

EXISTENCE AND INCOMPRESSIBLE LIMIT OF A TISSUE  
GROWTH MODEL WITH AUTOPHAGY\*JIAN-GUO LIU<sup>†</sup> AND XIANGSHENG XU<sup>‡</sup>

**Abstract.** In this paper we study a cross-diffusion system whose coefficient matrix is non-symmetric and degenerate. The system arises in the study of tissue growth with autophagy. The existence of a weak solution is established. We also investigate the limiting behavior of solutions as the pressure gets stiff. The so-called incompressible limit is a free boundary problem of Hele-Shaw type. Our key new discovery is that the usual energy estimate still holds as long as the time variable stays away from 0.

**Key words.** autophagy, existence, incompressible limit, tissue growth models

**AMS subject classifications.** 35B45, 35B65, 35Q92, 35K51

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**1. Introduction.** Let  $\Omega$  be a bounded domain in  $\mathbb{R}^N$  with Lipschitz boundary  $\partial\Omega$  and  $T$  any positive number. We consider the initial boundary value problem

$$\begin{aligned}
 (1.1) \quad & \partial_t n_1 - \operatorname{div}(n_1 \nabla p) = G(d)n_1 - K_1(d)n_1 + K_2(d)n_2 \equiv R_1 \\
 & \text{in } \Omega_T \equiv \Omega \times (0, T), \\
 (1.2) \quad & \partial_t n_2 - \operatorname{div}(n_2 \nabla p) = (G(d) - D)n_2 + K_1(d)n_1 - K_2(d)n_2 \equiv R_2 \text{ in } \Omega_T, \\
 (1.3) \quad & b\partial_t d - \Delta d = -\psi(d)n + an_2 \text{ in } \Omega_T, \\
 (1.4) \quad & n_1 \nabla p \cdot \mathbf{n} = n_2 \nabla p \cdot \mathbf{n} = 0 \text{ on } \Sigma_T \equiv \partial\Omega \times (0, T), \\
 (1.5) \quad & d = d_b \text{ on } \Sigma_T, \\
 (1.6) \quad & (n_1(x, 0), n_2(x, 0), d(x, 0)) = (n_{01}(x), n_{02}(x), d_0(x)) \text{ on } \Omega,
 \end{aligned}$$

where  $\mathbf{n}$  is the unit outward normal to  $\partial\Omega$  and

$$(1.7) \quad n = n_1 + n_2, \quad p = n^\gamma, \quad \gamma \geq 1.$$

This problem was proposed as a tissue growth model with autophagy in [9]. In the model, cells are classified into two phases: normal cells and autophagic cells, and  $n_1, n_2$  are their respective densities. The third unknown function  $d$  represents the concentration of nutrients. We assume that both cells have the same birth rate. Their death rates are different because autophagic cells have an extra death rate  $D$  due to the “self-eating” mechanism. Thus if  $G(d)$  is the net growth rate of normal cells, then  $G(d) - D$  gives the net growth rate for autophagic cells. Two types of cells can change from one to another. The transition rates are denoted by  $K_1(d), K_2(d)$ ,

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respectively. Since autophagy is a reversible process, we have

$$(1.8) \quad K_1(d) \geq 0, \quad K_2(d) \geq 0.$$

Both cells consume nutrients with the consumption rate  $\psi(d)$ . However, autophagic cells also provide nutrients by degrading its own constituents with a supply rate  $a$ . We assume

$$(1.9) \quad D, a \in (0, \infty).$$

Moreover,

$$(1.10) \quad \psi(0) = 0, \quad \psi(d) \text{ is increasing, and there is } d_0 > 0 \text{ such that } \psi(d_0) = a.$$

Condition in (1.10) means that when there is no nutrient the consumption rate should be zero. The number  $d_0$  is the so-called critical nutrient concentration. When  $d < d_0$ , autophagic cells supply more nutrients than they consume, while  $d > d_0$  indicates that autophagic cells consume more nutrients than they supply.

For the spatial motion of cells, we take a fluid mechanical point of view. That is, it is driven by a velocity field equal to the negative gradient of the pressure (Darcy's law) [15]. And the pressure arises from mechanical contact between cells. Denote by  $p$  the pressure. Then we can assume that (1.7), (1.1), and (1.2) hold.

One can also model tissue growth as free boundary problems [10]. They are also called geometric or incompressible models and describe tissue as a moving domain (see [6] and the references therein). Building a link between these two classes of models has attracted the attention of many researchers in recent years. The first result in this direction was obtained in [15] for a purely mechanical model. It indicates that the limit of the mechanical model gives rise to a free boundary problem as the pressure becomes stiff. Since then the same result has been achieved for a variety of models, which include active motion [16], viscosity [18], different laws of state [8], more than one species of cells [4], and multispace dimensions and viscosity [7]. In each case the limit model turns out to be a free boundary model of Hele-Shaw type.

The objective of this paper is to study the existence assertion for (1.1)–(1.6) and the limiting behavior of solutions as  $\gamma \rightarrow \infty$ .

We largely follow the approach adopted in [19] for the existence assertion. To understand the nature of the limiting model for our problem, we define a family of maximal monotone graphs [2] in  $\mathbb{R} \times \mathbb{R}$  by

$$\varphi_\gamma(s) = (s^+)^{\gamma+1} = \begin{cases} s^{\gamma+1} & \text{if } s \geq 0, \\ 0 & \text{if } s < 0. \end{cases}$$

Obviously,

$$(1.11) \quad \varphi_\gamma(s) \rightarrow \varphi_\infty(s) \equiv \begin{cases} [0, \infty) & \text{if } s = 1, \\ 0 & \text{if } s < 1 \end{cases}$$

in the sense of graphs as  $\gamma \rightarrow \infty$  [2]. The total density  $n = n^{(\gamma)}$  satisfies the problem

$$(1.12) \quad \begin{aligned} \partial_t n^{(\gamma)} - \frac{\gamma}{\gamma+1} \Delta v^{(\gamma)} &= G(d^{(\gamma)}) n^{(\gamma)} - D n_2^{(\gamma)} \equiv R^{(\gamma)} \quad \text{in } \Omega_T, \\ v^{(\gamma)} &= \left( n^{(\gamma)} \right)^{\gamma+1} \quad \text{a.e. on } \Omega_T, \\ \nabla v^{(\gamma)} \cdot \mathbf{n} &= 0 \quad \text{on } \Sigma_T, \\ n^{(\gamma)}(x, 0) &= n_0 \equiv n_{01} + n_{02} \quad \text{on } \Omega. \end{aligned}$$

Thus if we formally take  $\gamma \rightarrow \infty$ , we expect to arrive at the following problem:

$$(1.13) \quad \partial_t n^{(\infty)} - \Delta v^{(\infty)} = G(d^{(\infty)}) n^{(\infty)} - D n_2^{(\infty)} \equiv R^{(\infty)} \quad \text{in } \Omega_T,$$

$$(1.14) \quad v^{(\infty)} \in \varphi_\infty(n^{(\infty)}) \quad \text{a.e. on } \Omega_T,$$

$$(1.15) \quad \nabla v^{(\infty)} \cdot \mathbf{n} = 0 \quad \text{on } \Sigma_T,$$

$$(1.16) \quad n^{(\infty)}(x, 0) = n_0 \quad \text{on } \Omega.$$

If  $n_0 \leq 1$  a.e on  $\Omega$ , a result of [3] asserts that the limit problem (1.13)–(1.16) has an integral solution  $n^{(\infty)}$  and  $\lim_{\gamma \rightarrow \infty} n^{(\gamma)} = n^{(\infty)}$  in  $L^1(0, T; L^1(\Omega))$  (also see [23] for related results). If  $n_0 > 1$  on a set of positive measure, the initial condition is no longer compatible with  $\varphi_\infty$  and the resulting problem (1.13)–(1.16) becomes singular. Thus identifying the limit of the sequence  $\{n^{(\gamma)}\}$  is an interesting issue. When  $R^{(\gamma)} \equiv 0$ , this problem was solved in [5] through an application of the Aronson–Bénilan inequality [1]

$$(1.17) \quad \partial_t n^{(\gamma)} \geq -\frac{n^{(\gamma)}}{\gamma t}.$$

The precise result there is the following: If  $\Omega = \mathbb{R}^N$ ,  $n_0(x)$  has a star-shaped profile, and  $R^{(\gamma)} = 0$ , then  $n^{(\infty)} \equiv \lim_{\gamma \rightarrow \infty} n^{(\gamma)}$  exists and is given by

$$n^{(\infty)}(x) = \begin{cases} 1 & \text{if } x \in A, \\ n_0(x) & \text{if } x \notin A, \end{cases}$$

where  $A$  is the coincident set of the solution of the following variational inequalities:

$$-\Delta w \geq n_0 - 1, \quad w \geq 0, \quad (\Delta w + n_0 - 1) w = 0 \quad \text{in } \mathbb{R}^N.$$

A remarkable fact is that the limit  $n^{(\infty)}$  is a function of  $x$  only. A similar result was established for hyperbolic conservation laws in [22]. However, if  $R^{(\gamma)}$  changes sign, inequalities of the Aronson–Bénilan type no longer hold [17]. To circumvent this difficulty, the authors of [6] established a weaker version of (1.17) along with an  $L^4$  estimate for the gradient of the pressure. Our problem here does not quite fit the framework developed in [6]. This forces us to take a totally different approach. It seems more convenient for us to work with  $v^{(\gamma)} = (n^{(\gamma)})^{\gamma+1}$  instead of the pressure. Our key estimate is

$$\int_\tau^T \int_\Omega \left( v^{(\gamma)} \right)^2 dx dt + \int_\tau^T \int_\Omega \left| \nabla v^{(\gamma)} \right|^2 dx dt \leq \frac{c}{\tau} \quad \text{for all } \gamma \geq 1 \text{ and } \tau \in (0, T).$$

Here and in what follows the letter  $c$  denotes a generic positive constant whose value is determined by the given data. That is, the sequence  $\{v^{(\gamma)}\}$  is bounded in  $L^2(\tau, T; W^{1,2}(\Omega))$  for each  $\tau \in (0, T)$ .

Before we introduce our remaining results, we state the definition of a weak solution.

**DEFINITION 1.1.** *We say that  $(n_1, n_2, d)$  is a weak solution to (1.1)–(1.6) if the following:*

(D1)  $n_1, n_2, d$  are all nonnegative and bounded with

(1.18)

$$\partial_t n_1, \partial_t n_2, \partial_t d \in L^2(0, T; (W^{1,2}(\Omega))^*), \quad n^{\frac{\gamma+1}{2}}, d \in L^2(0, T; W^{1,2}(\Omega)),$$

where  $n$  is given as in (1.7) and  $(W^{1,2}(\Omega))^*$  denotes the dual space of  $W^{1,2}(\Omega)$ .

(D2) *There hold*

$$\begin{aligned}
& - \int_{\Omega_T} n_1 \partial_t \xi_1 dxdt + \int_{\Omega_T} n_1 \nabla n^\gamma \cdot \nabla \xi_1 dxdt \\
&= \int_{\Omega_T} R_1 \xi_1 dxdt - \langle n_1(\cdot, T), \xi_1(\cdot, T) \rangle + \int_{\Omega} n_{01}(x) \xi_1(x, 0) dx \\
&\quad \text{for each } \xi_1 \in H^1(0, T; W^{1,2}(\Omega)) \\
& - \int_{\Omega_T} n_2 \partial_t \xi_2 dxdt + \int_{\Omega_T} n_2 \nabla n^\gamma \cdot \nabla \xi_2 dxdt \\
&= \int_{\Omega_T} R_2 \xi_2 dxdt - \langle n_2(\cdot, T), \xi_2(\cdot, T) \rangle + \int_{\Omega} n_{02}(x) \xi_2(x, 0) dx \\
&\quad \text{for each } \xi_2 \in H^1(0, T; W^{1,2}(\Omega)), \text{ and} \\
& - b \int_{\Omega_T} d \partial_t \zeta dxdt + \int_{\Omega_T} \nabla d \cdot \nabla \zeta dxdt \\
&= \int_{\Omega_T} (-\psi(d)n + an_2) \zeta dxdt - b \langle d(\cdot, T), \zeta(\cdot, T) \rangle + b \int_{\Omega} d_0(x) \zeta(x, 0) dx \\
&\quad \text{for each } \zeta \in H^1(0, T; W_0^{1,2}(\Omega)) ,
\end{aligned}$$

where  $\langle \cdot, \cdot \rangle$  denotes the duality pairing between  $W^{1,2}(\Omega)$  and  $(W^{1,2}(\Omega))^*$  and  $H^1(0, T; W^{1,2}(\Omega)) = \{v \in L^2(0, T; W^{1,2}(\Omega)) : \partial_t v \in L^2(0, T; W^{1,2}(\Omega))\}$ .

(D3) (1.5) is satisfied.

To see that the three equations in (D2) make sense, we can conclude from (D1) that  $n_1, n_2, d \in C([0, T]; (W^{1,2}(\Omega))^*)$ . Since  $n$  is bounded and  $\gamma \geq \frac{\gamma+1}{2}$ , we also have  $n^\gamma \in L^2(0, T; W^{1,2}(\Omega))$ .

THEOREM 1.2. *Assume the following:*

- (H1)  $G, K_1, K_2, \psi$  are all continuous functions.
- (H2) (1.8), (1.9), and (1.10) hold.
- (H3)  $b \in (0, \infty)$  and  $\partial\Omega$  is Lipschitz.
- (H4)  $n_{01}, n_{02} \in W^{1,2}(\Omega) \cap L^\infty(\Omega)$ ,  $d_0 \in L^\infty(\Omega)$ , and  $d_b \in L^2(0, T; W^{1,2}(\Omega)) \cap L^\infty(\Omega_T)$ .

Then there is a weak solution to (1.1)–(1.6).

Set

$$(1.19) \quad L = \max\{\|d_b\|_{\infty, \Sigma_T}, \|d_0\|_{\infty, \Omega}, d_0\},$$

$$(1.20) \quad G_0 = \max_{s \in [0, L]} G(s).$$

THEOREM 1.3. *Let the assumptions of Theorem 1.2 hold. Assume the following:*

- (H5)  $G'(s)$  is bounded.
- (H6)  $d_b \in W^{1,s}(\Omega_T)$  for some  $s > N + 2$  and  $d_0 \in W^{1,\infty}(\Omega)$ .
- (H7)  $|\{n_0(x) \geq \sigma\}| \leq \frac{1}{e^{G_0 T} \|n_0\|_{\infty, \Omega}} |\Omega|$  for some  $\sigma \in (0, e^{-G_0 T})$ .
- (H8)  $\partial\Omega$  is  $C^{1,1}$ .

Denote by  $(n^{(\gamma)}, n_1^{(\gamma)}, n_2^{(\gamma)}, d^{(\gamma)})$  the solution obtained in Theorem 1.2. Then there is a subsequence of  $(n^{(\gamma)}, n_1^{(\gamma)}, n_2^{(\gamma)}, d^{(\gamma)})$ , which will not be relabeled, such that

$$(1.21) \quad \begin{aligned} (n^{(\gamma)}, n_1^{(\gamma)}, n_2^{(\gamma)}) &\rightarrow (n^{(\infty)}, n_1^{(\infty)}, n_2^{(\infty)}) \text{ weak* in } (L^\infty(\Omega_T))^3 \\ &\text{and strongly in } \left(C([\tau, T]; (W^{1,2}(\Omega))^*)\right)^3 \text{ for each } \tau \in (0, T), \end{aligned}$$

$$(1.22) \quad v^{(\gamma)} \rightarrow v^{(\infty)} \text{ weakly in } L^2(\tau, T; W^{1,2}(\Omega)) \text{ for each } \tau \in (0, T),$$

$$(1.23) \quad \nabla v^{(\gamma)} \rightarrow \nabla v^{(\infty)} \text{ strongly in } L^2(\tau, T; (L^2(\Omega))^N) \text{ for each } \tau \in (0, T),$$

$$(1.24) \quad \frac{n_2^{(\gamma)}}{n^{(\gamma)}} \rightarrow \eta^{(\infty)} \text{ weak* in } L^\infty(\Omega_T),$$

$$(1.25) \quad d^{(\gamma)} \rightarrow d^{(\infty)} \text{ weak* in } L^\infty(0, T; W^{1,\infty}(\Omega)) \text{ and strongly in } L^2(\Omega_T).$$

The limit  $(n^{(\infty)}, v^{(\infty)}, n_1^{(\infty)}, n_2^{(\infty)}, \eta^{(\infty)}, d^{(\infty)})$  satisfies

$$\begin{aligned} - \int_{\Omega_T} n^{(\infty)} \partial_t \xi_1 dx dt + \int_{\Omega_T} \nabla v^{(\infty)} \cdot \nabla \xi_1 dx dt &= \int_{\Omega_T} R^{(\infty)} \xi_1 dx dt \\ - \int_{\Omega_T} n_1^{(\infty)} \partial_t \xi_2 dx dt + \int_{\Omega_T} (1 - \eta^{(\infty)}) \nabla v^{(\infty)} \cdot \nabla \xi_2 dx dt &= \int_{\Omega_T} R_1^{(\infty)} \xi_2 dx dt \\ - \int_{\Omega_T} n_2^{(\infty)} \partial_t \xi_3 dx dt + \int_{\Omega_T} \eta^{(\infty)} \nabla v^{(\infty)} \cdot \nabla \xi_3 dx dt &= \int_{\Omega_T} R_2^{(\infty)} \xi_3 dx dt, \text{ and} \\ -b \int_{\Omega_T} d^{(\infty)} \partial_t \xi_4 dx dt + \int_{\Omega_T} \nabla d^{(\infty)} \cdot \nabla \xi_4 dx dt &= \int_{\Omega_T} (-\psi(d^{(\infty)}) n^{(\infty)} \\ &\quad + a n_2^{(\infty)}) \xi_4 dx dt \\ &\quad - b \langle d^{(\infty)}(\cdot, T), \xi_4(\cdot, T) \rangle \\ &\quad + b \int_{\Omega} d_0(x) \xi_4(x, 0) dx \end{aligned}$$

for each  $(\xi_1, \xi_2, \xi_3) \in (H^1(0, T; W^{1,2}(\Omega)))^3$  with  $(\xi_1, \xi_2, \xi_3) = 0$  near  $t = 0$  and  $(\xi_1, \xi_2, \xi_3)|_{t=T} = 0$  and each  $\xi_4 \in H^1(0, T; W_0^{1,2}(\Omega))$ , where  $R^{(\infty)}$  is given as in (1.13) and

$$\begin{aligned} R_1^{(\infty)} &= G(d^{(\infty)}) n_1^{(\infty)} - K_1(d^{(\infty)}) n_1^{(\infty)} + K_2(d^{(\infty)}) n_2^{(\infty)}, \\ R_2^{(\infty)} &= (G(d^{(\infty)}) - D) n_2^{(\infty)} + K_1(d^{(\infty)}) n_1^{(\infty)} - K_2(d^{(\infty)}) n_2^{(\infty)}. \end{aligned}$$

Moreover, (1.14) holds and

$$(1.26) \quad v^{(\infty)} \left( \Delta v^{(\infty)} + R^{(\infty)} \right) = 0.$$

If we compare the equations in (D2) with the ones here, two pieces are missing. One is that we are no longer able to identify the initial conditions for  $(n^{(\infty)}, n_1^{(\infty)}, n_2^{(\infty)})$ . This is to be expected due to the fact that  $\varphi_\infty$  is not defined on the set  $\{n_0 > 1\}$ . A redeeming feature is that we can view (1.26), the so-called complementary condition, as some kind of compensation for this lack of initial conditions. More significantly, this condition connects our limits to the geometric form of the Hele-Shaw problem [6]. At least formally, it says

$$-\Delta v^{(\infty)} = R^{(\infty)} \quad \text{on } \Omega(t) \equiv \{v^{(\infty)}(x, t) > 0\}.$$

The second one is that we have not been able to show

$$(1.27) \quad \eta^{(\infty)} = \frac{n_2^{(\infty)}}{n^{(\infty)}}.$$

This can be derived from the precompactness of  $\{n^{(\gamma)}\}$  in some  $L^q(\Omega_T)$  space with  $q \in [1, \infty)$  (see the proof of (2.59) in section 2 below). Unfortunately, this result is not available to us because in the generality considered here the sequence  $\{\nabla n^{(\gamma)}\}$  cannot be shown to be bounded in a function space. Furthermore, it does not seem to be possible to obtain any estimates on  $\partial_t v^{(\gamma)}$  that are uniform in  $\gamma$ . As a result, the precompactness of  $\{v^{(\gamma)}\}$  in some  $L^q(\Omega_T)$  space is also an issue. This is so in spite of the fact that we have (1.23).

We can easily see that (1.14) is equivalent to the following:

$$(1.28) \quad n^{(\infty)} \leq 1 \quad \text{on } \Omega_T \text{ and}$$

$$(1.29) \quad (1 - n^{(\infty)}) v^{(\infty)} = 0 \quad \text{on } \Omega_T.$$

Obviously, we can no longer expect  $n^{(\infty)}$  to be independent of  $t$  due to the presence of  $R^{(\infty)}$ . The term  $\Delta v^{(\infty)}$  may be a pure distribution. We define

$$v^{(\infty)} \Delta v^{(\infty)} = \operatorname{div} \left( v^{(\infty)} \nabla v^{(\infty)} \right) - \left| \nabla v^{(\infty)} \right|^2 \quad \text{in the sense of distributions.}$$

Also note that the assumption (H7) implies that  $n_0$  is close to 0 on a large set. The smaller  $T$  is, the easier it is for (H7) to hold.

The remainder of the paper is devoted to the proof of the above two theorems. To be specific, section 2 contains the proof of Theorem 1.2, while Theorem 1.3 is established in section 3.

**2. Existence of a global weak solution and proof of Theorem 1.2.** The proof will be divided into several lemmas. Before we begin, we state the following three well known results.

**LEMMA 2.1.** *Let  $h(s)$  be a convex and lower semicontinuous function on  $\mathbb{R}$  [13]. Assume that*

- (C1)  $f \in W_2(0, T) \equiv \{\varphi \in L^2(0, T; W^{1,2}(\Omega)) : \partial_t \varphi \in L^2(0, T; (W^{1,2}(\Omega))^*)\};$
- (C2)  $g \in L^2(0, T; W^{1,2}(\Omega))$  with the property  $g(x, t) \in \partial h(f(x, t))$  for a.e  $(x, t) \in \Omega_T$ , where  $\partial h$  is the subgradient of  $h$ .

*Then the function  $t \mapsto \int_{\Omega} h(f(x, t)) dx$  is absolutely continuous on  $[0, T]$  and*

$$(2.1) \quad \frac{d}{dt} \int_{\Omega} h(f) dx = \langle \partial_t f, g \rangle.$$

If  $h(s) = s^2$ , this lemma is a special case of the well known Lions–Magenes lemma ([21, pp. 176–177]). Formula (2.1) is trivial if  $f$  is smooth. The general case can be established by suitable approximation. See ([13, p. 101]) for the details.

**LEMMA 2.2 (Lions–Aubin).** *Let  $X_0, X$ , and  $X_1$  be three Banach spaces with  $X_0 \subseteq X \subseteq X_1$ . Suppose that  $X_0$  is compactly embedded in  $X$  and that  $X$  is continuously embedded in  $X_1$ . For  $1 \leq p, q \leq \infty$ , let*

$$W_{p,q}(0, T) = \{u \in L^p([0, T]; X_0) : \partial_t u \in L^q([0, T]; X_1)\}.$$

Then the following hold:

- (i) If  $p < \infty$ , then the embedding of  $W_{p,q}(0, T)$  into  $L^p([0, T]; X)$  is compact.
- (ii) If  $p = \infty$  and  $q > 1$ , then the embedding of  $W_{p,q}(0, T)$  into  $C([0, T]; X)$  is compact.

The proof of this lemma can be found in [20]. We mention in passing that Lemmas 2.1 and 2.2 imply that  $W_2(0, T)$  is contained in  $C([0, T]; L^2(\Omega))$ .

LEMMA 2.3. *Let  $\Omega$  be a bounded domain in  $\mathbb{R}^N$  with Lipschitz boundary and  $1 \leq p < N$ . Then there is a positive number  $c = c(N)$  such that*

$$\|u - u_S\|_{p^*} \leq \frac{cd^{N+1-\frac{p}{N}}}{|S|^{\frac{1}{p}}} \|\nabla u\|_p \quad \text{for each } u \in W^{1,p}(\Omega),$$

where  $S$  is any measurable subset of  $\Omega$  with  $|S| > 0$ ,  $u_S = \frac{1}{|S|} \int_S u dx$ , and  $d$  is the diameter of  $\Omega$ .

This lemma can be inferred from Lemma 7.16 in [12].

Our approximate problems are similar to those in [19]. For each  $\varepsilon > 0$ , we consider

$$(2.2) \quad \begin{aligned} \partial_t n - \varepsilon \Delta n &= \gamma \operatorname{div}(n^\gamma \nabla n) + G(d)n_1 + (G(d) - D)n_2 \quad \text{in } \Omega_T, \\ \partial_t n_1 - \varepsilon \Delta n_1 &= \gamma \operatorname{div}(n_1 n^{\gamma-1} \nabla n) + G(d)n_1 - K_1(d)n_1 \end{aligned}$$

$$(2.3) \quad \begin{aligned} &+ K_2(d)n_2 \quad \text{in } \Omega_T, \\ \partial_t n_2 - \varepsilon \Delta n_2 &= \gamma \operatorname{div}(n_2 n^{\gamma-1} \nabla n) + (G(d) - D)n_2 + K_1(d)n_1 \end{aligned}$$

$$(2.4) \quad - K_2(d)n_2 \quad \text{in } \Omega_T,$$

$$(2.5) \quad b \partial_t d - \Delta d = -\psi(d)n + a n_2 \quad \text{in } \Omega_T,$$

$$(2.6) \quad \nabla n \cdot \mathbf{n} = \nabla n_1 \cdot \mathbf{n} = \nabla n_2 \cdot \mathbf{n} = 0 \quad \text{on } \Sigma_T,$$

$$(2.7) \quad d = d_b \quad \text{on } \Sigma_T,$$

$$(2.8) \quad (n, n_1, n_2, d)|_{t=0} = (n_0(x), n_{01}(x), n_{02}(x), d_0(x)) \quad \text{on } \Omega.$$

LEMMA 2.4. *Assume that (H1)–(H4) hold. Then for each fixed  $\varepsilon > 0$  there exists a quadruplet  $(n, n_1, n_2, d)$  in the function space  $(W_2(0, T))^4 \cap (L^\infty(\Omega_T))^4$  such that (2.2)–(2.8) are all satisfied in the sense of Definition 1.1.*

*Proof.* This lemma will be established via the Leray–Schauder fixed point theorem ([12, p. 280]). For this purpose, we introduce a cut-off function

$$(2.9) \quad \theta_\ell(s) = \begin{cases} 0 & \text{if } s \leq 0, \\ s & \text{if } 0 < s < \ell, \\ \ell & \text{if } s \geq \ell, \end{cases}$$

where  $\ell > 0$  will be selected as below. We define an operator  $\mathbb{M}$  from  $(L^2(\Omega_T))^4$  into itself as follows: Let  $(w, v_1, v_2, u) \in (L^2(\Omega_T))^4$ . We first consider the initial boundary

value problem

$$(2.10) \quad \begin{aligned} \partial_t n - \operatorname{div} \left[ \varepsilon + \gamma (\theta_\ell(v_1) + \theta_\ell(v_2)) \theta_\ell^{\gamma-1}(w) \nabla n \right] &= \theta_\ell(v_1) G(\theta_\ell(u)) \\ &+ (G(\theta_\ell(u)) - D) \theta_\ell(v_2) \text{ in } \Omega_T, \end{aligned}$$

$$(2.11) \quad \begin{aligned} \nabla n \cdot \mathbf{n} &= 0 \text{ on } \Sigma_T, \\ n(x, 0) &= n_0(x) \text{ on } \Omega. \end{aligned}$$

For given  $(w, v_1, v_2, u)$  the above problem for  $n$  is linear and uniformly parabolic. Thus we can conclude from the classical result ([14, Chap. III]) that there is a unique weak solution  $n$  to (2.10)–(2.11) in the space  $W_2(0, T)$ . Use the function  $n$  so obtained to form the following two initial boundary problems:

$$(2.12) \quad \begin{aligned} \partial_t n_1 - \varepsilon \Delta n_1 &= \gamma \operatorname{div} \left[ \theta_\ell(v_1) \theta_\ell^{\gamma-1}(w) \nabla n \right] + (G(\theta_\ell(u)) - K_1(\theta_\ell(u))) \theta_\ell(v_1) \\ &+ \theta_\ell(v_2) K_2(\theta_\ell(u)) \text{ in } \Omega_T, \end{aligned}$$

$$\nabla n_1 \cdot \mathbf{n} = 0 \text{ on } \Sigma_T,$$

$$n_1(x, 0) = n_{01}(x) \text{ on } \Omega,$$

$$(2.13) \quad \begin{aligned} \partial_t n_2 - \varepsilon \Delta n_2 &= \gamma \operatorname{div} \left[ \theta_\ell(v_2) \theta_\ell^{\gamma-1}(w) \nabla n \right] + (G(\theta_\ell(u)) - K_2(\theta_\ell(u)) - D) \theta_\ell(v_2) \\ &+ \theta_\ell(v_1) K_1(\theta_\ell(u)) \text{ in } \Omega_T, \end{aligned}$$

$$(2.14) \quad \nabla n_2 \cdot \mathbf{n} = 0 \text{ on } \Sigma_T,$$

$$(2.15) \quad n_2(x, 0) = n_{02}(x) \text{ on } \Omega.$$

Each of the two problems here has a unique solution in  $W_2(0, T)$ . Then we solve the following linear problem:

$$b \partial_t d - \Delta d = -(\psi(\theta_\ell(u)) - a) \theta_\ell(w) - a \theta_\ell(v_1) \text{ in } \Omega_T,$$

$$d = d_b \text{ on } \Sigma_T,$$

$$d(x, 0) = d_0(x) \text{ on } \Omega.$$

We define  $(n, n_1, n_2, d) = \mathbb{M}(w, v_1, v_2, u)$ . Evidently,  $\mathbb{M}$  is well-defined.

**CLAIM 2.5.** *For each fixed pair  $\varepsilon > 0$  and  $\ell > 0$ , the operator  $\mathbb{M}$  is continuous and its range is precompact.*

*Proof.* The key observation here is that each initial boundary value problem in the definition of  $\mathbb{M}$  is linear and uniformly parabolic. This together with (H1) implies that  $\mathbb{M}$  is continuous. One can easily verify that the range of  $\mathbb{M}$  is bounded in  $(W_2(0, T))^4$ , which is compactly embedded in  $(L^2(\Omega_T))^4$ . It is similar to the proof of Lemma 2.4 in [19]. We shall omit the details.  $\square$

Now we are in a position to apply Corollary 11.2 in ([12, p. 280]), thereby obtaining that  $\mathbb{M}$  has a fixed point. That is, there is a  $(n, n_1, n_2, d)$  in  $(W_2(0, T))^4$  such that

$$(2.16) \quad \begin{aligned} \partial_t n - \varepsilon \Delta n &= \gamma \operatorname{div} \left[ (\theta_\ell(n_1) + \theta_\ell(n_2)) \theta_\ell^{\gamma-1}(n) \nabla n \right] + \theta_\ell(n_1) G(\theta_\ell(d)) \\ &\quad + (G(\theta_\ell(d)) - D) \theta_\ell(n_2) \text{ in } \Omega_T, \end{aligned}$$

$$(2.17) \quad \begin{aligned} \nabla n \cdot \mathbf{n} &= 0 \text{ on } \Sigma_T, \\ n(x, 0) &= n_0(x) \text{ on } \Omega, \\ \partial_t n_1 - \varepsilon \Delta n_1 &= \gamma \operatorname{div} \left[ \theta_\ell(n_1) \theta_\ell^{\gamma-1}(n) \nabla n \right] + (G(\theta_\ell(d)) - K_1(\theta_\ell(d))) \theta_\ell(n_1) \\ &\quad + \theta_\ell(n_2) K_2(\theta_\ell(d)) \text{ in } \Omega_T, \end{aligned}$$

$$(2.18) \quad \begin{aligned} \nabla n_1 \cdot \mathbf{n} &= 0 \text{ on } \Sigma_T, \\ n_1(x, 0) &= n_{01}(x) \text{ on } \Omega, \\ \partial_t n_2 - \varepsilon \Delta n_2 &= \gamma \operatorname{div} \left[ \theta_\ell(n_2) \theta_\ell^{\gamma-1}(n) \nabla n \right] + (G(\theta_\ell(d)) - K_2(\theta_\ell(d)) - D) \theta_\ell(n_2) \\ &\quad + \theta_\ell(n_1) K_1(\theta_\ell(d)) \text{ in } \Omega_T, \end{aligned}$$

$$(2.19) \quad \nabla n_2 \cdot \mathbf{n} = 0 \text{ on } \Sigma_T,$$

$$(2.20) \quad n_2(x, 0) = n_{02}(x) \text{ on } \Omega,$$

$$(2.21) \quad b \partial_t d - \Delta d = -(\psi(\theta_\ell(d)) - a) \theta_\ell(n) - a \theta_\ell(n_1) \text{ in } \Omega_T,$$

$$d = d_b \text{ on } \Sigma_T,$$

$$(2.22) \quad d(x, 0) = d_0(x) \text{ on } \Omega.$$

Now we pick

$$(2.23) \quad \ell \geq L,$$

where  $L$  is given as in (1.19). Note that

$$\theta_\ell(d) = \min\{d, \ell\}.$$

On account of (1.10), we have

$$(\psi(\theta_\ell(d)) - a)(d - L)^+ = (\psi(\theta_\ell(d)) - \psi(d_0))(d - L)^+ \geq 0 \text{ in } \Omega_T.$$

With this in mind, we use  $(d - L)^+$  as a test function in (2.21) to derive

$$\begin{aligned} &\frac{b}{2} \frac{d}{dt} \int_{\Omega} [(d - L)^+]^2 dx + \int_{\Omega} |\nabla(d - L)^+|^2 dx \\ &= \int_{\Omega} [-(\psi(\theta_\ell(d)) - a) \theta_\ell(n) - a \theta_\ell(n_1)] (d - L)^+ dx \leq 0. \end{aligned}$$

Integrate to obtain

$$(2.24) \quad d \leq L \text{ in } \Omega_T.$$

Note that

$$\theta_\ell(n_1) = 0 \text{ in } \{n_1 \leq 0\}.$$

With this in mind, we use  $n_1^-$  as a test function in (2.12) to derive

$$-\frac{1}{2} \frac{d}{dt} \int_{\Omega} (n_1^-)^2 dx - \varepsilon \int_{\Omega} |\nabla n_1^-|^2 dx = \int_{\Omega} \theta_\ell(n_2) K_2(\theta_\ell(d)) n_1^- dx \geq 0.$$

Consequently,

$$n_1 \geq 0.$$

By the same token,

$$n_2 \geq 0.$$

Use  $d^-$  as a test function in (2.21) to get

$$\begin{aligned} & -\frac{b}{2} \frac{d}{dt} \int_{\Omega} (d^-)^2 dx - \int_{\Omega} |\nabla d^-|^2 dx \\ &= \int_{\Omega} [-(\psi(\theta_{\ell}(d)) - a)\theta_{\ell}(n) - a\theta_{\ell}(n_1)] d^- dx \\ &= a \int_{\Omega} [\theta_{\ell}(n) - \theta_{\ell}(n_1)] d^- dx \geq 0. \end{aligned}$$

Here we have used the fact that  $\psi(0) = 0$ . Integrate to obtain

$$(2.25) \quad d \geq 0 \text{ in } \Omega_T.$$

This together with (2.24) implies

$$(2.26) \quad \theta_{\ell}(d) = d.$$

Add (2.17) to (2.18) and subtract the resulting equation from (2.16) to derive

$$\partial_t(n - (n_1 + n_2)) - \varepsilon \Delta(n - (n_1 + n_2)) = 0 \text{ in } \Omega_T.$$

Recall the initial boundary conditions for  $(n - (n_1 + n_2))$  to deduce

$$(2.27) \quad n = n_1 + n_2.$$

Let  $\lambda \in (0, \infty)$ , and define

$$(2.28) \quad w = e^{-\lambda t} n.$$

We easily check that  $w$  satisfies

$$\begin{aligned} & \partial_t w + \lambda w - \varepsilon \Delta w = \gamma \operatorname{div} \left[ (\theta_{\ell}(n_1) + \theta_{\ell}(n_2)) \theta_{\ell}^{\gamma-1} (e^{\lambda t} w) \nabla w \right] + e^{-\lambda t} \theta_{\ell}(n_1) G(d) \\ (2.29) \quad & + e^{-\lambda t} (G(d) - D) \theta_{\ell}(n_2) \text{ in } \Omega_T, \\ & \nabla w \cdot \mathbf{n} = 0 \text{ on } \Sigma_T, \\ & w(x, 0) = n_0(x) \text{ on } \Omega. \end{aligned}$$

Set

$$(2.30) \quad M_0 = \max \left\{ \max_{d \in [0, L]} |G(d)|, \max_{d \in [0, L]} |G(d) - D| \right\}.$$

Then the last two terms in (2.29) can be estimated as follows:

$$\begin{aligned} & |e^{-\lambda t} \theta_{\ell}(n_1) G(d) + e^{-\lambda t} (G(d) - D) \theta_{\ell}(n_2)| \\ & \leq e^{-\lambda t} \theta_{\ell}(n_1) |G(d)| + e^{-\lambda t} |G(d) - D| \theta_{\ell}(n_2) \\ & \leq M_0 e^{-\lambda t} (\theta_{\ell}(n_1) + \theta_{\ell}(n_2)) \\ & \leq 2M_0 e^{-\lambda t} \theta_{\ell}(n) \leq 2M_0 w. \end{aligned}$$

It immediately follows that

$$\partial_t w + (\lambda - 2M_0)w - \operatorname{div} \left[ \varepsilon + \gamma(\theta_\ell(n_1) + \theta_\ell(n_2))\theta_\ell^{\gamma-1}(e^{\lambda t}w)\nabla w \right] \leq 0 \text{ in } \Omega_T.$$

Choose  $\lambda = 2M_0$ . Then use  $(w - \|n_0\|_{\infty, \Omega})^+$  as a test function in the above differential inequality to derive

$$w \leq \|n_0\|_{\infty, \Omega} \text{ a.e. in } \Omega_T.$$

This immediately implies

$$(2.31) \quad n \leq e^{2M_0 T} \|n_0\|_{\infty, \Omega} \text{ a.e. in } \Omega_T.$$

Thus if, in addition to (2.23), we further require

$$(2.32) \quad \ell \geq e^{2M_0 T} \|n_0\|_{\infty, \Omega},$$

then

$$\theta_\ell(n) = n, \quad \theta_\ell(n_1) = n_1, \quad \theta_\ell(n_2) = n_2$$

and problem (2.16)–(2.22) reduces to problem (2.2)–(2.8). This completes the proof of Lemma 2.4.  $\square$

Let  $\varepsilon \in (0, 1)$ . Replace  $n_{01}(x)$  by  $n_{01}(x) + \varepsilon$  in (2.8) and denote the resulting solution to (2.2)–(2.8) by  $(n^{(\varepsilon)}, n_1^{(\varepsilon)}, n_2^{(\varepsilon)}, d^{(\varepsilon)})$ . That is, we have

$$(2.33) \quad \begin{aligned} \partial_t n^{(\varepsilon)} - \varepsilon \Delta n^{(\varepsilon)} &= \gamma \operatorname{div} \left[ \left( n^{(\varepsilon)} \right)^\gamma \nabla n^{(\varepsilon)} \right] + G(d^{(\varepsilon)}) n_1^{(\varepsilon)} \\ &\quad + (G(d^{(\varepsilon)}) - D) n_2^{(\varepsilon)} \text{ in } \Omega_T, \end{aligned}$$

$$(2.34) \quad \begin{aligned} \partial_t n_1^{(\varepsilon)} - \varepsilon \Delta n_1^{(\varepsilon)} &= \gamma \operatorname{div} \left[ n_1^{(\varepsilon)} \left( n^{(\varepsilon)} \right)^{\gamma-1} \nabla n^{(\varepsilon)} \right] \\ &\quad + G(d^{(\varepsilon)}) n_1^{(\varepsilon)} - K_1(d^{(\varepsilon)}) n_1^{(\varepsilon)} + K_2(d^{(\varepsilon)}) n_2^{(\varepsilon)} \text{ in } \Omega_T, \\ \partial_t n_2^{(\varepsilon)} - \varepsilon \Delta n_2^{(\varepsilon)} &= \gamma \operatorname{div} \left[ n_2^{(\varepsilon)} \left( n^{(\varepsilon)} \right)^{\gamma-1} \nabla n^{(\varepsilon)} \right] + (G(d^{(\varepsilon)}) - D) n_2^{(\varepsilon)} \end{aligned}$$

$$(2.35) \quad + K_1(d^{(\varepsilon)}) n_1^{(\varepsilon)} - K_2(d^{(\varepsilon)}) n_2^{(\varepsilon)} \text{ in } \Omega_T,$$

$$(2.36) \quad b \partial_t d^{(\varepsilon)} - \Delta d^{(\varepsilon)} = -\psi(d^{(\varepsilon)}) n^{(\varepsilon)} + a n_2^{(\varepsilon)} \text{ in } \Omega_T,$$

$$(2.37) \quad \nabla n^{(\varepsilon)} \cdot \mathbf{n} = \nabla n_1^{(\varepsilon)} \cdot \mathbf{n} = \nabla n_2^{(\varepsilon)} \cdot \mathbf{n} = 0 \text{ on } \Sigma_T,$$

$$(2.38) \quad d^{(\varepsilon)} = d_b \text{ on } \Sigma_T,$$

$$(2.39) \quad (n^{(\varepsilon)}, n_1^{(\varepsilon)}, n_2^{(\varepsilon)}, d^{(\varepsilon)}) \Big|_{t=0} = (n_0(x) + \varepsilon, n_{01}(x) + \varepsilon, n_{02}(x), d_0(x)) \text{ on } \Omega.$$

In addition, we have

$$(2.40) \quad \begin{aligned} n_1^{(\varepsilon)} &\geq 0, \quad n_2^{(\varepsilon)} \geq 0, \quad n^{(\varepsilon)} = n_1^{(\varepsilon)} + n_2^{(\varepsilon)} \leq c, \\ 0 &\leq d^{(\varepsilon)} \leq c. \end{aligned}$$

Here and in what follows the letter  $c$  is independent of  $\varepsilon$ . As we shall see, the addition of  $\varepsilon$  in (2.39) is to ensure that  $n^{(\varepsilon)}$  stays away from 0 below.

LEMMA 2.6. *We have*

$$\int_{\Omega_T} \left| \nabla \left( n^{(\varepsilon)} \right)^{\frac{\gamma+1}{2}} \right|^2 dxdt + \varepsilon \int_{\Omega_T} \left( \left| \nabla \sqrt{n_1^{(\varepsilon)}} \right|^2 + \left| \nabla \sqrt{n_2^{(\varepsilon)}} \right|^2 \right) dxdt \leq c.$$

*Proof.* Pick  $\tau > 0$ . Use  $\ln(n_1^{(\varepsilon)} + \tau)$  as a test function in (2.34) to derive

$$\begin{aligned} & \frac{d}{dt} \int_{\Omega} \left( (n_1^{(\varepsilon)} + \tau) \ln(n_1^{(\varepsilon)} + \tau) - n_1^{(\varepsilon)} \right) dx + \int_{\Omega} \frac{n_1^{(\varepsilon)}}{n_1^{(\varepsilon)} + \tau} \nabla \left( n^{(\varepsilon)} \right)^{\gamma} \nabla n_1^{(\varepsilon)} dx \\ & + \varepsilon \int_{\Omega} \frac{1}{n_1^{(\varepsilon)} + \tau} |\nabla n_1^{(\varepsilon)}|^2 \\ & = \int_{\Omega} \left( G(d^{(\varepsilon)}) n_1^{(\varepsilon)} - K_1(d^{(\varepsilon)}) n_1^{(\varepsilon)} + K_2(d^{(\varepsilon)}) n_2^{(\varepsilon)} \right) \ln(n_1^{(\varepsilon)} + \tau) dx \\ & \leq \int_{\Omega} \left| \left( G(d^{(\varepsilon)}) - K_1(d^{(\varepsilon)}) \right) n_1^{(\varepsilon)} \ln(n_1^{(\varepsilon)} + \tau) \right| dx \\ & + \int_{\{n_1^{(\varepsilon)} + \tau \geq 1\}} K_2(d^{(\varepsilon)}) n_2^{(\varepsilon)} \ln(n_1^{(\varepsilon)} + \tau) dx \\ & \leq 3C_0 \int_{\Omega} n^{(\varepsilon)} (n_1^{(\varepsilon)} + \tau) dx + 2C_0 \int_{\{n_1^{(\varepsilon)} + \tau \leq 1\}} |n_1^{(\varepsilon)} \ln n_1^{(\varepsilon)}| dx \leq c. \end{aligned}$$

Here

$$(2.41) \quad C_0 = \max \left\{ \max_{d \in [0, L]} |G(d)|, \max_{d \in [0, L]} K_1(d), \max_{d \in [0, L]} K_2(d) \right\}.$$

Integrate and take  $\tau \rightarrow 0$  to get

$$\int_{\Omega_T} \nabla \left( n^{(\varepsilon)} \right)^{\gamma} \cdot \nabla n_1^{(\varepsilon)} dxdt + 4\varepsilon \int_{\Omega_T} \left| \nabla \sqrt{n_1^{(\varepsilon)}} \right|^2 dxdt \leq c.$$

Similarly,

$$\int_{\Omega_T} \nabla \left( n^{(\varepsilon)} \right)^{\gamma} \cdot \nabla n_2^{(\varepsilon)} dxdt + 4\varepsilon \int_{\Omega_T} \left| \nabla \sqrt{n_2^{(\varepsilon)}} \right|^2 dxdt \leq c.$$

Add up the two preceding inequalities to obtain the desired result.  $\square$

LEMMA 2.7. *The sequences  $\{n^{(\varepsilon)}\}$  and  $\{d^{(\varepsilon)}\}$  are precompact in  $L^p(\Omega_T)$  for each  $p \geq 1$ .*

*Proof.* It follows from (2.30) and (2.33) that

$$(2.42) \quad \partial_t n^{(\varepsilon)} - \varepsilon \Delta n^{(\varepsilon)} \geq \gamma \operatorname{div} \left[ \left( n^{(\varepsilon)} \right)^{\gamma} \nabla n^{(\varepsilon)} \right] - M_0 n^{(\varepsilon)} \quad \text{in } \Omega_T.$$

Let  $w^{(\varepsilon)} = e^{M_0 t} n^{(\varepsilon)}$ . Then we have

$$(2.43) \quad \partial_t w^{(\varepsilon)} - \varepsilon \Delta w^{(\varepsilon)} \geq \gamma \operatorname{div} \left[ \left( n^{(\varepsilon)} \right)^{\gamma} \nabla w^{(\varepsilon)} \right] \quad \text{in } \Omega_T.$$

Use  $(\varepsilon - w^{(\varepsilon)})^+$  as a test function in (2.43) to get

$$(2.44) \quad -\frac{1}{2} \frac{d}{dt} \int_{\Omega} \left[ (\varepsilon - w^{(\varepsilon)})^+ \right]^2 dx - \gamma \int_{\Omega} \left( n^{(\varepsilon)} \right)^{\gamma} |\nabla (\varepsilon - w^{(\varepsilon)})^+|^2 dx - \varepsilon \int_{\Omega} |\nabla (\varepsilon - w^{(\varepsilon)})^+|^2 dx \geq 0.$$

Recall from (2.39) that  $w^{(\varepsilon)}(x, 0) = n^{(\varepsilon)}(x, 0) \geq \varepsilon$ . Integrate to obtain

$$(2.45) \quad n^{(\varepsilon)} \geq \varepsilon e^{-M_0 T}.$$

Consequently,  $(n^{(\varepsilon)})^r \in L^2(0, T; W^{1,2}(\Omega))$  for each  $r \in \mathbb{R}$ . We derive from (2.33) that

$$\begin{aligned} \partial_t \left( n^{(\varepsilon)} \right)^{\frac{\gamma+1}{2}} &= \frac{\gamma+1}{2} \left( n^{(\varepsilon)} \right)^{\frac{\gamma+1}{2}-1} \partial_t n^{(\varepsilon)} \\ &= \frac{\gamma+1}{2} \operatorname{div} \left[ \left( n^{(\varepsilon)} \right)^{\frac{\gamma+1}{2}} \nabla \left( n^{(\varepsilon)} \right)^\gamma \right] - \frac{\gamma+1}{2} \nabla \left( n^{(\varepsilon)} \right)^{\frac{\gamma+1}{2}} \cdot \nabla \left( n^{(\varepsilon)} \right)^\gamma \\ &\quad + \frac{(\gamma+1)\varepsilon}{2} \operatorname{div} \left[ \left( n^{(\varepsilon)} \right)^{\frac{\gamma+1}{2}-1} \nabla n^{(\varepsilon)} \right] - \frac{(\gamma+1)\varepsilon}{2} \nabla \left( n^{(\varepsilon)} \right)^{\frac{\gamma+1}{2}-1} \cdot \nabla n^{(\varepsilon)} \\ &\quad + \frac{\gamma+1}{2} \left( n^{(\varepsilon)} \right)^{\frac{\gamma+1}{2}-1} (G(d^{(\varepsilon)}) n_1^{(\varepsilon)} + (G(d^{(\varepsilon)}) - D) n_2^{(\varepsilon)}) \\ &= \gamma \operatorname{div} \left[ \left( n^{(\varepsilon)} \right)^\gamma \nabla \left( n^{(\varepsilon)} \right)^{\frac{\gamma+1}{2}} \right] - \frac{\gamma(\gamma-1)}{\gamma+1} \left( n^{(\varepsilon)} \right)^{\frac{\gamma+1}{2}-1} \left| \nabla \left( n^{(\varepsilon)} \right)^{\frac{\gamma+1}{2}} \right|^2 \\ &\quad + \varepsilon \Delta \left( n^{(\varepsilon)} \right)^{\frac{\gamma+1}{2}} - (\gamma^2 - 1)\varepsilon \left( n^{(\varepsilon)} \right)^{\frac{\gamma+1}{2}-1} \left| \nabla \sqrt{n^{(\varepsilon)}} \right|^2 \\ (2.46) \quad &\quad + \frac{\gamma+1}{2} \left( n^{(\varepsilon)} \right)^{\frac{\gamma+1}{2}-1} (G(d^{(\varepsilon)}) n_1^{(\varepsilon)} + (G(d^{(\varepsilon)}) - D) n_2^{(\varepsilon)}). \end{aligned}$$

Remember that  $\frac{\gamma+1}{2} - 1 > 0$ . We can conclude from Lemma 2.6 that the sequence  $\{\partial_t(n^{(\varepsilon)})^{\frac{\gamma+1}{2}}\}$  is bounded in  $L^2(0, T; (W^{1,2}(\Omega))^*) + L^1(\Omega_T) \equiv \{\psi_1 + \psi_2 : \psi_1 \in L^2(0, T; (W^{1,2}(\Omega))^*), \psi_2 \in L^1(\Omega_T)\}$ . Now we are in a position to use (i) in Lemma 2.2, thereby obtaining the precompactness of  $\{(n^{(\varepsilon)})^{\frac{\gamma+1}{2}}\}$  in  $L^2(\Omega_T)$ .

It is easy to see from (2.36) that  $\{d^{(\varepsilon)}\}$  is bounded in  $W_2(0, T)$ . The lemma follows from (2.40).  $\square$

We may extract a subsequence of  $\{(n^{(\varepsilon)}, n_1^{(\varepsilon)}, n_2^{(\varepsilon)}, d^{(\varepsilon)})\}$ , still denoted by the same notation, such that

$$(2.47) \quad n^{(\varepsilon)} \rightarrow n \text{ a.e. in } \Omega_T \text{ and strongly in } L^p(\Omega_T) \text{ for each } p \geq 1,$$

$$(2.48) \quad d^{(\varepsilon)} \rightarrow d \text{ a.e. in } \Omega_T \text{ and strongly in } L^p(\Omega_T) \text{ for each } p \geq 1,$$

$$n_1^{(\varepsilon)} \rightarrow n_1 \text{ weak* in } L^\infty(\Omega_T),$$

$$n_2^{(\varepsilon)} \rightarrow n_2 \text{ weak* in } L^\infty(\Omega_T), \text{ and}$$

$$(2.49) \quad \left( n^{(\varepsilon)} \right)^{\frac{\gamma+1}{2}} \rightarrow n^{\frac{\gamma+1}{2}} \text{ weakly in } L^2(0, T; W^{1,2}(\Omega)) \text{ as } \varepsilon \rightarrow 0.$$

Since  $\{n^{(\varepsilon)}\}$  is bounded, we also have

$$\left( n^{(\varepsilon)} \right)^p \rightarrow n^p \text{ weakly in } L^2(0, T; W^{1,2}(\Omega)) \text{ for each } p \geq \frac{\gamma+1}{2}.$$

This combined with (2.43) implies

$$\partial_t n^{(\varepsilon)} \rightarrow \partial_t n \text{ weakly in } L^2(0, T; (W^{1,2}(\Omega))^*).$$

Remember that  $G, K_1, K_2, \psi$  are all continuous functions. We also have

$$(2.50) \quad G(n^{(\varepsilon)}) \rightarrow G(n) \text{ strongly in } L^p(\Omega_T) \text{ for each } p \geq 1,$$

$$(2.51) \quad \psi(n^{(\varepsilon)}) \rightarrow \psi(n) \text{ strongly in } L^p(\Omega_T) \text{ for each } p \geq 1, \text{ and}$$

$$(2.52) \quad K_i(n^{(\varepsilon)}) \rightarrow K_i(n) \text{ strongly in } L^p(\Omega_T) \text{ for each } p \geq 1, i = 1, 2.$$

Our key result is the following.

LEMMA 2.8. *Passing to a subsequence if necessary, we have*

$$\left(n^{(\varepsilon)}\right)^{\gamma+1} \rightarrow n^{\gamma+1} \text{ strongly in } L^2(0, T; W^{1,2}(\Omega)).$$

*Proof.* We have

$$(2.53) \quad n^{(\varepsilon)} \nabla(n^{(\varepsilon)})^\gamma = \frac{\gamma}{\gamma+1} \nabla \left(n^{(\varepsilon)}\right)^{\gamma+1}.$$

Thus we can write (2.33) in the form

$$(2.54) \quad \partial_t n^{(\varepsilon)} - \frac{\gamma}{\gamma+1} \Delta w^{(\varepsilon)} = R^{(\varepsilon)},$$

where

$$\begin{aligned} w^{(\varepsilon)} &= \left(n^{(\varepsilon)}\right)^{\gamma+1} + \frac{\varepsilon(\gamma+1)}{\gamma} n^{(\varepsilon)}, \\ R^{(\varepsilon)} &= \left(G(d^{(\varepsilon)}) n_1^{(\varepsilon)} + (G(d^{(\varepsilon)}) - D) n_2^{(\varepsilon)}\right). \end{aligned}$$

We may assume that  $n^{(\varepsilon)}$  is a classical solution to (2.54) because it can be viewed as the limit of a sequence of classical approximate solutions. Use  $\partial_t w^{(\varepsilon)}$  as a test function in (2.54) to derive

$$(2.55) \quad \int_{\Omega} \partial_t n^{(\varepsilon)} \partial_t w^{(\varepsilon)} dx + \frac{\gamma}{\gamma+1} \int_{\Omega} \nabla w^{(\varepsilon)} \cdot \nabla \partial_t w^{(\varepsilon)} dx = \int_{\Omega} R^{(\varepsilon)} \partial_t w^{(\varepsilon)} dx.$$

We proceed to evaluate each integral in the above equation as follows:

$$\begin{aligned} \int_{\Omega} \partial_t n^{(\varepsilon)} \partial_t w^{(\varepsilon)} dx &= (\gamma+1) \int_{\Omega} \left(n^{(\varepsilon)}\right)^\gamma \left(\partial_t n^{(\varepsilon)}\right)^2 dx \\ &\quad + \frac{\varepsilon(\gamma+1)}{\gamma} \int_{\Omega} \left(\partial_t n^{(\varepsilon)}\right)^2 dx, \\ \int_{\Omega} \nabla w^{(\varepsilon)} \cdot \nabla \partial_t w^{(\varepsilon)} dx &= \frac{1}{2} \frac{d}{dt} \int_{\Omega} |\nabla w^{(\varepsilon)}|^2 dx, \\ \int_{\Omega} R^{(\varepsilon)} \partial_t w^{(\varepsilon)} dx &= (\gamma+1) \int_{\Omega} R^{(\varepsilon)} \left(n^{(\varepsilon)}\right)^\gamma \partial_t n^{(\varepsilon)} dx \\ &\quad + \frac{\varepsilon(\gamma+1)}{\gamma} \int_{\Omega} R^{(\varepsilon)} \partial_t n^{(\varepsilon)} dx \\ &\leq \frac{\gamma+1}{2} \int_{\Omega} \left(n^{(\varepsilon)}\right)^\gamma \left(\partial_t n^{(\varepsilon)}\right)^2 dx \\ &\quad + \frac{\gamma+1}{2} \int_{\Omega} \left(n^{(\varepsilon)}\right)^\gamma \left(R^{(\varepsilon)}\right)^2 dx \\ &\quad + \frac{\varepsilon(\gamma+1)}{2\gamma} \int_{\Omega} \left(\partial_t n^{(\varepsilon)}\right)^2 dx + \frac{\varepsilon(\gamma+1)}{2\gamma} \int_{\Omega} \left(R^{(\varepsilon)}\right)^2 dx. \end{aligned}$$

Plug the preceding three results into (2.55) and integrate to derive

$$\int_{\Omega_T} \left( \partial_t \left( n^{(\varepsilon)} \right)^{\frac{\gamma+2}{2}} \right)^2 dx dt + \varepsilon \int_{\Omega_T} \left( \partial_t n^{(\varepsilon)} \right)^2 dx dt + \sup_{0 \leq t \leq T} \int_{\Omega} \left| \nabla w^{(\varepsilon)} \right|^2 dx \leq c.$$

Note

$$\begin{aligned} \partial_t \left( n^{(\varepsilon)} \right)^{\gamma+1} &= 2 \left( n^{(\varepsilon)} \right)^{\frac{\gamma+2}{2}} \partial_t \left( n^{(\varepsilon)} \right)^{\frac{\gamma+2}{2}}, \\ \nabla \left( n^{(\varepsilon)} \right)^{\gamma+1} &= (\gamma+1) \left( n^{(\varepsilon)} \right)^{\gamma} \nabla n^{(\varepsilon)}. \end{aligned}$$

On account of (2.40),  $\{\partial_t(n^{(\varepsilon)})^{\gamma+1}\}$  is bounded in  $L^2(\Omega_T)$ , while  $\{(n^{(\varepsilon)})^{\gamma+1}\}$  is bounded in  $L^\infty(0, T; W^{1,2}(\Omega))$ . By (ii) in Lemma 2.2, the sequence  $\{(n^{(\varepsilon)})^{\gamma+1}\}$  is precompact in  $C([0, T], L^2(\Omega))$ . Consequently,  $\{(n^{(\varepsilon)})^{\gamma+1}\}$  is precompact in  $C([0, T], L^p(\Omega))$  for each  $p \geq 1$ . This asserts

$$(2.56) \quad \int_{\Omega} \left( n^{(\varepsilon)}(x, t) \right)^q dx \rightarrow \int_{\Omega} n^q(x, t) dx \quad \text{for each } t \in [0, T] \text{ and each } q \geq \gamma + 1$$

(pass to a subsequence if need be).

Take  $\varepsilon \rightarrow 0$  in (2.54) to obtain

$$\partial_t n - \frac{\gamma}{\gamma+1} \Delta n^{\gamma+1} = R \equiv G(d)n_1 + (G(d) - D)n_2.$$

Subtract this equation from (2.54) and keep (2.53) in mind to get

$$(2.57) \quad \partial_t(n^{(\varepsilon)} - n) - \frac{\gamma}{\gamma+1} \Delta \left[ \left( n^{(\varepsilon)} \right)^{\gamma+1} - n^{\gamma+1} \right] - \varepsilon \Delta n^{(\varepsilon)} = R^{(\varepsilon)} - R.$$

Use  $(n^{(\varepsilon)})^{\gamma+1} - n^{\gamma+1}$  as a test function in (2.57) to derive

$$\begin{aligned} & \frac{\gamma}{\gamma+1} \int_{\Omega_T} \left| \nabla \left[ \left( n^{(\varepsilon)} \right)^{\gamma+1} - n^{\gamma+1} \right] \right|^2 dx dt \\ & + \varepsilon \int_{\Omega_T} \nabla n^{(\varepsilon)} \cdot \nabla \left[ \left( n^{(\varepsilon)} \right)^{\gamma+1} - n^{\gamma+1} \right] dx dt \\ & = \int_{\Omega_T} (R^{(\varepsilon)} - R) \left[ \left( n^{(\varepsilon)} \right)^{\gamma+1} - n^{\gamma+1} \right] dx dt \\ (2.58) \quad & - \int_0^T \left\langle \partial_t(n^{(\varepsilon)} - n), \left( n^{(\varepsilon)} \right)^{\gamma+1} - n^{\gamma+1} \right\rangle dt. \end{aligned}$$

We will show that the last three terms in the above equation all go to 0 as  $\varepsilon \rightarrow 0$ . It is easy to see from Lemma 2.6 that

$$\begin{aligned} & \left| \varepsilon \int_{\Omega_T} \nabla n^{(\varepsilon)} \cdot \nabla \left[ \left( n^{(\varepsilon)} \right)^{\gamma+1} - n^{\gamma+1} \right] dx dt \right| \\ & = 4\varepsilon \left| \int_{\Omega_T} \sqrt{n^{(\varepsilon)}} \nabla \sqrt{n^{(\varepsilon)}} \cdot \left[ \left( n^{(\varepsilon)} \right)^{\frac{\gamma+1}{2}} \nabla \left( n^{(\varepsilon)} \right)^{\frac{\gamma+1}{2}} - n^{\frac{\gamma+1}{2}} \nabla n^{\frac{\gamma+1}{2}} \right] dx dt \right| \\ & \leq c\sqrt{\varepsilon} \rightarrow 0 \quad \text{as } \varepsilon \rightarrow 0. \end{aligned}$$

Obviously, we have

$$\int_{\Omega_T} (R^{(\varepsilon)} - R) \left[ (n^{(\varepsilon)})^{\gamma+1} - n^{\gamma+1} \right] dx dt \rightarrow 0 \text{ as } \varepsilon \rightarrow 0.$$

Finally, we compute from Lemma 2.1 and (2.56) that

$$\begin{aligned} & \int_0^T \left\langle \partial_t(n^{(\varepsilon)} - n), (n^{(\varepsilon)})^{\gamma+1} - n^{\gamma+1} \right\rangle dt \\ &= \frac{1}{\gamma+2} \int_0^T \left[ \frac{d}{dt} \int_{\Omega} (n^{(\varepsilon)})^{\gamma+2} dx + \frac{d}{dt} \int_{\Omega} n^{\gamma+2} dx \right] dt \\ & \quad - \int_0^T \left\langle \partial_t n^{(\varepsilon)}, n^{\gamma+1} \right\rangle dt - \int_0^T \left\langle \partial_t n, (n^{(\varepsilon)})^{\gamma+1} \right\rangle dt \\ &= \frac{1}{\gamma+2} \left[ \int_{\Omega} (n^{(\varepsilon)}(x, T))^{\gamma+2} dx + \int_{\Omega} n^{\gamma+2}(x, T) dx \right] \\ & \quad - \frac{2}{\gamma+2} \int_{\Omega} (n_0(x))^{\gamma+2} dx - \int_0^T \left\langle \partial_t n^{(\varepsilon)}, n^{\gamma+1} \right\rangle dt \\ & \quad - \int_0^T \left\langle \partial_t n, (n^{(\varepsilon)})^{\gamma+1} \right\rangle dt \\ & \rightarrow \frac{2}{\gamma+2} \int_{\Omega} n^{\gamma+2}(x, T) dx - \frac{2}{\gamma+2} \int_{\Omega} (n_0(x))^{\gamma+2} dx - 2 \int_0^T \left\langle \partial_t n, n^{\gamma+1} \right\rangle dt \\ &= 0. \end{aligned}$$

This completes the proof.  $\square$

*Proof of Theorem 1.2.* Equipped with the preceding lemmas, we can complete the proof of Theorem 1.2. Keeping (2.45) in mind, we can set

$$\eta_1^{(\varepsilon)} = \frac{n_1^{(\varepsilon)}}{n^{(\varepsilon)}}, \quad \eta_2^{(\varepsilon)} = \frac{n_2^{(\varepsilon)}}{n^{(\varepsilon)}}.$$

Suppose

$$\eta_1^{(\varepsilon)} \rightarrow \eta_1, \quad \eta_2^{(\varepsilon)} \rightarrow \eta_2 \text{ weak* in } L^\infty(\Omega_T).$$

We calculate

$$\begin{aligned} n_1^{(\varepsilon)} \nabla (n^{(\varepsilon)})^\gamma &= \eta_1^{(\varepsilon)} n^{(\varepsilon)} \nabla (n^{(\varepsilon)})^\gamma \\ &= \frac{\gamma}{\gamma+1} \eta_1^{(\varepsilon)} \nabla (n^{(\varepsilon)})^{\gamma+1} \\ &\rightarrow \frac{\gamma}{\gamma+1} \eta_1 \nabla n^{\gamma+1} = \eta_1 n \nabla n^\gamma \text{ weakly in } (L^2(\Omega_T))^N. \end{aligned}$$

We claim that

$$(2.59) \quad \eta_1 n = n_1 \text{ a.e. on } \Omega_T.$$

To see this, for each  $\delta > 0$  we deduce from Lemma 2.7 that

$$\eta_1^{(\varepsilon)} (n^{(\varepsilon)} - \delta)^+ \rightarrow \eta_1 (n - \delta)^+ \text{ weak* in } L^\infty(\Omega_T).$$

Note that  $\frac{(n^{(\varepsilon)} - \delta)^+}{n^{(\varepsilon)}} \leq 1$ . As a result, we have

$$\eta_1^{(\varepsilon)}(n^{(\varepsilon)} - \delta)^+ = n_1^{(\varepsilon)} \frac{(n^{(\varepsilon)} - \delta)^+}{n^{(\varepsilon)}} \rightarrow n_1 \frac{(n - \delta)^+}{n} \text{ weak* in } L^\infty(\Omega_T).$$

We obtain

$$n_1 \frac{(n - \delta)^+}{n} = \eta_1(n - \delta)^+ \text{ for each } \delta > 0.$$

This implies that

$$n_1 = n\eta_1 \text{ on the set } \{n > 0\}.$$

If  $n = 0$ , then  $n_1 = 0$ , and we still have  $n_1 = n\eta_1$ . This completes the proof of (2.59). Similarly, we can show

$$n_2^{(\varepsilon)} \nabla \left( n^{(\varepsilon)} \right)^\gamma \rightarrow n_2 \nabla n^\gamma \text{ weakly in } (L^2(\Omega_T))^N.$$

We are ready to pass to the limit in (2.34) and (2.35), thereby finishing the proof of Theorem 1.2.  $\square$

**3. The limit as  $\gamma \rightarrow \infty$  and proof of Theorem 1.3.** Once again, the proof will be divided into several lemmas. Now the solution to our problem (1.1)–(1.6) is denoted by  $(n^{(\gamma)}, n_1^{(\gamma)}, n_2^{(\gamma)}, d^{(\gamma)})$ . That is, we have

$$(3.1) \quad \partial_t n^{(\gamma)} - \frac{\gamma}{\gamma + 1} \Delta \left( n^{(\gamma)} \right)^{\gamma+1} = G(d^{(\gamma)}) n^{(\gamma)} - D n_2^{(\gamma)} \equiv R^{(\gamma)} \text{ in } \Omega_T,$$

$$\partial_t n_1^{(\gamma)} - \operatorname{div} \left( n_1^{(\gamma)} \nabla \left( n^{(\gamma)} \right)^\gamma \right) = G(d^{(\gamma)}) n_1^{(\gamma)} - K_1(d^{(\gamma)}) n_1^{(\gamma)}$$

$$(3.2) \quad + K_2(d^{(\gamma)}) n_2^{(\gamma)} \equiv R_1^{(\gamma)} \text{ in } \Omega_T,$$

$$\partial_t n_2^{(\gamma)} - \operatorname{div} \left( n_2^{(\gamma)} \nabla \left( n^{(\gamma)} \right)^\gamma \right) = (G(d^{(\gamma)}) - D) n_2^{(\gamma)} + K_1(d^{(\gamma)}) n_1^{(\gamma)}$$

$$(3.3) \quad - K_2(d^{(\gamma)}) n_2^{(\gamma)} \equiv R_2^{(\gamma)} \text{ in } \Omega_T,$$

$$(3.4) \quad b \partial_t d^{(\gamma)} - \Delta d^{(\gamma)} = -\psi(d^{(\gamma)}) n^{(\gamma)} + a n_2^{(\gamma)} \text{ in } \Omega_T,$$

$$(3.5) \quad n_1^{(\gamma)} \nabla \left( n^{(\gamma)} \right)^\gamma \cdot \mathbf{n} = n_2^{(\gamma)} \nabla \left( n^{(\gamma)} \right)^\gamma \cdot \mathbf{n} = 0 \text{ on } \Sigma_T \equiv \partial\Omega \times (0, T),$$

$$(3.6) \quad d^{(\gamma)} = d_b \text{ on } \Sigma_T,$$

$$(3.7) \quad \left. \left( n^{(\gamma)}, n_1^{(\gamma)}, n_2^{(\gamma)}, d^{(\gamma)} \right) \right|_{t=0} = \left( n_0(x) + \frac{1}{\gamma}, n_{01}(x) + \frac{1}{\gamma}, n_{02}(x), d_0(x) \right) \text{ on } \Omega.$$

As before, the term  $\frac{1}{\gamma}$  is added in (3.7) to ensure that  $n^{(\gamma)}$  stays away from 0 below. Therefore, it possesses enough regularity properties. We wish to find and identify the limit of solutions as  $\gamma \rightarrow \infty$ . By our analysis in the preceding section, we have

$$(3.8) \quad n_1^{(\gamma)} \geq 0, \quad n_2^{(\gamma)} \geq 0, \quad n^{(\gamma)} = n_1^{(\gamma)} + n_2^{(\gamma)} \leq c,$$

$$(3.9) \quad 0 \leq d^{(\gamma)} \leq L,$$

where  $L$  is given as in (1.19). In (3.8) and what follows, the generic positive number  $c$  is

independent of  $\gamma$ . We may assume that there is a subsequence of  $(n_1^{(\gamma)}, n_2^{(\gamma)}, n^{(\gamma)}, d^{(\gamma)})$ , not relabeled, such that

$$(3.10) \quad n_1^{(\gamma)} \rightarrow n_1^{(\infty)}, \quad n_2^{(\gamma)} \rightarrow n_2^{(\infty)}, \quad n^{(\gamma)} \rightarrow n^{(\infty)}, \quad d^{(\gamma)} \rightarrow d^{(\infty)} \quad \text{weak* in } L^\infty(\Omega_T).$$

LEMMA 3.1. *Assume that*

$$(3.11) \quad \partial_t d_b \in L^2(0, T; W^{1,2}(\Omega)), \quad d_0 \in W^{1,2}(\Omega).$$

*Then we have*

$$(3.12) \quad \int_{\Omega_T} \left( \partial_t d^{(\gamma)} \right)^2 dx dt \leq c.$$

*Furthermore, if (H6) and (H8) hold, then we have*

$$(3.13) \quad \|\nabla d^{(\gamma)}\|_{\infty, \Omega_T} \leq c.$$

*Proof.* Use  $\partial_t(d^{(\gamma)} - d_b)$  as a test function in (3.4) to get

$$\begin{aligned} & b \int_{\Omega} \left( \partial_t d^{(\gamma)} \right)^2 dx + \frac{1}{2} \frac{d}{dt} \int_{\Omega} |\nabla d^{(\gamma)}|^2 dx \\ &= b \int_{\Omega} \partial_t d^{(\gamma)} \partial_t d_b dx + \int_{\Omega} \nabla d^{(\gamma)} \cdot \nabla \partial_t d_b dx \\ & \quad + \int_{\Omega} \left( -\psi(d^{(\gamma)}) n^{(\gamma)} + a n_2^{(\gamma)} \right) \partial_t(d^{(\gamma)} - d_b) dx. \end{aligned}$$

Integrate to derive

$$(3.14) \quad \int_{\Omega_T} \left( \partial_t d^{(\gamma)} \right)^2 dx dt + \sup_{0 \leq t \leq T} \int_{\Omega} |\nabla d^{(\gamma)}|^2 dx \leq c.$$

With the aid of our assumptions (H6) and (H8), we can easily modify the proof of Proposition 2.3 in [24] to obtain (3.13). The basic strategy there is to derive an equation for  $\partial_{x_i} d^{(\gamma)}$  and then apply a parabolic version of the DeGiorgi iteration technique to the resulting equations. The boundary estimate is achieved by flattening the relevant portion of the boundary. All these steps can be carried out here. We shall omit the details. The proof is complete.  $\square$

Clearly, this lemma implies (1.25). Consequently,

$$(3.15) \quad R^{(\gamma)} \rightarrow R^{(\infty)} = G(d^{(\infty)}) n^{(\infty)} - D n_2^{(\infty)} \quad \text{weak* in } L^\infty(\Omega_T).$$

The core of our development is the following lemma.

LEMMA 3.2. *We have*

$$(3.16) \quad \int_{\Omega_T} t \left( v^{(\gamma)} \right)^2 dx dt + \int_{\Omega_T} t |\nabla v^{(\gamma)}|^2 dx dt \leq c.$$

*Proof.* Let  $G_0$  be given as in Theorem 1.3. Then

$$(3.17) \quad R^{(\gamma)} \leq G_0 n^{(\gamma)}.$$

Use this in (3.1) and multiply through the resulting inequality by  $e^{-G_0 t}$  to get

$$(3.18) \quad \partial_t w^{(\gamma)} - \frac{\gamma e^{\gamma G_0 t}}{\gamma + 1} \Delta (w^{(\gamma)})^{\gamma+1} \leq 0 \quad \text{in } \Omega_T,$$

where

$$w^{(\gamma)} = e^{-G_0 t} n^{(\gamma)}.$$

For each  $\varepsilon > 0$  we let

$$\eta_\varepsilon(s) = \begin{cases} 1 & \text{if } s > \varepsilon, \\ \frac{1}{\varepsilon} s & \text{if } 0 \leq s \leq \varepsilon, \\ 0 & \text{if } s < 0. \end{cases}$$

We can easily check that

$$\eta_\varepsilon(s) \rightarrow \text{sgn}_0^+(s) = \begin{cases} 1 & \text{if } s > 0, \\ 0 & \text{if } s \leq 0, \end{cases} \quad \text{as } \varepsilon \rightarrow 0.$$

Let  $\sigma \in (0, e^{-G_0 T})$  be given as in (H7). Clearly,  $\eta_\varepsilon(w^{(\gamma)} - \sigma) \geq 0$ . Multiply through (3.18) by this function to get

$$(3.19) \quad \int_{\Omega} \int_0^{w^{(\gamma)}(x,t)} \eta_\varepsilon(s - \sigma) ds dx \leq \int_{\Omega} \int_0^{w^{(\gamma)}(x,0)} \eta_\varepsilon(s - \sigma) ds dx.$$

Take  $\varepsilon \rightarrow 0$  in the above inequality to obtain

$$\begin{aligned} \int_{\Omega} (w^{(\gamma)}(x,t) - \sigma)^+ dx &\leq \int_{\Omega} (w^{(\gamma)}(x,0) - \sigma)^+ dx \\ &\leq \left( \|n_0\|_{\infty, \Omega} + \frac{1}{\gamma} - \sigma \right) \left| \left\{ n_0(x) + \frac{1}{\gamma} \geq \sigma \right\} \right|. \end{aligned}$$

Or equivalently,

$$(3.20) \quad \int_{\Omega} (n^{(\gamma)}(x,t) - \sigma e^{G_0 t})^+ dx \leq e^{G_0 t} \left( \|n_0\|_{\infty, \Omega} + \frac{1}{\gamma} - \sigma \right) \left| \left\{ n_0(x) + \frac{1}{\gamma} \geq \sigma \right\} \right|.$$

On the other hand,

$$\begin{aligned} \int_{\Omega} (n^{(\gamma)}(x,t) - \sigma e^{G_0 t})^+ dx &\geq \int_{\{n^{(\gamma)}(x,t) \geq 1\}} (n^{(\gamma)}(x,t) - \sigma e^{G_0 t})^+ dx \\ &\geq (1 - \sigma e^{G_0 t}) \left| \left\{ n^{(\gamma)}(x,t) \geq 1 \right\} \right|. \end{aligned}$$

This combined with (3.20) implies

$$\begin{aligned} \left| \left\{ n^{(\gamma)}(x,t) \geq 1 \right\} \right| &\leq \frac{e^{G_0 t} \left( \|n_0\|_{\infty, \Omega} + \frac{1}{\gamma} - \sigma \right)}{1 - \sigma e^{G_0 t}} \left| \left\{ n_0(x) + \frac{1}{\gamma} \geq \sigma \right\} \right| \\ &\rightarrow \frac{e^{G_0 t} (\|n_0\|_{\infty, \Omega} - \sigma)}{1 - \sigma e^{G_0 t}} |\{n_0(x) \geq \sigma\}| \quad (\text{as } \gamma \rightarrow \infty) \\ (3.21) \quad &\leq \frac{e^{G_0 t} (\|n_0\|_{\infty, \Omega} - \sigma)}{1 - \sigma e^{G_0 t}} \frac{1}{e^{G_0 T} \|n_0\|_{\infty, \Omega}} |\Omega|. \end{aligned}$$

The last step is due to our assumption (H7). We easily check

$$\frac{e^{G_0 t} (\|n_0\|_{\infty, \Omega} - \sigma)}{1 - \sigma e^{G_0 t}} < e^{G_0 t} \|n_0\|_{\infty, \Omega}.$$

Hence we can pick a number  $\sigma_0 \in (\frac{e^{G_0 t} (\|n_0\|_{\infty, \Omega} - \sigma)}{1 - \sigma e^{G_0 t}} \frac{1}{e^{G_0 T} \|n_0\|_{\infty, \Omega}}, 1)$ . Consequently,

$$(3.22) \quad \sup_{0 \leq t \leq T} \left| \left\{ n^{(\gamma)}(x, t) \geq 1 \right\} \right| \leq \sigma_0 |\Omega| \quad \text{at least for } \gamma \text{ sufficiently large.}$$

Using  $(w^{(\gamma)} - \|n_0\|_{\infty, \Omega} - \frac{1}{\gamma})^+$  as a test function in (3.18), we derive the weak maximum principle

$$(3.23) \quad w^{(\gamma)} \leq \|n_0\|_{\infty, \Omega} + \frac{1}{\gamma} \quad \text{in } \Omega_T.$$

This together with (3.17) implies

$$(3.24) \quad R^{(\gamma)} \leq G_0 e^{G_0 t} \left( \|n_0\|_{\infty, \Omega} + \frac{1}{\gamma} \right).$$

Let  $v^{(\gamma)}$  be given as in (1.12). Use  $tv^{(\gamma)}$  as a test function in (3.1) to deduce

$$(3.25) \quad \begin{aligned} & \frac{1}{\gamma+2} \frac{d}{dt} \int_{\Omega} t \left( n^{(\gamma)} \right)^{\gamma+2} dx + \frac{\gamma t}{\gamma+1} \int_{\Omega} \left| \nabla v^{(\gamma)} \right|^2 dx \\ &= \frac{1}{\gamma+2} \int_{\Omega} \left( n^{(\gamma)} \right)^{\gamma+2} dx + t \int_{\Omega} R^{(\gamma)} v^{(\gamma)} dx \\ &\leq \frac{e^{G_0 T} \left( \|n_0\|_{\infty, \Omega} + \frac{1}{\gamma} \right)}{\gamma+2} \int_{\Omega} v^{(\gamma)} dx + G_0 e^{G_0 T} \left( \|n_0\|_{\infty, \Omega} + \frac{1}{\gamma} \right) t \int_{\Omega} v^{(\gamma)} dx. \end{aligned}$$

Since

$$\left| \left\{ n^{(\gamma)}(x, t) \geq 1 \right\} \right| + \left| \left\{ n^{(\gamma)}(x, t) < 1 \right\} \right| = |\Omega|,$$

the inequality (3.22) implies

$$\left| \left\{ n^{(\gamma)}(x, t) < 1 \right\} \right| > (1 - \sigma_0) |\Omega|.$$

Evidently,

$$(v^{(\gamma)} - 1)^+ = 0 \quad \text{on } \{n^{(\gamma)}(x, t) < 1\}.$$

This puts us in a position to apply Lemma 2.3. Upon doing so, we arrive at

$$(3.26) \quad \int_{\Omega} (v^{(\gamma)} - 1)^+ dx \leq c \int_{\Omega} \left| \nabla (v^{(\gamma)} - 1)^+ \right| dx = c \int_{\{n^{(\gamma)}(x, t) \geq 1\}} \left| \nabla v^{(\gamma)} \right| dx.$$

To estimate the first term on the right-hand side of (3.25), we use  $(n^{(\gamma)} - 1)^+$  as a test function in (3.1) to get

$$(3.27) \quad \sup_{0 \leq t \leq T} \int_{\Omega} \left[ (n^{(\gamma)} - 1)^+ \right]^2 dx + \gamma \int_{\Omega_T} \left( n^{(\gamma)} \right)^{\gamma} \left| \nabla (n^{(\gamma)} - 1)^+ \right|^2 dx dt \leq c.$$

For each  $\varepsilon > 0$  we estimate

$$\begin{aligned}
\int_{\Omega} v^{(\gamma)} dx &= \int_{\{n^{(\gamma)}(x,t) \geq 1\}} v^{(\gamma)} dx + \int_{\{n^{(\gamma)}(x,t) < 1\}} v^{(\gamma)} dx \\
&\leq \int_{\Omega} (v^{(\gamma)} - 1)^+ dx + c \\
&\leq c \int_{\{n^{(\gamma)}(x,t) \geq 1\}} |\nabla v^{(\gamma)}| dx + c \\
&= c(\gamma + 1) \int_{\Omega} (n^{(\gamma)})^{\gamma} |\nabla(n^{(\gamma)} - 1)^+| dx + c \\
&\leq \varepsilon \int_{\Omega} (n^{(\gamma)})^{\gamma} dx + c(\varepsilon)(\gamma + 1)^2 \int_{\Omega} (n^{(\gamma)})^{\gamma} |\nabla(n^{(\gamma)} - 1)^+|^2 dx + c \\
&\leq \frac{\varepsilon}{\|n^{(\gamma)}\|_{\infty, \Omega_T}} \int_{\Omega} v^{(\gamma)} dx + c(\varepsilon)(\gamma + 1)^2 \int_{\Omega} (n^{(\gamma)})^{\gamma} |\nabla(n^{(\gamma)} - 1)^+|^2 dx + c.
\end{aligned}$$

By choosing  $\varepsilon$  suitably small, we immediately get

$$(3.28) \quad \int_{\Omega} v^{(\gamma)} dx \leq c(\gamma + 1)^2 \int_{\Omega} (n^{(\gamma)})^{\gamma} |\nabla(n^{(\gamma)} - 1)^+|^2 dx + c.$$

Use this in (3.25), then integrate, and apply (3.27) to obtain

$$\begin{aligned}
&\frac{1}{\gamma + 2} \sup_{0 \leq t \leq T} \int_{\Omega} t (n^{(\gamma)})^{\gamma+2} dx + \frac{\gamma}{\gamma + 1} \int_{\Omega_T} t |\nabla v^{(\gamma)}|^2 dx dt \\
&\leq c(\gamma + 1) \int_{\Omega_T} (n^{(\gamma)})^{\gamma} |\nabla(n^{(\gamma)} - 1)^+|^2 dx dt + c \int_{\Omega_T} t v^{(\gamma)} dx dt + c \\
&\leq c \int_{\Omega_T} t |\nabla v^{(\gamma)}| dx dt + c \\
&\leq \frac{\gamma}{2(\gamma + 1)} \int_{\Omega_T} t |\nabla v^{(\gamma)}|^2 dx dt + c.
\end{aligned}$$

Consequently,

$$\int_{\Omega_T} t |\nabla v^{(\gamma)}|^2 dx dt \leq c.$$

By a calculation similar to (3.26),

$$\begin{aligned}
\int_{\Omega_T} t (v^{(\gamma)})^2 dx dt &\leq \int_{\Omega_T} t [(v^{(\gamma)} - 1)^+]^2 dx dt + c \\
&\leq c \int_{\Omega_T} t |\nabla (v^{(\gamma)} - 1)^+|^2 dx dt + c \leq c.
\end{aligned}$$

This completes the proof of Lemma 3.2.  $\square$

We see that the sequence  $\{v^{(\gamma)}\}$  is bounded in  $L^2(\tau, T; W^{1,2}(\Omega))$  for each  $\tau \in (0, T)$ . Thus we may assume that (1.22) holds.

*Proof of (1.28) and (1.29).* We shall employ an argument from [11]. For each  $\delta > 0$  define

$$(3.29) \quad \Omega_{\delta}^{(\gamma)} = \{(x, t) \in \Omega_T : n^{(\gamma)}(x, t) \geq 1 + \delta\}.$$

We argue by contradiction. Suppose that (1.28) is not true. Then there is a  $\delta > 0$  such that

$$(3.30) \quad \left| \Omega_{2\delta}^{(\infty)} \right| > 0.$$

We claim

$$(3.31) \quad \liminf_{\gamma \rightarrow \infty} \left| \Omega_{\delta}^{(\gamma)} \right| \equiv c_0 > 0.$$

To see this, we estimate from (3.9) that

$$\begin{aligned} \int_{\Omega_T} n^{(\gamma)} n^{(\infty)} \chi_{\Omega_{2\delta}^{(\infty)}} dxdt &= \int_{\Omega_{2\delta}^{(\infty)} \cap \Omega_{\delta}^{(\gamma)}} n^{(\gamma)} n^{(\infty)} dxdt + \int_{\Omega_{2\delta}^{(\infty)} \setminus \Omega_{\delta}^{(\gamma)}} n^{(\gamma)} n^{(\infty)} dxdt \\ &\leq e^{2G_0 T} \left( \|n_0\|_{\infty, \Omega} + \frac{1}{\gamma} \right)^2 \left| \Omega_{\delta}^{(\gamma)} \right| + (1 + \delta) \int_{\Omega_{2\delta}^{(\infty)}} n^{(\infty)} dxdt. \end{aligned}$$

If  $c_0$  in (3.31) is 0, we take  $\gamma \rightarrow \infty$  in the above inequality to derive

$$(3.32) \quad \int_{\Omega_{2\delta}^{(\infty)}} n^{(\infty)} n^{(\infty)} dxdt \leq (1 + \delta) \int_{\Omega_{2\delta}^{(\infty)}} n^{(\infty)} dxdt.$$

This is possible only if  $\left| \Omega_{2\delta}^{(\infty)} \right| = 0$ . But this contradicts (3.30). Thus (3.31) holds. On the other hand, for each  $\tau \in (0, T)$  we have

$$(3.33) \quad c \geq \int_{\Omega_{\delta}^{(\gamma)}} t v^{(\gamma)} dxdt \geq \int_{\Omega_{\delta}^{(\gamma)} \cap (\Omega \times (\tau, T))} t v^{(\gamma)} dxdt \geq \tau (1 + \delta)^{\gamma+1} \left| \Omega_{\delta}^{(\gamma)} \cap (\Omega \times (\tau, T)) \right|.$$

That is,

$$\limsup_{\gamma \rightarrow \infty} \left| \Omega_{\delta}^{(\gamma)} \cap (\Omega \times (\tau, T)) \right| \leq 0 \quad \text{for each } \tau \in (0, T).$$

Obviously, this contradicts (3.31). This completes the proof of (1.28).

Fix  $\tau \in (0, T)$ . First, we claim

$$(3.34) \quad \lim_{\gamma \rightarrow \infty} \int_{\tau}^T \int_{\Omega} \left| 1 - n^{(\gamma)} \right| v^{(\gamma)} dxdt = 0.$$

To see this, let  $\varepsilon \in (0, 1)$  be given. We estimate from (3.16) that

$$\begin{aligned} \int_{\tau}^T \int_{\Omega} \left| 1 - n^{(\gamma)} \right| v^{(\gamma)} dxdt &= \int_{\{ |1 - n^{(\gamma)}| \leq \varepsilon \} \cap (\Omega \times (\tau, T))} \left| 1 - n^{(\gamma)} \right| v^{(\gamma)} dxdt \\ &\quad + \int_{\{ n^{(\gamma)} > 1 + \varepsilon \} \cap (\Omega \times (\tau, T))} \left| 1 - n^{(\gamma)} \right| v^{(\gamma)} dxdt \\ &\quad + \int_{\{ n^{(\gamma)} < 1 - \varepsilon \} \cap (\Omega \times (\tau, T))} \left| 1 - n^{(\gamma)} \right| v^{(\gamma)} dxdt \\ &\leq c\varepsilon + c \left| \{ n^{(\gamma)} > 1 + \varepsilon \} \cap (\Omega \times (\tau, T)) \right|^{\frac{1}{2}} + c(1 - \varepsilon)^{\gamma+1}. \end{aligned}$$

Consequently,

$$(3.35) \quad \limsup_{\gamma \rightarrow \infty} \int_{\tau}^T \int_{\Omega} \left| 1 - n^{(\gamma)} \right| v^{(\gamma)} dxdt \leq c\varepsilon.$$

Since  $\varepsilon$  can be arbitrarily small, we yield (3.34).

Observe from (3.1) that the sequence  $\{\partial_t n^{(\gamma)}\}$  is bounded in  $L^2(\tau, T; (W^{1,2}(\Omega))^*)$ . We can infer from the Lions–Aubin lemma that  $\{n^{(\gamma)}\}$  is precompact in  $C([\tau, T]; (W^{1,2}(\Omega))^*)$ . We may assume that

$$(3.36) \quad n^{(\gamma)} \rightarrow n^{(\infty)} \text{ strongly in } C([\tau, T]; (W^{1,2}(\Omega))^*).$$

Once again, we pass to a subsequence if need be. With this in mind, we can deduce from (1.22) that

$$\begin{aligned} \int_{\tau}^T \int_{\Omega} (1 - n^{(\gamma)}) v^{(\gamma)} dx dt &= \int_{\tau}^T \langle 1 - n^{(\gamma)}, v^{(\gamma)} \rangle dt \\ &= \int_{\tau}^T \langle 1 - n^{(\infty)}, v^{(\infty)} \rangle dt = \int_{\tau}^T \int_{\Omega} (1 - n^{(\infty)}) v^{(\infty)} dx dt. \end{aligned}$$

This together with (3.34) and (1.28) implies

$$(3.37) \quad (1 - n^{(\infty)}) v^{(\infty)} = 0,$$

from which (1.29) follows.  $\square$

Now we are ready to prove (1.23).

*Proof of (1.23).* Use  $t^2 \partial_t v^{(\gamma)}$  as a test function in (3.1) to get

$$\begin{aligned} &(\gamma + 1) t^2 \int_{\Omega} (n^{(\gamma)})^{\gamma} (\partial_t n^{(\gamma)})^2 dx + \frac{\gamma}{2(\gamma + 1)} \frac{d}{dt} \int_{\Omega} t^2 |\nabla v^{(\gamma)}|^2 dx \\ (3.38) \quad &= \frac{\gamma}{\gamma + 1} \int_{\Omega} t |\nabla v^{(\gamma)}|^2 dx + t^2 \int_{\Omega} R^{(\gamma)} \partial_t v^{(\gamma)} dx. \end{aligned}$$

To estimate the last integral in the above equation, we compute from (3.3) that

$$\begin{aligned} (3.39) \quad -Dt^2 \int_{\Omega} n_2^{(\gamma)} \partial_t v^{(\gamma)} dx &= -D \frac{d}{dt} \int_{\Omega} t^2 n_2^{(\gamma)} v^{(\gamma)} dx + 2Dt \int_{\Omega} n_2^{(\gamma)} v^{(\gamma)} dx \\ &\quad + Dt^2 \int_{\Omega} \partial_t n_2^{(\gamma)} v^{(\gamma)} dx \\ &= -D \frac{d}{dt} \int_{\Omega} t^2 n_2^{(\gamma)} v^{(\gamma)} dx + 2Dt \int_{\Omega} n_2^{(\gamma)} v^{(\gamma)} dx \\ &\quad - \frac{\gamma Dt^2}{\gamma + 1} \int_{\Omega} \frac{n_2^{(\gamma)}}{n^{(\gamma)}} |\nabla v^{(\gamma)}|^2 dx + Dt^2 \int_{\Omega} R_2^{(\gamma)} v^{(\gamma)} dx. \end{aligned}$$

Integrate and then apply (3.16) to deduce

$$(3.40) \quad -D \int_0^{\tau} t^2 \int_{\Omega} n_2^{(\gamma)} \partial_t v^{(\gamma)} dx dt \leq c.$$

Similarly,

$$\begin{aligned} t^2 \int_{\Omega} G(d^{(\gamma)}) n^{(\gamma)} \partial_t v^{(\gamma)} dx &= \frac{\gamma + 1}{\gamma + 2} \frac{d}{dt} \int_{\Omega} t^2 G(d^{(\gamma)}) (n^{(\gamma)})^{\gamma+2} dx \\ &\quad - \frac{2(\gamma + 1)t}{\gamma + 2} \int_{\Omega} G(d^{(\gamma)}) (n^{(\gamma)})^{\gamma+2} dx \\ &\quad - \frac{\gamma + 1}{\gamma + 2} \int_{\Omega} t^2 G'(d^{(\gamma)}) \partial_t d^{(\gamma)} (n^{(\gamma)})^{\gamma+2} dx. \end{aligned}$$

Integrate and then use (H5), (3.16), and (3.12) to derive

$$(3.41) \quad \int_0^\tau \int_\Omega t^2 G(d^{(\gamma)}) n^{(\gamma)} \partial_t v^{(\gamma)} dx dt \leq \frac{\gamma+1}{\gamma+2} \int_\Omega \tau^2 G(d^{(\gamma)}) n^{(\gamma)} v^{(\gamma)} dx + c.$$

Integrate (3.38) and then take into consideration (3.40) and (3.41) to obtain

$$(3.42) \quad \begin{aligned} & (\gamma+1) \int_0^\tau \int_\Omega t^2 (n^{(\gamma)})^\gamma (\partial_t n^{(\gamma)})^2 dx dt \\ & + \frac{\gamma}{2(\gamma+1)} \int_\Omega \tau^2 |\nabla v^{(\gamma)}|^2 dx \leq \frac{\gamma+1}{\gamma+2} \int_\Omega \tau^2 G(d^{(\gamma)}) n^{(\gamma)} v^{(\gamma)} dx + c. \end{aligned}$$

We easily infer from (3.26) that

$$(3.43) \quad \int_\Omega v^{(\gamma)} dx \leq c \int_\Omega |\nabla v^{(\gamma)}| dx + c \leq \varepsilon \int_\Omega |\nabla v^{(\gamma)}|^2 dx + c(\varepsilon), \quad \varepsilon > 0.$$

Use this in (3.42) and choose  $\varepsilon$  suitably small in the resulting inequality to derive

$$(3.44) \quad (\gamma+1) \int_{\Omega_T} t^2 (n^{(\gamma)})^\gamma (\partial_t n^{(\gamma)})^2 dx dt + \sup_{0 \leq t \leq T} \int_\Omega t^2 |\nabla v^{(\gamma)}|^2 dx \leq c.$$

This combined with (3.43) yields

$$(3.45) \quad \sup_{0 \leq t \leq T} \int_\Omega t^2 (v^{(\gamma)})^2 dx \leq c.$$

Use  $t^2 (v^{(\gamma)} - v^{(\infty)})$  as a test function in (3.1) to deduce

$$(3.46) \quad \begin{aligned} & \int_\Omega t^2 \partial_t n^{(\gamma)} (v^{(\gamma)} - v^{(\infty)}) dx \\ & + \frac{t^2 \gamma}{\gamma+1} \int_\Omega \nabla v^{(\gamma)} \cdot \nabla (v^{(\gamma)} - v^{(\infty)}) dx = t^2 \int_\Omega R^{(\gamma)} (v^{(\gamma)} - v^{(\infty)}) dx. \end{aligned}$$

Note that

$$(3.47) \quad \int_\Omega t^2 \partial_t n^{(\gamma)} v^{(\gamma)} dx = \frac{1}{\gamma+2} \frac{d}{dt} \int_\Omega t^2 (n^{(\gamma)})^{\gamma+2} dx - \frac{2t}{\gamma+2} \int_\Omega (n^{(\gamma)})^{\gamma+2} dx.$$

Integrate to get

$$\begin{aligned} \int_{\Omega_T} t^2 \partial_t n^{(\gamma)} v^{(\gamma)} dx dt &= \frac{1}{\gamma+2} \int_\Omega T^2 (n^{(\gamma)}(x, T))^{\gamma+2} dx \\ &\quad - \frac{2}{\gamma+2} \int_{\Omega_T} t (n^{(\gamma)})^{\gamma+2} dx dt \\ &\rightarrow 0 \quad \text{as } \gamma \rightarrow \infty. \end{aligned}$$

The last step is due to (3.45). Keeping this and (3.46) in mind, we calculate

$$(3.48) \quad \begin{aligned} & \limsup_{\gamma \rightarrow \infty} \int_{\Omega_T} t^2 |\nabla (v^{(\gamma)} - v^{(\infty)})|^2 dx dt \\ & \leq \limsup_{\gamma \rightarrow \infty} \int_{\Omega_T} t^2 \nabla v^{(\gamma)} \cdot \nabla (v^{(\gamma)} - v^{(\infty)}) dx dt \\ & \leq \int_0^T \langle t \partial_t n^{(\infty)}, t v^{(\infty)} \rangle dt + \limsup_{\gamma \rightarrow \infty} \int_{\Omega_T} t^2 R^{(\gamma)} (v^{(\gamma)} - v^{(\infty)}) dx dt. \end{aligned}$$

Observe that

$$R^{(\gamma)} = \left( G(d^{(\gamma)}) - G(d^{(\infty)}) \right) n^{(\gamma)} + G(d^{(\infty)}) n^{(\gamma)} - D n_2^{(\gamma)}.$$

Remember that  $\{tn^{(\gamma)}\}, \{tn_2^{(\gamma)}\}$  are precompact in  $C([0, T]; (W^{1,2}(\Omega))^*)$ . Furthermore, we have  $G(d^{(\infty)}) \in L^\infty(0, T; W^{1,\infty}(\Omega))$  due to (H5) and (3.13). Hence

$$\begin{aligned} & \lim_{\gamma \rightarrow \infty} \int_{\Omega_T} t^2 R^{(\gamma)} (v^{(\gamma)} - v^{(\infty)}) dx dt \\ &= \lim_{\gamma \rightarrow \infty} \int_0^T \left\langle tn^{(\gamma)}, tG(d^{(\infty)}) (v^{(\gamma)} - v^{(\infty)}) \right\rangle dt \\ (3.49) \quad & - D \lim_{\gamma \rightarrow \infty} \int_0^T \left\langle tn_2^{(\gamma)}, t (v^{(\gamma)} - v^{(\infty)}) \right\rangle dt = 0. \end{aligned}$$

Use this in (3.48) to obtain

$$(3.50) \quad \limsup_{\gamma \rightarrow \infty} \int_{\Omega_T} t^2 \left| \nabla (v^{(\gamma)} - v^{(\infty)}) \right|^2 dx dt \leq \int_0^T \langle t \partial_t n^{(\infty)}, t v^{(\infty)} \rangle dt.$$

We wish to show that the right-hand side of the above inequality is 0. To this end, we introduce a function

$$\Psi^{(\infty)}(s) = \begin{cases} 0 & \text{if } s \leq 1, \\ \infty & \text{if } s > 1. \end{cases}$$

Obviously,  $\Psi^{(\infty)}(s)$  is convex and lower semicontinuous ([13, p. 49]) and

$$(3.51) \quad \partial \Psi^{(\infty)}(s) = \varphi_\infty(s),$$

where  $\varphi_\infty(s)$  is given as in (1.11). We claim that  $t \mapsto \int_\Omega \Psi^{(\infty)}(n^{(\infty)}(x, t)) dx$  is an absolutely continuous function on  $(0, T)$  and

$$(3.52) \quad \frac{d}{dt} \int_\Omega \Psi^{(\infty)}(n^{(\infty)}(x, t)) dx = \langle \partial_t n^{(\infty)}, v^{(\infty)} \rangle \quad \text{for a.e. } t \in (0, T).$$

Note that Lemma 2.1 is not applicable here because we do not have

$$n^{(\infty)} \in L^2(\tau, T; W^{1,2}(\Omega)).$$

We shall give a direct proof. To do this, we infer from (1.14) that

$$\begin{aligned} & \int_\Omega \Psi^{(\infty)}(n^{(\infty)}(x, t + \varepsilon)) dx - \int_\Omega \Psi^{(\infty)}(n^{(\infty)}(x, t)) dx \\ & \geq \int_\Omega v^{(\infty)}(x, t) \left( n^{(\infty)}(x, t + \varepsilon) - n^{(\infty)}(x, t) \right) dx \\ (3.53) \quad & = \langle n^{(\infty)}(\cdot, t + \varepsilon) - n^{(\infty)}(\cdot, t), v^{(\infty)}(\cdot, t) \rangle, \quad \varepsilon > 0. \end{aligned}$$

Let  $\zeta(t) \in C_0^\infty(0, T)$  be such that  $\zeta(t) \geq 0$ . Multiply through (3.53) by  $\frac{1}{\varepsilon} \zeta(t)$ , integrate the resulting inequality over  $(0, T)$ , and thereby obtain

$$\begin{aligned} & \int_0^T \int_\Omega \Psi^{(\infty)}(n^{(\infty)}(x, t)) dx \frac{\zeta(t - \varepsilon) - \zeta(t)}{\varepsilon} dt \\ (3.54) \quad & \geq \int_0^T \left\langle \frac{n^{(\infty)}(\cdot, t + \varepsilon) - n^{(\infty)}(\cdot, t)}{\varepsilon}, v^{(\infty)}(\cdot, t) \right\rangle \zeta(t) dt \quad \text{for } \varepsilon \text{ suitably small.} \end{aligned}$$

We can easily take the limit  $\varepsilon \rightarrow 0$  on the left-hand side of the preceding inequality. To show that we can do the same for the right-hand side, we integrate (1.13) over  $(t, t + \varepsilon)$  to get

$$(3.55) \quad n^{(\infty)}(x, t + \varepsilon) - n^{(\infty)}(x, t) - \Delta \int_t^{t+\varepsilon} v^{(\infty)} ds = \int_t^{t+\varepsilon} R^{(\infty)}(x, s) ds \quad \text{in } \Omega.$$

Using  $\frac{1}{\varepsilon} v^{(\infty)}(x, t)$  as a test function in the above equation gives

$$(3.56) \quad \begin{aligned} & \left\langle \frac{n^{(\infty)}(\cdot, t + \varepsilon) - n^{(\infty)}(\cdot, t)}{\varepsilon}, v^{(\infty)}(\cdot, t) \right\rangle \\ & = - \int_{\Omega} \frac{1}{\varepsilon} \int_t^{t+\varepsilon} \nabla v^{(\infty)} ds \cdot \nabla v^{(\infty)} dx + \int_{\Omega} \frac{1}{\varepsilon} \int_t^{t+\varepsilon} R^{(\infty)}(x, s) ds v^{(\infty)} dx. \end{aligned}$$

We can verify

$$(3.57) \quad \frac{1}{\varepsilon} \int_t^{t+\varepsilon} \nabla v^{(\infty)}(x, s) ds \rightarrow \nabla v^{(\infty)}(x, t) \quad \text{a.e on } \Omega_T.$$

It follows from (3.44) that

$$(3.58) \quad \sup_{t \geq \tau} \left\| \frac{1}{\varepsilon} \int_t^{t+\varepsilon} \nabla v^{(\infty)}(x, s) ds \right\|_{2, \Omega} \leq c(\tau), \quad \tau \in (0, T).$$

This together with (3.57) implies

$$(3.59) \quad \frac{1}{\varepsilon} \int_t^{t+\varepsilon} \nabla v^{(\infty)}(x, s) ds \rightarrow \nabla v^{(\infty)}(x, t) \quad \text{weakly in } (L^2(\Omega \times (\tau, T)))^N.$$

Similarly,

$$\frac{1}{\varepsilon} \int_t^{t+\varepsilon} R^{(\infty)}(x, s) ds \rightarrow R^{(\infty)}(x, t) \quad \text{strongly in } L^q(\Omega_T) \text{ for each } q > 1.$$

We are ready to evaluate

$$(3.60) \quad \begin{aligned} & \lim_{\varepsilon \rightarrow 0} \int_0^T \left\langle \frac{n^{(\infty)}(\cdot, t + \varepsilon) - n^{(\infty)}(\cdot, t)}{\varepsilon}, v^{(\infty)}(\cdot, t) \right\rangle \zeta(t) dt \\ & = - \int_0^T \int_{\Omega} |\nabla v^{(\infty)}|^2 dx \zeta(t) dt + \int_0^T \int_{\Omega} R^{(\infty)} v^{(\infty)} dx \zeta(t) dt \\ & = \int_0^T \langle \partial_t n^{(\infty)}, v^{(\infty)}(\cdot, t) \rangle \zeta(t) dt. \end{aligned}$$

The last step is due to (1.13). Taking  $\varepsilon \rightarrow 0$  in (3.54) yields

$$\frac{d}{dt} \int_{\Omega} \Psi^{(\infty)}(n^{(\infty)}(x, t)) dx \geq \langle \partial_t n^{(\infty)}, v^{(\infty)} \rangle \quad \text{in the sense of distributions.}$$

Replacing each occurrence of  $\varepsilon$  by  $-\varepsilon$  in the preceding proof, we can derive

$$\frac{d}{dt} \int_{\Omega} \Psi^{(\infty)}(n^{(\infty)}(x, t)) dx \leq \langle \partial_t n^{(\infty)}, v^{(\infty)} \rangle \quad \text{in the sense of distributions.}$$

This completes the proof of (3.52).

With (3.52) in mind, we calculate

$$\begin{aligned}
 \int_0^T \langle t \partial_t n^{(\infty)}, t v^{(\infty)} \rangle dt &= \int_0^T t^2 \langle \partial_t n^{(\infty)}, v^{(\infty)} \rangle dt \\
 &= \int_0^T t^2 \frac{d}{dt} \int_{\Omega} \Psi^{(\infty)}(n^{(\infty)}(x, t)) dx dt \\
 &= \int_0^T \frac{d}{dt} \int_{\Omega} t^2 \Psi^{(\infty)}(n^{(\infty)}(x, t)) dx dt \\
 &\quad - 2 \int_0^T \int_{\Omega} t \Psi^{(\infty)}(n^{(\infty)}(x, t)) dx dt \\
 (3.61) \quad &= 0.
 \end{aligned}$$

The last step is due to the fact that  $\Psi^{(\infty)}(n^{(\infty)}(x, t)) \equiv 0$ . Combining (3.61) with (3.50) yields (1.23).

To complete the proof of Theorem 1.3, we still need to verify (1.26). To this end, we multiply through (3.1) by  $v^{(\gamma)}$  to get

$$\frac{1}{\gamma+2} \partial_t \left( n^{(\gamma)} \right)^{\gamma+2} - \frac{\gamma}{\gamma+1} \left( \operatorname{div}(v^{(\gamma)} \nabla v^{(\gamma)}) - |\nabla v^{(\gamma)}|^2 \right) = R^{(\gamma)} v^{(\gamma)}.$$

Even though it is not clear if  $\{t v^{(\gamma)}\}$  is precompact in  $L^2(\Omega_T)$  because we do not have any estimates on  $\partial_t v^{(\gamma)}$ , (1.23) and (3.49) are enough to justify passing to the limit in the above equation, thereby obtaining (1.26). This finishes the proof of Theorem 1.3.  $\square$

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