb{G}_0 \times \left[\frac{\mathbb{T}_1-\mathbb{T}^\dagger_1}{2i} - \mathbb{T}_1\mathrm{Im}\,[\mathbb{G}_0]\mathbb{T}^\dagger_1\right]\mathbb{G}^*_0\mathbb{W}^\dagger_{1,\bar{1}}\right\}. \tag{3}$$

The self-force is zero for an isolated symmetric object, but potentially nonzero in the presence of other objects. The sources in the environment yield the force component $\mathbf{F}^{(1)}_{\rm env}$, which is found to be

$$\mathbf{F}^{(1)}_{\rm env} = \frac{\hbar}{\pi}\int_0^\infty d\omega \left[\frac{1}{e^{\frac{\hbar\omega}{k_BT}}-1}+\frac{1}{2}\right] \times \mathrm{Re\,Tr}\{\boldsymbol{\nabla}(1+\mathbb{G}_0\mathbb{T}_1)\mathbb{G}_0 W_{\bar{1},1}(1+\mathbb{G}_0\mathbb{T}_{\bar{1}})\mathrm{Im}\,[\mathbb{G}_0] \times (1+\mathbb{T}^\dagger_{\bar{1}}\mathbb{G}^*_0)\mathbb{W}^\dagger_{\bar{1},1}\mathbb{G}^*_0\mathbb{T}^\dagger_1\}. \tag{4}$$

For reciprocal objects, the existence of a Casimir free energy for equilibrium systems is well established [34,40–42]. So, if $\mathbb{T}_i = \mathbb{T}_i^T$ for all i, Eqs. (2), (3), and (4) *must* sum up to an expression that is the gradient of the known free energy [34]. Here we show this for nonreciprocal objects. These pose additional challenges, because the operators to be traced do not enjoy the same symmetries as for reciprocal cases. For example, the term $\mathbb{T}_1\mathrm{Im}\,[\mathbb{G}_0]\mathbb{T}^\dagger_1$ is, for reciprocal cases, equal to the complex conjugate of $\mathbb{T}^\dagger_1\mathrm{Im}\,[\mathbb{G}_0]\mathbb{T}_1$. This property directly implies that heat transfer between reciprocal objects is symmetric [35]. Because this property is absent for nonreciprocal objects, the corresponding heat transfer is not necessarily symmetric [43]. It is thus not obvious *a priori* whether Eqs. (2), (3), and (4) sum up to a gradient force for nonreciprocal cases.

After some algebraic manipulations detailed in the Appendix, we find that the force is [44]

$$\mathbf{F}^{(1)}_{\rm eq} = -\boldsymbol{\nabla}_{\mathcal{O}_{(1)}}\frac{\hbar}{\pi}\int_0^\infty d\omega \left[\frac{1}{e^{\frac{\hbar\omega}{k_BT}}-1}+\frac{1}{2}\right] \times \mathrm{Im\,Tr}[\log(1-\mathbb{G}_0\mathbb{T}_1\mathbb{G}_0\mathbb{T}_{\bar{1}})], \tag{5}$$

which is the derivative of a function with respect to the position of the object 1. This implies that the force is conservative.

The function inside the derivative is the free energy $\mathcal{F}$,

$$\mathcal{F} = \frac{\hbar}{\pi}\int_0^\infty d\omega\left[\frac{1}{e^{\frac{\hbar\omega}{k_B\mathbb{T}}}-1}+\frac{1}{2}\right] \times \mathrm{Im}\,\mathrm{Tr}[\log(1-\mathbb{G}_0\mathbb{T}_1\mathbb{G}_0\mathbb{T}_{\bar{1}})]. \quad (6)$$

The conservative nature of the force rules out the possibility of building an engine at thermal equilibrium, as may be expected from thermodynamics: Extraction of mechanical work from a system at a single temperature is prohibited. In contrast, as shown in [12], for specific geometries nonreciprocity plays a key role for building a heat engine operating with bodies at different temperatures.

In a translationally invariant setup, such as the nonreciprocal plate considered in [9], the free energy will be independent of the invariant coordinate. This implies that the force on objects in the vicinity of this plate is zero.

Notably, the force in Eq. (5) and the free energy in Eq. (6) are of identical form to the corresponding expressions for reciprocal objects [34,35]. Nevertheless, as we shall see, their properties can be quite different for the nonreciprocal case.

C. Free energy as a Matsubara sum and large distance expansion

To make contact with typical formulations of equilibrium Casimir forces, we rewrite Eq. (6) in terms of a Matsubara sum. We start by extending the integration range of Eq. (6) to negative frequencies

$$\mathcal{F} = \frac{\hbar}{4i\pi}\int_{-\infty}^\infty d\omega\coth\left[\frac{\hbar\omega\beta}{2}\right] \times \mathrm{Tr}\{\log[1-\mathbb{G}_0\mathbb{T}_1\mathbb{G}_0\mathbb{T}_{\bar{1}}]\}. \quad (7)$$

Because the free Green's function and the T operator are causal response functions, the trace is an analytic function in the upper complex plane [39] and the integral is performed via the residue theorem,

$$\mathcal{F} = k_BT\sum_{n=0}^\infty \mathrm{Tr}\{\log[1-\mathbb{G}_0\mathbb{T}_1\mathbb{G}_0\mathbb{T}_{\bar{1}}]\}, \quad (8)$$

where we sum over Matsubara frequencies $\omega \to ic\kappa_n = \frac{2\pi ni}{\hbar\beta}$. The term with $n=0$ is counted only half in the sum [34], because the corresponding pole sits on the real frequency axis.

We will later investigate the so called one reflection approximation, where the log is expanded and the free energy is

$$\mathcal{F} \approx -k_BT\sum_{n=0}^\infty \mathrm{Tr}\{\mathbb{G}_0\mathbb{T}_1\mathbb{G}_0\mathbb{T}_{\bar{1}}\}. \quad (9)$$

This approximation becomes exact if the separation between objects is large compared to their size, as studied below for two spherical particles.

III. PROPERTIES FOR TWO OBJECTS

In this section, we analyze the properties of the free energy and the resulting forces for two objects. We use notation $\mathbb{T}_{\bar{1}} = \mathbb{T}_2$.

A. Force reciprocity

Even though we consider nonreciprocal materials, the generated Casimir equilibrium forces are reciprocal in the sense that they obey Newton's third law. The scenario here is thus different from cases were nonreciprocal forces or interactions are found [37]. This can be demonstrated by noticing that

$$-\boldsymbol{\nabla}_{\mathcal{O}_1}\mathrm{Tr}[\log(1-\mathbb{G}_0\mathbb{T}_1\mathbb{G}_0\mathbb{T}_2)] = \mathrm{Tr}\left[\frac{1}{1-\mathbb{G}_0\mathbb{T}_1\mathbb{G}_0\mathbb{T}_2}\mathbb{G}_0(\boldsymbol{\nabla}\mathbb{T}_1-\mathbb{T}_1\boldsymbol{\nabla})\mathbb{G}_0\mathbb{T}_2\right], \quad (10)$$

where we have used that $\boldsymbol{\nabla}_{\mathcal{O}_1}\mathbb{T}_1 = \boldsymbol{\nabla}\mathbb{T}_1 - \mathbb{T}_1\boldsymbol{\nabla}$ and that we can pass the derivative through the free Green's function, as shown in Eqs. (A5) and (A6). Here $\boldsymbol{\nabla}$ refers to the derivative acting to the right on the first argument of the subsequent expression. Applying $\boldsymbol{\nabla}\mathbb{T}_2 = \boldsymbol{\nabla}_{\mathcal{O}_2} + \mathbb{T}_2\boldsymbol{\nabla}$ to the last term and using the cyclic property of the trace we get that Eq. (10) is equal to

$$-\mathrm{Tr}\left[\frac{1}{1-\mathbb{G}_0\mathbb{T}_2\mathbb{G}_0\mathbb{T}_1}\mathbb{G}_0\mathbb{T}_1\boldsymbol{\nabla}_{\mathcal{O}_2}\mathbb{G}_0\mathbb{T}_2\mathbb{G}_0\mathbb{T}_1\right] = \boldsymbol{\nabla}_{\mathcal{O}_2}\mathrm{Tr}[\log(1-\mathbb{G}_0\mathbb{T}_2\mathbb{G}_0\mathbb{T}_1)]. \quad (11)$$

Combining (10) and (11), one obtains that the forces are reciprocal, that is $\mathbf{F}^{(2)}_{\mathrm{eq}} = -\mathbf{F}^{(1)}_{\mathrm{eq}}$ independently of the bodies' reciprocity. This is in agreement with the expectation that nonreciprocal forces require exchange of momentum with the environment, and can thus occur only in situations outside thermal equilibrium [32,45].

Generally, the statement of existence of a two-body potential already *requires* force reciprocity: For two objects in free space, the free energy must be invariant under simultaneous translation of both objects. Hence it must be a function of $\boldsymbol{\mathcal{O}}_1 - \boldsymbol{\mathcal{O}}_2$, and the forces derived from it must be reciprocal. We will below address the case of three bodies.

B. Only one non reciprocal object

Assume that object 1 is nonreciprocal while object 2 is reciprocal. In this case, using that $\mathrm{Tr}[\mathbb{O}] = \mathrm{Tr}[\mathbb{O}^T]$ for any operator $\mathbb{O}$, and using the cyclic property of the trace, we obtain

$$\mathrm{Tr}\{\mathbb{G}_0\mathbb{T}_1\mathbb{G}_0\mathbb{T}_2\} = \mathrm{Tr}\{\mathbb{G}_0\mathbb{T}_1^T\mathbb{G}_0\mathbb{T}_2\},$$

or, rewriting

$$\mathrm{Tr}\{\mathbb{G}_0(\mathbb{T}_1-\mathbb{T}_1^T)\mathbb{G}_0\mathbb{T}_2\} = 0. \quad (12)$$

We can thus write the contribution from the leading reflection, Eq. (9), as

$$\mathcal{F} = -\frac{1}{2}k_BT\sum_{n=0}^\infty \mathrm{Tr}\{\mathbb{G}_0\mathbb{T}_2\mathbb{G}_0(\mathbb{T}_1+\mathbb{T}_1^T)\}. \quad (13)$$

This means that the nonreciprocal part, represented by $\mathbb{T}_1 - \mathbb{T}_1^T$, does not contribute at one reflection to the free energy, if only a single object is nonreciprocal. For the limit of Eq. (9), the free energy will thus share all properties that are known for two reciprocal objects (see below). However, nonreciprocal properties enter at higher reflections, where the trace can be nonzero because it may contain an even power of the asymmetric (nonreciprocal) operator $\mathbb{T}_1$ at higher orders.

C. Two distinct terms

At one reflection, we can furthermore show that the force for two nonreciprocal objects is the sum of two distinct terms. We therefore write the scattering operators in terms of reciprocal and antireciprocal parts

$$\mathbb{A}_i^{(\pm)} = \tfrac{1}{2}\left(\mathbb{T}_i \pm \mathbb{T}_i^T\right) = \tfrac{1}{2}(\mathbb{T}_i \pm \mathbb{T}_i^\dagger).$$

The last equality follows because $\mathbb{T}_i$ is real for imaginary frequency, since it is a real function of time t. For imaginary frequency, $\mathbb{A}_i^{(+)}$ is thus Hermitian, and $\mathbb{A}_i^{(-)}$ is anti-Hermitian. Noting from Eq. (12) that there is no cross term between $\mathbb{A}_i^{(-)}$ and $\mathbb{A}_i^{(+)}$, the result for one reflection can be written

$$\begin{aligned}\mathcal{F} &= -k_B T \sum_{n=0}^{\infty} \mathrm{Tr}\{\mathbb{G}_0\mathbb{T}_1\mathbb{G}_0\mathbb{T}_2\} \\ &= -k_B T \sum_{n=0}^{\infty} [\mathrm{Tr}\{\mathbb{G}_0\mathbb{A}_1^{(+)}\mathbb{G}_0\mathbb{A}_2^{(+)}\} \\ &\quad + \mathrm{Tr}\{\mathbb{G}_0\mathbb{A}_1^{(-)}\mathbb{G}_0\mathbb{A}_2^{(-)}\}]. \end{aligned} \tag{14}$$

We see that, to leading order in the reflection expansion, the free energy is the sum of a free energy for two reciprocal objects and a free energy for two purely antireciprocal objects. We shall discuss their properties below.

D. Energy sign and force direction

The one-reflection approximation of Eq. (9) allows the determination of the sign of $\mathcal{F}$. Starting with two reciprocal objects, it is useful to note, based on basic considerations of statistical physics, that $\mathbb{T}_1$ evaluated at imaginary frequencies is a non-negative symmetric operator for nonmagnetic objects in vacuum [27,39]. $\mathbb{G}_0\mathbb{T}_2\mathbb{G}_0$ is thus non-negative and symmetric as well (because $\mathbb{G}_0$ is symmetric and real). Because the product of two symmetric non-negative operators has a non-negative trace, we thus have

$$\mathrm{Tr}\{\mathbb{G}_0\mathbb{T}_2\mathbb{G}_0\mathbb{T}_1\} \geqslant 0. \tag{15}$$

The free energy is thus a sum of nonpositive terms, so that, for two reciprocal objects in leading order of scattering events,

$$\mathcal{F} \leqslant 0. \tag{16}$$

This implies that, if the magnitude of $\mathcal{F}$ decreases with the objects separation d, the force has to be attractive. This statement of attraction is in agreement with previous studies of forces between reciprocal bodies [27,34]. Below, we will also consider the case of two bodies, which are each other's mirror images, in which case the force for reciprocal objects is attractive [27].

For the nonreciprocal case, the $\mathbb{T}$ operators are not symmetric, and Eq. (15) does not hold. It is thus possible to obtain repulsive forces (even at small separations) with nonreciprocal media, as exemplified below and seen in [11].

E. Anti reciprocal opposites repel

This statement can be strengthened by considering two objects that are each other's mirror images. We follow [27] to write $\mathbb{T}_2 = \mathbb{J}\mathbb{T}_1\mathbb{J}^\dagger$ with a unitary operator $\mathbb{J}$ that transforms between the original space and the mirror space. We also use the remarkable property [27] that $\mathbb{G}_0\mathbb{J}$ is a positive operator. We thus write $\mathbb{G}_0\mathbb{J} = \mathbb{C}^\dagger\mathbb{C} = \mathbb{B}\mathbb{B}^\dagger$ without specifying $\mathbb{C}$ or $\mathbb{B}$. The two terms in (14) are thus found to be

$$\begin{aligned}\mathrm{Tr}\{\mathbb{G}_0\mathbb{A}_1^{(\pm)}\mathbb{G}_0\mathbb{A}_2^{(\pm)}\} &= \mathrm{Tr}\{\mathbb{G}_0\mathbb{A}_1^{(\pm)}\mathbb{G}_0\mathbb{J}\,\mathbb{A}_1^{(\pm)}\mathbb{J}^\dagger\} \\ &= \mathrm{Tr}\{\mathbb{C}\mathbb{A}_1^{(\pm)}\mathbb{B}\mathbb{B}^\dagger\mathbb{A}_1^{(\pm)}\mathbb{C}^\dagger\} \\ &= \pm\mathrm{Tr}\{(\mathbb{C}\mathbb{A}_1^{(\pm)}\mathbb{B})(\mathbb{C}\mathbb{A}_1^{(\pm)}\mathbb{B})^\dagger\}. \end{aligned} \tag{17}$$

Because

$$\mathrm{Tr}\{(\mathbb{C}\mathbb{A}_1^{(\pm)}\mathbb{B})(\mathbb{C}\mathbb{A}_1^{(\pm)}\mathbb{B})^\dagger\} \geqslant 0, \tag{18}$$

we have thus shown that to leading order in reflections, for two objects that are mirror images of each other the free energy is the sum of a negative term due to the reciprocal parts of the objects, and a positive term due to the antireciprocal part of the objects.

Furthermore, we may also use that [27] $\partial_a\mathbb{G}_0\mathbb{J}$ is a negative operator, i.e., $\partial_a\mathbb{G}_0\mathbb{J} = -\mathbb{D}^\dagger\mathbb{D}$, with another unspecified operator $\mathbb{D}$, and where a is the distance between the mirror images [27] along the mirror axis. Using this result in the same manner as above, we find that, for the term composed of two purely antireciprocal objects,

$$\partial_a\mathcal{F} \leqslant 0. \tag{19}$$

This means that the corresponding force is repulsive. We have thus shown that nonreciprocal contributions for mirror images lead to repulsive terms in the force.

Such repulsion and attraction between different types of objects is somewhat reminiscent of critical Casimir forces [46].

F. Stability

Stability has been ruled out for equilibrium situations involving reciprocal objects [26]. We repeat this calculation here, again resorting to the simpler case of the leading order in scattering events. We then investigate this question for nonreciprocal objects. We have for the Laplacian of the free energy

$$\nabla^2_{\mathcal{O}_1}\mathcal{F}(\mathcal{O}_1,\mathcal{O}_2) = -k_B T \sum_{n=0}^{\infty} \mathrm{Tr}\left\{\mathbb{G}_0\left(\nabla^2_{\mathcal{O}_1}\mathbb{T}_1\right)\mathbb{G}_0\mathbb{T}_2\right\}. \tag{20}$$

We use $\nabla_{\mathcal{O}_1}\mathbb{T}_1 = \nabla\mathbb{T}_1 - \mathbb{T}_1\nabla$, and obtain three terms,

$$\begin{aligned}&\nabla^2_{\mathcal{O}_1}\mathcal{F}(\mathcal{O}_1,\mathcal{O}_2) \\ &\quad = -k_B T \sum_{n=0}^{\infty} \mathrm{Tr}\{\mathbb{G}_0[\nabla^2\mathbb{T}_1 - 2\nabla\mathbb{T}_1\nabla + \mathbb{T}_1\nabla^2]\mathbb{G}_0\mathbb{T}_2\}. \end{aligned} \tag{21}$$

We use that, up to a δ function in space, $\nabla^2\mathbb{G}_0 = \frac{\kappa_n^2}{c^2}\mathbb{G}_0$ for imaginary frequency [26]. This together with Eq. (15) shows that the last term is negative (including the overall minus

sign). For the first term we use that the operator $\boldsymbol{\nabla}^2$ can be moved past the free Green's function, with the same conclusion. The middle term appearing in Eq. (21) is

$$\mathrm{Tr}\{\mathbb{G}_0\boldsymbol{\nabla}\mathbb{T}_1\boldsymbol{\nabla}\mathbb{G}_0\mathbb{T}_2\} = -\mathrm{Tr}\{\boldsymbol{\nabla}\mathbb{T}_1\boldsymbol{\nabla}^T\mathbb{G}_0\mathbb{T}_2\mathbb{G}_0\} \leqslant 0, \quad (22)$$

where $\boldsymbol{\nabla}^T$ represents the derivative acting to the left on the second argument of the previous expression. By partial integration, we then have $\boldsymbol{\nabla} = -\boldsymbol{\nabla}^T$. To establish the inequality, we have used that the trace of the product of the symmetric non-negative operators $\boldsymbol{\nabla}\mathbb{T}_1\boldsymbol{\nabla}^T$ and $\mathbb{G}_0\mathbb{T}_2\mathbb{G}_0$ must be non-negative. The Laplacian of the free energy with respect to the position of either of the particles is thus nonpositive for reciprocal objects [26], and stability is ruled out.

For nonreciprocal objects, we cannot rely on the above arguments, so the Laplacian cannot be proven to be nonpositive. We will indeed provide a counterexample below, showing that the Laplacian can be positive and stable configurations with nonreciprocal systems appear possible in principle.

IV. THREE BODIES

Equation (6) shows that the force acting on body 2 in the presence of several other objects is the gradient of a free energy. Together with the statements of Sec. III A, the existence of a two-body potential for two objects is shown. However, to our knowledge, this statement does not imply that an N body potential exists for $N > 2$ objects. The case of three or more objects is especially interesting, since the persistent heat current discussed in Ref. [8] requires three or more objects. It is thus worthwhile to test whether for three bodies, a three-body potential exists. We do this here to leading order in scattering reflections. We therefore introduce $\mathbb{T}_{23}$, the composite operator for objects 2 and 3. The force acting on object 1 is then

$$\mathbf{F}^{(1,\mathrm{eq})} = -\boldsymbol{\nabla}_{\mathcal{O}_1}\frac{\hbar}{\pi}\int_0^\infty d\omega\left[\frac{1}{e^{\frac{\hbar\omega}{k_BT}}-1}+\frac{1}{2}\right] \times \mathrm{Im}\,\mathrm{Tr}\{\log[1-\mathbb{G}_0\mathbb{T}_1\mathbb{G}_0\mathbb{T}_{23}]\}. \quad (23)$$

We perform two series expansions. The first expands the log,

$$\log[1-\mathbb{G}_0\mathbb{T}_1\mathbb{G}_0\mathbb{T}_{23}] = -\mathbb{G}_0\mathbb{T}_1\mathbb{G}_0\mathbb{T}_{23}+\cdots. \quad (24)$$

The second expands the composite operator $\mathbb{T}_{23}$,

$$\mathbb{T}_{23} = \mathbb{T}_2+\mathbb{T}_3+\mathbb{T}_2\mathbb{G}_0\mathbb{T}_3+\mathbb{T}_3\mathbb{G}_0\mathbb{T}_2+\cdots \quad (25)$$

With this expansion, we obtain the force acting on object 1, to leading order in scattering operators

$$\mathbf{F}^{(1)}_{\mathrm{eq}} = \mathbf{F}^{(1,\mathrm{eq})}_2+\mathbf{F}^{(1,\mathrm{eq})}_3 + \boldsymbol{\nabla}_{\mathcal{O}_1}\frac{\hbar}{\pi}\int_0^\infty d\omega\left[\frac{1}{e^{\frac{\hbar\omega}{k_BT}}-1}+\frac{1}{2}\right] \times \mathrm{Im}\,\mathrm{Tr}\{\mathbb{G}_0\mathbb{T}_1\mathbb{G}_0(\mathbb{T}_2\mathbb{G}_0\mathbb{T}_3+\mathbb{T}_3\mathbb{G}_0\mathbb{T}_2)\}, \quad (26)$$

where $\mathbf{F}^{(1,\mathrm{eq})}_{2(3)}$ is the two-body force in the absence of object 3(2). We thus can write

$$\mathbf{F}^{(1)}_{\mathrm{eq}} = -\boldsymbol{\nabla}_{\mathcal{O}_1}\mathcal{F}_1, \quad (27)$$

where

$$\mathcal{F}_1 = \mathcal{F}_{12}+\mathcal{F}_{13}+\mathcal{F}_{23} - \frac{\hbar}{\pi}\int_0^\infty d\omega\left[\frac{1}{e^{\frac{\hbar\omega}{k_BT}}-1}+\frac{1}{2}\right] \times \mathrm{Im}\,\mathrm{Tr}\{\mathbb{G}_0\mathbb{T}_1\mathbb{G}_0(\mathbb{T}_2\mathbb{G}_0\mathbb{T}_3+\mathbb{T}_3\mathbb{G}_0\mathbb{T}_2)\}. \quad (28)$$

Here $\mathcal{F}_{12}$ is the two-body free energy in absence of object 3. $\mathcal{F}_{23}$ can be added because it drops out when taking the derivative $\boldsymbol{\nabla}_{\mathcal{O}_1}$.

The force on object 2 is found similarly,

$$\mathbf{F}^{(2)}_{\mathrm{eq}} = -\boldsymbol{\nabla}_{\mathcal{O}_2}\mathcal{F}_2, \quad (29)$$

with

$$\mathcal{F}_2 = \mathcal{F}_{12}+\mathcal{F}_{13}+\mathcal{F}_{23} - \frac{\hbar}{\pi}\int_0^\infty d\omega\left[\frac{1}{e^{\frac{\hbar\omega}{k_BT}}-1}+\frac{1}{2}\right] \times \mathrm{Im}\,\mathrm{Tr}\{\mathbb{G}_0\mathbb{T}_2\mathbb{G}_0(\mathbb{T}_1\mathbb{G}_0\mathbb{T}_3+\mathbb{T}_3\mathbb{G}_0\mathbb{T}_1)\}. \quad (30)$$

The same can be done for object 3. Using the cyclic property of the trace, we can see that the forces are found as gradients of the same free energy, i.e.,

$$\mathcal{F}_3 = \mathcal{F}_2 = \mathcal{F}_1 \equiv \mathcal{F}(\mathcal{O}_1,\mathcal{O}_2,\mathcal{O}_3) \quad (31)$$

and the force on object i is

$$\mathbf{F}^{(i)}_{\mathrm{eq}} = -\boldsymbol{\nabla}_{\mathcal{O}_i}\mathcal{F}(\mathcal{O}_1,\mathcal{O}_2,\mathcal{O}_3). \quad (32)$$

This demonstrates that, to leading order, a three-body potential $\mathcal{F}(\mathcal{O}_1,\mathcal{O}_2,\mathcal{O}_3)$ exists.

V. EXPLICIT EXAMPLES

In this section we discuss specific examples of the free energy for nonreciprocal objects.

A. Toy model with frequency independent permittivity

A simple toy model allows to illustrate many of the above general findings, and to demonstrate the existence of positive Laplacian and repulsive forces.

We start by considering two polarizable particles with separation d, which is large compared to the particles' sizes. To allow for analytic results, we assume that the particles' polarizability tensors are frequency-independent and have the following nonreciprocal form, with α_0 (units of volume) and b_i (dimensionless) real,

$$\alpha_i = \alpha_0\begin{pmatrix}1 & 0 & 0\\ 0 & 1 & -b_i\\ 0 & b_i & 1\end{pmatrix}. \quad (33)$$

This toy model, apart from frequency dependence, is of similar form to the response found for materials in the presence

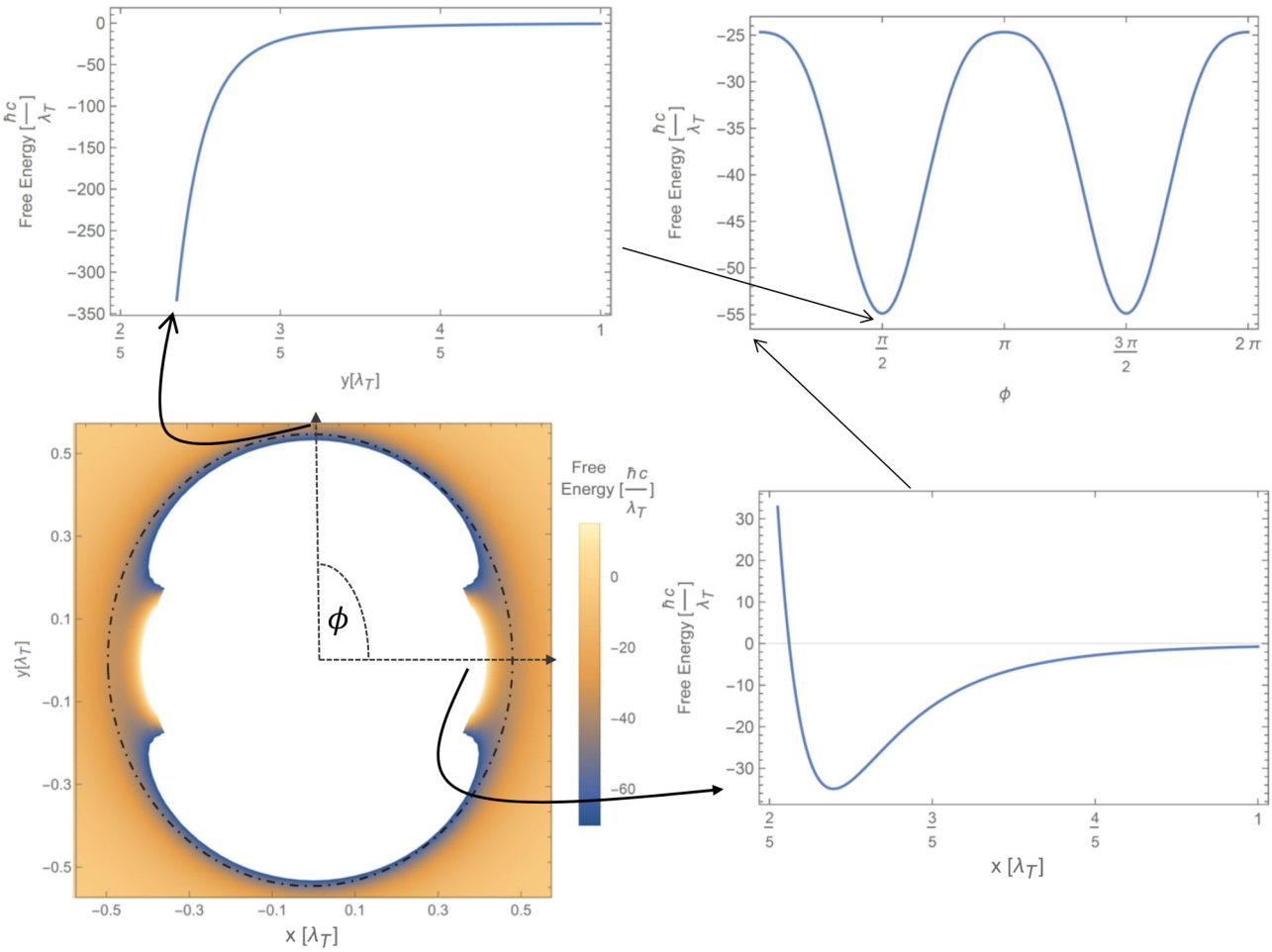

FIG. 2. Free energy of particle 2 as function of its position relative to particle 1 for a toy material. Both particles have same frequency-dependent polarizabilities, see Eq. (38) and [24], and the magnetic field points into the x direction. Left-Bottom: Free energy in the xy plane. Top left: Free energy as function of y for $x = z = 0$. Bottom right: Free energy as function of x for $y = z = 0$. Top right: Free energy as function of the angle ϕ for $z = 0$ and radius $0.47\lambda_T$ (dashed circle in bottom-left figure). At this separation, the free energy reaches a minimum along the x axis, but it shows a maximum along ϕ. Here $\omega_p = 2 \times 10^8$ Hz, $\omega_\tau = 2.46 \times 10^{14}$ Hz, $\omega_B = 1$ Hz, and $T = 300$ K.

of an external magnetic field pointing in x direction and with magnitude proportional to b_i. We will thus refer to b also as a magnetic field in the following.

Taking one particle at the origin and the second at position $d(\cos\phi \sin\theta, \sin\phi \sin\theta, \cos\theta)^T$ we find for $d \gg \frac{\hbar c}{k_B T}$

$$\mathcal{F} = \frac{k_B T \alpha_0^2}{32 d^6 \pi^2} \times [-12 - 5 b_1 b_2 - 3 b_1 b_2 (\cos[2\theta] - 2\cos[2\phi]\sin[\theta]^2)]. \tag{34}$$

For $b_i = 0$ this agrees with literature [34] and the free energy is negative, so the force is attractive. Notably, there exists no term linear in b_i, in agreement with Eq. (14): A term linear in b_i would result from a product of reciprocal and antireciprocal parts of α_i, and is thus ruled out.

The contribution from antireciprocal parts $\alpha_i - \alpha_i^T$ corresponds to terms quadratic in b. Indeed, these can be positive or negative (we will discuss the mirror symmetric case below). The free energy as a sum of both terms can indeed be positive, for example, for $\theta = \pi/2$, $\phi = 0$ and $b_1 b_2 > 3$.

For short separation, $d \ll \frac{\hbar c}{k_B T}$, we find

$$\mathcal{F} = \frac{\hbar c \alpha_0^2}{64 d^7 \pi^3} \times [-23 - 8 b_1 b_2 - 7 b_1 b_2 (\cos[2\theta] - 2\cos[2\phi]\sin[\theta]^2)]. \tag{35}$$

This limit shares the same general properties as the limit studied in Eq. (34). Here, for example, for $\theta = \pi/2$, $\phi = 0$ and $b_1 b_2 > \frac{23}{13}$, the free energy is positive, and the force is repulsive. Recall that this is impossible for reciprocal particles.

Continuing the example of $\theta = \pi/2$, $\phi = 0$, the free energy is positive for short distance and negative at large distance for $\frac{23}{13} < b_1 b_2 < 3$, which indicates the presence of a free energy minimum along the radial coordinate. Including displacements orthogonal to the radial direction, this point may be a minimum or a saddle point.

Exploring this numerically, we find that the Laplacian at this point is positive, but, for the given choice of system, these are indeed saddle points. Below we discuss an example of a frequency dependent permittivity.

Finally, we consider the case $\theta = 0$ and $\phi = \frac{\pi}{2}$, i.e., where the two particles are displaced along the z axis. They are then mirror images of each other, if we choose $b \equiv b_2 = -b_1$, meaning the magnetic field has to point in opposite directions. The free energy is then, for $d \gg \frac{\hbar c}{k_B T}$,

$$\mathcal{F} = \frac{k_B T \alpha_0^2}{32 d^6 \pi^2}[-12 + 8b^2], \tag{36}$$

obeying the properties found in Eq. (19). Similarly, for $d \ll \frac{\hbar c}{k_B T}$, we find

$$\mathcal{F} = \frac{\hbar c \alpha_0^2}{64 d^7 \pi^3}[-23 + 15b^2]. \tag{37}$$

Notably, if the particles are displaced along the x axis ($\theta = \pi/2$ and $\phi = 0$), mirror images are obtained without switching the direction of magnetic field, and the resulting antireciprocal part of free energy is positive in that case as well.

B. Frequency-dependent permittivity

Turning to a more realistic case, we consider a frequency-dependent permittivity of the form of a standard magneto-optical material. For a dc magnetic field pointing along the x direction [47] it has the following form:

$$\boldsymbol{\varepsilon} = \begin{pmatrix} \epsilon_p & 0 & 0 \\ 0 & \epsilon_d & -i\epsilon_f \\ 0 & i\epsilon_f & \epsilon_d \end{pmatrix}, \tag{38}$$

where $\epsilon_d = 1 - \frac{\omega_p^2(1+\frac{i\omega_\tau}{\omega})}{(\omega+i\omega_\tau)^2-\omega_B^2}$, $\epsilon_p = 1 - \frac{\omega_p^2}{\omega(\omega+i\omega_\tau)}$, and $\epsilon_f = -\frac{\omega_B\omega_p^2}{\omega((\omega+i\omega_\tau)^2-\omega_B^2)}$ [8]. Here, ω_p is the plasma frequency and ω_τ describes relaxation effects; the nonreciprocity ($\epsilon_f \neq 0$) due to the magnetic field is encoded via the cyclotron frequency ω_B. We assume that both particles have the same permittivity, which we use to calculate the polarizability [24]. The parameters used are given in the caption of Fig. 2.

The numerical results are presented in Figs. 2 and 3 for a toy material. The parameters were chosen to make the effect easier to see in the plots. Further investigation is required to determine the presence of saddle points and the positivity of the Laplacian in realistic materials. Notably, the discussion of frequency independent cases is applicable to a large extent here as well. We see that repulsion is possible, and the free energy, as a function of particle distance can have a minimum as before. Also, here, these minima in the radial direction are *saddle points* in full 3D space. As a result while we showed in Sec. III that Earnshaw's theorem cannot be relied on for nonreciprocal media, this does not necessarily establish the existence of a minimum. Figure 3 shows numerical results for the Laplacian of the free energy, displaying, as in the case of the frequency independent polarizability, it is positive around the saddle point.

The quest for minima in this situation requires further investigation that could include the case of bodies with different permittivities or several bodies.

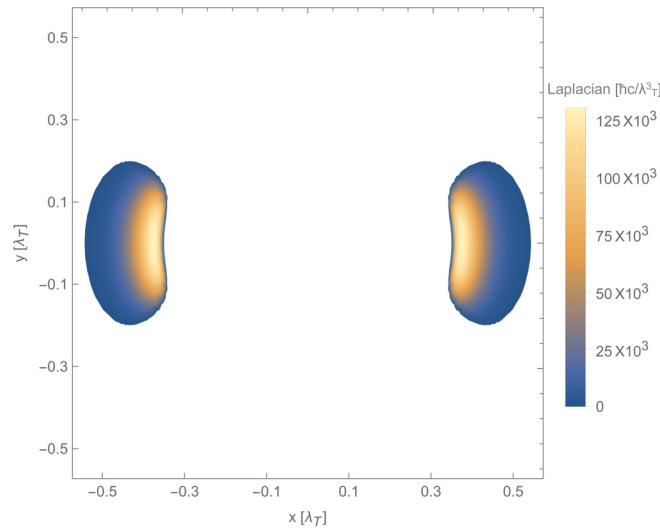

FIG. 3. Laplacian of the free energy of particle 2 as function of its position relative to particle 1. Only the positive regions are shown and they include the position of the minima in the radial direction ($z = 0$, $x = 0.47\lambda_T$, and $y = 0$), seen in Fig. 2. Same parameters as Fig. 2.

VI. SUMMARY

Casimir forces acting between nonreciprocal materials in thermal equilibrium are conservative and reciprocal. The corresponding potential or free energy has a similar mathematical structure as the known one for reciprocal materials. Despite that, the properties of the free energy and the forces can be quite different between the reciprocal and nonreciprocal cases [11,21], which we have discussed in the so-called one-reflection approximation. In contrast to reciprocal cases, the free energy can be positive, and it can display minima. Furthermore, reciprocal and antireciprocal contributions yield two distinct terms in the free energy, which is thus a sum of the free energy between two reciprocal objects and the free energy between two antireciprocal objects. For two objects that are each other's mirror images, the second term is positive and the corresponding force is repulsive. These properties can be demonstrated in calculations for two nonreciprocal polarizable particles.

For three objects, we perform an expansion in reflections, finding that to leading order, a three-body potential exists. This case is of interest because persistent heat currents have been found before in situations involving three bodies. Our calculations—to the given orders—thus show that, despite existence of mentioned currents, persistent torques are ruled out.

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APPENDIX: PROOF OF EQ. (5)

To obtain Eq. (5) in the main text we use the following identities:

$$\mathbb{T}_i\mathbb{G}_0\mathbb{W}_{ji} = \mathbb{W}_{ij}\mathbb{G}_0\mathbb{T}_i; \tag{A1}$$

$$\mathbb{G}_0\mathbb{T}_i\mathbb{G}_0\mathbb{T}_j\mathbb{G}_0\mathbb{W}_{ij} = \mathbb{G}_0\mathbb{W}_{ij}\mathbb{G}_0\mathbb{T}_i\mathbb{G}_0\mathbb{T}_j; \tag{A2}$$

$$\mathbb{G}_0\mathbb{T}_i\mathbb{G}_0\mathbb{T}_j\mathbb{G}_0\mathbb{W}_{ij} = \mathbb{G}_0\mathbb{W}_{ij} - 1. \tag{A3}$$

We get

$$\begin{aligned}\mathbf{F}^{(1)}_{\rm eq} = \frac{\hbar}{\pi}\int_0^\infty d\omega\left[\frac{1}{e^{\frac{\hbar\omega}{k_BT}}-1}+\frac{1}{2}\right]\\ \times\left[\frac{-1}{2i}\mathrm{Re\,Tr}\{\boldsymbol{\nabla}\mathbb{G}_0^\dagger\mathbb{W}_{\bar{1}1}^\dagger\mathbb{G}_0^*\mathbb{T}_1^\dagger\}\right.\\ +\mathrm{Re\,Tr}\left\{\boldsymbol{\nabla}(\mathbb{G}_0\mathbb{T}_1)\mathbb{G}_0\mathbb{W}_{\bar{1}1}\frac{1}{2i}\right\}\\ +\mathrm{Re\,Tr}\left\{\boldsymbol{\nabla}\mathbb{G}_0\mathbb{W}_{\bar{1}1}\frac{1}{2i}\right\}\\ \left.-\frac{1}{2i}\mathrm{Re\,Tr}\{\boldsymbol{\nabla}\mathbb{G}_0^\dagger\mathbb{W}_{1\bar{1}}^\dagger\}\right].\end{aligned} \tag{A4}$$

Taking the adjoint of the first term and using the cyclic properties of the trace we get

$$-\frac{1}{2i}\mathrm{Re\,Tr}\{\mathbb{G}_0\boldsymbol{\nabla}\mathbb{T}_1\mathbb{G}_0\mathbb{W}_{\bar{1}1}\}. \tag{A5}$$

We use that $\boldsymbol{\nabla}_r\mathbb{G}_0(r-r') = -\boldsymbol{\nabla}_{r'}\mathbb{G}(r-r')$. Furthermore, every operator product in the trace corresponds to the integral over a joint coordinate. We may thus use integration by parts (neglecting boundary terms), and we get for the above term,

$$-\frac{1}{2i}\mathrm{Re\,Tr}\{\boldsymbol{\nabla}\mathbb{G}_0\mathbb{T}_1\mathbb{G}_0\mathbb{W}_{\bar{1}1}\} \tag{A6}$$

and the first two terms of Eq. (A4) cancel.

Next we calculate the adjoint of the last term of Eq. (A4) and using a similar procedure as to pass from Eq. (A5) to Eq (A6) we get

$$\frac{-1}{2i}\mathrm{Re\,Tr}\{\boldsymbol{\nabla}\mathbb{G}_0^\dagger\mathbb{W}_{1\bar{1}}^\dagger\} = \frac{-1}{2i}\mathrm{Re\,Tr}\{\boldsymbol{\nabla}\mathbb{G}_0\mathbb{W}_{1\bar{1}}\}. \tag{A7}$$

Using identities (A3) we get

$$\begin{aligned}\frac{1}{2i}\mathrm{Re\,Tr}\{\boldsymbol{\nabla}\mathbb{G}_0\mathbb{W}_{\bar{1}1}\} &= \frac{1}{2i}\mathrm{Re\,Tr}\{\boldsymbol{\nabla}\mathbb{G}_0\mathbb{T}_{\bar{1}}\mathbb{G}_0\mathbb{T}_1\mathbb{G}_0\mathbb{W}_{\bar{1}1}\}\\ &= \frac{1}{2i}\mathrm{Re\,Tr}\{\mathbb{T}_1\boldsymbol{\nabla}\mathbb{G}_0\mathbb{T}_{\bar{1}}\mathbb{G}_0\mathbb{W}_{1\bar{1}}\mathbb{G}_0\}\\ &= \frac{1}{2i}\mathrm{Re\,Tr}\{\boldsymbol{\nabla}\mathbb{T}_1\mathbb{G}_0\mathbb{T}_{\bar{1}}\mathbb{G}_0\mathbb{W}_{1\bar{1}}\mathbb{G}_0\}\\ &\quad+\frac{1}{2i}\mathrm{Re\,Tr}\{\boldsymbol{\nabla}_{O_1}\mathbb{T}_1\mathbb{G}_0\mathbb{T}_{\bar{1}}\mathbb{G}_0\mathbb{W}_{1\bar{1}}\mathbb{G}_0\},\end{aligned} \tag{A8}$$

where in the last inequality we have used the identity

$$\mathbb{T}_1\boldsymbol{\nabla} = \boldsymbol{\nabla}\mathbb{T}_1 + \boldsymbol{\nabla}_{O_1}\mathbb{T}_1.$$

We now apply identities (A3) to (A7) and (A4)

$$\begin{aligned}&-\mathrm{Re\,Tr}\left\{\boldsymbol{\nabla}\mathbb{G}_0\mathbb{W}_{1\bar{1}}\frac{1}{2i}\right\}\\ &\quad=-\mathrm{Re\,Tr}\left\{\boldsymbol{\nabla}\mathbb{G}_0\mathbb{T}_1\mathbb{G}_0\mathbb{T}_{\bar{1}}\mathbb{G}_0\mathbb{W}_{1\bar{1}}\frac{1}{2i}\right\}.\end{aligned} \tag{A9}$$

Putting everything together we get that

$$\begin{aligned}&\frac{\hbar}{\pi}\int_0^\infty d\omega\left[\frac{1}{e^{\frac{\hbar\omega}{k_BT}}-1}+\frac{1}{2}\right]\\ &\quad\times\frac{1}{2i}\mathrm{Re\,Tr}\{\boldsymbol{\nabla}_{O_1}\mathbb{G}_0\mathbb{T}_1\mathbb{G}_0\mathbb{T}_{\bar{1}}\mathbb{G}_0\mathbb{W}_{1\bar{1}}\}\\ &=\frac{\hbar}{\pi}\int_0^\infty d\omega\left[\frac{1}{e^{\frac{\hbar\omega}{k_BT}}-1}+\frac{1}{2}\right]\\ &\quad\times\frac{1}{2i}\mathrm{Re\,Tr}\{\mathbb{G}_0\mathbb{W}_{1\bar{1}}\boldsymbol{\nabla}_{O_1}\mathbb{G}_0\mathbb{T}_1\mathbb{G}_0\mathbb{T}_{\bar{1}}\}\\ &=-\frac{\hbar}{\pi}\int_0^\infty d\omega\left[\frac{1}{e^{\frac{\hbar\omega}{k_BT}}-1}+\frac{1}{2}\right]\\ &\quad\times\frac{1}{2i}\mathrm{Re\,Tr}\boldsymbol{\nabla}_{O_1}\log(1-\mathbb{G}_0\mathbb{T}_1\mathbb{G}_0\mathbb{T}_{\bar{1}}).\end{aligned} \tag{A10}$$

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