

Using Network Analysis Techniques to Probe Student Understanding of Expressions Across Notations in Quantum Mechanics

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One important outcome of physics instruction is for students to be capable of relating physical concepts and phenomena to multiple mathematical representations. In quantum mechanics (QM), students are asked to work between multiple symbolic notations, some not previously encountered. To investigate student understanding of the relationships between expressions used in these various notations, many of which describe analogous physical concepts, a survey was distributed to students enrolled in upper-division QM courses at multiple institutions. Network analysis techniques were shown to be useful for gaining information about how students relate these expressions. Preliminary analysis suggests that students view Dirac bras and kets as more similar to generic vectors than to their physically analogous wave function counterparts, and that Dirac bras and kets serve as a bridge between vector and wave function expressions.

Keywords: Network Analysis, Quantum Mechanics, Notation

There has recently been a focus in research at the boundary of physics and mathematics in upper-division quantum mechanics (QM) (e.g., Wawro et al., 2020), including a focus on the three mathematical notations (Dirac, wave function, and vector-matrix) used to describe identical or analogous physical concepts and phenomena (Gire & Price, 2015; Schermerhorn et al., 2019). A comprehensive understanding of how expressions in these notations interrelate and how to translate between them is crucial for a deep understanding of QM (Wawro et al., 2020). One challenge facing upper-division education research is the relatively low sample size when compared to research conducted in the lower division, as this affects the generalizability of claims. This is particularly true in quantum mechanics, where the order in which these different notations are introduced—based primarily on the instructor’s choice of textbook—can drastically affect the focus of instruction and thus the eventual conceptual understanding of the students. We have implemented network analysis techniques to probe students’ conceptual connections between symbolic expressions, allowing for much larger sample sizes than is typically feasible for this research context. To that end, we address the following research questions: How can network analysis techniques be leveraged to study students’ conceptual connections between expressions in quantum mechanics, and what are the connections that these techniques show?

Prior Research on Quantum Notations and Network Analysis

The various notations used in upper-division quantum mechanics have different affordances and limitations for computation, both from an expert point of view (Gire & Price, 2015) as well as in students’ work (Schermerhorn et al., 2019). Additionally, incorrect translations between wave function and Dirac expressions causes students to struggle when developing models for determining probabilities (Wan et al., 2019). The ability to reason between and among different mathematical representations has been linked to understanding of QM concepts (Wawro et al., 2020), and work has been done to create instructional materials to aid students in working fluidly among multiple representations (Kohnle & Passante, 2017).

Network analysis techniques such as community detection and cluster analysis have recently been used extensively to study response groupings for various conceptual inventories in physics

education research (Brewer et al., 2016; Wells et al., 2019; Wells et al., 2020; Wells et al., 2021; Wheatley et al., 2021; Yang et al., 2020). Members of the research in undergraduate mathematics education community have used social networks among teachers to study community and coaching among educators (Hopkins et al., 2017; Smith et al., 2017), while students' social networks have been studied across multiple fields to study how they impact academic performance, persistence, self-efficacy, and anxiety (Hopkins et al., 2017; Thomas, 2000).

Study Design and Methodology

The survey was designed with two primary goals in mind: to easily collect and analyze responses from many students, and to create a dataset that allows for analysis of students' conceptual connections between mathematical expressions commonly used in QM—particularly those used to express probability concepts. To achieve the first goal, the number of free-response text entry questions were minimized to reduce participant attrition. This meant that the second goal would need to be achieved without much in the way of written responses showing explicit student reasoning. The questions therefore were designed as sorting tasks, where students were given a list of expressions as well as a single concept and asked to select all of the expressions which could represent that concept—see Figure 1 for an example. The survey consisted of 11 different concepts, with the same 16 expressions to choose from. This entirely relation-based dataset—between both expression-concept pairs as well as pairs of expressions used for a given concept—makes network analysis an ideal choice. The survey was distributed to three different institutions, including two public land-grant research universities in the American northeast and one private midwestern liberal arts college, for a total of 27 participants. Of these participants, 20 were in classes that taught Dirac notation prior to wave function notation (“spins first”), and seven were taught wave functions prior to Dirac notation (“functions first”).

Select which expression(s) (if any) are representations of the given concept, and drag them into the concept's box.

Items				Dot Product
$\langle E_4 \psi \rangle^2$	$\hat{j}$	$\psi(x)$	$\int \psi^*(x)\psi(x)dx$	
$\int \phi_1^*(x)\psi(x)dx$	$ \int \psi^*(x)\psi(x)dx ^2$	$\vec{u} \cdot \vec{v}$	$\phi_3(x)$	
$ \int \phi_2^*(x)\psi(x)dx ^2$	$\hat{S}_z$	$\langle E_3 \psi \rangle$	$\vec{v}$	
$ \psi\rangle$	$\langle \psi \psi \rangle$	$ E_2\rangle$	$\langle E_1 $	

Figure 1. Example of the survey task for the “Dot Product” concept.

Data Analysis and Results

The first step in data analysis was the creation of a weighted network with the 16 expressions as nodes, with the connections between them—known as edges—weighted by the number of students that used the two expressions for the same concept at least once in the survey (Figure 2). The larger network was then broken into communities of more closely connected expression nodes. The method chosen for detecting these communities involved measuring the number of geodesic paths between every pair of nodes in the network that pass through each edge, known as the edge's *betweenness* (Girvan & Newman, 2002). The edges with highest betweennesses are those that connect communities within the larger network. This is because if two communities exist within a network there will, in general, be fewer edges connecting between the two than

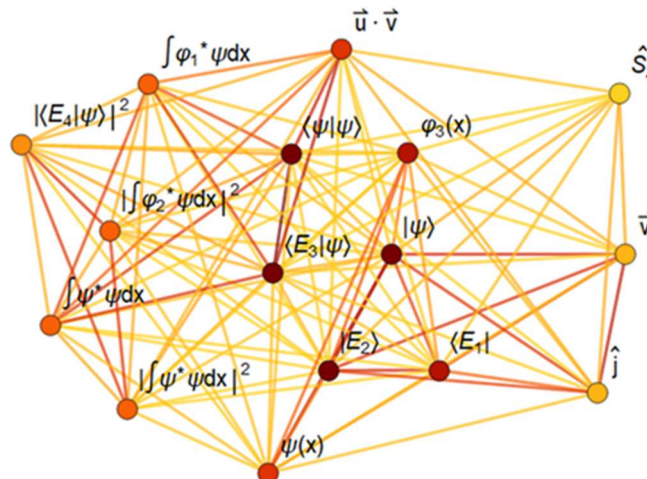


Figure 2. Network with expressions as nodes and occurrences of students using two expressions for a concept as edges. Edge weights are visualized with color, where red corresponds to higher weight and yellow to lower weight.

there are within each community—thus those few connecting edges will bear the load of all the geodesics traveling between the nodes in each community, causing them to have large betweennesses. In this community-detection algorithm, the betweenness of each edge in the network is calculated, and then the edge with the largest betweenness is removed. This calculation and subsequent removal of the edge with maximum betweenness repeats until all of the edges in the network are removed, eventually leaving all nodes fully disconnected. The process of removing one edge at a time gives a cascading hierarchy of communities in the network, with larger communities eventually being divided into constituent sub-communities that are themselves more strongly connected to their own members. The hierarchical structure of the communities found using this method can be visualized by a dendrogram (Figure 3).

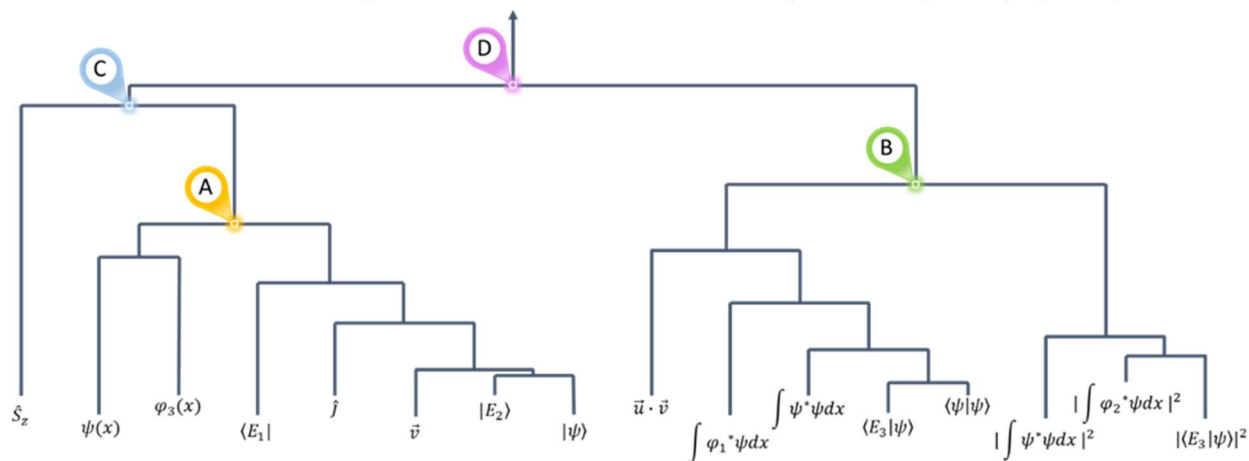


Figure 3 Dendrogram displaying the community structure of the network shown in Figure 2. Vertical height is proportional to number of edges removed between community divisions. Letters correspond to those on Figure 5.

Figure 3 suggests that the first distinction students drew was morphological: the first division of the network was into “single-term” and “double-term” communities. Individual functions, vectors, Dirac bras and kets, and quantum mechanical operators were grouped together, while expressions that contained Dirac bra-ket pairs, two vectors, or two functions—inner products—all shared their own community. However, these community separations clearly have conceptual

distinctions to the students as well. Almost immediately upon dividing into single- and double-term communities, the z -component spin operator ($\hat{S}_z$) was excluded from the other single-term expressions. This is of interest for two reasons. First, both physically and mathematically speaking, an operator is very much unlike any other single-term element. Second, $\hat{S}_z$ and $\hat{j}$ share many wholly morphological similarities; the fact that they were clearly not viewed as similar to students is an encouraging sign that the students were attending to actual physical and mathematical meaning and not merely focusing on morphological distinctions.

Figure 3 also gives insight into the relative strengths of conceptual connections between notations. The second division to occur within the single-term community is that of the two wave functions ($\psi(x)$, $\varphi_3(x)$) splitting off from both the Dirac bras and kets and the generic vector expressions. These edges and communities are, due to our survey design, entirely based on expressions that students view as conceptually similar, suggesting that Dirac bras and kets may be more closely associated with vector ideas than with concepts associated with wave functions.

Aside from the information to be gleaned from the hierarchical community structure alone, the structure within the communities as they are being separated into their sub-communities can also be used to investigate how certain expressions are connected, as well as what that means for the expressions' conceptual connections. This connective structure can be teased out by investigating the minimum vertex cut sets (MVCSs) between expression pairs throughout this cascading network decomposition. The MVCS between two nodes in a network is the smallest set of nodes that need to be removed to entirely disconnect the two nodes in question, and can be used to see which nodes tend to connect any pair of nodes. At the stage of the network decomposition shown in Figure 4a, for example, there are two single-node (size-1) MVCSs between $\langle\psi|\psi\rangle$ and $|\int\psi(x)^*\psi(x)dx|^2$: $\{|\langle E_3|\psi\rangle\rangle$ and $\{|\langle E_3|\psi\rangle|^2\}$. Likewise, in Figure 4b the $\hat{j}$ and $\varphi_3(x)$ nodes have one MVCS of size 3, made up of the Dirac bras and kets: $\{|\langle E_2\rangle, |\psi\rangle, \langle E_1|\}$. This is reflective of the apparent symmetry in the community's structure, where the Dirac expressions appear to serve as a sort of bridge between the wave function and Dirac expressions. The size, number, and elements contained within the MVCS(s) between any two nodes is liable to change as the edge-betweenness algorithm plays out and in fact the metric becomes entirely meaningless if the two nodes in question ever become connected directly, as at that point the only vertex that can be removed to separate the two would be one of the two nodes themselves.

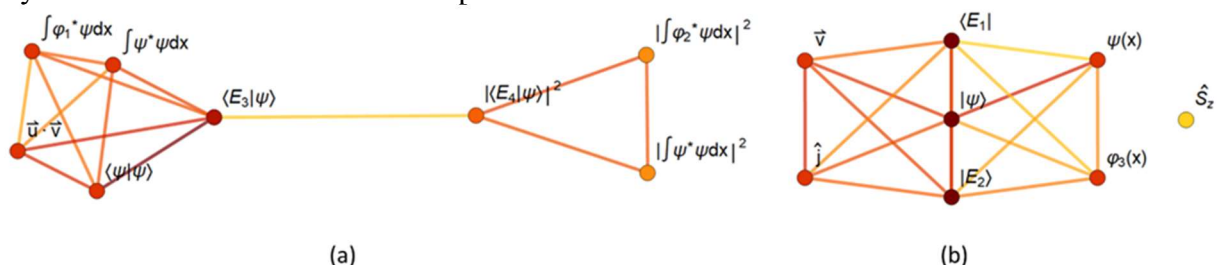


Figure 4. Network partway through betweenness algorithm showing (a) double- and (b) single-term communities.

To investigate whether this apparent Dirac-bridging is meaningful beyond the single cross-section of the betweenness algorithm shown in Figure 4, the MVCS between wave function and Dirac expressions can be examined throughout the community detection process. Moving upwards from the bottom of the dendrogram, where all edges have been deleted and thus all nodes are disconnected, can be thought of as playing the edge-betweenness algorithm backwards; this allows for the MVCSs between the pairs of vector and wave function expressions to be observed as the communities are being “formed.” The MVCSs between the $\hat{j}$

and $\varphi_3(x)$ nodes serve as an illustrative example of this, as shown in Figure 5. The 25th edge added into the network provides the first connection between $\hat{j}$ and $\varphi_3(x)$ (point A on Figs. 3 and 5), and the very next then expands the MVCS between the two from $\{|\psi\rangle\}$ to $\{|\psi\rangle, |E_2\rangle\}$. This remains the only MVCS for three more edge-additions before the MVCS expands again to $\{|\psi\rangle, |E_2\rangle, \langle E_1|\}$. This remains the stable MVCS as the next 21 edges are added, during which the two double-term communities merge (B), $\hat{S}_z$ rejoins the single-terms (C), and even the single- and double-term communities reconnect (D). The next edge added directly connects $\hat{j}$ and $\varphi_3(x)$, thus making the MVCS between them meaningless. The $\hat{j}$ - $\varphi_3(x)$ pair serves as an illustrative example: the Dirac expressions are always the most prominent connectors between the function and vector expressions, and remain so after all of the communities have connected.

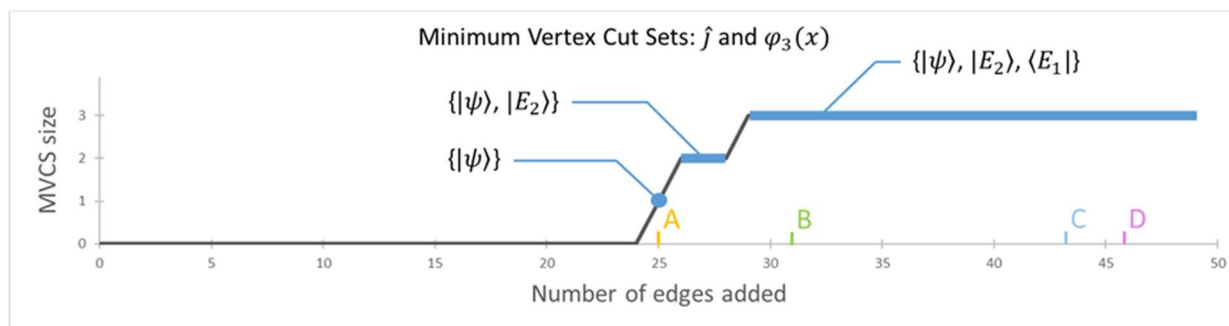


Figure 5. Graph displaying MVCSs for the $\hat{j}$ - $\varphi_3(x)$ node pair. Letters correspond with those on Figure 3.

Conclusions, Discussion, Implications for Further Research

Our survey and analysis using network techniques appears to be able to isolate students' conceptual knowledge as it applies to expressions in the various notations used in QM. This combined with the scalability of both the data collection and analysis methods is an encouraging sign of the ability of these techniques to study a large number of students at a large number of institutions. The methods described above will only improve with a larger sample size, and so there is likely more to be learned about students' conceptual connections between expressions across notations in QM if these techniques are applied more broadly. Our data suggests that students think of Dirac bras and kets as a blend of wave function and vector ideas. This is an encouraging finding, as that is effectively exactly why Dirac invented the notation in the first place. What is interesting, however, is that students appear to more closely link the Dirac bras and kets to vector ideas—likely due to their mathematical utility—than to ideas associated with wave functions, the connection to which is almost entirely grounded in a physical understanding.

Within this QM context, network analysis could be used to expose differences in students' thinking about various expressions due to either institutional context or pedagogical focus. We suspect that the networks formed by students in courses where Dirac notation is introduced first would differ greatly from those in courses focused largely on wave functions. Our current data pool is not large enough to make claims in this regard, but future work may show whether any distinctions become apparent. There are a number of areas where this type of data collection and analysis could bear future use, such as with expressions associated with integrals and sums, both in calculus as well as in physics contexts such as electromagnetism or thermodynamics.

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