

Covering action on Conley index theory

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(Received 3 May 2021 and accepted in revised form 18 January 2022)

Abstract. In this paper we apply Conley index theory in a covering space of an invariant set S , possibly not isolated, in order to describe the dynamics in S . More specifically, we consider the action of the covering translation group in order to define a topological separation of S which distinguishes the connections between the Morse sets within a Morse decomposition of S . The theory developed herein generalizes the classical connection matrix theory, since one obtains enriched information on the connection maps for non-isolated invariant sets, as well as for isolated invariant sets. Moreover, in the case of an infinite cyclic covering induced by a circle-valued Morse function, one proves that the Novikov differential of f is a particular case of the p -connection matrix defined herein.

Key words: Conley index, connection matrix, covering space, circle-valued function, Novikov complex

2020 Mathematics Subject Classification: 37B30, 14E20 (Primary); 57M10, 58E40, 37B35 (Secondary)

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1. Introduction

Conley index theory is concerned with the topological structure of invariant sets of a continuous flow on a topological space X and how they are connected to each other [4, 15, 20]. The foundation of this theory, introduced in [4], relies on the fact that there are two possibilities for the behavior of flow lines into an isolated invariant set: a point can either be chain recurrent or it can belong to a connecting orbit from one chain-recurrent piece to another. For instance, in the case of Morse–Novikov theory on a compact manifold, the chain-recurrent pieces are the rest points (and periodic orbits in the Novikov case) and the Morse–Novikov indices are related to the topology of the manifold. Furthermore, the Morse–Novikov inequalities impose the existence of connections between some pairs of rest points. On the other hand, in the case of Conley theory, for a flow not necessarily gradient-like, instead of connections between rest points, the global topology of the space forces connections between chain-recurrent pieces (isolated invariant sets) of the flow. Such information is encoded in a connection matrix (which corresponds to the boundary operator in Morse–Novikov theory).

More specifically, given an isolated invariant set $S \subseteq X$, the approach is to consider a decomposition $\mathcal{M}(S)$ of S into a family of compact invariant sets which contains the recurrent set and such that the flow on the rest of the space is gradient-like, that is, there is a continuous Lyapunov function which is strictly decreasing on orbits which are not chain recurrent. Such a decomposition is called a Morse decomposition of S and each set of the family is known as a Morse set. The Conley index of each Morse set carries some topological information about the local behavior of the flow near that set.

The connection matrix theory [8, 11, 19] was motivated by the desire to obtain information on the connections between the Morse sets within a Morse decomposition. The entries of a connection matrix are homomorphisms between the homology Conley indices of the Morse sets, hence it contains information about the distribution of the Morse sets within the Morse decomposition.

In the case of a Morse–Smale flow, connection matrices have a nice characterization. Suppose that φ is the negative gradient flow of a Morse function f on a closed manifold M , satisfying the Morse–Smale transversality condition. Consider the $\prec_f$ -Morse decomposition where each Morse set corresponds to a critical point of f and $\prec_f$ is the flow ordering. In this case, the connection matrix is unique and coincides with the differential of the Morse complex as proved in [21].

Another interesting situation is when φ is the negative gradient flow of a circle-valued Morse function f on a closed manifold M satisfying the Morse–Smale transversality condition. One can also define a chain complex, called the Novikov complex $(\mathcal{N}_*(f), \partial^{\text{Nov}})$, as in [5, 17]. However, ∂^{Nov} is not a connection matrix. For instance, the differential ∂^{Nov} corresponding to the example in Figure 1 is non-zero. In fact, $\partial^{\text{Nov}}(h_2^4, h_1^2) =$

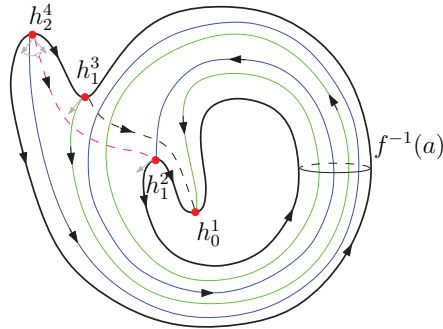


FIGURE 1. A flow on the torus.

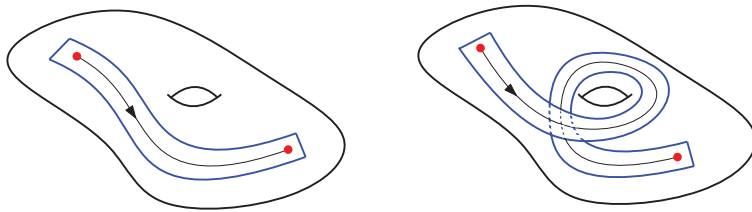


FIGURE 2. Flows lines on the torus.

$\partial^{\text{Nov}}(h_1^3, h_0^1) = 1 - t^2$. On the other hand, the connection matrix is the null map. Hence, the zero entries of the connection matrix do not give information about the connections between the corresponding Morse sets. In this particular setting, the Novikov differential gives more information than the connection matrix.

One approach to enriching the connection matrix is to consider a topological separation of the connecting sets to obtain an additive property of the connection map, as done by McCord in [12]. However, in this separation it is not possible to distinguish, for example, the connections in Figure 2, since both have the same connection maps. Therefore, one must consider an algebraic structure capable of capturing more information on those connections, as the Novikov differential does.

The goal in this paper is to define a chain complex associated to an invariant set S , not necessarily isolated, whose differential gives enriched information on the connections between the Morse sets of S .

In order to obtain information on the connecting orbits between critical points, the Novikov differential uses the Novikov ring and counts the orbits on the infinite cyclic covering on M . Inspired by the Novikov case, we will look for information about the connections between the Morse sets on the pullback flow defined on a regular cover $(\tilde{M}, p)$ of M , providing an algebraic setting that arises from the ambient space in order to distinguish those connections. More specifically, we use the covering action to distinguish all connections up to action of the covering translation group. For instance, the two connecting orbits in Figure 2 are different with respect to the covering action.

We introduce a chain complex $(NC(S), N\Delta)$ associated to a pair (S, p) , where S is an invariant set, p is a regular covering map and $M(S)$ is an attractor–repeller decomposition of S . We will assume coefficients in $\mathbb{Z}((G))$, where G is the group of covering translations of p . The map $N\Delta$ is called a p -connection matrix associated to $M(S)$ and contains enriched information on the connecting orbits. We prove the invariance of this chain complex under equivalent covering spaces.

Whenever S is an isolated invariant set and either p is the trivial covering map or G is projected into the trivial group, we recover the usual setting of Conley index theory. In other words, the p -connection matrix introduced herein coincides with the classical connection matrix defined by Franzosa in [11]. Moreover, when G is the infinite cyclic group and φ is the negative gradient flow of a circle-valued Morse function, we prove that the Novikov differential of f is a particular case of the p -connection matrix.

This paper is organized as follows. §2 recalls relevant elements of the connection matrix theory, as well as some basic facts about the Novikov chain complex. In §3 we prove some properties of invariant sets of a pullback flow on a regular covering space. Section 4 is the core of the paper, where we introduce the theory of p -connection matrices. In §4.1 we define p -attractor–repeller decompositions of invariant sets and we prove that S can be decomposed into smaller invariant sets where we can apply Conley index theory on the pullback flow. In §4.2 we state the algebraic structure $\mathbb{Z}((G))$ that enables us to count the flow lines connecting the Morse sets, distinguishing orbits according to the deck transformation group G . Moreover, we present the p -connection matrices for invariant sets. In §5, we consider the infinite cyclic covering induced by a circle-valued Morse function; in this case $\mathbb{Z}((G))$ is the Novikov ring and the p -connection matrix coincides with the Novikov differential.

2. Background

2.1. Attractor–repeller decompositions and connection matrices. Throughout this paper let $(P, <)$ be a partial ordered set with partial order $<$, where P is a finite set of indices. An *interval* in $<$ is a subset $I \subseteq P$ such that if $p, q \in I$ and $p < r < q$ then $r \in I$. The set of intervals in $<$ is denoted by $I(<)$.

An *adjacent n -tuple of intervals* in $<$ is an ordered collection $(I_1, \dots, I_n)$ of mutually disjoint non-empty intervals in $<$ satisfying:

- $\bigcup_{i=1}^n I_i \in I(<)$;
- $\pi \in I_j, \pi' \in I_k, j < k$ imply $\pi' \not\prec \pi$.

The collection of adjacent n -tuples of intervals in $<$ is denoted $I_n(<)$. An adjacent 2-tuple of intervals is also called an *adjacent pair* of intervals. If $<'$ is an extension of $<$, then $I_n(<') \subseteq I_n(<)$, for each n . If $<_I$ is a restriction of $<$ to an interval $I \in I(<)$, then $I_n(<_I) \subseteq I_n(<)$, for each n . If (I, J) is an adjacent pair (2-tuple) of intervals, then $I \cup J$ is denoted by IJ . If $(I_1, \dots, I_n) \in I_n(<)$ and $\bigcup_{i=1}^n I_i = I$, then $(I_1, \dots, I_n)$ is called a *decomposition* of I .

Let $\varphi : \mathbb{R} \times X \rightarrow X$ be a continuous flow on a locally compact Hausdorff space X and let $S \subseteq X$ be an invariant set under φ . We use the notation $x \cdot t := \varphi(t, x)$. For any set $Y \subseteq S$, the ω -limit and α -limit sets are given by $\omega(Y) = \bigcap_{t>0} \overline{Y \cdot [t, \infty)}$ and $\alpha(Y) =$

$\bigcap_{t>0} \overline{Y \cdot (-\infty, -t]}$, respectively. Both sets are closed, and if S is compact then they will be compact. An invariant set $A \subseteq S$ is an attractor in S if there exists a S -neighborhood U of A such that $\omega(U) = A$. A repeller in S is an invariant set $R \subseteq S$ such that there exists a S -neighborhood U of R with $\alpha(U) = R$. Whenever S is a compact set then A and R are also compact sets.

A ($<$ -ordered) *Morse decomposition* of S is a collection $\mathcal{M}(S) = \{M(\pi) \mid \pi \in P\}$ of mutually disjoint compact invariant subsets of S , indexed by a finite set P , such that if $x \in S \setminus \bigcup_{\pi \in P} M(\pi)$ then there exist $\pi < \pi'$ such that $\alpha(x) \subseteq M(\pi')$ and $\omega(x) \subseteq M(\pi)$. Each set $M(\pi)$ is called a *Morse set*. A partial order on P with this property induces a partial order on $\mathcal{M}(S)$ called an *admissible ordering* of the Morse decomposition.

The flow defines an admissible ordering on $\mathcal{M}(S)$, called the *flow ordering* of $\mathcal{M}(S)$, denoted $<_F$, such that $M(\pi) <_F M(\pi')$ if and only if there exists a sequence of distinct elements of $P : \pi = \pi_0, \dots, \pi_n = \pi'$, where the set of connecting orbits between $M(\pi_j)$ and $M(\pi_{j-1})$,

$$\begin{aligned} &C(M(\pi_j), M(\pi_{j-1})) \\ &= \{x \in S \setminus (M(\pi_j) \cup M(\pi_{j-1})) \mid \alpha(x) \subseteq M(\pi_j) \text{ and } \omega(x) \subseteq M(\pi_{j-1})\}, \end{aligned}$$

is non-empty for each $j = 1, \dots, n$. Note that every admissible ordering of $\mathcal{M}$ is an extension of $<_F$.

Given a Morse decomposition $\mathcal{M}(S)$ of S , the existence of an admissible ordering on $\mathcal{M}(S)$ implies that any recurrent dynamics in S must be contained within the Morse sets, thus the dynamics off the Morse sets must be gradient-like. For this reason, Conley index theory refers to the dynamics within a Morse set as local dynamics and off the Morse sets as global dynamics.

We briefly introduce the Conley index of an isolated invariant set and the connection matrix theory, which addresses this latter aspect. Recall that $S \subseteq X$ is an *isolated invariant set* if there exists a compact set $N \subseteq X$ such that $S \subseteq \text{int}(N)$ and

$$S = \text{Inv}(N, \varphi) = \{x \in N \mid \varphi(\mathbb{R}, x) \subseteq N\}.$$

In this case N is said to be an *isolating neighborhood* for S in X . Note that isolated invariant sets are compact sets. An *index pair* for an isolated invariant set S is a pair (N_1, N_0) of compact sets $N_0 \subseteq N_1$ such that: (i) $S \subseteq \text{int}(N_1 \setminus N_0)$ and $\text{cl}(N_1 \setminus N_0)$ is an isolating neighborhood for S ; (ii) N_0 is *positively invariant* in N_1 , that is, given $x \in N_0$ and $t > 0$ such that $x \cdot [0, t] \subseteq N_1$, then $x \cdot [0, t] \subseteq N_0$; (iii) N_0 is an *exit set* for N_1 , that is, given $x \in N_1$ and $t_1 > 0$ with $x \cdot t_1 \notin N_1$, there exists $t_0 \in [0, t_1]$ such that $x \cdot [0, t_0] \subseteq N_1$ and $x \cdot t_0 \in N_0$. The theorems of existence and equivalence of index pairs guarantee that given any isolating neighborhood $N \subseteq X$ of S and any neighborhood U of S , there exists an index pair (N_1, N_0) for S in X such that N_1 and N_0 are positively invariant in N and $\text{cl}(N_1 \setminus N_0) \subseteq U$. Moreover, the homotopy type of the pointed space N_1/N_0 is independent of the choice of the index pair and therefore it only depends on the behavior of the flow near the isolated invariant set S . For more details, see [4, 20].

The *homology Conley index* of S , $CH_*(S)$, is the homology (computed with coefficients in a module over a principal ideal domain) of the pointed space N_1/N_0 , where (N_1, N_0) is

an index pair for S . Setting

$$M(I) = \bigcup_{\pi \in I} M(\pi) \cup \bigcup_{\pi, \pi' \in I} C(M(\pi'), M(\pi)),$$

the Conley index $CH_*(M(I))$ of $M(I)$, denoted by $H_*(I)$, is well defined, since $M(I)$ is an isolated invariant set for all $I \in I(<)$. For more details, see [8].

The simplest case of a Morse decomposition of a compact invariant set S is an attractor–repeller pair (A, R) : A is an attractor in S and $R = \{x \in S \mid \omega(x) \cap A = \emptyset\}$ is its dual repeller. Note that, since S is compact the dual repeller is in fact a repeller; see [20]. Then S is decomposed into $A \cup C(R, A) \cup R$.

Given an attractor–repeller pair (A, R) of an isolated invariant set S , one obtains a long exact sequence, called the attractor–repeller sequence, which relates the Conley indices of the isolated invariant sets S , A and R , namely

$$\cdots \longrightarrow CH_k(A) \longrightarrow CH_k(S) \longrightarrow CH_k(R) \xrightarrow{\partial} CH_{k-1}(A) \longrightarrow \cdots$$

The map ∂ , in the previous sequence, is called the *connection homomorphism* or *connection map*. It has the property that if $\partial \neq 0$ then there exist connecting orbits from R to A in S . In many cases, it can give more information about the set of connecting orbits. For instance, if A and R are hyperbolic fixed points of indices k and $k - 1$, respectively, satisfying the transversality condition, then the connection map is equivalent to the intersection number between the stable and unstable spheres of A and R , respectively.

For a Morse decomposition $\mathcal{M}(S)$ with an admissible order $(P, <)$, there is an attractor–repeller sequence for every adjacent pair of intervals in P . Franzosa introduced in [11] connection matrices as devices that allow us to encode simultaneously the information in all of these sequences. Roughly speaking, connection matrices are boundary maps defined on the sum of the homology Conley indices of the Morse sets enabling each attractor–repeller sequence to be reconstructed.

More specifically, for each interval $I \subseteq P$, set $C_*\Delta(I) = \bigoplus_{\pi \in I} CH_*(M(\pi))$ and consider an upper triangular boundary map $\Delta(P) : C_*\Delta(P) \rightarrow C_*\Delta(P)$ with respect to the partial order $<$. Let $\Delta(I)$ be the submatrix of $\Delta(P)$ with respect to the interval I . Given an adjacent pair of intervals I, J in P , one can construct the commutative diagram

$$\begin{array}{ccccccc} 0 & \longrightarrow & C_k\Delta(I) & \xrightarrow{i} & C_k\Delta(IJ) & \xrightarrow{p} & C_k\Delta(J) \longrightarrow 0 \\ & & \downarrow \Delta(I) & & \downarrow \Delta(IJ) & & \downarrow \Delta(J) \\ 0 & \longrightarrow & C_{k-1}\Delta(I) & \xrightarrow{i} & C_{k-1}\Delta(IJ) & \xrightarrow{p} & C_{k-1}\Delta(J) \longrightarrow 0 \end{array}$$

where i and p are the inclusion and projection homomorphisms, respectively. In other words, we have a short exact sequence of chain complexes where the Δ s act as boundary homomorphisms. Since $(C_*\Delta(I), \Delta(I))$ is a chain complex, applying the homological functor H , the previous diagram produces a long exact sequence

$$\cdots \longrightarrow H_k\Delta(I) \xrightarrow{i_*} H_k\Delta(IJ) \xrightarrow{p_*} H_k\Delta(J) \xrightarrow{[\Delta(J,I)]} H_{k-1}\Delta(I) \longrightarrow \cdots$$

Therefore, for every adjacent pair of intervals, the upper triangular boundary map $\Delta(P)$ generates a long exact sequence. $\Delta(P)$ is called a *connection matrix* if all these sequences are canonically isomorphic to the corresponding attractor–repeller sequences. In other words, for each interval I , there is an isomorphism $\phi(I) : H_*\Delta(I) \rightarrow CH_*(M(I))$ such that $\phi(\{p\}) = \text{Id}$ for every $p \in P$, and for every adjacent pair of intervals (I, J) the following diagram commutes:

$$\begin{array}{ccccccc} \cdots & \longrightarrow & H_k\Delta(I) & \xrightarrow{i_*} & H_k\Delta(IJ) & \xrightarrow{p_*} & H_k\Delta(J) & \xrightarrow{\partial_*} & H_{k-1}\Delta(I) & \longrightarrow & \cdots \\ & & \downarrow \phi(I) & & \downarrow \phi(IJ) & & \downarrow \phi(J) & & \downarrow \phi(I) & & \\ \cdots & \longrightarrow & CH_k(M(I)) & \xrightarrow{i_*} & CH_k(M(IJ)) & \xrightarrow{p_*} & CH_k(M(J)) & \xrightarrow{\partial_*} & CH_{k-1}(M(I)) & \longrightarrow & \cdots \end{array}$$

Franzosa proved in [11] that, given a Morse decomposition $\mathcal{M}(S)$ of an isolated invariant set S , there exists a connection matrix for $\mathcal{M}(S)$. Moreover, he showed that non-zero entries in a connection matrix imply the existence of connecting orbits, that is, if $\Delta(p, q) \neq 0$ then $p < q$; in particular, for the flow defined order $<$ there is a sequence of connecting orbits from $M(q)$ to $M(p)$.

2.2. Dynamical chain complexes. In this subsection we present some background material on dynamical chain complexes associated to Morse–Smale functions and to circle-valued Morse functions. The main references for Morse chain complexes are [2, 21, 22] and for Novikov complexes are [5, 17, 18].

2.2.1. Morse chain complex. A *Morse–Smale function* (f, g) on a compact manifold $(M, \partial M)$ with boundary (possibly empty) is a function $f : M \rightarrow \mathbb{R}$ together with a Riemannian metric g such that:

- (1) the critical points are non-degenerate;
- (2) f is regular on each boundary component N of ∂M , that is, for all $x \in N$, $\nabla f(x) \notin T_x N \subseteq T_x M$;
- (3) for any two critical points $p, q \in M$, the stable and unstable manifolds $W^u(p)$ and $W^s(q)$ with respect to the negative gradient flow φ of f intersect transversely.

Let $\text{Crit}_k(f)$ be the set of critical points of f with Morse index k . Given $p \in \text{Crit}_k(f)$ and $q \in \text{Crit}_\ell(f)$, define $\mathcal{M}_{pq} = W^u(p) \cap W^s(q)$, the *connecting manifold* of p and q with respect to φ , and $\mathcal{M}_p^q = W^u(p) \cap W^s(q) \cap f^{-1}(a)$, the *moduli space* of p and q , that is, the space of connecting orbits from p to q , where a is some regular value of f with $f(q) < a < f(p)$. It is well known that $\mathcal{M}_p^q$ is a $(k - \ell - 1)$ -dimensional manifold. Moreover, when $\ell = k - 1$, $\mathcal{M}_p^q$ is a zero-dimensional compact manifold, hence it is a finite set.

Fix orientations of $T_p(W^u(p))$, for all $p \in \text{Crit}(f)$. Since $W^u(p)$ is contractible, these orientations induce orientations on the tangent spaces to the whole unstable manifolds. Also, since $W^s(p)$ is contractible, it follows that the normal space $\mathcal{V}_p W^s(p)$ is orientable and the orientation of $W^u(p)$ induces an obvious orientation on $\mathcal{V}_p W^s(p)$. Moreover, given $p, q \in \text{Crit}(f)$, the transversality condition implies that $T_{\mathcal{M}_{pq}} W^u(p)$ splits along $\mathcal{M}_{pq}$ as $T_{\mathcal{M}_{pq}} W^u(p) \simeq T\mathcal{M}_{pq} \oplus \mathcal{V}_{\mathcal{M}_{pq}} W^s(q)$, where the last term denotes the normal bundle of $W^s(q)$ restricted to $\mathcal{M}_{pq}$. Choose an orientation on $\mathcal{M}_{pq}$ such that this

isomorphism is orientation preserving. Whenever $p \in \text{Crit}_k(f)$ and $q \in \text{Crit}_{k-1}(f)$, the orientation on M_{pq} gives an orientation on the flow line associated to each $z \in M_p^q$. In this case, define $\epsilon(z) := +1$ if this orientation coincides with the one induced by the flow, otherwise define $\epsilon(z) := -1$. Finally, let

$$n(p, q; f) = \sum_{z \in M_p^q} \epsilon(z).$$

Given a Morse–Smale function $(f, g) : M \rightarrow \mathbb{R}$, the $\mathbb{Z}$ -coefficient *Morse group* is the free $\mathbb{Z}$ -module $C_*(f)$ generated by the critical points of f and graded by their Morse index, that is, $C_k(f) = \mathbb{Z}[\text{Crit}_k(f)]$.

The $\mathbb{Z}$ -coefficient *Morse boundary operator* ∂ of f is defined on a generator p by

$$\begin{aligned} \partial_k : C_k(f) &\longrightarrow C_{k-1}(f) \\ p &\longmapsto \sum_{q \in \text{Crit}_{k-1}(f)} n(p, q; f)q. \end{aligned}$$

The pair $(C_*(f), \partial_*)$ is called the *Morse chain complex* of the Morse–Smale function (f, g) .

Salamon proved in [21] that the Morse boundary operator is a special case of the connection matrix. More specifically, considering the $<_{\varphi_f}$ -ordered Morse decomposition $\mathcal{M}(M) = \{M_\pi\}_{\pi \in P}$ where each Morse set M_π is a critical point of f and $<_{\varphi_f}$ is the flow ordering, there exists a unique connection matrix for $\mathcal{M}(M)$, which coincides with the Morse boundary operator ∂ .

2.2.2. Novikov chain complex. Let $\mathbb{Z}[t, t^{-1}]$ be the Laurent polynomial ring. The *Novikov ring* $\mathbb{Z}((t))$ is the set consisting of all Laurent series

$$\lambda = \sum_{i \in \mathbb{Z}} a_i t^i$$

in one variable with coefficients $a_i \in \mathbb{Z}$, such that the part of λ with negative exponents is finite, that is, there is $n = n(\lambda)$ such that $a_k = 0$ if $k < n(\lambda)$. In fact, $\mathbb{Z}((t))$ has a natural Euclidean ring structure such that the inclusion $\mathbb{Z}[t, t^{-1}] \subseteq \mathbb{Z}((t))$ is a homomorphism.

Let M be a compact connected manifold and $f : M \rightarrow S^1$ be a smooth map. Given a point $x \in M$ and a neighborhood V of $f(x)$ in S^1 diffeomorphic to an open interval of $\mathbb{R}$, the map $f|_{f^{-1}(V)}$ is identified with a smooth map from $f^{-1}(V)$ to $\mathbb{R}$. Hence, one can define non-degenerate critical points and Morse indices in this context as in the classical case of smooth real-valued functions. A smooth map $f : M \rightarrow S^1$ is called a *circle-valued Morse function* if its critical points are non-degenerate. Denote by $\text{Crit}(f)$ the set of critical points of f and by $\text{Crit}_k(f)$ the set of critical points of f of index k .

Consider the exponential function $\text{Exp} : \mathbb{R} \rightarrow S^1$ given by $t \mapsto e^{2\pi i t}$. The structure group of this covering is the subgroup $\mathbb{Z} \subseteq \mathbb{R}$ acting on $\mathbb{R}$ by translations. It is convenient to use the multiplicative notation for the structure group and denote by t the generator corresponding to -1 in the additive notation. Let $(\bar{M}, p_E)$ be the infinite cyclic covering of M , where $\bar{M} = f^*(\mathbb{R}) = \{(x, t) \in M \times \mathbb{R} \mid f(x) = [t] \in S^1\}$ and $p_E : \bar{M} \rightarrow M$ is induced by the map $f : M \rightarrow S^1$ from the universal covering Exp . There exists a

$\mathbb{Z}$ -equivariant Morse–Smale function $F : \overline{M} \rightarrow \mathbb{R}$ which makes the following diagram commutative:

$$\begin{array}{ccc} \overline{M} & \xrightarrow{F} & \mathbb{R} \\ p_E \downarrow & & \downarrow \text{Exp} \\ M & \xrightarrow{f} & S^1 \end{array}$$

Note that if $\text{Crit}(F)$ is non-empty then it has infinite cardinality. Since $\overline{M}$ is non-compact, one cannot apply the classical Morse theory to study F . To overcome this, one can restrict F to a fundamental cobordism W of $\overline{M}$ with respect to the action of $\mathbb{Z}$. The *fundamental cobordism* W is defined as $W = F^{-1}([a-1, a])$, where a is a regular value of F . It can be viewed as the compact manifold obtained by cutting M along the submanifold $V = f^{-1}(\alpha)$, where $\alpha = \text{Exp}(a)$. Hence, $(W, V, t^{-1}V)$ is a cobordism with both boundary components diffeomorphic to V .

From now on, we consider circle-valued Morse functions f such that the vector field $-\nabla f$ satisfies the transversality condition, that is, the lift $-\nabla F$ of $-\nabla f$ to $\overline{M}$ satisfies the classical transversality condition on the unstable and stable manifolds. Denote by $\overline{\varphi}$ the pullback of φ , where φ is the flow associated to $-\nabla f$.

Fix lifts $\overline{p}, \overline{q} \in \text{Crit}(F)$ of $p, q \in \text{Crit}(f)$, respectively. Choosing arbitrary orientations for all unstable manifolds $W^u(p)$ of critical points of f , one considers the induced orientations on the unstable manifolds $W^u(t^\ell \overline{p})$ and $W^u(t^\ell \overline{q})$, for $\ell \in \mathbb{Z}$. As each path in M that originates at p lifts to a unique path in $\overline{M}$ originating at $\overline{p}$, the space $\bigcup_{\ell \in \mathbb{Z}} \mathcal{M}(\overline{p}, t^\ell \overline{q})$ of flow lines of $\overline{\varphi}$ that join $\overline{p}$ to one of the points $t^\ell \overline{q}$, $\ell \in \mathbb{Z}$, is bijective with $\mathcal{M}(p, q)$. In particular, for consecutive critical points p and q , by the equivariance of F ,

$$n(t^\ell \overline{p}, t^\ell \overline{q}; F) = n(\overline{p}, \overline{q}; F)$$

for all $\ell \in \mathbb{Z}$, where $n(t^\ell \overline{p}, t^\ell \overline{q}; F)$ is the intersection number between the critical points $t^\ell \overline{p}$ and $t^\ell \overline{q}$ of F .

Given $p \in \text{Crit}_k(f)$ and $q \in \text{Crit}_{k-1}(f)$, the *Novikov incidence coefficient* between p and q is defined as

$$N(p, q; f) = \sum_{\ell \in \mathbb{Z}} n(\overline{p}, t^\ell \overline{q}; F) t^\ell.$$

For more details, see [5, 17].

Let $\mathcal{N}_k$ be the $\mathbb{Z}((t))$ -module freely generated by the critical points of f of index k . Consider the k th boundary operator $\partial_k^{\text{Nov}} : \mathcal{N}_k \rightarrow \mathcal{N}_{k-1}$ which is defined on a generator $p \in \text{Crit}_k(f)$ by

$$\partial_k^{\text{Nov}}(p) = \sum_{q \in \text{Crit}_{k-1}(f)} N(p, q; f) q$$

and extended to all chains by linearity. In [17] it is proved that $\partial_k^{\text{Nov}} \circ \partial_{k+1}^{\text{Nov}} = 0$, hence $(\mathcal{N}_*(f), \partial^{\text{Nov}})$ is a chain complex which is called the *Novikov chain complex* associated to f .

3. Invariance properties of pullback flows on covering spaces

Consider a metric space X which admits a regular covering space $\tilde{X}$ with covering map $p : \tilde{X} \rightarrow X$ and let G be the group of the covering translations (deck transformation group). Thus, the action of G on each fiber is free and transitive and the quotient $\tilde{X}/G$ can be identified with X . Given a subset B of X and $g \in G$, we will denote by gB the set $\{gb \mid b \in B\}$ and if $e \in G$ is the trivial element, $B = eB$.

Let $\varphi : \mathbb{R} \times X \rightarrow X$ be a continuous flow on X . If $\tilde{X}$ is connected, locally path connected and $\varphi_{\#} \circ (\text{Id} \times p)_{\#}(\pi_1(\mathbb{R} \times \tilde{X})) \subseteq p_{\#}(\pi_1(\tilde{X}))$, then one can define the pullback flow of φ by p , denoted by $\tilde{\varphi}$, as the lifting of the map $\varphi \circ (\text{Id} \times p)$. Hence, we have the following commutative diagram:

$$\begin{array}{ccc} (\mathbb{R} \times \tilde{X}, (0, \tilde{x})) & \xrightarrow{\tilde{\varphi}} & (\tilde{X}, \tilde{x}) \\ \downarrow \text{Id} \times p & & \downarrow p \\ (\mathbb{R} \times X, (0, x)) & \xrightarrow{\varphi} & (X, x) \end{array}$$

Note that, as a consequence of the unique path lifting property of coverings, when p restricts to a homeomorphism from some subset $\tilde{Y}$ of $\tilde{X}$ onto an invariant subset of X , we have that $\tilde{Y}$ is also an invariant set.

If $x \in X$ and $\varphi(\mathbb{R}, x)$ is an aperiodic orbit, then the trajectories of the points of $p^{-1}(x)$ under the flow $\tilde{\varphi}$ are pairwise disjoint and aperiodic, and p restricted to any such trajectory is one-to-one. See [3].

Throughout this paper, let X be a locally compact metric space and $(\tilde{X}, p)$ be a regular cover of X , where $\tilde{X}$ is a connected, locally path connected metric space. Also, we use the following definition: a set $U \subseteq X$ is *evenly covered* by p if $p^{-1}(U)$ is a disjoint union of sets $\tilde{U}_{\lambda} \subset \tilde{X}$ such that $p|_{\tilde{U}_{\lambda}} : \tilde{U}_{\lambda} \rightarrow U$ is a homeomorphism for every λ . The homeomorphic copies in $\tilde{X}$ of an evenly covered set U are called *sheets* over U .

The next result is a characterization of evenly covered sets.

PROPOSITION 3.1. *Let $S \subseteq X$ be an evenly covered set. Then there exists $\tilde{S} \subset \tilde{X}$ such that $p^{-1}(S) = \bigsqcup_{g \in G} g\tilde{S}$ and $p|_{g\tilde{S}}$ is a homeomorphism, where G is the deck transformation group.*

Proof. Since S is an evenly covered set, there exists a sheet $\tilde{S} \subset \tilde{X}$ over S such that $p|_{\tilde{S}} : \tilde{S} \rightarrow S$ is a homeomorphism. It follows from the freeness and the transitivity of the action of G in $\tilde{X}$ that $p^{-1}(S) = \bigsqcup_{g \in G} g\tilde{S}$. Moreover, given a deck transformation $g \in G$, one has that $p|_{g\tilde{S}} = p|_{\tilde{S}} \circ g^{-1}|_{g\tilde{S}}$ is a homeomorphism. $\square$

The next result gives an important property of evenly covered sets on a regular covering space which is essential for this work.

THEOREM 3.2. *Let $S \subseteq X$ be an evenly covered compact set. Given a sheet $\tilde{S}$ over S , there exists a neighborhood W of $\tilde{S}$ such that $p|_W$ is a homeomorphism onto its image.*

Proof. It is sufficient to prove that there exists a compact neighborhood W of $\tilde{S}$ such that $p|_W$ is injective.

By Proposition 3.1, $p^{-1}(S) = \bigsqcup_{g \in G} g\tilde{S}$. Let $\mathcal{B} = (\bigsqcup_{g \in G} g\tilde{S}) \setminus \tilde{S}$. First, we prove that there exist closed disjoint neighborhoods of $\mathcal{B}$ and $\tilde{S}$.

Claim 1. $\mathcal{B}$ is a closed set.

If $\mathcal{B} = \emptyset$, the claim holds. Suppose that $\mathcal{B} \neq \emptyset$. Let $x \in \overline{\mathcal{B}}$ and let $(x_n)_{n \in \mathbb{N}}$ be a sequence in $\mathcal{B}$ such that $x_n \rightarrow x$. Then $(p(x_n))_{n \in \mathbb{N}}$ is a sequence in S and $p(x_n) \rightarrow p(x)$. Since $p^{-1}(S)$ is a closed set, $x \in p^{-1}(S)$. Suppose $x \in \tilde{S}$. Let V_x and $U_{p(x)}$ be neighborhoods of x and $p(x)$, respectively, such that $p : V_x \rightarrow U_{p(x)}$ is a homeomorphism. Then there is $n_0 \in \mathbb{N}$ such that $x_n \in V_x$ for all $n > n_0$. Moreover, since $p|_{\tilde{S}} : \tilde{S} \rightarrow S$ is a homeomorphism, there exists a sequence $(x'_n)_{n \in \mathbb{N}}$ in $\tilde{S}$ such that $p(x_n) = p(x'_n)$ and $x'_n \rightarrow x$. Then there is $n_1 \in \mathbb{N}$ such that $x'_n \in V_x$ for all $n > n_1$. Thus for all $n > \max\{n_0, n_1\}$, $x_n \neq x'_n$ and $p(x_n) = p(x'_n)$. This contradicts the fact that $p|_{V_x}$ is a homeomorphism, hence $x \notin \tilde{S}$. Consequently, $x \in \mathcal{B}$ and $\mathcal{B}$ is a closed set. $\square$

As a consequence of Claim 1 and the fact that $\tilde{X}$ is a metric space, there exist closed neighborhoods $F_{\tilde{S}}$ and $F_{\mathcal{B}}$ of $\tilde{S}$ and $\mathcal{B}$, respectively, such that $F_{\tilde{S}} \cap F_{\mathcal{B}} = \emptyset$.

Since $\tilde{X}$ is locally compact, for each $\tilde{s} \in \tilde{S}$, there is a compact neighborhood $V_{\tilde{s}}$ of $\tilde{s}$ such that $V_{\tilde{s}} \subset F_{\tilde{S}}$ and $U_s := p(V_{\tilde{s}})$ is an evenly covered neighborhood of $s := p(\tilde{s})$. By the compactness of $\tilde{S}$, there are $\tilde{s}_1, \dots, \tilde{s}_\ell \in \tilde{S}$ such that $\{\text{int}(V_{\tilde{s}_i})\}_{i=1}^\ell$ is a finite open cover of $\tilde{S}$ and hence $\{V_{\tilde{s}_i}\}_{i=1}^\ell$ is a finite compact cover of $\tilde{S}$, which will be denoted by $\{V_i\}_{i=1}^\ell$. Note that the correspondence between the collections $\{U_i\}_{i=1}^\ell$ and $\{V_i\}_{i=1}^\ell$ is bijective given that $\tilde{S}$ is homeomorphic to S via p .

Consider the sets $A_{ij} = \{y \in V_i \setminus V_j \mid \text{there exists } x \in V_j \text{ such that } p(x) = p(y)\}$, for $i, j \in \{1, \dots, \ell\}$. Since $A_{ij} \subset F_{\tilde{S}}$, it follows that $A_{ij} \cap \tilde{S} = \emptyset$.

Claim 2. $\tilde{S} \cap \overline{A_{ij}} = \emptyset$, for all i, j .

Suppose $\overline{A_{ij}} \cap \tilde{S} \neq \emptyset$, thus there is a sequence $(y_n)_{n \in \mathbb{N}}$ in A_{ij} such that $y_n \rightarrow \tilde{s}$, for some $\tilde{s} \in \tilde{S}$. By the definition of A_{ij} , there exists a sequence $(x_n)_{n \in \mathbb{N}}$ in V_j such that $x_n \neq y_n$ and $p(x_n) = p(y_n)$, hence $p(y_n) = p(x_n) \rightarrow p(\tilde{s})$. Since $(x_n)_{n \in \mathbb{N}}$ is a sequence in the compact set V_j , taking a subsequence if necessary, one can assume that $(x_n)_{n \in \mathbb{N}}$ converges to a certain $b \in p^{-1}(s)$, where $s := p(\tilde{s})$. Since $b \in V_j \subset F_{\tilde{S}}$, we have that $b = \tilde{s}$. Since p is locally injective and both sequences $(y_n)_{n \in \mathbb{N}}$ and $(x_n)_{n \in \mathbb{N}}$ converge to $\tilde{s}$, there exists $n_0 \in \mathbb{N}$ such that $x_n = y_n$ for all $n > n_0$. This contradicts the fact that $x_n \neq y_n$, for all n . Therefore $\tilde{S} \cap \overline{A_{ij}} = \emptyset$, for all i, j . $\square$

As a consequence of Claim 2 and the fact that $\tilde{X}$ is a metric space, there exists a closed neighborhood $F_{ij}^{\tilde{S}}$ of $\tilde{S}$ such that $F_{ij}^{\tilde{S}} \cap \overline{A_{ij}} = \emptyset$.

Finally, consider the compact neighborhood $W = (\bigcup_{i=1}^\ell V_i) \cap (\bigcup_{i,j=1}^\ell F_{ij}^{\tilde{S}})$ of $\tilde{S}$. Since W is compact and $p|_W$ is injective, $p|_W$ is a homeomorphism. $\square$

Although in the proof of Theorem 3.2 one extends the homeomorphism $p : \tilde{S} \rightarrow S$ to a compact neighborhood W of $\tilde{S}$, one could also extend it to an open neighborhood of $\tilde{S}$.

The following result is a direct consequence of the previous theorem.

PROPOSITION 3.3. *Let $S \subseteq X$ be an evenly covered compact set. Given a sheet $\tilde{S}$ over S , S is an isolated invariant set if and only if $\tilde{S}$ is an isolated invariant set. Moreover, in this case the homology Conley indices of S and $\tilde{S}$ coincide, that is, $CH_*(S) = CH_*(\tilde{S})$.*

Proof. By Theorem 3.2, there exists a neighborhood W of $\tilde{S}$ such that $p|_W$ is a homeomorphism. Since $\varphi \circ (\text{Id} \times p) = p \circ \tilde{\varphi}$, we have that $p \circ \tilde{\varphi}(t, \tilde{x}) = \varphi(t, p(\tilde{x}))$, which implies that the flows φ and $\tilde{\varphi}$ restricted to W are topologically equivalent by $p|_W$. Hence, $\tilde{S}$ is an isolated invariant set if and only if S is an isolated invariant set. The isomorphism between the homology Conley indices of S and $\tilde{S}$ follows from the existence of a neighborhood basis of index pairs for an isolated invariant set. $\square$

The invariant sets considered in the classical Conley theory (e.g. [4, 8, 11, 20]) are isolated invariant sets and hence compact. As a consequence, the attractors and repellers in these sets are always compact. Since the goal in this work is to study invariant sets which are not necessarily compact, one considers the following definitions of attractors and repellers. Given S an invariant set not necessarily compact, a compact invariant set $A \subseteq S$ is an attractor in S if there exists an open neighborhood U of A in S such that $A = \omega(U)$. A compact invariant set $R \subseteq S$ is a repeller in S if there exists an open neighborhood U of R in S such that $R = \alpha(U)$.

PROPOSITION 3.4. *Let S be an invariant set. Given $A \subseteq X$ an evenly covered attractor in S , if $\tilde{A}$ is a sheet over A , then $\tilde{A}$ is an attractor in $p^{-1}(S)$. Moreover $CH_*(A) = CH_*(\tilde{A})$.*

Proof. Since A is an evenly covered compact set, by Theorem 3.2, there exists a neighborhood W of $\tilde{A}$ such that $p|_W$ is a homeomorphism. The proof of Proposition 3.3 shows that the flows φ and $\tilde{\varphi}$ are topologically equivalent in W by $p|_W$. Therefore, $\tilde{A}$ is an attractor in $p^{-1}(S)$. $\square$

Analogously, if R is an evenly covered repeller in S , and $\tilde{R}$ is a sheet over R , then $\tilde{R}$ is a repeller in $p^{-1}(S)$ and $CH_*(R) = CH_*(\tilde{R})$.

PROPOSITION 3.5. *Let $S \subseteq X$ be an evenly covered compact invariant set and $\tilde{S}$ a sheet over S .*

- (1) *If (A, R) is an attractor–repeller pair of S , then $(p^{-1}(A) \cap \tilde{S}, p^{-1}(R) \cap \tilde{S})$ is an attractor–repeller pair of $\tilde{S}$.*
- (2) *If $(\tilde{A}, \tilde{R})$ is an attractor–repeller pair of $\tilde{S}$, then $(p(\tilde{A}) \cap S, p(\tilde{R}) \cap S)$ is an attractor–repeller pair of S .*

Remark 3.6. Assuming that $S \subseteq X$ is an isolated invariant set does not necessarily imply that $p^{-1}(S)$ is also an isolated invariant set. For instance, consider the flow on the torus $\mathbb{T}^2$ and the corresponding pullback flow on the infinite cyclic covering $\tilde{\mathbb{T}}^2$, as in Figure 3. The lift of the periodic orbit O is not an isolated invariant set.

The next proposition guarantees that all the properties of a given set $\tilde{S} \subseteq \tilde{X}$ which we are interested in, such as invariance and isolation, are preserved by equivalent coverings of X .

PROPOSITION 3.7. *Let $(\tilde{X}_i, p_i)$ be equivalent regular coverings of X , where $\tilde{X}_i$ is a connected, locally path connected metric space, for $i = 1, 2$. Let $S \subseteq X$ be an evenly covered set and $\tilde{S}$ a sheet over S with respect to p_1 . Let $h : \tilde{X}_2 \rightarrow \tilde{X}_1$ be a homeomorphism which provides an equivalence of the covering spaces.*

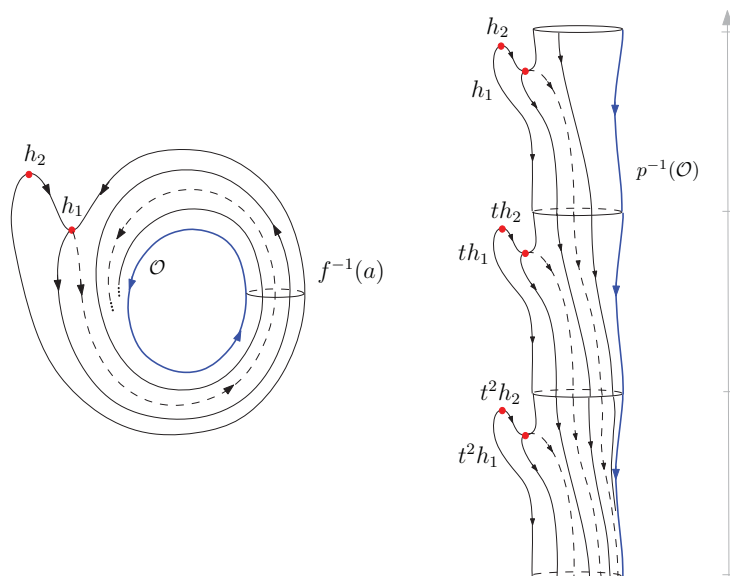


FIGURE 3. Infinite cyclic covering of the torus $\mathbb{T}^2$: $p^{-1}(O)$ is not isolated.

- (1) The set S is evenly covered with respect to p_2 .
- (2) If $\tilde{S}$ is an invariant set in $\tilde{X}_1$, then $h^{-1}(\tilde{S})$ is an invariant set in $\tilde{X}_2$.
- (3) If $\tilde{S}$ is compact and $(\tilde{A}, \tilde{R})$ is an attractor–repeller pair of $\tilde{S}$, then $(h^{-1}(\tilde{A}), h^{-1}(\tilde{R}))$ is an attractor–repeller pair of $h^{-1}(\tilde{S})$. Moreover, $CH_*(\tilde{A}) = CH_*(h^{-1}(\tilde{A}))$ and $CH_*(\tilde{R}) = CH_*(h^{-1}(\tilde{R}))$.
- (4) If $\tilde{S}$ is an isolated invariant set in $\tilde{X}_1$, then $h^{-1}(\tilde{S})$ is an isolated invariant set in $\tilde{X}_2$ and $CH_*(\tilde{S}) = CH_*(h^{-1}(\tilde{S}))$.

Proof. The proof of (1) is straightforward. Now consider the following diagram (where we omit the basepoints):

$$\begin{array}{ccc}
 \mathbb{R} \times \tilde{X}_2 & \xrightarrow{\tilde{\varphi}_2} & \tilde{X}_2 \\
 \downarrow \text{Id} \times h & & \downarrow h \\
 \mathbb{R} \times \tilde{X}_1 & \xrightarrow{\tilde{\varphi}_1} & \tilde{X}_1
 \end{array}$$

This diagram is commutative. In fact,

$$\begin{aligned}
 p_1 \circ h \circ \tilde{\varphi}_2 \circ (\text{Id} \times h^{-1}) &= p_2 \circ \tilde{\varphi}_2 \circ (\text{Id} \times h^{-1}) \\
 &= \varphi \circ (\text{Id} \times p_2) \circ (\text{Id} \times h^{-1}) = \varphi \circ (\text{Id} \times p_1),
 \end{aligned}$$

which implies, by the uniqueness of the lifting $\tilde{\varphi}_1$, that $\tilde{\varphi}_1 = h \circ \tilde{\varphi}_2 \circ (\text{Id} \times h^{-1})$.

If $\tilde{S} \subseteq \tilde{X}_1$ is an invariant set (respectively, isolated invariant set) with respect to $\tilde{\varphi}_1$ then $h^{-1}(\tilde{S})$ is also an invariant set (respectively, isolated invariant set) under $\tilde{\varphi}_2$, by the commutativity of the diagram above. This proves items (2) and (4).

In order to prove item (3), it is sufficient to show that $\omega(h(\tilde{U})) = h(\omega(\tilde{U}))$ and $\alpha(h(\tilde{U})) = h(\alpha(\tilde{U}))$, for all $\tilde{U} \subseteq \tilde{X}_2$. We have that

$$\begin{aligned}\omega(h(\tilde{U})) &= \bigcap_{t>0} \overline{\tilde{\varphi}_1([t, \infty), h(\tilde{U}))} \\ &= \bigcap_{t>0} \overline{h \circ \tilde{\varphi}_2 \circ (\text{Id} \times h^{-1})([t, \infty), h(\tilde{U}))} \\ &= h\left(\bigcap_{t>0} \overline{\tilde{\varphi}_2([t, \infty), \tilde{U})}\right) = h(\omega(\tilde{U})).\end{aligned}$$

Analogously, one proves that $\alpha(h(\tilde{U})) = h(\alpha(\tilde{U}))$. $\square$

4. *p*-connection matrices

In this section we will define a *p*-connection matrix for a *p*-Morse decomposition of an invariant set S . Its entries are homomorphisms which give dynamical information on the connecting orbits between *p*-Morse sets. In this setting, we only assume that S is an invariant set (possibly non-compact), dropping the assumption that S is isolated, even though the *p*-Morse sets are considered to be isolated invariant sets (hence, compact sets).

In §4.1 we define a *p*-attractor–repeller pair (A, R) for S as an attractor–repeller decomposition of S such that A and R are disjoint *p*-evenly covered isolated invariant sets; see Definition 4.1. Despite the fact that S may not be compact or *p*-evenly covered, we prove that, under some additional hypotheses, S can be decomposed into smaller invariant sets $S_{R,gA}$ which are compact evenly covered sets; see Theorem 4.7. In §4.2 we define a *p*-connection matrix for a *p*-attractor–repeller decomposition of an invariant set and we prove its invariance under equivalent regular covering spaces. Moreover, for the case of isolated invariant sets, we establish in Theorem 4.17 the relation between *p*-connection matrices and the classical connection matrices presented in [11], showing that the *p*-connection matrix generalizes the classical one. In §4.3 we present a way to use this theory to obtain information on the connections between Morse sets in a more general *p*-Morse decomposition. More specifically, given a *p*-Morse decomposition of S , we look at the maps between the *p*-Morse sets which are adjacent. In §4.4 we present some examples to illustrate the results obtained in the previous subsections.

4.1. *p*-attractor–repeller decomposition for invariant sets. It is well known that, when S is compact, each orbit has non-empty α - and ω -limit sets. However, this is not always the case when S is non-compact. For instance, the flow on $\mathbb{R}^2$ as in Figure 4 has a flow line γ whose α - and ω -limits are empty. In this case, if we consider the usual definition of connection between two invariant sets, then in Figure 4 the orbit γ would be a connection between R and A . In order to discard connections of these types, we restrict our analysis to the connecting orbits that have non-empty α - and ω -limit sets.

Recall that X is a locally compact metric space, $\varphi : \mathbb{R} \times X \rightarrow X$ is a continuous flow on X and $(\tilde{X}, p)$ is a regular cover of X , where $\tilde{X}$ is a connected, locally path connected metric space. Let S be an invariant set in X . Given A and R invariant sets of S , the set of

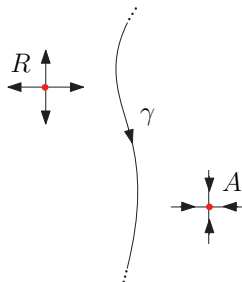


FIGURE 4. An orbit γ with $\alpha(\gamma) = \emptyset$ and $\omega(\gamma) = \emptyset$ that is not a connection between R and A .

connections between A and R is defined by

$$C^*(R, A) = \{x \in S \setminus (A \cup R) \mid \alpha(x) \neq \emptyset, \omega(x) \neq \emptyset, \alpha(x) \subseteq R \text{ and } \omega(x) \subseteq A\}.$$

Definition 4.1. Given an invariant set S in X , a pair of disjoint compact invariant sets (A, R) is a p -attractor–repeller pair for S if:

- (1) A and R are evenly covered;
- (2) A is an attractor in S and R is a repeller in S ;
- (3) given $x \in S$, then either $x \in A$ or $x \in R$ or $x \in C^*(R, A)$.

The decomposition $S = R \cup C^*(R, A) \cup A$ is called a p -attractor–repeller decomposition of S .

Given a p -attractor–repeller pair for S and $A^* := \{x \in S \mid \omega(x) \cap A = \emptyset\}$, note that $R = A^*$.

It is clear that there exist invariant sets which do not admit p -attractor–repeller decompositions for any covering p .

Remark 4.2. If S is an isolated invariant set, we have that $C^*(R, A) = C(R, A)$. Hence, a p -attractor–repeller pair of S is an attractor–repeller pair of S in the usual sense as defined in [4, 11, 20].

Note that, in the previous definition, it is not required that S is compact. In this paper we are interested in an attractor–repeller decomposition such that the deck transformation group ‘acts freely and transitively’ on the attractors and repellers. This property, which is a consequence of Proposition 3.1, is stated in the next result.

PROPOSITION 4.3. Given a p -attractor–repeller pair (A, R) for an invariant set S , there exist compact invariant sets $\tilde{A}, \tilde{R} \subseteq \tilde{X}$ such that:

- (1) $p^{-1}(A) = \bigsqcup_{g \in G} g\tilde{A}$ and $p|_{g\tilde{A}}$ is a homeomorphism;
- (2) $p^{-1}(R) = \bigsqcup_{g \in G} g\tilde{R}$ and $p|_{g\tilde{R}}$ is a homeomorphism,

where G is the deck transformation group.

It is well known that if S is a compact set and $Y \subseteq S$, then $\omega(Y)$ and $\alpha(Y)$ are compact invariant subsets of S . Furthermore, if U is a neighborhood of $\omega(Y)$, then there exists $t > 0$ such that $Y.[t, \infty) \subseteq U$. A similar statement holds for $\alpha(Y)$. Whenever S is not compact,

these properties do not necessarily hold. See the example in Figure 6. However, we can still retrieve some nice properties for subsets of S which admit compact neighborhoods. The next proposition states a property of $\alpha(Y)$ and $\omega(Y)$ when S is not necessarily compact.

LEMMA 4.4. *Let S and Y be invariant sets in X such that $Y \subseteq S$. If there exists a compact set K in X such that $Y \subseteq K \subseteq S$, then for every open neighborhood U of $\omega(Y)$ there exists t such that $Y \cdot [t, \infty) \subseteq U$.*

Proof. Assume that the claim is false. Then there is an open neighborhood U of $\omega(Y)$, a sequence of points $y_n \in Y$ and a sequence $t_n \in \mathbb{R}$ with $t_n \rightarrow \infty$ such that $y_n.t_n \notin U$ for all n . Since K is sequentially compact and $y_n.t_n \notin U$ for all n , there is a subsequence of $y_n.t_n$ that converges to some point $z \in K$ with $z \notin \omega(Y)$. However, $z \in \overline{Y \cdot [t, \infty)}$ for all $t > 0$ and hence $z \in \omega(Y) = \bigcap_{t>0} \overline{Y \cdot [t, \infty)}$. This contradiction establishes the result. $\square$

The following theorem is a generalization of the path lifting theorem for orbits which are contained in a compact set.

THEOREM 4.5. (Lifting of orbits) *Let (A, R) be a p -attractor–repeller pair for an invariant set S . Let γ be an orbit of φ such that $\alpha(\gamma) \subseteq R$, $\omega(\gamma) \subseteq A$ and there is a compact set $U \subseteq X$ containing γ . Fixing sheets $\tilde{R}$ over R and $\tilde{A}$ over A , there exist a unique $g \in G$ and a unique orbit $\tilde{\gamma}$ of $\tilde{\varphi}$ such that $\alpha(\tilde{\gamma}) \subseteq \tilde{R}$, $\omega(\tilde{\gamma}) \subseteq g\tilde{A}$ and $p(\tilde{\gamma}) = \gamma$.*

Proof. By Theorem 3.2, there are open neighborhoods $\tilde{V}_R$ and $\tilde{V}_A$ of $\tilde{R}$ and $\tilde{A}$ such that $p|_{\tilde{V}_R}$ and $p|_{\tilde{V}_A}$ are homeomorphisms. Let $V_R := p(\tilde{V}_R)$ and $V_A := p(\tilde{V}_A)$. Since the image of $\gamma : (-\infty, \infty) \rightarrow M$ is contained in a compact set, it follows from Lemma 4.4 that there exists t^* such that $\gamma((-\infty, -t^*]) \subset V_R$ and $\gamma([t^*, \infty)) \subset V_A$; see Figure 5.

Denote by a and b the lifts of $\gamma(-t^*)$ and $\gamma(t^*)$ which belong to $\tilde{V}_R$ and $\tilde{V}_A$, respectively. Considering the path $\gamma|_{[-t^*, t^*]}$, by the unique path lifting property, there is a unique path $\tilde{\beta} : [-t^*, t^*] \rightarrow \tilde{X}$ such that $\tilde{\beta}(-t^*) = a$ and $p \circ \tilde{\beta} = \gamma$. Since $\gamma(t^*) \in V_A$, we have that $\tilde{\beta}(t^*) \in p^{-1}(V_A)$. Therefore, by the transitivity and freeness of the action, there exists a unique $g \in G$ such that $\tilde{\beta}(t^*) = gb \in g\tilde{V}_A$.

The juxtaposition $\tilde{\gamma}$ of the paths $p^{-1}|_{\tilde{V}_R} \circ \gamma|_{(-\infty, -t^*]}$, $\tilde{\beta}$ and $p^{-1}|_{g\tilde{V}_A} \circ \gamma|_{[t^*, \infty)}$ is a lift of γ such that $\alpha(\tilde{\gamma}) \subseteq \tilde{R}$, $\omega(\tilde{\gamma}) \subseteq g\tilde{A}$; see Figure 5. The uniqueness of $\tilde{\gamma}$ follows from the uniqueness of each one of these paths. $\square$

The assumption that the orbit γ is contained in a compact set is necessary in the proof of Theorem 4.5. Figure 6 shows an orbit which is not contained in any compact set. Hence, one cannot apply Theorem 4.5.

Definition 4.6. Let (A, R) be a p -attractor–repeller pair for S where S is an invariant set. Fix sheets $\tilde{R}$ and $\tilde{A}$ over R and A , respectively.

- (1) Given $g \in G$, an orbit $\gamma \subset C^*(R, A)$ is said to be a g -orbit if there is a lift $\tilde{\gamma}$ of γ such that $\alpha(\tilde{\gamma}) \subseteq \tilde{R}$ and $\omega(\tilde{\gamma}) \subseteq g\tilde{A}$. The union of all g -orbits between R and A is denoted by $C_g^*(R, A)$.

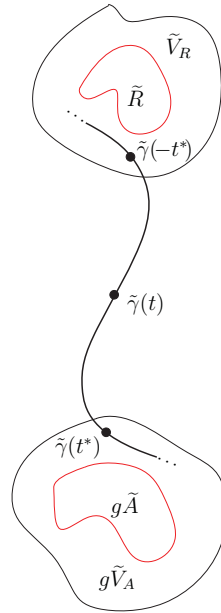
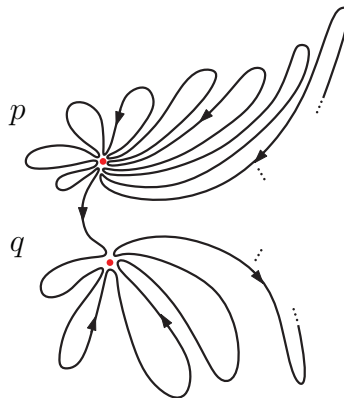

 FIGURE 5. Construction of a lift $\tilde{\gamma}$ of the orbit γ .


FIGURE 6. Example of an orbit which is not contained in a compact set.

(2) For each $g \in G$, define

$$S_{R,gA} = R \cup C_g^*(R, A) \cup A \quad \text{and} \quad \tilde{S}_{\tilde{R},g\tilde{A}} := \tilde{R} \cup C^*(\tilde{R}, g\tilde{A}) \cup g\tilde{A}.$$

It is important to keep in mind that $C_g^*(R, A)$, $S_{R,gA}$ and $\tilde{S}_{\tilde{R},g\tilde{A}}$ depend on the choice of the sheets $\tilde{R}$ and $\tilde{A}$. Note that, whenever $g \neq h$, we have $S_{R,gA} \cap S_{R,hA} = R \cup A$. By definition, a g -orbit always has non-empty α - and ω -limit sets.

It is clear that an invariant set S is not necessarily evenly covered, that is, $p^{-1}(S)$ is not necessarily a union of disjoint invariant sets homeomorphic to S , even though this

property holds, by assumption, for $p^{-1}(A)$ and $p^{-1}(R)$. However, the next theorem gives a sufficient condition for $S_{R,gA}$ to be evenly covered.

THEOREM 4.7. *Let (A, R) be a p -attractor–repeller pair for an invariant set S . Fix sheets $\tilde{R}$ and $\tilde{A}$ over R and A , respectively. Given $g \in G$, if $\tilde{S}_{\tilde{R},g\tilde{A}}$ is compact, then $p^{-1}(S_{R,gA}) = \bigsqcup_{h \in G} h\tilde{S}_{\tilde{R},g\tilde{A}}$ and, for all $h \in G$, $p|_{h\tilde{S}_{\tilde{R},g\tilde{A}}} : h\tilde{S}_{\tilde{R},g\tilde{A}} \rightarrow S_{R,gA}$ is a homeomorphism.*

In order to prove Theorem 4.7, one establishes some properties of $\tilde{S}_{\tilde{R},g\tilde{A}}$ in the next lemma.

LEMMA 4.8. *Let (A, R) be a p -attractor–repeller pair for an invariant set S . Fix sheets $\tilde{R}$ and $\tilde{A}$ over R and A , respectively. Given $g \in G$, we have that:*

- (1) *if $\tilde{x} \in C^*(\tilde{R}, g\tilde{A})$ then $h\tilde{x} \in C^*(h\tilde{R}, hg\tilde{A})$, for every $h \in G$;*
- (2) *$h\tilde{S}_{\tilde{R},g\tilde{A}} = \tilde{S}_{h\tilde{R},hg\tilde{A}}$, for all $h \in G$;*
- (3) *if $\tilde{x} \in C^*(\tilde{R}, g\tilde{A})$ then $h\tilde{x} \notin C^*(\tilde{R}, g\tilde{A})$, for all $h \in G$ with $h \neq e$.*

Proof. Item (1) follows from the fact that each element $h \in G$ is a covering space equivalence and therefore commutes with the dynamics. Items (2) and (3) follow directly from item (1). $\square$

Proof of Theorem 4.7. Since $\tilde{S}_{\tilde{R},g\tilde{A}}$ is compact, each g -orbit is contained in a compact set, namely $S_{R,gA}$, hence every g -orbit satisfies the hypotheses of Theorem 4.5. Moreover, $h\tilde{S}_{\tilde{R},g\tilde{A}}$ is compact for all $h \in G$ and $p|_{h\tilde{S}_{\tilde{R},g\tilde{A}}}$ is onto, by Theorem 4.5. Thus, in order to prove that $p|_{h\tilde{S}_{\tilde{R},g\tilde{A}}} : h\tilde{S}_{\tilde{R},g\tilde{A}} \rightarrow S_{R,gA}$ is a homeomorphism, it is enough to prove that $p|_{\tilde{S}_{\tilde{R},g\tilde{A}}}$ is injective. Let $\tilde{x}$ and $\tilde{y}$ be distinct points in $\tilde{S}_{\tilde{R},g\tilde{A}}$. We have the following cases to consider.

- (1) If $\tilde{x}$ or $\tilde{y}$ belongs to $\tilde{R}$ or $\tilde{A}$, then it is straightforward that $p(\tilde{x}) \neq p(\tilde{y})$.
- (2) If $\tilde{x}$ and $\tilde{y}$ do not belong to $\tilde{R}$ or $\tilde{A}$, then there are two possibilities:
 - (a) $\tilde{x}$ and $\tilde{y}$ belong to the same connecting orbit in $C^*(\tilde{R}, g\tilde{A})$, which is an aperiodic orbit. Then there is a one-to-one correspondence between this orbit and its projection via p , by Theorem 4.5. Therefore, $p(\tilde{x}) \neq p(\tilde{y})$.
 - (b) $\tilde{x}$ and $\tilde{y}$ belong to different connecting orbits in $C^*(\tilde{R}, g\tilde{A})$. Suppose that $p(\tilde{x}) = p(\tilde{y})$. Thus there exists $h' \in G$ such that $\tilde{x} = h'\tilde{y}$. By Lemma 4.8, $h' = e$. Hence $\tilde{x}$ and $\tilde{y}$ belong to the same orbit, which is a contradiction.

In all cases one verifies that $p(\tilde{x}) \neq p(\tilde{y})$, therefore $p|_{h\tilde{S}_{\tilde{R},g\tilde{A}}}$ is injective.

By Lemma 4.8, it follows that $p^{-1}(S_{R,gA}) = \bigsqcup_{h \in G} h\tilde{S}_{\tilde{R},g\tilde{A}}$. $\square$

The next proposition is a direct consequence of Theorems 4.7 and 3.2.

PROPOSITION 4.9. *Let (A, R) be a p -attractor–repeller pair for an invariant set S . Fix sheets $\tilde{R}$ and $\tilde{A}$ over R and A , respectively. Given $g \in G$, if $\tilde{S}_{\tilde{R},g\tilde{A}}$ is compact, then $S_{R,gA}$ is an isolated invariant set if and only if $\tilde{S}_{\tilde{R},g\tilde{A}}$ is an isolated invariant set. In this case, $CH_*(\tilde{S}_{\tilde{R},g\tilde{A}}) = CH_*(S_{R,gA})$.*

In general, $p^{-1}(S)$ is not an isolated invariant set even when S is an isolated invariant set. However, whenever S is an isolated invariant set and $\tilde{S}_{\tilde{R},g\tilde{A}}$ is compact for all $g \in G$, the next result guarantees that $p^{-1}(S)$ can be decomposed into a union of isolated invariant sets.

THEOREM 4.10. *Let S be an isolated invariant set and (A, R) a p -attractor–repeller pair of S . Fix sheets $\tilde{R}$ and $\tilde{A}$ over R and A , respectively. Given $g \in G$, if $\tilde{S}_{\tilde{R},g\tilde{A}}$ is compact, then $S_{R,gA}$ is an isolated invariant set.*

Proof. Since $\tilde{S}_{\tilde{R},g\tilde{A}}$ is compact, $S_{R,gA}$ is compact. Clearly $S_{R,gA}$ is an invariant set. Hence, one needs to prove that $S_{R,gA}$ is isolated.

By Theorem 4.7, $p|_{\tilde{S}_{\tilde{R},g\tilde{A}}} : \tilde{S}_{\tilde{R},g\tilde{A}} \rightarrow S_{R,gA}$ is a homeomorphism, which can be extended to a homeomorphism $p|_{\tilde{V}} : \tilde{V} \rightarrow V$ where $\tilde{V}$ is a compact neighborhood of $\tilde{S}_{\tilde{R},g\tilde{A}}$, by Theorem 3.2.

Given $h \in G$, let $\tilde{U}_h$ be an open neighborhood of $h\tilde{A}$ such that $\omega(\tilde{U}_h) = h\tilde{A}$. Now consider the set

$$\tilde{W} = \tilde{V} \setminus \bigcup_{h \in G \setminus \{g\}} \tilde{U}_h$$

which is still a compact neighborhood of $\tilde{S}_{\tilde{R},g\tilde{A}}$, since no g -orbit intersects the sets $\tilde{U}_h$, for $h \neq g$.

Let N be an isolating neighborhood of S . Then $N \cap p(\tilde{W})$ is a compact neighborhood of $S_{R,gA}$ and $S_{R,gA}$ is the maximal invariant set in $N \cap p(\tilde{W})$. Therefore, $S_{R,gA}$ is an isolated invariant set. $\square$

In general, we have that $\bigcup_{g \in G} S_{R,gA}$ is not equal to S . However, if S is a compact set (or an isolated invariant set), by Theorem 4.5 the equality holds and $p^{-1}(S) = \bigcup_{g, h \in G} h\tilde{S}_{\tilde{R},g\tilde{A}}$. Moreover, if $\tilde{S}_{\tilde{R},g\tilde{A}}$ is compact, Theorem 4.10 and Proposition 4.9 guarantee that $\tilde{S}_{\tilde{R},g\tilde{A}}$ is an isolated invariant set and $p^{-1}(S)$ is a disjoint union of isolated invariant sets.

In the remainder of this subsection, we establish some results for the case that $S = \bigcup_{g \in G} S_{R,gA}$.

LEMMA 4.11. *Let S be an invariant set which admits a p -attractor–repeller pair (A, R) such that $S = \bigcup_{g \in G} S_{R,gA}$. Given $g \in G$, if S is a closed set then $\tilde{S}_{\tilde{R},g\tilde{A}}$ is also closed.*

Proof. Suppose that $\tilde{S}_{\tilde{R},g\tilde{A}}$ is not closed. Let $y \in \text{cl}(\tilde{S}_{\tilde{R},g\tilde{A}}) \setminus \tilde{S}_{\tilde{R},g\tilde{A}}$. Since S is closed, $p^{-1}(S)$ is closed, hence $y \in p^{-1}(S)$. Clearly $y \notin \tilde{R}$, $y \notin p^{-1}(A)$. By hypothesis, there exists $h \in G$ such that $y \in \tilde{S}_{\tilde{R},h\tilde{A}}$. Of course $h \neq g$ and there is $t' \in \mathbb{R}$ such that $\tilde{\varphi}(t', y) \in \tilde{U}_h$, where $\tilde{U}_h$ is an open neighborhood of $h\tilde{A}$ such that $\omega(\tilde{U}_h) = h\tilde{A}$. Let $\tilde{B} \subset \tilde{U}_h$ be a neighborhood of $\tilde{\varphi}(t', y)$ and $T = \tilde{\varphi}([-t', 0], \tilde{B})$. Let (x_n) be a sequence in $\tilde{S}_{\tilde{R},g\tilde{A}} \setminus (\tilde{R} \cup g\tilde{A})$ converging to y . For n sufficiently large, $x_n \in T$, hence $\omega(x_n) \subset h\tilde{A}$. On the other hand, $x_n \in \tilde{S}_{\tilde{R},g\tilde{A}} \setminus (\tilde{R} \cup g\tilde{A})$, which means that $\omega(x_n) \subset g\tilde{A}$. That is a contradiction since $g \neq h$. $\square$

It follows from Lemma 4.11 that Theorem 4.10 holds assuming a weaker hypothesis: $\tilde{S}_{\tilde{R},g\tilde{A}}$ is contained in a compact set of $\tilde{X}$. In particular, it holds when $\bigcup_{g \in G} \tilde{S}_{\tilde{R},g\tilde{A}}$ is compact.

Considering sheets $\tilde{R}$ over R and $\tilde{A}$ over A , define

$$\tilde{C}_R^S := \bigcup_{g \in G} \tilde{S}_{\tilde{R},g\tilde{A}}.$$

Note that $p(\tilde{C}_R^S) = S$ and p restricted to $\tilde{C}_R^S \setminus p^{-1}(A)$ is a homeomorphism.

The next proposition gives a condition in order to guarantee that the set of all $g \in G$ such that $C_g(R, A) \neq \emptyset$ is finite, that is, $\tilde{C}_R^S$ can be written as a finite union of sets $\tilde{S}_{\tilde{R},g\tilde{A}}$.

PROPOSITION 4.12. *Let S be an invariant set which admits a p -attractor–repeller pair (A, R) such that $S = \bigcup_{g \in G} S_{R,gA}$. Then $\tilde{C}_R^S$ is compact if and only if $\tilde{S}_{\tilde{R},g\tilde{A}}$ is a compact invariant set for all $g \in G$ and there exists a finite subset Υ of G such that $\tilde{C}_R^S = \bigcup_{g \in \Upsilon} \tilde{S}_{\tilde{R},g\tilde{A}}$.*

Proof. If there exists $\Upsilon \subseteq G$ finite then $\tilde{C}_R^S$ is a finite union of compact sets, hence $\tilde{C}_R^S$ is compact. On the other hand, assume that $\tilde{C}_R^S$ is a compact set. It follows that S is a compact set and, by Lemma 4.11, $\tilde{S}_{\tilde{R},g\tilde{A}}$ is a compact invariant set, for all $g \in G$. Now suppose that there does not exist a finite set $\Upsilon \subseteq G$ such that $\tilde{C}_R^S = \bigcup_{g \in \Upsilon} \tilde{S}_{\tilde{R},g\tilde{A}}$. Let (x_n) be a sequence of points such that $x_n \in g_n \tilde{A}$. Since A is evenly covered, the sequence x_n does not have any accumulation point, which contradicts the fact that $\tilde{C}_R^S$ is compact. $\square$

COROLLARY 4.13. *Let S be an invariant set which admits a p -attractor–repeller pair (A, R) such that $S = \bigcup_{g \in G} S_{R,gA}$. If $\tilde{C}_R^S$ is compact then there exists a finite subset Υ of G such that $S = \bigcup_{g \in \Upsilon} S_{R,gA}$.*

In the next subsection we introduce p -connection matrices for invariant sets. Note that even when S is not an isolated invariant set but $\tilde{S}_{\tilde{R},g\tilde{A}}$ is an isolated invariant set for all $g \in G$, then $p^{-1}(S)$ can still be decomposed into a union of isolated invariant sets. We define a connecting map for this general setting. Proposition 4.9 and Theorem 4.10 guarantee that whenever S is an isolated invariant set and $\tilde{S}_{\tilde{R},g\tilde{A}}$ is compact then $\tilde{S}_{\tilde{R},g\tilde{A}}$ is an isolated invariant set. Hence, for this particular case, the connecting map for a p -attractor–repeller pair is well defined, as proved in the next section.

Remark 4.14. In [12], McCord decomposed the set of connections $C(R, A)$ for an isolated invariant set in a topological manner. Herein we decompose it by taking into account the covering action. Moreover, we do not require S to be an isolated set.

4.2. p -connection matrices for p -attractor–repeller decompositions. In this subsection we define a boundary map that ‘counts’ the flow lines between a repeller R and an attractor A by means of the lifts of these connections via the covering map p . In order to accomplish

that, one needs to have at hand an algebraic structure which makes it possible to ‘count’ these flow lines in a suitable way. In what follows, we define this structure, denoted by $\mathbb{Z}((G))$, where G is the deck transformation group associated to p .

- (H-1) If G is a finite group, we consider $\mathbb{Z}((G))$ as the group ring $\mathbb{Z}[G]$.
 (H-2) Assume that G is a totally ordered group, that is, G is equipped with a total ordering $\leq$ that is compatible with the multiplication of G (for all $x, y, z \in G$, $x \leq y$ implies that $zx \leq zy$ and $xz \leq yz$). Moreover, assume that the set $\{g \in G \mid C_g^*(R, A) \neq \emptyset\}$ is well ordered with respect to the order $\leq$. For every formal series $\eta \in \mathbb{Z}[G]$,

$$\eta = \sum_{g \in G} a_g g,$$

where $a_g \in \mathbb{Z}$, the support of η is defined as

$$\text{supp}(\eta) = \{g \in G \mid a_g \neq 0\}.$$

Let $\mathbb{Z}((G))$ be the ring of the formal series on G that have a well-ordered support. For more details, see [1].

- (H-3) For more general G , we will assume that $\#\{g \in G; C_g^*(R, A) \neq \emptyset\} < \infty$ and $\mathbb{Z}((G)) = \mathbb{Z}[G]$.

An important particular case of (H-2) is when G is an infinite cyclic group, namely $G = \langle g \rangle$. In this case, $\mathbb{Z}((G))$ is the Novikov ring $\mathbb{Z}((t))$, as defined in §2.2.

Note that any totally ordered group is torsion-free. The converse holds for abelian groups, that is, an abelian group admits a total ordering if and only if it is torsion-free.

The conditions on the flow in (H-2) and (H-3) are imposed to guarantee that there are no bi-infinite connections, and this fact is necessary in order to have a well-defined boundary map.

Let S be an invariant set and (A, R) a p -attractor–repeller pair of S . Fix sheets $\tilde{R}$ and $\tilde{A}$ over R and A , respectively. Consider the subset G' of G of all elements $g \in G$ such that $\tilde{S}_{\tilde{R}, g\tilde{A}}$ is an isolated invariant set. By Proposition 4.9, we have that $S_{R, gA} = R \cup C_g^*(R, A) \cup A$ is also an isolated invariant set.

Clearly, $(g\tilde{A}, \tilde{R})$ is an attractor–repeller pair for $\tilde{S}_{\tilde{R}, g\tilde{A}}$, for each $g \in G'$, thus the homology Conley exact sequence of the pair $(g\tilde{A}, \tilde{R})$ is

$$\cdots \longrightarrow CH_*(g\tilde{A}) \xrightarrow{i_*} CH_*(\tilde{S}_{\tilde{R}, g\tilde{A}}) \xrightarrow{p_*} CH_*(\tilde{R}) \xrightarrow{\tilde{\delta}_*(\tilde{R}, g\tilde{A})} CH_{*-1}(g\tilde{A}) \longrightarrow \cdots \quad (1)$$

One can build up an analogous exact sequence for any pair $(g\tilde{A}, h\tilde{R})$ whenever $h^{-1}g \in G'$. By the equivariance, we have that $\tilde{S}_{g\tilde{R}, g\tilde{A}} \cong \tilde{S}_{\tilde{R}, \tilde{A}}$ and $\tilde{S}_{h\tilde{R}, g\tilde{A}} \cong \tilde{S}_{\tilde{R}, h^{-1}g\tilde{A}}$, hence $\tilde{\delta}_*(\tilde{R}, \tilde{A}) = \tilde{\delta}_*(g\tilde{R}, g\tilde{A})$ and $\tilde{\delta}_*(h\tilde{R}, g\tilde{A}) = \tilde{\delta}_*(\tilde{R}, h^{-1}g\tilde{A})$. Note that, if $C_g^*(R, A)$ is empty, then $\tilde{\delta}_*(\tilde{R}, g\tilde{A})$ is the null map.

Fix the sets $B_k(R) = \{r_\alpha^k, \alpha \in \Lambda_k\}$ and $B_k(A) = \{a_\alpha^k, \alpha \in \Gamma_k\}$ of generators for $CH_k(R)$ and $CH_k(A)$, respectively. Using the isomorphisms $CH_*(h\tilde{R}) = CH_*(R)$

(respectively, $CH_*(h\tilde{A}) = CH_*(A)$), as in Proposition 3.4, one can define

$$\begin{aligned}\delta_k^N(R, A) : \mathbb{Z}((G)) \otimes_{\mathbb{Z}[G]} \mathbb{Z}[G][B_k(R)] &\longrightarrow \mathbb{Z}((G)) \otimes_{\mathbb{Z}[G]} \mathbb{Z}[G][B_{k-1}(A)] \\ h \otimes r_\alpha^k &\longmapsto \sum_{g \in G} h^{-1} g \otimes \tilde{\delta}_k(h\tilde{R}, g\tilde{A})(r_\alpha^k) \\ hr_\alpha^k &\longmapsto \sum_{g \in G} h^{-1} g \tilde{\delta}_k(h\tilde{R}, g\tilde{A})(r_\alpha^k)\end{aligned}\quad (2)$$

where $\mathbb{Z}[G][B_k(R)]$ is a free $\mathbb{Z}[G]$ -module generated by $B_k(R)$ and $\tilde{\delta}(h\tilde{R}, g\tilde{A})(r_\alpha^k)$ is the null map if $h^{-1}g \notin G'$. Note that there is an injective homomorphism from $CH_k(R)$ to $\mathbb{Z}[G][B_k(R)]$ given by the map

$$\begin{aligned}\xi : CH_k(R) &\longrightarrow \mathbb{Z}[G][B_k(R)] \\ tr_\alpha^k &\longmapsto ter_\alpha^k\end{aligned}$$

where $t \in \mathbb{Z}$, $e \in G$ is the identity element and $r_\alpha^k \in B_k(R)$.

Denoting $NCH_k(A) = \mathbb{Z}((G)) \otimes_{\mathbb{Z}[G]} \mathbb{Z}[G][B_k(A)]$ and $NCH_k(R) = \mathbb{Z}((G)) \otimes_{\mathbb{Z}[G]} \mathbb{Z}[G][B_k(R)]$, the map

$$N\Delta : NCH_*(A) \bigoplus NCH_*(R) \longrightarrow NCH_*(A) \bigoplus NCH_*(R)$$

defined by the matrix

$$\begin{pmatrix} 0 & \delta^N(R, A) \\ 0 & 0 \end{pmatrix}$$

is an upper triangular boundary map called a *p-connection matrix for the p-attractor–repeller decomposition of S*. Denoting $NC(S) = NCH_*(A) \bigoplus NCH_*(R)$, we have that $(NC(S), N\Delta)$ is a chain complex.

Whenever S is an isolated invariant set and $\tilde{S}_{h\tilde{R}, g\tilde{A}}$ is compact for all $g \in G$, by Theorem 4.10 and Proposition 4.9, we have that $G = G'$. Hence, the exact sequence in (1) is well defined for all $g \in G$. Therefore δ^N keeps track of all information on connections between adjacent invariant sets.

The next result shows that the entry $\delta^N(R, A)$ of a *p-connection matrix* gives dynamical information about the connecting orbits from the repeller R to the attractor A .

PROPOSITION 4.15. *If $\delta^N(R, A)$ is non-zero, then $C^*(R, A) \neq \emptyset$.*

Proof. Suppose that $C^*(R, A) = \emptyset$. Then $C(h\tilde{R}, g\tilde{A}) = \emptyset$, for each $g, h \in G'$, and hence $CH(\tilde{S}_{h\tilde{R}, g\tilde{A}}) = CH_*(g\tilde{A}) \oplus CH_*(h\tilde{R})$. It follows from the exactness of the long exact sequence in (1) that $\tilde{\delta}_*(h\tilde{R}, g\tilde{A}) = 0$ for each $g, h \in G'$. Therefore, $\delta^N(R, A) = 0$. $\square$

THEOREM 4.16. (Invariance of the *p-connection matrices*) *The chain complex $(NC(S), N\Delta)$ is invariant under equivalent regular covering spaces.*

Proof. Let $p_i : \tilde{X}_i \rightarrow X$ be equivalent regular covering spaces of X and G_i be the deck transformation groups of p_i , for $i = 1, 2$. Given $S \subset X$ an invariant set, a pair (A, R) is a p_1 -attractor–repeller pair of S if and only if it is a p_2 -attractor–repeller pair of S .

Let $h : \tilde{X}_2 \rightarrow \tilde{X}_1$ be a homeomorphism which provides an equivalence of the covering spaces and fix sheets $\tilde{R}, \tilde{A} \subset \tilde{X}_1$ over R and A with respect to p_1 . Given $g \in G_1$, it follows from Proposition 3.7 that $h^{-1}(\tilde{S}_{\tilde{R},g\tilde{A}}) = h^{-1}(\tilde{R}) \cup C(h^{-1}(\tilde{R}), h^{-1}(g\tilde{A})) \cup h^{-1}(g\tilde{A})$ is an isolated invariant set in $\tilde{X}_2$ if and only if $\tilde{S}_{\tilde{R},g\tilde{A}}$ is an isolated invariant set in $\tilde{X}_1$.

Let $\mathfrak{h}$ be the isomorphism between G_1 and G_2 induced by h . One has that $\mathfrak{h}$ induces an isomorphism

$$\begin{aligned} H : \mathbb{Z}((G_2)) \otimes_{\mathbb{Z}[G_2]} (\mathbb{Z}[G_2][B_k(A)] \oplus \mathbb{Z}[G_2][B_k(R)]) \\ \longrightarrow \mathbb{Z}((G_1)) \otimes_{\mathbb{Z}[G_1]} (\mathbb{Z}[G_1][B_k(A)] \oplus \mathbb{Z}[G_1][B_k(R)]). \end{aligned}$$

Note that H commutes with the boundary map, $H \circ \delta_2^N = \delta_1^N \circ H$, since $h_* \circ \tilde{\delta}_{2*} = \tilde{\delta}_{1*} \circ h_*$, where $\tilde{\delta}_{i*}$ is the connection map in the long exact sequence in (1). Then the chain complexes that arise from the covering spaces $(\tilde{X}_1, p_1)$ and $(\tilde{X}_2, p_2)$ are isomorphic. $\square$

In the case where S is an isolated invariant set and p is the trivial covering map, the next result proves that p -connection matrices coincide with the classical connection matrices defined by Franzosa in [11] when we consider the homology with coefficients in a field. In this sense, the p -connection matrix theory, developed herein, generalizes the classical connection matrix theory. More specifically, when one considers the trivial covering action or one projects the group G to the trivial group ($g \mapsto e$), the p -connection matrix is the usual connection matrix. In the next result, the coefficients of the homology Conley indices are assumed to be in a field.

THEOREM 4.17. *Let S be an isolated invariant set and (A, R) a p -attractor–repeller pair of S associated to a covering map p . If $\tilde{C}_R^S$ is compact, then the following diagram commutes:*

$$\begin{array}{ccc} NCH(A) \oplus NCH_*(R) & \xrightarrow{N\Delta} & NCH(A) \oplus NCH(R) \\ \downarrow \Pi & & \downarrow \Pi \\ CH(A) \oplus CH(R) & \xrightarrow{\Delta} & CH(A) \oplus CH(R) \end{array}$$

where Π is the following projection induced by the covering map p :

$$\begin{aligned} \Pi : NCH_*(A) \oplus NCH_*(R) &\longrightarrow CH_*(A) \oplus CH_*(R) \\ g_1 \otimes a \oplus g_2 \otimes r &\longmapsto a \oplus r. \end{aligned}$$

Proof. Fix sheets $\tilde{R}$ and $\tilde{A}$ over R and A , respectively, and let $\tilde{C}_R^S = \bigcup \tilde{S}_{\tilde{R},g\tilde{A}}$, where the union is over all $g \in G$ such that $C^*(\tilde{R}, g\tilde{A}) \neq \emptyset$. Since $\tilde{C}_R^S$ is compact, by Proposition 4.12 there exist $g_1, \dots, g_n \in G$ such that $\tilde{C}_R^S = \bigcup_{i=1}^n \tilde{S}_{\tilde{R},g_i\tilde{A}}$.

Also $\tilde{C}_R^S$ is an isolated invariant set. In fact, suppose that $\tilde{C}_R^S$ is not an isolated invariant set and let N be an isolating neighborhood for S . Let N' be a compact neighborhood of $\tilde{C}_R^S$ and $N'' = N' \cap p^{-1}(N)$. Since $p^{-1}(N)$ is closed, N'' is a compact neighborhood of $\tilde{C}_R^S$. By assumption, $\tilde{C}_R^S$ is not an isolated invariant set, hence there is $\tilde{x} \in N''$ such that $\tilde{x} \cdot \mathbb{R} \subseteq N''$ and $\tilde{x} \cdot \mathbb{R} \cap \tilde{C}_R^S = \emptyset$. Therefore $p(\tilde{x}) \cdot \mathbb{R} \subseteq N$ and $S \subseteq p(N'') \subseteq N$, which implies that S is not the maximal invariant set in N . This is a contradiction.

Consider the flow order $<$ for the Morse decomposition $\mathcal{M}(\tilde{C}_R^S) = \{g_1\tilde{A}, \dots, g_n\tilde{A}, \tilde{R}\}$ of $\tilde{C}_R^S$. Thus there is an index filtration $\{\tilde{N}_0, \tilde{N}_1, \dots, \tilde{N}_n, \tilde{N}\}$ for $(\mathcal{M}(S'), <)$ such that: $(\tilde{N}, \tilde{N}_0)$ is a regular index pair for $\tilde{C}_R^S$; $(\tilde{N}_i, \tilde{N}_0)$ is a regular index pair for $g_i\tilde{A}$; $(\tilde{N}, \tilde{N}_A)$ is a regular index pair for $\tilde{R}$, where $\tilde{N}_A = \bigcup_{i=0}^n \tilde{N}_i$; and $\tilde{N}_i \cap \tilde{N}_j = \tilde{N}_0$ for $i \neq j$ (see [8]).

Define the functions $\tau, \tau_i : \tilde{N} \rightarrow [0, \infty]$ by

$$\tau(\tilde{x}) = \begin{cases} \sup\{t > 0 \mid \tilde{x} \cdot [0, t] \subseteq \tilde{N} \setminus \tilde{N}_A\} & \text{if } \tilde{x} \in \tilde{N} \setminus \tilde{N}_A, \\ 0 & \text{if } \tilde{x} \in \tilde{N}_A, \end{cases}$$

and

$$\tau_i(\tilde{x}) = \begin{cases} \sup\{t > 0 \mid \tilde{x} \cdot [0, t] \subseteq \tilde{N} \setminus \tilde{N}_i\} & \text{if } \tilde{x} \in \tilde{N} \setminus \tilde{N}_i, \\ 0 & \text{if } \tilde{x} \in \tilde{N}_i. \end{cases}$$

Since $(\tilde{N}, \tilde{N}_A)$ is a regular index pair, τ is continuous. By Lemma 5.2 in [20], the index pair $(\tilde{N}, \tilde{N}_i)$ is also regular, hence τ and τ_i are continuous.

The connection maps between index pairs $\delta : \tilde{N}/\tilde{N}_A \rightarrow \Sigma \tilde{N}_A/\tilde{N}_0$ and $\delta_i : \tilde{N}/\tilde{N}_A \rightarrow \Sigma \tilde{N}_i/\tilde{N}_0$ are defined as

$$\delta([\tilde{x}]) = \begin{cases} [\tilde{x} \cdot \tau(\tilde{x}), 1 - \tau(\tilde{x})], & 0 \leq \tau(\tilde{x}) \leq 1, \\ [\tilde{N}_0 \times 0], & 1 \leq \tau(\tilde{x}) \leq \infty, \end{cases}$$

$$\delta_i([\tilde{x}]) = \begin{cases} [\tilde{x} \cdot \tau_i(\tilde{x}), 1 - \tau_i(\tilde{x})], & 0 \leq \tau_i(\tilde{x}) \leq 1, \\ [\tilde{N}_0 \times 0], & 1 \leq \tau_i(\tilde{x}) \leq \infty. \end{cases}$$

When $0 \leq \tau(\tilde{x}) \leq 1$, we have that $\delta([\tilde{x}]) = \delta_i([\tilde{x}])$, where i is such that $\tilde{x} \cdot \tau(\tilde{x}) \in \tilde{N}_i$. When $\tau(\tilde{x}) \geq 1$, we have that $\delta([\tilde{x}]) = \delta_i([\tilde{x}]) = \delta_j([\tilde{x}]) = [\tilde{N}_0 \times 0]$, for all i and j . Since $[\tilde{N}_0 \times 0] \in \Sigma \tilde{N}_i/\tilde{N}_0$ for all i , there is a homotopy equivalence

$$\Sigma \tilde{N}_A/\tilde{N}_0 \simeq \bigvee_i (\Sigma \tilde{N}_i/\tilde{N}_0),$$

where ' $\vee$ ' denotes the wedge sum with base point $[\tilde{N}_0 \times 0]$.

Note that $\delta_i(\tilde{N}/\tilde{N}_A) \subseteq \Sigma \tilde{N}_i/\tilde{N}_0$ and if $\tilde{x} \cdot \tau(\tilde{x}) \in \tilde{N}_i \cap \tilde{N}_j$, for $i \neq j$, then $\tilde{x} \cdot \tau(\tilde{x}) \in \tilde{N}_0$. Therefore $\delta = \vee_i \delta_i$. (Given maps $f : A \rightarrow B$ and $g : A \rightarrow C$, one defines $f \vee g : A \rightarrow B \vee C$ by $f \vee g : A \xrightarrow{f+g} A + B \rightarrow (A + B)/a \sim b = A \vee B$, where $+$ is the sum operation between topological spaces and $a \in A$ and $b \in B$ are base points.)

Applying the homological functor H on $\Sigma^{-1} \circ \delta = \Sigma^{-1} \circ \vee_i \delta_i$, we have the usual homological connection map $\delta^N = \oplus_i \delta_i^N$. By projecting with respect to the covering map p , we obtain $\delta(N, N_A) = \oplus_i \delta_i(N, N_A)$, since Π is induced by p . Hence, the following

diagram commutes:

$$\begin{array}{ccccccc}
 & & & \oplus_i NCH(\tilde{N}_i, \tilde{N}_0) & & & \\
 & & \nearrow \oplus \delta_i^N & \downarrow \simeq & & & \\
 \cdots & \longrightarrow & NCH(\tilde{N}, \tilde{N}_A) & \xrightarrow{\delta^N} & NCH(\tilde{N}_A, \tilde{N}_0) & \longrightarrow & \cdots \\
 & & \downarrow \Pi(\tilde{N}, \tilde{N}_A) & & \downarrow \Pi(\tilde{N}_A, \tilde{N}_0) & & \\
 \cdots & \longrightarrow & CH(N, N_A) & \xrightarrow{\delta(N, N_A)} & CH(N_A, N_0) & \longrightarrow & \cdots
 \end{array}$$

where (N, N_A, N_0) is an index filtration for (A, R) . $\square$

4.3. p -Morse decomposition. Let $(P, <)$ be a partial ordered set with partial order $<$, where P is a finite set of indices. One says that π and π' are *adjacent elements* with respect to $<$ if they are distinct and there is no element $\pi'' \in P$ satisfying $\pi'' \neq \pi, \pi'$ and $\pi < \pi'' < \pi'$ or $\pi' < \pi'' < \pi$.

In what follows, we define p -Morse decomposition for an invariant set S which is not necessarily isolated or even not compact.

Definition 4.18. Let S be an invariant set and $(P, <)$ be a partial ordered set. A family of disjoint isolated invariant sets $\mathcal{M}(S) = \{M_\pi\}_{\pi \in P}$ is a ($<$ -ordered) p -Morse decomposition for S if the sets M_π are evenly covered for all $\pi \in P$ and given $x \in S$, we have that either $x \in M_\pi$ for some $\pi \in P$ or $x \in C^*(M_{\pi'}, M_\pi)$, where $\pi, \pi' \in P$ and $\pi < \pi'$.

Each set M_π is called a p -Morse set. The partial order $<$ on P induces an obvious partial order on $\mathcal{M}(S)$, called an admissible ordering of the p -Morse decomposition. The flow defines an admissible ordering of $\mathcal{M}(S)$, called the *flow ordering* of $\mathcal{M}(S)$, denoted $<_f$, and such that $M_\pi <_f M_{\pi'}$ if and only if there exists a sequence of distinct elements of P : $\pi = \pi_0, \dots, \pi_n = \pi'$, where the set of connecting orbits $C^*(M_{\pi_j}, M_{\pi_{j-1}})$ between M_{π_j} and $M_{\pi_{j-1}}$ is non-empty, for each $j = 1, \dots, n$. Note that every admissible ordering of $\mathcal{M}(S)$ is an extension of $<_f$.

Given two adjacent elements π, π' define

$$M_{\pi, \pi'} = M_\pi \cup C^*(M_\pi, M_{\pi'}) \cup M_{\pi'}$$

which is an invariant set. Moreover, $(M_{\pi'}, M_\pi)$ is a p -attractor–repeller pair for $M_{\pi, \pi'}$. From now on fix sheets $\tilde{M}_\pi$ over M_π , for all $\pi \in P$. Consider the subset $G_{\pi\pi'}$ of G of all elements $g \in G$ such that

$$\tilde{M}_{\pi, g\pi'} := \tilde{M}_\pi \cup C^*(\tilde{M}_\pi, g\tilde{M}_{\pi'}) \cup g\tilde{M}_{\pi'}$$

is an isolated invariant set (hence, compact). Clearly, $(g\tilde{M}_{\pi'}, \tilde{M}_\pi)$ is an attractor–repeller pair in $\tilde{M}_{\pi, g\pi'}$ as in [11]. Hence, for each $g \in G_{\pi\pi'}$ there exists a long exact sequence

$$\cdots \longrightarrow CH_*(g\tilde{M}_{\pi'}) \xrightarrow{i_*} CH_*(\tilde{M}_{\pi, g\pi'}) \xrightarrow{p_*} CH_*(\tilde{M}_\pi) \xrightarrow{\tilde{\delta}_*(\tilde{M}_\pi, g\tilde{M}_{\pi'})} CH_{*-1}(g\tilde{M}_{\pi'}) \longrightarrow \cdots$$

By Proposition 4.9, $M_{\pi, g\pi'}$ is an isolated invariant set. Fix a set of generators $B_k(M_\pi)$ for $CH_k(M_\pi)$, for each $\pi \in P$.

Denoting $NCH_k(M_\pi) = \mathbb{Z}((G)) \otimes_{\mathbb{Z}[G]} \mathbb{Z}[G][B_k(M_\pi)]$, let

$$N\Delta : \bigoplus_{\pi \in P} NCH_*(M_\pi) \longrightarrow \bigoplus_{\pi \in P} NCH_*(M_\pi)$$

be the map defined by the upper triangular matrix

$$N\Delta = (\delta^N(\pi, \pi'))_{\pi, \pi' \in P},$$

where $\delta^N(\pi, \pi')$ is given by $\delta^N(M_\pi, M_{\pi'})$, as defined in (2), if π and π' are adjacent elements, and it is the null map otherwise. The possible non-zero entries of $N\Delta$ are always maps from $NCH(M_\pi)$ to $NCH(M_{\pi'})$, where π and π' are adjacent elements, and they give information on the orbits connecting M_π to $M_{\pi'}$.

The natural question herein is how to define a map from $NCH(M_\pi)$ to $NCH(M_{\pi'})$ when π and π' are not adjacent elements which would give more information than the null map. This question is related to a generalization of the work in this paper to the case of a p -Morse decomposition of an invariant set. The first step in this direction is to study the behavior of the Morse sets $M(I)$ for any interval I . Since we are considering S as an invariant set, not necessarily isolated, the description of the structure of $M(I)$ is a delicate and difficult problem. For instance, $M(I)$ is not necessarily an isolated invariant set and it may not be evenly covered. We will address this problem in a future work.

However, the p -connection matrix defined herein is rich enough to describe the behavior of the connecting orbits between the Morse sets M_π in the case of a p -Morse decomposition where each M_π is a critical point of a circle-valued Morse function, as we prove in §5.

4.4. Examples. In this subsection we present some examples where we describe the p -connection matrix $N\Delta$ for groups G that satisfy (H-1), (H-2) and (H-3).

Example 4.19. (Klein bottle) Let X be the Klein bottle. Consider a flow on X having one repelling singularity x , two saddle singularities y_1, y_2 and one attracting singularity z , as in Figure 7, where we consider the Klein bottle as the quotient space of $[0, 1] \times [0, 1]$ by the relations $(0, y) \sim (1, y)$ and $(x, 0) \sim (1 - x, 1)$. Consider the p -Morse decomposition where each Morse set is a singularity and the partial order is given by the flow.

The universal cover of X is the plane $\mathbb{R}^2$, and its deck transformations group G has the presentation $\langle a, b \mid ab = b^{-1}a \rangle$. In this case, one considers $\mathbb{Z}((G))$ as the group ring $\mathbb{Z}[G]$.

As usual in the Morse setting, one can associate the generator of the homology Conley index of each singularity with the singularity itself. With this notation and considering the unstable manifolds oriented as in Figure 7, the boundary operator is given by $\delta_2^N(x, y_1) = -y_1 + b.y_1$, $\delta_2^N(x, y_2) = y_2 + bab.y_2 = y_2 + a.y_2$, $\delta_1^N(y_1, z) = b.z - a.z$, $\delta_1^N(y_2, z) = -z + b.z$.

Example 4.20. (Double torus) Consider a flow on the double torus X having the invariant set as in Figure 8, where we present a saddle singularity y , an attracting periodic orbit γ_0 and a repeller singularity x .

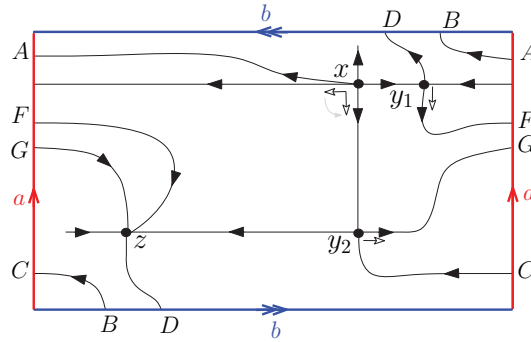


FIGURE 7. A flow on the Klein bottle.

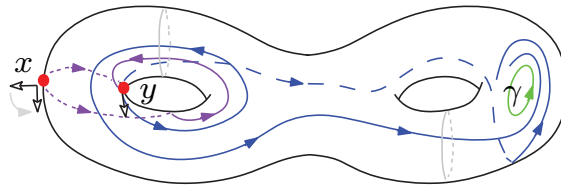
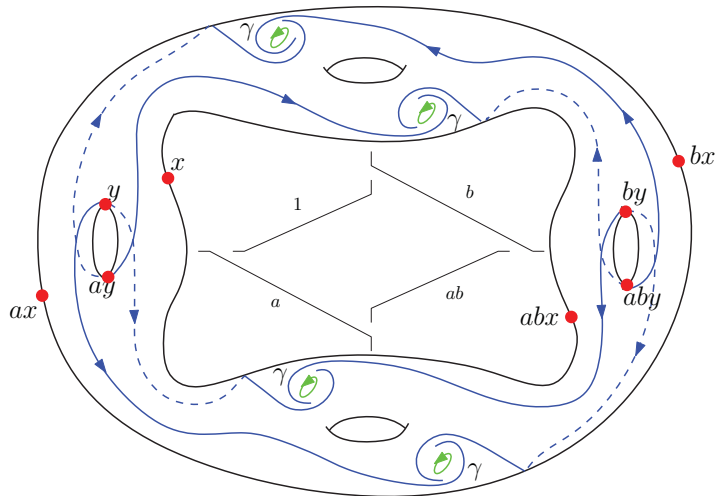


FIGURE 8. A flow on the double torus.


 FIGURE 9. A covering space of the double torus with deck transformation group isomorphic to $\langle a, b \mid a^2 = b^2 = aba^{-1}b^{-1} = 1 \rangle \cong \mathbb{Z}_2 \oplus \mathbb{Z}_2$.

Consider the 5-torus $\tilde{X}$ as a covering space of X with four leaves as in Figure 9. The deck transformation group is $G = \mathbb{Z}_2 \oplus \mathbb{Z}_2 = \{a, b \mid a^2 = 1, b^2 = 1\}$, which is a finite group, hence $\mathbb{Z}((G))$ is the group ring $\mathbb{Z}[G]$.

As usual in the Morse setting, one can associate the generator of the homology Conley index of each singularity with the singularity itself. Let r_1 and r_0 be the generators of $CH_i(\gamma)$, for $i = 1$ and $i = 0$, respectively. In what follows, we compute the boundary

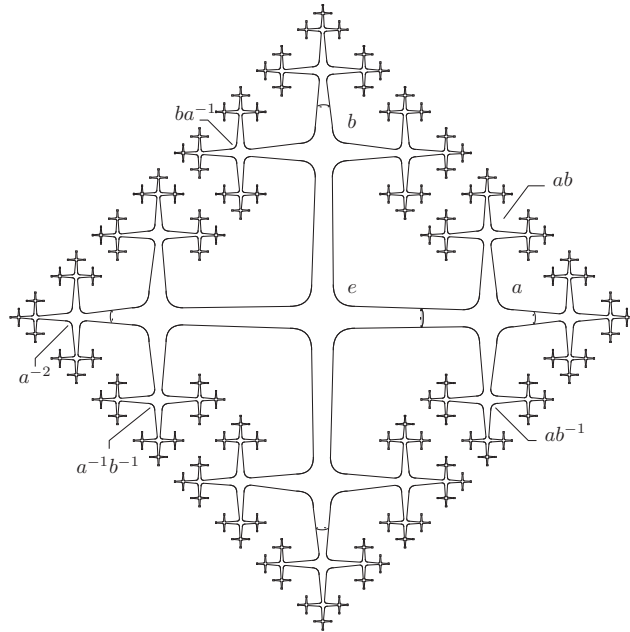


FIGURE 10. Universal covering of the solid double torus: Cayley graph of $\mathbb{Z} * \mathbb{Z}$ where every edge is a solid cylinder.

operator δ^N using this notation and the unstable manifolds oriented as in Figure 8. Consider the invariant set $S = \{y\} \cup C(y, \gamma) \cup \{\gamma\}$; the boundary operator is given by $\delta_1^N(y, \gamma) = -ar_0 + abr_0$ and $\delta_k^N = 0$ for $k \neq 1$. Now consider the invariant set $S = \{x\} \cup C(x, y) \cup \{y\}$; the boundary operator is given by $\delta_2^N(x, y) = y - ay$ and $\delta_k^N = 0$ for $k \neq 2$. See [6] for more details about the computation of the boundary map δ in the presence of periodic orbits.

Example 4.21. (Solid double torus) Consider a flow on the solid double torus M having two consecutive critical points p and q . Note that the Cayley graph of $\mathbb{Z} * \mathbb{Z}$ where every edge is a solid cylinder, as in Figure 10, is a universal covering of M . In this case $G = \pi_1(M) = \mathbb{Z} * \mathbb{Z}$. Assume that, for each $g \in G$ there is one isolated g -orbit between p and q , and hence there are infinite isolated connections between p and q . This is the case where $\mathbb{Z}((G))$ satisfies condition (H-2) in §4.2, where $\mathbb{Z} * \mathbb{Z}$ is equipped with the dictionary order. Note that dynamical systems such that $\{g \in G \mid C_g^*(R, A) \neq \emptyset\}$ is infinite are in general not trivial to understand completely; however, the machinery constructed in this paper contributes to a better understanding of the global behavior.

5. Novikov differential as a p -connection matrix

Let M be a compact Riemannian manifold. Consider the infinite cyclic covering $(\overline{M}, p_E)$ of M induced by a circle-valued Morse function $f : M \rightarrow S^1$. In this case, the deck transformation group is $G = \mathbb{Z}$, the group ring $\mathbb{Z}[G]$ is isomorphic to the polynomial ring $\mathbb{Z}[t]$ and $\mathbb{Z}((G))$ is isomorphic to the ring of the formal Laurent series $\mathbb{Z}((t))$.

Consider the flow ordering $<_f$ and a ($<_f$ -ordered) p -Morse decomposition $\mathcal{M}(M) = \{M_\pi\}_{\pi \in P}$ of M , where each p -Morse set M_π is a critical point of f . Throughout this section, fix sheets $\overline{M}_\pi$ over M_π , for all $\pi \in P$.

Given $M_\pi = \{h_k\}$ and $M_{\pi'} = \{h_{k-1}\}$, where h_k and h_{k-1} are consecutive critical points of f with Morse indices k and $k-1$, respectively, we have that π and π' are adjacent elements with respect to $<_f$. Moreover,

$$\overline{M}_{\pi, t^\ell \pi'} = \overline{M}_\pi \cup C^*(\overline{M}_\pi, t^\ell \overline{M}_{\pi'}) \cup t^\ell \overline{M}_{\pi'}$$

is an isolated invariant set, for all $\ell \in \mathbb{Z}$. Hence, $G_{\pi\pi'} = G = \mathbb{Z}$.

Consider the attractor–repeller pair $(t^\ell \overline{M}_{\pi'}, \overline{M}_\pi)$ of $\overline{M}_{\pi, t^\ell \pi'}$, for each $\ell \in \mathbb{Z}$. There exists a long exact sequence

$$\cdots \longrightarrow CH_*(t^\ell \overline{M}_{\pi'}) \xrightarrow{i_*} CH_*(\overline{M}_{\pi, t^\ell \pi'}) \xrightarrow{p_*} CH_*(\overline{M}_\pi) \xrightarrow{\bar{\delta}_*(\overline{M}_\pi, t^\ell \overline{M}_{\pi'})} CH_{*-1}(t^\ell \overline{M}_{\pi'}) \longrightarrow \cdots$$

In this case, the set of generators $B_k(M_\pi)$ of the homology Conley index $CH_*(M_\pi)$ has exactly one element and $B_i(M_\pi) = 0$, for all $i \neq k$. Thus $NCH_k(M_\pi) = \mathbb{Z}((t)) \otimes_{\mathbb{Z}[t]} \mathbb{Z}[t][B_k(M_\pi)]$ and

$$N\Delta : \bigoplus_{\pi \in P} NCH_*(M_\pi) \longrightarrow \bigoplus_{\pi \in P} NCH_*(M_\pi),$$

given by the matrix

$$N\Delta = (\delta^N(\pi, \pi'))_{\pi, \pi' \in P},$$

is an upper triangular map, where $\delta^N(\pi, \pi') := \delta^N(M_\pi, M_{\pi'})$ is the connecting map for the attractor–repeller pair introduced in §4.2. Note that $\delta^N(\pi, \pi') = 0$ whenever π and π' are not adjacent.

Example 5.1. Consider a flow φ on the solid torus which has two hyperbolic singularities p and q of indices 2 and 1, respectively. Moreover, for each $\ell \in \mathbb{Z}_+$ there is only one flow line joining p and q which intersects a given regular level set ℓ times (turns around ℓ times). Considering the invariant set $S = \{p\} \cup C^*(\{p\}, \{q\}) \cup \{q\}$, the collection $\mathcal{M}(S) = \{M_\pi = \{p\}; M_{\pi'} = \{q\}\}$ is a ($<_f$ -ordered) p -Morse decomposition of S . See Figure 11.

Fix sheets $\overline{M}_\pi = \{\overline{p}\}$ and $\overline{M}_{\pi'} = \{\overline{q}\}$ over M_π and $M_{\pi'}$. The set

$$\overline{M}_{\pi, t^\ell \pi'} = \{\overline{p}\} \cup C^*(\{\overline{p}\}, \{t^\ell \overline{q}\}) \cup \{t^\ell \overline{q}\}$$

is composed by two singularities and a unique orbit between them, hence it is an isolated invariant set, for all $\ell \in \mathbb{Z}$.

Even though $\overline{C}_{\{p\}}^S = \bigcup_{\ell \in \mathbb{Z}_+} \overline{M}_{\pi, t^\ell \pi'}$ is not compact, it can be decomposed into a union of isolated invariant sets. Therefore S is a union of evenly covered isolated invariant sets, that is, $S = \bigcup_{\ell \in \mathbb{Z}_+} M_{\pi, t^\ell \pi'}$.

Let $NCH_1(\{q\}) = \mathbb{Z}((t)) \otimes_{\mathbb{Z}[t]} \mathbb{Z}[t][a]$ and $NCH_2(\{p\}) = \mathbb{Z}((t)) \otimes_{\mathbb{Z}[t]} \mathbb{Z}[t][r]$, where $B_1(\{q\}) = \{a\}$ and $B_2(\{p\}) = \{r\}$ are generators of the homology Conley indices of $\{q\}$

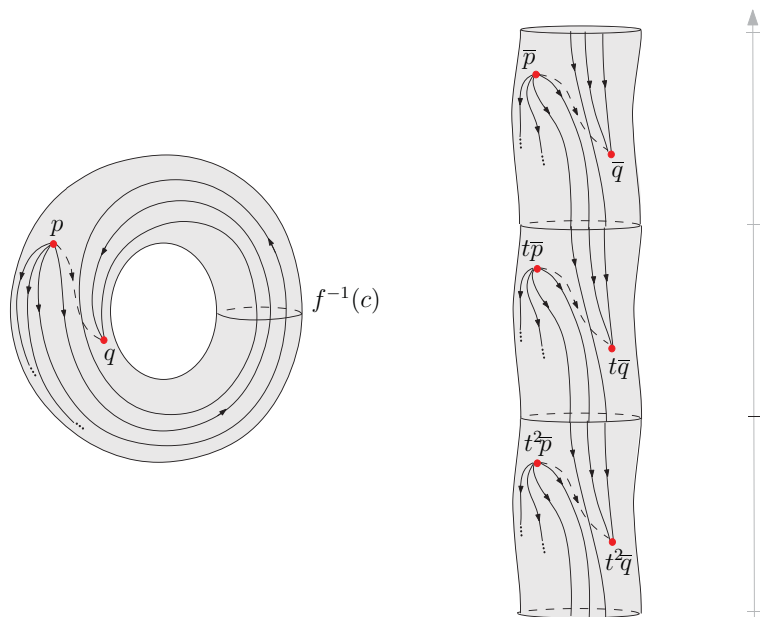


FIGURE 11. A flow on a solid torus.

and $\{p\}$, respectively. The map

$$N\Delta : NCH_*(\{q\}) \oplus NCH_*(\{p\}) \longrightarrow NCH_*(\{q\}) \oplus NCH_*(\{p\})$$

is defined by the matrix

$$\begin{pmatrix} 0 & \delta^N(\{p\}, \{q\}) \\ 0 & 0 \end{pmatrix}$$

where $\delta_2^N(\{p\}, \{q\})(r) = \sum_{\ell \in \mathbb{Z}_+} t^\ell \bar{\delta}_k(\bar{p}, t^\ell \bar{q})(r) = \sum_{\ell \in \mathbb{Z}_+} \pm t^\ell a$.

In order to prove that the Novikov boundary differential ∂^{Nov} is a p -connection matrix for the Morse decomposition $\mathcal{M}(M)$, we make use of Salamon's results in [21] and the characterization of the Novikov complex by direct limits given in [5].

THEOREM 5.2. *The Novikov differential ∂^{Nov} is the p -connection matrix for the $<_f$ -ordered p -Morse decomposition, where $(\bar{M}, p)$ is the infinite cyclic covering space.*

Proof. Assume that $1 \in S^1$ is a regular value of f . Denoting by V the set $f^{-1}(1) \subset M$ and cutting M along V , we obtain the fundamental cobordism (W, tV, V) for $\bar{M}$ and the Morse function $F_V : (W, tV, V) \rightarrow ([0, 1], 0, 1)$. See Figure 12.

Note that each Morse set M_π , which is a critical point of f , has a unique lift to a critical point of F_V in W which will be denoted by $\bar{M}_\pi$. Moreover, we have that

$$F = \bigcup_{j=-\infty}^{\infty} t^j F_V, \quad \bar{M} = \bigcup_{j=-\infty}^{\infty} t^j W \quad \text{and} \quad \text{Crit}_k(F) = \bigcup_{j=-\infty}^{\infty} t^j \text{Crit}_k(F_V).$$

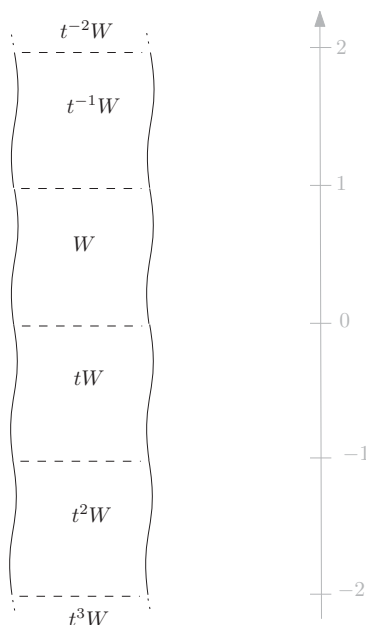


FIGURE 12. A fundamental domain.

Denote $W(\ell) = \bigcup_{j=0}^{\ell} t^j W$ and $F(\ell) = F|_{W(\ell)}$.

Choosing the lifts of the critical points of f that belong to W in the construction of the chain complex $(NC(M), \Delta)$, the coefficients of the differential are in $\mathbb{Z}[[t]] \subset \mathbb{Z}((t))$.

Define $[B_k^\ell(M_\pi)] := [B_k(\overline{M}_\pi), B_k(t\overline{M}_\pi), B_k(t^2\overline{M}_\pi), \dots, B_k(t^\ell\overline{M}_\pi)]$, where $B_k(t^i\overline{M}_\pi)$ is the set of generators of $CH_k(t^i\overline{M}_\pi)$. Note that $\{\mathbb{Z}[B_k^\ell(M_\pi)], \pi_j^\ell\}$ is an inverse system, where π_j^ℓ are the natural projections. Hence, the inverse limit is a based finitely generated free $\mathbb{Z}[[t]]$ -module and

$$\lim_{\leftarrow \ell} \mathbb{Z}[B_k^\ell(M_\pi)] = \mathbb{Z}[[t]][B_k(M_\pi)].$$

Therefore,

$$\begin{aligned} NCH_k(M_\pi) &= \mathbb{Z}((t)) \otimes_{\mathbb{Z}[[t]]} \mathbb{Z}[[t]][B_k^\ell(M_\pi)] \\ &= \mathbb{Z}((t)) \otimes_{\mathbb{Z}[[t]]} \lim_{\leftarrow \ell} \mathbb{Z}[B_k^\ell(M_\pi)]. \end{aligned}$$

Now, consider the upper triangular boundary map

$$N\Delta^\ell = (\bar{\delta}^\ell(\pi, \pi'))_{\pi, \pi' \in P}$$

where

$$\begin{aligned} \bar{\delta}_k^\ell(\pi, \pi') : \mathbb{Z}[B_k^\ell(M_\pi)] &\longrightarrow \mathbb{Z}[B_{k-1}^\ell(M_{\pi'})] \\ r_\alpha^k &\longmapsto \sum_{0 \leq i \leq \ell} \bar{\delta}_k(\overline{M}_\pi, t^i\overline{M}_{\pi'})(r_\alpha^k) \end{aligned}$$

is the connecting map for the attractor–repeller pair $(M_{\pi'}, M_{\pi})$, and $\bar{\delta}_k^{\ell}(\pi, \pi') = 0$ when π is not adjacent to π' . Therefore, $N\Delta$ can be rewritten as the inverse limit of the maps $N\Delta^{\ell}$, that is, for each $k \geq 0$,

$$N\Delta_k = 1 \otimes_{\mathbb{Z}[[t]]} \lim_{\leftarrow \ell} N\Delta_k^{\ell}.$$

Summarizing,

$$(NC(M), N\Delta) = \left(\bigoplus_P (\mathbb{Z}((t)) \otimes_{\mathbb{Z}[[t]]} \lim_{\leftarrow \ell} \mathbb{Z}[B_k^{\ell}(M_{\pi})]), 1 \otimes_{\mathbb{Z}[[t]]} \lim_{\leftarrow \ell} N\Delta^{\ell} \right).$$

On the other hand, for each $k \in \mathbb{Z}$ and $\ell \in \mathbb{N}$, we have that $\mathbb{Z}[B_k^{\ell}(M_{\pi})] = \bigoplus_{j=0}^{\ell} CH_k(t^j \overline{M}_{\pi})$, hence

$$NCH_k(M_{\pi}) = \mathbb{Z}((t)) \otimes_{\mathbb{Z}[[t]]} \lim_{\leftarrow \ell} \bigoplus_{j=0}^{\ell} CH_k(t^j \overline{M}_{\pi}).$$

Note that $N\Delta_*^{\ell}$ coincides with the Franzosa's connection matrix,

$$\Delta_k^{\ell} : \bigoplus_{\pi \in P} \bigoplus_{j=0}^{\ell} CH_k(t^j \overline{M}_{\pi}) \rightarrow \bigoplus_{\pi \in P} \bigoplus_{j=0}^{\ell} CH_{k-1}(t^j \overline{M}_{\pi}),$$

of the induced Morse decomposition for $\bigcup_{j=0}^{\ell} t^j W$.

Therefore, $N\Delta_k = 1 \otimes_{\mathbb{Z}[[t]]} \lim_{\leftarrow \ell} \Delta_k^{\ell}$ and

$$(NC(M), \Delta) = \left(\bigoplus_P (\mathbb{Z}((t)) \otimes_{\mathbb{Z}[[t]]} \lim_{\leftarrow \ell} \bigoplus_{j=0}^{\ell} CH_*(t^j \overline{M}_{\pi})), 1 \otimes_{\mathbb{Z}[[t]]} \lim_{\leftarrow \ell} \Delta_*^{\ell} \right).$$

Since $W(\ell)$ is a compact manifold with no critical points in the boundary, it follows from Lemma 2 in [21] that the connection matrix for a Morse flow, given by the negative gradient of the Morse–Smale function $F(\ell)$, coincides with the Morse differential of $F(\ell)$, that is,

$$\Delta_*^{\ell} = \partial_*(W(\ell), F(\ell)).$$

Since $\bigoplus_{j=0}^{\ell} CH_k(t^j \overline{M}_{\pi}) = \mathbb{Z}[\text{Crit}_k(F(\ell))]$, we have that

$$\begin{aligned} (NC(M), N\Delta) &= (\mathbb{Z}((t)) \otimes_{\mathbb{Z}[[t]]} \lim_{\leftarrow \ell} C_*(W(\ell), F(\ell)), 1 \otimes_{\mathbb{Z}[[t]]} \lim_{\leftarrow \ell} \partial_*(W(\ell), F(\ell))) \\ &= (C^{\text{Nov}}(M, f), \partial^{\text{Nov}}), \end{aligned}$$

where the second equality follows by Lemma 2.5 in [5]. □

As we proved in this section, the Novikov theory fits nicely as a special case of the covering action on Conley index theory. Consequently, it opens the possibility of making use of a variety of tools from Conley index theory in Novikov theory. For instance, one can study periodic orbits [13, 16], chaos [14], cancelations [7], and so forth. Furthermore, it enables us to apply transition matrices as in [9, 10] to understand bifurcations that may

occur when we consider a parameterized family of gradient flows of circle-valued Morse functions.

Acknowledgments. D. V. S. Lima would like to thank the São Paulo Research Foundation (FAPESP) for support under grants 2020/11326-8 and 2016/24707-4. M. R. da Silva would like to thank the São Paulo Research Foundation (FAPESP) for support under grants 2016/24707-4 and 2018/13481-0. E. R. Vieira would like to acknowledge the support of the National Science Foundation under grant HDR TRIPODS 1934924.

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