

On the Central Path of Semidefinite Optimization: Degree and Worst-Case Convergence Rate*

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Abstract. In this paper, we investigate the complexity of the central path of semidefinite optimization through the lens of real algebraic geometry. To that end, we propose an algorithm to compute real univariate representations describing the central path and its limit point, where the limit point is described by taking the limit of central solutions, as bounded points in the field of algebraic Puiseux series. As a result, we derive an upper bound $2^{O(m+n^2)}$ on the degree of the Zariski closure of the central path, when μ is sufficiently small, and for the complexity of describing the limit point, where m and n denote the number of affine constraints and size of the symmetric matrix, respectively. Furthermore, by the application of the quantifier elimination to the real univariate representations, we provide a lower bound $1/\gamma$, with $\gamma = 2^{O(m+n^2)}$, on the convergence rate of the central path.

Key words. semidefinite optimization, semi-algebraic sets, central path, real univariate representation, quantifier elimination

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1. Introduction. The main goal of this paper is to investigate the complexity of the central path of semidefinite optimization (SDO) through the lens of real algebraic geometry [6, 12]. Among other things, we address the following open problem, as stated in [13, p. 59].

Problem 1.1. *Derive a lower bound on the convergence rate of the central path.*

An SDO problem is defined as the minimization of a linear objective function on the cone of positive semidefinite matrices intersected with an affine subspace. SDO has been of great theoretical and practical interest, with numerous applications in theoretical computer science, control theory, optimization, and statistics [45]. There has been a growing interest in the study of SDO through the lens of convex algebraic geometry [10, Chapters 5 and 6] and polynomial optimization [26, 27], where SDO is an emerging computational tool.

Let \mathbb{S}^n denote the vector space of $n \times n$ real symmetric matrices endowed with an inner product $\langle X, S \rangle = \text{trace}(XS)$ for any $X, S \in \mathbb{S}^n$. Mathematically, an SDO problem is defined as

$$(P) \quad v_p^* := \inf_{X \in \mathbb{S}^n} \left\{ \langle C, X \rangle \mid \langle A^i, X \rangle = b_i, \quad i = 1, \dots, m, \quad X \succeq 0 \right\},$$

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where $C, A^i \in \mathbb{S}^n$ for $i = 1, \dots, m$ are $n \times n$ real symmetric matrices, and $b \in \mathbb{R}^m$. We assume that all the coefficients A^i , C , and b belong to \mathbb{Z} . In this context, $X \succeq 0$ means that X belongs to the cone of positive semidefinite matrices. The dual of (P) is given by

$$(D) \quad v_d^* := \sup_{(y, S) \in \mathbb{R}^m \times \mathbb{S}^n} \left\{ b^T y \mid \sum_{i=1}^m y_i A^i + S = C, S \succeq 0 \right\}.$$

A primal-dual vector is denoted by (X, y, S) , and it is called *feasible* if (X, y, S) satisfies the equalities and inequalities in (P) and (D). The sets of primal and dual solutions are denoted, respectively, by

$$\begin{aligned} \text{Sol(P)} &:= \left\{ X \in \mathbb{S}^n \mid \langle A^i, X \rangle = b_i, \quad i = 1, \dots, m, \quad X \succeq 0, \quad \langle C, X \rangle = v_p^* \right\}, \\ \text{Sol(D)} &:= \left\{ (y, S) \in \mathbb{R}^m \times \mathbb{S}^n \mid \sum_{i=1}^m y_i A^i + S = C, \quad S \succeq 0, \quad b^T y = v_d^* \right\}. \end{aligned}$$

The following conditions, which we assume throughout this paper, guarantee the existence of a primal-dual solution and the compactness of the primal and dual solution sets [44, Corollary 4.2].

Assumption 1.1. The matrices A^i for $i = 1, \dots, m$ are linearly independent, and there exists a feasible $(X^\circ, y^\circ, S^\circ)$ such that $X^\circ, S^\circ \succ 0$, where $\succ 0$ means positive definite.

Primal-dual interior point methods (IPMs) [1, 35] are among the most efficient methods for solving (P) and (D). However, unlike linear optimization (LO), the existence of a polynomial time algorithm for an exact solution of (P) and (D) is still an open problem; see [5, section 4.2] or [39, 40]. In the bit model of computation, a semidefinite feasibility problem either $\in \mathbf{NP} \cap \mathbf{co-NP}$ or $\notin \mathbf{NP} \cup \mathbf{co-NP}$ [39]. In the real number model of computation [11], a semidefinite feasibility problem belongs to $\mathbf{NP} \cap \mathbf{co-NP}$. The existence of a polynomial time algorithm for (P) and (D) with fixed dimension was proved by Porkolab and Khachiyan [38]; see also [22, 23].

Notation 1.1. We identify a (real or complex) symmetric matrix $X = (X_{ij})_{n \times n}$ by a vector x through the linear map

$$X \mapsto (X_{11}, X_{12}, \dots, X_{1n}, X_{22}, X_{23}, \dots, X_{2n}, \dots, X_{nn})^T,$$

where

$$t(n) := \binom{n+1}{2}.$$

For ease of exposition we introduce $\bar{n} := m + 2t(n)$. The notation $(., \dots, .)$ and $(.; \dots; .)$ is adopted for side-by-side arrangement of matrices and concatenation of column vectors, respectively. Accordingly, a primal-dual vector (X, y, S) is identified by $(x; y; s)$.

Besides the complexity in the bit/real number model of computation, it is possible to approach the complexity of SDO from the perspective of the so-called *central path*, which lies

at the heart of primal-dual path-following IPMs. The central path is a smooth¹ *semi-algebraic* function $\phi : (0, \infty) \rightarrow \mathbb{R}^{t(n)} \times \mathbb{R}^m \times \mathbb{R}^{t(n)}$ such that $\phi(\mu) : \mu \mapsto (x(\mu); y(\mu); s(\mu))$ and $(\mu, \phi(\mu))$ satisfies

$$(1.1) \quad \begin{aligned} \langle A^i, X \rangle &= b_i, & i &= 1, \dots, m, \\ \sum_{i=1}^m A^i y_i + S &= C, \\ XS &= \mu I_n, & X, S &\succ 0, \end{aligned}$$

where I_n is the identity matrix of size n ; see [13, p. 41] and [Notation 1.1](#). Generally speaking, the main idea of primal-dual path-following IPMs is to compute solutions in a contracting sequence of open sets, in the Euclidean topology, around the central path. For every fixed positive μ , $(X(\mu), y(\mu), S(\mu))$ is a so-called *central solution*. Given a fixed $\bar{\mu} > 0$, the central path restricted to $(0, \bar{\mu}]$ is bounded [13, Lemma 3.2]. It was proved in [20, Theorem A.3], on the basis of the curve selection lemma [30, Lemma 3.1], that the central path always converges with the limit point in the relative interior of the primal-dual solution set [17, Lemma 4.2]. Alternatively, the existence of a unique limit point follows from the fact that the semi-algebraic path $\phi|_{(0, \bar{\mu}]}$ is bounded [13, Lemma 3.2], and thus it can be continuously extended to $\mu = 0$ [6, Proposition 3.18].

Contribution. Algorithmic features of primal-dual path-following IPMs, such as search directions, neighborhoods, step lengths, etc., have been extensively studied for an approximate solution of (P) and (D); see e.g., [3, 35]. However, the analyticity or limiting behavior of the central path has received little attention in the absence of the strict complementarity condition; see, e.g., [20, 37] and section 2.3. In particular, the worst-case convergence rate of the central path is still unknown, and there are only a few partial characterizations for its limit point; see, e.g., [21, 42]. As illustrated by [Example 1.1](#), the distance of a central solution from the limit point can be as big as $\Omega(\mu^{2^{-n}})$.

Example 1.1 ([Example 3.3](#) in [13]). Consider the following SDO problem in dual form (D):

$$\max \left\{ -y_n \mid S = \begin{pmatrix} 1 & y_1 & y_2 & \cdots & y_{n-1} \\ y_1 & y_2 & 0 & \cdots & 0 \\ y_2 & 0 & y_3 & \ddots & \vdots \\ \vdots & \vdots & \ddots & \ddots & 0 \\ y_{n-1} & 0 & \cdots & 0 & y_n \end{pmatrix} \succeq 0 \right\},$$

where (y^{**}, S^{**}) with $y_i^{**} = 0$ for $i = 1, \dots, n$ is the unique solution. In this case, the central path converges with the order $\|S(\mu) - S^{**}\| = \Omega(\mu^{2^{-n}})$, where $\|\cdot\|$ denotes the Frobenius norm of a matrix.

The main contribution of this paper is to bound the degree and convergence rate of the central path for (P) and (D). To that end, we propose an algorithm to describe the central path

¹The analyticity of the central path follows from the nonsingularity of the Jacobian of the equations in (1.1) [13, Theorem 3.3] and the analytic implicit function theorem [16, Theorem 10.2.4].

and its limit point. The algorithm computes parametrized univariate representations which, for all $\mu \in (0, 1]$, represent a central solution. A parametrized univariate representation is the description of each coordinate of the central path as a rational function of μ and the roots of a univariate polynomial; see sections 2.1 and 3.1. Our algorithm invokes [6, Algorithm 12.18] (parametrized bounded algebraic sampling) which, for all $\mu \in (0, 1]$, describes X_1 -pseudocritical points [6, Definition 12.41] on an algebraic set formed by the sum of squares of the polynomials

$$(1.2) \quad \mathcal{P} := \left\{ \sum_{\substack{j, \ell=1 \\ j < \ell}}^n 2A_{j\ell}^i X_{j\ell} + \sum_{j=1}^n A_{jj}^i X_{jj} - b_i, \left(\sum_{i=1}^m A^i y_i + S - C \right)_{j\ell}, (XS + SX - 2\mu I_n)_{j\ell}, j \leq \ell \right\}$$

in the ring $\mathbb{Z}[\mu, V_1, \dots, V_n]$, where $v := (x; y; s)$, and $(\cdot)_{j\ell}$ refers to upper triangular entries of a symmetric matrix; see Remark 3.1 and Notation 1.1. The limit point of the central path is then described by taking the limits of bounded zeros of (1.2), as a μ -infinitesimally deformed polynomial system, whose coordinates belong to the field of algebraic Puiseux series. In doing so, our algorithm applies the subroutine [7, Algorithm 3] (limit of a bounded point) to the real univariate representations from the parametrized bounded algebraic sampling; see section 3.2. As a result, we derive an upper bound $2^{O(m+n^2)}$ on the degree of the Zariski closure of the central path, when μ is sufficiently small, and for the complexity of describing the limit point of the central path. Furthermore, the application of the quantifier elimination algorithm [6, Algorithm 14.5] to the real univariate representations gives rise to a bound on the convergence rate of the central path. The following theorem is one of the main results of this paper, answering the open question in [13, p. 59].

Theorem 1.1. *Let (X^{**}, y^{**}, S^{**}) be the limit point of the central path. Then the distance of a central solution from its limit point, when μ is sufficiently small, is bounded by*

$$\|X(\mu) - X^{**}\| = O(\mu^{1/\gamma}) \quad \text{and} \quad \|S(\mu) - S^{**}\| = O(\mu^{1/\gamma}),$$

where $\gamma = 2^{O(m+n^2)}$.

Related work. The central path of (convex) SDO is well studied in the literature; see, e.g., [13, 17, 19, 20, 21, 29, 31, 37, 42]. Under the strict complementarity condition only, Halická [19] extended the analyticity of the central path to $\mu = 0$. Halická, de Klerk, and Roos [20] showed that the convergence of the central path to the analytic center of the solution set (see section 2.3) is no longer guaranteed when the strict complementarity condition fails. Goldfarb and Scheinberg [17] proved, under the strict complementarity and primal-dual nondegeneracy conditions [2], that the first-order derivatives of the central path converge as $\mu \downarrow 0$. Sporre and Forsgren [42] characterized the limit point of the central path as the unique solution to an auxiliary convex optimization problem; see also [21]. Convergence of central solutions for smooth convex SDO problems was also established in [18]. Recently, Mohammad-Nezhad and Terlaky [31] provided bounds on the convergence rate of vanishing eigenvalues on the central path; see also [43].

The algebro-geometric properties of the central path were initially studied by Bayer and Lagarias [8, 9] for LO, where the central path was identified as an irreducible component of a

complete intersection. Furthermore, there are comprehensive studies on the total curvature² and Riemannian length of the central path [4, 14, 15, 32, 34, 36, 41, 47]. Techniques from differential and algebraic geometry were invoked by Dedieu, Malajovich, and Shub [15] to bound the total curvature of the central path. Under genericity assumptions, De Loera, Sturmfels, and Vinzant [14] applied algebraic geometry and matroid theory to describe the central path equations for LO and thus refine bounds on the total curvature and the degree of the central path. Very recently, using an algebraic geometry approach, Hoşten, Shankar, and Torres [24] provided a polynomial upper bound on the degree of the Zariski closure of the central path for generic SDO problems.

Organization of the paper. The rest of this paper is organized as follows. In section 2, we briefly review the required concepts in real algebraic geometry and some known results for the convergence of the central path. In section 3, we present our main results: We show that under a nonsingularity assumption, the polynomial system (1.2) is zero-dimensional at a given $\mu > 0$. In section 3.1, using the parametrized bounded algebraic sampling algorithm [6, Algorithm 12.18], we describe central solutions and provide an upper bound on the degree of the Zariski closure of the central path in the case when μ is sufficiently small. In section 3.2, we describe the limit point of the central path using [7, Algorithm 3] and provide a complexity bound for describing the limit point. In section 3.3, we apply the quantifier elimination algorithm [6, Algorithm 14.5] to real univariate representations of the central path and its limit point to bound the convergence rate of the central path. Finally, concluding remarks and topics for future research are stated in section 4.

Notation 1.2. Throughout this paper, \mathbb{S}_+^n denotes the cone of symmetric positive semidefinite matrices, $\mathbb{B}_n(x, r)$ is the closed ball of radius r centered at x in \mathbb{R}^n , and $\text{ri}(\cdot)$ denotes the relative interior of a convex set. For a symmetric matrix X , $\lambda_{[i]}(X)$ denotes the i th largest eigenvalue of X . Finally, the limit point of the central path is denoted by (X^{**}, y^{**}, S^{**}) or $(x^{**}; y^{**}; s^{**})$, and a central solution is denoted by $(X(\mu), y(\mu), S(\mu))$ or $(x(\mu); y(\mu); s(\mu))$.

2. Background. In this section, we provide a brief review of notions in real algebraic geometry, optimization theory, and the central path of SDO. We borrow our notation from [6, 13]. The reader is referred to [25] for a detailed discussion of algebraic notions in section 2.1.

2.1. Real algebraic geometry. Let C be an algebraically closed field, and let \mathcal{Q} be a finite subset of polynomials in $C[X_1, \dots, X_k]$, where $C[X_1, \dots, X_k]$ denotes the ring of polynomials with coefficients in C .³ Then the zero set of \mathcal{Q} in C^k is defined as

$$\text{Zer}(\mathcal{Q}, C^k) := \left\{ x \in C^k \mid \bigwedge_{P \in \mathcal{Q}} P(x) = 0 \right\},$$

²The Sonnevend total curvature was used in [47] to bound the iteration complexity of IPMs, i.e., the number of Newton steps to arrive at an approximate solution.

³In our notation, an indeterminate of a polynomial is shown by an uppercase letter, which should not be confused with a matrix.

which is called an *algebraic subset* of C^k . Let R be a field. Then R is called a *real field* if for every $x_1, \dots, x_k \in R$ we have

$$\sum_{i=1}^k x_i^2 = 0 \quad \Rightarrow \quad x_i = 0 \quad \forall i = 1, \dots, k.$$

A field R endowed with a total order \leq such that for any $x, y, z \in R$,

$$\begin{aligned} x \leq y &\Rightarrow x + z \leq y + z, \\ 0 \leq x, \quad 0 \leq y &\Rightarrow 0 \leq xy, \end{aligned}$$

is called an *ordered field*. A field R is called *real closed* if the field extension $R[X]/(X^2 + 1)$ is algebraically closed; see also [6, Theorem 2.11]. The set \mathbb{R} of real numbers and the set \mathbb{R}_{alg} of real algebraic numbers are both real closed fields. The field $R\langle\langle\varepsilon\rangle\rangle$ of *Puiseux series* in ε with coefficients in a real closed field R , i.e., a series of the form $\sum_{i \geq k} a_i \varepsilon^{i/q}$ with $a_i \in R$, $i, k \in \mathbb{Z}$, and q being a positive integer, is another example of a real closed field [6, Theorem 2.91]. The field of *algebraic Puiseux series*, denoted by $R\langle\varepsilon\rangle$, is a subfield of elements of $R\langle\langle\varepsilon\rangle\rangle$ which are algebraic over $R(\varepsilon)$. On the real closed field $R\langle\varepsilon\rangle$, the unique order $<_\varepsilon$ extends the order of R such that ε is infinitesimal over R , i.e., ε is positive and smaller than any positive element of R ; see [6, Notation 2.5]. For an element $0 \neq \kappa \in R\langle\varepsilon\rangle$, $o(\kappa)$ denotes the order of κ , and $\varepsilon^{o(\kappa)}$ indicates the leading monomial of κ with respect to the order $<_\varepsilon$. A *valuation ring* of $R\langle\varepsilon\rangle$ is a subring of algebraic Puiseux series with nonnegative order, and it is denoted by $R\langle\varepsilon\rangle_b$; see [6, Proposition 2.99]. On $R\langle\varepsilon\rangle$, \lim_ε is defined as a ring homomorphism from the valuation ring $R\langle\varepsilon\rangle_b$ to R , which maps a bounded algebraic Puiseux series $\sum_{i \in \mathbb{N}} a_i \varepsilon^{i/q}$ to a_0 ; see also [6, Notation 2.100 and Notation 12.23].

A *quantifier free formula* $\Phi(X_1, \dots, X_k)$ with coefficients in a real closed field R is the Boolean combination of *atoms*, where an atom is a polynomial equality or inequality defined by $P = 0$ or $P > 0$, respectively, for some $P \in R[X_1, \dots, X_k]$, and $\{X_1, \dots, X_k\}$ are called free variables. A quantified formula is defined as

$$(Q_1 Z_1) \dots (Q_\ell Z_\ell) \mathcal{B}(Z_1, \dots, Z_\ell, X_1, \dots, X_k),$$

where $Q_i \in \{\forall, \exists\}$ are quantifiers and \mathcal{B} is a quantifier free formula with polynomials in $R[Z_1, \dots, Z_\ell, X_1, \dots, X_k]$. The set of all $(x_1, \dots, x_k) \in R^k$ satisfying the formula Φ is called the *R-realization* of Φ . A *semi-algebraic* subset of R^k is the *R-realization* of a quantifier free formula $\Phi(X_1, \dots, X_k)$. In other words, a *semi-algebraic* subset of R^k is a subset of the form

$$\bigcup_{i=1}^s \bigcap_{j=1}^{r_i} \{x \in R^k \mid P_{ij}(x) \Delta_{ij} 0\},$$

where $P_{ij} \in R[X_1, \dots, X_k]$ and Δ_{ij} is either $<$ or $=$ for $i = 1, \dots, s$ and $j = 1, \dots, r_i$. The family of semi-algebraic subsets of R^k is closed under finite union, finite intersection, and complementation.

Let K be an ordered field contained in a real closed field R , $C = R[X]/(X^2 + 1)$, and let $\mathcal{Q} \subset K[X_1, \dots, X_k]$ be a zero-dimensional polynomial system, i.e., $\text{Zer}(\mathcal{Q}, C^k)$ is a finite set.

Coordinates of every $\bar{x} \in \text{Zer}(\mathcal{Q}, \mathbb{C}^k)$ can be described using a *k-univariate representation*, i.e., a $(k+2)$ -tuple of polynomials $u = (f, g_0, \dots, g_k) \in \mathbb{K}[T]^{k+2}$ such that

$$\bar{x} = \left(\frac{g_1(t)}{g_0(t)}, \dots, \frac{g_k(t)}{g_0(t)} \right) \in \mathbb{C}^k,$$

where $t \in \mathbb{C}$ is a root of $f(T)$, and f and g_0 are coprime; see [6, Proposition 12.16]. A *real k-univariate representation* of a real $\hat{x} \in \text{Zer}(\mathcal{Q}, \mathbb{R}^k)$ is a pair (u, σ) such that

$$\hat{x} = \left(\frac{g_1(t_\sigma)}{g_0(t_\sigma)}, \dots, \frac{g_k(t_\sigma)}{g_0(t_\sigma)} \right) \in \mathbb{R}^k,$$

where u is a *k-univariate representation* and σ is called the *Thom encoding* of a real root $t_\sigma \in \mathbb{R}$ of $f(T)$. Given a polynomial $P \in \mathbb{R}[T]$, a Thom encoding [6, Definition 2.29] of $t \in \mathbb{R}$ is a sign condition σ on the set of the derivatives of P , i.e., a mapping $\sigma : \{P, P^{(1)}, P^{(2)}, \dots, P^{(\deg(P))}\} \rightarrow \{0, 1, -1\}$, such that $\sigma(P) = 0$ and

$$\sigma(P^{(i)}) = \text{sign}(P^{(i)}(t)) := \begin{cases} 0 & P^{(i)}(t) = 0, \\ 1 & P^{(i)}(t) > 0, \\ -1 & P^{(i)}(t) < 0, \end{cases} \quad i = 1, \dots, \deg(P),$$

where $\deg(P)$ denotes the degree of the polynomial P , and $P^{(i)}$ for $i > 0$ denotes the i th-order derivative of P .

2.2. Optimality and complementarity. Assumption 1.1 guarantees that y is uniquely determined for a given dual vector S . Furthermore, Assumption 1.1 ensures that $\text{Sol}(P)$ and $\text{Sol}(D)$ are nonempty and bounded and that $v_p^* = v_d^*$ [44, Corollary 4.2]. As a consequence, $(\bar{X}, \bar{y}, \bar{S})$ is a primal-dual solution if and only if it satisfies

$$\begin{aligned} \langle A^i, X \rangle &= b_i, & i &= 1, \dots, m, \\ \sum_{i=1}^m y_i A^i + S &= C, \\ XS &= 0, & X, S &\succeq 0, \end{aligned}$$

where $XS = 0$ is called the *complementarity condition*. A primal-dual solution (X^*, y^*, S^*) is called *maximally complementary* if $X^* \in \text{ri}(\text{Sol}(P))$ and $(y^*, S^*) \in \text{ri}(\text{Sol}(D))$. Alternatively, (X^*, y^*, S^*) is called *maximally complementary* if $\text{rank}(X^*) + \text{rank}(S^*)$ is maximal on the solution set [13, Lemma 2.3]. A maximally complementary solution (X^*, y^*, S^*) is called *strictly complementary* if $X^* + S^* \succ 0$.

Remark 2.1. Throughout this paper, the strict complementarity condition is said to hold if there exists a strictly complementary solution. Equivalently, the strict complementarity condition holds if every maximally complementary solution is strictly complementary.

Under Assumption 1.1, both $\text{Sol}(P)$ and $\text{Sol}(D)$ are nonempty, and thus there always exists a maximally complementary solution. However, the following example shows that in contrast to an LO problem, an SDO problem may have no strictly complementary solution.

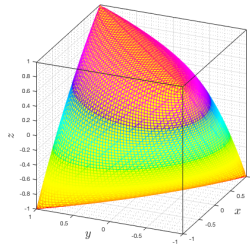


Figure 1. A 3-elliptope. A 3-elliptope is a convex semi-algebraic subset of \mathbb{R}^3 .

Example 2.1. Consider the minimization of a linear objective over a 3-elliptope (see Figure 1),

$$(2.1) \quad \min \left\{ 4x - 4y - 2z \mid \begin{pmatrix} 1 & x & y \\ x & 1 & z \\ y & z & 1 \end{pmatrix} \succeq 0 \right\},$$

which can be cast into the standard form (P) by introducing $b = (1, 1, 1)^T$, and

$$A^1 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad A^2 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad A^3 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad C = \begin{pmatrix} 0 & 2 & -2 \\ 2 & 0 & -1 \\ -2 & -1 & 0 \end{pmatrix}.$$

The unique solution of (2.1) is given by

$$(2.2) \quad X^* = \begin{pmatrix} 1 & -1 & 1 \\ -1 & 1 & -1 \\ 1 & -1 & 1 \end{pmatrix}, \quad y^* = (-4, -1, -1)^T, \quad S^* = \begin{pmatrix} 4 & 2 & -2 \\ 2 & 1 & -1 \\ -2 & -1 & 1 \end{pmatrix},$$

which fails $X^* + S^* \succ 0$.

2.3. Convergence of the central path. Recall from section 1 that the limit point of the central path is a maximally complementary solution, i.e., a point in the relative interior of the primal-dual solution set. The characterization of the limit point is well studied under the strict complementarity condition, where the limit point is the so-called *analytic center* of the solution set [13, Definition 3.1]. If we further assume the uniqueness of the solution, the limiting behavior can be described using the implicit function theorem. Let us assume, without loss of generality,⁴ that both primal and dual solutions are block diagonal,

$$X^* = \begin{pmatrix} U_{X^*} & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad S^* = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & U_{S^*} \end{pmatrix}, \quad U_{X^*}, U_{S^*} \succ 0,$$

where $\text{rank}(U_{X^*})$ and $\text{rank}(U_{S^*})$ are maximal on $\text{Sol}(P) \times \text{Sol}(D)$.⁵ The analytic center of the primal solution set is defined as the unique solution $X^a := \text{Diag}(U_{X^a}, 0, 0)$ such that

$$U_{X^a} := \arg \max_{U_X \succeq 0} \left\{ \det(U_X) \mid \text{Diag}(U_X, 0, 0) \in \text{Sol}(P) \right\}.$$

⁴This can be done using an orthogonal transformation from the optimal partition of the problem; see, e.g., [31].

⁵By definition, X^* and S^* have a common zero eigenvalue, only under the failure of the strict complementarity condition.

Analogously, the analytic center of the dual solution set is the unique solution (y^a, S^a) such that $S^a := \text{Diag}(0, 0, U_{S^a})$, and

$$U_{S^a} := \arg \max_{U_S \succeq 0} \left\{ \det(U_S) \mid (y, \text{Diag}(0, 0, U_S)) \in \text{Sol}(\text{D}) \right\}.$$

As a result of the strict complementarity condition, there exists a Lipschitzian bound on the distance of a central solution from the solution set.

Proposition 2.2 (Theorem 3.5 in [29]). *Suppose that the strictly complementarity condition holds. Then for $\mu \in (0, 1)$ we have*

$$\|X(\mu) - X^a\| = O(\mu) \quad \text{and} \quad \|S(\mu) - S^a\| = O(\mu),$$

where

$$X^a = \begin{pmatrix} U_{X^a} & 0 \\ 0 & 0 \end{pmatrix}, \quad S^a = \begin{pmatrix} 0 & 0 \\ 0 & U_{S^a} \end{pmatrix}, \quad U_{X^a}, U_{S^a} \succ 0.$$

As shown by [Example 1.1](#), the central path appears to have a complicated limiting behavior in the absence of the strict complementarity condition. If the strict complementarity condition fails, the central path does not necessarily converge to (X^a, y^a, S^a) [13, Example 3.1], and the Lipschitzian bounds in [Proposition 2.2](#) may fail to exist. For instance, this can be observed in [Table 1](#), where the central path of [Example 2.1](#) converges to the unique nonstrictly complementary solution at a rate of $\frac{1}{2}$.

Table 1

Convergence to the unique nonstrictly complementary solution.

μ	$\lambda_{[3]}(X(\mu))$	$\lambda_{[2]}(X(\mu))$	$\lambda_{[3]}(S(\mu))$	$\lambda_{[2]}(S(\mu))$	$\ X(\mu) - X^{**}\ $	$\ S(\mu) - S^{**}\ $
1.00E-09	1.66E-10	4.48E-05	3.33E-10	2.24E-05	6.72E-05	5.49E-05
1.00E-10	1.66E-11	1.42E-05	3.33E-11	7.08E-06	2.13E-05	1.74E-05
1.00E-11	1.66E-12	4.48E-06	3.33E-12	2.24E-06	6.72E-06	5.49E-06
1.00E-12	1.66E-13	1.42E-06	3.33E-13	7.08E-07	2.13E-06	1.74E-06
1.00E-13	1.70E-14	4.48E-07	3.30E-14	2.24E-07	6.72E-07	5.49E-07
1.00E-14	2.00E-15	1.42E-07	4.00E-15	7.12E-08	2.14E-07	1.74E-07

In general, a Hölderian (rather than Lipschitzian) bound exists on the distance of a central solution to the solution set.

Proposition 2.3 (Lemma 3.5 in [31]). *Let $(X(\mu), y(\mu), S(\mu))$ be a central solution. Then for sufficiently small μ we have*

$$\text{dist}(X(\mu), \text{Sol}(\text{P})) = O(\mu^{2^{1-n}}) \quad \text{and} \quad \text{dist}((y(\mu), S(\mu)), \text{Sol}(\text{D})) = O(\mu^{2^{1-n}}).$$

The magnitude of the positive and vanishing eigenvalues of $X(\mu)$ and $S(\mu)$ can be quantified using the bounds in [Proposition 2.3](#) and a condition number of (P) and (D); see [31, section 3.1]. By the analyticity of the central path, as $\mu \downarrow 0$, the eigenvalues of $X(\mu)$ and $S(\mu)$ naturally separate into the following three subsets:

1. $\lambda_{[i]}(X(\mu))$ converges to a positive value and $\lambda_{[n-i+1]}(S(\mu))$ converges to 0;

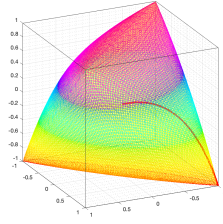


Figure 2. The central path converges tangentially to the unique nonstrictly complementary solution (2.2).

2. both $\lambda_{[i]}(X(\mu))$ and $\lambda_{[n-i+1]}(S(\mu))$ converge to 0;
 3. $\lambda_{[i]}(S(\mu))$ converges to a positive value and $\lambda_{[n-i+1]}(X(\mu))$ converges to 0.
- The bounds on the positive and vanishing eigenvalues are summarized as follows.

Proposition 2.4 (Theorem 3.8 in [31]). *Let $n_B := \text{rank}(X^{**})$, $n_N := \text{rank}(S^{**})$, and $n_T := n - n_B - n_N$. Then for all sufficiently small μ we have*

$$\begin{aligned} \lambda_{[n-i+1]}(S(\mu)) &= O(\mu), & \lambda_{[i]}(X(\mu)) &= \Theta(1), & i &= 1, \dots, n_B, \\ \lambda_{[n-i+1]}(X(\mu)) &= O(\mu), & \lambda_{[i]}(S(\mu)) &= \Theta(1), & i &= 1, \dots, n_N, \\ \lambda_{[i]}(X(\mu)) &= O(\mu^{2^{1-n}}), & \lambda_{[n-i+1]}(S(\mu)) &= O(\mu^{2^{1-n}}), & i &= n_B + 1, \dots, n_B + n_T, \\ \lambda_{[i]}(X(\mu)) &= \Omega(\mu^{1-2^{1-n}}), & \lambda_{[n-i+1]}(S(\mu)) &= \Omega(\mu^{1-2^{1-n}}), & i &= n_B + 1, \dots, n_B + n_T. \end{aligned}$$

Besides converging to a “nonanalytic center” of the solution set, the first-order derivatives of the central path fail to converge in the absence of the strict complementarity condition [17].

Example 2.2. The central path of Example 2.1 can be described by

$$\begin{aligned} X(\mu) &= \begin{pmatrix} 1 & X_{12}(\mu) & -X_{12}(\mu) \\ X_{12}(\mu) & 1 & -2X_{12}^2(\mu) + \mu X_{12}(\mu)/2 + 1 \\ -X_{12}(\mu) & -2X_{12}^2(\mu) + \mu X_{12}(\mu)/2 + 1 & 1 \end{pmatrix}, \\ y(\mu) &= (4X_{12}(\mu) - \mu, 2/X_{12}(\mu) + 1, 2/X_{12}(\mu) + 1)^T, \\ S(\mu) &= \begin{pmatrix} \mu - 4X_{12}(\mu) & 2 & -2 \\ 2 & -2/X_{12}(\mu) - 1 & -1 \\ -2 & -1 & -2/X_{12}(\mu) - 1 \end{pmatrix}, \end{aligned}$$

where $X_{12}(\mu)$ is the real root of

$$(2.3) \quad 2T^3 + (2 - \mu/2)T^2 - (\mu + 2)T - 2 = 0,$$

which makes $X(\mu)$ and $S(\mu)$ positive definite; see Figure 2. Since the limit point is the unique solution given in (2.2), we must have $X_{12}(\mu) \rightarrow -1$ as $\mu \downarrow 0$. Furthermore, it is easy to see from (2.3) that as $\mu \downarrow 0$,

$$\frac{dX_{12}(\mu)}{d\mu} = \frac{X_{12}^2(\mu)/2 + X_{12}(\mu)}{6X_{12}^2(\mu) + (4 - \mu)X_{12}(\mu) - (\mu + 2)} \rightarrow \infty,$$

i.e., the central path cannot be analytically extended to $\mu = 0$.

Example 2.3. The cubic polynomial in (2.3) has a discriminant equal to $\mu^4/4 + 6\mu^3 + 21\mu^2 + 128\mu$, implying that (2.3) has only three real isolated solutions for every $\mu > 0$ [6, Propositions 4.5 and 4.27]. However, the equation (2.3) at $\mu = 0$ has two distinct real roots $t = 1$ and $t = -1$ with multiplicity 2. All this means that as $\mu \downarrow 0$, two out of the three solutions of (2.3) converge to $t = -1$, yielding the singular unique solution (2.2), and the other solution converges to $t = 1$, resulting in an infeasible vector

$$\underline{X} = \begin{pmatrix} 1 & 1 & -1 \\ 1 & 1 & -1 \\ -1 & -1 & 1 \end{pmatrix}, \quad \underline{y} = (4, 3, 3)^T, \quad \underline{S} = \begin{pmatrix} -4 & 2 & -2 \\ 2 & -3 & -1 \\ -2 & -1 & -3 \end{pmatrix}.$$

3. On a semi-algebraic characterization of the central path. While Proposition 2.4 bounds the convergence rate of vanishing eigenvalues, it neither involves the limit point nor provides any quantitative bound on the distance to the limit point. To tackle this problem, we adopt a semi-algebraic approach to describe the limit point of the central path, which in turn allows us to bound the degree and convergence rate of the central path. Without loss of generality, we consider the restriction of the central path to the interval $(0, 1]$. Our derivation of bounds on the degree and convergence rate is based on parametrized univariate representations [6, p. 481], whose sets of associated points, for every $\mu \in (0, 1]$, contain a central solution.

Remark 3.1. By [3, p. 749], the centrality condition $XS = \mu I_n$ in (1.1) can be alternatively written as $XS + SX = 2\mu I_n$ in the presence of $X \succ 0$ and $S \succ 0$. From this point on, we consider this alternative system, which has a symmetric structure and still has a nonsingular Jacobian at a central solution [33, Theorem 2.1].

Recall from (1.2) the subset \mathcal{P} of polynomials in $\mathbb{Z}[\mu, V_1, \dots, V_{\bar{n}}]$, and let zeros of \mathcal{P} in \mathbb{C} and \mathbb{R} be denoted by

$$\begin{aligned} \mathcal{S}_\mu(\mathbb{C}) &:= \text{Zer}(\mathcal{P}, \mathbb{C}^{\bar{n}}), \\ \mathcal{S}_\mu(\mathbb{R}) &:= \text{Zer}(\mathcal{P}, \mathbb{R}^{\bar{n}}). \end{aligned}$$

Notice that $\mathcal{S}_\mu(\mathbb{R})$ has only a finite number of topological types over all $\mu \in \mathbb{R}$ [6, Theorem 5.47], and for every fixed positive μ , $(x(\mu); y(\mu); s(\mu))$ is an isolated solution of $\mathcal{S}_\mu(\mathbb{R})$ by the implicit function theorem [28, Theorem C.40]; see Remark 3.1. Furthermore, under a nonsingularity condition, both $\mathcal{S}_\mu(\mathbb{C})$ and $\mathcal{S}_\mu(\mathbb{R})$ are finite sets.

Proposition 3.2. Let $J(v, \mu)$ be the Jacobian of \mathcal{P} with respect to $\{V_1, \dots, V_{\bar{n}}\}$, and suppose that $\bar{\mu} \in \mathbb{C}$ such that $J(v, \bar{\mu})$ is nonsingular for every $v \in \mathcal{S}_{\bar{\mu}}(\mathbb{C})$. Then $\mathcal{S}_{\bar{\mu}}(\mathbb{C})$ is finite.

Proof. The result follows from [28, Theorem 5.12] by noting that $\mathcal{S}_{\bar{\mu}}(\mathbb{C})$ is a regular submanifold of $\mathbb{C}^{\bar{n}}$ of codimension \bar{n} , and thus it is a finite set. ■

3.1. Degree of the central path. The idea here is to invoke the parametrized bounded algebraic sampling algorithm [6, Algorithm 12.18] to describe, for all $\mu \in (0, 1]$, sample points in every semi-algebraically connected component of $\mathcal{S}_\mu(\mathbb{R})$. Although a central solution is always a nonsingular isolated solution of $\mathcal{S}_\mu(\mathbb{R})$, $\mathcal{S}_\mu(\mathbb{R})$ might have an unbounded solution in

$\mathbb{R}\langle\mu\rangle^{\bar{n}}$ to which [6, Algorithm 12.18] is not directly applicable because, in that case, the sample points need not meet every semi-algebraically connected component of $\mathcal{S}_\mu(\mathbb{R})$. Nevertheless, as mentioned in section 1, we should note that the central path is locally bounded at every positive μ , i.e., there exists⁶ a rational $\epsilon > 0$ such that

$$(3.1) \quad \|(x(\mu); y(\mu); s(\mu))\| \leq 1/\epsilon \quad \forall \mu \in (0, 1].$$

Therefore, the parametrized bounded algebraic sampling can be utilized to describe central solutions for every $\mu \in (0, 1]$, i.e., the tail end of the central path. To that end, we define polynomials $Q \in \mathbb{Z}[\mu, V_1, \dots, V_{\bar{n}}]$ and $\tilde{Q} \in \mathbb{Z}[\mu, V_1, \dots, V_{\bar{n}+1}]$ as

$$Q = \sum_{i=1}^m \left(\sum_{\substack{j,\ell=1 \\ j < \ell}}^n 2A_{j\ell}^i X_{j\ell} + \sum_{j=1}^n A_{jj}^i X_{jj} - b_i \right)^2 + \sum_{\substack{j,\ell=1 \\ j \leq \ell}}^n \left(\sum_{i=1}^m A^i y_i + S - C \right)_{j\ell}^2 + \left(XS + SX - 2\mu I_n \right)_{j\ell}^2$$

and $\tilde{Q} = Q^2 + (\epsilon^2(V_1^2 + \dots + V_{\bar{n}+1}^2) - 1)^2$, where $\text{Zer}(\tilde{Q}, \mathbb{R}^{\bar{n}+1})$ is the intersection of the cylinder based on $\mathcal{S}_\mu(\mathbb{R})$ and an \bar{n} -sphere centered at 0.

Remark 3.3. If $\mathcal{S}_\mu(\mathbb{R})$ is uniformly bounded over all μ , i.e., if there exists a rational $\hat{\epsilon} \leq 1$ such that

$$\|v\| \leq 1/\hat{\epsilon} \quad \forall v \in \mathcal{S}_\mu(\mathbb{R}), \quad \forall \mu \in \mathbb{R},$$

then the parametrized bounded algebraic sampling algorithm can be directly applied to Q .

Notice that for every $\mu \in (0, 1]$, $\text{Zer}(\tilde{Q}, \mathbb{R}^{\bar{n}+1})$ is nonempty and bounded. Furthermore,

$$\pi(\text{Zer}(\tilde{Q}, \mathbb{R}^{\bar{n}+1})) = \text{Zer}(Q, \mathbb{R}^{\bar{n}}) \cap \bar{\mathbb{B}}_{\bar{n}}(0, 1/\epsilon),$$

where $\pi(\cdot)$ denotes the projection from $\mathbb{R}^{\bar{n}+1}$ to the first \bar{n} coordinates. We now apply the parametrized bounded algebraic sampling with input \tilde{Q} to describe X_1 -pseudocritical points on $\text{Zer}(\tilde{Q}, \mathbb{R}^{\bar{n}+1})$. By the boundedness property (3.1), the set of X_1 -pseudocritical points on $\text{Zer}(\tilde{Q}, \mathbb{R}^{\bar{n}+1})$ meets every semi-algebraically connected component of $\text{Zer}(\tilde{Q}, \mathbb{R}^{\bar{n}+1})$ [6, Proposition 12.42]. Further, the set of the projection of the X_1 -pseudocritical points to the first \bar{n} coordinates contains a central solution for every $\mu \in (0, 1]$.

Consider a sequence of integers $(d_1, d_2, \dots, d_{\bar{n}+1})$ such that $d_1 \geq \dots \geq d_{\bar{n}+1}$, $\deg(\tilde{Q}) \leq d_1$, and $\text{tdeg}_{V_i}(\tilde{Q}) \leq d_i$ for $i = 2, \dots, \bar{n} + 1$, where $\text{tdeg}_{V_i}(\tilde{Q})$ denotes the maximal total degree of the monomials of \tilde{Q} which contain V_i . Furthermore, let \bar{d}_i be the smallest even integer greater than d_i for $i = 1, \dots, \bar{n} + 1$, and let π_1 denote the projection of $\mathbb{R}^{\bar{n}+1}$ to the first coordinate. Then X_1 -pseudocritical points on $\text{Zer}(\tilde{Q}, \mathbb{R}^{\bar{n}+1})$ —see also [6, section 12.6]—are defined as the limits of critical points of π_1 on a smooth submanifold of $\mathbb{R}^{\bar{n}+1}$ defined by a ξ -deformation

$$(3.2) \quad \text{Def}(\tilde{Q}, \xi) := \xi G_k(\bar{d}, \bar{\epsilon}) + (1 - \xi)\tilde{Q},$$

where

$$G_k(\bar{d}, \bar{\epsilon}) := \bar{\epsilon}^{\bar{d}_1} (V_1^{\bar{d}_1} + \dots + V_{\bar{n}+1}^{\bar{d}_{\bar{n}+1}} + V_2^2 + \dots + V_{\bar{n}+1}^2) - (2\bar{n} + 1),$$

⁶Note that $\|X(\mu)\|$ and $\|S(\mu)\|$ for $\mu \in (0, 1]$ are bounded by $\lceil \frac{2n}{\lambda_{\min}(S(1))} \rceil$ and $\lceil \frac{2n}{\lambda_{\min}(X(1))} \rceil$, respectively.

and $\bar{\epsilon} = \epsilon/2$, in which ϵ is defined as in (3.1). Deformation (3.2) and the bound (3.1) induce a nonsingular algebraic hypersurface $\text{Zer}(\text{Def}(\tilde{Q}, \xi), \mathbb{R}\langle \xi \rangle^{\bar{n}+1})$, whose coordinates are bounded over \mathbb{R} [6, Proposition 12.38]. Hence, the limits of points in $\text{Zer}(\text{Def}(\tilde{Q}, \xi), \mathbb{R}\langle \xi \rangle^{\bar{n}+1})$ are well defined.

Proposition 3.4 (Propositions 12.37 and 12.42 in [6]). *For every fixed $\mu \in (0, 1]$ we have*

$$\lim_{\xi} (\text{Zer}(\text{Def}(\tilde{Q}, \xi), \mathbb{R}\langle \xi \rangle^{\bar{n}+1})) = \text{Zer}(\tilde{Q}, \mathbb{R}^{\bar{n}+1}).$$

Furthermore, a central solution $(X(\mu), y(\mu), S(\mu))$ is obtained by forgetting the last coordinate of an X_1 -pseudocritical point on $\text{Zer}(\tilde{Q}, \mathbb{R}^{\bar{n}+1})$.

In view of Proposition 3.4 and its preceding discussion, the application of the parametrized bounded algebraic sampling to \tilde{Q} yields parametrized univariate representations

$$(3.3) \quad u := (f, g) := (f, (g_0, g_1, \dots, g_{\bar{n}+1})) \in \mathbb{Z}[\mu, T]^{\bar{n}+3}.$$

Then, for all sufficiently small positive μ , there exists a real root t_σ of $f(\mu, T)$ with Thom encoding σ such that

$$\begin{aligned} x_i(\mu) &= \frac{g_i(\mu, t_\sigma)}{g_0(\mu, t_\sigma)} \in \mathbb{R}, & i &= 1, \dots, t(n), \\ y_i(\mu) &= \frac{g_{t(n)+i}(\mu, t_\sigma)}{g_0(\mu, t_\sigma)} \in \mathbb{R}, & i &= 1, \dots, m, \\ s_i(\mu) &= \frac{g_{t(n)+m+i}(\mu, t_\sigma)}{g_0(\mu, t_\sigma)} \in \mathbb{R}, & i &= 1, \dots, t(n), \end{aligned}$$

where $g_0(\mu, t_\sigma) \neq 0$. Algorithm 3.1 presents the outline of our procedure for a real univariate representation of the central path.

Algorithm 3.1. Description of the central path.

Input: $\tilde{Q} \in \mathbb{Z}[\mu, V_1, \dots, V_{\bar{n}+1}]$; by (3.1), $\text{Zer}(\tilde{Q}, \mathbb{R}^{\bar{n}+1})$ is bounded over all μ .

return a set \mathcal{U} of parametrized univariate representations $(f, g) \in \mathbb{Z}[\mu, T]^{\bar{n}+3}$ and Thom encodings of real roots of $f \in \mathbb{Z}[\mu][T]$; for all sufficiently small μ , there exists $((f, g), \sigma)$ such that the central path is described by the projection of points associated to $((f, g), \sigma)$.

Procedure:

- Apply [6, Algorithm 12.18] (parametrized bounded algebraic sampling) with input \tilde{Q} and parameter μ , and output the set \mathcal{U} of parametrized univariate representations.
 - Apply [6, Algorithm 10.14] (Thom encoding) with input $f \in \mathbb{Z}[\mu][T]$ from the set \mathcal{U} : compute the ordered list of Thom encodings of the roots of f in $\mathbb{R}\langle \mu \rangle$; see [6, Remark 10.76].
-

Since a central solution is an isolated solution of $\text{Zer}(Q, \mathbb{R}^{\bar{n}})$, the set of all μ such that a given $((f, g), \sigma)$ with $(f, g) \in \mathcal{U}$ describes a central solution is a semi-algebraic subset of \mathbb{R} . In other words, Algorithm 3.1 partitions the parameter space $(0, 1]$ into subintervals \mathcal{I}_i such that

the central path restricted to each \mathcal{I}_i is represented by both a $u_i \in \mathcal{U}$ and a Thom encoding σ_i . Therefore, there must exist $((f, g), \sigma)$ which represents the central path when μ is sufficiently small.

From the complexity of the parametrized bounded algebraic sampling and Thom encoding in [Algorithm 3.1](#), the following results are immediate.

Lemma 3.5. *The polynomials (f, g) in (3.3) have degree $O(1)^{\bar{n}+1}$ in T and degree $2^{O(m+n^2)}$ in μ . Further, the complexity of describing the central path, when μ is sufficiently small, is $2^{O(m+n^2)}$.*

Proof. [Algorithm 3.1](#) outputs a set \mathcal{U} of $O(d_1) \dots O(d_{\bar{n}+1})$ polynomials (f, g) which are of degree $O(d_1) \dots O(d_{\bar{n}+1})$ in T and $\text{tdeg}_\mu(\tilde{Q})(d_1 \dots d_{\bar{n}+1})^{O(1)}$ in μ . Then the first part follows by noting that $(d_1, \dots, d_{\bar{n}+1}) = (8, \dots, 8)$ and $\text{tdeg}_\mu(\tilde{Q}) = 6$. The complexity of describing the central path is determined by the number of parametrized univariate representations (f, g) in \mathcal{U} and the complexity of [\[6, Algorithm 12.18\]](#) and [\[6, Algorithm 10.14\]](#) applied to every $(f, g) \in \mathcal{U}$. ■

Remark 3.6. Assume, without loss of generality, that $C_x, C_s \in \mathbb{Z}[\mu][T, \Lambda]$ are the characteristic polynomials of $\check{X}(\mu)$ and $\check{S}(\mu)$ associated to $((f, g), \sigma)$ from [Algorithm 3.1](#), which by [Lemma 3.5](#) are of maximum degree $2^{O(m+n^2)}$. Then $((f, g), \sigma)$ represents the central solution at a sufficiently small μ if the roots of $C_x(t_\sigma, \Lambda)$ and $C_s(t_\sigma, \Lambda)$ are all positive, i.e., if the following sentences are both true:

$$(3.4) \quad \begin{aligned} &(\exists T) (\forall \Lambda) \quad (\text{sign}(f^{(k)}(T)) = \sigma(f^{(k)}), \quad k = 0, 1, \dots) \wedge (\neg(C_x(T, \Lambda) = 0) \vee (\Lambda > 0)), \\ &(\exists T) (\forall \Lambda) \quad (\text{sign}(f^{(k)}(T)) = \sigma(f^{(k)}), \quad k = 0, 1, \dots) \wedge (\neg(C_s(T, \Lambda) = 0) \vee (\Lambda > 0)). \end{aligned}$$

There exists an algorithm with complexity $2^{O(m+n^2)}$ for deciding whether or not $((f, g), \sigma)$ represents the central solution at μ [\[6, Theorem 14.14\]](#).

Finally, using the degree of the polynomials in (3.3), we can derive a bound on the degree of the tail segment of the central path.

Theorem 3.7. *The degree of the Zariski closure of the central path restricted to $(0, \mu]$, when μ is sufficiently small, is bounded above by $2^{O(m+n^2)}$.*

Proof. The degree of the Zariski closure of $\phi((0, \mu])$ is bounded above by the number of points at which a generic hyperplane $a_0 + a_1 V_1 + \dots + a_{\bar{n}} V_{\bar{n}} = 0$ intersects the image $\phi((0, \mu])$, which is the zero set of $\{a_0 g_0(\mu, T) + a_1 g_1(\mu, T) + \dots + a_{\bar{n}} g_{\bar{n}}(\mu, T), f(\mu, T)\}$. By [\[6, Algorithm 11.1\]](#) (elimination), [\[6, Proposition 8.45\]](#), and also [Lemma 3.5](#), the elimination of T yields a polynomial of degree

$$\max\{\deg_\mu(f), \deg_\mu(g)\} \times (\deg_T(f) + \deg_T(g)) = 2^{O(m+n^2)}$$

in μ , which is also a bound on the degree of the Zariski closure of $\phi((0, \mu])$. ■

Remark 3.8. Notice that the degree bound in [Theorem 3.7](#) is valid in a general nongeneric situation. However, under genericity assumptions, a polynomial upper bound was given in [\[24\]](#) for the degree of the Zariski closure of the central path.

3.2. Limit point of the central path. The central path system (1.2) can be alternatively viewed as a μ -infinitesimal deformation of the polynomial system

$$(3.5) \quad \left\{ \sum_{\substack{j,\ell=1 \\ j < \ell}}^n 2A_{j\ell}^i X_{j\ell} + \sum_{j=1}^n A_{jj}^i X_{jj} - b_i, \left(\sum_{i=1}^m A^i y_i + S - C \right)_{j\ell}, (XS + SX)_{j\ell}, j \leq \ell \right\},$$

whose zeros, rather than $\mathbb{C}^{\bar{n}}$, belong to the extension $\mathbb{C}\langle\mu\rangle^{\bar{n}}$; see section 2.1. Let us denote the μ -infinitesimally deformed polynomial system by $\mathcal{G} \subset \mathbb{R}[\mu][V_1, \dots, V_{\bar{n}}]$. Bounded zeros of \mathcal{G} in $\mathbb{R}\langle\mu\rangle^{\bar{n}}$ and $\mathbb{C}\langle\mu\rangle^{\bar{n}}$ are defined as

$$\begin{aligned} \text{Zer}_b(\mathcal{G}, \mathbb{R}\langle\mu\rangle^{\bar{n}}) &:= \text{Zer}(\mathcal{G}, \mathbb{R}\langle\mu\rangle^{\bar{n}}) \cap \mathbb{R}\langle\mu\rangle_b^{\bar{n}}, \\ \text{Zer}_b(\mathcal{G}, \mathbb{C}\langle\mu\rangle^{\bar{n}}) &:= \text{Zer}(\mathcal{G}, \mathbb{C}\langle\mu\rangle^{\bar{n}}) \cap \mathbb{C}\langle\mu\rangle_b^{\bar{n}}, \end{aligned}$$

where $\mathbb{R}\langle\mu\rangle_b^{\bar{n}}$ and $\mathbb{C}\langle\mu\rangle_b^{\bar{n}}$ denote the subrings of bounded elements over \mathbb{R} and \mathbb{C} , respectively. As the infinitesimal μ goes to zero, the limits of points in $\text{Zer}_b(\mathcal{G}, \mathbb{R}\langle\mu\rangle^{\bar{n}})$ are well defined (see section 2.1) and they form a closed semi-algebraic set [6, Proposition 12.43]. Since \lim_{μ} is a ring homomorphism from $\mathbb{R}\langle\mu\rangle$ to \mathbb{R} , the limit of a bounded point of $\text{Zer}(\mathcal{G}, \mathbb{R}\langle\mu\rangle^{\bar{n}})$ is necessarily a zero of (3.5).

Our goal is to describe the limit point of the central path by taking the limits of bounded points described by Algorithm 3.1. More concretely, we describe $(x^{**}; y^{**}; s^{**})$ by applying [7, Algorithm 3] (limit of a bounded point)⁷ to the real univariate representations $((f, (g_0, g_1, \dots, g_{\bar{n}})), \sigma)$ from Algorithm 3.1,⁸ which all represent bounded points over \mathbb{R} . Given the input $((f, g), \sigma)$ with coefficients in $\mathbb{Z}[\mu]$, [7, Algorithm 3] computes an $(\bar{n} + 2)$ -tuple \bar{u} of polynomials and Thom encoding $\bar{\sigma}$, where

$$(3.6) \quad \begin{aligned} \bar{u} &= (\bar{f}, (\bar{g}_0, \bar{g}_1, \dots, \bar{g}_{\bar{n}})) \in \mathbb{Z}[T]^{\bar{n}+2}, \\ \bar{f}(T) &= \lim_{\mu} \mu^{-o(f(T))} f(T), \\ \bar{g}_i(T) &= \left(\lim_{\mu} \mu^{-o(g_i(T))} g_i(T) \right)^{(k_{\bar{\sigma}}-1)}, \quad i = 0, \dots, \bar{n}, \end{aligned}$$

in which $\lim_{\mu} \mu^{-o(f(T))} f(T)$ denotes the univariate polynomial whose coefficients are the limits of the coefficients in $\mu^{-o(f(T))} f(T)$, $t_{\bar{\sigma}}$ is the real root of $\bar{f}(T)$ with Thom encoding $\bar{\sigma}$, $k_{\bar{\sigma}}$ denotes the multiplicity of the real root $t_{\bar{\sigma}}$, and $\bar{g}_i(T)$ is the $(k_{\bar{\sigma}} - 1)$ th derivative of $\lim_{\mu} \mu^{-o(g_i)} g_i$ with respect to T ; see also [6, Notation 12.25]. Then $(\bar{u}, \bar{\sigma})$ represents the limit point of the central path if the ball of infinitesimal radius δ ($0 < \mu \ll \delta \ll 1$) centered at

$$\bar{x}_i = \frac{\bar{g}_i(t_{\bar{\sigma}})}{\bar{g}_0(t_{\bar{\sigma}})}, \quad \bar{y}_i = \frac{\bar{g}_{t(n)+i}(t_{\bar{\sigma}})}{\bar{g}_0(t_{\bar{\sigma}})}, \quad \bar{s}_i = \frac{\bar{g}_{t(n)+m+i}(t_{\bar{\sigma}})}{\bar{g}_0(t_{\bar{\sigma}})}$$

contains $(x(\mu); y(\mu); s(\mu))$. The outline of our procedure is summarized in Algorithm 3.2.

⁷We only invoke a simplified form of [7, Algorithm 3], which originally described the limit of a bounded point over a triangular Thom encoding. See [7, Definition 4.2] for details.

⁸From this point on, $(f, (g_0, g_1, \dots, g_{\bar{n}}))$ is simply denoted by (f, g) .

Algorithm 3.2. Description of the limit of the central path.

Input: Real univariate representations $((f, g), \sigma)$ from Algorithm 3.1.

return Real univariate representations $(\bar{u}, \bar{\sigma})$ given in (3.6) which represent the limits of bounded points associated to $((f, g), \sigma)$.

Procedure:

- Apply [7, Algorithm 3] to each $((f, g), \sigma)$ and output the real univariate representation $((\bar{f}, \bar{g}), \bar{\sigma})$.
-

As a consequence of Algorithm 3.2, the limits of the bounded points associated to \mathcal{U} , and the limit point of the central path in particular, can be described as a rational function of the real roots of $\bar{f}(T)$. The following theorem summarizes one of the main results of this paper.

Theorem 3.9. *Given the polynomial system (1.2), there exists an algorithm with complexity $2^{O(m+n^2)}$ for describing the limit point of the central path.*

Proof. The application of [7, Algorithm 3] to the real univariate representations $((f, g), \sigma)$ yields the bound $\max\{\deg_\mu(f), \deg_T(f), \deg_\mu(g), \deg_T(g)\}$ on the degrees of (\bar{f}, \bar{g}) and a complexity bound $(\max\{\deg_\mu(f), \deg_T(f), \deg_\mu(g), \deg_T(g)\})^{O(1)} = 2^{O(m+n^2)}$, which follows from [6, Algorithm 10.14] (Thom encoding) and Lemma 3.5. This also yields the overall complexity bound $2^{O(m+n^2)}$ for describing the limits of bounded points from the polynomial system (1.2). ■

Remark 3.10. We observe from (3.6) that if $((\bar{f}, \bar{g}), \bar{\sigma})$ represents a solution of (P) and (D), then both of the following analogues of (3.4) must be true:

$$\begin{aligned} (\exists T) (\forall \Lambda) \quad & (\text{sign}(\bar{f}^{(k)}(T)) = \bar{\sigma}(\bar{f}^{(k)}), k = 0, 1, \dots) \wedge (\neg(\bar{C}_x(T, \Lambda) = 0) \vee (\Lambda \geq 0)), \\ (\exists T) (\forall \Lambda) \quad & (\text{sign}(\bar{f}^{(k)}(T)) = \bar{\sigma}(\bar{f}^{(k)}), k = 0, 1, \dots) \wedge (\neg(\bar{C}_s(T, \Lambda) = 0) \vee (\Lambda \geq 0)), \end{aligned}$$

where $\bar{C}_x, \bar{C}_s \in \mathbb{Z}[T, \Lambda]$ are the characteristic polynomials of \bar{X} and \bar{S} consisting of the entries \bar{x}_i and \bar{s}_i associated to $((\bar{f}, \bar{g}), \bar{\sigma})$, and $\deg(\bar{C}_x), \deg(\bar{C}_s) = 2^{O(m+n^2)}$. Analogous to Remark 3.6, there exists an algorithm [6, Theorem 14.14] with complexity $2^{O(m+n^2)}$ in \mathbb{Z} for deciding whether or not the given $((\bar{f}, \bar{g}), \bar{\sigma})$ describes a solution of (P) and (D).

Remark 3.11. In the presence of the strict complementarity condition, Theorem 3.9 shows an improvement on the complexity of describing a strictly complementary solution when compared to the direct application of [6, Algorithm 13.2] (sampling). More precisely, the set of strictly complementary solutions is a bounded basic semi-algebraic set and can be described as the realization of a sign condition on the following set of $O(m+n^2)$ polynomials of degree n in $\mathbb{Z}[V_1, \dots, V_n]$:

$$\mathcal{R} = \left\{ \sum_{\substack{j, \ell=1 \\ j < \ell}}^n 2A_{j\ell}^i X_{j\ell} + \sum_{j=1}^n A_{jj}^i X_{jj} - b_i, \left(\sum_{i=1}^m A^i y_i + S - C \right)_{j\ell}, j \leq \ell, (XS), \det(X_{[i]} + S_{[i]}) \right\},$$

where (XS) stands for all entries of XS , and $X_{[i]}$ and $S_{[i]}$ denote the i th leading principal

submatrices of X and S for $i = 1, \dots, n^9$. In that case, the sampling algorithm applied to \mathcal{R} has a complexity $(2mn + 2n^3)^{O(m+n^2)}$ for describing a strictly complementary solution.

3.3. Worst-case convergence rate. We adopt the same approach as in Remark 3.6 to bound the convergence rate of the central path, i.e., the rate at which a central solution converges to (X^{**}, y^{**}, S^{**}) . Let $((f, g), \sigma)$ and $((\bar{f}, \bar{g}), \bar{\sigma})$ be the input and output of Algorithm 3.2 describing the central path for sufficiently small μ and its limit point, respectively, and define a polynomial $P_x \in \mathbb{Z}[\mu, D, T_1, T_2]$ as

$$P_x = D(g_0(\mu, T_1)\bar{g}_0(T_2))^2 - \sum_{i=1}^{t(n)} (g_i(\mu, T_1)\bar{g}_0(T_2) - \bar{g}_i(T_2)g_0(\mu, T_1))^2.$$

Given a sufficiently small μ , t_σ , and $t_{\bar{\sigma}}$, the real root of $P_x(\mu, D, t_\sigma, t_{\bar{\sigma}})$ is the distance of $x(\mu)$ from its unique limit point x^{**} , i.e.,

$$d_x = \|x(\mu) - x^{**}\|^2 = \sum_{i=1}^{t(n)} \left(\frac{g_i(\mu, t_\sigma)}{g_0(\mu, t_\sigma)} - \frac{\bar{g}_i(t_{\bar{\sigma}})}{\bar{g}_0(t_{\bar{\sigma}})} \right)^2,$$

where $f(\mu, t_\sigma) = 0$ and $\bar{f}(t_{\bar{\sigma}}) = 0$. Analogously, we can define $P_s \in \mathbb{Z}[\mu, D, T_1, T_2]$ as

$$P_s = D(g_0(\mu, T_1)\bar{g}_0(T_2))^2 - \sum_{i=m+t(n)+1}^{\bar{n}} (g_i(\mu, T_1)\bar{g}_0(T_2) - \bar{g}_i(T_2)g_0(\mu, T_1))^2,$$

which for a sufficiently small μ , t_σ , and $t_{\bar{\sigma}}$ has the real root

$$d_s = \|s(\mu) - s^{**}\|^2 = \sum_{i=m+t(n)+1}^{\bar{n}} \left(\frac{g_i(\mu, t_\sigma)}{g_0(\mu, t_\sigma)} - \frac{\bar{g}_i(t_{\bar{\sigma}})}{\bar{g}_0(t_{\bar{\sigma}})} \right)^2.$$

Notice that $\|X(\mu) - X^{**}\| \leq \sqrt{2}\|x(\mu) - x^{**}\|$ and $\|S(\mu) - S^{**}\| \leq \sqrt{2}\|s(\mu) - s^{**}\|$. In summary, for all sufficiently small μ , (μ, d_x) and (μ, d_s) belong to the \mathbb{R} -realization of the following quantified first-order formulas with integer coefficients:

$$(3.7) \quad \begin{aligned} \Psi_x(\mu, D) : & (\exists T_1) (\exists T_2) (P_x(\mu, D, T_1, T_2) = 0) \wedge (\text{sign}(f^{(k)}(\mu, T_1)) = \sigma(f^{(k)}), k = 0, 1, \dots) \\ & \wedge (\text{sign}(\bar{f}^{(k)}(T_2)) = \bar{\sigma}(\bar{f}^{(k)}), k = 0, 1, \dots), \\ \Psi_s(\mu, D) : & (\exists T_1) (\exists T_2) (P_s(\mu, D, T_1, T_2) = 0) \wedge (\text{sign}(f^{(k)}(\mu, T_1)) = \sigma(f^{(k)}), k = 0, 1, \dots) \\ & \wedge (\text{sign}(\bar{f}^{(k)}(T_2)) = \bar{\sigma}(\bar{f}^{(k)}), k = 0, 1, \dots). \end{aligned}$$

Now, we present the proof of the main result of this paper.

Proof of Theorem 1.1. The quantifier elimination algorithm [6, Algorithm 14.5] applied to the formulas Ψ_x and Ψ_s returns quantifier free formulas with polynomials $R \in \mathbb{R}[\mu, D]$ of degree $(\max\{\deg(P_x), \deg(P_s), \deg(f), \deg(\bar{f})\})^{O(1)}$, where

$$\max\{\deg(P_x), \deg(P_s), \deg(f), \deg(\bar{f})\} = 2^{O(m+n^2)}.$$

⁹Notice that the conditions $XS = 0$, $X + S \succ 0$ also imply $X \succeq 0$ and $S \succeq 0$.

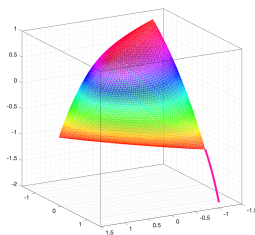


Figure 3. An exterior semi-algebraic path exists and converges to the unique nonstrictly complementary solution (2.2).

By the definitions of P_x and P_s , the \mathbb{R} -realization of (3.7) contains a unique (μ, d) for sufficiently small μ . Therefore, the quantifier free formula obtained from (3.7) must involve an atom $R = 0$ with $\deg(R) = 2^{O(m+n^2)}$. By the Newton–Puiseux theorem [46, Theorem 3.1 of Chapter IV], $R(\mu, D) = 0$ has a root $d \in \mathbb{C}\langle\mu\rangle$ with a positive valuation, since otherwise d would either be unbounded over \mathbb{C} or have a positive limit. Thus, the valuation of d is the negative of the slope of a segment from the Newton polygon of R [46, section 3.2 of Chapter IV], and it is bounded below by $1/\deg_D(R)$. Since $\deg_D(R) = 2^{O(m+n^2)}$, the proof is complete. ■

4. Concluding remarks and future research. In this paper, we investigated the degree and worst-case convergence rate of the central path of SDO problems. We described central solutions and the limit point of the central path as points associated to real univariate representations $((f, g), \sigma)$ and $((\bar{f}, \bar{g}), \bar{\sigma})$ from Algorithm 3.1 and Algorithm 3.2, respectively. As a result, we derived an upper bound $2^{O(m+n^2)}$ on the degree of the Zariski closure of the central path, when μ is sufficiently small, and a complexity bound $2^{O(m+n^2)}$ for describing the limit point of the central path. Additionally, by applying the quantifier elimination algorithm to $((f, g), \sigma)$ and $((\bar{f}, \bar{g}), \bar{\sigma})$, we provided a lower bound $1/\gamma$, with $\gamma = 2^{O(m+n^2)}$, on the convergence rate of the central path. It is worth mentioning that the worst-case convergence rate of the central path could serve as a quantitative measure for the hardness of solving (P) and (D) using primal-dual path-following IPMs; see, e.g., [29].

Exterior semi-algebraic paths. As Remark 3.10 suggests, Algorithm 3.1 and Algorithm 3.2 describe not only the central path and its limit point but also other semi-algebraic paths, arising from the parametrized univariate representations in \mathcal{U} , which may converge to a point in the solution set. By the centrality condition $XS = \mu I_n$ and the positive sign of the eigenvalues of $X(\mu)$ and $S(\mu)$, the central path is the only semi-algebraic path which converges from the interior of \mathbb{S}_+^n . However, it turns out that a solution could be approached from a semi-algebraic path, the so-called *exterior semi-algebraic path*, which converges from the exterior of \mathbb{S}_+^n . This can be observed in Example 2.3, where the solutions of (2.3) are all bounded over \mathbb{R} ; see also Figure 3. The existence of an exterior semi-algebraic path is particularly important when the strict complementarity condition fails. Such a path may exhibit numerical behavior superior to the central path, which suffers from a slower convergence rate than 1 in the absence of the strict complementarity condition. See, e.g., Example 2.1 and Table 1 where the central path converges to the unique solution at a rate of $\frac{1}{2}$.

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