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STRATIFIED SURGERY AND K-THEORY INVARIANTS OF THE SIGNATURE OPERATOR

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ABSTRACT. – In the works of Higson-Roe the fundamental role of the signature as a homotopy and bordism invariant for oriented manifolds is the starting point for an investigation of the relationships between analytic and topological invariants of smooth orientable manifolds. The signature and related K-theory invariants, primary and secondary, are used to define a natural transformation between the (Browder-Novikov-Sullivan-Wall) surgery exact sequence and a long exact sequence of C^*-algebra K-theory groups.

In recent years the primary signature invariants have been extended from closed oriented manifolds to a class of stratified spaces known as L-spaces or Cheeger spaces. In this paper we show that secondary invariants, such as the ρ-class, also extend from closed manifolds to Cheeger spaces. We give a rigorous account of a surgery exact sequence for stratified spaces originally introduced by Browder-Quinn and obtain a natural transformation analogous to that of Higson-Roe. We also discuss geometric applications.

RÉSUMÉ. – Dans les travaux de Higson-Roe le rôle fondamental de la signature comme invariant par homotopie et par bordisme de variétés orientées est le point de départ des recherches sur les liens entre les invariants analytiques et topologiques des variétés régulières orientées. La signature et certains invariants de K-théorie associés, primaires et secondaires, définissent une transformation naturelle entre la suite exacte de chirurgie de Browder-Novikov-Sullivan-Wall et une suite exacte longue des groupes de K-théorie pour des algèbres C^*.

Dans les dernières années l'étude des invariants de signature primaires des variétés orientées a été étendue à une classe d'espaces stratifiés connue sous le nom de L-espaces ou espaces de Cheeger. Dans ce papier, nous démontrons que les invariants secondaires, tels que la classe ρ, peuvent être étendus aux espaces de Cheeger. Nous traitons rigoureusement une suite exacte de chirurgie pour espaces stratifiés introduite originalement par Browder-Quinn et nous obtenons une transformation naturelle analogue à celle de Higson-Roe. Nous discutons aussi des applications géométriques.

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1. Introduction

The discovery by Milnor of smooth manifolds that are homemorphic to $\mathbb{S}^7$ but not diffeomorphic to it, a milestone of modern mathematics, gave rise to the development of methods for classifying smooth manifolds within a given homotopy class. (The undecidability of the word problem makes an unrestricted classification impossible.) The fundamental object to be studied in this context is the *structure set* $\mathrm{S}(X)$ associated to a smooth compact manifold X. The set $\mathrm{S}(X)$ is defined as the quotient of the set of triples $(M \xrightarrow{h} X)$, with M a smooth compact manifold and h an homotopy equivalence, modulo h-cobordism: $(M_0 \xrightarrow{h_0} X)$ is h-cobordant to $(M_1 \xrightarrow{h_1} X)$ if there exists a bordism $F : W \to X\times[0,1]$ with $\partial W = M_0 \sqcup M_1$, F restricting to f_j on M_j and F a homotopy equivalence. Notice that $\mathrm{S}(X)$ is a pointed set, with $[X \xrightarrow{\mathrm{Id}} X]$ as a distinguished point. It is in general very difficult to compute explicitly the structure set associated to X, a notable exception being the structure set of the spheres, $\mathrm{S}(\mathbb{S}^n)$. In this particular case, $\mathrm{S}(\mathbb{S}^n)$ can be identified with Θ_n, the Kervaire-Milnor group of h-cobordism classes of oriented homotopy n-spheres [33, 51]. Θ_n is a finite Abelian group, of cardinality 1 for $n \leq 6$, and, for example, cardinality 28 for $n = 7$. (The structure set of other simple spaces such as complex projective spaces, tori, and lens spaces are also known [57, Part 3].) In general there is no group structure on $\mathrm{S}(X)$ (1).

Even if an explicit computation is often out of reach, one would like to determine, for example, the cardinality of $\mathrm{S}(X)$, in particular whether it is greater than one, finite or infinite. A smooth manifold with $|\,\mathrm{S}(X)| = 1$ is said to be *rigid* and so $\mathrm{S}(X)$ is a measure of the non-rigidity of X.

A powerful method to get information about the structure set is provided by the surgery exact sequence of Browder, Novikov, Sullivan, and Wall which, roughly speaking, relates the structure set $\mathrm{S}(X)$ with the set $\mathrm{N}(X)$ of degree one maps preserving normal bundle information, known as 'normal invariants', (also with a bordism equivalence relation) and an algebraically defined L-group depending only on $\Gamma = \pi_1 X$, the fundamental group of X,

$$\cdots \longrightarrow \mathrm{L}_{m+1}(\mathbb{Z}\Gamma) \dashrightarrow \mathrm{S}(X) \longrightarrow \mathrm{N}(X) \longrightarrow \mathrm{L}_m(\mathbb{Z}\Gamma)\ . \tag{1.1}$$

with $m = \dim M$. (See below and, e.g., [57, 51, 36, 19] for more on the surgery exact sequence.)

In a series of papers Higson and Roe [27, 28, 29] established the remarkable result that there are natural maps out of the surgery sequence (1.1), into a long exact sequence of K-theory groups of certain C^*-algebras and that these maps make the resulting diagram commute. The C^*-algebras in question are $C^*(X_\Gamma)^\Gamma$ and $D^*(X_\Gamma)^\Gamma$, obtained as the closures of the Γ-equivariant operators on the universal cover X_Γ of X that satisfy a finite propagation property and, in addition, are respectively 'locally compact' or 'pseudolocal'. The former C^*-algebra is an ideal in the latter so we have a short exact sequence

$$0 \to C^*(X_\Gamma)^\Gamma \to D^*(X_\Gamma)^\Gamma \to D^*(X_\Gamma)^\Gamma / C^*(X_\Gamma)^\Gamma \to 0,$$

(1) The analogous set in the topological category, $\mathrm{S}^{\mathrm{top}}(X)$, X a topological manifold, can be given a group structure through the Siebenmann periodicity theorem, see for example [15].

which gives rise to a long exact sequence in K-theory known as the *analytic surgery sequence* of Higson and Roe. Making use of the canonical isomorphisms

$$K_{*+1}(D^*(X_\Gamma)^\Gamma/C^*(X_\Gamma)^\Gamma) = K_*(X) \quad \text{and} \quad K_*(C^*(X_\Gamma)^\Gamma) = K_*(C^*_r\Gamma),$$

with the K-homology of X and the K-theory of the reduced C^*-algebra of Γ, the long exact sequence reads

$$(1.2) \qquad \cdots \to K_{m+1}(C^*_r\Gamma) \to K_{m+1}(D^*(X_\Gamma)^\Gamma) \to K_m(X) \to K_m(C^*_r\Gamma) \to \cdots .$$

The result of Higson and Roe is thus a commutative diagram of long exact sequences

$$(1.3) \qquad \begin{array}{ccccccc} \mathrm{L}_{m+1}(\mathbb{Z}\Gamma) & \dashrightarrow & \mathrm{S}(X) & \longrightarrow & \mathrm{N}(X) & \longrightarrow & \mathrm{L}_m(\mathbb{Z}\Gamma) \\ \downarrow \gamma & & \downarrow \rho & & \downarrow \beta & & \downarrow \gamma \\ K_{m+1}(C^*_r\Gamma)[\frac{1}{2}] & \longrightarrow & K_{m+1}(D^*(X_\Gamma)^\Gamma)[\frac{1}{2}] & \longrightarrow & K_m(X)[\frac{1}{2}] & \longrightarrow & K_m(C^*_r\Gamma)[\frac{1}{2}], \end{array}$$

where we use the short-hand $A[\frac{1}{2}]$ to indicate $A \otimes_{\mathbb{Z}} \mathbb{Z}[\frac{1}{2}]$ whenever A is an Abelian group. These maps were recast by the second author and Schick [46] in a more index-theoretic light, using in a crucial way properties of the signature operator on Galois Γ-coverings. In particular, the homomorphism γ is shown to be realized by an Atiyah-Patodi-Singer index map. The approach by Piazza and Schick also allowed for a treatment of the Stolz surgery sequence for positive scalar curvature metrics and its mapping to the Higson-Roe surgery sequence using the spin-Dirac operator:

(1.4)

$$\begin{array}{ccccccc} R^{\mathrm{spin}}_{m+1}(B\Gamma) & \dashrightarrow & \mathrm{Pos}^{\mathrm{spin}}_m(B\Gamma) & \longrightarrow & \Omega^{\mathrm{spin}}_m(B\Gamma) & \longrightarrow & R^{\mathrm{spin}}_{m+1}(B\Gamma) \\ \downarrow g & & \downarrow \rho & & \downarrow b & & \downarrow g \\ K_{m+1}(C^*_r\Gamma)[\frac{1}{2}] & \longrightarrow & K_{m+1}(D^*(E\Gamma)^\Gamma)[\frac{1}{2}] & \longrightarrow & K_m(B\Gamma)[\frac{1}{2}] & \longrightarrow & K_m(C^*_r\Gamma)[\frac{1}{2}], \end{array}$$

see [45]. For alternative treatments see also [63, 66, 67, 68, 65]. While the vertical maps in these diagrams are in general not known to be injective or surjective (though see [64, Corollary 1.3] where this is related to the Baum-Connes conjecture), it is still possible to get interesting geometric applications from the interplay between the geometric sequence upstairs and the analytic sequence downstairs. This is true both for (1.3) and (1.4). See for example, [29, 17, 56, 66, 61, 60, 65, 10].

In this paper we generalize (1.3) to the setting of stratified spaces.

THEOREM 1.1. – *Every m-dimensional, oriented, smoothly stratified Cheeger space, $\widehat{X}$, with fundamental group Γ, gives rise to a commutative diagram*

$$(1.5) \quad \begin{array}{ccccccc} \mathrm{L}_{\mathrm{BQ},d_{\widehat{X}\times I}}(\widehat{X}\times I) & \dashrightarrow & \mathrm{S}_{\mathrm{BQ}}(\widehat{X}) & \longrightarrow & \mathrm{N}_{\mathrm{BQ}}(\widehat{X}) & \longrightarrow & \mathrm{L}_{\mathrm{BQ},d_{\widehat{X}}}(\widehat{X}) \\ \downarrow \mathrm{Ind}_{\mathrm{APS}} & & \downarrow \rho & & \downarrow \beta & & \downarrow \mathrm{Ind}_{\mathrm{APS}} \\ K_{m+1}(C^*_r\Gamma)[\frac{1}{2}] & \longrightarrow & K_{m+1}(D^*(\widehat{X}_\Gamma)^\Gamma)[\frac{1}{2}] & \longrightarrow & K_m(\widehat{X})[\frac{1}{2}] & \longrightarrow & K_m(C^*_r\Gamma)[\frac{1}{2}] \end{array}$$

between the Browder-Quinn surgery exact sequence and the Higson-Roe analytic surgery sequence. Here $I := [0,1]$ *and* $d_{\widehat{X}}$, $d_{\widehat{X}\times I}$ *are dimension functions associated to the stratifications of* $\widehat{X}$ *and* $\widehat{X}\times I$ *respectively. Our result holds, in particular, if* $\widehat{X}$ *is a Witt space.*

The group $\mathrm{L}_{\mathrm{BQ},d_{\widehat{X}}}(\widehat{X})$ is the analogue of $\mathrm{L}_d(\mathbb{Z}\pi_1(\widehat{X}))$ in the Browder-Novikov-Sullivan-Wall surgery sequence (1.1). The subscript $d_{\widehat{X}}$ refers to the dimension function of $\widehat{X}$, the function on the poset of strata that assigns to each stratum its dimension. If $\widehat{X}$ is smooth of dimension d, $\widehat{X}\equiv X$, then $\mathrm{L}_{\mathrm{BQ},d}(X) = \mathrm{L}_d(\mathbb{Z}\pi_1(X))$, where on the left we now have the dimension function that assigns to the unique (smooth) stratum X the value $d = \dim X$. In the general stratified case $\mathrm{L}_{\mathrm{BQ},d_{\widehat{X}}}(\widehat{X})$ depends in principle on all of $\widehat{X}$ and not only on its fundamental group.

We prove a better version of this diagram in §6.6 involving the signature operator on all of the strata of $\widehat{X}$ and their fundamental groups, but refer the reader to the text so as not to introduce more notation.

As a geometric application of our techniques we prove:

COROLLARY 1.2. – *Let* $\widehat{X}$ *be a Cheeger space of dimension* $4\ell-1$, $\ell>1$, *with smooth stratum equal to* X, $X\overset{i}{\hookrightarrow}\widehat{X}$. *Assume that* $\pi_1(X)$ *has an element of finite order and that* $i_*:\pi_1(X)\longrightarrow\pi_1(\widehat{X})$ *is injective. Then*

$$|\,\mathrm{S}_{\mathrm{BQ}}(\widehat{X})| = \infty.$$

The top row of (1.5) is the surgery sequence for stratified spaces of Browder-Quinn [13]. One of our contributions in this paper is a detailed treatment in §2 of the Browder-Quinn surgery exact sequence in the setting of smoothly stratified spaces (a.k.a. Thom-Mather stratified spaces). The original treatment in [13] is quite sparse and its generalization in [59] uses algebraic tools applicable in its setting of homotopically stratified spaces. For our purposes it is necessary to have geometric proofs that stay within the category of smooth stratifications. Our treatment naturally draws heavily from these two sources.

A feature of the Browder-Quinn surgery sequence is that if $\widehat{X}$ is a Witt space [53] or a Cheeger space [9, 2, 5] then all of the spaces that arise in the surgery sequence are also Witt spaces, respectively Cheeger spaces. This allows us to bring to bear the analysis that we have developed in joint work with Eric Leichtnam and Rafe Mazzeo [3, 5, 4] to define the vertical maps in (1.5) and to show that the diagram commutes. Notice that while the vertical maps are defined in analogy with [46], there are substantial technical differences, especially in the Cheeger case, where ideal boundary conditions must be chosen.

In detail, the vertical maps out of the Browder-Quinn L-groups are Atiyah-Patodi-Singer index classes; the map out of the normal invariants $\mathrm{N}_{\mathrm{BQ}}(\widehat{X})$ is given in terms of the fundamental class, in K-homology, associated to the signature operator on a Cheeger space; finally the rho-map is a true secondary invariant associated to a suitable perturbation of the signature operator. In the smooth case this rho-map is directly connected with well-known numeric rho invariants; we comment on the validity of this principle in the singular case at the end of the paper. As already remarked, all of these constructions depend upon the definition of ideal boundary conditions; these depend, in turn, on the choice of a *mezzoperversity* and a major theme in this article is the detailed analysis of the dependence of these classes on the choice of a mezzoperversity and the proof of the remarkable fact that our maps are in fact all independent of the choice of a mezzoperversity.

The paper is organized as follows. In Section 2 we give a rigorous and detailed treatment of the relevant results stated in the paper by Browder and Quinn. In Section 3 we specialize to Cheeger spaces and give a coarse theoretic treatment of some of the results in [3, 5, 4]; in particular we define the fundamental K-homology class of a Cheeger space without boundary and the associated index class. Finally, in the invertible case, we introduce the rho class of an invertible perturbation of the signature operator. In Section 4 we pass to manifolds with boundary, with a particular emphasis on the notion of Cheeger space bordism. It is in this section that we explain the statement of the delocalized Atiyah-Patodi-Singer index theorem on Cheeger spaces, a key tool in our analysis, and we illustrate its proof, building on [45, 46]. In Section 5 we recall and expand results around the Hilsum-Skandalis perturbation for the signature operator on the disjoint union of two Cheeger spaces that are stratified homotopy equivalent. In Section 6 we finally define the vertical maps in the diagram that maps the Browder-Quinn surgery sequence to the Higson-Roe surgery sequence; we prove the well-definedness of these maps and that they are independent of the choice of a mezzoperversity. We then prove the commutativity of the squares of the diagram. We end this section by observing that it is in fact possible to consider different diagrams, each one associated to an individual closed stratum. Section 7, the last section of the paper, presents some geometric applications of our main result, in the spirit of [17].

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Notation

Below we will occasionally use diagrams such as

$$(\widehat{M};\partial_1\widehat{M},\partial_2\widehat{M}) \xrightarrow{\ f\ } (\widehat{Y};\partial_1\widehat{Y},\partial_2\widehat{Y}).$$

This should be understood to imply that $\partial\widehat{M} = \partial_1\widehat{M}\cup\partial_2\widehat{M}$, $\partial\widehat{Y} = \partial_1\widehat{Y}\cup\partial_2\widehat{Y}$, also that $\partial_1\widehat{M}\cap\partial_2\widehat{M} = \partial(\partial_i\widehat{M})$ (and similarly for $\widehat{Y}$), and that f restricts to maps

$$f| : \partial_1\widehat{M} \longrightarrow \partial_1\widehat{Y}, \quad f| : \partial_2\widehat{M} \longrightarrow \partial_2\widehat{Y},$$

which we sometimes denote $\partial_1 f$, $\partial_2 f$, respectively. Occasionally it will be useful to decompose the boundary of a space into more than two pieces, in which case similar conventions are in effect.

Our main object of study will be smoothly stratified spaces. As reviewed below, this will mean Thom-Mather stratified pseudomanifolds. A bordism between two stratified spaces will be a stratified space with boundary and a bordism between two stratified spaces with boundary will be a stratified space with corners (as is well known, e.g., [58, §8.3], this is only useful if restrictions are placed on part of the boundary). For a careful discussion of these concepts we refer the reader to [55] (see also [1, §6]). While we do not use the language of 'n-ads' as in [57], it is clear that the constructions below extend to 'n-ads of stratifed spaces'.

Note that the boundary of a manifold with corners is not itself a manifold with corners, but rather a union of manifolds with corners with various identifications of boundary faces. Following Melrose (see, e.g., [40]), an 'articulated manifold (without boundary)' is a finite union of connected components of the boundary of a compact manifold with corners (thus guaranteeing that the identifications of boundary faces are consistent). More generally, an 'articulated manifold with corners' is a finite union of boundary hypersurfaces of the boundary of a compact manifold with corners (not necessarily making up full connected components of the boundary).

Similarly an 'articulated stratified space (without boundary)' refers to a finite union of connected components of the boundary of a compact stratified space with corners, and an 'articulated stratified space with corners' is a finite union of boundary hypersurfaces of a stratified space with corners.

Working with 'articulated stratified spaces' is analogous to working with n-ads in the category of stratified spaces; for instance, if $(\widehat{M}; \partial_1\widehat{M}, \partial_2\widehat{M})$ is as above and $\widehat{M}$ is a stratified space with corners then $\partial_i\widehat{M}$ are articulated stratified spaces with corners. An alternate approach, see, e.g., [58, §2.6], is to 'round the corners' and work with spaces with boundary.

2. Browder-Quinn Surgery

We will make use of the surgery theory for stratified spaces of Browder and Quinn [13]. Some of the results we need for the purpose of defining maps into K-theory are only implicit in their exposition so we have decided to include a more explicit description of this surgery theory. In carrying this out we have benefitted from Weinberger's exposition [59, §7.1] where some of the proofs below are sketched (e.g., the Π-Π theorem, Theorem 2.1), as well as from [57, 51, 20, 19, 6, 49] and [21] (from which we adapted the proof of the Wall realization theorem, Theorem 2.5). We are also happy to acknowledge useful conversations and email exchanges with Markus Banagl, Jim Davis, Wolfgang Lück, Tibor Macko and Shmuel Weinberger.

2.1. Browder-Quinn stratified spaces and transverse maps

Although there are many notions of stratified spaces, perhaps the most common is that of a Whitney stratified space. If L is a smooth manifold then a Whitney stratification of a subset $\widehat{X} \subseteq L$ is a locally finite collection of pairwise disjoint smooth submanifolds covering $\widehat{X}$, known as strata, satisfying the 'frontier condition'

$$Y \cap \overline{Y'} \neq \emptyset \implies Y \subseteq \overline{Y'}$$

and 'Whitney's condition (B)' concerning the relations of the tangent spaces of the strata. For example, Whitney showed [62] that algebraic varieties admit Whitney stratifications. It was subsequently shown by Thom and Mather [39] that in a Whitney stratified space neighborhoods of the strata have geometric structure and this was abstracted in the notion of Thom-Mather stratified space.

A further abstraction was given by Browder and Quinn [13] (cf. [32, 22, 59]). They fix a category $\mathscr{F}$ of 'manifolds with fibrations' such as smooth manifolds and locally trivial smooth fiber bundles, PL manifolds and block bundles with manifold fibers, topological manifolds and locally trivial topological fiber bundles, or Poincaré spaces and maps whose homotopy fiber satisfies Poincaré duality. Although Browder-Quinn do not specify what properties are necessary in the category $\mathscr{F}$, an important property is that there be a notion of pull-back in the category $\mathscr{F}$.

An $\mathscr{F}$-stratified space is a topological space $\widehat{X}$ filtered by subsets X_a indexed by a partially ordered set A satisfying the following. If for each $a \in A$ we let

$$X_{\partial a} = \bigcup \{X_b : b \in A, b < a\},$$

then each X_a is equipped with a closed neighborhood $N_a = N(X_a)$ of $X_{\partial a}$ in X_a and a projection $\nu_a : \partial N_a \longrightarrow X_{\partial a}$ such that

(i) $X_a \setminus X_{\partial a}$ and ∂N_a are manifolds in $\mathscr{F}$,

(ii) N_a is the mapping cylinder of ν_a (with ∂N_a and $X_{\partial a}$ corresponding to the top and bottom of the cylinder),

(iii) If $a, b \in A$, $b < a$, $W_b = X_b \setminus \mathrm{int}(N_b)$, then

$$\nu_a| : \nu_a^{-1}(W_b) \longrightarrow W_b$$

is a fibration in $\mathscr{F}$.

If $\widehat{X}$ and $\widehat{M}$ are two $\mathscr{F}$-stratified spaces whose filtrations are indexed by the same partially ordered set A, then a filtration-preserving map $f : \widehat{X} \longrightarrow \widehat{M}$ of $\mathscr{F}$-stratified sets is said to be *transverse* if each fibration in $\widehat{X}$ is the pull-back along f of the corresponding fibration in $\widehat{M}$.

REMARK 1. – *When $\mathcal{F}$ is equal to the category of smooth manifolds and locally trivial smooth fiber bundles, Browder-Quinn $\mathcal{F}$-stratified spaces are the same as Thom-Mather stratified spaces. One could show this by, for example, proceeding as in* [3] *and proving that any Browder-Quinn $\mathscr{F}$-stratified space can be 'resolved' to an $\mathscr{F}$-space 'with corners' and iterated fibration structures, and any such can be collapsed to a Browder-Quinn $\mathscr{F}$-stratified space. Instead of developing this, we will work directly with Thom-Mather stratified spaces in establishing the Browder-Quinn surgery sequence below.*

2.2. Smoothly stratified spaces and transverse maps

From now on we will only work with $\mathscr{F}$ equal to smooth manifolds and locally trivial smooth fiber bundles, i.e., the setting of Thom-Mather stratified spaces. For this class of spaces there is a construction going back to Thom [54] and carried out in [3] that replaces a stratified space, $\widehat{X}$, with its 'resolution', $\widetilde{X}$, a manifold with corners and an iterated fibration structure. We now recall this construction.

Let $\widehat{X}$ be a stratified space with singular strata

$$\mathcal{S}(\widehat{X}) = \{Y_1, Y_2, \dots, Y_\ell\}.$$

Each Y_i is subset of $\widehat{X}$ that inherits the structure of an open manifold (indeed, of the interior of a manifold with corners). We write $Y_i < Y_j$ if the closure of Y_j in $\widehat{X}$ contains Y_i. The closure of Y_j in $\widehat{X}$ is given by

$$\widehat{Y_j} = \bigcup \{Y_i : Y_i \leq Y_j\}$$

and is itself a stratified space. Every point in Y_i has a neighborhood in $\widehat{X}$ homeomorphic to a ball in $\mathbb{R}^{\dim Y_i}$ times the cone over a stratified space $\widehat{Z}_i$, known as the link of Y_i in $\widehat{X}$.

The resolution of $\widehat{X}$, denoted $\widetilde{X}$, is a smooth manifold with corners. Each stratum Y_i of $\widehat{X}$ corresponds to a collective boundary hypersurface $\mathfrak{B}_{Y_i}$ of $\widetilde{X}$, by which we mean a collection of boundary hypersurfaces no two of which intersect. Each collective boundary hypersurface participates in a fiber bundle,

$$\widetilde{Z}_i - \mathfrak{B}_{Y_i} \xrightarrow{\phi_{Y_i}} \widetilde{Y}_i,$$

where the base is the resolution of $\widehat{Y_i}$ and the typical fiber is the resolution of $\widehat{Z}_i$. If Y_i and Y_j are strata of $\widehat{X}$ with $Y_i < Y_j$ then $\mathfrak{B}_{Y_i} \cap \mathfrak{B}_{Y_j} \neq \emptyset$ and we have a commutative diagram of fiber bundle maps

$$\begin{array}{ccc} \mathfrak{B}_{Y_i} \cap \mathfrak{B}_{Y_j} & \xrightarrow{\phi_{Y_j}} & \mathfrak{B}_{Y_iY_j} \subseteq \mathfrak{B}_{Y_j} \\ & {\scriptstyle \phi_{Y_i}} \searrow \quad \swarrow {\scriptstyle \phi_{Y_iY_j}} & \\ & \widetilde{Y}_i, & \end{array}$$

where $\mathfrak{B}_{Y_iY_j}$ is a collective boundary hypersurface of $\mathfrak{B}_{Y_j}$. We refer to a manifold with corners together with these collective boundary hypersurface fiber bundle maps as a manifold with an iterated fibration structure.

There is a canonical 'blow-down map' between a manifold with corners and an iterated fibration structure $\widetilde{X}$ and a stratified space $\widehat{X}$,

$$\beta : \widetilde{X} \longrightarrow \widehat{X},$$

which collapses the fibers of the boundary fiber bundles to their base. Note that β is a diffeomorphism between the interior of $\widetilde{X}$ and the regular part of $\widehat{X}$.

A continuous map between stratified spaces is *stratum preserving* if the inverse image of a stratum is a union of strata. A stratum preserving map $\widehat{M} \xrightarrow{\widehat{F}} \widehat{X}$ is *smooth* if it lifts to a smooth map $\widetilde{M} \xrightarrow{\widetilde{F}} \widetilde{X}$. We denote the space of such maps by

$$\mathcal{C}^\infty_\Phi(\widetilde{M}, \widetilde{X}) \subseteq \mathcal{C}^\infty(\widetilde{M}, \widetilde{X})$$

and the corresponding maps between $\widehat{M}$ and $\widehat{X}$ by $\mathcal{C}^\infty(\widehat{M}, \widehat{X})$. Directly from the definition we have a natural identification

$$\beta_* : \mathcal{C}^\infty_\Phi(\widetilde{M}, \widetilde{X}) \longrightarrow \mathcal{C}^\infty(\widehat{M}, \widehat{X}).$$

The smooth map $\widetilde{F}$ necessarily induces a fiber bundle map between the collective boundary hypersurfaces of $\widetilde{M}$ and those of $\widetilde{X}$,

$$\begin{array}{ccccccc} \widetilde{M} & \supseteq & \mathfrak{B}_N & \xrightarrow{\widetilde{F}} & \mathfrak{B}_Y & \subseteq & \widetilde{X} \\ & & {\scriptstyle \phi_N}\big\downarrow & & \big\downarrow{\scriptstyle \phi_Y} & & \\ & & \widetilde{N} & \xrightarrow{\widetilde{F|_{\widehat{N}}}} & \widetilde{Y}. & & \end{array} \tag{2.1}$$

We will say that $\widehat{M} \xrightarrow{\widehat{F}} \widehat{X}$ and $\widetilde{M} \xrightarrow{\widetilde{F}} \widetilde{X}$ are *transverse* if the commutative diagram of fiber bundles (2.1) is a pull-back diagram, that is,

$$\mathfrak{B}_N = (\widetilde{F|_{\widehat{N}}})^* \mathfrak{B}_Y. \tag{2.2}$$

We denote the class of such maps by $\mathcal{C}^\infty_\pitchfork(\widetilde{M}, \widetilde{X})$ and $\mathcal{C}^\infty_\pitchfork(\widehat{M}, \widehat{X})$; the identification β_* restricts to an identification

$$\beta_* : \mathcal{C}^\infty_\pitchfork(\widetilde{M}, \widetilde{X}) \longrightarrow \mathcal{C}^\infty_\pitchfork(\widehat{M}, \widehat{X}).$$

Note that $\mathcal{C}^\infty_\pitchfork(\widehat{M}, \widehat{X})$ are the transverse maps of Browder-Quinn. (Indeed, the fibrations ν_a of a Browder-Quinn stratified space correspond in the smooth category to the fiber bundle maps ϕ_Y.)

2.3. Examples of transverse maps

Here is a list of examples of transverse maps.

— Transverse maps are used in [23, Part I, §4] and [24, §5.4] where they are called 'normal non-singular'. A weaker notion called 'homotopy transverse' was used by Weinberger [59, §5.2].

— An example from [24] is the inclusion $H \cap \widehat{X} \hookrightarrow \widehat{X}$ where $\widehat{X}$ is a stratified subset of a smooth manifold and H is a smooth submanifold transverse to the strata of $\widehat{X}$.

— Another example from the same source is the fiber bundle projection map for a fiber bundle over a stratified space with fiber a smooth manifold.

— Browder-Quinn considered these maps first in an equivariant situation. If G is a compact Lie group and L, L' are spaces with G-actions then a map $f : L \longrightarrow L'$ is *isovariant* if, for any $x \in L$, $g \in G$,

$$f(gx) = gf(x), \text{ and } gf(x) = f(x) \iff gx = x.$$

An isovariant map is *transverse linear* if whenever $H \subseteq G$ is a subgroup, and $L^{\{H\}}$ denotes the subset of L consisting of points whose isotropy group is conjugate to H, there are G-vector bundle tubular neighborhoods

$$L^{\{H\}} \subseteq U, \quad (L')^{\{H\}} \subseteq U'$$

such that f restricts to a G-linear vector bundle map $U \longrightarrow U'$. Transverse linear isovariant maps are examples of transverse maps for the stratification of a space into the orbit types of a group action.

2.4. Properties preserved by transverse maps.

There are some properties of a stratified space $\widehat{X}$ such that the existence of a transverse map $\widehat{f} : \widehat{M} \longrightarrow \widehat{X}$ implies that $\widehat{M}$ also has this property. For example, if the dimensions of all of the links of $\widehat{X}$ are odd, then the dimensions of all of the links of $\widehat{M}$ must be odd as well. A class of stratified spaces determined by such a property will be said to be *preserved by transverse maps*. As examples of such classes let us mention, in order of increasing generality: IP spaces, Witt spaces, and Cheeger spaces or L-spaces. Recall that a stratified space is a Witt space if, whenever the link of a stratum is even dimensional, its middle degree middle perversity intersection homology vanishes. A Witt space is an IP space (intersection Poincaré space) if, whenever the link $\widehat{Z}$ of a stratum is odd dimensional, its middle perversity homology in degree $\frac{1}{2}(\dim \widehat{Z} - 1)$ is torsion-free. A stratified space is an L-space if there is a self-dual sheaf compatible with the intersection homology sheaves of upper and lower middle perversity [9]. Smoothly stratified L-spaces are known as Cheeger spaces [5].

A property of transverse maps that we will use repeatedly below is that the restriction of a transverse map to the closure of a stratum is again a transverse map.

2.5. Surgery definitions

When carrying out surgery constructions we will need to use stratified spaces with corners. By a stratified space with corners we mean a stratified space with collared corners [1, Definition 2] or an 'abstract stratification with faces' in the sense of Verona [55, §5]. An 'articulated stratified space with corners' is a union of boundary hypersurfaces of a stratified space with corners; thus it is a union of stratified spaces with corners together with identifications of certain of their boundary hypersurfaces. Smoothness of maps to and from articulated spaces is defined in the natural way, i.e., continuity on the whole and smoothness on each stratified space with corners.

DEFINITION 1. – *Let $\widehat{M}$ and $\widehat{X}$ be oriented stratified spaces.*

(i) *A* BQ-transverse map $f : \widehat{M} \longrightarrow \widehat{X}$ *is a transverse map that is orientation preserving and restricts to a diffeomorphism between strata of dimension less than five.*

(ii) *A* BQ-normal map $f : \widehat{M} \longrightarrow \widehat{X}$ *is a BQ-transverse map such that, in the notation of §2.1, for each $a \in A$, f restricts to a degree one normal map*

$$f_a : M_a \setminus \mathrm{int}(N(M_a)) \longrightarrow X_a \setminus \mathrm{int}(N(X_a)),$$

meaning that there is a smooth vector bundle $\tau \longrightarrow X_a \setminus \mathrm{int}(N(X_a))$ and a stable bundle isomorphism of the stable normal bundle $\mathrm{Nor}_{M_a \setminus \mathrm{int}(N(M_a))}$, $b : \mathrm{Nor}_{M_a \setminus \mathrm{int}(N(M_a))} \longrightarrow f_a^* \tau$, *covering f_a. (We will not explicitly keep track of the bundle data as it will not affect our analytic maps.)*

(iii) *A* BQ-equivalence $f : \widehat{M} \longrightarrow \widehat{X}$ *is a BQ-transverse map whose restriction to each stratum is a homotopy equivalence. (By Miller's criterion* [41] *(see* [4, Corollary 1.11]*), f is a BQ-equivalence if and only if there is a BQ-transverse map $g : \widehat{X} \longrightarrow \widehat{M}$ and homotopies of $f \circ g$ and $g \circ f$ to the respective identities through BQ-transverse maps.)*

One could also work with simple homotopy equivalences and there is an s-cobordism theorem in this context [13, pg. 34].

Given a stratified space, its dimension function will refer to the function on the poset of strata that assigns to each stratum its dimension.

DEFINITION 2. – *Let $\widehat{X}$ be a smooth oriented stratified space, possibly with boundary. (Our convention is that the spaces below are allowed to be empty and a map between empty sets counts as an equivalence.)*

(a) *Let $\mathcal{L}_{\mathrm{BQ},d}(\widehat{X})$ denote the set of diagrams*

$$(\widehat{M};\partial\widehat{M}) \xrightarrow{\phi} (\widehat{Y};\partial\widehat{Y}) \xrightarrow{\omega} \widehat{X} \tag{2.3}$$

where $\widehat{M}$ and $\widehat{Y}$ are oriented stratified spaces with corners, d is the dimension function of $\widehat{M}$, ϕ is BQ-normal, $\partial\phi$ is a BQ-equivalence (between articulated stratified spaces with corners), and ω is BQ-transverse.

The set $\mathcal{L}_{\mathrm{BQ},d_{\widehat{X}}}(\widehat{X})$, where $d_{\widehat{X}}$ denotes the dimension function of $\widehat{X}$, will sometimes be abbreviated as $\mathcal{L}_{\mathrm{BQ}}(\widehat{X})$, with the dimension function omitted.

We refer to these diagrams as (Browder-Quinn) $\mathcal{L}$-cycles over $\widehat{X}$.

A null bordism of an $\mathcal{L}$-cycle over $\widehat{X}$ *as above will mean a diagram*

$$(\widehat{N};\partial_1\widehat{N},\partial_2\widehat{N}) \xrightarrow{\Phi} (\widehat{Z};\partial_1\widehat{Z},\partial_2\widehat{Z}) \xrightarrow{\Omega} \widehat{X}\times I$$

between stratified spaces with corners, with the same dimension function as $\widehat{M}\times I$, where Φ is BQ-normal, $\Phi| : \partial_2\widehat{N} \longrightarrow \partial_2\widehat{Z}$ is a BQ-equivalence, Ω is BQ-transverse, and

$$\left((\partial_1\widehat{N},\partial_{12}\widehat{N}) \xrightarrow{\Phi|} (\partial_1\widehat{Z},\partial_{12}\widehat{Z}) \xrightarrow{\pi\circ\Omega|} \widehat{X}\right) = \left((\widehat{M};\partial\widehat{M}) \xrightarrow{\phi} (\widehat{Y};\partial\widehat{Y}) \xrightarrow{\omega} \widehat{X}\right)$$

(with $\pi : \widehat{X}\times I \longrightarrow \widehat{X}$ the projection). In this case we say that the $\mathcal{L}$-cycle is null bordant.

The $\mathcal{L}$-cycles over $\widehat{X}$ naturally form an Abelian monoid with addition induced by disjoint union and zero given by the diagram with $\widehat{M} = \widehat{Y} = \emptyset$. We say that two $\mathcal{L}$-cycles over $\widehat{X}$, α, β are equivalent if $\alpha + \beta^{\mathrm{op}}$ is null bordant, where β^{op} denotes β with orientations reversed. This is an equivalence relation, known as $\mathcal{L}$-bordism, and the set of equivalence classes, denoted $\mathrm{L}_{\mathrm{BQ},d}(\widehat{X})$, is known as the Browder-Quinn L-group of $\widehat{X}$. *(The equivalence classes form a group with the inverse of $[\alpha]$ being $[\alpha^{\mathrm{op}}]$.)*

(b) *The (Browder-Quinn)* normal invariants, *denoted $\mathcal{N}_{\mathrm{BQ}}(\widehat{X})$ are the subset of $\mathcal{L}_{\mathrm{BQ}}(\widehat{X})$ in which, with notation as above, $\widehat{Y} = \widehat{X}$ and $\omega = \mathrm{id}$. A* normal bordism *is an $\mathcal{L}$-bordism between normal invariants which, in the notation above, has $\widehat{Z} = \widehat{X}\times I$ and $\Omega = \mathrm{id}$. The set of normal invariants modulo normal bordisms is denoted $\mathrm{N}_{\mathrm{BQ}}(\widehat{X})$.*

If $\widehat{X}$ has boundary we denote by $\mathcal{N}_{\mathrm{BQ}}(\widehat{X},\partial\widehat{X})$ the subset of $\mathcal{N}_{\mathrm{BQ}}(\widehat{X})$ in which, with notation as above, $\phi|_{\partial\widehat{M}}$ is a (stratum preserving) diffeomorphism. A normal bordism, relative to $\partial\widehat{X}$, will be a normal bordism as above in which $\partial_2\widehat{N} = \widehat{K}\times I$ and $\Phi|_{\partial_2\widehat{N}} = \phi|_{\partial\widehat{M}}\times\mathrm{id}$. The corresponding set of normal invariants is denoted $\mathrm{N}_{\mathrm{BQ}}(\widehat{X},\partial\widehat{X})$.

(c) *The (Browder-Quinn)* Thom-Mather structures, *denoted $\mathcal{S}_{\mathrm{BQ}}(\widehat{X})$ are the subset of $\mathcal{N}_{\mathrm{BQ}}(\widehat{X})$ in which, with notation as above, ϕ is a BQ-equivalence. Two such objects are equivalent if there is a normal bordism between them which, in the notation above, has Φ*

a BQ-equivalence. The set of equivalence classes is denoted $\mathrm{S}_{\mathrm{BQ}}(\widehat{X})$ *and is known as the (Browder-Quinn)* structure set.

Similarly, if $\widehat{X}$ *has boundary then* $\mathcal{S}_{\mathrm{BQ}}(\widehat{X}, \partial\widehat{X})$ *is the subset of* $\mathcal{N}(\widehat{X}, \partial\widehat{X})$ *in which* ϕ *is a BQ-equivalence. Equivalence classes modulo normal bordism relative to* $\partial\widehat{X}$ *in which* Φ *is a BQ-equivalence form the set* $\mathrm{S}_{\mathrm{BQ}}(\widehat{X}, \partial\widehat{X})$.

REMARK 2. – *A BQ-transverse map* $f : \widehat{X} \longrightarrow \widehat{W}$ *induces a homomorphism*

$$f_* : \mathcal{L}_{\mathrm{BQ},d}(\widehat{X}) \longrightarrow \mathcal{L}_{\mathrm{BQ},d}(\widehat{W}).$$

(Recall that d *is the dimension function of* $\widehat{M}$ *in* (2.3)*.) It is easy to see that if* f *is a BQ-equivalence then it induces an isomorphism* $\mathrm{L}_{\mathrm{BQ},d}(\widehat{X}) \cong \mathrm{L}_{\mathrm{BQ},d}(\widehat{W})$*. Browder and Quinn point out* [13, Proposition 4.7] *that this is true for any BQ-transverse* f *that satisfies the* Π-Π *condition below. For example,* $\mathrm{L}_{\mathrm{BQ},d}(\widehat{X}) \cong \mathrm{L}_{\mathrm{BQ},d}(\widehat{X} \times I)$.

REMARK 3. – *The Pontrjagin-Thom constructions on the various strata can be fit together using transversality to identify* $\mathcal{N}_{\mathrm{BQ}}(\widehat{X})$ *with homotopy classes of maps from* $\widehat{X}$ *into* G/O *that are constant on strata of dimension less than five. Since we do not use this we do not elaborate, see* [59, pg. 140].

REMARK 4. – *If* $\widehat{X} \equiv X$ *is a smooth manifold of dimension* d*, then* $\mathrm{L}_{\mathrm{BQ},d}(X)$*, with* d *the dimension function that assigns to the unique stratum* X *its dimension, is precisely equal to* $\mathrm{L}_d(\mathbb{Z}\pi_1(X))$*. Indeed, our definition of* $\mathrm{L}_{BQ,d}(X)$ *coincides with Wall's* $L^1_d(X)$ *from* [57, Chapter 9] *save that we required* Y *in Definition 2 to be smooth (by the transversality requirement on* ω*). However* X *smooth and Wall's realization theorem (*[57, Theorem 10.4] *and Theorem 2.5 below) shows that this does not change the group we obtain.*

The inclusion maps between the sets above descend to maps between the equivalence classes

$$\mathrm{S}_{\mathrm{BQ}}(\widehat{X}) \xrightarrow{\eta} \mathrm{N}_{\mathrm{BQ}}(\widehat{X}) \xrightarrow{\theta} \mathrm{L}_{\mathrm{BQ},d_{\widehat{X}}}(\widehat{X}),$$

where $d_{\widehat{X}}$ is the dimension function of $\widehat{X}$. We will show that this sequence is exact in an appropriate sense and extends to the left

$$\mathrm{L}_{\mathrm{BQ},d_{\widehat{X}\times I}}(\widehat{X} \times I) \dashrightarrow \mathrm{S}_{\mathrm{BQ}}(\widehat{X}) ,$$

in that there is an action of the L-group of $\widehat{X} \times I$ on the structure set of $\widehat{X}$. Moreover the extended sequence is exact in that two elements of $\mathrm{S}_{\mathrm{BQ}}(\widehat{X})$ have the same image under η if and only if they are in the same orbit of the L-group. For stratified spaces with boundary there are analogous sequences relative to the boundary. In fact the extension to the left of the sequence above, for a stratified space without boundary, is

$$\mathrm{S}_{\mathrm{BQ}}(\widehat{X} \times I, \widehat{X} \times \partial I) \xrightarrow{\eta} \mathrm{N}_{\mathrm{BQ}}(\widehat{X} \times I, \widehat{X} \times \partial I) \xrightarrow{\theta} \mathrm{L}_{\mathrm{BQ},d_{\widehat{X}\times I}}(\widehat{X} \times I) \dashrightarrow \mathrm{S}_{\mathrm{BQ}}(\widehat{X}) \xrightarrow{\eta} \mathrm{N}_{\mathrm{BQ}}(\widehat{X}) \xrightarrow{\theta} \mathrm{L}_{\mathrm{BQ},d_{\widehat{X}}}(\widehat{X}).$$

2.6. The Π-Π condition

DEFINITION 3. – *We say that a map* $h : \widehat{M} \longrightarrow \widehat{N}$ *between stratified spaces with corners satisfies* the Π-Π condition *if:*

For every connected component of a stratum of $\widehat{N}$, S_N, *there is exactly one connected component of a stratum of* $\widehat{M}$, S_M, *such that* $h(S_M) \cap S_N \neq \emptyset$. *Moreover,* $h(S_M) \subseteq S_N$ *and* $h_* : \pi_1(S_M) \longrightarrow \pi_1(S_N)$ *is an isomorphism.*

The π-π theorem (or surgery lemma) in our context is implicit in [13] and presented by Weinberger in [59, pg. 140], where a proof is also sketched. We formulate it as in Quinn's thesis [50, Theorem 2.4.4] and prove it following [59].

THEOREM 2.1 (BQ Π-Π theorem). – *Let* $\widehat{M}$, $\widehat{Y}$, $\widehat{X}$ *be stratified spaces with boundary, with the same dimension function, together with decompositions of their boundaries, e.g.,* $\partial\widehat{X} = \partial_0\widehat{X} \cup \partial_1\widehat{X}$, *into two codimension zero stratified spaces with common boundary. Consider a diagram*

$$(\widehat{M}; \partial_0\widehat{M}, \partial_1\widehat{M}) \xrightarrow{f} (\widehat{Y}; \partial_0\widehat{Y}, \partial_1\widehat{Y}) \xrightarrow{\omega} \widehat{X}, \tag{2.4}$$

in which f *is BQ-normal,* $f| : \partial_0\widehat{M} \longrightarrow \partial_0\widehat{Y}$ *is a BQ-equivalence,* ω *is BQ-transverse and orientation preserving.*

If the inclusion of $\partial_1\widehat{Y}$ *into* $\widehat{Y}$ *satisfies the* Π-Π *condition then there is a bordism between* (2.4) *and*

$$(\widehat{M}'; \partial_0\widehat{M}, \partial_1\widehat{M}') \xrightarrow{f'} (\widehat{Y}; \partial_0\widehat{Y}, \partial_1\widehat{Y}) \xrightarrow{\omega} \widehat{X}, \tag{2.5}$$

where f' *satisfies the same properties as* f *but is moreover a BQ-equivalence. Explicitly this bordism is a diagram of oriented stratified spaces with corners*

$$\widehat{N} \xrightarrow{F} \widehat{Y} \times I \xrightarrow{\omega\times\mathrm{id}} \widehat{X} \times I,$$

in which $\partial\widehat{N} = \widehat{M} \cup \widehat{M}' \cup \partial_0\widehat{M} \times I \cup \widehat{P}$, *with* F *a BQ-normal map satisfying*

$$(\widehat{N}; \widehat{M}, \widehat{M}', \partial_0\widehat{M}\times I, \widehat{P}) \xrightarrow{(F;f,f',f|\times\mathrm{id},F|)} (\widehat{Y}\times I; \widehat{Y}\times\{0\}, \widehat{Y}\times\{1\}, \partial_0\widehat{Y}\times I, \partial_1\widehat{Y}\times I). \tag{2.6}$$

Proof. – We proceed by induction on the depth of the stratified space. Our base case is when $\widehat{Y}$, $\widehat{M}$, and $\widehat{X}$ are smooth manifolds with boundary and this is Theorem 3.3 in Wall's book [57], since in dimension less than five our maps are diffeomorphisms by definition.

Suppose the theorem is established for all stratified spaces with boundary whose stratification has depth less than k and consider (2.4) where $\widehat{Y}$ (and hence $\widehat{M}$, $\widehat{X}$) has a stratification of depth k. Denote the subsets of depth k by a † decoration and note that these are smooth manifolds and that transversality of the maps in (2.4) implies that these subsets are preserved by these maps. Thus we obtain

$$(M^\dagger; \partial_0 M^\dagger, \partial_1 M^\dagger) \xrightarrow{f^\dagger} (Y^\dagger; \partial_0 Y^\dagger, \partial_1 Y^\dagger) \xrightarrow{\omega^\dagger} X^\dagger \tag{2.7}$$

satisfying the same conditions as the diagram (2.4). Since the Π-Π condition holds, there is a bordism satisfying the same conditions as (2.6),

$$N^\dagger \xrightarrow{F^\dagger} Y^\dagger \times I \xrightarrow{\omega^\dagger\times\mathrm{id}} X^\dagger \times I \tag{2.8}$$

between (2.7) and

$$(M^{\dagger\prime}; \partial_0 M^{\dagger\prime}, \partial_1 M^{\dagger\prime}) \xrightarrow{f^{\dagger\prime}} (Y^\dagger; \partial_0 Y^\dagger, \partial_1 Y^\dagger) \xrightarrow{\omega^\dagger} (X^\dagger; \partial_0 X^\dagger, \partial_1 X^\dagger),$$

with $f^{\dagger\prime}$ a homotopy equivalence.

Transversality of f and ω guarantees that we can find neighborhoods $\mathcal{T}_{M^\dagger} \subseteq \widehat{M}$, $\mathcal{T}_{Y^\dagger} \subseteq \widehat{Y}$, and $\mathcal{T}_{X^\dagger} \subseteq \widehat{X}$ that fiber over $M^\dagger$, $Y^\dagger$, and $X^\dagger$ respectively, such that each square in

(2.9)
$$\begin{array}{ccccc} \mathcal{T}_{M^\dagger} & \xrightarrow{f|} & \mathcal{T}_{Y^\dagger} & \xrightarrow{\omega|} & \mathcal{T}_{X^\dagger} \\ \downarrow{\scriptstyle \phi^\dagger_M} & & \downarrow{\scriptstyle \phi^\dagger_Y} & & \downarrow{\scriptstyle \phi^\dagger_X} \\ M^\dagger & \xrightarrow{f^\dagger} & Y^\dagger & \xrightarrow{\omega^\dagger} & X_\dagger, \end{array}$$

where $f|$ and $\omega|$ denote the restrictions of f and ω respectively, is a pull-back square. Note that for $i = 0, 1$ we have

$$\partial_i \widehat{M} \cap \mathcal{T}_{M^\dagger} = (\phi^\dagger_M)^{-1}(\partial_i M^\dagger)$$

and similarly for $\widehat{Y}$ and $\widehat{X}$, so the top row of (2.9) is a diagram satisfying conditions analogous to (2.4).

We have an extension of this diagram to a similar diagram over the bordism (2.8)

(2.10)
$$\begin{array}{ccccc} F^{\dagger *}(\mathcal{T}_{Y^\dagger} \times I) & \xrightarrow{\overline{F}^\dagger} & \mathcal{T}_{Y^\dagger} \times I & \xrightarrow{\omega|\times \mathrm{id}} & \mathcal{T}_{X^\dagger} \times I \\ \downarrow & & \downarrow & & \downarrow \\ N^\dagger & \xrightarrow{F^\dagger} & Y^\dagger \times I & \xrightarrow{\omega^\dagger \times \mathrm{id}} & X_\dagger \times I \end{array}$$

with $F^{\dagger *}(\mathcal{T}_{Y^\dagger})|_{M^\dagger} = \mathcal{T}_{M^\dagger}$ and $\overline{F}^\dagger$ the induced map on the pull-back. Restricting this diagram to the 'fixed' ∂_0 part of the boundary we get

$$\begin{array}{ccccc} (\partial_0 \widehat{M} \cap \mathcal{T}_{M^\dagger}) \times I & \xrightarrow{f|\times \mathrm{id}} & (\partial_0 \widehat{Y} \cap \mathcal{T}_{Y^\dagger}) \times I & \xrightarrow{\omega|\times \mathrm{id}} & (\partial_0 \widehat{X} \cap \mathcal{T}_{X^\dagger}) \times I \\ \downarrow & & \downarrow & & \downarrow \\ \partial_0 M^\dagger \times I & \xrightarrow{f^\dagger|\times \mathrm{id}} & \partial_0 Y^\dagger \times I & \xrightarrow{\omega^\dagger|\times \mathrm{id}} & \partial_0 X^\dagger \times I \end{array}$$

and so the top row of (2.10) is a bordism, satisfying conditions analogous to (2.6), from the top row of (2.9) to

(2.11)
$$(f^{\dagger\prime})^* \mathcal{T}_{Y^\dagger} \xrightarrow{\overline{f}^{\dagger\prime}} \mathcal{T}_{Y^\dagger} \xrightarrow{\omega|} \mathcal{T}_{X^\dagger},$$

in which $\overline{f}^{\dagger\prime}$ is a BQ-equivalence.

Now, as in [12, Theorem 2.14] [59, §4.3], we multiply each space in (2.4) by the unit interval and attach the top row of (2.10) to get a bordism

$$\widehat{M}\times I \bigcup_{\mathcal{T}_{M^\dagger}\times\{1\}\sim F^{\dagger *}(\mathcal{T}_{Y^\dagger})|_{M^\dagger}} F^{\dagger *}(\mathcal{T}_{Y^\dagger}) \xrightarrow{(f\times \mathrm{id})\cup \overline{F}^\dagger} \widehat{Y}\times I \bigcup_{\mathcal{T}_{Y^\dagger}\times\{1\}\sim \mathcal{T}_{Y^\dagger}\times\{0\}} \mathcal{T}_{Y^\dagger}\times I$$
$$\xrightarrow{\omega\times\mathrm{id}\,\cup\,\omega|\times\mathrm{id}} \widehat{X}\times I \bigcup_{\mathcal{T}_{X^\dagger}\times\{1\}\sim \mathcal{T}_{X^\dagger}\times\{0\}} \mathcal{T}_{X^\dagger}\times I$$

from (2.4) to a similar diagram over $\widehat{X}$ which we denote

$$(\widehat{P};\partial_0\widehat{P},\partial_1\widehat{P}) \xrightarrow{g} (\widehat{Y};\partial_0\widehat{Y},\partial_1\widehat{Y}) \xrightarrow{\omega} (\widehat{X};\partial_0\widehat{X},\partial_1\widehat{X}) \tag{2.12}$$

and which restricts to (2.11) in a neighborhood of the subsets of depth k.

Now we remove these neighborhoods of the subsets of depth k to form

$$\widehat{P}^+ = \widehat{P}\setminus (f^{\dagger\prime})^*\mathcal{T}_{Y^\dagger},\quad \widehat{Y}^+ = \widehat{Y}\setminus \mathcal{T}_{Y^\dagger},\quad \widehat{X}^+ = \widehat{X}\setminus \mathcal{T}_{X^\dagger}.$$

These are stratified spaces with corners (see, e.g., [1, §6], [55]) and we define

$$\partial_0\widehat{Y}^+ = (\partial_0\widehat{Y}\cap\widehat{Y}^+)\cup\partial\mathcal{T}_{Y^\dagger},\quad \partial_1\widehat{Y}^+ = \partial_1\widehat{Y}\cap\widehat{Y}^+$$

and similarly for $\widehat{P}^+$ and $\widehat{X}^+$. Note that the restrictions of g and ω to $\partial_0\widehat{P}^+$ and $\partial_0\widehat{Y}^+$ are BQ-equivalences so we have a diagram

$$(\widehat{P}^+;\partial_0\widehat{P}^+,\partial_1\widehat{P}^+) \xrightarrow{g|} (\widehat{Y}^+;\partial_0\widehat{Y}^+,\partial_1\widehat{Y}^+) \xrightarrow{\omega|} (\widehat{X}^+;\partial_0\widehat{X}^+,\partial_1\widehat{X}^+)$$

satisfying conditions analogous to (2.4). (Note that though the stratified space now has corners of codimension two, one can 'smooth out the corners' as in [58, §2.6], [25, §3].)

Moreover, the compatibility between the stratifications and the boundary faces implies that each stratum of $\widehat{Y}^+$ is homotopy equivalent to the corresponding stratum of $\widehat{Y}$, and the same is true for the strata of $\partial_1\widehat{Y}^+$. (Indeed iterating this process of removing tubular neighborhoods of deepest strata produces the resolution of $\widehat{Y}$ which does not change the homotopy type of the strata.) Significantly, the inclusion of $\partial_1\widehat{Y}^+$ into $\widehat{Y}^+$ satisfies the Π-Π condition and since $\widehat{Y}^+$ has depth less than k we can apply our inductive hypothesis to find a bordism, satisfying conditions analogous to (2.6),

$$\widehat{N}^+ \xrightarrow{G} \widehat{Y}^+\times I \xrightarrow{\Omega} \widehat{X}^+\times I$$

between

$$(\widehat{P^+};\partial_0\widehat{P^+},\partial_1\widehat{P^+}) \xrightarrow{g|} (\widehat{Y^+};\partial_0\widehat{Y^+},\partial_1\widehat{Y^+}) \xrightarrow{\omega|} (\widehat{X^+};\partial_0\widehat{X^+},\partial_1\widehat{X^+})$$
$$\text{and } (\widehat{P^{+\prime}};\partial_0\widehat{P^{+\prime}},\partial_1\widehat{P^{+\prime}}) \xrightarrow{g'} (\widehat{Y^+};\partial_0\widehat{Y^+},\partial_1\widehat{Y^+}) \xrightarrow{\omega|} (\widehat{X^+};\partial_0\widehat{X^+},\partial_1\widehat{X^+}),$$

with g' a BQ-equivalence. Since the bordism does not change the spaces $\partial((f^{\dagger\prime})^*\mathcal{T}_{Y^\dagger})$, $\partial\mathcal{T}_{Y^\dagger}$, $\partial\mathcal{T}_{X^\dagger}$ or the maps between them, we can glue in the bordism (2.11) to finally obtain a bordism between (2.4) and (2.5) with

$$\widehat{M}' = P^{+\prime}\cup (f^{\dagger\prime})^*\mathcal{T}_{Y^\dagger}$$

and $f' = g'\cup \overline{f}^{\dagger\prime}$ a BQ-equivalence as required. □

The key to applying the Π-Π theorem is a result of Wall that allows us to represent every class in $\mathrm{L}_{\mathrm{BQ}}(\widehat{X})$ by a 'restricted' representative. This is sometimes referred to as "$L^1 = L^2$" evoking the notation of [57, Chapter 9].

Definition 4. – *Let $\widehat{X}$ be a stratified space (possibly with boundary). An $\mathcal{L}$-cycle over $\widehat{X}$,*

$$(\widehat{M};\partial\widehat{M}) \xrightarrow{\phi} (\widehat{Y};\partial\widehat{Y}) \xrightarrow{\omega} \widehat{X}$$

is a restricted $\mathcal{L}$-cycle *if $\omega : \widehat{Y} \longrightarrow \widehat{X}$ satisfies the* Π-Π *condition.*

A null bordism of a restricted $\mathcal{L}$-cycle over $\widehat{X}$,

$$(\widehat{N};\partial_1\widehat{N},\partial_2\widehat{N}) \xrightarrow{\Phi} (\widehat{Z};\partial_1\widehat{Z},\partial_2\widehat{Z}) \xrightarrow{\Omega} \widehat{X}\times I$$

is a restricted null bordism *if $\Omega : \widehat{Z} \longrightarrow \widehat{X}\times I$ satisfies the* Π-Π *condition.*

Remark 5. – *If $\widehat{X}$ has depth zero then these are the restricted cycles of* [57, Chapter 9]*, see Remark 4.*

Theorem 2.2. – *Let $\widehat{X}$ be a stratified space (possibly with boundary). Every element of $\mathrm{L}_{\mathrm{BQ}}(\widehat{X})$ is $\mathcal{L}$-bordant, relative to the boundary, to a restricted $\mathcal{L}$-cycle over $\widehat{X}$. If a restricted $\mathcal{L}$-cycle over $\widehat{X}$ is null bordant, then it participates in a restricted null bordism.*

Proof. – Our proof is parallel to that of the Π-Π theorem. When possible we will simply refer back to the latter proof.

We will prove by induction on the depth of the stratifications that, whenever we have

$$\widehat{M} \xrightarrow{\phi} \widehat{Y} \xrightarrow{\omega} \widehat{X} \tag{2.13}$$

with ϕ BQ-normal, ω BQ-transverse, and $\widehat{M}$, $\widehat{Y}$, $\widehat{X}$ stratified spaces *with corners* with the same dimension functions, there is a bordism relative to the boundary to a similar diagram

$$\widehat{M}' \xrightarrow{\phi'} \widehat{Y}' \xrightarrow{\omega'} \widehat{X},$$

in which ω' satisfies the Π-Π condition. Specifically, there is a diagram

$$(\widehat{N};\widehat{M},\widehat{M}',\partial\widehat{M}\times I) \xrightarrow{(\Phi;\phi,\phi',\phi|\times\mathrm{id})} (\widehat{Z};\widehat{Y},\widehat{Y}',\partial\widehat{Y}\times I) \xrightarrow{(\Omega;\omega,\omega',\omega|\times\mathrm{id})} \widehat{X}\times I,$$

in which Φ is BQ-normal and Ω is BQ-transverse and moreover

$$\widehat{M}\cap\widehat{M}' = \emptyset, \quad \widehat{M}\cap(\partial\widehat{M}\times I) = \partial\widehat{M}, \quad \widehat{M}'\cap(\partial\widehat{M}\times I) = \partial\widehat{M}'$$

and similarly for $\widehat{Z}$.

We proceed by induction on the depth of the stratified space. Our base case is for smooth manifolds with corners. If $\dim Y < 5$ then the theorem is automatic since f and ω are then both diffeomorphisms. Assuming $\dim Y \geq 5$, this case is handled by Wall in [57, Theorems 9.4, 9.5], where he shows that this can be arranged by carrying out surgery on Y along 1-handles and 2-handles and then modifying M along the preimages of the corresponding embeddings. As pointed out in [42, Proof of Lemma 3], a theorem of Whitney implies that for $\dim Y > 4$ any homotopy class of maps from $\mathbb{S}^1$ or $\mathbb{S}^2$ into Y contains an embedding with image contained in the interior of Y. Thus all of the modifications in M and Y can be carried out in their interiors.

Suppose the theorem is established for all stratified spaces with boundary whose stratification has depth less than k and consider (2.13) where $\widehat{Y}$ (and hence $\widehat{M}$, as they have the same dimension function) has a stratification of depth k. Denote the subsets of depth k by a † decoration and note that these are smooth manifolds and that transversality of the maps in (2.13) implies that these subsets are preserved by these maps. Thus we obtain

$$(\widehat{M^\dagger}, \partial\widehat{M^\dagger}) \xrightarrow{f^\dagger} (\widehat{Y^\dagger}, \partial\widehat{Y^\dagger}) \xrightarrow{\omega^\dagger} \widehat{X}^\dagger,$$

an element of $\mathcal{L}_{\mathrm{BQ}}(\widehat{X^\dagger})$. Applying the base case to this situation we obtain an $\mathcal{L}$-bordism, relative to the boundary, to an element satisfying the desired Π-Π condition. Proceeding as in the proof of Theorem 2.1 we can lift this $\mathcal{L}$-bordism to neighborhoods of these subsets and then graft it onto the product of (2.13) with the unit interval to obtain an $\mathcal{L}$-bordism, relative to the boundary, between (2.13) and an element of $\mathcal{L}_{\mathrm{BQ}}(\widehat{X})$ analogous to (2.12),

$$(\widehat{P}, \partial\widehat{P}) \xrightarrow{g} (\widehat{V}, \partial\widehat{V}) \xrightarrow{\alpha} \widehat{X},$$

where $\alpha : \widehat{V} \longrightarrow \widehat{X}$ satisfies the Π-Π condition for strata of depth k.

Now we remove consistent tubular neighborhoods of the subsets of depth k to form $\widehat{P}^+$, $\widehat{V}^+$, and $\widehat{X}^+$ as in the proof of Theorem 2.1. This gives a diagram

$$(\widehat{P^+}; \partial_0\widehat{P^+}, \partial_1\widehat{P^+}) \xrightarrow{g|} (\widehat{V^+}; \partial_0\widehat{V^+}, \partial_1\widehat{V^+}) \xrightarrow{\alpha|} (\widehat{X}; \partial_0\widehat{X}, \partial_1\widehat{X})^+$$

in which the stratification on $\widehat{V}^+$ has depth less than k. By our inductive hypothesis there is a bordism, relative to the boundary, between this and another diagram

$$(\widehat{P^{+\prime}}; \partial_0\widehat{P^{+\prime}}, \partial_1\widehat{P^{+\prime}}) \xrightarrow{g'} (\widehat{V^{+\prime}}; \partial_0\widehat{V^{+\prime}}, \partial_1\widehat{V^{+\prime}}) \xrightarrow{\alpha'} (\widehat{X}; \partial_0\widehat{X}, \partial_1\widehat{X})^+$$

for which $\alpha' : V^{+\prime} \longrightarrow \widehat{X}^+$ satisfies the Π-Π condition. Since the bordism is relative to the boundary we may glue in the previous bordism between neighborhoods of the strata of depth k to obtain a bordism, relative to the boundary, between our original diagram and

$$\widehat{P^{+\prime}} \cup \mathcal{T}_{P^\dagger} \xrightarrow{g' \cup (g|)} \widehat{V^{+\prime}} \cup \mathcal{T}_{V^\dagger} \xrightarrow{\alpha' \cup (\alpha|)} \widehat{X}.$$

Finally, because of the compatibility between the stratifications and the boundary faces, the fact that α satisfies the Π-Π condition between the strata of depth k and α' satisfies the Π-Π condition between strata of depth less than k means that $\alpha' \cup (\alpha|)$ satisfies the Π-Π condition on all strata. □

Our final preliminary result is to point out that the sum in $\mathrm{L}^{\mathrm{BQ}}(\widehat{X} \times I)$ which is induced by disjoint union can, when appropriate, be carried out by identifying boundary faces.

LEMMA 2.3. – *Let*

$$\alpha : (\widehat{M}; \partial\widehat{M}) \xrightarrow{\phi} (\widehat{Y}; \partial\widehat{Y}) \xrightarrow{\omega} \widehat{X} \times I$$

$$\beta : (\widehat{L}; \partial\widehat{L}) \xrightarrow{\psi} (\widehat{W}; \partial\widehat{W}) \xrightarrow{\theta} \widehat{X} \times I$$

be two $\mathcal{L}$-cycles over $\widehat{X} \times I$. Suppose that the pre-images in α lying above $\widehat{X} \times \{0\}$ coincide with the pre-images in β lying above $\widehat{X} \times \{1\}$, both equal to

$$\gamma : (\widehat{P}; \partial\widehat{P}) \xrightarrow{\rho} (\widehat{V}; \partial\widehat{V}) \xrightarrow{\xi} \widehat{X}.$$

The class of $\alpha+\beta$ *in* $\mathrm{L}_{\mathrm{BQ}}(\widehat{X}\times I)$ *is represented by the union of the diagrams along their common boundary,* $\alpha\cup_\gamma\beta$.

Proof. – Let $\widehat{N}=(\widehat{M}\cup_{\widehat{P}}\widehat{L})\times I$, $\widehat{Z}=(\widehat{Y}\cup_{\widehat{V}}\widehat{W})\times I$, and consider a diagram

$$\widehat{N}\xrightarrow{\Phi}\widehat{Z}\xrightarrow{\Omega}\widehat{X}\times I, \tag{2.14}$$

where Φ has the form $\Phi(x,t)=(\Phi_t(x),t)$ and similarly $\Omega(x,t)=(\Omega_t(x),t)$. By definition, $\widehat{P}$ has a collar neighborhood in each of $\widehat{M}$ and $\widehat{L}$ and gluing these together we have a neighborhood of the form $(-\varepsilon,\varepsilon)\times\widehat{P}$ in $\widehat{M}\cup_{\widehat{P}}\widehat{L}$ and of the form $(-\varepsilon,\varepsilon)\times\widehat{P}\times I$ in $\widehat{N}$. Similarly, we have a neighborhood of the form $(-\varepsilon,\varepsilon)\times\widehat{V}$ in $\widehat{Y}\cup_{\widehat{V}}\widehat{W}$ and of the form $(-\varepsilon,\varepsilon)\times\widehat{V}\times I$ in $\widehat{Z}$. Note that

$$\widehat{M}\cup_{\widehat{P}}\widehat{L}\setminus((-\varepsilon,\varepsilon)\times P)=\widehat{M}\sqcup\widehat{L},\quad \widehat{Y}\cup_{\widehat{V}}\widehat{W}\setminus((-\varepsilon,\varepsilon)\times V)=\widehat{Y}\sqcup\widehat{W}.$$

We choose Φ so that $\Phi_0=\phi\cup\psi$ and for $t>0$, $\Phi_t|_{(-t\varepsilon,t\varepsilon)\times P}=\mathrm{id}\times\rho$, while off of this neighborhood Φ is essentially $\phi\sqcup\psi$. We similarly choose Ω_t. With these choices, since $\phi|_{\partial\widehat{M}}$ and $\psi|_{\partial\widehat{L}}$ are BQ-equivalences, we recognize (2.14) as an $\mathcal{L}$-bordism between the disjoint union of α and β and their union along γ. □

2.7. Surgery theorem

With the preliminary results out of the way, we can establish the fundamental result of surgery: a normal map is normal bordant to an equivalence precisely when its surgery obstruction vanishes.

THEOREM 2.4 (Exactness part 1). – *Let* $\widehat{X}$ *be an oriented stratified space with boundary (possibly empty) and let*

$$h:\widehat{K}\longrightarrow\partial\widehat{X}$$

be a (stratum preserving) diffeomorphism.

Given a stratified space $\widehat{M}$ *with boundary* $\partial\widehat{M}=\widehat{K}$, *and a BQ-normal map*

$$\phi:\widehat{M}\longrightarrow\widehat{X}$$

extending h, there is a normal bordism relative to h between ϕ *and a BQ-equivalence if and only if* ϕ *is* $\mathcal{L}$*-null-bordant.*

We briefly encode this property by saying that the sequence of pointed sets (not groups)

$$\mathrm{S}_{\mathrm{BQ}}(\widehat{X},\partial\widehat{X})\xrightarrow{\eta}\mathrm{N}_{\mathrm{BQ}}(\widehat{X},\partial\widehat{X})\xrightarrow{\theta}\mathrm{L}_{\mathrm{BQ}}(\widehat{X}) \tag{2.15}$$

is exact.

Proof. – If there is a normal bordism, relative to h, between ϕ and a BQ-equivalence, $\phi':\widehat{M}'\longrightarrow\widehat{X}$, say

$$(\widehat{N};\widehat{M},\widehat{M}',\widehat{K}\times I)\xrightarrow{\Phi}(\widehat{X}\times I;\widehat{X}\times\{0\},\widehat{X}\times\{1\},\partial\widehat{X}\times I)\xrightarrow{\mathrm{id}}\widehat{X}\times I,$$

then this bordism witnesses the triviality of ϕ in $\mathrm{L}_{\mathrm{BQ}}(\widehat{X})$.

On the other hand, if

$$(\widehat{M};\widehat{K})\xrightarrow{(\phi;h)}(\widehat{X};\partial\widehat{X})\xrightarrow{\mathrm{id}}\widehat{X}$$

is a null bordant $\mathcal{L}$-cycle over $\widehat{X}$ then, since it is a restricted cycle, there is by Theorem 2.2 a restricted null bordism. That is to say, there are maps of stratified spaces with corners

$$(\widehat{N};\partial_1\widehat{N},\partial_2\widehat{N}) \xrightarrow{\Phi} (\widehat{Z};\partial_1\widehat{Z},\partial_2\widehat{Z}) \xrightarrow{\Omega} \widehat{X}\times I$$

with Φ BQ-normal, $\Phi| : \partial_2\widehat{N} \longrightarrow \partial_2\widehat{Z}$ a BQ-equivalence, Ω BQ-transverse,

$$\left((\partial_1\widehat{N},\partial_{12}\widehat{N}) \xrightarrow{\Phi|} (\partial_1\widehat{Z},\partial_{12}\widehat{Z}) \xrightarrow{\pi\circ\Omega|} \widehat{X}\right) = \left((\widehat{M};\partial\widehat{M}) \xrightarrow{\phi} (\widehat{X};\partial\widehat{X}) \xrightarrow{\mathrm{id}} \widehat{X}\right),$$

and Ω satisfies the Π-Π condition. Now $\partial_1\widehat{Z} = \widehat{X}$ and $\Omega|_{\partial_1\widehat{Z}} = \mathrm{id}$, so the inclusion of $\partial_1\widehat{Z}$ into $\widehat{Z}$ satisfies the Π-Π condition. Since $\Phi|_{\partial_2\widehat{N}}$ is a BQ-equivalence we can apply the Π-Π theorem and find a bordism (not necessarily an $\mathcal{L}$-bordism) between $\widehat{N} \xrightarrow{\Phi} \widehat{Z} \xrightarrow{\Omega} \widehat{X}\times I$ and

$$(\widehat{N}';\partial_1\widehat{N}',\partial_2\widehat{N}') \xrightarrow{\Phi'} (\widehat{Z};\partial_1\widehat{Z},\partial_2\widehat{Z}) \xrightarrow{\Omega} \widehat{X}\times I$$

with Φ' a BQ-equivalence, and

$$\left(\partial_2\widehat{N}' \xrightarrow{\Phi'|} \partial_2\widehat{Z}\right) = \left(\partial_2\widehat{N} \xrightarrow{\Phi|} \partial_2\widehat{Z}\right).$$

The bordism itself has the form

$$\widehat{L} \xrightarrow{\Psi} \widehat{Z}\times I \xrightarrow{\Gamma} \widehat{X}\times I^2$$

or, more explicitly,

$$(\widehat{L};\widehat{N},\widehat{N}',\partial_2\widehat{N}\times I,\widehat{P}) \xrightarrow{(\Psi;\Phi,\Phi',\Phi|\times I,\Psi|)} (\widehat{Z}\times I;\widehat{Z}\times\{0\},\widehat{Z}\times\{1\},\partial_2\widehat{Z}\times I,\partial_1\widehat{Z}\times I) \xrightarrow{\Omega\times\mathrm{id}} \widehat{X}\times I^2,$$

where this diagram serves to define $\widehat{P}$. Note that

$$\partial\widehat{P} = \widehat{P}\cap(\widehat{N}\cup\widehat{N}'\cup\partial_2\widehat{N}\times I) = \partial_1\widehat{N}\cup\partial_1\widehat{N}'\cup\partial_{12}\widehat{N}\times I = \widehat{M}\cup\partial_1\widehat{N}'\cup\widehat{K}\times I$$

and recall that $\partial_1\widehat{Z} = \widehat{X}$, so if we restrict this diagram to the boundary face $\widehat{P}$, we find

$$(\widehat{P};\widehat{M},\partial_1\widehat{N}',\widehat{K}\times I) \xrightarrow{(\Psi|;\phi,\Phi'|,h\times\mathrm{id})} (\widehat{X}\times I;\widehat{X}\times\{0\},\widehat{X}\times\{1\},\partial\widehat{X}\times I) \xrightarrow{\mathrm{id}} \widehat{X}\times I.$$

We recognize this as a normal bordism, relative to $h : \widehat{K} \longrightarrow \partial\widehat{X}$, (i.e., without changing h) between

$$(\widehat{M};\widehat{K}) \xrightarrow{(\phi;h)} (\widehat{X};\partial\widehat{X}) \xrightarrow{\mathrm{id}} \widehat{X} \text{ and } (\partial_1\widehat{N}';\widehat{K}) \xrightarrow{(\Phi'|;h)} (\widehat{X};\partial\widehat{X}) \xrightarrow{\mathrm{id}} \widehat{X}$$

which, since $\Phi'|$ is a BQ-equivalence, proves the theorem. □

2.8. Wall realization

In this section we follow Dovermann-Rothenberg [21, §8].

THEOREM 2.5 (Wall realization). – *Let $\widehat{X}$ be a stratified space (without boundary) and $\widehat{L} \xrightarrow{f} \widehat{X}$ a BQ-equivalence. Every element $\alpha \in \mathrm{L}_{\mathrm{BQ}}(\widehat{X}\times I)$ has a representative of the form*

$$(\widehat{W};\partial_-\widehat{W},\partial_+\widehat{W}) \xrightarrow{F} (\widehat{X}\times I;\widehat{X}\times\{0\},\widehat{X}\times\{1\}) \xrightarrow{\mathrm{id}} \widehat{X}\times I$$

with

$$\left[\partial_-\widehat{W} \xrightarrow{F|} \widehat{X}\times\{0\}\right] = \left[\widehat{L} \xrightarrow{f} \widehat{X}\right].$$

Note that this representative is an element of $\mathcal{N}_{\mathrm{BQ}}(\widehat{X} \times I)$ and that the restriction

$$\partial_+ \widehat{W} \xrightarrow{F|} \widehat{X} \times \{1\}$$

gives another Thom-Mather structure on $\widehat{X}$.

Proof. – Choose an $\mathcal{L}$-cycle representing α,

$$(\widehat{M}; \partial \widehat{M}) \xrightarrow{\phi} (\widehat{Y}; \partial \widehat{Y}) \xrightarrow{\omega} \widehat{X} \times I$$

and consider the null bordant $\mathcal{L}$-cycle obtained from f,

$$(\widehat{L} \times I; \widehat{L} \times \partial I) \xrightarrow{f \times \mathrm{id}} (\widehat{X} \times I; \widehat{X} \times \partial I) \xrightarrow{\mathrm{id}} \widehat{X} \times I.$$

Adding these $\mathcal{L}$-cycles together produces another representative of α,

$$(\widehat{M} \sqcup \widehat{L} \times I; \partial \widehat{M} \sqcup \widehat{L} \times \partial I) \xrightarrow{\phi \sqcup f \times \mathrm{id}} (\widehat{Y} \sqcup \widehat{X} \times I; \partial \widehat{Y} \sqcup \widehat{X} \times \partial I) \xrightarrow{\omega \sqcup \mathrm{id}} \widehat{X} \times I.$$

We can improve this representative using Theorem 2.2 to obtain

$$(\widehat{M}'; \partial \widehat{M} \sqcup \widehat{L} \times \partial I) \xrightarrow{\phi'} (\widehat{Y}'; \partial \widehat{Y} \sqcup \widehat{X} \times \partial I) \xrightarrow{\omega'} \widehat{X} \times I,$$

with ϕ' and ω' equal to ϕ and ω when restricted to the boundary, and with ω' satisfying the Π-Π condition. Let us write

$$\partial_1 \widehat{M}' = \partial \widehat{M} \sqcup \widehat{L} \times \{1\}, \quad \partial_2 \widehat{M}' = \widehat{L} \times \{0\} = \widehat{L}$$
$$\partial_1 \widehat{Y}' = \partial \widehat{Y} \sqcup \widehat{X} \times \{1\}, \quad \partial_2 \widehat{Y}' = \widehat{X} \times \{0\} = \widehat{X}.$$

By commutativity of

$$\begin{array}{ccc} \widehat{X} \times \{0\} & \hookrightarrow & \widehat{Y}' \\ \downarrow{\scriptstyle \omega'|} & & \downarrow{\scriptstyle \omega'} \\ \widehat{X} \times \{0\} & \hookrightarrow & \widehat{X} \times I \end{array}$$

we see that the inclusion $\partial_2 \widehat{Y}' \hookrightarrow \widehat{Y}'$ satisfies the Π-Π condition. Since moreover the map

$$\phi'| : \partial_1 \widehat{M}' \longrightarrow \partial_1 \widehat{Y}'$$

is a BQ-equivalence, we can apply Theorem 2.1, the Π-Π theorem, relative to this part of the boundary to find a BQ-equivalence

$$\widehat{M}'' \xrightarrow{\phi''} \widehat{Y}',$$

where $\partial \widehat{M}'' = \partial_1 \widehat{M}'' \sqcup \partial_2 \widehat{M}''$, with $\partial_1 \widehat{M}'' = \partial_1 \widehat{M}'$ and ϕ'' is equal to ϕ' on $\partial_1 \widehat{M}'$, and $\phi'' : \partial_2 \widehat{M}'' \longrightarrow \partial_2 \widehat{Y}''$ a BQ-equivalence. The BQ-equivalence ϕ'' is related to ϕ' by a bordism

$$\widehat{N} \xrightarrow{\Phi} \widehat{Y}' \times I,$$

where $\partial \widehat{N} = \widehat{M}' \cup \widehat{M}'' \cup (\partial_1 \widehat{M}' \times I) \cup \widehat{P}$,

$$\widehat{M}' \cap \widehat{M}'' = \emptyset, \quad \widehat{M}' \cap (\partial_1 \widehat{M}' \times I) = \partial_1 \widehat{M}' = \widehat{M}'' \cap (\partial_1 \widehat{M}' \times I), \quad \partial \widehat{P} = \partial_2 \widehat{M}' \sqcup \partial_2 \widehat{M}''.$$

The restriction of Φ to $\widehat{P}$ yields an element γ of $\mathcal{L}_{\mathrm{BQ}}(\widehat{X} \times I)$

$$(\widehat{P}; \widehat{L}, \partial_2 \widehat{M}'') \xrightarrow{\Phi|} (\partial_2 \widehat{Y}' \times I = \widehat{X} \times I; \widehat{X} \times \{0\}, \widehat{X} \times \{1\}) \xrightarrow{\mathrm{id}} \widehat{X} \times I$$

of the kind required in the statement of the theorem. (Incidentally, note that the fact that $\Phi|_{\widehat{P}}$ is BQ-normal and not a BQ-equivalence is why the bordism $\widehat{N}$ is not a null bordism for α.)

The BQ-normal map $\Phi : \widehat{N} \longrightarrow \widehat{Y}' \times I$ is a null bordism of the $\mathcal{L}$-cycle

$$\widehat{M}' \cup_{\widehat{L}} \widehat{P} \longrightarrow \widehat{Y}' \cup_{\widehat{X}} \widehat{X} \times I \longrightarrow \widehat{X} \times I,$$

which shows, by Lemma 2.3, that α and γ represent the same class in $\mathrm{L}_{\mathrm{BQ}}(\widehat{X} \times I)$. □

COROLLARY 2.6. – *Given* $[\alpha] \in \mathrm{L}_{\mathrm{BQ}}(\widehat{X} \times I)$ *and* $[\beta] \in \mathrm{S}_{\mathrm{BQ}}(\widehat{X})$ *we can use the theorem to choose representatives of the form*

$$\beta : \widehat{M} \xrightarrow{f} \widehat{X}$$

$$\alpha : (\widehat{W}; \widehat{M}, \widehat{M}') \xrightarrow{(\phi;\mathrm{id},\phi_2)} (\widehat{M} \times [0,1]; \widehat{M} \times \{0\}, \widehat{M} \times \{1\}) \xrightarrow{\mathrm{id}} \widehat{M} \times I,$$

and then the class of $f \circ \phi_2 : \widehat{M}' \longrightarrow \widehat{X}$ *in* $\mathrm{S}_{\mathrm{BQ}}(\widehat{X})$ *is well-defined and denoted* $\partial(\alpha)(\beta)$. *The map*

$$\begin{aligned} \mathrm{L}_{\mathrm{BQ}}(\widehat{X} \times I) \times \mathrm{S}_{\mathrm{BQ}}(\widehat{X}) &\longrightarrow \mathrm{S}_{\mathrm{BQ}}(\widehat{X}) \\ ([\alpha],[\beta]) &\longmapsto \partial(\alpha)(\beta) \end{aligned}$$

defines a group action of the Browder-Quinn L-group of $\widehat{X} \times I$ *on the structure set of* $\widehat{X}$.

Proof. – Since $f : \widehat{M} \longrightarrow \widehat{X}$ is a BQ-equivalence, $\mathrm{L}_{\mathrm{BQ}}(\widehat{X} \times I) = \mathrm{L}_{\mathrm{BQ}}(\widehat{M} \times I)$ and we can use the Wall representation theorem starting with the BQ-equivalence $\widehat{M} \xrightarrow{\mathrm{id}} \widehat{M}$ to represent α as above.

If we fix the representative β then any two representatives α, α' of $[\alpha]$ as above can be glued together along their common boundary and the result $\gamma \in \mathcal{L}_{\mathrm{BQ}}(M \times I)$ represents the zero element of $\mathrm{L}_{\mathrm{BQ}}(M \times I)$. It follows, from Theorem 2.4 applied to $\widehat{M} \times I$, that γ is normal bordant relative to the boundary to a BQ-equivalence. Thus $f \circ \phi_2$ and $f \circ \phi_2'$ represent the same element of $\mathrm{S}_{\mathrm{BQ}}(M)$.

If the BQ-equivalence $\beta' : \widehat{L} \xrightarrow{f'} \widehat{X}$ represents the same class as β, then there is a bordism between them

$$(\widehat{N}; \widehat{M}, \widehat{L}) \xrightarrow{(F;f,f')} (\widehat{X} \times I; \widehat{X} \times \{0\}, \widehat{X} \times \{1\}).$$

Using the theorem we can find a representative of $[\alpha]$ of the form

$$\alpha' : (\widehat{V}; \widehat{L}, \widehat{L}') \xrightarrow{(\psi:\mathrm{id},\psi_2)} (\widehat{L} \times [0,1]; \widehat{L} \times \{0\}, \widehat{L} \times \{1\}).$$

Now let us glue these, and α, together in the following order by matching the 'lower boundary' of one row with the 'upper boundary' of the following row,

$$\begin{array}{lll} (\widehat{V}; \widehat{L}, \widehat{L}') \xrightarrow{(\psi:\mathrm{id},\psi_2)} (\widehat{L} \times [0,1]; \widehat{L} \times \{0\}, \widehat{L} \times \{1\}) \xrightarrow{f'\times\mathrm{id}} \widehat{X} \times I \\ (\widehat{N}; \widehat{M}, \widehat{L}) \xrightarrow{\mathrm{id}} (\widehat{N}; \widehat{M}, \widehat{L}) \xrightarrow{(F;f,f')} \widehat{X} \times I \\ (\widehat{W}^{\mathrm{op}}; \widehat{M}', \widehat{M}) \xrightarrow{(\phi;\phi_2,\mathrm{id})} (\widehat{M} \times [0,1]; \widehat{M} \times \{0\}, \widehat{M} \times \{1\}) \xrightarrow{f\times\mathrm{id}} \widehat{X} \times I \end{array}$$

We end up with a cycle in $\mathcal{L}_{\mathrm{BQ}}(X \times I)$ with $\partial(\alpha')(\beta')$ along the upper boundary and $\partial(\alpha)(\beta)$ along the lower boundary. Moreover, this cycle is null bordant since by Lemma 2.3, it represents the class $[\alpha] + 0 - [\alpha] = 0$. It follows as in the previous case that $\partial(\alpha')(\beta')$ and $\partial(\alpha)(\beta)$ represent the same element in $\mathrm{S}_{\mathrm{BQ}}(\widehat{X})$.

Compatibility with the group operation on $\mathrm{L}_{\mathrm{BQ}}(\widehat{X} \times I)$ is easy as the operation is given by stacking normal bordisms together as in Lemma 2.3. □

COROLLARY 2.7 (Exactness part 2). – *Let $\widehat{X}$ be a stratified space (without boundary). The sequence*

$$\mathrm{N}_{\mathrm{BQ}}(\widehat{X} \times I, \widehat{X} \times \partial I) \xrightarrow{\theta} \mathrm{L}_{\mathrm{BQ}}(\widehat{X} \times I) \overset{\partial}{\dashrightarrow} \mathrm{S}_{\mathrm{BQ}}(\widehat{X}) \xrightarrow{\eta} \mathrm{N}_{\mathrm{BQ}}(\widehat{X})$$

is exact in that two elements of the L-group have the same action on the class of the identity map precisely when their difference is in the image of θ, and two elements in the structure set are in the same orbit precisely when they have the same image under η.

Proof. – Given $x_1, x_2 \in \mathrm{L}_{\mathrm{BQ}}(\widehat{X} \times I)$ such that $\partial(x_1)(\mathrm{id}) = \partial(x_2)(\mathrm{id})$ we can use the Wall representation theorem to find

$$(\widehat{W}_i; \widehat{X}, \partial_+ \widehat{W}_i) \xrightarrow{F_i} (\widehat{X} \times I; \widehat{X} \times \{0\}, \widehat{X} \times \{1\}) \xrightarrow{\mathrm{id}} \widehat{X} \times I$$

representing x_i. Without loss of generality $\partial_+ \widehat{W}_1 \xrightarrow{F_1} \widehat{X}$ and $\partial_+ \widehat{W}_2 \xrightarrow{F_2} \widehat{X}$ can be taken to be the same representative of $\partial(x_1)(\mathrm{id})$, so that we may form

$$(\widehat{W}_1 \sqcup_{\partial_+ \widehat{W}_i} -\widehat{W}_2; \widehat{X}, \widehat{X}) \xrightarrow{F_1 \sqcup F_2} (\widehat{X} \times I; \widehat{X} \times \{0\}, \widehat{X} \times \{1\}) \xrightarrow{\mathrm{id}} \widehat{X} \times I$$

and recognize this as a representative of a class in $\mathrm{N}_{\mathrm{BQ}}(\widehat{X} \times I, \widehat{X} \times \partial I)$ representing $x_1 - x_2$. The converse follows by similar reasoning.

If $[\beta], [\beta'] \in \mathrm{S}_{\mathrm{BQ}}(\widehat{X})$ have the same image under η then there is a normal bordism α between β and β'. This normal bordism defines an element of $\mathrm{L}_{\mathrm{BQ}}(\widehat{X} \times I)$ whose action on $\mathrm{S}_{\mathrm{BQ}}(\widehat{X})$ sends $[\beta]$ to $[\beta']$. For the same reason elements in the structure set that are in an orbit of the action of an element of the L-group have the same image under η. □

3. K-theory classes associated to the signature operator on Witt and Cheeger spaces

3.1. Metric structures

In order to do analysis we endow a stratified space with a Riemannian metric. Let $\widehat{X}$ be a smoothly stratified space and $\widetilde{X}$ its resolution to a manifold with corners and an iterated fibration structure.

Recall from, e.g., [3, 5, 4], that an *iterated incomplete edge metric* (briefly, an iie-metric) is a Riemannian metric on the interior of $\widetilde{X}$ (or, better, a bundle metric on the iterated incomplete edge tangent bundle over all of $\widetilde{X}$) that in a collar neighborhood of each collective boundary hypersurface $\mathfrak{B}_Y$ takes the form

$$dx^2 + x^2 g_Z + \phi_Y^* g_Y.$$

Here x is a boundary defining function for $\mathfrak{B}_Y$, i.e., a smooth non-negative function on $\widetilde{X}$ that is positive except at $\mathfrak{B}_Y = \{x = 0\}$ where it vanishes to exactly first order, and g_Z

and g_Y are metrics with the same structure on the spaces Z and Y. (Thus this is really an inductive definition over the depth of a stratified space, with spaces of depth zero being assigned smooth Riemannian metrics, see *loc. cit.*.)

In particular we point out that an iie-metric on $\widehat{X}$ includes a Riemannian metric on each stratum of $\widehat{X}$ and that these metrics fit together continuously (even smoothly in that they lift to a smooth section over $\widetilde{X}$). Thus endowing $\widehat{X}$ with an iie-metric in particular gives $\widehat{X}$ the structure of a 'Riemannian Whitney (A) space' in the sense of Pflaum [43, §2.4]. (Note that the latter concept is more general, e.g., if we were working with metrics that were asymptotically of the form $dx^2 + x^{2\ell} g_Z + \phi_Y^* g_Y$ for any $\ell > 0$ we would still get a 'Riemannian Whitney (A) space'.) In particular, by Theorem 2.4.7 in [43], the topology on $\widehat{X}$ is that of the metric space with distance between two points given by taking the infimum over rectifiable curves joining them. As a metric space, $\widehat{X}$ is complete and locally compact [43, Theorem 2.4.17] and hence a 'length space'.

3.2. Galois coverings

Let $\widehat{X}$ be a smoothly stratified pseudomanifold of arbitrary depth. Consider a Galois covering $\pi : \widehat{X}_\Gamma \to \widehat{X}$ with Galois group Γ and fundamental domain $\widehat{\mathscr{F}}_\Gamma$. There is a natural way to define a topological stratification on $\widehat{X}_\Gamma$. Decompose $\widehat{X}_\Gamma$ into the preimages under π of the strata in $\widehat{X}$. Surjectivity of π ensures that each stratum in the covering is mapped surjectively onto the corresponding stratum in $\widehat{X}$. Since π is a local homeomorphism, it is straightforward to check that $\widehat{X}_\Gamma$ and its fundamental domain are again topological stratified spaces.

In fact, more is true: by using these local homeomorphisms we can induce a *smooth* stratification on $\widehat{X}_\Gamma$ by simply pulling it up from the base, in either the Whitney as well as the Thom-Mather cases. It is important to point out that, by definition, the link of a point $\widetilde{p} \in \widehat{X}_\Gamma$ is equal to the link of its image, p, in the base. This construction exhibits the covering map π as a transverse map and thus *if $\widehat{X}$ belongs to a class $\mathscr{C}$ as above, then so does $\widehat{X}_\Gamma$*.

Needless to say, if $\widehat{X}_\Gamma$ is the universal covering space of $\widehat{X}$, the individual strata in $\widehat{X}_\Gamma$ need not be the universal covering of the corresponding strata in the base. We denote by X_Γ the regular stratum of $\widehat{X}_\Gamma$ and observe that it is a Galois covering of the regular stratum X of $\widehat{X}$ with fundamental domain $\mathscr{F}_\Gamma$ equal to the regular part of $\widehat{\mathscr{F}}_\Gamma$. Let g be an admissible incomplete edge metric on X. We can lift g to the Galois covering X_Γ where it becomes a Γ-invariant admissible incomplete edge metric $\widetilde{g}$. Moreover, there is an isometric embedding of $\mathscr{F}_\Gamma$ into X with complement of measure zero. We denote by D_Γ the signature operator on X_Γ associated to such a metric.

3.3. C^* and D^* algebras

First of all, we need to fix an Hilbert space H with a unitary action of Γ and a C^*-representation from $C_0(\widehat{X}_\Gamma)$ to $\mathscr{B}(H)$ intertwining the two actions of Γ. Notice that the representation is associated to the *stratified* Galois covering $\widehat{X}_\Gamma$ (and not to its regular part X_Γ). We take $H = L^2(X_\Gamma, \Lambda^* X_\Gamma)$; the representation is given by the multiplication operator associated to the restriction of a function to the regular part X_Γ. To these data we can associate two C^*-algebras: the Roe algebra $C^*(\widehat{X}_\Gamma, H)^\Gamma$, obtained as the closure of the Γ-equivariant finite propagation locally compact bounded operators on H, and the Higson-Roe algebra $D^*(\widehat{X}_\Gamma, H)^\Gamma$, obtained as the closure of the Γ-equivariant finite propagation pseudolocal bounded operators on H. Since we shall be eventually interested in the K-theory groups of these C^*-algebras and since the K-theory groups are independent of the choice of the (adequate) Γ equivariant $C_c(\widehat{X}_\Gamma)$-module H, we shall adopt the notation $C^*(\widehat{X}_\Gamma)^\Gamma$ and $D^*(\widehat{X}_\Gamma)^\Gamma$ for these two C^*-algebras. (2)

We shall also use the universal versions of these algebras, defined as

$$C^*_\Gamma := C^*(E\Gamma)^\Gamma, \quad D^*_\Gamma := D^*(E\Gamma)^\Gamma. \tag{3.1}$$

See for example [45, Definition 2.19].

We have the following fundamental

Proposition 3.1. – *Let $(\widehat{X}, g)$ and $(\widehat{X}_\Gamma, \widetilde{g})$ as above. Assume that $\widehat{X}$ is a Cheeger-space and let $\mathcal{W}$ be a self-dual mezzoperversity for D. Then:*

(1) *there exists a closed Γ-equivariant self adjoint extension of D_Γ associated to $\mathcal{W}$, denoted $D^{\mathcal{W}}_\Gamma$;*

(2) *if $\phi \in C_0(\mathbb{R})$, then $\phi(D^{\mathcal{W}}_\Gamma) \in C^*(\widehat{X}_\Gamma)^\Gamma$.*

(3) *if χ is a chopping function (i.e., $\chi : \mathbb{R} \to [-1, 1]$ is odd and $\lim_{x\to\pm\infty} \chi(x) = \pm 1$); then $\chi(D^{\mathcal{W}}_\Gamma) \in D^*(\widehat{X}_\Gamma)^\Gamma$.*

Proof. – (1) The pull-back of the mezzoperversity $\mathcal{W}$ along the covering map π is a mezzoperversity on X_Γ, which we briefly denote $\mathcal{W}_\Gamma$. The definition of the domain associated to a mezzoperversity in [5] applies in the setting of X_Γ, as the asymptotic expansions on which it relies are carried out in distinguished neighborhoods of points on the singular strata and these are the same on $\widehat{X}$ or $\widehat{X}_\Gamma$. Similarly we can define $\mathcal{D}_{\mathcal{W}_\Gamma}(d)$ and $\mathcal{D}_{\mathcal{W}_\Gamma}(\delta)$ as in [5] and see that they are mutually adjoint and that

$$\mathcal{D}_{\mathcal{W}_\Gamma}(D^{\mathcal{W}}_\Gamma) = \mathcal{D}_{\mathcal{W}_\Gamma}(d) \cap \mathcal{D}_{\mathcal{W}_\Gamma}(\delta)$$

so that $D^{\mathcal{W}}_\Gamma$ with this domain is self-adjoint.

The analysis of [5] that establishes that $\mathcal{D}_{\mathcal{W}}(D)$ includes compactly into $L^2(X; \Lambda^* X)$ implies that, for any compact subset $K \subseteq \widehat{X}_\Gamma$,

$$\{u \in \mathcal{D}_{\mathcal{W}_\Gamma}(D^{\mathcal{W}}_\Gamma) : \mathrm{supp}(u) \subseteq K\}$$

includes compactly into H. See [5, Section 5.1].

(2) As far as the second item is concerned we initially tackle the local compactness of $\phi(D^{\mathcal{W}}_\Gamma)$. We have to prove that if $g \in C_c(\widehat{X}_\Gamma)$ then $g\phi(D^{\mathcal{W}}_\Gamma)$ and $\phi(D^{\mathcal{W}}_\Gamma)g$ are compact operators. By taking adjoints it suffices to see that $g\phi(D^{\mathcal{W}}_\Gamma)$ is compact. Using the Stone-Weierstrass theorem it suffices to establish this property for the function $\phi(x) = (i + x)^{-1}$. As this maps H into $\mathcal{D}_{\mathcal{W}_\Gamma}(D^{\mathcal{W}}_\Gamma)$, the local compactness of the inclusion of the latter into H implies that of $\phi(D^{\mathcal{W}}_\Gamma)$.

Next we consider the finite propagation property: by a density argument it suffices to see such a property for smooth functions ϕ that are of rapid decay and have compactly supported Fourier transform. Thus, let $\widehat{\phi}$ be the Fourier transform of a smooth rapidly decaying ϕ and

(2) For technical reasons having to do with functoriality one actually takes $H = L^2(X_\Gamma, \Lambda^* X_\Gamma) \otimes \ell^2(\mathbb{N})$.

let us assume that the support of $\widehat{\phi}$ is contained in $[-R/2, R/2]$. We must show that there exists $S \in \mathbb{R}^+$ such that $f\phi(D_\Gamma^{\mathcal{W}})g = 0$ whenever the distance between the support of f and g is greater than S. Proceeding precisely as in [4, Theorem 5.3] we know that there exists a δ such that $\exp(isD_\Gamma^{\mathcal{W}})$ has propagation $|s|$ if $|s| < \delta$; thus $f \exp(isD_\Gamma^{\mathcal{W}})g = 0$ if the distance of the supports of f and g is greater than $|s|$, with s in the range $(-\delta, \delta)$. Recall now that, by functional calculus, we can write

$$\phi(D_\Gamma^{\mathcal{W}}) = \frac{1}{2\pi}\int \exp(isD_\Gamma^{\mathcal{W}})\widehat{\phi}(s)ds$$

where the integral converges weakly:

$$\langle \phi(D_\Gamma^{\mathcal{W}})u, v\rangle = \frac{1}{2\pi}\int \langle \exp(isD_\Gamma^{\mathcal{W}})u, v\rangle\widehat{\phi}(s)ds$$

for each pair of compactly supported sections on X_Γ. Assume initially that $R < \delta$. Then, from the above integral representation, we see that $\phi(D_\Gamma^{\mathcal{W}})$ has finite propagation (in fact, propagation R) which is what we wanted to prove. For the general case we use a trick from [52]. Write $\phi = \sum_j f_j$ where the sum is finite and where f_j has Fourier transform supported in $(T_j - \delta/2, T_j + \delta/2)$. Consider $g_j(x) = \exp(-iT_j x) f_j(x)$. Then $g_j(D_\Gamma^{\mathcal{W}})$ has propagation δ by what we have just seen. Write now

$$\exp(iT_j x) = \prod_{\ell=1}^{k} \exp(i\tau_\ell x) \quad \text{with} \quad |\tau_\ell| < \delta.$$

We have then $f_j(x) = \prod_{\ell=1}^{k} \exp(i\tau_\ell x) g_j(x)$ and thus

$$f_j(D_\Gamma^{\mathcal{W}}) = \exp(i\tau_1 D_\Gamma^{\mathcal{W}}) \circ \cdots \circ \exp(i\tau_k D_\Gamma^{\mathcal{W}}) \circ g_j(D_\Gamma^{\mathcal{W}}).$$

All the operators appearing on the right hand side have finite propagations and we know that the composition of two operators of finite propagation is again of finite propagation. Thus $f_j(D_\Gamma^{\mathcal{W}})$ has finite propagation. The proof of item 2 is complete.

(3) Let χ be a chopping function. Recall from [26, Section 10.6] that for every $t > 0$ there exists a chopping function χ with (distributional) Fourier transform supported in $(-t, t)$. Moreover, if χ_0 and χ are two arbitrary chopping functions, then $\chi_0 - \chi = \phi$, with $\phi \in C_0(\mathbb{R})$. This implies immediately that if $\chi_0(D_\Gamma^{\mathcal{W}})$ is of finite propagation then $\chi_1(D_\Gamma^{\mathcal{W}})$ is a limit of finite propagation operators; moreover $\chi_0(D_\Gamma^{\mathcal{W}})g - \chi(D_\Gamma^{\mathcal{W}})g$ is a compact operator for any $g \in C_c(\widehat{X}_\Gamma)$. We choose a chopping function χ_0 with Fourier transform supported in $(-\delta/2, \delta/2)$. Then we know, from the previous arguments, that $\chi_0(D_\Gamma^{\mathcal{W}})$ is of propagation δ. Hence $\chi(D_\Gamma^{\mathcal{W}})$ is a limit of finite propagation operators for each chopping function χ. It remains to see that $\chi(D_\Gamma^{\mathcal{W}})$ is pseudolocal, i.e., $[f, \chi(D_\Gamma^{\mathcal{W}})]$ is compact for any $f \in C_0(\widehat{X}_\Gamma)$. By Kasparov's Lemma, see [26, Lemma 5.4.7] and [26, Lemma 10.6.4], we know that $\chi(D_\Gamma^{\mathcal{W}})$ is pseudolocal if and only if $f\chi(D_\Gamma^{\mathcal{W}})g$ is compact for any choice of $f \in C(\widehat{X}_\Gamma)$ bounded and $g \in C_c(\widehat{X}_\Gamma)$ with disjoint supports. Now, if $\eta > 0$ is the distance between the support of f and the support of g and if we choose a chopping function χ_0 with Fourier transform supported in $(-\eta/2, \eta/2)$ then we know that $f\chi_0(D_\Gamma^{\mathcal{W}})g = 0$. But then for an arbitrary chopping function χ we have

$$f\chi(D_\Gamma^{\mathcal{W}})g = f\chi_0(D_\Gamma^{\mathcal{W}})g + f(\chi(D_\Gamma^{\mathcal{W}}) - \chi_0(D_\Gamma^{\mathcal{W}}))g = 0 + f(\chi(D_\Gamma^{\mathcal{W}}) - \chi_0(D_\Gamma^{\mathcal{W}}))g$$

and since f is bounded and $(\chi(D^{\mathcal{W}}_{\Gamma}) - \chi_0(D^{\mathcal{W}}_{\Gamma}))g$ is compact we see that on the right hand side we do have a compact operator as required. The proof of item 3 is now complete. □

3.4. K-homology classes

Proposition 3.1 allows us to recover, in the bounded picture, the fundamental classes that were defined in [3, Theorem 6.2] for Witt spaces and in [4, Theorem 5.3] for Cheeger spaces. More precisely:

PROPOSITION 3.2. – *If $\widehat{X}$ is an n-dimensional Cheeger space endowed with a rigid iterated conic metric g and if $\mathcal{W}$ is a self-dual mezzoperversity adapted to g then there is a well defined K-homology signature class $[D^{\mathcal{W}}] \in K_n(\widehat{X})$.*

Proof. – Let χ be a chopping function; then χ^2-1 is an element in $C_0(\mathbb{R})$ and thus $\chi(D^{\mathcal{W}}_{\Gamma})$ is an involution in the quotient $D^*(\widehat{X}_\Gamma)^\Gamma/C^*(\widehat{X}_\Gamma)^\Gamma$. Thus, using also the grading in even dimension, one defines an element in $K_{n+1}(D^*(\widehat{X}_\Gamma)^\Gamma/C^*(\widehat{X}_\Gamma)^\Gamma)$ which is precisely $K_n(\widehat{X})$ by Paschke duality. □

REMARK 6. – *The class we have just defined does coincide with the one defined in Theorem 5.3 in* [4]*: this follows from the proof of* [26, Theorem 10.6.5] *and the correspondence between the unbounded and bounded picture for K-homology.*

REMARK 7. – *The class $[D^{\mathcal{W}}]_{\mathbb{Q}} \in K_n(\widehat{X}) \otimes \mathbb{Q}$ is independent of the choice of self-dual mezzoperversity $\mathcal{W}$; indeed the homological Chern character of $[D^{\mathcal{W}}]_{\mathbb{Q}}$, in $H_*(\widehat{X},\mathbb{Q})$ is equal to the homology L-class of the Cheeger space, see* [4, Thorem 5.6]*, and we know that the L-class is independent of the choice of $\mathcal{W}$, see* [4, Section 5.1]*. We shall come back to this point later on.*

3.5. Higson-Roe sequences associated to a Thom-Mather space

If $\widehat{X}$ is a Thom-Mather stratified space and $\widehat{X}_\Gamma$ is a Galois covering with structure group Γ, then there is a short exact sequence of C^*-algebras

$$0 \to C^*(\widehat{X}_\Gamma)^\Gamma \to D^*(\widehat{X}_\Gamma)^\Gamma \to D^*(\widehat{X}_\Gamma)^\Gamma/C^*(\widehat{X}_\Gamma)^\Gamma \to 0$$

and thus a 6-term long exact sequence in K-theory:

(3.2)

$$\cdots \to K_{m+1}(C^*(\widehat{X}_\Gamma)^\Gamma) \to K_{m+1}(D^*(\widehat{X}_\Gamma)^\Gamma) \to K_{m+1}(D^*(\widehat{X}_\Gamma)^\Gamma/C^*(\widehat{X}_\Gamma)^\Gamma) \to K_m(C^*(\widehat{X}_\Gamma)^\Gamma) \to \cdots .$$

This is the *analytic surgery sequence* of Higson and Roe associated to the Γ-compact Γ-space $\widehat{X}_\Gamma$. Since we have the canonical isomorphism $K_{*+1}(D^*(\widehat{X}_\Gamma)^\Gamma/C^*(\widehat{X}_\Gamma)^\Gamma) = K_*(\widehat{X})$ we can also rewrite (3.2) as

$$\cdots \to K_{m+1}(C^*(\widehat{X}_\Gamma)^\Gamma) \to K_{m+1}(D^*(\widehat{X}_\Gamma)^\Gamma) \to K_m(\widehat{X}) \to K_m(C^*(\widehat{X}_\Gamma)^\Gamma) \to \cdots . \tag{3.3}$$

Moreover, since $\widehat{X}$ is compact, we recall that there exists a canonical isomorphism

$$K_*(C^*(\widehat{X}_\Gamma)^\Gamma) = K_*(C^*_r\Gamma).$$

Now, in particular, all of the above is true with $\widehat{X}_\Gamma$ equal to the universal covering of $\widehat{X}$ and $\Gamma = \pi_1(\widehat{X})$. Recall from §2.2 that the closure of a stratum Y in $\widehat{X}$ is itself a stratified space, equal to

$$\widehat{Y} = \bigcup \{Y_i : Y_i \leq Y\}.$$

These are often referred to as the 'closed strata'. Consider $\Gamma(\widehat{Y}) := \pi_1(\widehat{Y})$; then we have a 6-term exact sequence similar to (3.3) but associated to the universal covering, $\Gamma(\widehat{Y}) - \widehat{Y}_{\Gamma(\widehat{Y})} \to \widehat{Y}$ of $\widehat{Y}$.

3.6. Index classes

Let now $\widehat{X}$ be an n-dimensional Cheeger space and let us choose a self-dual mezzoperversity $\mathcal{W}$. Then, by Proposition (3.2), we have a K-homology class $[D^{\mathcal{W}}] \in K_n(\widehat{X}) = K_{n+1}(D^*(\widehat{X}_\Gamma)^\Gamma / C^*(\widehat{X}_\Gamma)^\Gamma)$ and thus an index class

$$\operatorname{Ind}(D^{\mathcal{W}}_\Gamma) := \partial[D^{\mathcal{W}}] \in K_n(C^*(\widehat{X}_\Gamma)^\Gamma), \tag{3.4}$$

with ∂ the connecting homomorphism in the Higson-Roe surgery sequence. Following the proof given in [45, Proposition 2.1] this class corresponds to the one considered in [4] through the canonical isomorphism $K_*(C^*(\widehat{X}_\Gamma)^\Gamma) = K_*(C^*_r\Gamma)$; notice that the class defined in [4] is a Mishchenko class, obtained by twisting the signature operator by the Mishchenko bundle $\widehat{X}_\Gamma \times_\Gamma C^*_r\Gamma$.

Both in the Higson-Roe formalism, see [30], and in the Mishchenko formalism, we can also consider the index class with values in the maximal version of our C^*-algebras. We denote the maximal group C^*-algebra associated to Γ as $C^*\Gamma$.

Finally, if $\widehat{Y}$ is a closed m-dimensional stratum with fundamental group $\Gamma(\widehat{Y})$ then $\mathcal{W}$ induces a self-dual mezzoperversity $\mathcal{W}(\widehat{Y})$ for $\widehat{Y}$ and we obtain a K-homology class $[D^{\mathcal{W}(\widehat{Y})}] \in K_m(\widehat{Y})$ and thus an Index class $\operatorname{Ind}(D^{\mathcal{W}(\widehat{Y})}_{\Gamma(\widehat{Y})}) \in K_m(C^*_r(\Gamma(\widehat{Y}))$.

3.7. Rho classes associated to trivializing perturbations

Let $\widehat{X}$ be a Cheeger space endowed with an iie metric g. We initially assume that $\widehat{X}$ is odd dimensional. Let $\mathcal{W}$ be a self-dual mezzoperversity for $\widehat{X}$ and let $D^{\mathcal{W}}$ be the corresponding signature operator, an unbounded self-adjoint operator on $L^2(X, \Lambda^* X)$. (Recall that $D^{\mathcal{W}}$ is a short notation for the pair $(D, \mathcal{D}_{\mathcal{W}}(D))$, the (extension of the) signature operator on (X, g), the regular part of $\widehat{X}$ endowed with the Riemannian metric g, with domain defined by the self-dual mezzoperversity $\mathcal{W}$.) Given a Galois Γ-covering $\widehat{X}_\Gamma$ of $\widehat{X}$, we also have the Γ-equivariant signature operator $D^{\mathcal{W}}_\Gamma$, a self-adjoint unbounded operator on $L^2(X_\Gamma, \Lambda^* X_\Gamma)$. Let now A be a bounded Γ-equivariant self-adjoint operator on $L^2(X_\Gamma, \Lambda^* X_\Gamma)$. Then $D^{\mathcal{W}}_\Gamma + A$, with domain equal to the domain of $D^{\mathcal{W}}_\Gamma$, is also self-adjoint. Following [46, Section 2B] we make the assumption that $D^{\mathcal{W}}_\Gamma + A$ is L^2-invertible and that $A \in \mathfrak{M}(C^*(\widehat{X}_\Gamma)^\Gamma)$, the multiplier algebra of $C^*(\widehat{X}_\Gamma)^\Gamma$. We refer to A as a *trivializing perturbation*. Then, using Proposition 3.1 and [46, Proposition 2.8], we see that

$$\frac{D^{\mathcal{W}}_\Gamma + A}{|D^{\mathcal{W}}_\Gamma + A|} \text{ is an element in } D^*(\widehat{X}_\Gamma)^\Gamma. \tag{3.5}$$

Moreover, $D^{\mathcal{W}}_{\Gamma} + A/|D^{\mathcal{W}}_{\Gamma} + A|$ is clearly an involution and thus

$$\frac{1}{2}\left(\frac{D^{\mathcal{W}}_{\Gamma} + A}{|D^{\mathcal{W}}_{\Gamma} + A|} + 1\right)$$

is a projection in $D^*(\widehat{X}_\Gamma)^\Gamma$. We define the rho class associated to $D^{\mathcal{W}}_{\Gamma} + A$ as

$$\rho(D^{\mathcal{W}}_{\Gamma} + A) := \left[\frac{1}{2}\left(\frac{D^{\mathcal{W}}_{\Gamma} + A}{|D^{\mathcal{W}}_{\Gamma} + A|} + 1\right)\right] \text{ in } K_0(D^*(\widehat{X}_\Gamma)^\Gamma). \tag{3.6}$$

In the even dimensional case we consider the grading associated to the Hodge $\star$ operator and we demand that the trivializing perturbation $A \in \mathfrak{M}(C^*(\widehat{X}_\Gamma)^\Gamma)$ be odd with respect to this grading; thus $D^{\mathcal{W}}_{\Gamma} + A$ can be written as

$$\begin{pmatrix} 0 & D^{\mathcal{W},-}_{\Gamma} + A^- \\ D^{\mathcal{W},+}_{\Gamma} + A^+ & 0 \end{pmatrix}.$$

We now fix a chopping function χ equal to the sign function on the spectrum of the invertible operator $D^{\mathcal{W}}_{\Gamma} + A$; we also fix a Γ-equivariant isometry $u : \Lambda_-({}^{\mathrm{iie}}T^*X_\Gamma) \to \Lambda_+({}^{\mathrm{iie}}T^*X_\Gamma)$ and consider the induced bounded Γ-equivariant operator on the space of L^2 sections of these bundles, call it U. Observing that $\chi(D^{\mathcal{W}}_{\Gamma} + A)$ is also odd (see [26, Lemma 10.6.2]) we consider $U\chi(D^{\mathcal{W}}_{\Gamma} + A)_+$ which is a unitary in $D^*(\widehat{X}_\Gamma)^\Gamma$. We then define the rho class in the even dimensional case as

$$\rho(D^{\mathcal{W}}_{\Gamma} + A) := [U\chi(D^{\mathcal{W}}_{\Gamma} + A)_+] \text{ in } K_1(D^*(\widehat{X}_\Gamma)^\Gamma). \tag{3.7}$$

As explained in [46, page 118] this is well defined, independent of the choice of u.

4. Bordisms and associated K-theory classes

4.1. Bordisms of Cheeger spaces

We recall here some fundamental facts established in [4]. Assume that M is a topological space and denote by $\mathrm{Sig}_n(M)$ the bordism group of four-tuples $(\widehat{X}, g, \mathcal{W}, r : \widehat{X} \longrightarrow M)$ where $\widehat{X}$ is an oriented Cheeger space of dimension n, g is an adapted iterated incomplete conic metric (briefly an iie metric), $\mathcal{W}$ is a self-dual Hodge mezzoperversity adapted to g and $r : \widehat{X} \longrightarrow M$ is a continuous map. An admissible bordism between $(\widehat{X}, g, \mathcal{W}, r : \widehat{X} \longrightarrow M)$ and $(\widehat{X}', g', \mathcal{W}', r' : \widehat{X}' \longrightarrow M)$ is a four-tuple $(\widehat{\mathscr{X}}, G, \mathscr{W}, R : \widehat{\mathscr{X}} \longrightarrow M)$ consisting of:

(i) a smoothly stratified, oriented, compact pseudomanifold with boundary $\widehat{\mathscr{X}}$, whose boundary is $\widehat{X} \sqcup (-\widehat{X}')$, and whose strata near the boundary are collars of the strata of $\widehat{X}$ or $\widehat{X}'$,

(ii) an iie metric G on $\widehat{\mathscr{X}}$ that near the boundary is of the collared form $dx^2 + g$ or $dx^2 + g'$,

(iii) an adapted self-dual mezzoperversity $\mathscr{W}$ that extends, in a collared way, that of $\widehat{X}$ and $\widehat{X}'$,

(iv) a map $R : \widehat{\mathscr{X}} \longrightarrow M$ that extends r and r'.

We shall briefly say that $(\widehat{X}, g, \mathcal{W}, r : \widehat{X} \longrightarrow M)$ and $(\widehat{X}', g', \mathcal{W}', r' : \widehat{X}' \longrightarrow M)$ are Cheeger-bordant through $(\widehat{\mathscr{X}}, G, \mathscr{W}, R : \widehat{\mathscr{X}} \longrightarrow M)$. We are mainly interested in the case $M = B\Gamma$, so that a map $r : \widehat{X} \to B\Gamma$ defines a Galois Γ-covering $\widehat{X}_\Gamma$. We have the following important results, see [4, Sections 5.3 and 5.4] for proofs:

THEOREM 4.1. – *If $(\widehat{X}, g, \mathcal{W}, r)$ and $(\widehat{X}', g', \mathcal{W}', r')$ are n-dimensional and Cheeger-bordant through $(\widehat{\mathscr{X}}, G, \mathscr{W}, R : \widehat{\mathscr{X}} \longrightarrow M)$ then:*

1] *the numeric Fredholm indices associated to $D^{\mathcal{W}}$ and $(D')^{\mathcal{W}'}$ are equal;*

2] *there exists a well defined relative K-homology class $[D^{\mathscr{W}}] \in K_{n+1}(\widehat{\mathscr{X}}, \partial\widehat{\mathscr{X}})$;*

3] *if $\partial : K_{n+1}(\widehat{\mathscr{X}}, \partial\widehat{\mathscr{X}}) \to K_n(\partial\widehat{\mathscr{X}}) \equiv K_n(\widehat{X} \cup (-\widehat{X}'))$ is the connecting homomorphism associated to the long exact sequence of the pair $(\widehat{\mathscr{X}}, \partial\widehat{\mathscr{X}})$ then*

$$\partial[D^{\mathscr{W}}] = [D^{\mathcal{W}}] - [(D')^{\mathcal{W}'}] \quad \textit{in} \quad K_n(\partial\widehat{\mathscr{X}}) \otimes_{\mathbb{Z}} \mathbb{Z}[\frac{1}{2}];$$

4] *the signature index classes associated to $(\widehat{X}, g, \mathcal{W}, r : \widehat{X} \longrightarrow B\Gamma)$ and $(\widehat{X}', g', \mathcal{W}', r' : \widehat{X}' \longrightarrow B\Gamma)$ are equal in $K_*(C_r^*\Gamma) \otimes_{\mathbb{Z}} \mathbb{Z}[\frac{1}{2}]$.*

5] *If $\mathcal{W}$ and $\mathcal{W}'$ are adapted to g and g' on the same Cheeger space $\widehat{X}$ and $r : X \to M$ is a continuous map then $(\widehat{X}, g, \mathcal{W}, r : \widehat{X} \to M)$ is Cheeger-bordant to $(\widehat{X}, g', \mathcal{W}', r : \widehat{X} \to M)$. In particular, the numeric Fredholm index, in $\mathbb{Z}$, and the signature index class, in $K_*(C_r^*\Gamma) \otimes_{\mathbb{Z}} \mathbb{Z}[\frac{1}{2}]$, are independent of the choice of self-dual mezzoperversity.*

REMARK 8. – *The statements in* [4] *are given with values in $K_0(C_r^*\Gamma) \otimes_{\mathbb{Z}} \mathbb{Q}$ but it is easy to see that the arguments given there establish the same results in $K_0(C_r^*\Gamma) \otimes_{\mathbb{Z}} \mathbb{Z}[\frac{1}{2}]$.*

The main idea behind the formulation and the proof of item 5] is due to Markus Banagl, see [9].

Notation. – Since the signature index class $\mathrm{Ind}(D_\Gamma^{\mathcal{W}}) \in K_*(C_r^*\Gamma) \otimes_{\mathbb{Z}} \mathbb{Z}[\frac{1}{2}]$ associated to a Galois covering $r : \widehat{X} \to B\Gamma$ is in fact independent of the choice of $\mathcal{W}$, we shall often denote it simply by $\mathrm{Ind}(D_\Gamma)$ or even $\mathrm{Ind}(\widehat{X}_\Gamma)$.

4.2. The signature operator on Cheeger spaces with cylindrical ends

Let $\Gamma - \widehat{\mathscr{X}_\Gamma} \to \widehat{\mathscr{X}}$ be a Galois Γ-covering of an even dimensional Cheeger space with boundary. We consider a rigid iie metric g on the regular part $\mathscr{X}$ which is collared near $\partial\mathscr{X}$ and we lift it to a Γ-equivariant rigid iie metric on $\mathscr{X}_\Gamma$. We also consider the Cheeger spaces with cylindrical ends, $\widehat{\mathscr{X}_\infty}$, $\widehat{\mathscr{X}_{\Gamma,\infty}}$, obtained by attaching $(-\infty, 0] \times \partial\widehat{\mathscr{X}}$ and $(-\infty, 0] \times \partial\widehat{\mathscr{X}_\Gamma}$ to $\widehat{\mathscr{X}}$, $\widehat{\mathscr{X}_\Gamma}$ respectively. We endow $(-\infty, 0] \times \partial\mathscr{X}$ and $(-\infty, 0] \times \partial\mathscr{X}_\Gamma$ with product metrics and we obtain in this way global metrics on $\mathscr{X}_\infty$ and $\mathscr{X}_{\infty,\Gamma}$. If $\mathscr{W}$ is a self-dual mezzoperversity on $\mathscr{X}$ then we obtain in a natural way a self-dual mezzoperversity $\mathscr{W}_\infty$ on X_∞ and thus, by lifting, a Γ-equivariant self-dual mezzoperversity $\mathscr{W}_{\Gamma,\infty}$ on $X_{\infty,\Gamma}$. We denote by $\partial\mathscr{W}$ the self-dual mezzoperversity induced on the boundary, see Theorem 4.1. Finally, we denote by P_0 the multiplication operator by the characteristic function of the attached semi-cylinder.

PROPOSITION 4.2. – *Let D_∞ and $D_{\infty,\Gamma}$ be the the signature operators on X_∞ and $X_{\infty,\Gamma}$ respectively. By employing $\mathscr{W}_\infty$ and $\mathscr{W}_{\Gamma,\infty}$ we can define self-adjoint extensions $D_\infty^{\mathscr{W}}$ and $D_{\infty,\Gamma}^{\mathscr{W}}$.*

Proof. – Extend the iterated fibration structure from X to X_∞ by including the cylindrical direction in the base of each fiber bundle. Define

$$D^{\mathscr{W}}_\infty = (D_\infty, \mathcal{D}_{\mathscr{W}_\infty}(D_\infty)),\ \text{ where } \mathcal{D}_{\mathscr{W},\infty}(D) = \{u \in \mathscr{D}_{\max}(D_\infty) :\ \text{at each singular stratum, } u \text{ satisfies the ideal boundary condition corresponding to } \mathscr{W}_\infty\}.$$

It is easy to see that this is a self-adjoint domain. Indeed, this domain is localizable (an element is in the domain if and only if it is in the domain after multiplying by any function in $\mathcal{C}^\infty_\Phi(X)$, see [5, §2] and, e.g., the discussion after assumption 3.8 in [5]) and so it suffices to show that the corresponding domain on the full cylinder $\partial X \times \mathbb{R}$ is self-adjoint; here we could either use Fourier transform in the $\mathbb{R}$-factor to reduce to the self-adjointness of $D^{\partial\mathscr{W}}$, or alternately recognize $\partial X \times \mathbb{R}$ as a cover of $\partial X \times \mathbb{S}^1$, consider the pull-back of $\partial\mathscr{W}$ to this product, and then appeal to Proposition 3.1 above. □

4.3. Perturbations and coarse APS-index classes

In this subsection we shall define APS-index classes associated to the self-adjoint operator $D^{\mathscr{W}}_{\infty,\Gamma}$. First we recall, for example from [45, Definition 1.7], the definition of relative C^*-algebra .

Definition 5. – *The subalgebra*

$$C^*_c(\widehat{\mathscr{X}_\Gamma} \subset \widehat{\mathscr{X}_{\Gamma,\infty}})^\Gamma \subset C^*_c(\widehat{\mathscr{X}_{\Gamma,\infty}})^\Gamma$$

is defined by imposing on an operator T in $C^*_c(\widehat{\mathscr{X}_{\Gamma,\infty}})^\Gamma$ the additional property that $\exists\ R > 0$ such that $\phi T = 0 = T\phi$ whenever $\phi \in C_c(\widehat{\mathscr{X}_{\Gamma,\infty}})$ and $d(\mathrm{supp}(\phi), \widehat{\mathscr{X}_\Gamma}) > R$. The C^*-algebra $C^*(\widehat{\mathscr{X}_\Gamma} \subset \widehat{\mathscr{X}_{\Gamma,\infty}})^\Gamma$ is obtained by closing the subalgebra $C^*_c(\widehat{\mathscr{X}_\Gamma} \subset \widehat{\mathscr{X}_{\Gamma,\infty}})^\Gamma$ in the operator norm.

A similar definition can be given for $D^*(\widehat{\mathscr{X}_\Gamma} \subset \widehat{\mathscr{X}_{\Gamma,\infty}})^\Gamma$. One can prove, see [45, Lemma 1.8], that the inclusion $c : \widehat{\mathscr{X}_\Gamma} \hookrightarrow \widehat{\mathscr{X}_{\Gamma,\infty}}$ induces K-theory isomorphisms:

$$K_*(C^*(\widehat{\mathscr{X}_\Gamma})^\Gamma) \simeq K_*(C^*(\widehat{\mathscr{X}_\Gamma} \subset \widehat{\mathscr{X}_{\Gamma,\infty}})^\Gamma); \quad K_*(D^*(\widehat{\mathscr{X}_\Gamma})^\Gamma) \simeq K_*(D^*(\widehat{\mathscr{X}_\Gamma} \subset \widehat{\mathscr{X}_{\Gamma,\infty}})^\Gamma). \tag{4.1}$$

Notice that $C^*(\widehat{\mathscr{X}_\Gamma} \subset \widehat{\mathscr{X}_{\Gamma,\infty}})^\Gamma)$ and $D^*(\widehat{\mathscr{X}_\Gamma} \subset \widehat{\mathscr{X}_{\Gamma,\infty}})^\Gamma$ are ideals in $D^*(\widehat{\mathscr{X}_{\Gamma,\infty}})^\Gamma$.

We assume the existence of a trivializing perturbation C_∂ for the signature operator $D^{\partial\mathscr{W}}_\Gamma$ on $\partial\mathscr{X}_\Gamma$: this means, as before, that C_∂ is bounded, that $D^{\partial\mathscr{W}}_\Gamma + C_\partial$ (with domain equal to the domain of $D^{\partial\mathscr{W}}_\Gamma$) is L^2-invertible and that $C_\partial \in \mathfrak{M}(C^*(\partial\widehat{\mathscr{X}_\Gamma})^\Gamma)$. $C_\partial \otimes \mathrm{Id}_{\mathbb{R}}$ then defines a bounded operator on $L^2(\partial\widehat{\mathscr{X}_\Gamma} \times \mathbb{R})$. We can then graft this perturbation on $\mathscr{X}_{\Gamma,\infty}$ and obtain a bounded perturbation C_∞ for $D^{\mathscr{W}}_{\infty,\Gamma}$. In the case of interest to us it will be the case that C_∞ is a limit in the norm topology of finite propagation operators and so we assume this property in what follows. In fact, we might more generally consider a perturbation B_∞ which is a limit of finite propagation operators and such that

$$P_0 B_\infty P_0 - P_0 C_\infty P_0 \in C^*(\widehat{\mathscr{X}_\Gamma} \subset \widehat{\mathscr{X}_{\Gamma,\infty}})^\Gamma, \tag{4.2}$$

with P_0 the operator defined by multiplication by the characteristic function of the cylindrical end $(-\infty, 0] \times \partial\widehat{\mathscr{X}_\Gamma} \subset \widehat{\mathscr{X}_{\Gamma,\infty}}$.

Proposition 4.3. – *Let C_∂, C_∞ and B_∞ be as above. If $\phi \in C_0(\mathbb{R})$ and if χ is a chopping function equal to the sign function on the spectrum of $D_\Gamma^{\partial\mathscr{W}} + C_\partial$, then:*

1] $\phi(D_{\Gamma,\infty}^{\mathscr{W}} + B_\infty) \in C^*(\widehat{\mathscr{X}_{\Gamma,\infty}})^\Gamma$*;*

2] $\chi(D_{\Gamma,\infty}^{\mathscr{W}} + B_\infty) \in D^*(\widehat{\mathscr{X}_{\Gamma,\infty}})^\Gamma$*;*

3] $\chi(D_{\Gamma,\infty}^{\mathscr{W}} + B_\infty)$ *is an involution modulo* $C^*(\widehat{\mathscr{X}_\Gamma} \subset \widehat{\mathscr{X}_{\Gamma,\infty}})^\Gamma$.

Proof. – For 1] and 2] we use Proposition 3.1 and the purely functional analytic arguments given in [46, Lemma 2.25]. For the third item we use the proof of Proposition 2.26 in [46], which is once again purely functional analytic. □

Given C_∂ as above, choosing $B_\infty = C_\infty$ and using Proposition 4.3 we can define a coarse relative index class

$$\mathrm{Ind}^{\mathrm{rel}}(D_{\Gamma,\infty}^{\mathscr{W}} + C_\infty) := \partial[\chi(D_{\Gamma,\infty}^{\mathscr{W}} + C_\infty)] \in K_*(C^*(\widehat{\mathscr{X}_\Gamma} \subset \widehat{\mathscr{X}_{\Gamma,\infty}})^\Gamma)$$

and thus, using (4.1), a coarse APS-index class

$$\mathrm{Ind}(D_\Gamma^{\mathscr{W}}, C) := c_*^{-1}(\mathrm{Ind}^{\mathrm{rel}}(D_{\Gamma,\infty}^{\mathscr{W}} + C_\infty)) \in K_*(C^*(\widehat{\mathscr{X}_\Gamma})^\Gamma) \simeq K_*(C_r^*\Gamma).$$

Notice that the left hand side is just notation; we have not really defined a perturbation C on $\widehat{\mathscr{X}_\Gamma}$.

One can prove, following the arguments in the proof of [46, Proposition 2.33], that for C_∂, C_∞ and B_∞ as above:

$$\mathrm{Ind}^{\mathrm{rel}}(D_{\Gamma,\infty}^{\mathscr{W}} + C_\infty) = \mathrm{Ind}^{\mathrm{rel}}(D_{\Gamma,\infty}^{\mathscr{W}} + B_\infty) \in K_*(C^*(\widehat{\mathscr{X}_\Gamma} \subset \widehat{\mathscr{X}_{\Gamma,\infty}})^\Gamma),$$

where the right hand side is well defined because of item 3] of Proposition 4.3.

4.4. The delocalized APS index theorem

Let $\mathscr{X}$, $\mathscr{X}_\Gamma$, $\mathscr{X}_\infty$, $\mathscr{X}_{\Gamma,\infty}$, $D^{\mathscr{W}}$, $D_\Gamma^{\mathscr{W}}$, $D_\infty^{\mathscr{W}}$, $D_{\Gamma,\infty}^{\mathscr{W}}$, $D^{\partial\mathscr{W}}$, $D_\Gamma^{\partial\mathscr{W}}$ and C_∂ be as in the previous subsections. We assume $\mathscr{X}$ to be even dimensional. By assumption C_∂ is a trivializing perturbation for $D_\Gamma^{\partial\mathscr{W}}$; assume that $C_\partial \in C^*(\partial\mathscr{W})^\Gamma)$, so that C_∂ is a norm limit of finite propagation operators. Consequently C_∞ is also a norm limit of finite propagation operators. We can consider the rho class $\rho(D_\Gamma^{\partial\mathscr{W}} + C_\partial) \in K_0(D^*(\partial\mathscr{X}_\Gamma)^\Gamma)$ and the coarse-APS index class $\mathrm{Ind}(D_\Gamma^{\mathscr{W}}, C) \in K_0(C^*(\mathscr{X}_\Gamma)^\Gamma)$. Let $\iota : C^*(\mathscr{X}_\Gamma)^\Gamma \hookrightarrow D^*(\mathscr{X}_\Gamma)^\Gamma$ be the natural inclusion and consider $j_* : K_0(D^*(\partial\mathscr{X}_\Gamma)^\Gamma)) \to K_0(D^*(\mathscr{X}_\Gamma)^\Gamma)$ induced by the inclusion of $\partial\mathscr{X}_\Gamma$ into $\mathscr{X}_\Gamma$. Our main tool in the next section will be the *delocalized APS index theorem for perturbed signature operators on Cheeger spaces*:

Theorem 4.4 (Delocalized APS index theorem). – *If the trivializing perturbation C_∂ is a norm limit of finite propagation operators, then the following equality holds*

$$\iota_*(\mathrm{Ind}(D_\Gamma^{\mathscr{W}}, C)) = j_*(\rho(D_\Gamma^{\partial\mathscr{W}} + C_\partial)) \quad \text{in} \quad K_0(D^*(\mathscr{X}_\Gamma)^\Gamma). \tag{4.3}$$

Proof. – All the arguments given in [45, Theorem 1.14] and then [46, Theorem 3.1] are functional analytic with the exception of the proof of Proposition 5.33 in [46]. However, the alternative proof of this particular proposition given by Zenobi in the context of Lipschitz manifolds, see Proposition 3.20 in [66], applies verbatim to the present context. □

Let now $\mathscr{X}$ be odd dimensional. After inverting 2 we can reduce the delocalized APS index theorem on $\mathscr{X}$ to the one on $\mathscr{X} \times S^1$ by a suspension argument. This is discussed carefully in [66, §5] where a different description of the group $\mathrm{K}_*(D^*(\widehat{X}_\Gamma)^\Gamma)$ is given for metric spaces with Γ-actions. These arguments apply in our situation largely unchanged.

In summary, the delocalized APS index theorem holds in every dimension.

5. Stratified homotopy equivalences and associated perturbations

5.1. The Hilsum-Skandalis replacement

Let $\widehat{X}$ be a Cheeger space, $r : \widehat{X} \longrightarrow B\Gamma$ the classifying map for the universal cover of $\widehat{X}$, $\mathscr{G}(r)$ the Mishchenko bundle associated to r, and $\mathcal{W}_X$ a self-dual mezzoperversity on $\widehat{X}$. If $\widehat{M}$ is another Cheeger space and $f : \widehat{M} \longrightarrow \widehat{X}$ a stratified homotopy equivalence then (see [4, Theorem 4.6]), there is a 'Hilsum-Skandalis replacement' for the pull-back of differential forms by f,

$$HS(f) : L^2(X; \Lambda^{*\mathrm{iie}}T^*X \otimes \mathscr{G}(r)) \longrightarrow L^2(M; \Lambda^{*\mathrm{iie}}T^*M \otimes \mathscr{G}(r \circ f)),$$

that we can use to define a self-dual mezzoperversity $\mathcal{W}_M = f^\sharp(\mathcal{W}_X)$ on $\widehat{M}$. These data satisfy

— $HS(f)d^{\mathscr{G}(r)} = d^{\mathscr{G}(r\circ f)}HS(f)$ and $HS(f)(\mathcal{D}_{\mathcal{W}_X}(d^{\mathscr{G}(r)})) \subseteq \mathcal{D}_{\mathcal{W}_M}(d^{\mathscr{G}(f\circ r)})$.

— There is an L^2-bounded operator Υ acting on $\mathcal{D}_{\mathcal{W}_X}(d^{\mathscr{G}(r)})$, such that

$$\mathrm{Id} - HS(f)'HS(f) = d_{\mathscr{G}(r)}\Upsilon + \Upsilon d_{\mathscr{G}(r)},$$

where $HS(f)'$ denotes the adjoint with respect to the quadratic form defined by the Hodge operator.

We point out that the boundedness of $HS(f)$ on $L^2(X; \Lambda^{*\mathrm{iie}}T^*X \otimes \mathscr{G}(r))$, together with the first of these properties, implies that $HS(f)$ is bounded as a map

$$HS(f) : \mathcal{D}_{\mathcal{W}_X}(d^{\mathscr{G}(r)}) \longrightarrow \mathcal{D}_{\mathcal{W}_M}(d^{\mathscr{G}(f\circ r)}),$$

when these spaces are endowed with the respective d-graph norm. Similarly Υ is bounded as an operator on the Hilbert space $\mathcal{D}_{\mathcal{W}_X}(d^{\mathscr{G}(r)})$. Note however that $HS(f)$ does not map L^2 differential forms into the maximal domain of d; indeed, if a differential form extends to be smooth on the closure of $\widetilde{X}$ and its exterior derivative fails to be in L^2, then the same will be true of its image under $HS(f)$.

5.2. The compressed Hilsum-Skandalis replacement

Following [44], we will also make use of a compressed version of the Hilsum-Skandalis replacement. In this case the replacement will make use of a fixed mezzoperversity and will have the property that it maps all of the L^2 differential forms into the domain of d. Recall that one of the main results in [4] is that the resolvents of $D^{\mathcal{W}}_{G(r)}$ and $D^{f^\sharp\mathcal{W}}_{G(r\circ f)}$ are $C^*_r\Gamma$-compacts.

DEFINITION 6. – *Let $\widehat{X}$ be a Cheeger space, $\mathcal{W}_X$ a self-dual mezzoperversity, $r : \widehat{X} \longrightarrow B\Gamma$ the classifying map for the universal cover of $\widehat{X}$ and $f : \widehat{M} \longrightarrow \widehat{X}$ be a smooth stratified map. For each $\mu : \mathbb{R} \longrightarrow \mathbb{R}$ an even, rapidly decreasing function, we define the* compressed Hilsum-Skandalis replacement of f *to be the operator*

$$HS_\mu(f) : L^2(X; \Lambda^{*\mathrm{iie}}T^*X \otimes \mathscr{G}(r)) \longrightarrow L^2(M; \Lambda^{*\mathrm{iie}}T^*M \otimes \mathscr{G}(f \circ r)),$$
$$HS_\mu(f) = \mu(D^{f^\sharp \mathcal{W}_X}) \circ HS(f) \circ \mu(D^{\mathcal{W}_X}).$$

As elements of the functional calculus we know that, e.g.,

$$\mu(D^{\mathcal{W}_X}) : L^2(X; \Lambda^{*\mathrm{iie}}T^*X \otimes \mathscr{G}(r)) \longrightarrow L^2(X; \Lambda^{*\mathrm{iie}}T^*X \otimes \mathscr{G}(r))$$

commutes with $D^{\mathcal{W}_X}$ and is a bounded operator with range contained in the domain of $D^{\mathcal{W}_X}$. In fact the range is contained in the domain

$$\mathcal{D}^\infty_{\mathcal{W}_X}(D) = \bigcap_{\ell \in \mathbb{N}} \{\omega \in \mathcal{D}_{\mathcal{W}_X}(D) : D\omega, \dots, D^\ell \omega \in \mathcal{D}_{\mathcal{W}_X}(D)\}$$

as $x^\ell \mu(x)$ is a rapidly decreasing function for any $\ell \in \mathbb{N}$. Since this domain is compactly included in $L^2(X; \Lambda^{*\mathrm{iie}}T^*X \otimes \mathscr{G}(r))$, it follows that $\mu(D^{\mathcal{W}_X})$ is a compact operator. Moreover since μ is even and d commutes with $(d+\delta)^2$, d commutes with $\mu(D)$. Thus,

$$HS_\mu(f) \text{ is a compact operator and } HS_\mu(f) d^{\mathcal{W}_X} = d^{f^\sharp \mathcal{W}_X} HS_\mu(f).$$

The compressed Hilsum-Skandalis replacement satisfies properties similar to those of $HS(f)$, see Lemma 9.7 in [44].

5.3. The Hilsum-Skandalis perturbation

On $X \sqcup -M$ consider the operators

$$d_{X \sqcup -M} = \begin{pmatrix} d_X & 0 \\ 0 & d_M \end{pmatrix}, \quad \tau_{X \sqcup -M} = \begin{pmatrix} \tau_X & 0 \\ 0 & -\tau_M \end{pmatrix}$$

and, for $t \in [0,1]$, the operator

$$\mathcal{L}_t : \mathcal{D}_{\mathcal{W}_X \sqcup f^\sharp \mathcal{W}_X}(d_{X \sqcup -M}) \longrightarrow \mathcal{D}_{\mathcal{W}_X \sqcup f^\sharp \mathcal{W}_X}(d_{X \sqcup -M})$$
$$\mathcal{L}_t = \begin{pmatrix} \mathrm{Id} - HS(f)' HS(f) & (1 - it\gamma\Upsilon) \circ HS(f)' \\ HS(f) \circ (1 + it\gamma\Upsilon) & \mathrm{Id} \end{pmatrix}. \tag{5.1}$$

We point out that $\mathcal{L}_t$ is bounded as an operator on the space $\mathcal{D}_{\mathcal{W}_X \sqcup f^\sharp \mathcal{W}_X}(d_{X \sqcup -M})$ endowed with its $d_{X \sqcup -M}$-graph norm, and let $|\mathcal{L}_t| = \sqrt{\mathcal{L}_t^* \mathcal{L}_t}$ denote the operator defined by the functional calculus on this Hilbert space (or equivalently as a bounded operator on L^2-differential forms).

As in [31], the Hilsum-Skandalis replacement can be used to construct a perturbation of the signature operator

$$D^{\mathcal{W}_X \sqcup f^\sharp \mathcal{W}_X} = \begin{pmatrix} D^{\mathcal{W}_X} & 0 \\ 0 & -D^{\mathcal{W}_M} \end{pmatrix} \text{ on } \widehat{X} \sqcup (-\widehat{M})$$

that results in an invertible operator. Indeed, for sufficiently small t, the operator

$$D^{\mathcal{W}_X \sqcup f^\sharp \mathcal{W}_X} + \mathcal{C}_t(f) = \tfrac{1}{i} U_t \circ \underline{D}_t \circ U_t^{-1}$$

is invertible, where $\underline{D}_t$ is the operator obtained from $D^{\mathcal{W}_X \sqcup f^\sharp \mathcal{W}_X}$ by making two replacements:

$$d_{X\sqcup -M} \mapsto \begin{pmatrix} d_X & tHS(f)' \\ 0 & d_M \end{pmatrix}, \quad \tau_{X\sqcup -M} \mapsto \operatorname{sign}(\tau_{X\sqcup -M} \circ \mathcal{L}_t) = \tau_{X\sqcup -M} \circ \operatorname{sign}(\mathcal{L}_t)$$

and

$$U_t = |\tau_{X\sqcup -M} \circ \mathcal{L}_t|^{1/2}.$$

LEMMA 5.1. – *The operator $\mathcal{C}_t(f)$ is a bounded operator relative to $D^{\mathcal{W}_X \sqcup f^\sharp \mathcal{W}_X}$; that is, $\mathcal{C}_t(f)$ is bounded as a map*

$$\mathcal{C}_t(f) : \mathcal{D}_{\mathcal{W}_X \sqcup f^\sharp \mathcal{W}_X}(D) \longrightarrow L^2(X \sqcup -M; \Lambda_*(X \sqcup -M)).$$

The operator $D^{\mathcal{W}_X \sqcup f^\sharp \mathcal{W}_X} + \mathcal{C}_t(f)$ is invertible for small enough $t > 0$.

Proof. – The boundedness of $\mathcal{C}_t(f)$ relative to $D^{\mathcal{W}_X \sqcup f^\sharp \mathcal{W}_X}$ follows from the fact that $\mathcal{L}_t$ is a bounded operator on $\mathcal{D}_{\mathcal{W}_X}(d)$. With notation similar to [66, Proof of Proposition 3.4], we can write

$$E_t = \begin{pmatrix} 0 & tHS(f)' \\ 0 & 0 \end{pmatrix}, \quad \operatorname{sign}(\mathcal{L}_t) = \operatorname{Id} + G_t, \quad U_t = \operatorname{Id} + H_t', \quad U_t^{-1} = \operatorname{Id} + F_t'$$

with E_t, G_t, H_t', F_t' bounded operators on $\mathcal{D}_{\mathcal{W}_X \sqcup f^\sharp \mathcal{W}_X}(d)$, with its d-graph norm as well as on $L^2(X \sqcup -M; \Lambda_*(X \sqcup -M))$. Then, e.g., in the even dimensional case we can write

$$D_t = \tfrac{1}{i}(1 + F_t') \circ ((d + E_t) + \tau_{X\sqcup -M} \circ (1 + G_t) \circ (d + E_t) \circ \tau_{X\sqcup -M} \circ (1 + G_t)) \circ (1 + H_t') = D + \mathcal{C}_t(f)$$

and it follows that $\mathcal{C}_t(f)$ is bounded as a map from $\mathcal{D}_{\mathcal{W}_X \sqcup f^\sharp \mathcal{W}_X}(D)$ to

$$L^2(X \sqcup -M; \Lambda_*(X \sqcup -M)).$$

Note that $\mathcal{L}_0$ satisfies

$$\mathcal{L}_0 = R'R, \quad R = \begin{pmatrix} \operatorname{Id} & 0 \\ HS(f) & \operatorname{Id} \end{pmatrix}$$

and, since R is invertible, this shows that $\mathcal{L}_0$ is invertible and hence $\mathcal{L}_t$ is invertible for small enough t. The invertibility of $D^{\mathcal{W}_X \sqcup f^\sharp \mathcal{W}_X} + \mathcal{C}_t(f)$ as an unbounded operator with domain $\mathcal{D}_{\mathcal{W}_X \sqcup f^\sharp \mathcal{W}_X}(D_{X\sqcup -M})$ now follows as in [31, Lemme 2.1], [56, §3]. □

A similar result holds for the signature operator on $X \sqcup (-M)$, with mezzoperversity given by $\mathcal{W}_X$ and $f^\sharp \mathcal{W}_X$ and twisted by the Mishchenko bundle $\mathscr{G}(r)$ on X and $\mathscr{G}(f \circ r)$ on M. In this case we use the Hilsum-Skandalis replacement $HS(f) : \mathcal{D}_{\mathcal{W}_X}(d^{\mathscr{G}(r)}) \longrightarrow \mathcal{D}_{\mathcal{W}_M}(d^{\mathscr{G}(f\circ r)})$.

5.4. The compressed Hilsum-Skandalis perturbation

We can repeat the argument from the previous subsection replacing $HS(f)$ by $HS_\mu(f)$. The resulting perturbation, which we denote $\mathcal{C}_{t,\mu}(f)$ and refer to as the *compressed Hilsum-Skandalis perturbation*, satisfies an improved version of Lemma 5.1.

LEMMA 5.2. – *The operator $\mathcal{C}_{t,\mu}(f)$ extends from $\mathcal{D}_{\mathcal{W}_X \sqcup f^\sharp \mathcal{W}_X}(D_{X\sqcup -M})$ to a compact operator*

$$\mathcal{C}_{t,\mu}(f) : L^2(X \sqcup -M; \Lambda_*(X \sqcup -M)) \longrightarrow L^2(X \sqcup -M; \Lambda_*(X \sqcup -M)).$$

The operator $D^{\mathcal{W}_X \sqcup f^\sharp \mathcal{W}_X} + \mathcal{C}_{t,\mu}(f)$ is invertible.

Proof. – If $\mathcal{L}_{t,\mu}$ is the operator obtained as in (5.1) but using $HS_\mu(f)$, then it is an invertible operator of the form $\mathrm{Id} + H_{t,\mu}$ with $H_{t,\mu}$ a compact operator such that both $H_{t,\mu}$ and its adjoint map send $L^2(X \sqcup -M; \Lambda_*(X \sqcup -M))$ into $\mathcal{D}_{\mathcal{W}_X \sqcup f^\sharp \mathcal{W}_X}(D^\infty_{X\sqcup -M})$. It follows from, e.g., the argument used in Lemma A.12 of [44], see also [66, Proposition 3.4], that each of the operators $E_{t,\mu}$, $G_{t,\mu}$, $H'_{t,\mu}$, $F'_{t,\mu}$ defined as in the proof of Lemma 5.1 will also have this property. Hence $\mathcal{C}_{t,\mu}(f)$ will be a compact operator.

The invertibility of the perturbed signature operator follows from [31, Lemme 2.1]. □

Also in this case we can extend the whole analysis to the signature operators twisted by the appropriate Mishchenko bundles; we state and use this result in Proposition 5.3 below.

5.5. Passing to the Roe algebra

Let $\mathbb{B}(\mathcal{E})$ denote the operators acting on the Hilbert $C^*_r\Gamma$-module

$$\mathcal{E} := L^2(X, \Lambda^* X \otimes \mathcal{G}(r)) \oplus L^2(M, \Lambda^* M \otimes G(r \circ f)).$$

Recall that there is a C^*-homomorphism

$$L_\pi : \mathbb{B}(\mathcal{E}) \to \mathcal{B}(L^2(X_\Gamma, \Lambda_* X_\Gamma) \oplus L^2(M_\Gamma, \Lambda_* M_\Gamma))$$

and that L_π induces an isomorphism between $\mathbb{K}(\mathcal{E})$ and the Roe algebra $C^*(\widehat{X}_\Gamma \sqcup (-\widehat{M}_\Gamma))^\Gamma$ and between $\mathbb{B}(\mathcal{E})$ and the multiplier algebra $\mathfrak{M}(C^*(\widehat{X}_\Gamma \sqcup (-\widehat{M}_\Gamma))^\Gamma)$ of the Roe algebra.

PROPOSITION 5.3. – *The compressed Hilsum-Skandalis perturbation $\mathcal{C}_{\mu,t}(f)$ is an element in $\mathbb{K}(\mathcal{E})$. Consequently, if $C_{\mu,t}(f) := L_\pi(\mathcal{C}_{\mu,t}(f))$, then*

$$C_{\mu,t}(f) \in C^*(\widehat{X}_\Gamma \sqcup (-\widehat{M}_\Gamma))^\Gamma. \tag{5.2}$$

5.6. APS-index classes associated to $\mathcal{L}_{\mathrm{BQ}}$-cycles

Let

$$\alpha : (\mathscr{M}, \partial\mathscr{M}) \xrightarrow{F} (\mathscr{Y}, \partial\mathscr{Y}) \xrightarrow{\omega} \mathscr{X}$$

be an $\mathcal{L}_{\mathrm{BQ}}$-cycle with $\mathscr{M}$, $\mathscr{Y}$, and $\mathscr{X}$ Cheeger spaces with boundary. We denote $\widehat{M} := \partial\mathscr{M}$ and $\widehat{Y} := \partial\mathscr{Y}$. We let $\mathscr{Z} := \mathscr{Y} \sqcup (-\mathscr{M})$. Recall that $F : \mathscr{M} \to \mathscr{Y}$ is a smooth transverse stratified map that restricts to a BQ-equvalence between $\widehat{M}$ and $\widehat{Y}$. The map ω and the classifying map for the universal cover of $\mathscr{X}$ induce a classifying map $\mathscr{Y} \longrightarrow B\Gamma$, where $\Gamma = \pi_1\mathscr{X}$. Together with F this defines a Γ-covering $\Gamma \to \mathscr{Z}_\Gamma \to \mathscr{Z}$. We fix a self-dual mezzoperversity $\mathscr{W}$ on $\mathscr{X}$ and consider the induced mezzo-perversities $\omega^\sharp\mathscr{W}$ on $\mathscr{Y}$ and $(\omega\circ F)^\sharp\mathscr{W}$ on $\mathscr{M}$. This gives $\mathscr{Z}$, and thus $\mathscr{Z}_\Gamma$, a self-dual mezzo-perversity $\omega^\sharp\mathscr{W}\sqcup(\omega\circ F)^\sharp\mathscr{W}$. We consider now $\widehat{Z} := \partial\mathscr{Z} \equiv \widehat{Y} \sqcup (-\widehat{M})$ and $\widehat{Z}_\Gamma := \partial\mathscr{Z}_\Gamma$; this gives a Galois Γ-covering of Cheeger-spaces without boundary

$$\Gamma - \widehat{Z}_\Gamma \to \widehat{Z} \;\equiv\; \Gamma - \partial\mathscr{Z}_\Gamma \to \partial\mathscr{Z}.$$

By our discussion above there is a well defined (compressed) Hilsum-Skandalis perturbation $C(F_\partial) \in C^*(\widehat{Z}_\Gamma)^\Gamma$ (for simplicity, we will no longer include the t,μ sub-indices in the notation for the perturbation); this is a trivializing perturbation for the signature operator on $\partial\mathscr{Z}_\Gamma$ with domain fixed by $\partial\omega^\sharp\mathscr{W} \sqcup \partial(\omega\circ F)^\sharp\mathscr{W}$. By grafting this perturbation on $\mathscr{Z}_\Gamma$ and extending it in the obvious way on the associated pseudomanifold with cylindrical ends, $\mathscr{Z}_{\Gamma,\infty}$, we thus obtain a well defined APS coarse index class that we shall denote as $\mathrm{Ind}_{\mathrm{APS}}(D_\Gamma^{\omega^\sharp\mathscr{W}\sqcup(\omega\circ F)^\sharp\mathscr{W}}, C(F_\partial))$ in $K_*(C^*(\mathscr{Z}_\Gamma)^\Gamma)$. This class can be further pushed forward to $K_*(C^*(\mathscr{X}_\Gamma)^\Gamma)$ using the maps F and ω.

Summarizing: to an $\mathcal{L}_{\mathrm{BQ}}$-cycle $\alpha : (\mathscr{M}, \partial\mathscr{M}) \xrightarrow{F} (\mathscr{Y}, \partial\mathscr{Y}) \xrightarrow{\omega} \mathscr{X}$ and the choice of a mezzoperversity $\mathscr{W}$ on $\mathscr{X}$, we can associate an APS-index class in $K_*(C^*(\mathscr{X}_\Gamma)^\Gamma)$, with $\mathscr{X}_\Gamma$ equal to the universal cover of $\mathscr{X}$.

Notation: we denote this index class by $\mathrm{Ind}_{\mathrm{APS}}(\alpha,\mathscr{W}) \in K_*(C^*(\mathscr{X}_\Gamma)^\Gamma)$.

Proposition 5.4. – *If F is a global stratified transverse homotopy equivalence, then*

$$\mathrm{Ind}_{\mathrm{APS}}(D_\Gamma^{\omega^\sharp\mathscr{W}\sqcup(\omega\circ F)^\sharp\mathscr{W}}, C(F_\partial)) = 0. \tag{5.3}$$

Consequently, if F is a global stratified transverse homotopy equivalence then

$$\mathrm{Ind}_{\mathrm{APS}}(\alpha,\mathscr{W}) = 0 \;\; in \;\; K_*(C^*(\mathscr{X}_\Gamma)^\Gamma). \tag{5.4}$$

Proof. – We can and we shall assume that F is of product type near the boundary; thus in a collar neighborhood of the boundary, $U \equiv \partial\mathscr{M} \times [0,1]$, we have $F|_U = F_\partial \otimes \mathrm{Id}_{[0,1]}$. The Hilsum-Skandalis method [31] can be extended to manifolds with cylindrical ends as in [56, Proposition 8.1]; combining these arguments with the ones given above and in [4, Theorem 4.6], we can thus prove that associated to F there is a well-defined equivariant Hilsum-Skandalis perturbation $C_u(F)$ on the pseudomanifold with cylindrical ends $\mathscr{Z}_{\Gamma,\infty}$. It is important to notice that this is an 'un-compressed' perturbation, hence the subscript, and that it is defined on the whole $\mathscr{Z}_{\Gamma,\infty}$. Notice for later use that because of the structure of F near the boundary, $C_u(F)$ is equal to $C_u(F_\partial) \otimes \mathrm{Id}$ on the cylindrical end, with $C_u(F_\partial)$ the un-compressed Hilsum-Skandalis perturbation associated to the homotopy equivalence

$F_\partial : \partial\mathscr{M} \to \partial\mathscr{Y}$. It follows, as in [56, Proposition 8.1], that the associated perturbed signature operator $D_\Gamma^{\omega^\sharp\mathscr{W}\sqcup(\omega\circ F)^\sharp\mathscr{W}} + C_u(F)$ is *invertible* and hence

$$\mathrm{Ind}_{\mathrm{APS}}(D_\Gamma^{\omega^\sharp\mathscr{W}\sqcup(\omega\circ F)^\sharp\mathscr{W}} + C_u(F)) = 0.$$

In order to prove the proposition it therefore suffices to show that

$$\mathrm{Ind}_{\mathrm{APS}}(D_\Gamma^{\omega^\sharp\mathscr{W}\sqcup(\omega\circ F)^\sharp\mathscr{W}}, C(F_\partial)) = \mathrm{Ind}_{\mathrm{APS}}(D_\Gamma^{\omega^\sharp\mathscr{W}\sqcup(\omega\circ F)^\sharp\mathscr{W}} + C_u(F)). \tag{5.5}$$

Recall that on the left hand side the have the APS-index class associated to

$$D_\Gamma^{\omega^\sharp\mathscr{W}\sqcup(\omega\circ F)^\sharp\mathscr{W}} + C_\infty(F_\partial)$$

with $C_\infty(F_\partial)$ the perturbation obtained by grafting to $\mathscr{Z}_{\Gamma,\infty}$ the *compressed* Hilsum-Skandalis perturbation $C(F_\partial)$ on the boundary $\partial\mathscr{Z}_\Gamma$; this index class is well defined, given that the associated boundary operator,

$$B_0 := D_{\Gamma,\partial}^{\partial\omega^\sharp\mathscr{W}\sqcup\partial(\omega\circ F)^\sharp\mathscr{W}} + C(F_\partial), \tag{5.6}$$

is invertible. Notice that the boundary operator of $D_\Gamma^{\omega^\sharp\mathscr{W}\sqcup(\omega\circ F)^\sharp\mathscr{W}} + C_u(F)$ is instead equal to

$$B_1 := D_{\Gamma,\partial}^{\partial\omega^\sharp\mathscr{W}\sqcup\partial(\omega\circ F)^\sharp\mathscr{W}} + C_u(F_\partial). \tag{5.7}$$

In order to establish (5.5) we now follow closely the proof given in [56, Theorem 8.4], where the same property is proved in the smooth context. Thus we first consider the cylinder $\partial\mathscr{Z}_\Gamma \times [0, 1]$. There is a natural and explicit 1-parameter family of perturbed operators $\{B_x\}_{x\in[0,1]}$ interpolating between B_0 and B_1. Consider $\partial_x - B_x$ on $\partial\mathscr{Z}_\Gamma \times [0, 1]$, where the minus sign comes from the sign-conventions in [56]. There is a well-defined APS index class associated to this operator on $\partial\mathscr{Z}_\Gamma \times [0, 1]$; indeed, the boundary operator is invertible. Moreover, we know that this APS-index class is equal to the higher spectral flow of $\{B_x\}_{x\in[0,1]}$, see [35, Theorem 10]. Wahl proves that this higher spectral flow, and thus this APS-index class, is equal to 0 and exactly the same argument applies here. We now attach this cylinder to $\mathscr{Z}_\Gamma$ and obtain a pseudomanifold which is clearly stratified diffeomorphic to $\mathscr{Z}_\Gamma$; we shall not distinguish between these two pseudomanifolds and work exclusively with the one with longer collar neighborhood. There is a natural perturbed Dirac operator on this pseudomanifold, call it $D_\Gamma^{\omega^\sharp\mathscr{W}\sqcup(\omega\circ F)^\sharp\mathscr{W}} + R$, defined by

$$D_\Gamma^{\omega^\sharp\mathscr{W}\sqcup(\omega\circ F)^\sharp\mathscr{W}} + C_u(F)$$

and by $\partial_x - B_x$. By the gluing formula proved in Proposition 6.4 below, we know that the index class associated to this operator is equal to

$$\mathrm{Ind}_{\mathrm{APS}}(D_\Gamma^{\omega^\sharp\mathscr{W}\sqcup(\omega\circ F)^\sharp\mathscr{W}} + C_u(F)) + \mathrm{Ind}_{\mathrm{APS}}(\partial_x - B_x),$$

which is again $\mathrm{Ind}_{\mathrm{APS}}(D_\Gamma^{\omega^\sharp\mathscr{W}\sqcup(\omega\circ F)^\sharp\mathscr{W}} + C_u(F))$ given that $\mathrm{Ind}_{\mathrm{APS}}(\partial_x - B_x) = 0$. Now, following again Wahl, we can construct an homotopy between the operator on $\mathscr{Z}_{\Gamma,\infty}$ associated to $D_\Gamma^{\omega^\sharp\mathscr{W}\sqcup(\omega\circ F)^\sharp\mathscr{W}} + R$ and the operator on $\mathscr{Z}_{\Gamma,\infty}$ associated to $D_\Gamma^{\omega^\sharp\mathscr{W}\sqcup(\omega\circ F)^\sharp\mathscr{W}} + C_\infty(F_\partial)$. The associated boundary operator is invertible along the whole homotopy and so, consequently, the index class associated to $D_\Gamma^{\omega^\sharp\mathscr{W}\sqcup(\omega\circ F)^\sharp\mathscr{W}} + R$ is equal to the index class associated to $D_\Gamma^{\omega^\sharp \mathscr{W} \sqcup (\omega\circ F)^\sharp \mathscr{W}} + C_\infty(F_\partial)$. Summarizing:

$$0 = \mathrm{Ind}_{\mathrm{APS}}(D_\Gamma^{\omega^\sharp \mathscr{W} \sqcup (\omega\circ F)^\sharp \mathscr{W}} + C_u(F))$$

because of the extension of Hilsum-Skandalis to pseudomanifolds with cylindrical ends;

$$\mathrm{Ind}_{\mathrm{APS}}(D_\Gamma^{\omega^\sharp \mathscr{W} \sqcup (\omega\circ F)^\sharp \mathscr{W}} + C_u(F)) = \mathrm{Ind}_{\mathrm{APS}}(D_\Gamma^{\omega^\sharp \mathscr{W} \sqcup (\omega\circ F)^\sharp \mathscr{W}} + R)$$

because of the gluing and the spectral flow argument, and

$$\mathrm{Ind}_{\mathrm{APS}}(D_\Gamma^{\omega^\sharp \mathscr{W} \sqcup (\omega\circ F)^\sharp \mathscr{W}} + R) = \mathrm{Ind}_{\mathrm{APS}}(D_\Gamma^{\omega^\sharp \mathscr{W} \sqcup (\omega\circ F)^\sharp \mathscr{W}} + C_\infty(F_\partial))$$

because of the homotopy constructed in Wahl. Since the right hand side of the last equation is precisely $\mathrm{Ind}_{\mathrm{APS}}(D_\Gamma^{\omega^\sharp \mathscr{W} \sqcup (\omega\circ F)^\sharp \mathscr{W}}, C(F_\partial))$ we conclude that (5.3) is now established. □

6. Mapping the Browder-Quinn surgery sequence to analysis

6.1. The rho class of a stratified homotopy equivalence

Let $f : \widehat{M} \to \widehat{X}$ be a transverse stratified homotopy equivalence. Let Γ be $\pi_1(\widehat{X})$. Let $\widehat{Z} := (-\widehat{M}) \sqcup \widehat{X}$. The Cheeger space $\widehat{Z}$ comes equipped with two maps induced respectively by f and the identity and by f and the classifying map for $\widehat{X}$:

$$\phi : \widehat{Z} \to \widehat{X}, \quad u : \widehat{Z} \to B\Gamma.$$

In particular, there is a well defined Γ covering $\Gamma - \widehat{Z}_\Gamma \to \widehat{Z}$ induced by u. We let $u_\Gamma : \widehat{Z}_\Gamma \to E\Gamma$ be the Γ-equivariant lift of u. We also let ϕ_Γ be the Γ-equivariant lift of ϕ,

$$\phi_\Gamma : \widehat{Z}_\Gamma \to \widehat{X}_\Gamma.$$

We fix a self-dual mezzoperversity $\mathcal{W}$ on $\widehat{X}$ and consider the associated self-dual mezzoperversity $f^\sharp \mathcal{W}$ on $-\widehat{M}$. We call $\mathcal{W} \sqcup f^\sharp \mathcal{W}$ the resulting self-dual mezzoperversity on $\widehat{Z}$. We then have self-adjoint extensions $D^{\mathcal{W} \sqcup f^\sharp \mathcal{W}}$ on Z, $D_\Gamma^{\mathcal{W} \sqcup f^\sharp \mathcal{W}}$ on Z_Γ and, by Proposition 5.3, a well defined (compressed) Hilsum-Skandalis perturbation $C_f \in C^*(\widehat{X}_\Gamma \sqcup (-\widehat{M}_\Gamma))^\Gamma \equiv C^*(\widehat{Z}_\Gamma)^\Gamma$. Summarizing, we have a well-defined class

$$\rho(D_\Gamma^{\mathcal{W} \sqcup f^\sharp \mathcal{W}} + C_f)) \in K_{\dim X+1}(D^*(\widehat{Z}_\Gamma)^\Gamma).$$

Recall, see for example [45, Subsection 1.2], that ϕ_Γ induces a well-defined K-theory morphism

$$(\phi_\Gamma)_* : K_*(D^*(\widehat{Z}_\Gamma)^\Gamma) \to K_*(D^*(\widehat{X}_\Gamma)^\Gamma). \tag{6.1}$$

We recall briefly the definition and refer for example to [45, Definition 1.6] for more details. It is implicit in the definition of $D^*(\widehat{Z}_\Gamma)^\Gamma$ and $D^*(\widehat{X}_\Gamma)^\Gamma$ that there is a $C_0(\widehat{Z}_\Gamma)^\Gamma$-module H and a $C_0(\widehat{X}_\Gamma)^\Gamma$ module L on which the group Γ acts by isometries; these are the Hilbert spaces on which the finite-propagation operators belonging to $D^*(\widehat{Z}_\Gamma)^\Gamma$ and $D^*(\widehat{X}_\Gamma)^\Gamma$ respectively act upon. One proves the existence of an operator $W : H \to L$, commuting with the action of Γ and covering ϕ_Γ in a suitable sense. This operator W defines an adjoint morphism

$$D^*(\widehat{Z}_\Gamma)^\Gamma \xrightarrow{\mathrm{Ad}(W)} D^*(\widehat{X}_\Gamma)^\Gamma, \quad \mathrm{Ad}(W)(T) := W \circ T \circ W^*$$

and one sets

$$(\phi_\Gamma)_* := \mathrm{Ad}(W).$$

This algebra homomorphism induces the K-theory morphism in (6.1).

DEFINITION 7. – *The rho-class* $\rho(\widehat{M} \xrightarrow{f} \widehat{X}, \mathcal{W})$ *associated to* $f : \widehat{M} \to \widehat{X}$ *and the self-dual mezzoperversity* $\mathcal{W}$ *is given by*

$$\rho(\widehat{M} \xrightarrow{f} \widehat{X}, \mathcal{W}) := (\phi_\Gamma)_*(\rho(D_\Gamma^{\mathcal{W} \sqcup f^\sharp \mathcal{W}} + C_f)) \in K_{\dim X+1}(D^*(\widehat{X}_\Gamma)^\Gamma). \tag{6.2}$$

The universal rho class is, by definition,

$$\rho_\Gamma(\widehat{M} \xrightarrow{f} \widehat{X}, \mathcal{W}) := (u_\Gamma)_*(\rho(D_\Gamma^{\mathcal{W} \sqcup f^\sharp \mathcal{W}} + C_f)) \in K_{\dim X+1}(D^*_\Gamma). \tag{6.3}$$

We shall see in the next subsection that the rho class of a stratified homotopy equivalence is independent of $\mathcal{W}$ and descends to $\mathrm{S_{BQ}}(\widehat{X})$.

6.2. The rho map from $\mathrm{S_{BQ}}(\widehat{X})$ to $K_{\dim \widehat{X}+1}(D^*(\widehat{X}_\Gamma)^\Gamma)$

PROPOSITION 6.1. – *The rho class associated to a transverse stratified homotopy equivalence* $f : \widehat{M} \to \widehat{X}$ *and a self-dual mezzoperversity* $\mathcal{W}$ *on* $\widehat{X}$ *satisfies the following properties:*

1] *it is independent of the choice of* $\mathcal{W}$;

2] *it gives a well-defined map*

$$\rho : \mathrm{S_{BQ}}(\widehat{X}) \longrightarrow K_{\dim \widehat{X}+1}(D^*(\widehat{X}_\Gamma)^\Gamma). \tag{6.4}$$

We denote by $\rho[\widehat{M} \xrightarrow{f} \widehat{X}]$ *the image of* $[\widehat{M} \xrightarrow{f} \widehat{X}]$ *through the rho map.*

Proof. – Let g and g' be two iie-metrics on $\widehat{X}$ and let $\mathcal{W}$ and $\mathcal{W}'$ be two self-dual mezzoperversities adapted respectively to g and g'. Let $r : \widehat{X} \to B\Gamma$ be a classifying map. Recall, following Banagl, how it is proved that $(\widehat{X}, g, \mathcal{W}, r)$ is Cheeger-bordant to $(\widehat{X}, g', \mathcal{W}', r)$; we refer the reader to [4, Section 4.4] for the details. We consider the pseudomanifold with boundary

$$\mathscr{X} = \widehat{X} \times [0,1]_t.$$

Instead of the product stratification, we stratify $\mathscr{X}$ using the strata of $\widehat{X}$ as follows:

i) The regular stratum X of $\widehat{X}$ contributes $X \times [0,1]$.

ii) Every singular stratum of $\widehat{X}$, Y^k, contributes three strata to $\mathscr{X}$,

$$Y^k \times [0,1/2), \quad Y^k \times (1/2,1], \quad Y^k \times \{1/2\}.$$

The link of $\mathscr{X}$ at $Y^k \times [0,1/2)$ and $Y^k \times (1/2,1]$ is equal to Z^k, while the link of $\mathscr{X}$ at $Y^k \times \{1/2\}$ is seen to be the (unreduced) suspension of Z^k, SZ^k. Since the lower middle perversity intersection homology of SZ^k, when $\dim Z^k = 2j-1$, is given by

$$I^{\overline{m}}H_i(SZ^k) = \begin{cases} I^{\overline{m}}H_{i-1}(Z^k) & i > j, \\ 0 & i = j, \\ I^{\overline{m}}H_i(Z^k) & i < k, \end{cases}$$

we see that $\mathscr{X}$ satisfies the Witt condition at the strata $Y^k \times \{1/2\}$. Put it differently, we do not need to fix a self-dual mezzoperversity at this stratum.

Let us endow $\mathscr{X}$ with any iie metric G such that, for some $t_0 > 0$,

$$G|_{X \times [0,t_0)} = g + dt^2, \quad G|_{X \times (1-t_0,1]} = g' + dt^2.$$

Next we endow $\mathscr{X}$ with a self-dual mezzoperversity $\mathscr{W}$ as follows: let $Y^1, \ldots, Y^T$ be an ordering of the strata of $\widehat{X}$ with non-decreasing depth. Denote

$$\mathcal{W} = \{W^1 \longrightarrow Y^1, \ldots, W^T \longrightarrow Y^T\}, \quad \mathcal{W}' = \{(W^1)' \longrightarrow Y^1, \ldots, (W^T)' \longrightarrow Y^T\}$$

and denote the fiber of, e.g., $W^j \longrightarrow Y^j$ at the point $q \in Y^j$, by W^j_q. Let us define

$$W^1_- \longrightarrow Y^1 \times [0, 1/2)$$

by requiring that the Hodge-de Rham isomorphism identifies all of the fibers. Once this is done, we can define $W^2_- \longrightarrow Y^2 \times [0, 1/2)$ in the same way, and inductively define $W^3_- \longrightarrow Y^3 \times [0, 1/2), \ldots, W^T_- \longrightarrow Y^T \times [0, 1/2)$.

We define $W^j_+ \longrightarrow Y^j \times (1/2 \times 1]$ in the same way to obtain

$$\mathscr{W} = \{W^1_- \longrightarrow Y^1 \times [0, 1/2), W^1_+ \longrightarrow Y^1 \times (1/2, 1], \ldots,$$
$$W^T_- \longrightarrow Y^T \times [0, 1/2), W^T_+ \longrightarrow Y^T \times (1/2, 1]\},$$

a self-dual mezzoperversity over $\mathscr{X}$. So, in words, we extend the metrics g and g' arbitrarily to an iie metric G without changing them in collar neighborhoods of the boundary, and then we choose a Hodge mezzoperversity by extending the de Rham mezzoperversities trivially from Y^i to $Y^i \times [0, 1/2)$ on the left and from Y^i to $Y^i \times (1/2, 1]$ on the right. Since the strata induced by $Y^k \times [0, 1/2)$ are disjoint from the strata induced by $Y^k \times (1/2, 1]$, there is no compatibility required between the corresponding mezzoperversities.

Finally, define $R : \mathscr{X} \longrightarrow B\Gamma$ by $R(\zeta, t) = r(\zeta)$. The result is a Cheeger-bordism

$$(\mathscr{X}, G, \mathscr{W}, R : \mathscr{X} \longrightarrow B\Gamma)$$

between $(\widehat{X}, g, \mathcal{W}, r : \widehat{X} \longrightarrow B\Gamma)$ and $(\widehat{X}', g', \mathcal{W}', r : \widehat{X} \longrightarrow B\Gamma)$.

Let us go back to the proof of our proposition. Let $f : \widehat{M} \to \widehat{X}$ be a transverse stratified homotopy equivalence. We want to show that the rho class is independent of the choice of the self-dual mezzoperversity $\mathcal{W}$ on $\widehat{X}$. Let $g, \mathcal{W}$ and $g', \mathcal{W}'$ as above and consider f^*g, $f^\sharp W$ and $f^*(g')$, $f^\sharp\mathcal{W}'$ on $\widehat{M}$. We can consider $\mathscr{M} := \widehat{M} \times [0, 1]$, stratified as above. Remark now that, by definition, the map $F : \mathscr{M} \to \mathscr{X}$, $F(\zeta, t) = f(\zeta)$ is such that $F^\sharp\mathscr{W}$, adapted to F^*G, is precisely equal to the self-dual mezzoperversity producing the Cheeger bordism between $(\widehat{M}, f^*g, f^\sharp W, (r \circ f) : \widehat{M} \to B\Gamma)$ and $(\widehat{M}, f^*(g'), f^\sharp\mathcal{W}', (r \circ f) : \widehat{M} \to B\Gamma)$. Moreover, F is a (transverse) stratified homotopy equivalence between $\mathscr{M}$ and $\mathscr{X}$.

We thus have a stratified Cheeger-space with boundary,

$$\mathscr{Z} := (-\mathscr{M}) \sqcup \mathscr{X},$$

which is the disjoint union of two stratified Cheeger spaces with boundary, endowed with a stratified homotopy equivalence $F : \mathscr{M} \to \mathscr{X}$, with self-dual mezzoperversities $F^\sharp\mathscr{W}$ on $\mathscr{M}$ and $\mathscr{W}$ on $\mathscr{X}$ and with a classifying map into $B\Gamma$, the latter producing a Galois Γ-covering $\Gamma - \mathscr{Z}_\Gamma \to \mathscr{Z}$; moreover, by construction, the self-dual mezzoperversity on the manifold with boundary $(-\mathscr{M}) \sqcup \mathscr{X}$ restricts to give $f^\sharp\mathcal{W} \sqcup \mathcal{W}$ on one boundary, the one corresponding to $t = 0$, and $f^\sharp\mathcal{W}' \sqcup \mathcal{W}'$ on the other boundary, the one corresponding to $t = 1$. For later use we denote by j_0 and j_1 the obvious inclusions of $(-\widehat{M}_\Gamma) \sqcup \widehat{X}_\Gamma$ into $\mathscr{Z}_\Gamma$ as the $t = 0$ and $t = 1$ boundary respectively. We now apply Proposition 5.4 and obtain that

$$\operatorname{Ind}(D^{\mathscr{W} \sqcup F^\sharp\mathscr{W}}, C(F_\partial)) = 0 \quad \text{in} \quad K_*(C^*(\mathscr{Z}_\Gamma)^\Gamma).$$

By applying the delocalized APS-index theorem we then obtain that

(6.5) $0 = (j_0)_*(\rho(D^{\mathcal{W}\sqcup f^\sharp\mathcal{W}} + C_f)) - (j_1)_*(\rho(D^{\mathcal{W}'\sqcup f^\sharp\mathcal{W}'} + C'_f))$ in $K_*(C^*(\mathscr{Z}_\Gamma)^\Gamma)$.

Observe now that there is an obvious Γ-equivariant map $\mathscr{Z}_\Gamma \to \mathscr{X}_\Gamma = \widehat{X}_\Gamma \times [0,1]$, induced by F and the identity, and thus, by projecting onto the first factor, a Γ-equivariant map $\mathscr{Z}_\Gamma \to \widehat{X}_\Gamma$. We can push-forward the equality (6.5) through this map and use functoriality in order to obtain

$$0 = \rho(\widehat{M} \xrightarrow{f} \widehat{X}, \mathcal{W}) - \rho(\widehat{M} \xrightarrow{f} \widehat{X}, \mathcal{W}') \in K_{*+1}(D^*(\widehat{X}_\Gamma)^\Gamma) \text{ with } * = \dim \widehat{X};$$

this shows indeed that the rho class is independent of the choice of self-dual mezzoperversity.

The proof of item 2] is very similar. □

6.3. The map from $\mathrm{N_{BQ}}(\widehat{X})$ to $K_{\dim \widehat{X}}(\widehat{X})$

We have defined $\mathrm{N_{BQ}}(\widehat{X})$ as equivalence classes of *transverse* degree one normal maps into $\widehat{X}$ which are diffeomorphisms when restricted to strata of dimension less than five. Our task is to map an element $[\widehat{M} \xrightarrow{f} \widehat{X}] \in \mathrm{N_{BQ}}(\widehat{X})$ to $K_{\dim \widehat{X}}(\widehat{X})$, or, more precisely, to $K_{\dim \widehat{X}}(\widehat{X}) \otimes \mathbb{Z}[1/2]$. Following the original treatment of Higson and Roe in the smooth setting, we shall in fact forget about the normal data encoded in $[\widehat{M} \xrightarrow{f} \widehat{X}] \in \mathrm{N_{BQ}}(\widehat{X})$.

$$\beta[\widehat{M} \xrightarrow{f} \widehat{X}] := f_*[D^{f^\sharp\mathcal{W}}] - [D^{\mathcal{W}}] \quad \in \quad K_{\dim \widehat{X}}(\widehat{X}) \otimes \mathbb{Z}[1/2]. \tag{6.6}$$

We then have the following

PROPOSITION 6.2. – 1] *The right hand side of* (6.6) *is independent of the choice of self-dual mezzoperversity* $\mathcal{W}$.

2] *The map* β *is well defined: if* $[\widehat{M}_0 \xrightarrow{f_0} \widehat{X}] = [\widehat{M}_1 \xrightarrow{f_1} \widehat{X}]$ *in* $\mathrm{N_{BQ}}(\widehat{X})$, *then*

$$(f_0)_*[D^{f_0^\sharp\mathcal{W}}] - [D^{\mathcal{W}}] = (f_1)_*[D^{f_1^\sharp\mathcal{W}}] - [D^{\mathcal{W}}] \quad \in \quad K_{\dim \widehat{X}}(\widehat{X}) \otimes \mathbb{Z}[1/2]. \tag{6.7}$$

Proof. – We establish both statements by adapting an argument due to Higson and Roe and by making use of Theorem 4.1 above, item 3.

Thus let $\mathcal{W}$ and $\mathcal{W}'$ be two self-dual mezzoperversities, adapted to iie metrics g and g' respectively. We must show that

$$f_*[D^{f^\sharp\mathcal{W}}] - [D^{\mathcal{W}}] - (f_*[D^{f^\sharp\mathcal{W}'}] - [D^{\mathcal{W}'}]) = 0 \quad \in \quad K_{\dim \widehat{X}}(\widehat{X}) \otimes \mathbb{Z}[1/2].$$

We initially follow the construction exploited in the previous subsection. Thus we consider $\mathscr{X} := \widehat{X} \times [0,1]$ and $\mathscr{M} := \widehat{M} \times [0,1]$, both stratified à la Banagl. We consider the transverse map $F : \mathscr{M} \to \mathscr{X}$, $F(\zeta, t) = f(\zeta)$ and consider G, F^*G, $\mathscr{W}$ and $F^\sharp\mathscr{W}$ as in the previous subsection. We thus have a stratified Cheeger-space with boundary,

$$\mathscr{Z} := (-\mathscr{M}) \sqcup \mathscr{X} \equiv (-(\widehat{M} \times [0,1])) \sqcup (\widehat{X} \times [0,1]),$$

which is the disjoint union of two stratified Cheeger spaces with boundary, endowed with a stratified transverse map $F : \mathscr{M} \to \mathscr{X}$, with self-dual mezzoperversities $F^\sharp\mathscr{W}$ on $\mathscr{M}$ and $\mathscr{W}$ on $\mathscr{X}$; moreover, by construction, the self-dual mezzoperversity $F^\sharp\mathscr{W} \sqcup \mathscr{W}$ on the manifold with boundary $(-\mathscr{M}) \sqcup \mathscr{X}$ restricts to give $f^\sharp\mathcal{W} \sqcup \mathcal{W}$ on one boundary, the one corresponding to $t = 0$, and $f^\sharp\mathcal{W}' \sqcup \mathcal{W}'$ on the other boundary, the one corresponding to $t = 1$.

Remark first of all that the K-homology group of a disjoint union of two spaces $A \sqcup B$ is equal to the direct sum of the individual K-homology groups. We define two group homomorphisms

$$\Phi : K_*(\mathscr{Z}, \partial\mathscr{Z}) \to K_*(\mathscr{X}, \partial\mathscr{X}) = K_*(\widehat{X} \times [0,1], \widehat{X} \times \{0,1\}),$$
$$\phi : K_*(\partial\mathscr{Z}) \to K_*(\partial\mathscr{X}) = K_*(\widehat{X} \times \{0,1\})$$

as follows:

$$\Phi(\alpha_{\mathscr{M}}, \beta_{\mathscr{X}}) = F_*\alpha_{\mathscr{M}} - \beta_{\mathscr{X}}, \quad \phi(\alpha_0, \alpha_1, \beta_0, \beta_1) = (f_*\alpha_0 - \beta_0, f_*\alpha_1 - \beta_1).$$

It is easy to check, using the functoriality properties of the connecting homomorphism in the long exact sequence of a pair, that the following diagram is commutative:

$$\begin{array}{ccc} K_{*+1}(\mathscr{Z}, \partial\mathscr{Z}) & \xrightarrow{\partial_\sqcup} & K_*(\partial\mathscr{Z}) \\ \Big\downarrow{\scriptstyle\Phi} & & \Big\downarrow{\scriptstyle\phi} \\ K_{*+1}(\mathscr{X}, \partial\mathscr{X}) & \xrightarrow{\partial} & K_*(\partial\mathscr{X}). \end{array}$$

The bottom horizontal homomorphism is part of the long exact sequence

$$K_{*+1}(\mathscr{X}, \partial\mathscr{X}) \xrightarrow{\partial} K_*(\partial\mathscr{X}) \xrightarrow{\iota} K_*(\mathscr{X}),$$

which can be rewritten as

$$K_{*+1}(\widehat{X} \times [0,1], \widehat{X} \times \{0,1\}) \xrightarrow{\partial} K_*(\widehat{X} \times \{0,1\}) \xrightarrow{\iota} K_*(\widehat{X} \times [0,1]). \tag{6.8}$$

Notice that there is a natural group homomorphism

$$\psi : K_*(\widehat{X} \times \{0,1\}) \to K_*(\widehat{X}), \quad \psi(\gamma_0, \gamma_1) = \gamma_0 - \gamma_1$$

and that ψ factors as follows:

$$\begin{array}{ccc} K_*(\widehat{X} \times \{0,1\}) & & \\ \Big\downarrow{\scriptstyle\iota} & \searrow^{\psi} & K_*(\widehat{X}) \\ K_*(\widehat{X} \times [0,1]), & \nearrow_{\pi} & \end{array}$$

with π induced by the projection onto the first factor. Using these remarks and Theorem 4.1, which in the present context states that

$$\partial_\sqcup([D^{F^\sharp\mathscr{W} \sqcup \mathscr{W}}]) = ([D^{f^\sharp\mathcal{W}}], [D^{f^\sharp\mathcal{W}'}], [D^{\mathcal{W}}], [D^{\mathcal{W}'}]),$$

we then have

$$\begin{aligned} f_*[D^{f^\sharp\mathcal{W}}] - [D^{\mathcal{W}}] - (f_*[D^{f^\sharp\mathcal{W}'}] - [D^{\mathcal{W}'}]) &= \psi(f_*[D^{f^\sharp\mathcal{W}}] - [D^{\mathcal{W}}], f_*[D^{f^\sharp\mathcal{W}'}] - [D^{\mathcal{W}'}]) \\ &= \pi \circ \iota(f_*[D^{f^\sharp\mathcal{W}}] - [D^{\mathcal{W}}], f_*[D^{f^\sharp\mathcal{W}'}] - [D^{\mathcal{W}'}]) \\ &= \pi \circ \iota \circ \phi([D^{f^\sharp\mathcal{W}}], [D^{f^\sharp\mathcal{W}'}], [D^{\mathcal{W}}], [D^{\mathcal{W}'}]) \\ &= \pi \circ \iota \circ \phi \circ \partial_\sqcup([D^{F^\sharp\mathscr{W} \sqcup \mathscr{W}}]) \\ &= \pi \circ \iota \circ \partial \circ \Phi[D^{F^\sharp\mathscr{W} \sqcup \mathscr{W}}] = 0, \end{aligned}$$

where in the last step we have used the exactness of (6.8).

This establishes item 1]. Item 2] is similar, but easier. □

6.4. The index map from $\mathrm{L}_{\mathrm{BQ}}(\widehat{X})$ to $K_*(C^*(\widehat{X}_\Gamma)^\Gamma)$

We finally consider the (APS) index homomorphisms

$$\mathrm{L}_{\mathrm{BQ}}(\widehat{X}\times[0,1]) \xrightarrow{\mathrm{Ind_{APS}}} K_{\dim\widehat{X}+1}(C^*(\widehat{X}_\Gamma)^\Gamma), \qquad \mathrm{L}_{\mathrm{BQ}}(\widehat{X}) \xrightarrow{\mathrm{Ind_{APS}}} K_{\dim\widehat{X}}(C^*(\widehat{X}_\Gamma)^\Gamma).$$

We shall treat in detail the first homomorphism, the second one is similar (in fact easier).

Recall from §5.6 that to each $\mathcal{L}_{\mathrm{BQ}}$-cycle α,

$$\alpha : (\mathscr{M},\partial\mathscr{M}) \xrightarrow{F} (\mathscr{Y},\partial\mathscr{Y}) \xrightarrow{\omega} \mathscr{X} \equiv \widehat{X}\times[0,1],$$

and each choice of mezzoperversity $\mathcal{W}$ on $\widehat{X}\times[0,1]$ we have defined an APS-index class $\mathrm{Ind_{APS}}(\alpha,\mathscr{W}) \in K_*(C^*(\mathscr{X}_\Gamma)^\Gamma) \equiv K_*(C^*(\widehat{X}_\Gamma)^\Gamma)$, with $\widehat{X}_\Gamma$ the universal cover of $\widehat{X}$, see §5.6 for the details. In what follows we shall use the canonical isomorphism $K_*(C^*(\widehat{X}_\Gamma)^\Gamma) \simeq K_*(C^*_r\Gamma)$. We will show that this class is independent of the choice of mezzoperversity and well-defined on $\mathrm{L}_{\mathrm{BQ}}(\widehat{X}\times[0,1])$. First we establish the independence with respect to the choice of the mezzoperversity.

LEMMA 6.3. – *Let $\mathcal{W}$ and $\mathcal{W}'$ be two mezzoperversities on $\widehat{X}\times[0,1]$. Then*

$$\mathrm{Ind_{APS}}(\alpha,\mathcal{W}) = \mathrm{Ind_{APS}}(\alpha,\mathcal{W}') \quad \textit{in} \quad K_*(C^*_r\Gamma).$$

Proof. – Let $\alpha : (\mathscr{M},\partial\mathscr{M}) \xrightarrow{F} (\mathscr{Y},\partial\mathscr{Y}) \xrightarrow{\omega} \mathscr{X} \equiv \widehat{X}\times[0,1]$ be a $\mathcal{L}_{\mathrm{BQ}}$-cycle as above. Consider the stratified manifolds with corners

$$\mathscr{M}\times[0,1]_t \quad \text{and} \quad \mathscr{Y}\times[0,1]_t.$$

We stratify $\mathscr{Y}\times[0,1]_t$ as we did in the proof of Proposition 6.1; the mezzoperversity $\mathcal{W}$ on $\widehat{X}\times[0,1]$ induces through ω a mezzoperversity $\omega^\sharp\mathcal{W}$ on $\mathscr{Y}\times\{0\}$; similarly, the mezzoperversity $\mathcal{W}'$ on $\widehat{X}\times[0,1]$ induces through ω a mezzoperversity $\omega^\sharp\mathcal{W}'$ on $\mathscr{Y}\times\{1\}$. We know that there is a mezzoperversity $\mathscr{W}_{\mathscr{Y}\times[0,1]}$ on $\mathscr{Y}\times[0,1]_t$ interpolating between $\omega^\sharp\mathcal{W}$ and $\omega^\sharp\mathcal{W}'$. Similarly, we stratify $-(\mathscr{M}\times[0,1]_t)$ as in Proposition 6.1. Let $\Phi : \mathscr{M}\times[0,1]_t \to \mathscr{Y}\times[0,1]_t$ the map $\Phi(m,t) = (F(m),t)$; the mezzoperversity $\Phi^\sharp\mathscr{W}_{\mathscr{Y}\times[0,1]}$ interpolates between $F^\sharp(\omega^\sharp\mathcal{W})$ and $F^\sharp(\omega^\sharp\mathcal{W}')$. Consider now $\mathscr{Z} := -\mathscr{M}\sqcup\mathscr{X}$ and

$$\mathscr{Z}\times[0,1]_t := -(\mathscr{M}\times[0,1]_t)\sqcup(\mathscr{Y}\times[0,1]_t).$$

This is a Cheeger space with corners, with boundary hypersurfaces:

$$F = (-\mathscr{M}\times\{t=0\}\sqcup(\mathscr{Y}\times\{t=0\}))\sqcup(-\mathscr{M}\times\{t=1\}\sqcup(\mathscr{Y}\times\{t=1\}))$$

and

$$G = -(\partial\mathscr{M}\times[0,1]_t)\sqcup(\partial\mathscr{Y}\times[0,1]_t).$$

Consider the signature operator on $\mathscr{Z}\times[0,1]_t$, together with the mezzoperversity that has been fixed above; using appropriate Hilsum-Skandalis perturbations as in §5.6 we can perturb this operator and make it invertible at G. To fix notation, let us assume that $\widehat{X}$ is odd dimensional, so that $\widehat{X}\times[0,1]$, $\mathscr{M}$ and $\mathscr{Y}$ are even dimensional. We can define a bivariant class $B\in KK_1(C_F(\mathscr{Z}\times[0,1]), C^*_r\Gamma)$, with $C_F(\mathscr{Z}\times[0,1])$ denoting the continuous functions on $\mathscr{Z}\times[0,1]$ which vanish on F. Notice, crucially, that without further hypothesis we could only define a bivariant class in $KK_1(C_{\partial(\mathscr{X}\times[0,1])}(\mathscr{X}\times[0,1]), C_r^*\Gamma)$ with $C_{\partial(\mathscr{X}\times[0,1])}(\mathscr{X}\times[0,1])$ denoting the continuous function on $\mathscr{X}\times[0,1]$ vanishing on the whole boundary. Consider $\pi^F : F \to$ point and $\pi^{\mathscr{X}\times[0,1]} : \mathscr{X}\times[0,1] \to$ point. Denote by ι the natural inclusion $F \hookrightarrow \mathscr{X}\times[0,1]$ and by $q : \mathscr{X}\times[0,1] \to F$ the restriction map to F. Obviously $\pi^F = \pi^{\mathscr{X}\times[0,1]}\circ\iota$. From the semi-split short exact sequence

$$0 \to C_F(\mathscr{X}\times[0,1]) \xrightarrow{j} C(\mathscr{X}\times[0,1]) \xrightarrow{q} C(F) \to 0$$

we obtain the following portion of the associated long exact sequence in KK-theory

$$KK_1(C_F(\mathscr{X}), C_r^*\Gamma) \xrightarrow{\delta} KK_0(C(F), C_r^*\Gamma) \xrightarrow{\iota_*} KK_0(C(\mathscr{X}), C_r^*\Gamma).$$

In particular, by exactness, $\iota_*\circ\delta = 0$. Then, on the one hand a classic argument based on the principle that *boundary of Dirac is Dirac* [(3)], see for example [3, Section 7.1] [4, Theorem 5.8], shows that

$$\pi_*^F(\delta B) = \mathrm{Ind}_{\mathrm{APS}}(\alpha, \mathcal{W}) - \mathrm{Ind}_{\mathrm{APS}}(\alpha, \mathcal{W}') \quad \text{in} \quad KK_0(\mathbb{C}, C_r^*\Gamma) = K_0(C_r^*\Gamma)$$

and, on the other hand, $\pi_*^F(\delta B) = \pi_*^{\mathscr{X}\times[0,1]}\circ\iota_*(\delta B) = \pi_*^{\mathscr{X}\times[0,1]}\circ\iota_*\circ\delta(B) = 0$ by exactness. Thus

$$\mathrm{Ind}_{\mathrm{APS}}(\alpha, \mathcal{W}) - \mathrm{Ind}_{\mathrm{APS}}(\alpha, \mathcal{W}') = 0,$$

as required. □

We now describe a gluing theorem following Bunke [14].

Let $\widehat{Z}$ be a stratified space with boundary and $\widehat{H}$ a compact hypersurface transverse to the stratification that does not meet the boundary of $\widehat{Z}$. We can view $\widehat{Z}$ as two stratified spaces with boundary glued along $\widehat{H}$,

$$\widehat{Z} = \widehat{Z}^1 \bigcup_{\widehat{H}} \widehat{Z}^2.$$

We decompose a Γ-cover of $\widehat{Z}$ accordingly:

$$\widehat{Z}_\Gamma = \widehat{Z}_\Gamma^1 \bigcup_{\widehat{H}_\Gamma} \widehat{Z}_\Gamma^2.$$

We assume that $\widehat{Z}$ is Cheeger, we fix an iterated incomplete edge metric g which is of product type near $\widehat{H}$; finally, we fix a selfdual mezzoperversity $\mathcal{W}$ adapted to g. This restricts to a selfdual mezzoperversity on the hypersurface $\widehat{H}$ which we denote $\mathcal{W}^H$. Similarly, we obtain selfdual mezzoperversities $\mathscr{W}^1$ on $\widehat{Z}^1$ and $\mathscr{W}^2$ on $\widehat{Z}^2$. We lift all these structures to the Γ-covers with minimal change of notation.

Let D_Γ be the signature operator on Z_Γ, the regular part of $\widehat{Z}_\Gamma$. We assume that a trivializing perturbation Q_∂ of the boundary operator has been fixed; the latter gives a grafted perturbation Q_∞ on the associated manifold with cylindrical ends and thus an index class $\mathrm{Ind}_{\mathrm{APS}}(D_\Gamma^{\mathcal{W}}, Q_\partial) \in K_*(C_r^*\Gamma)$, where the canonical isomorphism $K_*(C^*(\widehat{Z}_\Gamma)^\Gamma) \simeq K_*(C_r^*\Gamma)$ has been used. We can assume, without loss of generality, that the perturbation Q_∞ is localized away from $\widehat{H}_\Gamma$.

(3) Here *boundary of Dirac* refers to the image of the Dirac class in K-homology under the boundary homomorphism δ.

The signature operator near H_Γ, the regular part of $\widehat{H}_\Gamma$, will decompose in the usual way, given that the metric is of product type near H_Γ. Let C_H be a perturbation of $D_\Gamma^{\mathcal{W}^H}$ such that $D_\Gamma^{\mathcal{W}^H} + C_H$ is invertible [4] and let

$$D_{\Gamma,\infty}^{\mathscr{W}^1} + C_{H,\infty}^1, \quad D_{\Gamma,\infty}^{\mathscr{W}^2} + C_{H,\infty}^2$$

be perturbed differential operators on the spaces obtained from $\widehat{Z}_\Gamma^1$, $\widehat{Z}_\Gamma^2$ by attaching an infinite half-cylinder along $\widehat{H}_\Gamma$. We obtain in this way well defined index classes

$$\mathrm{Ind}_{\mathrm{APS}}(D_\Gamma^{\mathscr{W}^1}, Q_\partial^1 \sqcup C_H), \ \mathrm{Ind}_{\mathrm{APS}}(D_\Gamma^{\mathscr{W}^2}, Q_\partial^2 \sqcup C_H) \quad \text{in} \quad K_*(C_r^*\Gamma)$$

where Q_∂^j is Q_∂ restricted to $\partial\widehat{Z}_\Gamma \cap \widehat{Z}_\Gamma^j$.

Proposition 6.4 (Gluing). – *With notation as above, the index classes satisfy*

$$\mathrm{Ind}_{\mathrm{APS}}(D_\Gamma^{\mathcal{W}}, Q_\partial) = \mathrm{Ind}_{\mathrm{APS}}(D_\Gamma^{\mathscr{W}^1}, Q_\partial^1 \sqcup C_H) + \mathrm{Ind}_{\mathrm{APS}}(D_\Gamma^{\mathscr{W}^2}, Q_\partial^2 \sqcup C_H) \quad \textit{in} \quad K_*(C_r^*\Gamma).$$

If $\widehat{Z}$ is without boundary, then

$$\mathrm{Ind}(D_\Gamma^{\mathcal{W}}) = \mathrm{Ind}_{\mathrm{APS}}(D_\Gamma^{\mathscr{W}^1}, C_H) + \mathrm{Ind}_{\mathrm{APS}}(D_\Gamma^{\mathscr{W}^2}, C_H) \quad \textit{in} \quad K_*(C_r^*\Gamma).$$

Proof. – A happy byproduct of the functional analytic nature of Bunke's proof is that it applies almost unchanged to our setting. Let D denote either of $D_{\Gamma,\infty}^{\mathscr{W}^j} + C_{H,\infty}^j$, $j \in \{1,2\}$, with $\mathcal{D}(D)$ its self-adjoint domain. Replace the definition of the spaces H^ℓ, $\ell \geq 0$, in [14, (2)] by

$$H^\ell = \mathcal{D}(D^\ell), \quad \|\phi\|_\ell^2 = \sum_{k=0}^{\ell} \|D^k\phi\|_{L^2}^2,$$

where the norm on the right hand side is the pointwise norm coming from the Hilbert C^*-module structure.

Note that since $D^2 + \mathrm{Id}$ is invertible we can take a compact exhaustion of the regular part of our space, approximate the constant function one, and find a non-negative $f \in \mathcal{C}_c^\infty(Z_{\Gamma,\infty}^j)$ such that $D^2 + f$ is invertible, thereby satisfying Bunke's assumption 1.

With these conventions, the analytical results in §1.2 of Bunke now hold verbatim save that the expressions $R(\lambda)\mathrm{grad}(f)R(\lambda)$ should be replaced by

$$R(\lambda)(\mathrm{grad}(f) + [C, f])R(\lambda)$$

where C is the perturbation at H. This replacement is still a compact operator and hence the argument in Proposition 1.13 of Bunke yields well-defined index classes. The argument in Theorem 1.15 of Bunke then yields the equality of the index classes we seek, once we take into account that the index of a translation invariant operator on the infinite cylinder vanishes. □

(4) A simple argument using the cobordism invariance of the signature index class with Cheeger boundary conditions shows that such a perturbation always exists.

Recall that an articulated stratified space without boundary $\widehat{L}$ is the (entirety of the) boundary of a stratified space with corners. Thus $\widehat{L}$ is a finite union of stratified spaces with corners together with identifications of their boundary faces and the absence of boundary says that there are no unmatched faces. If $\widehat{L}$ is the boundary of a Cheeger space, $\widehat{L} = \partial\widehat{X}$, so in particular each of its constituent stratified spaces with corners is a Cheeger space, then a choice of mezzoperversity on $\widehat{X}$ induces compatible mezzoperversities on the constituents of $\widehat{L}$, and the boundary identifications (which are stratified diffeomorphisms) give rise to Hilsum-Skandalis perturbations, and so we have an index class,

$$\mathrm{Ind}_{\mathrm{APS}}((\partial\widehat{X})_\Gamma) \in K_*(C_r^*\Gamma),$$

where Γ is the fundamental group of $\widehat{X}$.

Lemma 6.5. – *If $\widehat{X}$ is a Cheeger space with corners then*

$$\mathrm{Ind}_{\mathrm{APS}}((\partial\widehat{X})_\Gamma) = 0 \text{ in } K_*(C_r^*\Gamma)[\tfrac{1}{2}].$$

Proof. – Our convention is that every boundary hypersurface $\widehat{M}$ of $\widehat{X}$ is collared, i.e., has a neighborhood of the form $[0,1)_{\rho_{\widehat{M}}} \times \widehat{M}$ in $\widehat{X}$, consistent with the stratification of $\widehat{X}$. We refer to $\rho_{\widehat{M}}$ as a boundary defining function for $\widehat{M}$. By a 'total boundary defining function' for $\widehat{X}$, we mean a function $\rho_{\partial\widehat{X}}$ obtained by taking the product of boundary defining functions, one per boundary hypersurface of $\widehat{X}$. Since the boundary hypersurfaces are collared, for all $\varepsilon > 0$ sufficiently small the set $\{\rho_{\partial\widehat{X}} \geq \varepsilon\}$ is a stratified space with boundary and $\partial\widehat{X}$ can be obtained from $\partial\{\rho_{\partial\widehat{X}} \geq \varepsilon\}$ by 'introducing corners' (i.e., partitioning it and considering as an articulated manifold). Cobordism invariance of the signature on Cheeger spaces [4, Theorem 4.8] implies that the signature of $\partial\{\rho_{\partial\widehat{X}} \geq \varepsilon\}$ vanishes in $K_*(C_r^*\Gamma)[\frac{1}{2}]$ (see Remark 8) and then Proposition 6.4 implies that the signature of $\partial\widehat{X}$ vanishes. □

The following proposition is the main result of this Subsection:

Proposition 6.6. – *The APS-index map defined on $\mathcal{L}_{\mathrm{BQ}}(\widehat{X} \times [0,1])$ descends to a map*

$$\mathrm{L}_{BQ}(\widehat{X} \times [0,1]) \ni \zeta \longrightarrow \mathrm{Ind}_{\mathrm{APS}}(\zeta) \in K_{\dim \widehat{X}+1}(C_r^*\Gamma)$$

by setting $\mathrm{Ind}_{\mathrm{APS}}(\zeta) := \mathrm{Ind}_{\mathrm{APS}}(\alpha, \mathcal{W})$ for any representative α of ζ, $[\alpha] = \zeta$, and any choice of mezzoperversity $\mathcal{W}$. This map is a homomorphism of abelian groups.

Proof. – It suffices to show that if α is null bordant then $\mathrm{Ind}_{\mathrm{APS}}(\alpha, \mathcal{W}) = 0$.

Let

$$(\widehat{N}; \partial_1\widehat{N}, \partial_2\widehat{N}) \xrightarrow{\Phi} (\widehat{Z}; \partial_1\widehat{Z}, \partial_2\widehat{Z}) \xrightarrow{\Omega} (\widehat{X} \times I) \times I$$

be a null bordism of α. Thus, Φ is BQ-normal, $\Phi| : \partial_2\widehat{N} \longrightarrow \partial_2\widehat{Z}$ is a BQ-equivalence, Ω is BQ-transverse, and

$$\left((\partial_1\widehat{N}, \partial_{12}\widehat{N}) \xrightarrow{\Phi|} (\partial_1\widehat{Z}, \partial_{12}\widehat{Z}) \xrightarrow{\pi\circ\Omega|} \widehat{X} \right) = \alpha.$$

By Lemma 6.5, we know that

$$\mathrm{Ind}((\partial_1\widehat{N} \xrightarrow{\partial_1\Phi} \partial_1\widehat{Z}) \cup (\partial_2\widehat{N} \xrightarrow{\partial_2\Phi} \partial_2\widehat{Z}), (\partial_1\Phi^\sharp(\mathcal{W}^{\partial_1\widehat{Z}}) \sqcup \mathcal{W}^{\partial_1\widehat{Z}}) \sqcup (\partial_2\Phi^\sharp(\mathcal{W}^{\partial_2\widehat{Z}}) \sqcup \mathcal{W}^{\partial_2\widehat{Z}})) = 0$$

for any mezzoperversity $\mathcal{W}^{\widehat{Z}}$ on $\widehat{Z}$. By Proposition 6.4, we can write this as the sum of two APS indices,

$$\mathrm{Ind}_{APS}(\partial_1 \widehat{N} \xrightarrow{\partial_1 \Phi} \partial_1 \widehat{Z}, \partial_1 \Phi^\sharp(\mathcal{W}^{\partial_1 \widehat{Z}}) \sqcup \mathcal{W}^{\partial_1 \widehat{Z}}) + \mathrm{Ind}_{APS}(\partial_2 \widehat{N} \xrightarrow{\partial_2 \Phi} \partial_2 \widehat{Z}, \partial_2 \Phi^\sharp(\mathcal{W}^{\partial_2 \widehat{Z}}) \sqcup \mathcal{W}^{\partial_2 \widehat{Z}}).$$

However, the second summand is equal to zero since $\Phi| : \partial_2 \widehat{N} \longrightarrow \partial_2 \widehat{Z}$ is a BQ-equivalence, and hence so is the first summand. The fact that $\mathrm{Ind}_{\mathrm{APS}}$ is a homomorphism of Abelian groups follows from the fact that addition in L_{BQ} is induced by disjoint union. The proposition is proved. □

We have defined the APS-index homomorphism for general cycles, because this is useful, for example, in studying its behavior under the homomorphism induced on the BQ-L groups by a transverse map. However, for certain purposes, it is more convenient to be able to handle this homomorphism exclusively on special cycles. This is the case, for example, when we need to check the compatibility of the APS-index homomorphism with the rho homomorphism and with the action of the BQ-L group on the BQ-structure set, given that this action is defined in terms of special cycles. Another example where this is useful is given in Proposition 6.9 below. The next lemma and the following proposition clarify that it is indeed possible to work exclusively with special cycles.

LEMMA 6.7. – *Every* $[\alpha] \in \mathrm{L}_{\mathrm{BQ}}(\widehat{X} \times [0,1])$ *can be represented by a diagram of the form*

$$\alpha : (\widehat{M}; \widehat{X}, \widehat{X}') \xrightarrow{(\phi;\mathrm{id},\psi)} (\widehat{X} \times [0,1]; \widehat{X} \times \{0\}, \widehat{X} \times \{1\}) \xrightarrow{\mathrm{id}} \widehat{X} \times I.$$

Let α_1 *and* α_2 *be two such diagrams representing the same class and* β *the diagram obtained by gluing* α_1 *and* $-\alpha_2$,

$$\beta : (\widehat{W}; \widehat{X}_2', \widehat{X}_1') \xrightarrow{(\Phi;\psi_2,\psi_1)} (\widehat{X} \times [0,1]; \widehat{X} \times \{0\}, \widehat{X} \times \{1\}) \xrightarrow{\mathrm{id}} \widehat{X} \times I,$$

then there are stratified spaces with corners $\widehat{P}$, $\widehat{Q}$ *together with a BQ-normal map* Θ,

$$(\widehat{P}; \partial_0 \widehat{P}, \partial_1 \widehat{P}) \xrightarrow{(\Theta;\theta_0,\theta_1)} (\widehat{Q}; \partial_0 \widehat{Q}, \partial_1 \widehat{Q})$$

such that

$$\left(\partial_0 \widehat{P} \xrightarrow{\theta_0} \partial_0 \widehat{Q}\right) = \left(\widehat{W} \xrightarrow{\Phi} \widehat{X} \times I\right)$$

and θ_1 *is a BQ-equivalence.*

Proof. – We can represent $[\alpha]$ in this way directly from the Wall representation theorem (as in Corollary 2.6). Taking

$$\widehat{W} = \widehat{M_1} \bigsqcup_{\widehat{X}} -\widehat{M_2}$$

and gluing α_1 and $-\alpha_2$ along their common boundary yields β, an element of $\mathcal{N}_{BQ}(\widehat{X} \times [0,1])$. Since Lemma 2.3 implies that β represents $[\alpha_1] - [\alpha_2] = 0$ in $\mathrm{L}_{BQ}(\widehat{X} \times [0,1])$, Theorem 2.4, implies that the class of β in $\mathrm{N}_{BQ}(\widehat{X} \times [0,1], \partial(\widehat{X} \times [0,1]))$ is in the image of $\mathrm{S}_{BQ}(\widehat{X} \times [0,1], \partial(\widehat{X} \times [0,1]))$. It follows that there is a normal bordism between β and an element in $\mathrm{S}_{\mathrm{BQ}}(\widehat{X} \times [0,1], \partial(\widehat{X} \times [0,1]))$, which yields $\Theta : \widehat{P} \longrightarrow \widehat{Q}$ as above. □

Proceeding exactly as in the proof of Proposition 6.6 and using crucially the above lemma we thus have the following;

PROPOSITION 6.8. – *Let $\mathcal{W}$ be a mezzoperversity for $\widehat{X} \times [0,1]$, for example the product mezzoperversity associated to a mezzoperversity on $\widehat{X}$. Let*

$$\alpha_1 = \Big((\widehat{M_1}; \widehat{X}, \widehat{X}_1') \xrightarrow{(\phi_1; \mathrm{id}, \psi_1)} (\widehat{X} \times [0,1]; \widehat{X} \times \{0\}, \widehat{X} \times \{1\}) \xrightarrow{\mathrm{id}} \widehat{X} \times I \Big)$$
$$\alpha_2 = \Big((\widehat{M_2}; \widehat{X}, \widehat{X}_2') \xrightarrow{(\phi_2; \mathrm{id}, \psi_2)} (\widehat{X} \times [0,1]; \widehat{X} \times \{0\}, \widehat{X} \times \{1\}) \xrightarrow{\mathrm{id}} \widehat{X} \times I \Big)$$

be two elements in $\mathcal{L}_{BQ}(\widehat{X} \times [0,1])$. Assume that $[\alpha_1] = [\alpha_2]$ in $\mathrm{L}_{BQ}(\widehat{X} \times [0,1])$. Then

$$\mathrm{Ind}_{\mathrm{APS}}(\alpha_1, \mathcal{W}) = \mathrm{Ind}_{\mathrm{APS}}(\alpha_2, \mathcal{W})$$

and this common value is independent of the choice of the mezzoperversity $\mathcal{W}$. Moreover, the APS-index of the sum of two such elements is the APS-index of the element obtained by stacking the two cycles as in Lemma 2.3.

Proof. – Only the last sentence needs to be justified and this follows immediately from the gluing theorem. □

We can use the gluing result in Proposition 6.4 to simplify the APS index map from an $\mathcal{L}$-cycle $\alpha \in \mathcal{L}_{\mathrm{BQ}}(\widehat{X} \times I)$ if it restricts to be a diffeomorphism on the boundary.

PROPOSITION 6.9. – *Let $\alpha \in \mathcal{L}_{\mathrm{BQ}}(\widehat{X} \times I)$ be given by*

$$\alpha = \Big((\widehat{M}; \widehat{X}, \widehat{X}') \xrightarrow{(\phi; \mathrm{id}, \psi)} (\widehat{X} \times [0,1]; \widehat{X} \times \{0\}, \widehat{X} \times \{1\}) \xrightarrow{\mathrm{id}} \widehat{X} \times I \Big),$$

where ψ is a diffeomorphism $\widehat{X}' \longrightarrow \widehat{X}$. Let $G(\alpha) \in \mathcal{L}_{\mathrm{BQ}}(\widehat{X} \times \mathbb{S}^1)$ be the $\mathcal{L}_{\mathrm{BQ}}$-cycle given by

$$G(\alpha) = \Big(G(\widehat{M}) = \widehat{M}/(\widehat{X} \sim_\psi \widehat{X}') \xrightarrow{G(\phi)} \widehat{X} \times [0,1]/(\widehat{X} \times \{0\} \sim \widehat{X} \times \{1\}) \xrightarrow{\mathrm{id}} \widehat{X} \times \mathbb{S}^1 \Big).$$

Given a mezzoperversity $\mathcal{W}$ on $\widehat{X}$ we have

$$\mathrm{Ind}_{\mathrm{APS}}(\alpha, \mathcal{W}) = \mathrm{Ind}(G(\alpha), \mathcal{W}) \text{ in } K_*(C_r^*\Gamma),$$

where on the left $\mathcal{W}$ is lifted to $\widehat{X} \times [0,1]$ and pulled-back to $\widehat{M}$, while on the right $\mathcal{W}$ is lifted to $\widehat{X} \times \mathbb{S}^1$ and pulled-back to $G(\widehat{M})$. Note that the index map on the right does not require boundary conditions; put differently, this is the index class of a cycle involving Cheeger spaces without boundary.

REMARK 9. – *The index of $(G(\alpha), \mathcal{W})$ is, by definition, the index of the signature operator on*

$$G(\widehat{M}) \cup (\widehat{X} \times \mathbb{S}^1)$$

(twisted using a reference map to $B\Gamma$, with $\Gamma = \pi_1 \widehat{X}$). Another application of Proposition 6.4 shows that this index coincides with that of the signature operator on

$$\widehat{M} \bigsqcup_{\substack{\widehat{X} \sim \widehat{X} \times \{0\} \\ \widehat{X}' \sim_\psi \widehat{X} \times \{1\}}} \widehat{X} \times I,$$

as in the original definition of Higson-Roe [29].

Proof. – Let $\widehat{N} \subseteq G(\widehat{M})$ and $\widehat{Y} \subseteq \widehat{X} \times \mathbb{S}^1$ be the images of the boundary of $\widehat{M}$, respectively of $\widehat{X} \times [0,1]$, under the identification maps $\widehat{M} \longrightarrow G(\widehat{M})$, $\widehat{X} \times [0,1] \longrightarrow \widehat{X} \times \mathbb{S}^1$. Without loss of generality we assume that $G(\phi)$ is collared near these subsets, i.e., that there are neighborhoods on which

$$G(\phi)| : \widehat{N} \times (-1,1) \longrightarrow \widehat{Y} \times (-1,1)$$

is the identity on the second factor. We assume that the stratifications respect the product structure of these neighborhoods.

We will apply our gluing result to this situation. Let $\widehat{Z} = G(\widehat{M}) \sqcup (\widehat{X} \times \mathbb{S}^1)$ endowed with the natural map to $B\Gamma$, $\Gamma = \pi_1 \widehat{X}$, and let

$$\widehat{H} = (\widehat{N} \times \{-\tfrac{1}{2}, \tfrac{1}{2}\}) \sqcup (\widehat{Y} \times \{-\tfrac{1}{2}, \tfrac{1}{2}\}).$$

Given a mezzoperversity $\mathscr{W}$ on $\widehat{X}$ adapted to a wedge metric on $\widehat{Z}$ that respects the product decomposition of the neighborhood $(\widehat{N} \sqcup \widehat{Y}) \times (-1,1)$, we can apply Proposition 6.4 to see that

$$\mathrm{Ind}(D_\Gamma^{\mathcal{W}}; \widehat{Z}) = \mathrm{Ind}_{\mathrm{APS}}(D_\Gamma^{\mathcal{W}^1}, C_H; \widehat{M} \sqcup (\widehat{X} \times [0,1])) + \mathrm{Ind}_{\mathrm{APS}}(D_\Gamma^{\mathcal{W}^1}, C_H; (\widehat{N} \sqcup \widehat{Y}) \times (-\tfrac{1}{2}, \tfrac{1}{2})),$$

where C_H is the Hilsum-Skandalis perturbation and we have used more explicit notation than is our wont. Finally note that the first summand on the right hand side is $\mathrm{Ind}(\alpha, \mathcal{W})$, and the second term vanishes, as it is the index of an invertible translation-invariant operator on an infinite cylinder (indeed, as we have already remarked, even though we denote our classes as APS classes, they are really classes on manifolds with cylindrical ends). □

6.5. Mapping stratified surgery to analysis

At this point we have shown that all of the maps in the following diagram are well defined and independent of the choice of mezzoperversity:

(6.9)

$$\begin{array}{ccccccc}
\mathrm{L}_{\mathrm{BQ}}(\widehat{X} \times I) & \dashrightarrow & \mathrm{S}_{\mathrm{BQ}}(\widehat{X}) & \xrightarrow{\ \eta\ } & \mathrm{N}_{\mathrm{BQ}}(\widehat{X}) & \xrightarrow{\ \theta\ } & \mathrm{L}_{\mathrm{BQ}}(\widehat{X}) \\
\downarrow{\scriptstyle \mathrm{Ind}_{\mathrm{APS}}} & & \downarrow{\scriptstyle \rho} & & \downarrow{\scriptstyle \beta} & & \downarrow{\scriptstyle \mathrm{Ind}_{\mathrm{APS}}} \\
\mathrm{K}_{\dim \widehat{X}+1}(C_r^*\Gamma)[\tfrac{1}{2}] & \longrightarrow & \mathrm{K}_{\dim \widehat{X}+1}(D^*(\widehat{X}_\Gamma)^\Gamma)[\tfrac{1}{2}] & \longrightarrow & \mathrm{K}_{\dim \widehat{X}}(\widehat{X})[\tfrac{1}{2}] & \longrightarrow & \mathrm{K}_{\dim \widehat{X}}(C_r^*\Gamma)[\tfrac{1}{2}].
\end{array}$$

We now establish the commutativity of this diagram.

The key fact for establishing commutativity of the first square is the behavior of the rho class under composition ([56, Theorem 9.1], [46, (4.14)]):

Proposition 6.10. – *Let $\widehat{L}$, $\widehat{M}$, $\widehat{V}$, be Cheeger spaces and*

$$\widehat{M} \xrightarrow{f} \widehat{V}, \quad \widehat{L} \xrightarrow{g} \widehat{M}$$

transverse stratified homotopy equivalences. We fix a self-dual mezzoperversity $\mathcal{W}$ on $\widehat{V}$ and we consider the induced mezzoperversities $f^\sharp\mathcal{W}$ on $\widehat{M}$ and $g^\sharp(f^\sharp\mathcal{W})$ on $\widehat{L}$. If $\widehat{V}_\Gamma$ is a Γ-covering of $\widehat{V}$ then we lift these mezzoperversities to the induced coverings $f^*\widehat{V}_\Gamma$ on M and $g^*(f^*(\widehat{V}_\Gamma))$ on L. The following identity holds in $K_*(D^*(\widehat{V}_\Gamma)^\Gamma)$:

$$\rho(\widehat{L} \xrightarrow{f \circ g} \widehat{V}, \mathcal{W})] + \widetilde{f}_*(\rho(\widehat{M} \xrightarrow{\mathrm{id}} \widehat{M}, f^\sharp\mathcal{W})) = \widetilde{f}_*(\rho(\widehat{L} \xrightarrow{g} \widehat{M}, f^\sharp\mathcal{W})) + \rho(\widehat{M} \xrightarrow{f} \widehat{V}, \mathcal{W}],$$

where in the first summand on the right-hand side it is the rho class of the perturbed signature operator on the covering $g^*(f^*(\widehat{V}_\Gamma)) \sqcup f^*(\widehat{V}_\Gamma)$ that appears.

Consequently, with a small abuse of notation, we have

$$\rho[\widehat{L} \xrightarrow{f\circ g} \widehat{V}] + \widetilde{f}_*(\rho[\widehat{M} \xrightarrow{\mathrm{id}} \widehat{M}]) = \widetilde{f}_*(\rho[\widehat{L} \xrightarrow{g} \widehat{M}]) + \rho[\widehat{M} \xrightarrow{f} \widehat{V}]. \tag{6.10}$$

REMARK 10. – *It can be shown that the ρ-invariant of the identity map vanishes, but we will not need this here.*

Proof. – The proof given in [46], based in turn on the proof of [56, Proposition 7.1] and on the delocalized APS index theorem, applies to the present situation. □

Recall now how $\mathrm{L}_{\mathrm{BQ}}(\widehat{X} \times I)$ acts on $\mathrm{S}_{\mathrm{BQ}}(\widehat{X})$. If $[\alpha] \in \mathrm{L}_{\mathrm{BQ}}(\widehat{X} \times I)$ and $[\beta] \in \mathrm{S}_{\mathrm{BQ}}(\widehat{X})$ then we can choose representatives of the form

$$\beta : \widehat{M} \xrightarrow{f} \widehat{X}$$
$$\alpha : (\widehat{W}; \widehat{M}, \widehat{M}') \xrightarrow{(\phi;\mathrm{id},\phi_2)} (\widehat{M} \times [0,1]; \widehat{M} \times \{0\}, \widehat{M} \times \{1\}) \xrightarrow{\mathrm{id}} \widehat{M} \times I,$$

and then the class of $f \circ \phi_2 : \widehat{M}' \longrightarrow \widehat{X}$ in $\mathrm{S}_{\mathrm{BQ}}(\widehat{X})$ is well-defined and denoted $\partial(\alpha)(\beta)$. The map

$$\begin{aligned} \mathrm{L}_{\mathrm{BQ}}(\widehat{X} \times I) \times \mathrm{S}_{\mathrm{BQ}}(\widehat{X}) &\longrightarrow \mathrm{S}_{\mathrm{BQ}}(\widehat{X}) \\ ([\alpha], [\beta]) &\longmapsto \partial(\alpha)(\beta) \end{aligned}$$

defines the group action of the Browder-Quinn L-group of $\widehat{X} \times I$ on the structure set of $\widehat{X}$. In order to show that the first diagram in (6.9) commutes we need to show that

$$\rho(\partial(\alpha)(\beta)) - \rho(\beta) = \iota_*(\mathrm{Ind}_{\mathrm{APS}}(\alpha)) \tag{6.11}$$

with $\mathrm{Ind}_{\mathrm{APS}}(\alpha) \in \mathrm{K}_{\dim \widehat{X}+1}(C^*(\widehat{X}_\Gamma)^\Gamma)[\frac{1}{2}] = \mathrm{K}_{\dim \widehat{X}+1}(C^*_r\Gamma)[\frac{1}{2}]$ and $\iota : C^*(\widehat{X}_\Gamma)^\Gamma \to D^*(\widehat{X}_\Gamma)^\Gamma$ the natural inclusion. The left hand side of (6.11) is, by definition,

$$\rho[\widehat{M}' \xrightarrow{f\circ\phi_2} \widehat{X}] - \rho[\widehat{M} \xrightarrow{f} \widehat{X}].$$

We now apply Proposition 6.10 and obtain that this difference equals:

$$\widetilde{f}_*\rho[\widehat{M}' \xrightarrow{\phi_2} \widehat{M}] - \widetilde{f}_*\rho[\widehat{M} \xrightarrow{\mathrm{id}} \widehat{M}]$$

and a direct application of the delocalized APS index theorem shows that this difference is precisely equal to $\iota_*(\mathrm{Ind}_{\Gamma,\mathrm{APS}}(\alpha))$. This establishes the commutativity of the first square in the diagram.

The second square in the diagram is proved to commute exactly as in [46].

Finally, for the third square, we observe that the image of a class in $\mathrm{N}_{\mathrm{BQ}}(\widehat{X})$ is a union of two closed Cheeger spaces and that for such an element in $\mathrm{L}_{\mathrm{BQ}}(\widehat{X})$ the APS-index class is just the index class of Subsection 3.6; the commutativity of the third square then follows by the functoriality of the boundary map in the Higson-Roe surgery sequence.

6.6. Mapping stratified surgery to analysis on all strata

The use of transverse maps in the definition of the Browder-Quinn surgery sequence implies that there are well-defined restriction maps from the long exact sequence of a stratified space to the corresponding sequence of a singular stratum.

Recall from §2.2 that if $Y \in \mathcal{S}(\widehat{X})$ then the closure of Y in $\widehat{X}$ is a stratified space denoted $\widehat{Y}$. An $\mathcal{L}$-cycle over $\widehat{X}$ restricts to an $\mathcal{L}$-cycle over $\widehat{Y}$, and a null bordism over $\widehat{X}$ restricts to a null bordism over $\widehat{Y}$. The restriction of a normal invariant or a Thom-Mather structure from $\widehat{X}$ to $\widehat{Y}$ is an $\mathcal{L}$-cycle of the same type, as the normal conditions or homotopy equivalence conditions are imposed on each stratum. Thus we have commutative diagrams

$$\begin{array}{ccccccc}
\mathrm{L}_{\mathrm{BQ}}(\widehat{X}\times I) & \dashrightarrow & \mathrm{S}_{\mathrm{BQ}}(\widehat{X}) & \xrightarrow{\eta} & \mathrm{N}_{\mathrm{BQ}}(\widehat{X}) & \xrightarrow{\theta} & \mathrm{L}_{\mathrm{BQ}}(\widehat{X}) \\
\downarrow & & \downarrow & & \downarrow & & \downarrow \\
\mathrm{L}_{\mathrm{BQ}}(\widehat{Y}\times I) & \dashrightarrow & \mathrm{S}_{\mathrm{BQ}}(\widehat{Y}) & \xrightarrow{\eta} & \mathrm{N}_{\mathrm{BQ}}(\widehat{Y}) & \xrightarrow{\theta} & \mathrm{L}_{\mathrm{BQ}}(\widehat{Y}),
\end{array}$$

which we can extend arbitrarily to the left. Note that the vertical arrows are generally neither injective nor surjective.

Let us introduce the abbreviations,

$$\mathrm{K}_{[j]}(C^*;\mathcal{S}(\widehat{X}))[\tfrac{1}{2}] = \mathrm{K}_{\dim\widehat{X}+j}(C_r^*\Gamma)[\tfrac{1}{2}] \oplus \bigoplus_{Y\in\mathcal{S}(X)} \mathrm{K}_{\dim\widehat{Y}+j}(C_r^*\Gamma(\widehat{Y}))[\tfrac{1}{2}],$$

$$\mathrm{K}_{[j]}(D^*;\mathcal{S}(\widehat{X}))[\tfrac{1}{2}] = \mathrm{K}_{\dim\widehat{X}+j}(D^*(\widehat{X}_\Gamma)^\Gamma)[\tfrac{1}{2}] \oplus \bigoplus_{Y\in\mathcal{S}(X)} \mathrm{K}_{\dim\widehat{Y}+j}(D^*(\widehat{Y}_{\Gamma(\widehat{Y})})^{\Gamma(\widehat{Y})})[\tfrac{1}{2}]$$

$$\mathrm{K}_{[j]}(D^*/C^*;\mathcal{S}(\widehat{X}))[\tfrac{1}{2}] = \mathrm{K}_{\dim\widehat{X}+j-1}(\widehat{X})[\tfrac{1}{2}] \oplus \bigoplus_{Y\in\mathcal{S}(X)} \mathrm{K}_{\dim\widehat{Y}+j-1}(\widehat{Y})[\tfrac{1}{2}].$$

By restricting to each singular stratum and making use of their respective commutative diagram (6.9) we end up with a combined diagram
(6.12)

$$\begin{array}{ccccccc}
\mathrm{L}_{\mathrm{BQ}}(\widehat{X}\times I) & \dashrightarrow & \mathrm{S}_{\mathrm{BQ}}(\widehat{X}) & \xrightarrow{\eta} & \mathrm{N}_{\mathrm{BQ}}(\widehat{X}) & \xrightarrow{\theta} & \mathrm{L}_{\mathrm{BQ}}(\widehat{X}) \\
\downarrow{\scriptstyle\oplus\mathrm{Ind}} & & \downarrow{\scriptstyle\oplus\rho} & & \downarrow{\scriptstyle\oplus\beta} & & \downarrow{\scriptstyle\oplus\mathrm{Ind}} \\
\mathrm{K}_{[1]}(C^*;\mathcal{S}(\widehat{X}))[\tfrac{1}{2}] & \longrightarrow & \mathrm{K}_{[1]}(D^*;\mathcal{S}(\widehat{X}))[\tfrac{1}{2}] & \longrightarrow & \mathrm{K}_{[1]}(D^*/C^*;\mathcal{S}(\widehat{X}))[\tfrac{1}{2}] & \longrightarrow & \mathrm{K}_{[0]}(C^*;\mathcal{S}(\widehat{X}))[\tfrac{1}{2}].
\end{array}$$

REMARK 11. – *In* [59, §12.4], *for Witt spaces with simply connected links (also known as 'supernormal spaces', see* [16]*) we find that*

$$\mathrm{L}_{\mathrm{BQ}}(\widehat{X})[\tfrac{1}{2}] = \mathrm{L}(\mathbb{Z}\pi_1(\widehat{X}))[\tfrac{1}{2}] \oplus \bigoplus_{Y\in\mathcal{S}(X)} \mathrm{L}(\mathbb{Z}\pi_1(\widehat{Y}))[\tfrac{1}{2}],$$

(where $\widehat{Y}$, the 'closed stratum', is the closure of Y in $\widehat{X}$ with the induced stratification) so that in this case the vertical arrows in (6.12) *map from the* algebraic *L-groups.*

7. Further considerations

In [17], Chang and Weinberger use the surgery exact sequence of a manifold to show that torsion in its fundamental group implies the existence of infinitely many homotopy equivalent manifolds that are homeomorphically distinct.

In this section we use the argument in *loc. cit.* as a launching pad to discuss related topics. First, as an answer to 'how to map in to/out of a BQ L-group?' we establish a couple of long exact sequences. Secondly we establish Atiyah's L^2-signature theorem for Cheeger spaces. Finally we combine these to discuss a version of the result of [17] for Cheeger spaces.

7.1. A long exact sequence for the Browder-Quinn L-groups

It is interesting to connect the Browder-Quinn L-groups with the usual (Wall) L-groups of a smooth manifold. If $\widehat{X}$ is a smoothly stratified space then, on the one hand, recall that we can identify the regular part of $\widehat{X}$ with the interior of the resolution $\widetilde{X}$ of $\widehat{X}$,

$$X = \widetilde{X}^{\circ} = \widehat{X}^{\mathrm{reg}},$$

and so the inclusion, i, of the regular part induces

$$\mathrm{L}_k(X) \xrightarrow{i_*} \mathrm{L}_{\mathrm{BQ},d^{\widetilde{X}}(k)}(\widehat{X}),$$

where $d^{\widetilde{X}}(k)$ is the dimension function that is equal to k on the inverse image of the regular part of $\widehat{X}$. (Note that this determines all of the dimension function since, in the notation of (2.3), by transversality of ϕ and ω the codimension of a stratum in $\widehat{M}$ or $\widehat{Y}$ is equal to the codimension of the stratum it maps into in $\widehat{X}$.) On the other hand, if $X^{\dagger}$ is a minimal stratum (a stratum of greatest depth), and hence a closed manifold, then the fact that the Browder-Quinn L-groups are defined using transverse maps means that we have a natural restriction map

$$\mathrm{L}_{\mathrm{BQ},d}(\widehat{X}) \xrightarrow{R} \mathrm{L}_{d_{X^{\dagger}}}(X^{\dagger}),$$

where $d_{X^{\dagger}}$ is the restriction of the dimension function to $X^{\dagger}$.

Both of these maps fit into a long exact sequence of Browder-Quinn L-groups. An example of the former is found in [13] and of the latter in [59, §6]. We treat these as special cases of long exact sequences in L-groups associated with inclusions.

We recall from, e.g., [58, Lemma 8.3.1], a standard construction of long exact sequences in cobordism. Suppose that α and β denote two possible types of structure a manifold can have, and that a β structure implies an α structure (for example, if $Y \subseteq X$ and α structure could be 'is endowed with a map to X' and an β structure 'is endowed with a map to Y'). Denote by Ω_n^{α}, Ω_n^{β} the cobordism groups of n-dimensional manifolds with the corresponding structure, and denote by $\Omega_n^{\alpha,\beta}$ the cobordism group of manifolds with boundary with an α structure and a compatible β structure on the boundary then, with the obvious maps, there is a long exact sequence

$$\cdots \longrightarrow \Omega_n^{\beta} \longrightarrow \Omega_n^{\alpha} \longrightarrow \Omega_n^{\alpha,\beta} \xrightarrow{\partial} \Omega_{n-1}^{\beta} \longrightarrow \cdots .$$

The proof of exactness in *loc. cit.* does not depend on the specific structures α and β and adapts easily to the situations we consider below.

One type of relative L-group. In [57], Wall explains how to associate to a map $h : V \longrightarrow W$ between two manifolds an L-group that moreover fits into a long exact sequence with the L-groups of V and W. We now observe that the same is true if $\widehat{V}$ and $\widehat{W}$ are smoothly stratified spaces and $h : \widehat{V} \longrightarrow \widehat{W}$ is a transverse map between them.

Let us denote by $\mathcal{L}_{\mathrm{BQ},d}(\widehat{V} \xrightarrow{h} \widehat{W})$ the set of commutative diagrams of the form

$$
\begin{array}{ccccc}
(\widehat{S};\partial_1\widehat{S},\partial_2\widehat{S}) & \xrightarrow{\phi} & (\widehat{T};\partial_1\widehat{T},\partial_2\widehat{T}) & \xrightarrow{\eta} & \widehat{W}\\
\uparrow & & \uparrow & & \uparrow h\\
\partial_2\widehat{S} & \xrightarrow{\phi|} & \partial_2\widehat{T} & \xrightarrow{\eta|} & \widehat{V},
\end{array}
\tag{7.1}
$$

where $\widehat{S}$ is a smoothly stratified space with dimension function d, and boundary $\partial_1\widehat{S} \cup \partial_2\widehat{S}$, and similarly $\widehat{T}$, ϕ is a BQ-normal map and restricts to a BQ-equivalence between $\partial_1\widehat{S}$ and $\partial_1\widehat{T}$ and η is BQ-transverse. As usual $\mathrm{L}_{\mathrm{BQ},d}(\widehat{V} \xrightarrow{h} \widehat{W})$ then denotes cobordism classes of such cycles. As above, these groups fit into a long exact sequence

$$
\cdots \xrightarrow{\partial} \mathrm{L}_{\mathrm{BQ},d}(\widehat{V}) \xrightarrow{h_*} \mathrm{L}_{\mathrm{BQ},d}(\widehat{W}) \xrightarrow{j} \mathrm{L}_{\mathrm{BQ},d}(\widehat{V} \xrightarrow{h} \widehat{W}) \xrightarrow{\partial} \mathrm{L}_{\mathrm{BQ},d-1}(\widehat{V}) \xrightarrow{h_*} \cdots,
$$

where the maps are given by

$$
h_*\left[(\widehat{S}';\partial\widehat{S}') \xrightarrow{\phi'} (\widehat{T}',\partial\widehat{T}') \xrightarrow{\eta'} \widehat{V}\right] = \left[(\widehat{S}';\partial\widehat{S}') \xrightarrow{\phi'} (\widehat{T}',\partial\widehat{T}') \xrightarrow{h\circ\eta'} \widehat{W}\right],
$$

$$
j\left[(\widehat{S}'';\partial\widehat{S}'') \xrightarrow{\phi''} (\widehat{T}'',\partial\widehat{T}'') \xrightarrow{\eta''} \widehat{W}\right] = \left[\begin{array}{ccccc}
(\widehat{S}'';\partial\widehat{S}'',\emptyset) & \xrightarrow{\phi''} & (\widehat{T}'';\partial\widehat{T}'',\emptyset) & \xrightarrow{\eta''} & \widehat{W}\\
\uparrow & & \uparrow & & \uparrow h\\
\emptyset & \longrightarrow & \emptyset & \longrightarrow & \widehat{V}
\end{array}\right],
$$

$$
\partial\left[\begin{array}{ccccc}
(\widehat{S};\partial_1\widehat{S},\partial_2\widehat{S}) & \xrightarrow{\phi} & (\widehat{T};\partial_1\widehat{T},\partial_2\widehat{T}) & \xrightarrow{\eta} & \widehat{W}\\
\uparrow & & \uparrow & & \uparrow h\\
\partial_2\widehat{S} & \xrightarrow{\phi|} & \partial_2\widehat{T} & \xrightarrow{\eta|} & \widehat{V}
\end{array}\right] = \left[(\partial_2\widehat{S};\partial(\partial_2\widehat{S})) \xrightarrow{\phi|} (\partial_2\widehat{T};\partial(\partial_2\widehat{T})) \xrightarrow{\eta|} \widehat{V}\right].
$$

If we apply this to the inclusion of the regular part, $i : X \longrightarrow \widehat{X}$, we obtain

$$
\cdots \xrightarrow{\partial} \mathrm{L}_k(X) \xrightarrow{i_*} \mathrm{L}_{\mathrm{BQ},d^{\widehat{X}}(k)}(\widehat{X}) \xrightarrow{j} \mathrm{L}_{\mathrm{BQ},d^{\widehat{X}}(k)}(X \xrightarrow{i} \widehat{X}) \xrightarrow{\partial} \mathrm{L}_{k-1}(X) \xrightarrow{i_*} \cdots
\tag{7.2}
$$

(cf. [13, Proposition 4.8]).

Another type of relative L-group. – (For this sequence cf. [22, Theorem 5.4].) Let $\widehat{X}$ be a stratified space, $\widehat{\Sigma} \subseteq \widehat{X}$ a closed subset of $\widehat{X}$ made up of a union of strata. Let $\mathcal{L}_{\mathrm{BQ},d}(\widehat{X};\widehat{\Sigma})$ denote the $\mathcal{L}$-cycles over X with dimension function d whose restriction to $\widehat{\Sigma}$ is a BQ-equivalence, and $\mathrm{L}_{\mathrm{BQ},d}(\widehat{X};\widehat{\Sigma})$ the corresponding bordism classes. We allow $\widehat{\Sigma} = \emptyset$ for which

$$
\mathrm{L}_{\mathrm{BQ},d}(\widehat{X};\emptyset) = \mathrm{L}_{\mathrm{BQ},d}(\widehat{X}).
$$

Analogously to the above, if $\widehat{\Sigma}' \subseteq \widehat{X}$ is another closed subset of $\widehat{X}$ made up of a union of strata, with $\widehat{\Sigma} \subseteq \widehat{\Sigma}'$ then there is a relative group $\mathrm{L}_{\mathrm{BQ},d}(\widehat{X};\widehat{\Sigma} \subseteq \widehat{\Sigma}')$ with classes represented by diagrams of the form

$$(7.3)\qquad (\widehat{M};\partial_0\widehat{M},\partial_1\widehat{M}) \xrightarrow{\phi} (\widehat{N};\partial_0\widehat{N},\partial_1\widehat{N}) \xrightarrow{\omega} \widehat{X},$$

such that ϕ is BQ-normal, ω is BQ-transverse, $\partial_0\phi$ is a BQ-equivalence, $\partial_1\phi$ restricted to the preimage of $\widehat{\Sigma}'$ is a BQ-equivalence, and ϕ restricted to the preimage of $\widehat{\Sigma}$ is a BQ-equivalence.

There are natural inclusion maps

$$\mathrm{L}_{\mathrm{BQ},d}(\widehat{X};\widehat{\Sigma}') \longrightarrow \mathrm{L}_{\mathrm{BQ},d}(\widehat{X};\widehat{\Sigma}), \quad \mathrm{L}_{\mathrm{BQ},d}(\widehat{X};\widehat{\Sigma}) \longrightarrow \mathrm{L}_{\mathrm{BQ},d}(\widehat{X};\widehat{\Sigma}\subseteq\widehat{\Sigma}'),$$

which fit into a long exact sequence

$$\cdots \longrightarrow \mathrm{L}_{\mathrm{BQ},d}(\widehat{X};\widehat{\Sigma}') \longrightarrow \mathrm{L}_{\mathrm{BQ},d}(\widehat{X};\widehat{\Sigma}) \longrightarrow \mathrm{L}_{\mathrm{BQ},d}(\widehat{X};\widehat{\Sigma}\subseteq\widehat{\Sigma}') \xrightarrow{\partial_1} \mathrm{L}_{\mathrm{BQ},d-1}(\widehat{X};\widehat{\Sigma}') \longrightarrow \cdots.$$

Exactness of this sequence follows from the usual construction of relative sequences in bordism, see [58].

Let us consider the case where $\widehat{\Sigma}=\emptyset$ and $\widehat{\Sigma}'=X^\dagger$, a minimal stratum of $\widehat{X}$. We point out that there are compatible restriction maps

$$\mathrm{L}_{\mathrm{BQ},d}(\widehat{X};\emptyset) \longrightarrow \mathrm{L}_{d(X^\dagger)}(X^\dagger), \quad \mathrm{L}_{\mathrm{BQ},d}(\widehat{X};\emptyset\subseteq X^\dagger) \longrightarrow \mathrm{L}_{d(X^\dagger)}(X^\dagger),$$

both of which are onto, since any $\mathcal{L}$-cycle over $X^\dagger$ has a lift to an $\mathcal{L}$-cycle over $\widehat{X}$,

$$\Big((M;\partial M) \xrightarrow{\phi} (N,\partial N) \xrightarrow{\omega} X^\dagger\Big) \mapsto \Big((\omega\circ\phi)^*\mathcal{T}_{X^\dagger} \longrightarrow \omega^*\mathcal{T}_{X^\dagger} \longrightarrow (\mathcal{T}_{X^\dagger}\hookrightarrow)\,\widehat{X}\Big)$$

(where $\mathcal{T}_{X^\dagger}$ is a tubular neighborhood of $X^\dagger$ in $\widehat{X}$).

The restriction map $\mathrm{L}_{\mathrm{BQ},d}(\widehat{X};\emptyset\subseteq X^\dagger) \longrightarrow \mathrm{L}_{d(X^\dagger)}(X^\dagger)$ is also injective Indeed, assume that (7.3) is such that ϕ restricts to $X^\dagger$ to be a BQ-equivalence and consider

$$(7.4)\qquad M\times[0,1] \xrightarrow{\phi\times\mathrm{id}} N\times[0,1] \xrightarrow{\omega\times\mathrm{id}} X\times[0,1].$$

Since $\phi\times\mathrm{id}$ restricted to $\partial M\times[0,1]\cup M\times\{1\}$ is a BQ-equivalence over $X^\dagger$, we recognize (7.4) as a null bordism for (7.3). Thus we have established the long exact sequence

$$(7.5)\quad \cdots \longrightarrow \mathrm{L}_{\mathrm{BQ},d}(\widehat{X};X^\dagger) \longrightarrow \mathrm{L}_{\mathrm{BQ},d}(\widehat{X}) \longrightarrow \mathrm{L}_{d(X^\dagger)}(X^\dagger) \longrightarrow \mathrm{L}_{\mathrm{BQ},d-1}(\widehat{X};X^\dagger) \longrightarrow \cdots.$$

7.2. Atiyah's L^2 signature theorem

If M is a closed even-dimensional manifold and $M_\Gamma \longrightarrow M$ is a regular cover with transformation group Γ then given any elliptic differential operator on M, Atiyah's L^2-index theorem asserts the equality of its index with the Γ-equivariant L^2-von Neumann index of its lift to M_Γ. When applied to the signature operator, Atiyah's theorem gives us the equality of the L^2-von Neumann signature of M_Γ and the signature of M:

$$\sigma_{(2)}(M_\Gamma)=\sigma(M).$$

There are many equivalent definitions of the two members of this fundamental equality (see [38] for a thorough discussion); in this article we see the above equality as an equality between the index and the L^2-von Neumann index of the signature operator, as in the original treatment of Atiyah.

In [5, Theorem 6.5], it is shown that the signature operator admits a parametrix which is ε-local; once we have this key information, Atiyah's original proof (for the signature operator) carries over to the setting of Cheeger spaces with minor modifications. However, as some of the arguments will be useful below, we present instead a proof that follows [17, Appendix].

Recall that whenever $\widehat{X}$ is an even-dimensional Cheeger space and $r : \widehat{X} \longrightarrow B\Gamma$ is the classifying map of a regular Γ-cover $\widehat{X}_\Gamma$ we have numeric signatures $\sigma(\widehat{X})$ and $\sigma_{(2)}(\widehat{X}_\Gamma)$. These are defined, respectively, as the index of the signature operator on X and the von Neumann Γ-index of the Γ-equivariant signature operator on X_Γ with respect to a choice of mezzoperversity (but independent of which one). There are (classic) K-theoretic descriptions of these two fundamental numbers that we now proceed to describe. To this end, we first recall some classic material for which we refer, for example, to [38, 44].

Let $C^*\Gamma$ denote the *maximal* group C^*-algebra associated to a discrete finitely generated group Γ. Recall that we have two trace-homomorphisms on $K_0(C^*\Gamma)$, induced respectively from two traces on $C^*\Gamma$. The first one is the canonical trace $\tau_\Gamma : C^*\Gamma \to \mathbb{C}$, obtained by extending the trace $\tau_\Gamma : \mathbb{C}\Gamma \to \mathbb{C}$ given by

$$\tau_\Gamma(\sum_\gamma \alpha_\gamma \, \gamma) := \alpha_e.$$

τ_Γ induces a well-defined trace-homomorphism, still denoted τ_Γ, from $K_0(C^*\Gamma)$ to $\mathbb{C}$, assigning to an idempotent matrix (a_{ij}) the complex number $\sum_i \tau_\Gamma(a_{ii})$. Notice that this trace-homomorphism factors through the K-theory of the group von Neumann algebra $\mathcal{N}\Gamma$:

(7.6)
$$\begin{array}{ccc} K_0(C^*\Gamma) & \xrightarrow{\quad\tau_\Gamma\quad} & \mathbb{C} \\ & {\scriptstyle \iota_*}\searrow \quad \nearrow{\scriptstyle \tau_\Gamma} & \\ & K_0(\mathcal{N}\Gamma). & \end{array}$$

The second trace, $\tau : C^*\Gamma \to \mathbb{C}$, is obtained by extending the trace $\tau : \mathbb{C}\Gamma \to \mathbb{C}$,

$$\tau(\sum_\gamma \alpha_\gamma \, \gamma) := \sum_\gamma \alpha_\gamma$$

(it is for this second trace that we need the maximal completion; the first trace is in fact already well-defined on $C^*_r\Gamma$.) The trace-homomorphism $\tau : K_0(C^*\Gamma) \to \mathbb{C}$ is obtained as before, but using τ instead of τ_Γ.

For any group Γ we can define homomorphisms

(7.7)
$$\begin{array}{ccc} & & \mathbb{Z} \\ & \nearrow{\scriptstyle \sigma} & \\ \Omega^{\mathrm{Che}}(B\Gamma) & & \\ & \searrow{\scriptstyle \sigma_{(2)}} & \\ & & \mathbb{R} \end{array}$$

by assigning to a representative $Y \xrightarrow{\omega} B\Gamma$ of a class in $\Omega^{\mathrm{Che}}(B\Gamma)$ either the signature of Y or the L^2-signature of the pull-back of $E\Gamma$ to Y along ω, to obtain the homomorphism σ, $\sigma_{(2)}$ respectively.

The homomorphism $\Omega^{\mathrm{Che}}(B\Gamma) \xrightarrow{\sigma_{(2)}} \mathbb{R}$ has a K-theoretic description that we now describe.

Recall that whenever $\widehat{X}$ is an even-dimensional Cheeger space and $r : \widehat{X} \longrightarrow B\Gamma$ is the classifying map of a regular Γ-cover we have a signature class

$$\mathrm{Ind}(\widehat{X}_\Gamma) := \mathrm{Ind}(D^{\mathscr{G}(r)}) \in K_0(C^*\Gamma) \otimes_{\mathbb{Z}} \mathbb{Z}[\tfrac{1}{2}] \equiv K_0(C^*\Gamma)[\tfrac{1}{2}],$$

where $\mathscr{G}(r)$ denotes the flat bundle of $C^*\Gamma$-modules corresponding to r, and $D^{\mathscr{G}(r)}$ is the twisted signature operator.

Notation. – As already remarked around Theorem 4.1, the index class in $K_0(C^*\Gamma)$ is defined using a choice of a mezzoperversity but it is in fact independent of this choice in $K_0(C^*\Gamma)[\frac{1}{2}]$; this is why here and in what follows we omit the mezzoperversity from the notation.

The signature class $\mathrm{Ind}(D^{\mathscr{G}(r)})$ only depends on the bordism class

$$[(\widehat{X}, r : \widehat{X} \longrightarrow B\Gamma)] \in \Omega^{\mathrm{Che}}_{\dim \widehat{X}}(B\Gamma)$$

modulo 2-torsion, and defines a group homomorphism [4, Corollary 5.11]

$$\sigma_\Gamma : \Omega^{\mathrm{Che}}_{\dim \widehat{X}}(B\Gamma) \longrightarrow K_0(C^*\Gamma)[\frac{1}{2}].$$

Now we can repeat this construction but instead twist the signature operator D by the von Neumann Mishchenko bundle

$$r^* E\Gamma \times_\Gamma \mathcal{N}\Gamma.$$

We obtain in this way $\sigma_{\mathcal{N}\Gamma} : \Omega^{\mathrm{Che}}_{\dim \widehat{X}}(B\Gamma) \longrightarrow K_0(\mathcal{N}\Gamma)[\frac{1}{2}]$ and $\sigma_{(2)}(\widehat{X}, r)$ is obtained by applying τ_Γ to the von Neumann signature class $\sigma_{\mathcal{N}\Gamma}(\widehat{X}, r) \in K_0(\mathcal{N}\Gamma)[\frac{1}{2}]$:

$$\sigma_{(2)}(\widehat{X}, r) = \tau_\Gamma(\sigma_{\mathcal{N}\Gamma}(\widehat{X}, r)) \equiv \tau_\Gamma(\mathrm{Ind}(D^{\mathscr{G}(r)})), \tag{7.8}$$

where, with a small abuse of notation, we still denote by $\mathscr{G}(r)$ the von Neumann Mishchenko bundle associated to a continuous map $r : \widehat{X} \to B\Gamma$.

THEOREM 7.1 (Atiyah's L^2-signature theorem for Cheeger spaces).
If $\widehat{X}$ is an even-dimensional Cheeger space and $r : \widehat{X} \longrightarrow BG$ is the classifying space of a regular cover $\widehat{X}_\Gamma$, then

$$\sigma(\widehat{X}) = \sigma_{(2)}(\widehat{X}_\Gamma).$$

Proof. – (We follow [17, Appendix].) Recall the two homomorphisms (7.7). Given a group homomorphism, $f : \Gamma_1 \longrightarrow \Gamma_2$, there is an induced map

$$\Omega^{\mathrm{Che}}(B\Gamma_1) \xrightarrow{Bf_*} \Omega^{\mathrm{Che}}(B\Gamma_2)$$

that sends a representative $Y \xrightarrow{\omega} B\Gamma_1$ to a representative $Y \xrightarrow{Bf \circ \omega} B\Gamma_2$. This map trivially commutes with σ. Assume now that f is injective. We are interested in showing that Bf_* commutes with $\sigma_{(2)}$. This is equivalent to showing that

$$\tau_{\Gamma_2}(\mathrm{Ind}(D^{\mathscr{G}(Bf\circ\omega)})) = \tau_{\Gamma_1}(\mathrm{Ind}(D^{\mathscr{G}(\omega)})). \tag{7.9}$$

We thus want to relate $\mathscr{G}(\omega) \to \widehat{X}$ and $\mathscr{G}(Bf \circ \omega) \to \widehat{X}$. Let

$$\mathcal{N}\Gamma_i - \mathcal{U}_{\Gamma_i} \longrightarrow B\Gamma_i$$

be the two universal Mishchenko bundles and note that

$$(Bf)^*\mathcal{U}_{\Gamma_2} = \mathcal{U}_{\Gamma_1} \otimes_{\mathcal{N}\Gamma_1} \mathcal{N}\Gamma_2,$$

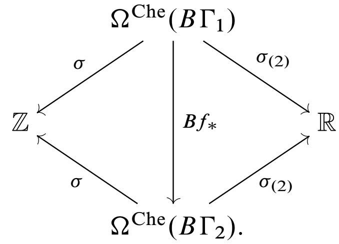
where the tensor product makes use of the map induced by f on the group von Neumann algebra (cf. [34, Appendix B, pg. 378]). Correspondingly

$$\mathcal{G}(Bf \circ \omega) = \mathcal{G}(\omega) \otimes_{\mathcal{N}\Gamma_1} \mathcal{N}\Gamma_2. \tag{7.10}$$

This means that $\mathrm{Ind}(D^{\mathcal{G}(Bf\circ\omega)})$, a formal difference of two finitely generated projective $\mathcal{N}\Gamma_2$-modules, is obtained by applying the induction homomorphism associated to f. Consequently, by [18, (2.3)], [37, Lemma 1.24]), we have that $\tau_{\Gamma_2}(\mathrm{Ind}(D^{\mathcal{G}(Bf\circ\omega)})) = \tau_{\Gamma_1}(\mathrm{Ind}(D^{\mathcal{G}(\omega)}))$ as required.

Summarizing, if f is injective then we have a commutative diagram

(7.11)



Now let

$$\{e\} \xrightarrow{\ i\ } G \xrightarrow{\ h\ } A$$

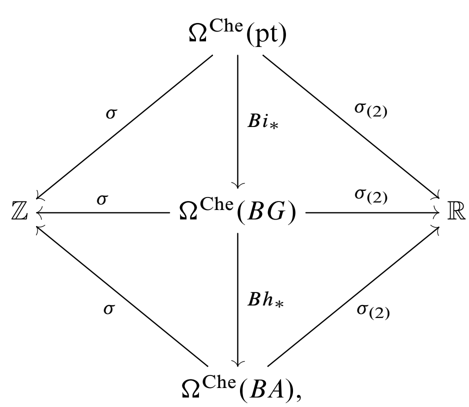
be the inclusion of the identity into G and an injection of G into an acyclic group A. (Recall that a group A is called acyclic if BA is an acyclic space and that any group has an injective homomorphism into an acyclic group, discrete if G is discrete.)

As BA is acyclic its suspension is contractible and since, by the Eilenberg-Steenrod axioms, generalized homology theories are stably invariant, they must vanish on BA. From [4] we know that Ω_*^{Che} is a generalized homology theory and so

$$(B(hi))_* : \Omega_*^{\mathrm{Che}}(\mathrm{pt}) \longrightarrow \Omega_*^{\mathrm{Che}}(BA)$$

is an isomorphism.

Thus from the commutative diagram



the equality $\sigma = \sigma_{(2)}$ on $\Omega^{\mathrm{Che}}(BG)$ reduces to the equality on $\Omega^{\mathrm{Che}}(\mathrm{pt})$ where it is immediate. □

7.3. Torsion elements and the cardinality of the BQ-structure set.

In this subsection we adapt an argument of [17] to the setting of Cheeger spaces. For any Cheeger space $\widehat{X}$ of odd dimension with $\pi_1 \widehat{X} = \Gamma$ we can use these traces and the APS-index homomorphism in order to define a group homomorphism α:

$$\begin{array}{ccc} \mathrm{L}_{\mathrm{BQ}}(\widehat{X} \times I) & \xrightarrow{\quad\alpha\quad} & \mathbb{C} \\ & {\scriptstyle \mathrm{Ind}_{\mathrm{APS}}} \searrow \quad \nearrow {\scriptstyle \tau_\Gamma - \tau} & \\ & K_0(C^*\Gamma). & \end{array} \tag{7.12}$$

Notice that α is actually valued in $\mathbb{R}$, because the index class is "self-adjoint". We write

$$\alpha = \xi_\Gamma - \xi$$

with $\xi_\Gamma := \tau_\Gamma \circ \mathrm{Ind}_{\mathrm{APS}}$ and $\xi := \tau \circ \mathrm{Ind}_{\mathrm{APS}}$. The homomorphisms ξ_Γ and ξ can be explicitly described as follows. Recall that if $\gamma \in \mathrm{L}_{\mathrm{BQ}}(\widehat{X} \times I)$ is represented by

$$(\widehat{M}; \widehat{X}, \widehat{X}') \xrightarrow{(\phi;\mathrm{id},\psi)} (\widehat{X} \times [0,1]; \widehat{X} \times \{0\}, \widehat{X} \times \{1\}) \xrightarrow{\mathrm{id}} \widehat{X} \times I,$$

then $\mathrm{Ind}_{\mathrm{APS}}(\gamma)$ is obtained from $\mathrm{Ind}_{\mathrm{APS}}(D^{\mathcal{G}(r)} + \mathcal{C})$ where D is the signature operator on $\widehat{Z} = (-\widehat{M}) \sqcup (\widehat{X} \times I)$, and, if $R : \widehat{X} \longrightarrow B\Gamma$ is the classifying map of the universal cover of $\widehat{X}$ then

$$r : \widehat{Z} \longrightarrow B\Gamma \text{ is given by } (R \circ \phi) \sqcup R,$$

where we do not distinguish between R and $R \times \mathrm{id}_{[0,1]}$. Using r we have a well-defined Γ-covering $\widehat{Z}_\Gamma$ and a Mishchenko bundle $\mathcal{G}(r) \to \widehat{Z}$; $\mathcal{C}$ denotes the Hilsum-Skandalis perturbation associated to the stratified homotopy equivalence from $\partial\widehat{M}$ to $\partial(\widehat{X} \times [0,1])$ induced by ψ and $\mathrm{id}_{\widehat{X}}$. By functoriality

$$\xi_\Gamma(\gamma) = \tau_\Gamma(\mathrm{Ind}_{\mathrm{APS}}(D^{\mathcal{G}(r)} + \mathcal{C})), \tag{7.13}$$

where the index class on the right hand side can be taken to be a von Neumann class and $\tau_\Gamma : K_0(\mathcal{N}\Gamma) \to \mathbb{C}$ is the canonical von Neumann trace. The right hand side of (7.13) is nothing but the Γ-von Neumann index of the perturbed operator on the covering $\widehat{Z}_\Gamma$, with the perturbation on the covering induced by $\mathcal{C}$ through the regular representation as in [46, Section 1.2B]. Notice once again that it is implicit here the choice of a mezzoperversity and the statement that this number, in either of the two equivalent descriptions, is in fact independent of this choice.

Similarly, see again [44], $\xi(\gamma)$ is equal to the index of the perturbed operator on $\widehat{Z}$, with the perturbation induced once again by $\mathcal{C}$ but through the trivial representation instead of the regular representation; this perturbation is in fact equal to the Hilsum-Skandalis perturbation built with ordinary differential forms (i.e., without taking the differential forms with values in the Mishchenko bundle). A similar description can be given if γ is represented by a general cycle, as in Subsection 5.6.

With these preliminaries out of the way, we now finally tackle the geometric applications we want to give. First, combining Proposition 6.9 with Atiyah's L^2-signature theorem (Theorem 7.1) we have the following result:

PROPOSITION 7.2. – *The homomorphism α vanishes on the image of the map*

$$\theta : \mathrm{N}_{\mathrm{BQ}}(\widehat{X} \times I, \widehat{X} \times \partial I) \longrightarrow \mathrm{L}_{\mathrm{BQ}}(\widehat{X} \times I)$$

from the surgery exact sequence of $\widehat{X}$.

By exactness of the surgery sequence this proposition says that α vanishes on those elements of $\mathrm{L}_{\mathrm{BQ}}(\widehat{X} \times I)$ that act trivially on $\mathrm{S}_{\mathrm{BQ}}(\widehat{X})$. Conversely, if $x \in \mathrm{L}_{\mathrm{BQ}}(\widehat{X} \times I)$ is such that $\alpha(x) \neq 0$, then we can show that x acts non-trivially on $\mathrm{S}_{\mathrm{BQ}}(\widehat{X})$.

Indeed, let $\rho : \mathrm{S}_{\mathrm{BQ}}(\widehat{X}) \longrightarrow K_{\dim \widehat{X}+1}(D^*(\widehat{X}_\Gamma)^\Gamma)$ be the ρ-map from §6.2 and let ρ_Γ be the composition with the natural map induced by the classifying map

$$\rho_\Gamma : \mathrm{S}_{\mathrm{BQ}}(\widehat{X}) \longrightarrow K_0(D^*_\Gamma).$$

In [48] it is shown that $\rho_\Gamma(\iota) = 0$, where ι denotes the class in $\mathrm{S}_{\mathrm{BQ}}(\widehat{X})$ represented by the identity map (the context in [48] is that of differentiable or topological manifold, but it is easy to see that the same arguments establish the more general statement given here). On the other hand, Benameur-Roy in [11], have defined a homomorphism

$$\beta_{\mathrm{CG}} : K_0(D^*_\Gamma) \longrightarrow \mathbb{R}, \tag{7.14}$$

for which our main result, together with Corollary 3.20 in [11], implies the following.

PROPOSITION 7.3. – *The map $\rho_{\mathrm{CG}} = \beta_{\mathrm{CG}} \circ \rho_\Gamma : \mathrm{S}_{\mathrm{BQ}}(\widehat{X}) \to \mathbb{R}$ satisfies*

$$\rho_{\mathrm{CG}}(\iota) = 0 \quad \textit{and} \quad \rho_{\mathrm{CG}}(\partial(x)(\iota)) = \alpha(x),$$

where $\partial(x)(\iota)$ denotes the action of $\mathrm{L}_{\mathrm{BQ}}(\widehat{X} \times I)$ on $\mathrm{S}_{\mathrm{BQ}}(\widehat{X})$ defined in Corollary 2.6.

In particular, if $\alpha(x) \neq 0$, then x acts non-trivially on ι.

It is pointed out in [17] that, since α is a homomorphism into $\mathbb{R}$, the existence of a non-zero element in its range implies that its range has infinite cardinality. They also point out that, if C_k is the cyclic group of order k then, for any $\ell \in \mathbb{N}$, the homomorphism $\alpha_k : L_{4\ell}(\mathbb{Z}C_k) \to \mathbb{R}$ defined as in (7.12) has range of infinite cardinality. Thus the idea is to use these elements to find elements in the range of α.

Assume that the dimension of $\widehat{X}$ is $4\ell - 1$, for some $\ell > 1$. Let $i : X \longrightarrow \widehat{X}$ be the inclusion of the regular part, which we recall is a BQ-transverse map, and let $d^{\widetilde{X}}(4\ell)$ be the dimension function for $\widehat{X}$ that is equal to 4ℓ on the inverse image of the regular part of $\widehat{X}$, so that as in §7.1 we have a homomorphism

$$i_* : L_{4\ell}(X) = L_{4\ell}(\mathbb{Z}\pi_1(X)) \longrightarrow \mathrm{L}_{\mathrm{BQ},d^{\widetilde{X}}(4\ell)}(\widehat{X}) = \mathrm{L}_{\mathrm{BQ},d^{\widetilde{X}}(4\ell)}(\widehat{X} \times I).$$

(Note that we could equally well use $\widetilde{X}$ instead of X as they are homotopy equivalent and so have the same fundamental group.) Note that we also have a homomorphism $\widetilde{\alpha} : \mathrm{L}_{4\ell}(\mathbb{Z}\pi_1(X)) \longrightarrow \mathbb{R}$ defined as above.

PROPOSITION 7.4. – *If the map $i_* : \pi_1(X) \longrightarrow \pi_1(\widehat{X})$ is injective and there is a monomorphism $p : C_k \longrightarrow \pi_1(X)$ then the following diagram commutes*

$$\begin{array}{ccccc} L_{4\ell}(\mathbb{Z}C_k) & \xrightarrow{p_*} & L_{4\ell}(\mathbb{Z}\pi_1(X)) & \xrightarrow{i_*} & L_{BQ,d^{X}(4\ell)}(\widehat{X} \times I) \\ & \searrow^{\alpha_k} & \downarrow \widetilde{\alpha} & \swarrow^{\alpha} & \\ & & \mathbb{R}. & & \end{array}$$

Proof. – The commutativity of the left triangle is a classical result, already used by Chang-Weinberger. The commutativity of the second triangle, i.e., that $\widetilde{\alpha} = \alpha \circ i_*$, follows from the more general result proved in the next proposition. □

PROPOSITION 7.5. – *If $F : \widehat{X}_1 \longrightarrow \widehat{X}_2$ is a BQ-transverse map between Cheeger spaces inducing an injection on π_1 then we have a commutative diagram*

(7.15)
$$\begin{array}{ccc} L_{BQ,d}(\widehat{X}_1 \times I) & \xrightarrow{\alpha_{\Gamma_1}} & \\ F_* \downarrow & & \mathbb{R} \\ L_{BQ,d}(\widehat{X}_2 \times I), & \xrightarrow{\alpha_{\Gamma_2}} & \end{array}$$

with $\Gamma_j := \pi_1(\widehat{X}_j)$

Proof. – With a small abuse of notation we do not distinguish between $F : \widehat{X}_1 \longrightarrow \widehat{X}_2$ and $F \times \mathrm{Id} : \widehat{X}_1 \times I \longrightarrow \widehat{X}_2 \times I$. Let $\gamma \in L_{BQ}(\widehat{X}_1 \times I)$ be represented by

$$(\widehat{M}; \widehat{X}_1, \widehat{X}'_1) \xrightarrow{(\phi;\mathrm{id},\psi)} (\widehat{X}_1 \times [0,1]; \widehat{X}_1 \times \{0\}, \widehat{X}_1 \times \{1\}) \xrightarrow{\mathrm{id}} \widehat{X}_1 \times I.$$

Then, by definition, $F_*\gamma \in L_{BQ}(\widehat{X}_2 \times I)$ is represented by

$$(\widehat{M}; \widehat{X}_1, \widehat{X}'_1) \xrightarrow{(\phi;\mathrm{id},\psi)} (\widehat{X}_1 \times [0,1]; \widehat{X}_1 \times \{0\}, \widehat{X}_1 \times \{1\}) \xrightarrow{F} \widehat{X}_2 \times I.$$

Our goal is to show that

$$\xi_{\Gamma_2}(\mathrm{Ind}_{APS}(F_*(\gamma)) = \xi_{\Gamma_1}(\mathrm{Ind}_{APS}(\gamma))$$

the equality involving ξ being trivial, as we shall explain.

Call $R_1 : \widehat{X}_1 \to B\Gamma_1$ and $R_2 : \widehat{X}_2 \to B\Gamma_2$ the classifying maps for the respective universal coverings and let $f : \Gamma_1 \to \Gamma_2$ be the homomorphism induced by F,

$$f = F_* : \pi_1(\widehat{X}_1) \equiv \Gamma_1 \to \pi_1(\widehat{X}_2) \equiv \Gamma_2.$$

We are assuming that f is *injective*. Notice that the diagram

(7.16)
$$\begin{array}{ccc} \widehat{X}_1 & \xrightarrow{F} & \widehat{X}_2 \\ R_1 \downarrow & & \downarrow R_2 \\ B\Gamma_1 & \xrightarrow{Bf} & B\Gamma_2 \end{array}$$

commutes up to homotopy. As for F and $F \times \mathrm{Id}$, we will not distinguish between R_j and $R_j \times \mathrm{Id}$ on $\widehat{X}_j \times [0,1]$. Consider now

$$\widehat{Z} := (-\widehat{M}) \sqcup (\widehat{X}_1 \times [0,1])$$

and the classifying map

$$r_1 := (R_1 \circ \phi) \sqcup R_1 : Z \to B\Gamma_1.$$

Recall our explicit description of $\xi_{\Gamma_1}(\gamma)$, given in (7.13):

$$\xi_{\Gamma_1}(\gamma) = \tau_{\Gamma_1}(\mathrm{Ind}_{\mathrm{APS}}(D^{\mathcal{G}(r_1)} + \mathcal{C}(r_1)) \tag{7.17}$$

with $\mathcal{G}(r_1)$ the von Neumann Mishchenko bundle associated to r_1 and $\mathcal{C}_1$ the Hilsum-Skandalis perturbation associated to

$$\psi : \widehat{X}_1' \to \widehat{X}_1 \ \text{ and } \ \mathrm{id} : \widehat{X}_1 \to \widehat{X}_1$$

and with values in the flat bundle $\mathscr{G}(r_1)$.[5] Consider now $\xi_{\Gamma_2}(F_*\gamma)$. By definition $F_*\gamma$ is defined in terms of a Γ_2-covering of $\widehat{Z}$. We pause a moment and describe this Γ_2-covering: it is obtained by pulling-back $E\Gamma_2$ through the map

$$((R_2 \circ F) \circ \phi) \sqcup (R_2 \circ F) : \widehat{Z} \to B\Gamma_2.$$

By (7.16) above, this is the same as the Γ_2-covering on $\widehat{Z}$ obtained by pulling-back $E\Gamma_2$ through

$$r_2 := ((Bf \circ R_1) \circ \phi) \sqcup (Bf \circ R_1) : \widehat{Z} \to B\Gamma_2.$$

Put it differently:

$$r_2 = Bf \circ r_1 : Z \to B\Gamma_2. \tag{7.18}$$

Thus, by definition,

$$\xi_{\Gamma_2}(F_*\gamma) = \tau_{\Gamma_2}(\mathrm{Ind}_{\mathrm{APS}}(D^{\mathcal{G}(Bf \circ r_1)} + \mathcal{C}(Bf \circ r_1). \tag{7.19}$$

The Hilsum-Skandalis perturbation is associated once again to (ψ, id) but with values now in the Mishchenko bundle $\mathscr{G}(Bf \circ r_1)$. We want to argue that these two Γ_j-von Neumann indices (two real numbers) are equal, viz.

$$\tau_{\Gamma_1}(\mathrm{Ind}_{\mathrm{APS}}(D^{\mathcal{G}(r_1)} + \mathcal{C}(r_1)) = \tau_{\Gamma_2}(\mathrm{Ind}_{\mathrm{APS}}(D^{\mathcal{G}(Bf \circ r_1)} + \mathcal{C}(Bf \circ r_1)).$$

We remark, crucially, that thanks to (7.18) we are in exactly the same situation as in the proof of Atiyah's theorem, see (7.9), but for the presence of the two perturbations $\mathcal{C}(r_1)$ and $\mathcal{C}(Bf \circ r_1)$. However the two perturbations are associated to the *same* homotopy equivalence on the boundary, i.e.,

$$\psi : \widehat{X}_1' \to \widehat{X}_1 \ \text{ and } \ \mathrm{id} : \widehat{X}_1 \to \widehat{X}_1,$$

and so the difference is all in the Mishchenko bundles $\mathscr{G}(r_1) \to Z$ and $\mathscr{G}(Bf \circ r_1) \to Z$. We have already remarked, see (7.10), that

$$\mathscr{G}(Bf \circ r_1) = \mathscr{G}(r_1) \otimes_{\mathcal{N}\Gamma_1} \mathcal{N}\Gamma_2.$$

Thus, as for the proof of Atiyah's theorem, $\mathrm{Ind}_{\mathrm{APS}}(D^{\mathcal{G}(Bf \circ r_1)} + \mathcal{C}(Bf \circ r_1))$ is obtained from $\mathrm{Ind}_{\mathrm{APS}}(D^{\mathcal{G}(r_1)} + \mathcal{C}(r_1)$ by applying the induction homomorphism associated to $f : \Gamma_1 \to \Gamma_2$. Thus, applying once again [18, (2.3)], [37, Lemma 1.24]), we get finally

$$\tau_{\Gamma_1}(\mathrm{Ind}_{\mathrm{APS}}(D^{\mathcal{G}(r_1)} + \mathcal{C}(r_1))) = \tau_{\Gamma_2}(\mathrm{Ind}_{\mathrm{APS}}(D^{\mathcal{G}(Bf \circ r_1)} + \mathcal{C}(Bf \circ r_1)),$$

[5] As already remarked around (7.13), this is the Γ_1-von Neumann index of a perturbed signature operator on the Γ_1-covering of $\widehat{Z}$ obtained by pulling back $E\Gamma_1$ through r_1. The perturbation, call it C_1, is induced through the regular representation by $\mathcal{C}(r_1)$ as in [46, Section 2B1].

that is,

$$\xi_{\Gamma_1}(\gamma) = \xi_{\Gamma_2}(F_*\gamma).$$

The commutativity involving ξ is clear: indeed, by definition, $\xi(\gamma)$ is the Fredholm index of the signature operator on $\widehat{Z}$ perturbed by the Hilsum-Skandalis perturbation C associated to

$$\psi : \widehat{X}_1' \to \widehat{X}_1 \text{ and } \mathrm{id} : \widehat{X}_1 \to \widehat{X}_1,$$

but with values in the trivial bundle $\mathbb{C} \times \widehat{Z} \to \widehat{Z}$. However, this is also $\xi(F_*\gamma)$. Thus

$$\xi_{\Gamma_1}(\gamma) = \xi_{\Gamma_2}(F_*\gamma) \text{ and } \xi(\gamma) = \xi(F_*\gamma),$$

so that

$$\alpha_{\Gamma_1}(\gamma) = \alpha_{\Gamma_2}(F_*\gamma),$$

as required. □

An immediate consequence of the discussion above is the infinite cardinality of the structure set.

Corollary 7.6. – *Let $\widehat{X}$ be a Cheeger space of dimension $4\ell - 1$, $\ell > 1$, such that $\pi_1(X)$ has an element of finite order and $i_* : \pi_1(X) \longrightarrow \pi_1(\widehat{X})$ is injective. There exist elements $x_j \in \mathrm{L}_{\mathrm{BQ},d}(\widehat{X} \times I)$, $j \in \mathbb{Z}$, such that $\alpha(x_i) \neq \alpha(x_j)$ for $i \neq j$. Consequently, the elements $\partial(x_j)(\iota)$ are all distinct in $\mathrm{S}_{\mathrm{BQ}}(\widehat{X})$. In particular*

$$|\,\mathrm{S}_{\mathrm{BQ}}(\widehat{X})| = \infty.$$

If $\widehat{X}$ is Witt and has depth one we can prove a sharper result:

Remark 12. – Write $\partial(x_i)(\iota) = [\widehat{M_j} \xrightarrow{f_i} \widehat{X}] \in \mathrm{S}_{\mathrm{BQ}}(\widehat{X})$ with f_i a transverse stratified homotopy equivalence. If $\widehat{X}$ is a Witt space of depth one, then we claim that $\widehat{M_i}$ is not stratified diffeomorphic to $\widehat{M_k}$ for $i \neq k$. Indeed, let

$$x_i = [(\widehat{W}_i; \widehat{X}, \widehat{M_i}) \xrightarrow{(\phi;\mathrm{id}, f_i)} (\widehat{X} \times [0,1]; \widehat{X} \times \{0\}, \widehat{X} \times \{1\}) \xrightarrow{\mathrm{id}} \widehat{X} \times [0,1]]$$

and consider $\alpha(x_i)$. This is the difference of two numbers: one is the Von Neumann-index on the total space of a Galois Γ-covering with boundary and the other is the usual index on the base of such Galois covering. We point out that the operators we are considering are invertible on the boundary because they have been perturbed by the Hilsum-Skandalis perturbation. We now write the APS-index formula, upstairs and downstairs, following [47], and take the difference; we find ourselves with the Cheeger-Gromov rho invariant of the signature operator of $\widehat{M_i} \sqcup (-\widehat{X})$ pertubed by the Hilsum-Skandalis perturbation associated to f_i. Now we proceed as in [44, Section 10], taking an ε-concentrated Hilsum-Skandalis perturbation and letting $\varepsilon \downarrow 0$. We then obtain, finally, that $\alpha(x_i)$ is equal to the difference of the Cheeger-Gromov rho-invariants of $\widehat{M_i}$ and $\widehat{X}$. Since $\alpha(x_i) \neq \alpha(x_k)$ for $i \neq k$ we can conclude that the Cheeger-Gromov rho-invariants of $\widehat{M_i}$ and $\widehat{M_k}$ are indeed different, and the statement follows from the stratified diffeomorphism invariance of the Cheeger-Gromov rho invariant on Witt spaces of depth 1, established in [47]. For more on numeric rho invariants we refer the reader to [7, 8].

REMARK 13. – If $\widehat{X}$ has depth one we can use van Kampen's theorem to see that the map $\pi_1(\widetilde{X}) \longrightarrow \pi_1(\widehat{X})$ induced by inclusion is an isomorphism if and only if the link of the singular stratum of $\widehat{X}$ is simply connected (i.e., $\widehat{X}$ is supernormal). For $\widehat{X}$ of arbitrary depth, using Remark 11 in the setting of simply connected links lets us argue as above to see that torsion in the fundamental group of any 'closed stratum' $\widehat{Y}$ forces $|\,\mathrm{S}_{\mathrm{BQ}}(\widehat{X})| = \infty$. Indeed note that the condition on fundamental groups is superseded by the injective map

$$\mathrm{L}(\mathbb{Z}\pi_1(\widehat{Y}))[\tfrac{1}{2}] \longrightarrow \mathrm{L}_{\mathrm{BQ}}(\widehat{X})[\tfrac{1}{2}].$$

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