REFLECTED BSDES IN NON-CONVEX DOMAINS

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Abstract. This paper establishes the well-posedness of reflected backward stochastic differential equations in non-convex domains that satisfy a weak version of the star-shaped property. The main results are established (i) in a Markovian framework with Hölder-continuous generator and terminal condition and (ii) in a general setting under a smallness assumption on the input data. We also investigate the connections between this well-posedness result and the theory of martingales on manifolds, which, in particular, illustrates the sharpness of some of our assumptions.

1. Introduction

Backward stochastic differential equations (BSDEs), originally introduced in [2] and fully developed in [43, 41], can be viewed as the probabilistic analogues of semi-linear partial differential equations (PDEs). In particular, BSDEs are used to describe the solutions of stochastic control problems (see, among many others, [42, 16, 25]). If the control variable of such an optimization problem has a discrete component – e.g., an option to switch the state process to a different regime or to terminate the process and obtain an instantaneous payoff – then, the associated PDE obtains a free-boundary feature and the associated BSDE becomes reflected: i.e., its solution lives inside a given domain and is reflected at the boundary of this domain. The theory of reflected BSDEs in dimension one – i.e., when the reflected process is one-dimensional – is well developed in a very high generality: see, e.g. [15, 11, 14, 23, 24]. However, the multidimensional case presents significant additional challenges, e.g., due to the lack of the comparison principle. To date, the well-posedness of general multidimensional reflected BSDEs (or, systems of reflected BSDEs) has only been established in the case of convex reflection domains: see, e.g., [22, 34, 10, 19]. The systems of reflected BSDEs in convex domains appear in certain types of stochastic control problems, such as the switching problems: see, among others, [26, 29, 9, 8, 38, 1]. On the other hand, [21] describes a class of control-stopping stochastic differential games where the equilibria are described by the systems of reflected BSDEs

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in non-convex domains (see also the closely related [20]). We also refer to [6], which considers another example of a system of reflected BSDEs in a non-convex domain. This paper presents the first general well-posedness result for the systems of reflected BSDEs in non-convex domains under the assumption of a weak star-shape property (see Assumption 1.1 below).

In addition to the control-stopping games, the reflected BSDEs in non-convex domains have a direct connection to the theory of martingales on manifolds. We refer to [17] for an introduction and an overview of this theory. One of the key questions therein is the following: given a random variable ξ with values in a manifold M, is it possible to define a martingale Y in M such that the terminal value of this martingale (at time $T > 0$) is given by ξ (i.e., $Y_T = \xi$), and is such a martingale unique? A positive answer to this question, in particular, allows one to define the notions of conditional expectation and barycenter for a manifold (see e.g. [18, 46]). We refer to [32, 33, 45, 12] for other applications, and to [3, 4] for the theory of BSDEs on manifolds. As explained in [12], it is possible to give a positive answer to the above question by solving a BSDE with quadratic non-linearities with respect to the z-variable, stated in $\mathbb{R}^d$ – the Euclidean space in which the manifold is embedded. It turns out that for a certain class of non-convex reflection domains $\mathcal{D}$, the reflected BSDE in $\mathcal{D}$ gives rise to a martingale on the manifold $\partial\mathcal{D}$, see Section 5. In particular, our results provide a new proof of the existence and uniqueness of a martingale with a prescribed terminal value in a given strict sub-sector of a hemisphere of $\mathbb{S}^{d-1}$, in the Markovian framework or under the appropriate smallness assumptions (see the example in Section 5).

On a technical level, our analysis is connected to the theory of BSDEs with quadratic growth in the z-variable. This connection is made precise in Section 3, but it can also be seen if one attempts to map a given non-convex domain into a convex one: the resulting reflected BSDE in a convex domain will have quadratic terms in z. Thus, the reflected BSDEs in non-convex domains can be viewed as the quadratic reflected BSDEs in convex domains. This observation also explains the additional mathematical challenges caused by the non-convexity of the reflection domain – these challenges are similar to those arising in the well-posedness theory for the systems of quadratic BSDEs [47, 30, 48, 27]. The present work uses some of the results developed in the latter theory: in particular, the results of [48] are crucial for our analysis.

Another important connection is to the methods of [36, 37], which establish the well-posedness of the forward stochastic differential equations (SDEs) reflected at the boundary of a given domain. In particular, we use the arguments of [37] to establish the stability of solutions to the reflected BSDEs considered herein, see Section 2. It is important to mention, however, that many crucial arguments used in the proof of the well-posedness of a reflected (forward) SDE cannot be applied to the case of a reflected BSDE due to the adaptedness issues which, in

particular, prohibit the application of the Skorokhod's mapping, used in [37], and of the standard localization methods.

The remainder of this paper is organized as follows. Section 1.1 states the reflected BSDE (equation (1.2)) and the main assumptions (Assumptions 1.1 and 1.2) which hold throughout the paper. Section 2 describes various auxiliary properties and a priori estimates, as well as the stability (Proposition 2.2) and uniqueness (Corollary 2.2) of solutions to the reflected BSDE in a certain class. Section 3 describes a sequence of penalized quadratic BSDEs in a Markovian framework, shows that their solutions converge to a solution of the reflected BSDE, and verifies that this solution belongs to the class in which the uniqueness holds, thus establishing the well-posedness of the target reflected BSDE in a Markovian framework (Theorem 3.1). In Section 4, we approximate a general reflected BSDE by the Markovian ones, to obtain the well-posedness of the former (Theorem 4.2) under an additional smallness assumption (Assumption 2.1). Finally, Section 5 provides a more detailed description of the connection between the reflected BSDEs in non-convex domains and the martingales on manifolds, which, in particular, illustrates the sharpness of some of our assumptions.

1.1. **The setup and main assumptions.** Let $\mathcal{D}$ be a subset of $\mathbb{R}^d$ given by

$$\mathcal{D} = \{y \in \mathbb{R}^d : \phi(y) < 0\},$$

with a function $\phi : \mathbb{R}^d \to \mathbb{R}$. We denote by ∇ the gradient, and by ∇^2 the Hessian, of a given function. For any subset A of a Euclidean space, we denote its closure by $\bar{A}$ and, if $A \neq \varnothing$, we denote by $d(., A)$ the distance function to A.

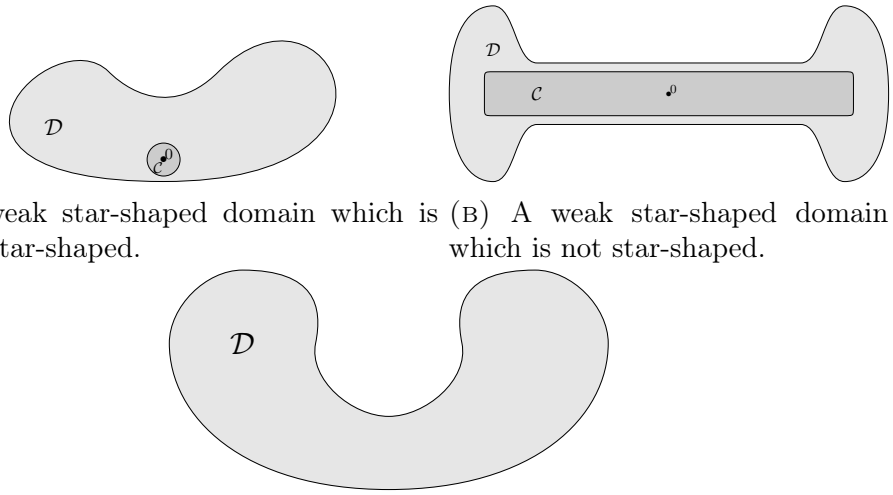
Assumption 1.1. *We assume that ϕ satisfies the following:*

- *(Compactness) There exists $R > 0$, such that $\phi(y) > 0$ for all $|y| \geqslant R$.*
- *(Smoothness) $\phi \in C^2(\mathbb{R}^d)$, $|\nabla\phi(y)| > 0$ for all $y \in \partial\mathcal{D}$, and $\nabla^2\phi$ is locally Lipschitz.*
- *(Weak star-shape property) There exists a non-empty open convex set $\mathcal{C} \subset \mathcal{D}$ such that*
  - *$0 \in \mathcal{C}$,*
  - *there exists a convex function $\phi_{\mathcal{C}} : \mathbb{R}^d \to \mathbb{R}$ satisfying: $\phi_{\mathcal{C}} \in C^2(\mathbb{R}^d)$,*

    $$\mathcal{C} = \{y \in \mathbb{R}^d : \phi_{\mathcal{C}}(y) < 0\},$$

    $\phi_{\mathcal{C}} \geqslant \phi_{\mathcal{C}}(0)$, and $\phi_{\mathcal{C}}(y) = |y - \mathfrak{P}_{\bar{\mathcal{C}}}(y)|$ for all $y \in \mathbb{R}^d \backslash \mathcal{C}$, where $\mathfrak{P}_{\bar{\mathcal{C}}}$ denotes the projection onto $\bar{\mathcal{C}}$,
  - *it holds that*

$$\gamma := \inf_{y \in \partial\mathcal{D}} \nabla\phi_{\mathcal{C}}(y) \cdot \frac{\nabla\phi(y)}{|\nabla\phi(y)|} > 0. \tag{1.1}$$



(A) A weak star-shaped domain which is strictly star-shaped.

(B) A weak star-shaped domain which is not star-shaped.

(C) A simply connected domain which is not weak star-shaped.

FIGURE 1. Examples of domains with and without the (weak) star-shaped property.

Remark 1.1. *(i) If $\mathcal{D}$ is a star-shaped domain with respect to 0, i.e., if it satisfies*

$$\inf_{y\in\partial\mathcal{D}} \frac{y}{|y|}\cdot\frac{\nabla\phi(y)}{|\nabla\phi(y)|} > 0,$$

then the weak star-shape property also holds for $\mathcal{D}$, with $\mathcal{C}$ being a ball of radius $\varepsilon > 0$ centered at 0, and with

$$\phi_{\mathcal{C}}(y) = \varrho_\varepsilon(|y| - \varepsilon),$$

where $\varrho_\varepsilon : \mathbb{R} \to \mathbb{R}$ is a convex increasing function satisfying $\varrho_\varepsilon \in C^2(\mathbb{R})$, $\varrho_\varepsilon(x) = -\varepsilon/2$ for $x < -\varepsilon$ and $\varrho_\varepsilon(x) = x$ for $x > 0$.
(ii) As shown in Figure 1, a weak star-shaped domain is not necessarily star-shaped.

All stochastic processes and random variables appearing in this paper are constructed on a fixed stochastic basis $(\Omega, \mathbb{F}, \mathbb{P})$, with the filtration $\mathbb{F}$ being a completion of the natural filtration of a multidimensional Brownian motion W in $\mathbb{R}^{d'}$ on a time interval $[0, T]$.

For $p \geqslant 1$, we denote by $\mathcal{L}^p$ the space of (classes of equivalence of)[1] $\mathcal{F}_T$-measurable random variables ξ (with values in a Euclidean space), such that

[1]We drop this clarification in further definitions.

$\|\xi\|_{\mathcal{L}^p} := \mathbb{E}[|\xi|^p]^{1/p} < \infty$. The space $\mathcal{L}^\infty$ stands for all $\mathcal{F}_T$-measurable essentially bounded random variables. We also define $\mathscr{H}^2$ as the space of progressively measurable processes Z (with values in a Euclidean space), such that $\|Z\|_{\mathscr{H}^2} := \mathbb{E}\Big[\int_0^T |Z_t|^2 \mathrm{d}t\Big]^{1/2} < \infty$. Next, for $p \geqslant 1$, we define $\mathcal{M}^p$ as the space of all continuous local martingales M with $\|M\|_{\mathcal{M}^p} := \mathbb{E}\Big[\langle M\rangle_T^{p/2}\Big]^{1/p} < \infty$. For $p \in [1,\infty]$, we denote by $\mathscr{S}^p$ the set of continuous adapted process U such that $\big\|\sup_{t\in[0,T]} |U_t|\big\|_{\mathscr{L}^p} < \infty$. We also denote by $\mathrm{Var}_t(K)$ the variation of a process K. (with values in a Euclidean space) on the time interval $[0,t]$ and by $\mathscr{K}^p$, for $p \in [1,\infty]$, the set of all finite-variation process K such that $\big\|\mathrm{Var}_{[0,T]}(K)\big\|_{\mathscr{L}^p} < \infty$ and $K_0 = 0$. Finally, we denote by $\mathscr{B}^2$ the set of processes $V \in \mathscr{H}^2$, satisfying

$$\|V\|_{\mathscr{B}^2} := \Big\|\sup_{t\in[0,T]} \mathbb{E}\Big[\int_t^T |V_s|^2 \mathrm{d}s|\mathcal{F}_t\Big]\Big\|_{\mathscr{L}^\infty}^{\frac{1}{2}} < +\infty.$$

Let us remark that $V \in \mathscr{B}^2$ implies that the martingale $\int_0^\cdot V_s \mathrm{d}W_s$ is a BMO martingale, and $\|V\|_{\mathscr{B}^2}$ is the BMO norm of $\int_0^\cdot V_s \mathrm{d}W_s$. We refer to [31] for further details about BMO martingales.

We are investigating the well-posedness of the following reflected BSDE $(Y,Z,K) \in \mathscr{S}^2 \times \mathscr{H}^2 \times \mathscr{K}^1$

$$\begin{cases} (i)\ Y_t = \xi + \displaystyle\int_t^T f(s,Y_s,Z_s)\mathrm{d}s - \int_t^T \mathrm{d}K_s - \int_t^T Z_s \mathrm{d}W_s, \quad 0 \leqslant t \leqslant T, \\ (ii)\ Y_t \in \bar{\mathcal{D}} \text{ a.s.}, \quad K_t = \displaystyle\int_0^t \mathfrak{n}(Y_s)\mathrm{dVar}_s(K), \quad 0 \leqslant t \leqslant T, \end{cases} \tag{1.2}$$

where $\mathfrak{n}$ is the unit outward normal to $\partial\mathcal{D}$, extended as zero into $\mathcal{D}$:

$$\mathfrak{n}(y) = \frac{\nabla\phi(y)}{|\nabla\phi(y)|}, \quad \forall y \in \partial\mathcal{D} \quad \text{and} \quad \mathfrak{n}(y) = 0, \quad \forall y \in \mathcal{D}.$$

Assumption 1.2. *We assume that ξ takes values in $\bar{\mathcal{D}}$, $f(\cdot,y,z)$ is progressively measurable, $f(t,\cdot,\cdot)$ is globally Lipschitz ($K_{f,y}$-Lipschitz in y and $K_{f,z}$-Lipschitz in z), uniformly in (t,ω), and $\||f(\cdot,0,0)|\|_{\mathscr{L}^\infty} < \infty$. In addition, without loss of generality (in view of the boundedness of $\mathcal{D}$), we assume that there exists a compact $\mathcal{K} \subset \mathbb{R}^d$, such that $f(t,y,z) = 0$ whenever $y \notin \mathcal{K}$.*

Assumptions 1.1 and 1.2 hold throughout the rest of the paper even if not cited explicitly.

2. Geometric properties and a priori estimates

In this section, we derive cetain useful geometric properties of the domain $\mathcal{D}$, expressed via the corresponding properties of the function ϕ. We construct an auxiliary function ψ which is used in the next section to define a sequence of approximating equations to (1.2). We also present some key *a priori* estimates and properties of the solutions to the RBSDEs (1.2).

2.1. Absolute continuity of the process K.

As noticed in [22], we can take advantage of the smoothness of $\mathcal{D}$ to show that the process K is absolutely continuous with respect to the Lebesgue measure.

Lemma 2.1. *Assume that $(Y, Z, K) \in \mathscr{S}^2 \times \mathscr{H}^2 \times \mathscr{K}^1$ solves (1.2). Then, almost every path of K is absolutely continuous with respect to the Lebesgue measure.*

Proof. Applying Itô's formula to $t \mapsto \phi(Y_t)$, we obtain

$$\begin{aligned} \mathrm{d}\phi(Y_t) &= \left(-\nabla\phi(Y_t)\cdot f(t,Y_t,Z_t) + \frac{1}{2}\mathrm{Tr}[Z_t^\top \nabla^2\phi(Y_t)Z_t]\right)\mathrm{d}t \\ &\quad + \nabla\phi(Y_t)\cdot \mathrm{d}K_t + \nabla\phi(Y_t)\cdot Z_t\mathrm{d}W_t \end{aligned} \tag{2.1}$$

Then, the Itô-Tanaka formula applied to the positive part of the semi-martingale $-\phi(Y_t)$ reads

$$\mathrm{d}[-\phi(Y_t)]^+ = \mathbf{1}_{\{-\phi(Y_t)>0\}}\mathrm{d}[-\phi(Y_t)] + \frac{1}{2}\mathrm{d}L_t^0, \tag{2.2}$$

where L^0 is the local time of the semi-martingale $-\phi(Y)$ at zero. Since $\phi(Y_t) \leqslant 0$, we have $\mathrm{d}[-\phi(Y_t)]^+ = -\mathrm{d}\phi(Y_t)$ which yields, combining (2.1)–(2.2),

$$\begin{aligned} &\mathbf{1}_{\{\phi(Y_t)=0\}}\left(-\nabla\phi(Y_t)\cdot f(t,Y_t,Z_t) + \frac{1}{2}\mathrm{Tr}[Z_t^\top \nabla^2\phi(Y_t)Z_t]\right)\mathrm{d}t + |\nabla\phi(Y_t)|\mathrm{dVar}_t(K) \\ &\quad + \mathbf{1}_{\{\phi(Y_t)=0\}}\nabla\phi(Y_t)\cdot Z_t\mathrm{d}W_t + \frac{1}{2}\mathrm{d}L_t^0 = 0\,. \end{aligned}$$

In particular, we deduce that

$$|\nabla\phi(Y_t)|\mathrm{dVar}_t(K) \leqslant \mathbf{1}_{\{\phi(Y_t)=0\}}\left[\nabla\phi(Y_t)\cdot f(t,Y_t,Z_t) - \frac{1}{2}\mathrm{Tr}[Z_t^\top \nabla^2\phi(Y_t)Z_t]\right]^+\mathrm{d}t, \tag{2.3}$$

which proves the absolute continuity of K. □

2.2. The exterior sphere property.

The following lemma states the well known observation that, for any boundary point of a smooth domain, there exists a small enough tangent external sphere, see e.g. [37].

Lemma 2.2. *There exists $R_0 > 0$, such that*

$$(y - y')\cdot \mathfrak{n}(y) + \frac{1}{2R_0}|y-y'|^2 \geqslant 0\,, \quad \forall\, y \in \partial\mathcal{D},\ y' \in \bar{\mathcal{D}}. \tag{2.4}$$

Proof. Due to the smoothness of ϕ, for any $y \in \partial\mathcal{D}$ and $y' \in \bar{\mathcal{D}}$, there exists $\lambda \in [0,1]$, such that

(2.5)
$$0 \geqslant \phi(y') = \phi(y) + (y'-y)\cdot \mathfrak{n}(y)|\nabla\phi(y)| + \frac{1}{2}(y-y')^\top \nabla^2\phi(\lambda y + (1-\lambda)y')(y-y'),$$

It only remains to notice that: $\phi = 0$ and $|\nabla\phi|$ is bounded away from zero on $\partial\mathcal{D}$, and $|\nabla^2\phi|$ is bounded from above on $\bar{\mathcal{D}}$. Thus, we obtain the statement of the lemma. □

Using the above lemma, we can define the projection operator that is used in the subsequent sections. To this end, we first introduce the set

$$Q = \{y \in \mathbb{R}^d : d(y,\mathcal{D}) < R_0\},$$

and the set-valued projection operator

$$\mathfrak{P}(y) = \operatorname{argmin}_{x\in\bar{\mathcal{D}}}|x-y|, \quad y \in \mathbb{R}^d.$$

Corollary 2.1. *For any $y \in Q$, $\mathfrak{P}(y)$ is a singleton.*

Proof. It is easy to see that, for a ball $B_r(y) \subset \mathbb{R}^d$, with radius $r > 0$ and center at y, we have:

$$(x-x')\cdot\frac{y-x}{|y-x|} + \frac{1}{2r}|x-x'|^2 = 0, \quad \forall\, x,x' \in \partial B_r(y). \tag{2.6}$$

Next, assume that there exist $y \in \mathbb{R}^d\backslash\bar{\mathcal{D}}$ and $x \neq x' \in \bar{\mathcal{D}}$, such that

$$|x-y| = |x'-y| = \operatorname{argmin}_{z\in\bar{\mathcal{D}}}|z-y|.$$

Then, it is clear that $x, x' \in \partial B_r \cap \partial\mathcal{D}$, with $r = \min_{z\in\bar{\mathcal{D}}}|z-y| < R_0$, and the equations (2.4), (2.6) yield a contradiction. □

Without loss of generality, we will identify the value of $\mathfrak{P}(y)$ with its only element, for any $y \in Q$.

Remark 2.1. *Using (2.5) and (2.3), we easily deduce that, for any solution $(Y,Z,K) \in \mathscr{S}^2 \times \mathscr{H}^2 \times \mathscr{K}^1$ of (1.2),*

$$\mathrm{dVar}_t(K) \leqslant \mathbf{1}_{\{\phi(Y_t)=0\}}\left(\left[\frac{\nabla\phi(Y_t)}{|\nabla\phi(Y_t)|}\cdot f(t,Y_t,Z_t)\right]^+ + \frac{1}{2R_0}|Z_t|^2\right)\mathrm{d}t,$$

with R_0 satisfying (2.4).

The following lemma provides an alternative to (1.2)(ii), and it becomes useful in the subsequent sections.

Lemma 2.3. *Assume that $(Y,Z,K) \in \mathscr{S}^2 \times \mathscr{H}^2 \times \mathscr{K}^1$ solves (1.2)(i) and that $Y_t \in \bar{\mathcal{D}}$ a.s. for all $t \in [0,T]$. Then*

$$K_t = \int_0^t \mathfrak{n}(Y_s)\mathrm{dVar}_s(K), \quad t \in [0,T],$$

holds if and only if there exists a constant $c > 0$, depending only on $\mathcal{D}$, such that for all essentially bounded continuous adapted process V in $\mathcal{D}$, we have

$$\int_0^T \big((Y_s - V_s) + c|Y_s - V_s|^2 \mathfrak{n}(Y_t)\big)\, \mathrm{d}K_s \geqslant 0 \,. \tag{2.7}$$

Proof. One implication is a direct consequence of Lemma 2.2. The other implication is a minor extension of Lemma 2.1 in [22]. □

2.3. **The pseudo-distance function.** In this subsection, we modify the function ϕ in order to construct a new smooth function ψ which satisfies the inequality (1.1) in $\mathbb{R}^d \backslash \mathcal{D}$ instead of $\partial\mathcal{D}$. We denote by $\vartheta : \mathbb{R} \to [0,1]$ an infinitely smooth nondecreasing function which is equal to zero on $(-\infty, 0]$ and to one on $[1, \infty)$. We also choose a large enough $R > 1$, such that $\mathcal{D} \subset B_{R-1}(0)$, and a small enough $\epsilon \in (0,1)$, such that, for all $y \in B_{R+1}(0) \backslash \mathcal{D}$, we have:

$$\phi(y) \leqslant \epsilon \quad \Rightarrow \quad y \in B_R(0), \quad \nabla\phi_{\mathcal{C}}(y) \cdot \nabla\phi(y) > 0.$$

Then, we define

$$\tilde{\phi}(y) := \phi^+(y)(1 - \vartheta(|y| - R - 1)) + \vartheta(|y| - R), \quad \psi(y) := \tilde{\phi}(y) + \kappa|y|\vartheta(\tilde{\phi}(y)/\epsilon), \tag{2.8}$$

for an arbitrary constant $\kappa > 0$.

We refer to ψ as the pseudo-distance function.

Notice that

$$\begin{aligned}\nabla\phi_{\mathcal{C}}(y) \cdot \nabla\psi(y) &= \nabla\phi_{\mathcal{C}}(y) \cdot \nabla\tilde{\phi}(y) + \kappa\nabla\phi_{\mathcal{C}}(y) \cdot \frac{y}{|y|}\vartheta(\tilde{\phi}(y)/\epsilon) \\ &\quad + \kappa\nabla\phi_{\mathcal{C}}(y) \cdot \nabla\tilde{\phi}(y)|y|\vartheta'(\tilde{\phi}(y)/\epsilon)/\epsilon \\ &= \nabla\phi_{\mathcal{C}}(y) \cdot \nabla\tilde{\phi}(y)\left(1 + \kappa|y|\vartheta'(\tilde{\phi}(y)/\epsilon)/\epsilon\right) + \kappa\nabla\phi_{\mathcal{C}}(y) \cdot \frac{y}{|y|}\vartheta(\tilde{\phi}(y)/\epsilon).\end{aligned}$$

It is clear that $\psi \in C^2(\mathbb{R}^d \backslash \bar{\mathcal{D}})$ and that its derivatives up to the second order are locally Lipschitz-continuous on $\mathbb{R}^d \backslash \mathcal{D}$. It is also easy to see that $\tilde{\phi}(y) \in (0, \epsilon]$ if and only if $y \in B_{R+1}(0) \backslash \mathcal{D}$ and $\phi(y) \leqslant \epsilon$, in which case $y \in B_R(0)$, $\tilde{\phi}(y) = \phi(y)$, $\nabla\tilde{\phi}(y) = \nabla\phi(y)$, and

$$\nabla\phi_{\mathcal{C}}(y) \cdot \nabla\psi(y) \geqslant \nabla\phi_{\mathcal{C}}(y) \cdot \nabla\phi(y) > 0,$$

where we also observed that $\inf_{y \in \mathbb{R}^d \backslash \mathcal{D}} \nabla\phi_{\mathcal{C}}(y) \cdot y/|y| > 0$, which follows from the convexity of $\mathcal{C}$ and from the fact that $0 \in \mathcal{C}$. If $\tilde{\phi}(y) \leqslant 0$, then $y \in \bar{\mathcal{D}}$. If $\tilde{\phi}(y) > \epsilon$, then

$$\nabla\phi_{\mathcal{C}}(y) \cdot \nabla\psi(y) = \nabla\phi_{\mathcal{C}}(y) \cdot \nabla\tilde{\phi}(y) + \kappa\nabla\phi_{\mathcal{C}}(y) \cdot \frac{y}{|y|},$$

which can be made positive for all $y \in \mathbb{R}^d \backslash \mathcal{D}$ by choosing large enough $\kappa > 0$, as $|\nabla\tilde{\phi}|$ is bounded on $\mathbb{R}^d \backslash \mathcal{D}$ and $\inf_{y \in \mathbb{R}^d \backslash \mathcal{D}} \nabla\phi_{\mathcal{C}}(y) \cdot y/|y| > 0$.

The following lemma summarizes the above properties of ψ and states several additional properties which can be easily verified.

Lemma 2.4. *There exist constants $R, \epsilon, \kappa > 0$, such that the function ψ defined in (2.8) satisfies the following properties.*

(1) ψ is globally Lipschitz-continuous in $\mathbb{R}^d$.
(2) There exist constants $c, C > 0$, such that $c\, d(y, \mathcal{D}) \leqslant \psi(y) \leqslant C\, d(y, \mathcal{D})$ for $y \in \mathbb{R}^d$.
(3) $\psi \in C^2(\mathbb{R}^d \backslash \bar{\mathcal{D}})$, and its derivatives up to the second order are globally Lipschitz-continuous in $\mathbb{R}^d \backslash \mathcal{D}$.
(4) $\inf_{y \in \mathbb{R}^d \backslash \mathcal{D}} \nabla \phi_{\mathcal{C}}(y) \cdot \nabla \psi(y) > 0$.
(5) $\inf_{y \in \mathbb{R}^d \backslash \mathcal{D}} |\nabla \psi(y)| > 0$.
(6) $\psi(y) = \phi(y) = 0$, $\nabla \psi(y) = \nabla \phi(y)$, and $\nabla^2 \psi(y) = \nabla^2 \phi(y)$, for $y \in \partial \mathcal{D}$.

In the remainder of the paper, we fix (R, ϵ, κ) as in the above lemma and consider the associated pseudo-distance function ψ. For convenience, we also extend the vector-valued function $\mathfrak{n}$ to $\mathbb{R}^d$ as follows:

$$\mathfrak{n}(y) = \frac{1}{|\nabla \psi(y)|} \nabla \psi(y)\, \mathbf{1}_{\{\mathbb{R}^d \backslash \mathcal{D}\}}(y).$$

2.3.1. *Asymptotic convexity of the squared pseudo-distance.* Due to Lemma 2.4, the Hessian of ψ^2, denoted $\nabla^2 \psi^2$, is well defined in $\mathbb{R}^d \backslash \mathcal{D}$ (it is extended to the boundary of the latter set by continuity). The following lemma shows that $\nabla^2 \psi^2$, viewed as a bilinear form, becomes positive semidefinite close to $\mathcal{D}$.

Lemma 2.5. *There exists a constant $C > 0$, such that, for all $y \in \mathbb{R}^d \backslash \mathcal{D}$ and $z \in \mathbb{R}^d$,*

$$z^\top \nabla^2 \psi^2(y) z \geqslant -C \psi(y) |z|^2.$$

Proof. Notice that, for $y \in \mathbb{R}^d \backslash \mathcal{D}$ and $z \in \mathbb{R}^d$,

$$\nabla^2 \psi^2(y) = 2 \nabla \psi(y) \nabla^\top \psi(y) + 2 \psi(y) \nabla^2 \psi(y),$$

$$z^\top \nabla^2 \psi^2(y) z = 2 (\nabla \psi(y) \cdot z)^2 + 2 \psi(y) z^\top \nabla^2 \psi(y) z \geqslant 2 \psi(y) z^\top \nabla^2 \psi(y) z.$$

Using the fact that $\nabla^2 \psi$ is bounded (cf. the third property in Lemma 2.4) and the second property in Lemma 2.4, we complete the proof. □

2.4. ***A priori* estimates.** In this subsection, we establish *a priori* estimates of the solutions to (1.2) for general terminal condition ξ and generator f. We first introduce the appropriate "smallness assumption".

Assumption 2.1. *We assume that at least one of the following four conditions is fulfilled with some $\theta \geqslant 1$:*

(i) $|\phi_{\mathcal{C}}^+(\xi)|_{\mathscr{L}^\infty} < \frac{\gamma R_0}{\theta}$ and $\nabla \phi_{\mathcal{C}}(y) \cdot f(s, y, z) \leqslant 0, \quad \forall s, y, z \in [0, T] \times \bar{\mathcal{D}} \backslash \mathcal{C} \times \mathbb{R}^{d \times d'}$,
(ii) or $\sup_{x \in \mathcal{D}} \phi_{\mathcal{C}}^+(x) < \frac{\gamma R_0}{\theta}$,
(iii) or $\mathcal{C}$ is the Euclidean ball centered at 0 with radius $\lambda > 0$, and

$$|\xi|_{\mathscr{L}^\infty}^2 < \lambda^2 + \frac{2 R_0^2}{\theta}, \quad \nabla \phi_{\mathcal{C}}(y) \cdot f(s, y, z) \leqslant 0, \quad \forall s, y, z \in [0, T] \times \bar{\mathcal{D}} \backslash \mathcal{C} \times \mathbb{R}^{d \times d'},$$

(iv) or $\mathcal{C}$ is the Euclidean ball centered at 0 with radius $\lambda > 0$, and

$$\sup_{x\in\mathcal{D}} |x|^2 < \lambda^2 + \frac{2R_0^2}{\theta},$$

with R_0 satisfying (2.4) and γ appearing in Assumption 1.1.

It is worth mentioning that Assumption 2.1 is not a standing assumption and is cited explicitly whenever it is invoked. In particular, the well-posedness results in the Markovian framework do not require the smallness assumption, see Section 3 .

Next, we consider the following class of solutions.

Definition 2.1. *For any $\theta \geqslant 1$, we denote by $\mathscr{U}(\theta, \xi, f, T)$ the set of all solutions $(Y, Z, K) \in \mathscr{S}^2 \times \mathscr{H}^2 \times \mathscr{K}^1$ to (1.2) such that*

$$\mathbb{E}\left[e^{\frac{\theta p}{R_0}\mathrm{Var}_T(K)}\right] < \infty, \tag{2.9}$$

with some $p > 1$ and with R_0 satisfying (2.4).

In what follows, we often drop (ξ, f, T) in the notation for the class $\mathscr{U}(\theta)$. Note also that we mainly consider $\theta = 1$ and $\theta = 2$.

The following proposition clarifies the link between Assumption 2.1 and the class $\mathscr{U}(\theta)$.

Proposition 2.1. *Let $(Y, Z, K) \in \mathscr{S}^2 \times \mathscr{H}^2 \times \mathscr{K}^1$ be a solution to the RBSDE (1.2). Then, $Z \in \mathscr{B}^2$. Moreover, if Assumption 2.1 holds for some $\theta \geqslant 1$, then, there exist constants C and $p > 1$, which depend only on $K_{f,y}$, $K_{f,z}$, γ, λ, $\sup_{y\in\mathcal{D}} |y|$, $\sup_{y\in\bar{\mathcal{D}}} \phi_{\mathcal{C}}^+(y)$, $\|\phi_{\mathcal{C}}^+(\xi)\|_{\mathscr{L}^\infty}$, $\|f(.,0,0)\|_{\mathscr{L}^\infty}$ and R_0 (recall Assumption 1.1 and (2.4)), such that*

$$\mathbb{E}\left[e^{\frac{\theta p}{R_0}\mathrm{Var}_T(K)}\right] \leqslant C. \tag{2.10}$$

Thus, under Assumption 2.1, any solution (Y, Z, K) belongs to $\mathscr{U}(\theta, \xi, f, T)$.

Proof. 1. We start by applying Itô-Tanaka's formula to $\phi_{\mathcal{C}}^+(Y_t)$ (note that $\phi_{\mathcal{C}}^+$ is convex): for all $t \leqslant t'$,

$$\mathbb{E}_t\left[\int_t^{t'} \nabla\phi_{\mathcal{C}}^+(Y_s)\cdot \mathrm{d}K_s\right] \leqslant \mathbb{E}_t\left[\phi_{\mathcal{C}}^+(Y_{t'}) + \int_t^{t'} \nabla\phi_{\mathcal{C}}^+(Y_s)\cdot f(s, Y_s, Z_s)\mathrm{d}s\right]. \tag{2.11}$$

In the equation above and in the remainder of the proof, we use the shorter notation $\mathbb{E}_t[\cdot]$ for $\mathbb{E}[\cdot|\mathcal{F}_t]$.

Recalling Assumption 1.1, we obtain

$$\gamma\mathbb{E}_t\left[\int_t^{t'} \mathrm{dVar}_s(K)\right] \leqslant \mathbb{E}_t\left[\int_t^{t'} \nabla\phi_{\mathcal{C}}^+(Y_s)\cdot \mathfrak{n}(Y_s)\mathrm{dVar}_s(K)\right] = \mathbb{E}_t\left[\int_t^{t'} \nabla\phi_{\mathcal{C}}^+(Y_s)\cdot \mathrm{d}K_s\right]. \tag{2.12}$$

This yields, for $t' = T$,

$$\gamma \mathbb{E}_t \left[\int_t^T \mathrm{dVar}_s(K) \right] \leqslant \mathbb{E}_t \left[\phi_{\mathcal{C}}^+(\xi) + \int_t^T \nabla \phi_{\mathcal{C}}^+(Y_s) \cdot f(s, Y_s, Z_s) \mathrm{d}s \right]. \tag{2.13}$$

Next, we consider an arbitrary $\varepsilon > 0$ and apply Itô's formula to $\varepsilon |Y_t|^2$ between t and t', to obtain:

$$\begin{aligned} \varepsilon \mathbb{E}_t \left[\int_t^{t'} |Z_s|^2 \mathrm{d}s \right] \leqslant & \varepsilon \mathbb{E}_t \left[|Y_{t'}|^2 \right] + C_1 \varepsilon \mathbb{E}_t \left[\int_t^{t'} (1 + |f(s,0,0)| + |Z_s|) \mathrm{d}s \right] \\ & + \varepsilon C_1 \mathbb{E} \left[\int_t^{t'} \mathrm{dVar}_s(K) \right], \end{aligned}$$

where we recalled that $|Y|$ is bounded. The above inequality implies

$$\frac{\varepsilon}{2} \mathbb{E}_t \left[\int_t^{t'} |Z_s|^2 \mathrm{d}s \right] \leqslant \varepsilon \mathbb{E}_t \left[|Y_{t'}|^2 \right] + C_\varepsilon \mathbb{E}_t \left[\int_t^{t'} (1 + |f(s,0,0)|) \mathrm{d}s \right] + \varepsilon C_1 \mathbb{E} \left[\int_t^{t'} \mathrm{dVar}_s(K) \right]. \tag{2.14}$$

Setting $t' = T$ and $\varepsilon = 1$ in (2.14) and combining it with (2.13), we obtain

$$\begin{aligned} \frac{1}{2} \mathbb{E}_t \left[\int_t^T |Z_s|^2 \mathrm{d}s \right] \leqslant & \mathbb{E}_t \left[|\xi|^2 + \frac{C_1}{\gamma} \phi_{\mathcal{C}}^+(\xi) \right] + C \mathbb{E}_t \left[\int_t^T (1 + |f(s,0,0)|) \mathrm{d}s \right] \\ & + \frac{C}{\gamma} \mathbb{E} \left[\int_t^T \nabla \phi_{\mathcal{C}}^+(Y_s) \cdot f(s, Y_s, Z_s) \mathrm{d}s \right]. \end{aligned} \tag{2.15}$$

Next, we observe that

$$\frac{C_1}{\gamma} \mathbb{E} \left[\int_t^T \nabla \phi_{\mathcal{C}}^+(Y_s) \cdot f(s, Y_s, Z_s) \mathrm{d}s \right] \leqslant C \mathbb{E}_t \left[\int_t^{t'} (1 + |f(s,0,0)|) \mathrm{d}s \right] + \frac{1}{4} \mathbb{E}_t \left[\int_t^T |Z_s|^2 \mathrm{d}s \right].$$

Inserting the above estimate into (2.15), we obtain

$$\frac{1}{4} \mathbb{E}_t \left[\int_t^T |Z_s|^2 \mathrm{d}s \right] \leqslant \mathbb{E}_t \left[|\xi|^2 + \frac{C}{\gamma} \phi_{\mathcal{C}}^+(\xi) \right] + C \mathbb{E}_t \left[\int_t^T (1 + |f(s,0,0)|) \mathrm{d}s \right],$$

which proves that $Z \in \mathscr{B}^2$.

2. We now turn to the estimation of the exponential moments of $\mathrm{Var}_T(K)$, under the smallness Assumption 2.1.

2.a Combining Assumption 2.1(i) with (2.13), we obtain

$$\frac{\theta}{R_0} \left\| \sup_{t \in [0,T]} \mathbb{E}_t \left[\int_t^T \mathrm{dVar}_s(K) \right] \right\|_{\mathscr{L}^\infty} < 1. \tag{2.16}$$

Then, we apply the energy inequalities for non-decreasing processes with bounded potential (see, e.g., (105.1)–(105.2) in [13]) to obtain (2.10) in this case.

2.b Let Assumption 2.1-(ii) hold. Using (2.11)–(2.12) and recalling that $|Y|$ is bounded, we obtain, for all $0 \leqslant t < t' \leqslant T$ and for any $\varepsilon > 0$,

$$\gamma \mathbb{E}_t \left[\int_t^{t'} \mathrm{dVar}_s(K) \right] \leqslant \sup_{y \in \bar{\mathcal{D}}} \phi_{\mathcal{C}}^+(y) + C'_\varepsilon (t'-t)(1 + |f(.,0,0)|_{\mathscr{L}^\infty}) + \frac{\varepsilon}{2} \mathbb{E}_t \left[\int_t^{t'} |Z_s|^2 \mathrm{d}s \right].$$

Using the above inequality and (2.14) (with the same $\varepsilon > 0$), we obtain:

$$(\gamma - C\varepsilon) \mathbb{E}_t \left[\int_t^{t'} \mathrm{dVar}_s(K) \right] \leqslant \varepsilon \sup_{y \in \bar{\mathcal{D}}} |y|^2 + \sup_{y \in \bar{\mathcal{D}}} \phi_{\mathcal{C}}^+(y) + C''_\varepsilon (t'-t)(1 + |f(.,0,0)|_{\mathscr{L}^\infty}).$$

In particular, by taking ε small enough, we conclude that, for any $\varepsilon' > 0$, there exists $C_{\varepsilon'} > 0$ such that

$$\mathbb{E}_t \left[\int_t^{t'} \mathrm{dVar}_s(K) \right] \leqslant \frac{\sup_{y \in \bar{\mathcal{D}}} \phi_{\mathcal{C}}^+(y)}{\gamma} (1 + \varepsilon') + C_{\varepsilon'}(t'-t). \tag{2.17}$$

Next, using (2.17) and Assumption 2.1(ii), we conclude that there exist $0 < \varepsilon'' < 1$, $p > 1$, and $N \geqslant 1$, depending only on $K_{f,y}$, $K_{f,z}$, γ, $\sup_{y \in \mathcal{D}} |y|$, $\sup_{y \in \bar{\mathcal{D}}} \phi_{\mathcal{C}}(y)^+$, $\|f(.,0,0)\|_{\mathscr{L}^\infty}$ and R_0, such that, a.s.:

$$\mathbb{E}_t \left[\int_t^{T(k+1)/N} \mathrm{dVar}_s(K) \right] \leqslant \frac{R_0}{\theta p}(1 - \varepsilon''), \quad \forall 0 \leqslant k < N, \ \forall t \in [Tk/N, T(k+1)/N]. \tag{2.18}$$

Then, we apply the energy inequalities for non-decreasing processes with bounded potential (see, e.g., (105.1)–(105.2) in [13]), to obtain, for all $0 \leqslant k < N$,

$$\mathbb{E}_{Tk/N} \left[e^{\frac{\theta p}{R_0} \int_{Tk/N}^{T(k+1)/N} \mathrm{dVar}_s(K)} \right] \leqslant \tilde{C}, \tag{2.19}$$

with $\tilde{C}$ that depends only on $K_{f,y}$, $K_{f,z}$, γ, $\sup_{y \in \mathcal{D}} |y|$, $\sup_{y \in \bar{\mathcal{D}}} \phi_{\mathcal{C}}(y)^+$, $\|f(.,0,0)\|_{\mathscr{L}^\infty}$ and R_0. We now observe that

$$\begin{aligned} \mathbb{E}\Big[e^{\frac{\theta p}{R_0} Var_T(K)} \Big] &= \mathbb{E}\Big[e^{\frac{\theta p}{R_0} \mathrm{Var}_{T(N-1)/N}(K)} \mathbb{E}_{T(N-1)/N} \left[e^{\frac{\theta p}{R_0} \int_{T(N-1)/N}^{T} \mathrm{dVar}_s(K)} \right] \Big] \\ &\leqslant \tilde{C} \mathbb{E}\Big[e^{\frac{\theta p}{R_0} \mathrm{Var}_{T(N-1)/N}(K)} \Big], \end{aligned}$$

where we used (2.19) with $k = N-1$ to obtain the last inequality. Iterating the above procedure, we conclude the proof of (2.10) in this case.

2.c Let Assumption 2.1(iii) hold. Using (2.3), the linear growth of f and Young's inequality, we deduce, for all $\varepsilon > 0$,

$$\mathrm{Var}_T(K) \leqslant C_\varepsilon + \frac{1+\varepsilon}{2R_0} \int_0^T \mathbf{1}_{\{\phi(Y_t)=0\}} |Z_t|^2 \mathrm{d}t. \tag{2.20}$$

Moreover, we apply Itô-Tanaka formula to $(|Y_s|^2-\lambda^2)^+$ to obtain, for all $t\leqslant t'$,

$$\begin{aligned}\mathbb{E}_t\left[\int_t^{t'}\mathbf{1}_{\{\phi(Y_s)=0\}}|Z_s|^2\mathrm{d}s\right]&\leqslant\mathbb{E}_t\left[\int_t^{t'}\mathbf{1}_{\{\phi_{\mathcal{C}}(Y_s)>0\}}|Z_s|^2\mathrm{d}s\right]\\ \leqslant&\mathbb{E}_t\left[(|Y_{t'}|^2-\lambda^2)^+ + 2\int_t^{t'}\mathbf{1}_{\{\phi_{\mathcal{C}}(Y_s)>0\}}|Y_s|\nabla\phi_{\mathcal{C}}(Y_s)\cdot f(s,Y_s,Z_s)\mathrm{d}s\right],\end{aligned}\tag{2.21}$$

where we also recall that $\mathbf{1}_{\{\phi_{\mathcal{C}}(Y_s)>0\}}|Y_s|\nabla\phi_{\mathcal{C}}(Y_s)=\mathbf{1}_{\{\phi_{\mathcal{C}}(Y_s)>0\}}Y_s$ since $\mathcal{C}$ is a Euclidean ball centered at zero. Then, by taking $t'=T$ in (2.21) and using Assumption 2.1(iii), we obtain, for $\varepsilon>0$ and $p>1$ small enough,

$$\frac{\theta p(1+\varepsilon)}{2R_0^2}\left\|\sup_{t\in[0,T]}\mathbb{E}_t\left[\int_t^T\mathbf{1}_{\{\phi(Y_s)=0\}}|Z_s|^2\mathrm{d}s\right]\right\|_{\mathscr{L}^\infty}<1.$$

It remains to apply the John-Nirenberg inequality for BMO Martingales (see Theorem 2.2 in [31]) and recall (2.20), to conclude that

$$\mathbb{E}\left[e^{\frac{\theta p}{R_0}\mathrm{Var}_T(K)}\right]\leqslant C_\varepsilon\mathbb{E}\left[e^{\frac{\theta p(1+\varepsilon)}{2R_0^2}\int_0^T\mathbf{1}_{\{\phi(Y_s)=0\}}|Z_s|^2\mathrm{d}s}\right]<+\infty,$$

which yields (2.10).

2.d The proof of (2.10) in the case of Assumption 2.1(iv) follows from (2.20) and (2.21), by partitioning $[0,T]$ into small time intervals as in step 2.b. For brevity, we skip these routine calculations. □

2.5. **Stability and uniqueness in $\mathscr{U}(\theta)$.** Using a priori estimates established in the previous subsection, we prove the following stability property of solutions to (1.2).

Proposition 2.2. *Let us consider $(Y,Z,K)\in\mathscr{S}^2\times\mathscr{H}^2\times\mathscr{K}^1$ (resp. $(Y',Z',K')\in\mathscr{S}^2\times\mathscr{H}^2\times\mathscr{K}^1$) which solve the RBSDE (1.2) with a domain $\mathcal{D}$ (resp. $\mathcal{D}'$), with a terminal condition ξ (resp. ξ'), and with a generator f (resp. f'). Assume, moreover, that there exists $p>1$ such that*

$$\kappa:=\mathbb{E}\left[e^{\frac{p}{R_0}(\mathrm{Var}_T(K)+\mathrm{Var}_T(K'))}\right]<+\infty,\tag{2.22}$$

with R_0 satisfying (2.4) for $\mathcal{D}$ and $\mathcal{D}'$. Let us denote by $\bar{\mathfrak{P}}$ (resp. $\bar{\mathfrak{P}}'$) a measurable selection of the projection operator onto $\mathcal{D}$ (resp. $\mathcal{D}'$). Then, the following stability result holds: there exists a constant $C>0$, which depends only on $K_{f,y}$, $K_{f',y}$,

$K_{f,z}$, $K_{f',z}$ (recall Assumption 1.2), $\sup_{y\in\mathcal{D}\cup\mathcal{D}'}|y|$, R_0, and on κ, and is such that

$$\begin{aligned}
&\|Y-Y'\|_{\mathscr{S}^2}+\|Z-Z'\|_{\mathscr{H}^2}+\|K-K'\|_{\mathscr{S}^2}\\
\leqslant& C\mathbb{E}[|\xi-\xi'|^{2p/(p-1)}]^{(p-1)/(2p)}+C\mathbb{E}\left[\left(\int_0^T|f(s,Y_s,Z_s)-f'(s,Y_s,Z_s)|\mathrm{d}s\right)^{2p/(p-1)}\right]^{(p-1)/(2p)}\\
&+C\mathbb{E}\left[\sup_{s\in[0,T]}|Y_s-\bar{\mathfrak{P}}'(Y_s)|^{p/(p-1)}\right]^{(p-1)/(2p)}+C\mathbb{E}\left[\sup_{s\in[0,T]}|Y'_s-\bar{\mathfrak{P}}(Y'_s)|^{p/(p-1)}\right]^{(p-1)/(2p)}.
\end{aligned}$$

Proof. We apply Itô's formula to the process

$$e^{\beta t+\frac{1}{R_0}(\mathrm{Var}_t(K)+\mathrm{Var}_t(K'))}|Y_t-Y'_t|^2,$$

with the constant β to be determined later. By denoting

$$\delta f_t:=f(t,Y_t,Z_t)-f'(t,Y'_t,Z'_t),\quad \delta\xi:=\xi-\xi',$$

$$\Gamma_t:=e^{\beta t+\frac{1}{R_0}(\mathrm{Var}_t(K)+\mathrm{Var}_t(K'))},\quad \delta Y:=Y-Y',\quad \delta Z=Z-Z',$$

we obtain

$$\begin{aligned}
&\Gamma_t|\delta Y_t|^2+\int_t^T\Gamma_s|\delta Z_s|^2\mathrm{d}s\\
=&\Gamma_T|\delta\xi|^2+2\int_t^T\Gamma_s\delta Y_s\cdot\delta f_s\mathrm{d}s-2\int_t^T\Gamma_s\delta Y_s\cdot\mathrm{d}K_s+2\int_t^T\Gamma_s\delta Y_s\cdot\mathrm{d}K'_s\\
&-\beta\int_t^T\Gamma_s|\delta Y_s|^2\mathrm{d}s-\frac{1}{R_0}\int_t^T\Gamma_s|\delta Y_s|^2\mathrm{dVar}_s(K)-\frac{1}{R_0}\int_t^T\Gamma_s|\delta Y_s|^2\mathrm{dVar}_s(K')\\
&-2\int_t^T\Gamma_s\delta Y_s\cdot\delta Z_s\mathrm{d}W_s. \qquad (2.23)
\end{aligned}$$

Using Burkholder-Davis-Gundy inequality, as well as Hölder inequality (with $q=p/(p-1)>1$ being the conjugate exponent) and the fact that $|\delta Y|$ is bounded, we obtain:

$$\begin{aligned}
\mathbb{E}\left[\sup_{t\in[0,T]}\left|\int_0^t\Gamma_s\delta Y_s\cdot\delta Z_s\mathrm{d}W_s\right|\right]&\leqslant C\mathbb{E}\left[\left(\int_0^T|\Gamma_s\delta Z_s|^2\,\mathrm{d}s\right)^{\frac12}\right]\\
&\leqslant C\mathbb{E}[(\Gamma_T)^p]^{\frac1p}\,\mathbb{E}\left[\left(\int_0^T|\delta Z_s|^2\mathrm{d}s\right)^{\lceil q/2\rceil}\right]^{\frac1q}<\infty,
\end{aligned}$$

where the last inequality is due to (2.22) and to the Energy Inequality (since $Z,Z'\in\mathscr{B}^2$). Hence, we conclude that the local martingale term in the right hand side of (2.23) is a true martingale.

Next, we estimate the second term in the right hand side of (2.23) using the Lipschitz property of f':

$$\delta Y_s \cdot \delta f_s \leqslant |\delta Y_s||f(s,Y_s,Z_s) - f'(s,Y_s,Z_s)| + \beta|\delta Y_s|^2 + \frac{1}{4}|\delta Z_s|^2,$$

provided $\beta > 0$ is large enough. In addition, the condition (1.2)(ii) and the exterior sphere property (recall (2.4)) yield

$$\begin{aligned}
&-2\int_t^T \Gamma_s \delta Y_s \cdot \mathrm{d}K_s - \frac{1}{R_0}\int_t^T \Gamma_s|\delta Y_s|^2 \mathrm{dVar}_s(K)\\
=&-2\int_t^T \Gamma_s(\bar{\mathfrak{P}}(Y_s') - Y_s')\cdot \mathrm{d}K_s - 2\int_t^T \Gamma_s\left(Y_s - \bar{\mathfrak{P}}(Y_s')\right)\cdot \mathrm{d}K_s - \frac{1}{R_0}\int_t^T \Gamma_s|\delta Y_s|^2 \mathrm{dVar}_s(K)\\
=&-2\int_t^T \Gamma_s\left(Y_s - \bar{\mathfrak{P}}(Y_s')\right)\cdot \mathrm{d}K_s - \frac{1}{R_0}\int_t^T \Gamma_s|Y_s - \bar{\mathfrak{P}}(Y_s')|^2 \mathrm{dVar}_s(K)\\
&+\frac{1}{R_0}\int_t^T \Gamma_s(|\bar{\mathfrak{P}}(Y_s') - Y_s'|^2 - |\delta Y_s|^2)\mathrm{dVar}_s(K) - 2\int_t^T \Gamma_s(\bar{\mathfrak{P}}(Y_s') - Y_s')\cdot \mathrm{d}K_s\\
\leqslant& C\int_t^T \Gamma_s|\bar{\mathfrak{P}}(Y_s') - Y_s'|\mathrm{dVar}_s(K) \leqslant C\Gamma_T \sup_{s\in[0,T]}|\bar{\mathfrak{P}}(Y_s') - Y_s'|,
\end{aligned}$$

where we used $\int_t^T \exp\left(\frac{\mathrm{Var}_s(K)}{R_0}\right)\mathrm{dVar}_s(K) \leqslant R_0 \exp\left(\frac{\mathrm{Var}_T(K)}{R_0}\right)$ to establish the last inequality. Using the same arguments, we obtain

$$2\int_t^T \Gamma_s \delta Y_s \cdot \mathrm{d}K_s' - \frac{1}{R_0}\int_t^T \Gamma_s|\delta Y_s|^2 \mathrm{dVar}_s(K')) \leqslant C\Gamma_T \sup_{s\in[0,T]}|\bar{\mathfrak{P}}'(Y_s) - Y_s|.$$

Using the above estimates, we take expectations on both sides of 2.23, with $t = 0$, and apply Hölder inequality to obtain

$$\begin{aligned}
\|\Gamma^{1/2}\delta Z\|_{\mathscr{H}^2} \leqslant& \mathbb{E}\left[\Gamma_T|\delta\xi|^2 + 2\int_0^T \Gamma_s|\delta Y_s||f(s,Y_s,Z_s) - f'(s,Y_s,Z_s)|\mathrm{d}s\right]^{1/2}\\
&+\mathbb{E}\left[\Gamma_T\left(\sup_{s\in[0,T]}|\bar{\mathfrak{P}}'(Y_s) - Y_s| + |\bar{\mathfrak{P}}(Y_s') - Y_s'|\right)\right]^{1/2}\\
\leqslant& C\mathbb{E}[|\delta\xi|^{2q}]^{1/(2q)}\\
&+2\mathbb{E}\left[\sup_{s\in[0,T]}(\Gamma_s^{1/2}|\delta Y_s|)\Gamma_T^{1/2}\int_0^T |f(s,Y_s,Z_s) - f'(s,Y_s,Z_s)|\mathrm{d}s\right]^{1/2}\\
&+C\mathbb{E}\left[\sup_{s\in[0,T]}\left(|\bar{\mathfrak{P}}'(Y_s) - Y_s| + |\bar{\mathfrak{P}}(Y_s') - Y_s'|\right)^q\right]^{1/(2q)}. \qquad (2.24)
\end{aligned}$$

Using (2.23) and (2.24), we apply Burkholder-Davis-Gundy, Hölder and Young's inequalities to obtain

$$\begin{aligned}\|\Gamma^{1/2}\delta Y\|_{\mathscr{S}^2} &\leqslant C\mathbb{E}[|\delta\xi|^q]^{1/(2q)} \\ &\quad + C\mathbb{E}\left[\sup_{s\in[0,T]}(\Gamma_s^{1/2}|\delta Y_s|)\Gamma_T^{1/2}\int_0^T|f(s,Y_s,Z_s)-f'(s,Y_s,Z_s)|\mathrm{d}s\right]^{1/2} \\ &\quad + C\mathbb{E}\left[\sup_{s\in[0,T]}\left(|\bar{\mathfrak{P}}'(Y_s)-Y_s|+|\bar{\mathfrak{P}}(Y'_s)-Y'_s|\right)^q\right]^{1/(2q)} \\ &\leqslant C\mathbb{E}[|\delta\xi|^q]^{1/(2q)}+\frac{1}{2}\|\Gamma^{1/2}\delta Y\|_{\mathscr{S}^2}+C\mathbb{E}\left[\left(\int_0^T|f(s,Y_s,Z_s)-f'(s,Y_s,Z_s)|\mathrm{d}s\right)^{2q}\right]^{1/(2q)} \\ &\quad + C\mathbb{E}\left[\sup_{s\in[0,T]}\left(|\bar{\mathfrak{P}}'(Y_s)-Y_s|+|\bar{\mathfrak{P}}(Y'_s)-Y'_s|\right)^q\right]^{1/(2q)}. \end{aligned} \tag{2.25}$$

Then, combining (2.24), Young's inequality, and (2.25), yields

$$\begin{aligned}&\|Y-Y'\|_{\mathscr{S}^2}+\|Z-Z'\|_{\mathscr{H}^2}\leqslant\|\Gamma^{1/2}\delta Y\|_{\mathscr{S}^2}+\|\Gamma^{1/2}\delta Z\|_{\mathscr{H}^2} \\ &\leqslant C\mathbb{E}[|\xi-\xi'|^{2q}]^{1/(2q)}+C\mathbb{E}\left[\left(\int_0^T|f(s,Y_s,Z_s)-f'(s,Y_s,Z_s)|\mathrm{d}s\right)^{2q}\right]^{1/(2q)} \\ &\quad + C\mathbb{E}\left[\sup_{s\in[0,T]}\left(|\bar{\mathfrak{P}}'(Y_s)-Y_s|+|\bar{\mathfrak{P}}(Y'_s)-Y'_s|\right)^q\right]^{1/(2q)}. \end{aligned} \tag{2.26}$$

Finally, we recall that

$$K_t-K'_t=\delta Y_t-\delta Y_0+\int_0^t f(s,Y_s,Z_s)-f'(s,Y'_s,Z'_s)\mathrm{d}s-\int_0^t\delta Z_s\mathrm{d}W_s.$$

Then, the Burkholder-Davis-Gundy inequality inequality, the Lipschitz property of f', as well as (2.26), yield

$$\begin{aligned}\|K-K'\|_{\mathscr{S}^2}\leqslant&C\mathbb{E}[|\xi-\xi'|^{2q}]^{1/(2q)}+C\mathbb{E}\left[\left(\int_0^T|f(s,Y_s,Z_s)-f'(s,Y_s,Z_s)|\mathrm{d}s\right)^{2q}\right]^{1/(2q)} \\ &+C\mathbb{E}\left[\sup_{s\in[0,T]}\left(|\bar{\mathfrak{P}}'(Y_s)-Y_s|+|\bar{\mathfrak{P}}(Y'_s)-Y'_s|\right)^q\right]^{1/(2q)},\end{aligned}$$

which completes the proof of the proposition. □

In a general non-Markovian framework, we obtain the following uniqueness result as a direct consequence of Proposition 2.2.

Corollary 2.2. *The reflected BSDE (1.2) has at most one solution in the class $\mathscr{U}(2)$.*

Proof. Indeed, it suffices to check that, for any two solutions in the class $\mathscr{U}(2)$, (2.22) holds. This follows directly from the Cauchy-Schwarz inequality. □

This uniqueness result is improved in the Markovian setting: see Theorem 3.1 and Remark 3.6.

3. Well-posedness in a Markovian framework

In this section, we establish the existence and uniqueness of the solution to (3.3) under the assumption that the terminal condition and the generator of the reflected BSDE are functions of a Markov diffusion process X in $\mathbb{R}^{d'}$:

$$X_t = x + \int_0^t b(s, X_s)\mathrm{d}s + \int_0^t \sigma(s, X_s)\mathrm{d}W_s, \quad x \in \mathbb{R}^{d'}. \tag{3.1}$$

Namely, we make the following assumptions.

Assumption 3.1. *We assume that (b, σ) are bounded measurable functions, uniformly Lipschitz with respect to x, and such that $\sigma^\top\sigma$ is uniformly positive definite (i.e. uniformly elliptic).*

Note that Assumption 3.1, in particular, implies that the matrix σ is invertible.

Assumption 3.2. *We assume that*

$$\xi := g(X_T) \quad \text{and} \quad f(t, y, z) := F(t, X_t, y, z),$$

where g is α-Hölder and $\bar{\mathcal{D}}$-valued, F is measurable in all variables, globally Lipschitz in (y, z), and such that $|F(\cdot, \cdot, 0, 0)|$ is bounded.

Recall that Assumptions 1.1 and 1.2 hold throughout the paper, even if they are not cited explicitly.

3.1. Penalized equation.

We begin by noticing that $\psi^2 \in C^1(\mathbb{R}^d)$ and denoting

$$\Psi(y) := \frac{1}{2}\nabla\psi(y)^2 = \psi(y)\nabla\psi(y), \quad y \in \mathbb{R}^d,$$

where we extend (naturally) $\nabla\psi$ to $\mathcal{D}$ by zero. We also extend $\nabla^2\psi^2$ to $\mathcal{D}$ by zero.

It is useful to note that there exist constants c, C, such that

$$0 < c\psi \leqslant |\Psi| \leqslant C\psi. \tag{3.2}$$

Next, we consider the following penalized equation:

$$Y_t^n = \xi + \int_t^T f(s, Y_s^n, Z_s^n)\mathrm{d}s - \int_t^T n\Psi(Y_s^n)(1 + |Z_s^n|^2)\mathrm{d}s - \int_t^T Z_s^n \mathrm{d}W_s. \tag{3.3}$$

Let us remark that, in contrast with the convex framework tackled in [22], it is natural (and necessary) to include $|z|^2$ in the penalization term, as can be seen, e.g., from (2.3). For convenience, we introduce

$$\Phi_t^n := \int_0^t n\Psi(Y_s^n)\mathrm{d}s, \quad \Theta_t^n := \int_0^t n\Psi(Y_s^n)|Z_s^n|^2\mathrm{d}s, \qquad K_t^n := \Phi_t^n + \Theta_t^n. \tag{3.4}$$

3.2. **Existence of a solution to the penalized equation.** We start by considering the following family of approximating BSDEs, indexed by a pair of positive integers $M = (M_1, M_2)$:

$$Y_t^{n,M} = g(X_T) + \int_t^T F^{n,M}(s, X_s, Y_s^{n,M}, Z_s^{n,M})\mathrm{d}s - \int_t^T Z_s^{n,M}\mathrm{d}W_s, \tag{3.5}$$

with

$$F^{n,M}(t,x,y,z) := f(t,x,y,z) - n\rho_{M_1}(\psi(y))\nabla\psi(y)(1+\rho_{M_2}(|z|^2)), \quad \rho_k(x) := x \wedge k.$$

The above BSDE has a globally Lipschitz generator and, therefore, is known to have a unique Markovian solution $(Y^{n,M}, Z^{n,M}) \in \mathscr{S}^2 \times \mathscr{H}^2$ (see, e.g., Theorem 4.1 in [16]). The following Proposition uses the weak star-shape property of $\mathcal{D}$, stated in Assumption 1.1, to establish a uniform estimate on $(Y^{n,M}, Z^{n,M})$.

Lemma 3.1. *There exists a constant $C > 0$, such that, for any $n \geqslant 1$, any $M = (M_1, M_2)$, and any $t \in [0,T]$, the following holds a.s.:*

$$|Y_t^{n,M}|^2 + \mathbb{E}_t\bigg[\int_t^T |Y_s^{n,M}|^2 + |Z_s^{n,M}|^2\mathrm{d}s\bigg] \leqslant C\,\mathbb{E}_t\bigg[|\xi|^2 + \int_t^T (1 + |f(s,0,0)|^2)\mathrm{d}s\bigg], \tag{3.6}$$

$$\mathbb{E}_t\bigg[\int_t^T n\rho_{M_1}(\psi(Y_s^{n,M}))(1 + \rho_{M_2}(|Z_s^{n,M}|^2))\mathrm{d}s\bigg] \leqslant C\,\mathbb{E}_t\bigg[|\xi|^2 + \int_t^T (1 + |f(s,0,0)|^2)\mathrm{d}s\bigg]. \tag{3.7}$$

Proof. Without loss of generality, we assume that $\phi_{\mathcal{C}}$ attains its minimum at zero. Then, we consider arbitrary $t \in [0,T]$ and constants $\alpha > 0$, $\beta > 0$, to be determined later, and define

$$[t,T] \times \mathbb{R}^d \ni (s,y) \mapsto h(s,y) := e^{\beta(s-t)}\left(\alpha|y|^2 + (\phi_{\mathcal{C}}(y) - \phi_{\mathcal{C}}(0))^2\right) \in \mathbb{R}.$$

We observe that $(\phi_{\mathcal{C}} - \phi_{\mathcal{C}}(0))^2$ is convex and that $h(s,y) \leqslant e^{\beta(T-s)}c_0|y|^2$, for some positive constant c_0. Then, we apply Itô's formula to the process $h(s, Y_s^{n,M})$

(recalling (3.3)), to obtain

$$
\begin{aligned}
\alpha|Y_t^{n,M}|^2 &\leqslant h(t,Y_t^{n,M}) \leqslant h(T,\xi) \\
&\quad + 2\int_t^T e^{\beta(s-t)}(\alpha Y_s^{n,M} + (\phi_{\mathcal{C}}(Y_s^{n,M}) - \phi_{\mathcal{C}}(0))\nabla\phi_{\mathcal{C}}(Y_s^{n,M}))\cdot f(s,Y_s^{n,M},Z_s^{n,M})\mathrm{d}s \\
&\quad - \int_t^T n\rho_{M_1}(\psi(Y_s^{n,M}))\nabla_y h(s,Y_s^{n,M})\cdot\ \nabla\psi(Y_s^{n,M})(1+\rho_{M_2}(|Z_s^{n,M}|^2))\mathrm{d}s \\
&\quad - 2\int_t^T \nabla_y h(s,Y_s^{n,M})\cdot Z_s^{n,M}\mathrm{d}W_s - \alpha\int_t^T e^{\beta(s-t)}|Z_s^{n,M}|^2\mathrm{d}s - \beta\int_t^T e^{\beta(s-t)}|Y_s^{n,M}|^2\mathrm{d}s.
\end{aligned}
\tag{3.8}
$$

As $Y^{n,M}\in\mathscr{S}^2$ and $Z^{n,M}\in\mathscr{H}^2$, the local martingale in the above representation is in $\mathcal{M}^1$ and, hence, is a true martingale.

Next, we notice that the fourth property in Lemma 2.4 implies the existence of a constant $c_1>0$, such that

$$
\nabla\phi_{\mathcal{C}}(Y_s^{n,M})\cdot\nabla\psi(Y_s^{n,M}) \geqslant c_1\mathbf{1}_{\{Y_s^{n,m}\notin\mathcal{D}\}}.
$$

Then, there exist constants $c_2,c_3>0$ such that

$$
\begin{aligned}
&- \int_t^T n\rho_{M_1}(\psi(Y_s^{n,M}))\nabla_y h(s,Y_s^{n,M})\cdot\ \nabla\psi(Y_s^{n,M})(1+\rho_{M_2}(|Z_s^{n,M}|^2))\mathrm{d}s \\
&\leqslant -2\int_t^T ne^{\beta(s-t)}\rho_{M_1}(\psi(Y_s^{n,M}))(c_1(\phi_{\mathcal{C}}(Y_s^{n,M})-\phi_{\mathcal{C}}(0)) - \alpha c_2|Y_s^{n,M}|)(1+\rho_{M_2}(|Z_s^{n,M}|^2))\mathrm{d}s \\
&\leqslant -2\int_t^T ne^{\beta(s-t)}\rho_{M_1}(\psi(Y_s^{n,M}))\left[c_1(\phi_{\mathcal{C}}(Y_s^{n,M})-\phi_{\mathcal{C}}(0))\right. \\
&\qquad\qquad \left. - \alpha c_2(\phi_{\mathcal{C}}(Y_s^{n,M}) + |\mathfrak{P}_{\bar{\mathcal{C}}}(Y_s^{n,M})|)\right]\left(1+\rho_{M_2}(|Z_s^{n,M}|^2)\right)\mathrm{d}s \\
&\leqslant -c_3\int_t^T ne^{\beta(s-t)}\rho_{M_1}(\psi(Y_s^{n,M}))(1+\rho_{M_2}(|Z_s^{n,M}|^2))\mathrm{d}s,
\end{aligned}
\tag{3.9}
$$

provided α is small enough. In the rest of the proof, we assume that α is chosen so that the above inequality holds.

Next, we remark that

$$
\begin{aligned}
&\left|2\int_t^T e^{\beta(s-t)}\left(\alpha Y_s^{n,M} + (\phi_{\mathcal{C}}(Y_s^{n,M})-\phi_{\mathcal{C}}(0))\nabla\phi_{\mathcal{C}}(Y_s^{n,M})\right)\cdot f(s,Y_s^{n,M},Z_s^{n,M})\mathrm{d}s\right| \\
&\leqslant C_1\int_t^T e^{\beta(s-t)}\left((\alpha+1)|Y_s^{n,M}| - \phi_{\mathcal{C}}(0)\right)\left(|f(s,0,0)| + C_2|Y_s^{n,M}| + C_2|Z_s^{n,M}|\right)\mathrm{d}s \\
&\leqslant \int_t^T e^{\beta(s-t)}\left(C_3|Y_s^{n,M}|^2 + C_3 + |f(s,0,0)|^2 + \frac{\alpha}{2}|Z_s^{n,M}|^2\right)\mathrm{d}s.
\end{aligned}
$$

Combining the above estimates and (3.8), we conclude that, for a large enough $\beta > 0$, there exists a constant $C_4 > 0$, such that

$$\alpha|Y_t^{n,M}|^2 + \mathbb{E}_t\Big[c_3\int_t^T n\rho_{M_1}(\psi(Y_s^{n,M}))\left(1+\rho_{M_2}(|Z_s^{n,M}|^2)\right)\mathrm{d}s + \frac{\alpha}{2}\int_t^T |Y_s^{n,M}|^2 + |Z_s^{n,M}|^2\mathrm{d}s\Big]$$
$$\leqslant e^{\beta(T-t)}\mathbb{E}_t\Big[c_0|\xi|^2 + \int_t^T (C_4 + |f(s,0,0)|^2)\mathrm{d}s\Big],$$

which yields the statement of the lemma. □

Proposition 3.1. *Under Assumptions 3.1 and 3.2, for any $n \geqslant 1$, the BSDE (3.3) has a Markovian solution (Y^n, Z^n). In particular, there exists a measurable function u^n such that $Y_t^n = u^n(t, X_t)$. Moreover, the estimates (3.6)–(3.7) hold with $(Y^{n,M}, Z^{n,M})$ and ρ_{M_i} replaced, respectively, by any solution (Y^n, Z^n) of (3.3) and by the identity function.*

Proof. The main statement of the proposition follows from Theorem 2.8 in [48] (without the localization used in [48]). To be able to apply the latter theorem, we first consider the following auxiliary BSDE, which can be viewed as a middle ground between (3.3) and (3.5):

$$\tilde{Y}_t^{n,M_1} = g(X_T) + \int_t^T \tilde{F}^{n,M_1}(s, X_s, \tilde{Y}_s^{n,M_1}, \tilde{Z}_s^{n,M_1})\mathrm{d}s - \int_t^T \tilde{Z}_s^{n,M_1}\mathrm{d}W_s, \tag{3.10}$$

with

$$\tilde{F}^{n,M_1}(t,x,y,z) := f(t,x,y,z) - n\rho_{M_1}(\psi(y))\nabla\psi(y)(1+|z|^2)$$

and recalling that $\rho_{M_1}(x) = x \wedge M_1$. We claim that the unique solution $(Y^{n,M}, Z^{n,M})$ of (3.5) converges (along a subsequence) to a Markovian solution $(\tilde{Y}^{n,M_1}, \tilde{Z}^{n,M_1})$ of (3.10), as $M_2 \to \infty$. Indeed, this claim follows directly from Theorem 2.8 in [48]. To verify the assumptions of the latter theorem, we first notice that, due to (3.6), there exists a constant $c > 0$ such that $|Y_t^{n,M}| \leqslant c$, for all $t \in [0,T]$ and n, M. Moreover, for large enough $C > 0$ (independent of n and M), $h(y) := C\left(\alpha|y|^2 + (\phi_{\mathcal{C}}(y) - \phi_{\mathcal{C}}(0))^2\right)$ is a global c-Lyapunov function for $(F^{n,M})_M$, in the sense of Definition 2.3 in [48], where α is the constant chosen in the proof of Lemma 3.1. Indeed, there exists a large enough $C > 0$, such that, for all $|y| \leqslant c$, we have:

$$\begin{aligned}
&\frac{1}{2}C\,\mathrm{Tr}[(z\sigma)^\top(\nabla^2 h(y))z\sigma] - C\nabla h(y)\cdot F^{n,M}\\
&\geqslant C\alpha\,\mathrm{Tr}[(z\sigma)^\top z\sigma] - 2C\left[\alpha y + (\phi_{\mathcal{C}}(y) - \phi_{\mathcal{C}}(0))\nabla\phi_{\mathcal{C}}(y)\right]\cdot f(t,x,y,z)\\
&\quad + 2Cn\left[\alpha y + (\phi_{\mathcal{C}}(y) - \phi_{\mathcal{C}}(0))\nabla\phi_{\mathcal{C}}(y)\right]\cdot\nabla\psi(y)\rho_{M_1}(\psi(y))(1+\rho_{M_2}(|z|^2))\\
&\geqslant |z|^2 - C',
\end{aligned}$$

where we used the uniform ellipticity of $\sigma^\top\sigma$, Assumption 1.2, and the fourth property in Lemma 2.4, and repeated the estimates used in (3.9). In addition,

we have $|F^{n,M}(t,x,y,z)| \leqslant C + C_n|z|^2$, with the constants (C, C_n) independent of M_2. Observing that $F^{n,M}$ converges to $\tilde{F}^{n,M_1}$ locally uniformly, as $M_2 \to \infty$, we conclude that the assumptions of Theorem 2.8 in [48] are satisfied and that (3.10) has a Markovian solution $(\tilde{Y}^{n,M_1}, \tilde{Z}^{n,M_1})$ which is a limit point of $\{(Y^{n,M}, Z^{n,M})\}_{M_2}$.

Next, we recall that, due to (3.6), $|Y^{n,M}|$ is bounded uniformly over M. Hence, $|\tilde{Y}^{n,M_1}|$ can be bounded uniformly over $M_1 \geqslant 1$, and, in turn, $(\tilde{Y}^{n,M_1}, \tilde{Z}^{n,M_1})$ solve (3.3) for any large enough $M_1 > 0$.

The estimates (3.6)–(3.7) are obtained by repeating the proof of Lemma 3.1 for the equation (3.3) in place of (3.5). □

3.3. **A priori estimates.** The following result relies on the asymptotic convexity of the squared pseudo-distance function, stated in Lemma 2.5.

Lemma 3.2. *Under Assumptions 3.1 and 3.2, there exists a constant $C > 0$, such that, for any $n \geqslant 1$, any solution (Y^n, Z^n) of (3.3), and any $t \in [0,T]$, the following holds a.s.:*

$$n\psi^2(Y_t^n) + \mathbb{E}_t\left[\int_t^T n^2|\Psi(Y_s^n)|^2\left(1+|Z_s^n|^2\right)\mathrm{d}s\right] \leqslant C\,\mathbb{E}_t\left[|\xi|^2 + \int_t^T |f(s,0,0)|^2\mathrm{d}s\right],$$

and, in particular,

$$d(Y_t^n, \mathcal{D}) \leqslant Cn^{-1/2}.$$

Proof. We begin by applying Itô's formula to $|\psi(Y_t^n)|^2$, to obtain

$$\begin{aligned}\psi^2(Y_t^n) &= 2\int_t^T \Psi(Y_s^n)\cdot f(s,Y_s^n,Z_s^n)\mathrm{d}s - 2\int_t^T n|\Psi(Y_s^n)|^2\mathrm{d}s \\ &\quad - 2\int_t^T n|\Psi(Y_s^n)|^2|Z_s^n|^2\mathrm{d}s - 2\int_t^T \Psi(Y_s^n)\cdot Z_s^n\mathrm{d}W_s \\ &\quad - \frac{1}{2}\int_t^T \mathrm{Tr}[(Z_s^n)^\top\nabla^2\psi^2(Y_s^n)Z_s^n]\mathrm{d}s\end{aligned} \tag{3.11}$$

Remark 3.1. *Note that the Hessian of ψ^2 has a discontinuity at $\partial\mathcal{D}$. To justify the use of Itô's formula, we approximate ψ^2 by a sequence of C^2 functions $\{g^m\}$, such that g^m, ∇g^m and $\nabla^2 g^m$ converge, respectively, to ψ^2, $\nabla\psi^2$ and $\nabla^2\psi^2$ everywhere in $\mathbb{R}^d$, and $|\nabla g^m|$, $|\nabla^2 g^m|$ are locally bounded uniformly over m. To construct such a sequence, we first define*

$$\hat{\phi}(y) := \phi(y)(1-\vartheta(|y|-R-1)) + \vartheta(|y|-R), \quad \hat{\psi}(y) := \hat{\phi}(y) + \kappa|y|\vartheta(\hat{\phi}(y)/\epsilon), \quad y \in \mathbb{R}^d,$$

where we recall the original function ϕ, appearing in Assumption 1.1, and use the same ϑ, R, and ϵ, as the ones used in Subsection 2.3 to define ψ (see (2.8)). It is clear that $\hat{\psi}(y) = \psi(y)$, for $y \in \mathbb{R}^d \backslash \mathcal{D}$, and that $\hat{\psi}(y) = \phi(y)$, for $y \in \mathcal{D}$. Thus, $\hat{\psi}$ is a smooth extension of ψ into $\mathcal{D}$. Next, we consider an infinitely smooth nondecreasing function $\rho : \mathbb{R} \to \mathbb{R}$, such that $\rho(x) = -1$ for $x \leqslant -1$ and $\rho(x) = x$ for $x \geqslant 0$, and define

$$g^m(y) := \frac{1}{m^2}\rho^2\left(m\hat{\psi}(y)\right), \quad y \in \mathbb{R}^d.$$

It is easy to check by a direct computation that $g^m(y)$, $\nabla g^m(y)$ and $\nabla^2 g^m(y)$ converge to zero as $m \to \infty$, for any $y \in \mathcal{D}$. On the other hand, $g^m(y)$ and its first two derivatives coincide with $\psi^2(y)$ and with its respective derivatives, for all $y \in \mathbb{R}^d \backslash \mathcal{D}$ and all m. Thus, we obtain the desired sequence $\{g^m\}$. Applying Itô's formula to $g^m(Y_t^n)$ and using the dominated convergence theorem to pass to the limit as $m \to \infty$, we establish (3.11).

As $|\Psi|$ is linearly bounded (see Lemma 2.4), we conclude, as in the proof of Lemma 3.1, that the local martingale in (3.11) is a true martingale.

Next, we note that

$$2\Psi(Y_s^n) \cdot f(s, Y_s^n, Z_s^n) \leqslant n|\Psi(Y_s^n)|^2 + n^{-1}|f(s, Y_s^n, Z_s^n)|^2$$

and use Lemma 3.1, to obtain:

(3.12)
$$\mathbb{E}_t \int_t^T 2\Psi(Y_s^n) \cdot f(s, Y_s^n, Z_s^n)\mathrm{d}s \leqslant \mathbb{E}_t \int_t^T n|\Psi(Y_s^n)|^2\mathrm{d}s + Cn^{-1}\mathbb{E}_t\left[|\xi|^2 + \int_t^T (1 + |f(s,0,0)|^2)\mathrm{d}s\right].$$

In addition, Lemmas 2.4 and 2.5 yield

$$\mathrm{Tr}[(Z_s^n)^\top \nabla^2\psi^2(Y_s^n)Z_s^n] \geqslant -C\Psi(Y_s^n)|Z_s^n|^2.$$

Then,

$$-n\left(\frac{1}{2}\mathrm{Tr}[(Z_s^n)^\top \nabla^2\psi^2(Y_s^n)Z_s^n] + 2n|\Psi(Y_s^n)|^2|Z_s^n|^2\right) \leqslant \left(Cn|\Psi(Y_s^n)| - cn^2|\Psi(Y_s^n)|^2\right)|Z_s^n|^2. \tag{3.13}$$

Next, we observe that

$$\left(Cn|\Psi(Y_s^n)| - Cn^2|\Psi(Y_s^n)|^2\right)|Z_s^n|^2 \leqslant C|Z_s^n|^2. \tag{3.14}$$

Collecting (3.13)–(3.14) and using Lemma 3.1, we obtain

$$
\begin{aligned}
(3.15)\qquad & -\mathbb{E}_t\int_t^T \left(\mathrm{Tr}[(Z_s^n)^\top\nabla^2\psi^2(Y_s^n)Z_s^n]+2n|\Psi(Y_s^n)|^2|Z_s^n|^2\right)\mathrm{d}s \\
& \leqslant Cn^{-1}\mathbb{E}_t\int_t^T |Z_s^n|^2\mathrm{d}s \leqslant Cn^{-1}\mathbb{E}_t\bigg[|\xi|^2+\int_t^T(1+|f(s,0,0)|^2)\mathrm{d}s\bigg].
\end{aligned}
$$

Taking the conditional expectation in (3.11), multiplying both sides by n, and using (3.12), (3.15), we complete the proof. □

The following proposition improves the rate of convergence of Y^n to $\mathcal{D}$.

Proposition 3.2. *Under Assumptions 3.1 and 3.2, there exist $\mathfrak{N}, C>0$, such that, for any $n\geqslant\mathfrak{N}$, any solution (Y^n,Z^n) of (3.3), and any $t\in[0,T]$, the following holds a.s.:*

$$n\psi(Y_t^n)\leqslant C.$$

Proof. First, we denote by $\|\nabla^2\psi(y)\|_*$ the maximum absolute value across all negative parts of the entries of the matrix $\nabla^2\psi(y)$. Then, we fix arbitrary $\epsilon,\varepsilon>0$ satisfying

$$\epsilon\leqslant\left(\varepsilon+\sup_{y\in\partial\mathcal{D}}\frac{\|\nabla^2\psi(y)\|_*}{|\nabla\psi(y)|^2}\right)^{-1},\quad \epsilon\leqslant\left(\varepsilon+2\sup_{y\in\partial\mathcal{D},\,z\in\mathbb{R}^{d\times m},\,s\in[0,T]}\frac{\|\mathfrak{n}(y)\cdot f(s,y,z)\|_{\mathscr{L}^\infty}}{|\nabla\psi(y)|}\right)^{-1},$$

and define

$$\Psi^n(y):=(\psi(y)-1/(\epsilon n))^+\nabla\psi(y)=\frac{1}{2}\nabla\left((\psi(y)-1/(\epsilon n))^+\right)^2,$$

$$\tilde{H}(y):=\nabla^2\left((\psi(y)-1/(\epsilon n))^+\right)^2=2\nabla\psi(y)\nabla^\top\psi(y)\,\mathbf{1}_{\psi\geqslant 1/(\epsilon n)}+2(\psi(y)-1/(\epsilon n))^+\nabla^2\psi(y).$$

Next, we apply Itô's formula to $((\psi(Y_t^n)-1/(\epsilon n))^+)^2$ (the validity of Itô's formula for the function $((\psi-1/(\epsilon n))^+)^2$ is justified similarly to Remark 3.1), to obtain

$$
\begin{aligned}
((\psi(Y_t^n)-1/(\epsilon n))^+)^2 &= 2\int_t^T\Psi^n(Y_s^n)\cdot f(s,Y_s^n,Z_s^n)\mathrm{d}s-2\int_t^T n|\Psi(Y_s^n)||\Psi^n(Y_s^n)|\mathrm{d}s \\
(3.16)\qquad & -2\int_t^T n|\Psi(Y_s^n)||\Psi^n(Y_s^n)||Z_s^n|^2\mathrm{d}s-2\int_t^T\Psi^n(Y_s^n)\cdot Z_s^n\mathrm{d}W_s \\
& -\frac{1}{2}\int_t^T\mathrm{Tr}[(Z_s^n)^\top\tilde{H}(Y_s^n)Z_s^n]\mathrm{d}s.
\end{aligned}
$$

As $|\Psi^n|$ is linearly bounded (see Lemma 2.4), we conclude, as in the proof of Proposition 3.1, that the local martingale in the above representation is a true martingale.

Next, we note that

$$\begin{aligned}&2\Psi^n(Y^n_s)\cdot f(s,Y^n_s,Z^n_s)-n|\Psi(Y^n_s)||\Psi^n(Y^n_s)|\\&\leqslant|\Psi^n(Y^n_s)|\left(2|\mathfrak{n}(Y^n_s)\cdot f(s,Y^n_s,Z^n_s)|-n\psi(Y^n_s)|\nabla\psi(Y^n_s)|\right).\end{aligned}$$

Notice that, whenever $\Psi^n(Y^n_s)>0$, we have $\psi(Y^n_s)\geqslant 1/(\epsilon n)$ and, hence, $n\psi(Y^n_s)\geqslant 1/\epsilon$. Then, since $\epsilon>0$ satisfies

$$\epsilon\leqslant\left(\varepsilon+2\sup_{y\in\partial\mathcal{D},\,z\in\mathbb{R}^{d\times m},\,s\in[0,T]}\frac{\|\mathfrak{n}(y)\cdot f(s,y,z)\|_{\mathscr{L}^\infty}}{|\nabla\psi(y)|}\right)^{-1},$$

and since Y^n_s is close to $\mathcal{D}$ for large enough n (due to Lemma 3.2), we conclude that

$$\begin{aligned}n\psi(Y^n_s)|\nabla\psi(Y^n_s)|&\geqslant|\nabla\psi(Y^n_s)|\left(\varepsilon+2\sup_{y\in\partial\mathcal{D},\,z\in\mathbb{R}^{d\times m},\,s\in[0,T]}\frac{\|\mathfrak{n}(y)\cdot f(s,y,z)\|_{\mathscr{L}^\infty}}{|\nabla\psi(y)|}\right)\\&\geqslant|\nabla\psi(Y^n_s)|\varepsilon/2+2|\mathfrak{n}(Y^n_s)\cdot f(s,Y^n_s,Z^n_s)|\end{aligned}$$

and, in turn,

$$2\Psi^n(Y^n_s)\cdot f(s,Y^n_s,Z^n_s)-n|\Psi(Y^n_s)||\Psi^n(Y^n_s)|\leqslant 0. \tag{3.17}$$

Next, we recall that

$$\frac{1}{2}\mathrm{Tr}[(Z^n_s)^\top\tilde{H}(Y^n_s)Z^n_s]\geqslant(\psi(Y^n_s)-1/(\epsilon n))^+\,\mathrm{Tr}[(Z^n_s)^\top\nabla^2\psi(Y^n_s)Z^n_s]$$

and, hence,

$$-\frac{1}{2}\mathrm{Tr}[(Z^n_s)^\top\tilde{H}(Y^n_s)Z^n_s]\leqslant\|\nabla^2\psi(Y^n_s)\|_*\,(\psi(Y^n_s)-1/(\epsilon n))^+\,|Z^n_s|^2.$$

In addition,

$$-|\Psi(Y^n_s)||\Psi^n(Y^n_s)||Z^n_s|^2=-|\nabla\psi(Y^n_s)|^2\psi(Y^n_s)\,(\psi(Y^n_s)-1/(\epsilon n))^+\,|Z^n_s|^2.$$

Collecting the two equations above, we deduce

$$\begin{aligned}&-\frac{1}{2}\mathrm{Tr}[(Z^n_s)^\top\tilde{H}(Y^n_s)Z^n_s]-n|\Psi(Y^n_s)||\Psi^n(Y^n_s)||Z^n_s|^2\\&\qquad\leqslant(\psi(Y^n_s)-1/(\epsilon n))^+\,|Z^n_s|^2\left(\|\nabla^2\psi(Y^n_s)\|_*-|\nabla\psi(Y^n_s)|^2n\psi(Y^n_s)\right).\end{aligned}$$

Recall that, whenever $\psi(Y^n_s)\geqslant 1/(\epsilon n)$, we have $n\psi(Y^n_s)\geqslant 1/\epsilon$. Then, since $\epsilon>0$ satisfies

$$\epsilon\leqslant\left(\varepsilon+\sup_{y\in\partial\mathcal{D}}\frac{\|\nabla^2\psi(y)\|_*}{|\nabla\psi(y)|^2}\right)^{-1},$$

and since Y_s^n is close to $\mathcal{D}$ for large enough n, we conclude that

$$|\nabla\psi(Y_s^n)|^2 n\psi(Y_s^n) \geqslant |\nabla\psi(Y_s^n)|^2 \left(\varepsilon + \sup_{y\in\partial\mathcal{D}} \frac{\|\nabla^2\psi(y)\|_*}{|\nabla\psi(y)|^2}\right)$$
$$\geqslant |\nabla\psi(Y_s^n)|^2\varepsilon/2 + \|\nabla^2\psi(Y_s^n)\|_*$$

and, in turn,

$$-\frac{1}{2}\mathrm{Tr}[(Z_s^n)^\top \tilde{H}(Y_s^n)Z_s^n] - n|\Psi(Y_s^n)||\Psi^n(Y_s^n)||Z_s^n|^2 \leqslant 0. \tag{3.18}$$

Taking the conditional expectation in (3.16), we make use of equations (3.17) and (3.18), and of the fact that $|\Psi^n| \leqslant |\Psi|$, to obtain

$$((\psi(Y_t^n) - 1/(\epsilon n))^+)^2 + \mathbb{E}_t \int_t^T n|\Psi^n(Y_s^n)|^2 \left(1 + |Z_s^n|^2\right) \mathrm{d}s \leqslant 0$$

and complete the proof. □

Using Proposition 3.2, we can improve the statement of Proposition 3.1 and deduce that the Hölder norms of the Markovian solutions of the penalized BSDEs are bounded uniformly over n.

Corollary 3.1. *Under Assumptions 3.1 and 3.2, there exist constants $\mathfrak{N} \geqslant 1$, $\alpha' \in (0,1]$, and $C > 0$ (independent of n), such that, for any $n \geqslant \mathfrak{N}$, the BSDE (3.3) has a Markovian solution (Y^n, Z^n), with $Y_t^n = u^n(t, X_t)$, and any such solution satisfies*

$$\sup_{(t,x)\neq(t',x')} \frac{|u^n(t,x) - u^n(t',x')|}{|t-t'|^{\alpha'/2} + |x-x'|^{\alpha'}} \leqslant C. \tag{3.19}$$

Proof. The statement of the corollary follows from Theorem 2.5 in [48] (without the localization used in [48]). To verify the assumptions of the latter theorem, we consider the following capped version of (3.3):

$$\hat{Y}_t^{n,N} = g(X_T) + \int_t^T \hat{F}^{n,N}(s, X_s, \hat{Y}_s^{n,N}, \hat{Z}_s^{n,N})\mathrm{d}s - \int_t^T \hat{Z}_s^{n,N}\mathrm{d}W_s, \tag{3.20}$$

with

$$\hat{F}^{n,N}(t,x,y,z) := f(t,x,y,z) - \rho_N(n\psi(y))\nabla\psi(y)(1+|z|^2)$$

and recalling $\rho_N(x) = x \wedge N$. Propositions 3.1 and 3.2 imply the existence of (large enough) $N, \mathfrak{N} > 0$, such that, for every $n \geqslant \mathfrak{N}$, there exists a Markovian solution (Y^n, Z^n) of (3.3), with $Y_t^n = u^n(t, X_t)$, and any such solution also solves (3.20). Moreover, there exists $c > 0$ such that $|u^n| \leqslant c$ for all n.

Next, we fix N as above and verify easily (as in the proof of Proposition 3.1) that, for large enough $C > 0$ and small enough $\alpha > 0$ (independent of n), $C(\alpha|y|^2 + (\phi_{\mathcal{C}}(y) - \phi_{\mathcal{C}}(0))^2)$ is a global c-Lyapunov function for $(\hat{F}^{n,N})_n$, in the sense of Definition 2.3 in [48]. In addition, $|\hat{F}^{n,N}(t,x,y,z)| \leqslant C + C_N|z|^2$,

with the constants (C, C_N) independent of n. Thus, Theorem 2.5 in [48] yields the uniform boundedness of the Hölder norm of u^n. □

Without loss of generality, we assume that the statements of Proposition 3.2 and Corollary 3.1 hold with $\mathfrak{N} = 1$. From Corollary 3.1, we deduce that there exists a subsequence of $\{u^n\}_{n\geqslant 1}$ converging locally uniformly to a function u satisfying (3.19). To alleviate the notation, this subsequence is still denoted $(u^n)_{n\geqslant 1}$. Recalling that $Y^n_t = u^n(t, X_t)$ and introducing $Y_t := u(t, X_t)$, for $t \in [0, T]$, we observe that

$$\mathbb{E}\left[\sup_{t\in[0,T]} |Y^n_t - Y_t|^2\right] \underset{n\to+\infty}{\longrightarrow} 0\,, \tag{3.21}$$

since $t \mapsto (t, X_t)$ is a.s. continuous and $\{|Y^n|\}$ is bounded uniformly by a constant, see Lemma 3.2.

We conclude this section with the following lemma, which is used in the next section. This lemma provides a uniform upper bound on the second moment of the auxiliary process

$$\Gamma^{n,m}_t := \exp\left(\int_0^t \left(1 + |\dot{K}^n_s| + |\dot{K}^m_s|\right) \mathrm{d}s\right), \quad t \in [0, T],$$

where we recall (3.4).

Lemma 3.3. *Under Assumptions 3.1 and 3.2, for any $\varepsilon > 0$, there exists $N \geqslant 1$ (independent of n) such that, for all $n \geqslant 1$ and all $0 \leqslant k < N$, we have, a.s.:*

$$\mathbb{E}_t\left[\int_t^{T(k+1)/N} |Z^n_s|^2 + |\dot{K}^n_s| \mathrm{d}s\right] \leqslant \varepsilon, \quad \forall t \in [Tk/N, T(k+1)/N].$$

In particular, for any $\beta > 0$, there exists a constant $C = C(\beta)$ (independent of (n, m)) such that

$$\mathbb{E}[(\Gamma^{n,m}_T)^\beta] \leqslant C, \quad \forall\, n, m \geqslant 1.$$

Remark 3.2. *It is worth noticing that the constant C appearing in Lemma 3.3 does not depend on the initial value x of the diffusion X, as follows from the proof of the lemma.*

Proof. The proof of the first statement of the lemma is an improvement of the estimates in the proof of Lemma 3.1, with the use of Corollary 3.1. We fix $t < t' \in [0, T]$, $\beta' > 0$ and $\alpha > 0$, and apply Itô's formula to the process $(e^{\beta'(s-t)}(\alpha|Y^n_s|^2 + (\phi_{\mathcal{C}}(Y^n_s) - \phi_{\mathcal{C}}(0))^2)_{s\in[t,t']}$ (recall (3.3)), to obtain, as in the proof of Lemma 3.1,

$$|Y^n_t|^2 + c\mathbb{E}_t\left[\int_t^{t'} |Z^n_s|^2 + |\dot{K}^n_s| \mathrm{d}s\right] \leqslant \mathbb{E}_t\left[e^{\beta'(t'-t)}|Y^n_{t'}|^2 + C\int_t^{t'} e^{\beta'(s-t)}(1 + |F(s, X_s, 0, 0)|^2)\mathrm{d}s\right],$$

which holds for large enough β' and small enough α.

Then, by using the upper bounds on $|Y^n|$ (see Proposition 3.2) and on $|F(.,.,0,0)|$ (see Assumption 3.2), we obtain:

$$\begin{aligned}
&\mathbb{E}_t\left[\int_t^{t'} |Z^n_s|^2 + |\dot{K}^n_s|\mathrm{d}s\right] \\
\leqslant &\mathbb{E}_t\left[e^{\beta'(t'-t)}|Y^n_{t'}|^2 - |Y^n_t|^2 + C\int_t^{t'} e^{\beta'(s-t)}(1+|F(s,X_s,0,0)|^2)\mathrm{d}s\right] \\
\leqslant &\mathbb{E}_t\left[(e^{\beta'(t'-t)}-1)|Y^n_{t'}|^2 + |Y^n_{t'}+Y^n_t||u^n(t',X_{t'})-u^n(t,X_t)| + \frac{C}{\beta'}(e^{\beta'(t'-t)}-1)\right] \\
\leqslant &C(\beta')(t'-t) + C\mathbb{E}_t\left[(t'-t)^{\alpha'/2} + |X_{t'}-X_t|^{\alpha'}\right] \leqslant C'(\beta')(t'-t)^{\alpha'/2},
\end{aligned}$$

where C' is independent of n, and we made use of Jensen's inequality and of standard SDE estimates on X in the last inequality. The above proves the first statement of the lemma.

To prove the second statement, we fix an arbitrary $\beta > 0$ and consider N corresponding to $\varepsilon = 1/(8\beta)$. Then, the first statement of the lemma and the John-Nirenberg inequality yield:

$$\begin{aligned}
\mathbb{E}\left[e^{2\beta\int_0^T |\dot{K}^n_s|+|\dot{K}^m_s|\mathrm{d}s}\right] &\leqslant \mathbb{E}\left[e^{2\beta\int_0^{T(N-1)/N} |\dot{K}^n_s|+|\dot{K}^m_s|\mathrm{d}s}\mathbb{E}_{T(N-1)/N}\left[e^{2\beta\int_{T(N-1)/N}^T |\dot{K}^n_s|+|\dot{K}^m_s|\mathrm{d}s}\right]\right] \\
&\leqslant 2\mathbb{E}\left[e^{2\beta\int_0^{T(N-1)/N} |\dot{K}^n_s|+|\dot{K}^m_s|\mathrm{d}s}\right].
\end{aligned}$$

Iterating the above, we obtain the desired estimate. □

3.4. **Existence and uniqueness.** We denote by $\{(Y^n, Z^n)\}_{n\geqslant 1}$ a sequence of Markovian solutions to (3.3) satisfying (3.21) (whose existence is established in the previous subsection). The goal of this subsection is to establish that $\{(Y^n, Z^n, K^n) \in \mathscr{S}^2 \times \mathscr{H}^2 \times \mathscr{K}^1\}_{n\geqslant 1}$ (with K^n defined in (3.4))[2] converges to a solution of the reflected BSDE (1.2) and that this solution is unique in the appropriate class.

Theorem 3.1. *Let Assumptions 3.1 and 3.2 hold. Then, there exists a triplet $(Y, Z, K) \in \mathscr{S}^2 \times \mathscr{H}^2 \times \mathscr{K}^1$, such that*

$$\lim_{n\to\infty}(\|Y^n - Y\|_{\mathscr{S}^2}, \|Z^n - Z\|_{\mathscr{H}^2}, \|K^n - K\|_{\mathscr{S}^2}) = 0\ ,$$

and (Y, Z, K) solves (1.2). The process K is absolutely continuous and satisfies, for all $\beta > 0$:

$$\mathbb{E}\big[e^{\beta \mathrm{Var}_T(K)}\big] < \infty\ . \tag{3.22}$$

[2]The fact that $K^n \in \mathscr{K}^1$ follows from the inequality (3.7) and from the second statement of Proposition 3.1.

Moreover, this solution to (1.2) is unique in the class $\mathscr{U}(1)$ (recall Definition 2.1).

Remark 3.3. *If, in addition to Assumptions 3.1 and 3.2, g and F are globally Lipschitz in x (i.e., $\alpha = 1$ in Assumption 3.2), then there exists a constant C such that*

$$|Z| \leqslant C, \quad \mathrm{d}t \times \mathrm{d}\mathbb{P}\text{-}a.e.$$

Indeed, using the same arguments as in the proof of Corollary 3.1, we conclude that the conditions of Theorem 2.16 in [27] are satisfied. The latter theorem yields the existence of a constant C, such that $|Z_t^n| \leqslant C$ for a.e. (t, ω) and for all n. It follows then that $|Z| \leqslant C$.

Remark 3.4. *It is worth noticing that every exponential moment of $\mathrm{Var}_T(K)$ can be bounded by a constant that does not depend on the initial value x of the diffusion X, as follows from Remark 3.2 and from the proof of Theorem 3.1.*

Remark 3.5. *As explained in the discussion preceding (3.21), there exists a measurable function u such that $Y_t = u(t, X_t)$. In addition, since (Y^n, Z^n) is Markovian (see Corollary 3.1), there exists a measurable function v^n such that $Z_t^n = v^n(t, X_t)$. Then, using the convergence of Z^n (see Theorem 3.1) and the strict ellipticity of X, we easily deduce the existence of a measurable v, such that $Z_t = v(t, X_t)$. In this sense, the solution constructed in Theorem 3.1 is Markovian. One may naturally wonder if this Markovian solution yields a solution to an associated partial differential equation (PDE). Adapting the PDE formulation provided in [35] for a convex reflection domain, we conjecture the following PDE for the Markovian solution u constructed in Theorem 3.1: for all admissible test functions $w : [0, T] \times \mathbb{R}^{d'} \to \mathcal{D}$,*

$$\begin{cases} (i) \left\langle \dfrac{\partial u}{\partial t} + \mathscr{L}_t u + f(t, x, u, \sigma^\top \nabla u), u - w + c|w - u|^2 \mathfrak{n}(u) \right\rangle \geqslant 0, \quad 0 \leqslant t \leqslant T, \\ (ii)\ u(T, .) = g, \quad u(t, .) \in \mathcal{D}, \quad 0 \leqslant t \leqslant T, \end{cases}$$

with $\mathscr{L}_t$ being the infinitesimal generator of X at time t and with c being a large enough constant that appears in Lemma 2.3. An alternative, though related, formulation can be obtained using [39]. In any case, studying the precise connection between the Markovian solution to (1.2) constructed in Theorem 3.1 and the associated PDE is outside of the scope of this article and is left for future research.

Proof. 1.a We first prove the uniqueness of the solution in the desired class. For any solution (Y', Z', K') in $\mathscr{U}(1)$, we have

$$\mathbb{E}\left[e^{\frac{p'}{R_0} \mathrm{Var}_T(K')} \right] < +\infty, \tag{3.23}$$

for some $p' > 1$. Setting $1 < p := (1+p')/2 < p'$, $q' := p'/p > 1$ and $q = q'/(q'-1)$, we obtain, using Hölder inequality:

$$\mathbb{E}\Big[e^{\frac{p}{R_0}(\mathrm{Var}_T(K)+\mathrm{Var}_T(K'))}\Big] \leqslant \mathbb{E}\Big[e^{\frac{qp}{R_0}\mathrm{Var}_T(K)}\Big]^{\frac{1}{q}}\,\mathbb{E}\Big[e^{\frac{q'p}{R_0}\mathrm{Var}_T(K')}\Big]^{\frac{1}{q'}}. \tag{3.24}$$

By (3.23), we have $\mathbb{E}\Big[e^{\frac{q'p}{R_0}\mathrm{Var}_T(K)}\Big] = \mathbb{E}\Big[e^{\frac{p'}{R_0}\mathrm{Var}_T(K)}\Big] < \infty$. Then, using (3.22), which is proved below, we obtain

$$\mathbb{E}\Big[e^{\frac{p}{R_0}(\mathrm{Var}_T(K)+\mathrm{Var}_T(K'))}\Big] < +\infty. \tag{3.25}$$

Proposition 2.2, then, yields the desired uniqueness stated in the theorem.

1.b The fact that K is absolutely continuous is proved in Lemma 2.1.

2. Turning to the existence part of the proof, we recall that the convergence of $\{Y^n\}$ is established in (3.21). Moreover, it follows easily from Proposition 3.2 that, with probability one, Y_t takes values in $\bar{\mathcal{D}}$ for all $t \in [0,T]$.

We now turn to the convergence of $\{Z^n\}$. For $n, m \geqslant 1$, we denote

$$\delta f_t := f(t, Y_t^n, Z_t^n) - f(t, Y_t^m, Z_t^m), \quad \delta K := K^n - K^m,$$

$$\delta Y := Y^n - Y^m, \quad \delta Z = Z^n - Z^m.$$

Applying Itô's formula to $(e^{\beta' s}|\delta Y_s|^2)_{s\in[t,T]}$, we obtain

$$\begin{aligned} |\delta Y_t|^2 + \int_t^T e^{\beta'(s-t)}|\delta Z_s|^2 \mathrm{d}s &= 2\int_t^T e^{\beta'(s-t)}\delta Y_s \cdot \delta f_s \mathrm{d}s - 2\int_t^T e^{\beta'(s-t)}\delta Y_s \cdot \delta \dot{K}_s \mathrm{d}s \\ &- 2\int_t^T e^{\beta'(s-t)}\delta Y_s \cdot \delta Z_s \mathrm{d}W_s - \beta' \int_t^T e^{\beta'(s-t)}|\delta Y_s|^2 \mathrm{d}s. \end{aligned} \tag{3.26}$$

Choosing a large enough $\beta' > 0$ and using the standard estimates, we deduce

$$\mathbb{E}\Big[\int_0^T |\delta Z_s|^2 \mathrm{d}s\Big] \leqslant C\mathbb{E}\Big[\int_0^T \Big|\delta Y_s \cdot \delta \dot{K}_s\Big| \mathrm{d}s\Big]. \tag{3.27}$$

Note that Lemma 3.3 yields the existence of a constant C, such that $\mathbb{E}\Big[\Big(\int_0^T |\delta \dot{K}_s| \mathrm{d}s\Big)^2\Big] \leqslant C$ for all n, m. Then, using Cauchy-Schwartz inequality, we obtain

$$\mathbb{E}\Big[\int_0^T |\delta Y_s \cdot \delta \dot{K}_s| \mathrm{d}s\Big] \leqslant \mathbb{E}\Big[\sup_{s\in[0,T]} |\delta Y_s|^2\Big]^{\frac{1}{2}} \Big[\mathbb{E}\Big(\int_0^T |\delta \dot{K}_s| \mathrm{d}s\Big)^2\Big]^{\frac{1}{2}}.$$

The above estimate, along with (3.27) and (3.21), implies that $\{Z^n\}_{n\geqslant 1}$ is a Cauchy sequence. Thus, there exists $(Y, Z) \in \mathscr{S}^2 \times \mathscr{H}^2$ such that $(Y^n, Z^n) \to (Y, Z)$.

Next, we recall that

$$K_t^n = Y_t^n - Y_0^n + \int_0^t f(s, Y_s^n, Z_s^n)\mathrm{d}s - M_t^n, \quad M_t^n := \int_0^t Z_s^n \mathrm{d}W_s.$$

Doob's maximal inequality implies that $\{M^n\}$ converges to M in $\mathscr{S}^2$, with $M_t := \int_0^t Z_s \mathrm{d}W_s$. As $f(t,\cdot,\cdot)$ is Lipschitz, we conclude that

$$\|K^n - K\|_{\mathscr{S}^2} \to 0, \tag{3.28}$$

with the continuous process K defined as

$$K_t := Y_t - Y_0 + \int_0^t f(s, Y_s, Z_s)\mathrm{d}s - \int_0^t Z_s \mathrm{d}W_s.$$

Let us now prove that $K \in \mathscr{K}^1$, and that $\mathrm{d}K_t$ is directed along $\mathfrak{n}$ and is active only when Y touches the boundary. To this end, we define the auxiliary nondecreasing processes

$$\hat{K}_t^n := \int_0^t n\psi(Y_s^n)(1 + |Z_s^n|^2)\mathrm{d}s, \quad t \in [0,T].$$

From Lemma 3.1 we deduce the existence of a constant C, such that $\mathbb{E}\Big[\hat{K}_T^n\Big] \leqslant C$ for all n. Then, using Proposition 3.4 in [7], we know that there exists a nondecreasing nonnegative process $\hat{K}$, two sequences of integers $\{p \leqslant N_p\}$, with $p \to \infty$, and a family of numbers $\{\lambda_r^p\}$, with $\sum_{r=p}^{N_p} \lambda_r^p = 1$, such that

$$\mathbb{P}\left({}^p\hat{K}_t := \sum_{r=p}^{N_p} \lambda_r^p \hat{K}_t^r \to \hat{K}_t\,,\ \forall t \in [0,T] \right) = 1\,. \tag{3.29}$$

The above implies that the measure induced by $\mathrm{d}\,{}^p\hat{K}_t$ on $[0,T]$ converges a.s. to $\mathrm{d}\hat{K}_t$. Then, for any bounded continuous process χ and any $0 \leqslant t_1 < t_2 \leqslant T$,

$$\eta^p(t_1,t_2) := \int_{t_1}^{t_2} \chi_t \sum_{r=p}^{N_p} \lambda_r^p r\psi(Y_t^r)(1 + |Z_t^r|^2)\mathrm{d}t = \int_{t_1}^{t_2} \chi_t \mathrm{d}\,{}^p\hat{K}_t \to \int_0^T \chi_t \mathrm{d}\hat{K}_t, \quad a.s. \tag{3.30}$$

From the first statement of Lemma 3.3 (with the use of Proposition 3.2), we conclude that, for any $\varepsilon > 0$, there exists $N \geqslant 1$ (independent of n) such that for all p and all $0 \leqslant k < N$ we have, a.s.:

$$\mathbb{E}_t[|\eta^p(t, T(k+1)/N)|] \leqslant \varepsilon, \quad \forall t \in [Tk/N, T(k+1)/N].$$

Then, repeating the last part of the proof of Lemma 3.3, we conclude that, for any $\beta > 0$, there exists a constant C such that

$$\mathbb{E}\Big[e^{\beta\eta^p(0,T)}\Big] \leqslant C, \quad \forall\, p.$$

Thus, the family $\{\exp(\beta\eta^p(0,T))\}_p$ is uniformly integrable. The latter implies, in particular, that the convergence in (3.30) holds in $\mathcal{L}^1$ and that all exponential moments of $\hat{K}_T$ are finite.

Next, we define

$$^pK_t := \sum_{r=p}^{N_p} \lambda_r^p K_t^r, \quad t \in [0,T].$$

We also denote by $\widetilde{\nabla\psi}$ a Lipschitz extension of $\nabla\psi$ into $\mathcal{D}$ (constructed as in Remark 3.1). Then, for any event A and any $t \in [0,T]$, we have:

$$\begin{aligned}
\mathbb{E}[^pK_t\,\mathbf{1}_A] &= \mathbb{E}\left[\int_0^t \sum_{r=p}^{N_p} \lambda_r^p \widetilde{\nabla\psi}(Y_s^r) r\psi(Y_s^r)(1+|Z_s^r|^2)\mathrm{d}s\,\mathbf{1}_A\right] \\
&= \mathbb{E}\left[\int_0^t \widetilde{\nabla\psi}(Y_s) \sum_{r=p}^{N_p} \lambda_r^p r\psi(Y_s^r)(1+|Z_s^r|^2)\mathrm{d}s\,\mathbf{1}_A\right] \\
&\quad + O\left(\mathbb{E}\left[\sum_{r=p}^{N_p} \lambda_r^p \sup_{s\in[0,t]} |Y_s^r - Y_s| \int_0^t (1+|Z_s^r|^2)\mathrm{d}s\right]\right) \\
&\to \mathbb{E}\left[\int_0^t \widetilde{\nabla\psi}(Y_s)\mathrm{d}\hat{K}_s\mathrm{d}s\,\mathbf{1}_A\right],
\end{aligned}$$

where we used (3.30), along with its $\mathcal{L}^1$ version, and the estimate

$$\mathbb{E}\left[\sup_{s\in[0,t]} |Y_s^r - Y_s| \int_0^t (1+|Z_s^r|^2)\mathrm{d}s\right] \leqslant C\|Y^r - Y\|_{\mathscr{S}^2}\mathbb{E}\left[e^{\int_0^t (1+|Z_s^r|^2)}\right]^{1/2} \leqslant C\|Y^r - Y\|_{\mathscr{S}^2},$$

which follows from Lemma 3.3.

On the other hand, as K^n converges to K in $\mathscr{S}^2$, $\mathbb{E}[^pK_t\,\mathbf{1}_A]$ converges to $\mathbb{E}[K_t\mathbf{1}_A]$. Since A is arbitrary and $K_\cdot$ is continuous, we conclude:

$$\mathbb{P}\left(K_t = \int_0^t \widetilde{\nabla\psi}(Y_s)\mathrm{d}\hat{K}_s\mathrm{d}s, \quad \forall\, t \in [0,T]\right) = 1. \tag{3.31}$$

Note that the integrability of $\hat{K}_T$ and the above representation, in particular, imply $K \in \mathscr{K}^1$.

It only remains to show that

$$\int_0^T \mathbf{1}_{\mathcal{D}}(Y_t)\mathrm{d}\hat{K}_t = 0. \tag{3.32}$$

To this end, we choose an arbitrary Lipschitz f supported in $\mathcal{D}$ and any event A, to obtain:

$$\mathbb{E}\left[\int_0^T f(Y_t)\mathrm{d}\hat{K}_t\,\mathbf{1}_A\right] = \lim_{n\to\infty} \mathbb{E}\left[\int_0^T f(Y_t) \sum_{r=p}^{N_p} \lambda_r^p r\psi(Y_t^r)(1+|Z_t^r|^2)\mathrm{d}t\,\mathbf{1}_A\right]$$

$$= \lim_{n\to\infty} \mathbb{E}\left[\int_0^T \sum_{r=p}^{N_p} \lambda_r^p f(Y_t^r) r\psi(Y_t^r)(1+|Z_t^r|^2)\mathrm{d}t\, \mathbf{1}_A\right]$$

$$+O\left(\mathbb{E}\left[\sum_{r=p}^{N_p} \lambda_r^p \sup_{t\in[0,T]} |Y_t^r - Y_t| \int_0^T (1+|Z_t^r|^2)\mathrm{d}t\right]\right) = 0.$$

As A is arbitrary, we conclude that, for any Lipschitz f supported in $\mathcal{D}$, we have $\int_0^T f(Y_t)\mathrm{d}\hat{K}_t = 0$ a.s.. Approximating $\mathbf{1}_{\mathcal{D}}$ with a sequence of such f, we use the monotone convergence theorem to deduce (3.32). Combining the latter with (3.31), we obtain (1.2)(ii) and conclude the proof of the first part of Theorem 3.1. □

Remark 3.6. *Theorem 3.1 implies that, under Assumptions 3.1, 3.2, and 2.1 with $\theta = 1$, there exists a solution (Y, Z, K) to (1.2) that is unique in $\mathscr{S}^2 \times \mathscr{H}^2 \times \mathscr{K}^1$.*

4. Well-posedness beyond Markovian framework

4.1. **Discrete path-dependent framework.** In this subsection, we extend the existence and uniqueness result obtained in a Markovian framework (see Theorem 3.1) to a discrete path-dependent framework.

Assumption 4.1. *Let ℓ be an arbitrary strictly positive integer and consider the partition $0 = t_0 < t_1 < ... < t_\ell = T$ of $[0, T]$. We assume that*

$$\xi = g(X_{t_1}, ..., X_{t_\ell}) \quad \text{and} \quad f(s, y, z) = F(s, X_{t_1\wedge s}, \cdots, X_{t_\ell\wedge s}, y, z),$$

where

(i) g is α-Hölder and takes values in $\bar{\mathcal{D}}$,
(ii) F is measurable in all variables, globally Lipschitz in (y, z) uniformly over $(x_1, ..., x_\ell)$, globally α-Hölder in $(x_1, ..., x_\ell)$ uniformly over (y, z), and $|F(\cdot, \cdots, 0, 0)|$ is bounded.

We note that $\ell = 1$ corresponds to the Markovian framework of the previous section, with an extra regularity assumption on the generator with respect to x. We also recall that Assumptions 1.1 and 1.2 hold throughout Section 4 even if not cited explicitly.

Theorem 4.1. *Let Assumptions 3.1 and Assumption 4.1 hold. Then, there exists a triplet $(Y, Z, K) \in \mathscr{S}^2 \times \mathscr{H}^2 \times \mathscr{K}^1$ that solves (1.2). Moreover, all exponential moments of $\mathrm{Var}_T(K)$ are finite, and this solution is unique in the class $\mathscr{U}(1)$ (recall Definition 2.1).*

Proof. Once the finiteness of the exponential moments of $\mathrm{Var}_T(K)$ is proven, the uniqueness of the solution in the class $\mathscr{U}(1)$ follows from the same arguments as in step 1.a of the proof of Theorem 3.1. Let us now prove the existence part of

the theorem. To this end, we use the backward recursion to construct a solution on each interval $[t_i, t_{i+1}]$ for $0 \leqslant i \leqslant \ell - 1$.

Since the case $\ell = 1$ corresponds to the Markovian framework of the previous section, we assume that $\ell > 1$ and consider the time interval $[t_{\ell-1}, T]$. For any $(t, x) \in [0, T] \times \mathbb{R}^{d'}$, we denote by $X^{t,x}$ the unique solution of (3.1) on $[t, T]$, which starts from x at time t. We shall use the notation X for the original diffusion started at time zero. For any $\mathbf{x} = (\mathbf{x}_1, ..., \mathbf{x}_{\ell-1}) \in (\mathbb{R}^{d'})^{\ell-1}$, we denote by $(Y^{\mathbf{x}}, Z^{\mathbf{x}}, K^{\mathbf{x}})$ the solution of (1.2) on $[t_{\ell-1}, T]$, with the terminal condition $g(\mathbf{x}, X_T^{t_{\ell-1},\mathbf{x}_{\ell-1}})$ and with the generator $F(., \mathbf{x}, X_.^{t_{\ell-1},\mathbf{x}_{\ell-1}}, ., .)$, whose existence follows from Theorem 3.1 and whose uniqueness in the appropriate class follows from Theorem 2.2.

Next, we denote by $(Y^{\mathbf{x},n}, Z^{\mathbf{x},n})$ a Markovian solution of the penalized BSDE (3.3) on $[t_{\ell-1}, T]$, whose existence follows from Proposition 3.1. In particular, there exist measurable functions $u^n(\mathbf{x}, ., .)$ and $v^n(\mathbf{x}, ., .)$ such that

$$Y_t^{\mathbf{x},n} = u^n(\mathbf{x}, t, X_t^{t_{\ell-1},\mathbf{x}_{\ell-1}}), \quad Z_t^{\mathbf{x},n} = v^n(\mathbf{x}, t, X_t^{t_{\ell-1},\mathbf{x}_{\ell-1}}).$$

By considering a sequence of Lipschitz approximations of (3.3), given by (3.5), we apply Theorem 5.4 in [28] and, passing to the limit for the Lipschitz approximations as in the proof of Proposition 3.1, we conclude that a Markovian solution to (3.3) can be constructed so that u^n and v^n are jointly measurable in all variables. Passing to the limit in n along a subsequence, we use Theorem 3.1 and the uniform Hölder estimate in Corollary 3.1, to deduce the existence of jointly measurable functions u and v satisfying

$$Y_t^{\mathbf{x}} = u(\mathbf{x}, t, X_t^{t_{\ell-1},\mathbf{x}_{\ell-1}}), \quad Z_t^{\mathbf{x}} = v(\mathbf{x}, t, X_t^{t_{\ell-1},\mathbf{x}_{\ell-1}}). \tag{4.1}$$

Then, by denoting $\mathbf{X} = (X_{t_1}, ...X_{t_{\ell-1}})$, we consider the progressively measurable processes $(Y_t^{\mathbf{X}}, Z_t^{\mathbf{X}})_{t\in[t_{\ell-1},T]}$ and define

$$K_t^{\mathbf{X}} := Y_t^{\mathbf{X}} - Y_{t_{\ell-1}}^{\mathbf{X}} + \int_{t_{\ell-1}}^t F(s, \mathbf{X}, X_s^{t_{\ell-1},X_{t_{\ell-1}}}, Y_s^{\mathbf{X}}, Z_s^{\mathbf{X}})\mathrm{d}s - \int_{t_{\ell-1}}^t Z_s^{\mathbf{X}}\mathrm{d}W_s, \quad t_{\ell-1} \leqslant t \leqslant T.$$

We note that $X_s^{t_{\ell-1},X_{t_{\ell-1}}} = X_s$ and that $(Y_t^{\mathbf{X}}, Z_t^{\mathbf{X}}, K_t^{\mathbf{X}})_{t\in[t_{\ell-1},T]}$ is a solution of (1.2) on the time interval $[t_{\ell-1}, T]$, satisfying $K_{t_{\ell-1}}^{\mathbf{X}} = 0$.

In order to iterate this construction and to extend the solution to the time interval $[t_{\ell-2}, t_{\ell-1}]$, we have to verify that the associated terminal condition $Y_{t_{\ell-1}}^{\mathbf{X}}$ of the reflected BSDE (1.2) on $[t_{\ell-2}, t_{\ell-1}]$ is an α-Hölder function of $\mathbf{X}$. To this end, we recall the function u in (4.1) and define, for all $\tilde{\mathbf{x}} = (\mathbf{x}_1, ..., \mathbf{x}_{\ell-2}) \in (\mathbb{R}^{d'})^{\ell-2}$ and $\mathbf{x}_{\ell-1} \in \mathbb{R}^{d'}$, the deterministic function

$$\tilde{g}(\tilde{\mathbf{x}}, \mathbf{x}_{\ell-1}) := u(\tilde{\mathbf{x}}, \mathbf{x}_{\ell-1}, t_{\ell-1}, \mathbf{x}_{\ell-1}) = Y_{t_{\ell-1}}^{\tilde{\mathbf{x}},\mathbf{x}_{\ell-1}}.$$

Let us prove that this function is α-Hölder. Indeed, for any $\mathbf{x} := (\tilde{\mathbf{x}}, \mathbf{x}_{\ell-1}) \in (\mathbb{R}^{d'})^{\ell-1}$ and $\mathbf{x}' := (\tilde{\mathbf{x}}', \mathbf{x}'_{\ell-1}) \in (\mathbb{R}^{d'})^{\ell-1}$, Proposition 2.2 with $p = 2$ yields

$$
\begin{aligned}
&|\tilde{g}(\mathbf{x}) - \tilde{g}(\mathbf{x}')| \leqslant \|Y^{\mathbf{x}} - Y^{\mathbf{x}'}\|_{\mathscr{S}^2} \\
\leqslant & C\mathbb{E}\left[\left|g(\mathbf{x}, X_T^{t_{\ell-1},\mathbf{x}_{\ell-1}}) - g(\mathbf{x}', X_T^{t_{\ell-1},\mathbf{x}'_{\ell-1}})\right|^4\right]^{1/4} \\
&+ C\mathbb{E}\left[\left(\int_0^T |F(s, \mathbf{x}, X_s^{t_{\ell-1},\mathbf{x}_{\ell-1}}, Y_s^{\mathbf{x}}, Z_s^{\mathbf{x}}) - F(s, \mathbf{x}', X_s^{t_{\ell-1},\mathbf{x}'_{\ell-1}}, Y_s^{\mathbf{x}}, Z_s^{\mathbf{x}})|\mathrm{d}s\right)^4\right]^{1/4} \\
\leqslant & C\left(|\mathbf{x} - \mathbf{x}'|^\alpha + \mathbb{E}\left[\sup_{0\leqslant s\leqslant T} |X_s^{t_{\ell-1},\mathbf{x}_{\ell-1}} - X_s^{t_{\ell-1},\mathbf{x}'_{\ell-1}}|^{4\alpha}\right]^{1/4}\right),
\end{aligned}
$$

with a constant C that does not depend on x (see Remark 3.4). Then, the Jensen's inequality and the standard SDE estimates yield

$$|\tilde{g}(\mathbf{x}) - \tilde{g}(\mathbf{x}')| \leqslant C|\mathbf{x} - \mathbf{x}'|^\alpha,$$

which proves the α-Hölder property of $\tilde{g}$. Considering the reflected BSDE (1.2) on $[t_{\ell-2}, t_{\ell-1}]$, with the terminal condition $Y_{t_{\ell-1}}^{\mathbf{x}} = \tilde{g}(X_{t_1}, ..., X_{t_{\ell-1}})$ and with the generator

$$F(s, X_{t_1\wedge s}, ..., X_{t_{\ell-2}\wedge s}, X_{t_{\ell-1}\wedge s}, X_{t_{\ell-1}\wedge s}, y, z),$$

we deduce, as in the first part of the proof, that it has a solution in the form (4.1).

Finally, iterating the above construction, we concatenate the "Y" and "Z" parts of the solutions constructed on the individual sub-intervals, and we sum up the "K" parts (assuming that every individual "K" part is extended continuously as a constant to the left and to the right of the associated sub-interval). It is easy to see that the resulting process $(Y, Z, \tilde{K}) \in \mathscr{S}^2 \times \mathscr{H}^2 \times \mathscr{K}^1$ is a solution of (1.2) on $[0, T]$. □

4.2. **General case.**

Theorem 4.2. *Let Assumption 2.1 hold with $\theta = 2$. Then, there exists a triplet $(Y, Z, K) \in \mathscr{S}^2 \times \mathscr{H}^2 \times \mathscr{K}^1$ that solves (1.2), and this solution is unique in the class $\mathscr{U}(2)$.*

Proof. The uniqueness part of the theorem is a direct consequence of Proposition 2.1 and Corollary 2.2. Let us prove the existence part. To this end, we construct a Cauchy sequence of solutions to the approximating reflected BSDEs.

First, we observe that the terminal condition ξ can be approximated by a sequence of random variables of the form $\xi^n := g_n(W_{t_1}, ..., W_{t_n})$, where g_n is infinitely differentiable. The sequence $(\xi^n)_{n\in\mathbb{N}^*}$ can be chosen so that it converges to ξ in $\mathscr{L}^q$, for any $q \geqslant 1$ (see, e.g., [40]). In particular,

$$\lim_{n\to\infty} \mathbb{E}\left[|\xi - \xi^n|^{2p/(p-1)}\right] = 0, \tag{4.2}$$

with $p > 1$ appearing in Proposition 2.1. Replacing g_n by $g_n \wedge \|\xi\|_{\mathscr{L}^\infty}$, we can assume $\|\xi^n\|_{\mathscr{L}^\infty} \leqslant \|\xi\|_{\mathscr{L}^\infty}$. We observe that ξ^n satisfies Assumption 4.1(i) with $X = W$.

Second, to approximate the generator, for every $n \in \mathbb{N}^*$, we denote by $\mathcal{K}^n$ the closed ball in $\mathbb{R}^{d\times d'}$ of radius n centered at zero, and choose a sequence of numbers $\epsilon_n \downarrow 0$. We set

$$\ell_n := \|f(\cdot,0,0)\|_{\mathscr{L}^\infty} + K_{f,y}\sup_{y\in\bar{\mathcal{D}}}|y| + nK_{f,z},$$

recalling Assumption 1.2. For each n, we denote by $\mathfrak{L}^n$ the compact convex subset of $\mathscr{C}(\bar{\mathcal{D}}\times\mathcal{K}^n)$ (the space of continuous function endowed with the uniform norm $\|\cdot\|_s$) consisting of all Lipschitz functions with the Lipschitz coefficients $K_{f,y}$ and $K_{f,z}$ in the $y \in \bar{\mathcal{D}}$ and $z \in \mathcal{K}^n$ variables, respectively, and with the (uniform) norm bounded by ℓ_n. Note that the stochastic process $f_{|\bar{\mathcal{D}}\times\mathcal{K}^n}$ takes values in $\mathfrak{L}^n$.

Let us now construct an approximation of $f_{|\bar{\mathcal{D}}\times\mathcal{K}^n}$ in $\mathfrak{L}^n$ that satisfies Assumption 4.1 (for $X = W$). To this end, we denote by $\{\phi_n^m\}_{m=1}^{M_n}$ an ϵ_n-cover of the compact set $\mathfrak{L}^n$, with M_n being a positive integer. We denote by $\tilde{f}^n(t,\cdot)$ the (measurable selection of the) proximal projection of $f_{|\bar{\mathcal{D}}\times\mathcal{K}^n}(t,\cdot)$ on $\{\phi_n^m\}_{m=1}^{M_n}$. It satisfies

$$\tilde{f}^n(t,\cdot) = \sum_{m=1}^{M_n}\phi_n^m(\cdot)(\tilde{\eta}_t^n)^m =: \phi_n(\cdot)\tilde{\eta}_t^n \quad \text{and} \quad \|\tilde{f}^n(t,\cdot) - f_{|\bar{\mathcal{D}}\times\mathcal{K}^n}(t,\cdot)\|_s \leqslant \epsilon_n \quad \text{a.s.},$$

where $\tilde{\eta}^n$ is a progressively measurable process taking values in the (non-empty) set of extremal points of $\mathcal{S}_{M_n} := \{x \in \mathbb{R}^{M_n} \,|\, 0 \leqslant x^m \leqslant 1, \sum_{m=1}^{M_n} x^m \leqslant 1\}$. Then, using the dominated convergence theorem, we obtain

$$\mathbb{E}\bigg[\int_0^T \|f_{|\bar{\mathcal{D}}\times\mathcal{K}^n}(t,\cdot) - \tilde{f}^n(t,\cdot)\|_s^{2p/(p-1)}\mathrm{d}t\bigg] \leqslant T(\epsilon_n)^{2p/(p-1)}. \tag{4.3}$$

Next, we consider a standard approximation of $(\tilde{\eta}_t^n)_{t\in[0,T]}$ by an adapted process $(\hat{\eta}_t^n)_{t\in[0,T]}$ that is piecewise constant on the time grid $\Pi_n := \{t_0 = 0 < \cdots < t_k^n < \cdots < t_{\kappa_n}^n = T\}$. This process can be chosen to be $\mathcal{S}_{M_n}$-valued and satisfying

$$\mathbb{E}\bigg[\int_0^T |\tilde{\eta}_t^n - \hat{\eta}_t^n|^{2p/(p-1)}\mathrm{d}t\bigg] \leqslant \frac{\epsilon_n}{(M_n\ell_n^2)^{p/(p-1)}}\,.$$

Setting

$$\hat{f}^n(t,\cdot) = \sum_{k=0}^{\kappa_n-1}\phi_n(\cdot)\hat{\eta}_{t_k^n}^n\mathbf{1}_{(t_k^n,t_{k+1}^n]}(t), \tag{4.4}$$

which is $\mathfrak{L}^n$-valued, as a random convex combination of $\{\phi_n^m\}_{m=1}^{M_n}$, we deduce

$$\mathbb{E}\bigg[\int_0^T \|\tilde{f}^n(t,\cdot) - \hat{f}^n(t,\cdot)\|_s^{2p/(p-1)}\mathrm{d}t\bigg] \leqslant \epsilon_n. \tag{4.5}$$

Next, we apply the approximation result of [40] for each $\hat{\eta}^n_{t^n_k}$. Introducing, if necessary, a finer grid $\Re_n \subset \Pi_n$, we set

$$\eta^n_{t^n_k} := \mathfrak{P}_{\mathcal{S}}\left[r^n_k\left((W_r)_{r\in\Re_n, r\leqslant t^n_k}\right)\right],$$

where r^n_k is a smooth function with values in $\mathbb{R}^{M_n}$ and $\mathfrak{P}_{\mathcal{S}}$ the (orthogonal) projection onto $\mathcal{S}_{M_n}$. We can chose r^n_k so that

$$\mathbb{E}\big[|\eta^n_{t_k} - \hat{\eta}^n_{t_k}|^{2p/(p-1)}\big] \leqslant \frac{\epsilon_n}{(M_n\ell_n^2)^{p/(p-1)}}\,. \tag{4.6}$$

Setting $f^n(t,\cdot) = \sum_{k=0}^{\kappa_n-1}\phi_n(\cdot)\eta^n_{t^n_k}\mathbf{1}_{(t^n_k,t^n_{k+1}]}(t)$, which belongs to $\mathfrak{L}^n$, we have

$$\mathbb{E}\bigg[\int_0^T \|\hat{f}^n(t,\cdot) - f^n(t,\cdot)\|_s^{2p/(p-1)}\mathrm{d}t\bigg] \leqslant T\epsilon_n. \tag{4.7}$$

Collecting the above, we conclude that

$$\lim_{n\to\infty}\mathbb{E}\int_0^T \sup_{y\in\bar{\mathcal{D}},\, z\in\mathcal{K}^n}|f(t,y,z) - f^n(t,y,z)|^{2p/(p-1)}\mathrm{d}t = 0. \tag{4.8}$$

We extend $f^n(t,y,\cdot)$ to $\mathbb{R}^{d\times d'}\backslash\mathcal{K}^n$ as a constant in each radial direction, so that its uniform norm and the Lipschitz coefficient do not change.

It is easy to see that, if f satisfies Assumption 2.1-(i) (resp. Assumption 2.1-(iii)), the above construction allows us to build an approximating sequence f^n having the same properties. Indeed, we simply work with $\tilde{\mathfrak{L}}^n$ instead of $\mathfrak{L}^n$, where $\tilde{\mathfrak{L}}^n$ is the closed convex subset of $\mathfrak{L}^n$ whose elements satisfy Assumption 2.1-(i) (resp. Assumption 2.1-(iii)).

Thus, for any $n\in\mathbb{N}^*$, we have constructed the approximations ξ^n and f^n that satisfy Assumption 4.1. Therefore, we can invoke Theorem 4.1 to obtain the unique solution $(Y^n, Z^n, K^n)\in\mathscr{S}^2\times\mathscr{H}^2\times\mathscr{K}^1$ of (1.2) associated with the input data (ξ^n, f^n). Thanks to Proposition 2.1, we can apply Proposition 2.2, to deduce that, for all $n, m\in\mathbb{N}^*$,

$$\begin{aligned}
&\|Y^n - Y^m\|_{\mathscr{S}^2} + \|Z^n - Z^m\|_{\mathscr{H}^2} + \|K^n - K^m\|_{\mathscr{S}^2}\\
&\leqslant C\mathbb{E}\left[|\xi^n - \xi^m|^{2p/(p-1)}\right]^{(p-1)/(2p)}\\
&+ C\mathbb{E}\left[\left(\int_0^T |f^n(s,Y^n_s,Z^n_s) - f^m(s,Y^n_s,Z^n_s)|\mathrm{d}s\right)^{2p/(p-1)}\right]^{(p-1)/(2p)},
\end{aligned} \tag{4.9}$$

with a constant C that does not depend on n and m.

Applying Cauchy-Schwartz, Jensen's and Chebyshev's inequalities, we obtain

$$\begin{aligned}
&\mathbb{E}\bigg[\bigg(\int_0^T (1+|Z_t^n|)\mathbf{1}_{\{|Z_t^n|>n\}}\mathrm{d}t\bigg)^{2p/(p-1)}\bigg] \\
&\leqslant T^{p/(p-1)-1/2}\mathbb{E}\bigg[\bigg(\int_0^T (1+|Z_t^n|)^2\mathrm{d}t\bigg)^{2p/(p-1)}\bigg]^{1/2}\mathbb{E}\bigg[\int_0^T \mathbf{1}_{\{|Z_t^n|>n\}}\mathrm{d}t\bigg]^{1/2} \\
&\leqslant \frac{T^{p/(p-1)-1/2}}{n}\mathbb{E}\bigg[\bigg(\int_0^T (1+|Z_t^n|)^2\mathrm{d}t\bigg)^{2p/(p-1)}\bigg]^{1/2}\mathbb{E}\bigg[\int_0^T |Z_t^n|^2\mathrm{d}t\bigg]^{1/2}.
\end{aligned}$$

Using Proposition 2.1 and the energy inequality for BMO martingales, we bound $\mathbb{E}\left(\int_0^T (1+|Z_t^n|)^2\mathrm{d}t\right)^{2p/(p-1)}$ uniformly over n. Then, for all $m \geqslant n$, we obtain from the above estimate:

$$\begin{aligned}
&\mathbb{E}\left[\left(\int_0^T |f^n(t,Y_t^n,Z_t^n)-f^m(t,Y_t^n,Z_t^n)|\mathrm{d}t\right)^{2p/(p-1)}\right]^{(p-1)/(2p)} \\
&\leqslant C\mathbb{E}\left[\left(\int_0^T (1+|Z_t^n|)\mathbf{1}_{\{|Z_t^n|>n\}}\mathrm{d}t\right)^{2p/(p-1)}\right]^{(p-1)/(2p)} \\
&\quad + C\mathbb{E}\left[\left(\int_0^T |f^n(t,Y_t^n,Z_t^n)-f^m(t,Y_t^n,Z_t^n)|\mathbf{1}_{\{|Z_t^n|\leqslant n\}}\mathrm{d}t\right)^{2p/(p-1)}\right]^{(p-1)/(2p)} \\
&\leqslant \frac{C}{n^{(p-1)/(2p)}} + C\mathbb{E}\bigg[\int_0^T |f^n(t,Y_t^n,Z_t^n)-f(t,Y_t^n,Z_t^n)|^{2p/(p-1)}\mathbf{1}_{\{Y_t^n\in\bar{\mathcal{D}},|Z_t^n|\leqslant n\}}\mathrm{d}t\bigg]^{(p-1)/(2p)} \\
&\quad + C\mathbb{E}\bigg[\int_0^T |f^m(t,Y_t^n,Z_t^n)-f(t,Y_t^n,Z_t^n)|^{2p/(p-1)}\mathbf{1}_{\{Y_t^n\in\bar{\mathcal{D}},|Z_t^n|\leqslant m\}}\mathrm{d}t\bigg]^{(p-1)/(2p)}.
\end{aligned} \tag{4.10}$$

In view of (4.8), the right hand side of the above vanishes as $n, m \to \infty$. Collecting (4.2), (4.9) and (4.10), we conclude:

$$\|Y^n - Y^m\|_{\mathscr{S}^2} + \|Z^n - Z^m\|_{\mathscr{H}^2} + \|K^n - K^m\|_{\mathscr{S}^2} \xrightarrow{n,m\to+\infty} 0.$$

In other words, $(Y^n, Z^n, K^n)_{n\in\mathbb{N}^*}$ is a Cauchy sequence in $\mathscr{S}^2 \times \mathscr{H}^2 \times \mathscr{S}^2$. Then, there exists $(Y, Z, K) \in \mathscr{S}^2 \times \mathscr{H}^2 \times \mathscr{S}^2$ such that $(Y^n, Z^n, K^n) \xrightarrow{n\to+\infty} (Y, Z, K)$. Moreover, Y takes values in $\bar{\mathcal{D}}$. Recall that (Y^n, Z^n, K^n) is the unique solution to (1.2) associated with the terminal condition ξ^n and the generator f^n. In addition,

we have

$$\begin{aligned}&\mathbb{E}\left[\int_t^T |f^n(s,Y_s^n,Z_s^n)-f(s,Y_s,Z_s)|\mathrm{d}s\right]\\ \leqslant&\mathbb{E}\left[\int_t^T |f^n(s,Y_s,Z_s)-f(s,Y_s,Z_s)|\mathrm{d}s\right]+C\mathbb{E}\left[\int_t^T |Y_s^n-Y_s|+|Z_s^n-Z_s|\mathrm{d}s\right].\end{aligned}$$

Then, we can easily pass to the limit in (1.2)(i) to show that (Y,Z,K) satisfies (1.2)(i).

It remains to prove that $K\in\mathscr{K}^1$, that $\mathrm{d}K_t$ is directed along $\mathfrak{n}(Y_t)$, and that it is active only when Y touches the boundary (the latter two properties will be shown via the alternative characterization given by Lemma 2.3). Repeating the derivation of (2.11)-(2.12) for (Y^n,Z^n,K^n), but without taking the conditional expectations and with $\beta=0$, we obtain:

$$\int_0^T \mathrm{dVar}_s(K^n)\leqslant C\left(|\xi^n|^2+\int_0^T 2Y_s^n\cdot f(s,Y_s^n,Z_s^n)\mathrm{d}s-\int_0^T 2Y_s^nZ_s^n\mathrm{d}W_s\right),$$

where the constant C does not depend on n. The right hand side of the above inequality converges in probability, as $n\to\infty$, hence it also converges a.s. up to a subsequence which we still denote $\{(Y^n,Z^n,K^n)\}$. Then, $\{\mathrm{Var}_T(K^n)\}_{n\in\mathbb{N}^*}$ is a.s. bounded uniformly over n, and Fatou's lemma yields that $\mathrm{Var}_T(K)$ is a.s. bounded – i.e., K is a bounded variation process. Thanks to Proposition 2.1, $\{\mathrm{Var}_T(K^n)\}_{n\in\mathbb{N}^*}$ is uniformly integrable and, hence, $K\in\mathscr{K}^1$. As (Y^n,Z^n,K^n) solves (1.2) with the terminal condition ξ^n and the generator f^n, Lemma 2.3 yields the existence of a constant c, independent of n, such that, for all continuous adapted process V with values in $\bar{\mathcal{D}}$, we have

$$\int_0^T (Y_s^n-V_s)\mathrm{d}K_s^n+c|Y_s^n-V_s|^2\mathfrak{n}(Y_s^n)\mathrm{d}K_s^n\geqslant 0\quad\text{a.s.}$$

Finally, we use Lemma 5.8 in [22] to pass to the limit in the above inequality and obtain

$$\int_0^T (Y_s-V_s)\mathrm{d}K_s+c|Y_s-V_s|^2\mathfrak{n}(Y_s)\mathrm{d}K_s\geqslant 0\quad\text{a.s.},$$

which completes the proof of the theorem via another application of Lemma 2.3. □

5. Connection to Brownian Γ-martingales

It turns out that the solutions to reflected BSDEs in non-convex domains, defined via (1.2) and constructed in the previous sections, are naturally connected to the notion of martingales on manifolds (also named Γ-martingales – see [17]). In this section, we investigate this connection more closely, in particular, discovering a new proof of the existence and uniqueness of a Brownian martingale with a

prescribed terminal value on a section of a sphere and illustrating the sharpness of the weak star-shape assumption on $\mathcal{D}$ (see Assumption 1.1).

The connection to martingales on manifolds is made precise by the following proposition, which states that, under certain assumptions, one can ensure that the Y-component of the solution to (1.2) always stays on the boundary of the domain $\mathcal{D}$. Treating $\partial\mathcal{D}$ as a manifold and expressing $\mathrm{d}K_t$ via $\nabla^2\phi(Y_t)$ and Z_t, we discover that Y satisfies the definition of a Brownian Γ-martingale on the manifold $\partial\mathcal{D}$, given in [17].

Proposition 5.1. *Assume the following:*

- *there exists a convex domain $\mathcal{A}$, satisfying $\bar{\mathcal{A}} \cap \bar{\mathcal{D}} \subset \partial\mathcal{D}$,*
- $\mathbf{1}_{\{y\in\partial\mathcal{D}\setminus\bar{\mathcal{A}}\}}\nabla d(y,\mathcal{A})\cdot\nabla\phi(y) \geqslant 0$,
- *$f \equiv 0$ and $\xi \in \bar{\mathcal{A}} \cap \partial\mathcal{D}$ almost surely,*
- *$(Y,Z,K) \in \mathscr{S}^2 \times \mathscr{H}^2 \times \mathscr{K}^1$ solve (1.2).*

Then, $Y \in \bar{\mathcal{A}} \cap \bar{\mathcal{D}} \subset \partial\mathcal{D}$ almost surely. Moreover, we have

$$\mathrm{dVar}_t(K) = \left[-\frac{1}{2}\mathrm{Tr}[Z_t^\top \nabla^2\phi(Y_t)Z_t]\right]^+ \mathrm{d}t. \tag{5.1}$$

Finally, Y is a Γ-martingale with the terminal value ξ on the manifold $\partial\mathcal{D}$ endowed with the Riemannian structured inherited from $\mathbb{R}^d$ and its canonical connection Γ, as defined in [17].

Remark 5.1. *It is worth mentioning that the assumptions made in Proposition 5.1 imply that the set $\mathcal{A}$ cannot be smooth. To obtain an intuitive understanding of what the set $\mathcal{A}$ may look like, we refer the reader to the example that follows.*

Proof. We apply Itô's formula for general convex functions (in the form of an inequality, as in [5]) to the process $d(Y_t,\mathcal{A})$ to obtain

$$0 \leqslant d(Y_t,\mathcal{A}) \leqslant \mathbb{E}_t\left[d(\xi,\mathcal{A}) - \int_t^T \mathbf{1}_{\{Y_s\in\partial\mathcal{D}\setminus\bar{\mathcal{A}}\}}\nabla d(Y_s,\mathcal{A})dK_s\right] \leqslant 0, \quad t\in[0,T],$$

which yields $Y \in \bar{\mathcal{A}} \cap \mathcal{D} \subset \partial\mathcal{D}$. Applying Itô's formula to $\phi(Y_t)$ yields (5.1). Finally, using (5.1), the fact that dK_t is orthogonal to the tangent space of $\partial\mathcal{D}$ at the point Y_t, as well as (4.9), (4.10), and (5.6)(ii) from [17], we conclude that Y is a Γ-martingale on $\partial\mathcal{D}$. □

In the remainder of this section, we assume that $f = 0$ and present a simple example of the domains $\mathcal{D}$ and $\mathcal{A}$ for which the assumptions of Proposition 5.1 hold. This example allows us to obtain an alternative proof of a known result on Γ-martingales using the reflected BSDEs and to illustrate the sharpness of the weak star-shape assumption (see Assumption 1.1).

In this example, we first construct the functions ϕ and $\phi_{\mathcal{C}}$, which define the domains $\mathcal{D}$ and $\mathcal{C}$ as in Assumption 1.1, on the plane $\mathcal{P} := \mathbb{R}\times\{0\}^{d-2}\times\mathbb{R}$ of $\mathbb{R}^d$. These functions and the associated domains are designed to be symmetric with

respect to the y_d-axis – see the precise description below. Then, we extend these functions and domains to $\mathbb{R}^d$ via

$$\phi(y) = \phi((r(y), 0, ..., 0, y_d)), \quad \phi_{\mathcal{C}}(y) = \phi_{\mathcal{C}}((r(y), 0, ..., 0, y_d)),$$

with $r(y) := \left(\sum_{i=1}^{d-1} |y^i|^2\right)^{1/2}$. For convenience, we use the same symbols $\mathcal{D}$ and $\mathcal{C}$ to denote the desired domains in $\mathbb{R}^d$ and their intersections with $\mathcal{P}$.

Consider the three parameters $\alpha \in (0, \pi/2)$, $\eta > 0$, $\varepsilon \in (0, \pi/2 - \alpha)$, and the domains $\mathcal{D}_{\alpha,\eta,\varepsilon}$, $\mathcal{C}_{\alpha,\eta}$, $\mathcal{A}_{\alpha,\varepsilon}$ given in Figure 2, which satisfy the following properties:

- $\mathcal{C}_{\alpha,\eta}$ is obtained from a square centered at $(0, -1 - \eta - \sin(\alpha))$, with the sides being parallel to the axes and having length $2\sin(\alpha)+2\eta$, by rounding its corners (in their η-neighborhoods), such that $\partial\mathcal{C}_{\alpha,\eta}$ is a C^2 curve and $\mathcal{C}_{\alpha,\eta}$ is convex,
- $\mathcal{D}_{\alpha,\eta,\varepsilon}$ is symmetric with respect to the axis y_d.
- $\partial\mathcal{D}_{\alpha,\eta,\varepsilon}$ is C^2 and is made up of the following pieces:
  - the arc $\mathcal{S}_\alpha$ of angle 2α, symmetric with respect to the axis y_d, of the circle centered at zero and with the radius one,
  - the arc of angle 2α, symmetric with respect to the axis y_d, of the circle centered at zero and with the radius $(2\sin(\alpha) + 2\eta + 1)/\cos(\alpha)$,
  - and two smooth curves $\mathcal{L}_1$ and $\mathcal{L}_{2,}$, symmetric to each other with respect to the axis y_d, which connect the two arcs described above forming a C^2 closed curve that does not intersect itself nor $\mathcal{C}_{\alpha,\eta}$.
- We denote by A^1 (respectively, A^2) the end point of the curve $\mathcal{S}_\alpha$ that belongs to the right (respectively, left) half-plane with respect to the axis y_d.
- Let us assume that $\mathcal{L}_1$ (respectively, $\mathcal{L}_2$) belongs to the right (respectively, left) half-plane with respect to the axis y_d. We also assume that the curve $\mathcal{L}_1$ is constructed so that, in its natural parameterization with the starting point A^1, the slope of its tangent vector has exactly one change of monotonicity. Namely, we assume that there exists a point B^1_ε, such that the angle between B^1_ε and A^1 relative to the origin is ε and such that the derivative of the slope of the aforementioned tangent vector is continuous, nonincreasing, and equal to zero at B^1_ε. The curve $\mathcal{L}_2$, then, satisfies the analogous property due to symmetry, with the associated point B^2_ε.
- As the curve $\partial\mathcal{D}_{\alpha,\eta,\varepsilon}$ is C^2, closed, and without self-intersections, we construct ϕ as the signed distance to $\partial\mathcal{D}_{\alpha,\eta,\varepsilon}$ in a neighborhood of $\partial\mathcal{D}_{\alpha,\eta,\varepsilon}$ and, then, extend it in a smooth way to $\mathbb{R}^2$. $\phi_{\mathcal{C}}$ is constructed similarly.
- We define $\mathcal{S}_{\alpha,\varepsilon}$ as the concatenation of the curves $\overset{\frown}{B^2_\varepsilon A^2}$, $\mathcal{S}_\alpha$, $\overset{\frown}{A^1 B^1_\varepsilon}$, and we define $\mathcal{A}_{\alpha,\varepsilon}$ as the interior of the convex hull of $\mathcal{S}_{\alpha,\varepsilon}$.

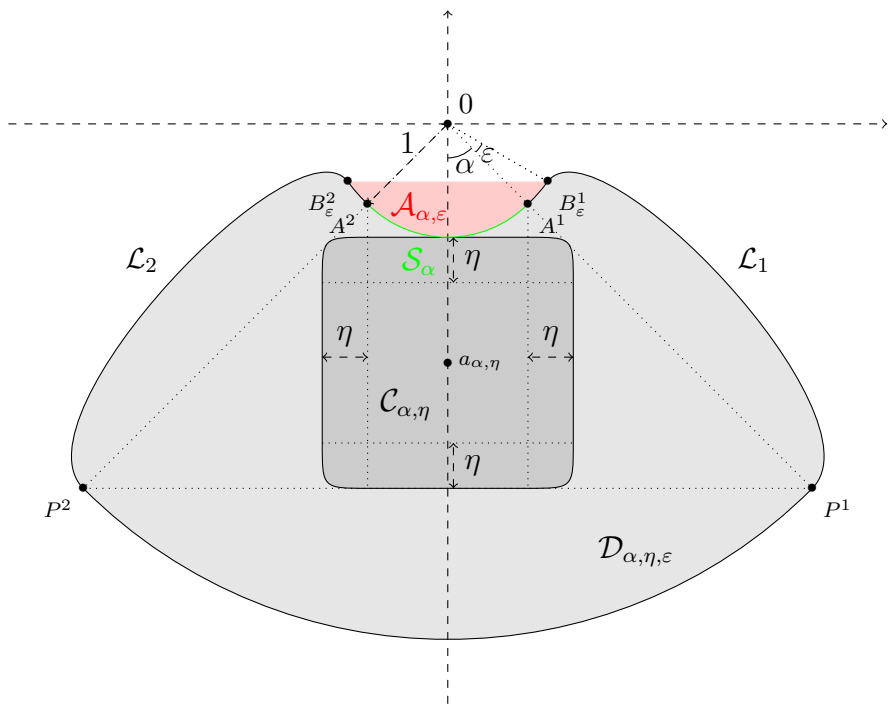


FIGURE 2. Domains $\mathcal{D}_{\alpha,\eta,\varepsilon}$, $\mathcal{C}_{\alpha,\eta}$, $\mathcal{A}_{\alpha,\varepsilon}$

- Finally, we assume that $\eta > 0$ is small enough, so that $\mathcal{C}_{\alpha,\eta}$ is included in the triangle with vertices P^1, P^2 and the origin, as shown in Figure 2. This ensures that $\bar{\mathcal{C}}_{\alpha,\eta} \subset \mathcal{D}_{\alpha,\eta,\varepsilon}$ for any $\varepsilon > 0$.

Let us now consider a terminal condition $\xi \in \mathcal{S}_\alpha$ and verify that $\mathcal{D}_{\alpha,\eta,\varepsilon}$, $\mathcal{A}_{\alpha,\varepsilon}$ and ξ satisfy the desired assumptions. We easily deduce that $R_0 = 1$. Then, for any $\alpha \in (0, \pi/2)$ and $\eta > 0$, there exists $\varepsilon_0 \in (0, \pi/2 - \alpha)$, such that, for all $0 < \varepsilon < \varepsilon_0$, the condition (1.1) holds up to the shift of coordinates in $\mathbb{R}^d$ that maps the origin to $a_{\alpha,\eta} := (0, ..., 0, -1-\eta-\sin(\alpha))$. The other conditions of Assumption 1.1 follow easily.

Next, we notice that, in the discrete path-dependent framework and under Assumption 4.1, we can apply Theorem 4.1 to conclude that there exists a unique (in $\mathscr{U}(1)$) triplet $(Y^\varepsilon, Z^\varepsilon, K^\varepsilon) \in \mathscr{S}^2 \times \mathscr{H}^2 \times \mathscr{K}^1$ that solves (1.2) in the domain $\mathcal{D} = \mathcal{D}_{\alpha,\eta,\varepsilon}$ (we suppress the dependence of the solution on η and α as they are fixed in what follows). An application of Proposition 5.1 to $\mathcal{D} = \mathcal{D}_{\alpha,\eta,\varepsilon}$ and $\mathcal{A} = \mathcal{A}_{\alpha,\varepsilon}$ (the assumptions of the proposition are satisfied by the construction of $\mathcal{A}_{\alpha,\varepsilon}$, $\mathcal{L}_1$ and $\mathcal{L}_2$) yields that Y^ε takes all its values in $\mathcal{S}_{\alpha,\varepsilon}$. Then, the stability result of Proposition 2.2 implies that $\{(Y^{1/n}, Z^{1/n}, K^{1/n})\}_{n=1}^\infty$ is a Cauchy sequence and, hence, has a limit (Y, Z, K). It is clear that Y stays in $\mathcal{S}_\alpha$. Then, applying the arguments similar to those used in the proof of Theorem 3.1, one can deduce

that (Y, Z, K) solves the reflected BSDE (1.2) in the domain $\mathcal{D} = \mathcal{D}_{\alpha,\eta,\varepsilon'}$, for any $\varepsilon' \in (0, \varepsilon_0)$. Applying Proposition 5.1 once more and recalling that Y takes all its values in $\mathcal{S}_\alpha$, we conclude that Y is a Γ-martingale on the manifold $\mathcal{S}_\alpha$ with the terminal condition ξ. The uniqueness part of Theorem 4.1 yields that such a Γ-martingale is unique (in $\mathscr{U}(1)$).

We now study the case of a general terminal condition. We first notice that Proposition 2.1 holds for any solution $(Y, Z, K) \in \mathscr{S}^2 \times \mathscr{H}^2 \times \mathscr{K}^1$ of (1.2) that stays in $\mathcal{S}_\alpha$ and satisfies Assumption 2.1(i) with γ replaced by

$$\gamma_\alpha := \inf_{y \in \mathcal{S}_\alpha} \nabla\phi_{\mathcal{C}}(y) \cdot \frac{\nabla\phi(y)}{|\nabla\phi(y)|}.$$

We can easily compute $\gamma_\alpha = \cos(\alpha)$. Moreover, we have $R_0 = 1$ and

$$|\phi^+_{\mathcal{C}}(\xi)|_{\mathscr{L}^\infty} \leqslant 1 - \cos(\alpha).$$

Thus, we conclude that Assumption 2.1(i) is fulfilled with $\theta = 2$ as long as $\cos(\alpha) > 2/3$. Considering a sequence of discrete path-dependent terminal conditions that approximate the given (general) terminal condition and take values in $\mathcal{S}_\alpha$, we repeat the proof of Theorem 4.2 obtaining the unique (in $\mathscr{U}(2)$) triplet $(Y, Z, K) \in \mathscr{S}^2 \times \mathscr{H}^2 \times \mathscr{K}^1$ that solves (1.2) in the domain $\mathcal{D} = \mathcal{D}_{\alpha,\eta,\varepsilon'}$, for any $\varepsilon' \in (0, \varepsilon_0)$, and is such that Y stays in $\mathcal{S}_\alpha$. Applying Proposition 5.1 once more, we conclude that Y is a Γ-martingale on the manifold $\mathcal{S}_\alpha$ with the terminal condition ξ. The uniqueness part of Theorem 4.1 yields that such a Γ-martingale is unique in $\mathscr{U}(2)$.

To sum up, the above construction proves the existence and uniqueness of a Brownian Γ-martingale with a prescribed discrete path-dependent terminal condition ξ, satisfying Assumption 4.1, on any sector of the sphere $\mathbb{S}^{d-1}$ (we understand a sector as an intersection of a sphere and a half-space) that is strictly contained in a hemisphere. For a general terminal condition ξ, we are only able to tackle the case $\alpha < \arccos(2/3)$. These results provide an alternative proof of some of the facts established in [32, 45], where the existence and uniqueness is shown for any $\alpha < \pi/2$. Considering the case $\alpha = \pi/2$, we notice that, for any $\mathcal{D}$ that is included in the compliment of an open ball and whose boundary contains a hemisphere (on the boundary of this ball), it is impossible to find a convex domain $\mathcal{C} \subset \mathcal{D}$ that can "see" all points on the boundary of this hemisphere with a strictly positive angle: in other words, (1.1) can not be fulfilled. In particular, our existence and uniqueness results fail for such $\mathcal{D}$. On the other hand, considering directly the problem of existence and uniqueness of a Brownian Γ-martingale with a prescribed terminal condition on a closed hemisphere of $\mathbb{S}^{d-1}$, we notice a major challenge that stems from the non-uniqueness of geodesics, when $d \geqslant 3$. Indeed, let us assume that ξ takes its values in the set $\{z_1, z_2\}$ consisting of two antipodes on the sphere: i.e., the line connecting the two points goes through the center of the sphere. Note that $\mathcal{S}_{\pi/2}$ does contain such points. Then, for any

shortest arc $\widehat{z_1 z_2} \subset \mathcal{S}_{\pi/2}$, there exists a Γ-martingale on the manifold $\widehat{z_1 z_2}$ with the terminal condition ξ. As any such arc $\widehat{z_1 z_2}$ is a geodesic, we conclude that the resulting Γ-martingale is also a Γ-martingale in the larger manifold $\mathcal{S}_{\pi/2}$. Assuming that ξ takes each of its two values with a strictly positive probability and recalling that there are infinitely many geodesic arcs $\widehat{z_1 z_2}$ on $\mathcal{S}_{\pi/2}$, we conclude that the uniqueness of a Γ-martingale on $\mathbb{S}^{d-1}$ with the terminal condition ξ does not hold. Proposition 5.1, in turn, implies that the uniqueness fails for solutions to (1.2) with the terminal condition ξ, with $f \equiv 0$, and with $\mathcal{D}$ described above. This observation, in particular, illustrates the sharpness of the weak star-shape assumption (condition (1.1) in Assumption 1.1) for general terminal conditions and general $d \geqslant 2$.

Let us also mention that the non-uniqueness described above does not occur for $d = 2$, which indicates that it may be possible to relax our assumptions for reflected BSDEs in planar non-convex domains. In particular, we refer to [44] for a complete treatment of Γ-martingales on $\mathbb{S}^1$. The latter result also yields the existence and uniqueness of a solution to the reflected BSDE in the domain $\mathcal{D} = \{y \in \mathbb{R}^2, 1 < |y| < 2\}$, which does not possess the weak star-shape property, with zero generator and with a terminal condition satisfying $|\xi| = 1$.

Moreover, in Section 3 of [45], Picard was able to prove the existence and uniqueness of a Brownian Γ-martingale with a prescribed terminal condition in a closed hemisphere of $\mathbb{S}^{d-1}$, and in an even bigger domain, for a small enough T and under a smoothness assumption on the terminal condition[3]. The latter indicates that in a smooth Markovian or discrete path-dependent framework, under an additional smallness assumption, it may also be possible to relax the requirement of a weak star-shape property even for $d > 2$.

Finally, let us give a simple example showing that *a priori* estimates of Proposition 2.1 are not sharp.[4] Mimicking [44], we consider a $\mathcal{F}_T$-measurable random variable ν with values in $[-\alpha, \alpha]$, where $0 < \alpha < \pi/2$ is a given parameter, and let $(\theta_t, \eta_t)_{t\in[0,T]}$ be the solution of the BSDE $\theta_t = \nu - \int_t^T \eta_s \mathrm{d}W_s$ for $t \in [0, T]$. We set $Y_t = (\cos(\theta_t), \sin(\theta_t))^\top$ for all $t \in [0, T]$, and we easily check that Y is a solution to the BSDE

$$Y_t = \xi + \int_t^T \frac{|Z_s|^2}{2} Y_s \mathrm{d}s - \int_t^T Z_s \mathrm{d}W_s, \quad 0 \leqslant t \leqslant T,$$

where $\xi = (\cos(\nu), \sin(\nu))^\top$ and $Z_t = (-\eta_t \sin(\theta_t), \eta_t \cos(\theta_t))^\top$. Notice that this multidimensional quadratic BSDE can also be seen as a reflected BSDE in the domain $\mathcal{D}_{\alpha,\eta,\varepsilon}$, with sufficiently small $\eta, \varepsilon > 0$, rotated by $\pi/2$. Indeed, Y takes

[3]To be precise, it is assumed that the process Z, defined by $\xi = \mathbb{E}[\xi] + \int_0^T Z_s \mathrm{d}W_s$, has sufficiently small $\int_0^T \operatorname{ess\,sup}_\Omega |Z_s|^2 \mathrm{d}s$.

[4]Note that these estimates are not needed in a Markovian or discrete path-dependent case.

all its values in (rotated) $\mathcal{S}_\alpha$, and its drift points along the outer normal vector to (rotated) $\mathcal{S}_\alpha$. Recall that $\mathcal{D}_{\alpha,\eta,\varepsilon}$ satisfies the weak star-shape property and note that $\mathrm{dVar}_t(K) = \frac{|\eta_t|^2}{2}$. Then, an application of Itô's formula to θ_t^2 yields

$$\mathbb{E}_t\left[\int_t^T \mathrm{dVar}_s(K)\right] = \frac{1}{2}\mathbb{E}_t\left[\int_t^T |\eta_s|^2 \mathrm{d}s\right] = \frac{1}{2}\mathbb{E}_t\left[\nu^2 - (\mathbb{E}_t\nu)^2\right] \leqslant \frac{\alpha^2}{2}.$$

Moreover, the above becomes an equality for $t = 0$ and $\nu = \mathrm{sign}(W_T)\alpha$. Then, recalling that $R_0 = 1$ for $\mathcal{D}_{\alpha,\eta,\varepsilon}$, we deduce from John-Nirenberg inequality that

$$\mathbb{E}\left[e^{\frac{2p}{R_0}\mathrm{Var}_T(K)}\right] < \infty,$$

for some $p > 1$, provided $\alpha < 1$, which is weaker than the condition $\alpha < \arccos(2/3) < 1$ required by Assumption 2.1(i) with $\theta = 2$, as computed earlier in this subsection.

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