Empirical Likelihood Method for Complete Independence Test on High Dimensional Data

Yongcheng Qi[1] and Yingchao Zhou[1]

[1]Department of Mathematics and Statistics, University of Minnesota Duluth, 1117 University Drive, Duluth, MN 55812, USA

ABSTRACT

Given a random sample of size n from a p dimensional random vector, where both n and p are large, we are interested in testing whether the p components of the random vector are mutually independent. This is the so-called complete independence test. In the multivariate normal case, it is equivalent to testing whether the correlation matrix is an identity matrix. In this paper, we propose a one-sided empirical likelihood method for the complete independence test for multivariate normal data based on squared sample correlation coefficients. The limiting distribution for our one-sided empirical likelihood test statistic is proved to be $Z^2 I(Z > 0)$ when both n and p tend to infinity, where Z is a standard normal random variable. In order to improve the power of the empirical likelihood test statistic, we also introduce a rescaled empirical likelihood test statistic. We carry out an extensive simulation study to compare the performance of the rescaled empirical likelihood method and two other statistics which are related to the sum of squared sample correlation coefficients.

---

1. Introduction

Statistical inference on high dimensional data has gained a wide range of applications in recent years. New techniques generate a vast collection of data sets with high dimensions, for example, trading data from financial market, social network data and biological data like microarray and DNA data. The dimension of these types of data is not small compared with sample size, and typically of the same order as sample size or even larger. Yet classical multivariate statistics usually deal with data from normal distributions with a large sample size n and a fixed dimension p, and one can easily find some classic treatments in reference books such as Anderson [1], Morrison [10] and Muirhead [11].

Under multivariate normality settings, the likelihood ratio test statistic converges in distribution to a chi-squared distribution when p is fixed. However, when p changes with n and tends to infinity, this conclusion is no longer true as discovered in Bai et al. [2], Jiang et al. [6], Jiang and Yang [8], Jiang and Qi [7], Qi et al. [16], among others. The results in these papers indicate that the chi-square approximation fails when p diverges as n goes to infinity.

The test of complete independence of a random vector is to test whether all the components of the random vector are mutually independent. In the multivariate normal case, the test of complete independence is equivalent to the test whether covariance matrix is a diagonal matrix, or whether the correlation matrix is the identity matrix.

For more details, we assume $X = (X_1, \cdots, X_p)$ is a random vector from a p-dimensional multivariate normal distribution $N_p(\boldsymbol{\mu}, \boldsymbol{\Sigma})$, where $\boldsymbol{\mu}$ denotes the mean vector, and $\boldsymbol{\Sigma}$ is a $p \times p$ covariance matrix. Given a random sample of size n from the normal distribution, $\mathbf{x}_1, \mathbf{x}_2, \cdots, \mathbf{x}_n$, where $\mathbf{x}_i = (x_{i1}, x_{i2}, \cdots, x_{ip})$ for $1 \leq i \leq n$, Pearson's correlation coefficient between the i-th and j-th components is given by

$$r_{ij} = \frac{\sum\limits_{k=1}^{n} (x_{ki} - \bar{x}_i)(x_{kj} - \bar{x}_j)}{\sqrt{\sum\limits_{k=1}^{n} (x_{ki} - \bar{x}_i)^2 \cdot \sum\limits_{k=1}^{n} (x_{kj} - \bar{x}_j)^2}}, \tag{1}$$

where $\bar{x}_i = \frac{1}{n} \sum\limits_{k=1}^{n} x_{ki}$ and $\bar{x}_j = \frac{1}{n} \sum\limits_{k=1}^{n} x_{kj}$ for $1 \leq i, j \leq p$. Now we set $\mathbf{R}_n = (r_{ij})_{p \times p}$ as the sample correlation coefficient matrix.

2

The complete independence test for the normal random vector is

$$H_0 : \mathbf{\Gamma} = \mathbf{I_p} \quad vs \quad H_a : \mathbf{\Gamma} \neq \mathbf{I}_p, \tag{2}$$

where $\mathbf{\Gamma}$ is the population correlation matrix and $\mathbf{I}_p$ is $p \times p$ identity matrix. When $p < n$, the likelihood ratio test statistic for (2) is a function of $|\mathbf{R}_n|$, the determinant of $\mathbf{R}_n$, from Bartlett [3] or Morrison [10]. In traditional multivariate analysis, when p is a fixed integer, we have under the null hypothesis in (2) that

$$-(n - 1 - \frac{2p + 5}{6}) \log |\mathbf{R}_n| \xrightarrow{d} \chi^2_{p(p-1)/2} \quad \text{as } n \to \infty,$$

where χ^2_f denotes a chi-square distribution with f degrees of freedom.

When $p = p_n$ depends on n with $2 \leq p_n < n$ and $p_n \to \infty$, the likelihood ratio method can still be applied to test (2). The limiting distributions of the likelihood ratio test statistics in this case have been discussed in the aforementioned papers. It is worth mentioning that Qi et al. [16] propose an adjusted likelihood ratio test statistic and show that the distribution of the adjusted likelihood test statistic can be well approximated by a chi-squared distribution whose number of degrees of freedom depends on p regardless of whether p is fixed or divergent.

The limitation of the likelihood ratio test is that the dimension p of the data must be smaller than the sample size n. Many other likelihood tests related to the sample covariance matrix or sample correlation matrix have the same problem as the sample covariance matrices are degenerate when $p \geq n$. In order to relax this constraint, a new test statistic using the sum of squared sample correction coefficients is proposed by Schott [17] as follows

$$t_{np} = \sum_{1 \leq j < i \leq p} r_{ij}^2.$$

Assume that the null hypothesis of (2) holds. Under assumption $\lim_{n \to \infty} p_n/n = \gamma \in (0, \infty)$, Schott [17] proves that $t_{np} - \frac{p(p-1)}{2(n-1)}$ converges in distribution to a normal distribution with

mean 0 and variance γ^2, that is,

$$t_{np}^* := \frac{t_{np} - \frac{p(p-1)}{2(n-1)}}{\sigma_{np}} \xrightarrow{d} N(0,1), \tag{3}$$

where $\sigma_{np}^2 = \frac{p(p-1)(n-2)}{(n-1)^2(n+1)}$.

Recently, Mao [9] proposes a different test for complete independence. His test statistic is closely related to Schott's test and is defined by

$$T_{np} = \sum_{1 \le j < i \le p} \frac{r_{ij}^2}{1 - r_{ij}^2}.$$

It has been proved in Mao [9] that T_{np} is asymptotically normal under the null hypothesis of (2) and the assumption that $\lim_{n \to \infty} p_n/n = \gamma \in (0, \infty)$.

Very recently, Chang and Qi [4] investigate the limiting distributions for the two test statistics above under less restrictive conditions on n and p. Chang and Qi [4] show that (3) is also valid under the general condition that $p_n \to \infty$ as $n \to \infty$, regardless of the convergence rate of p_n. Thus, the normal approximation in (3) based on t_{np}^* yields an approximate level α rejection region

$$\mathcal{R}_t^*(\alpha) = \left\{ t_{np} \ge \frac{p(p-1)}{2(n-1)} + z_{1-\alpha} \sqrt{\frac{p(p-1)(n-1)}{(n-1)^2(n+1)}} \right\}, \tag{4}$$

where z_α is a α level critical value of the standard normal distribution.

Furthermore, Chang and Qi [4] propose adjusted test statistics whose distribution can be fitted by chi-squared distribution regardless of how p changes with n as long as n is large. Chang and Qi's [4] adjusted test statistics t_{np}^c is defined as

$$t_{np}^c = \sqrt{p(p-1)} t_{np}^* + \frac{p(p-1)}{2}. \tag{5}$$

Chang and Qi show that

$$\sup_x \left| P(t_{np}^c \le x) - P(\chi_{p(p-1)/2}^2 \le x) \right| \to 0$$

4

as long as $p_n \to \infty$ as $n \to \infty$. Let $\chi_f^2(\alpha)$ denote the α level critical value of χ_f^2. Then an approximate level α rejection region based on t_{np}^c is given by

$$\mathcal{R}_t^c(\alpha) = \left\{ t_{np} \geq \frac{p(p-1)}{2}(1 - \sqrt{\frac{n-2}{n+1}}) + \chi_{\frac{p(p-1)}{2}}^2(\alpha)\sqrt{\frac{n-2}{(n-1)^2(n+1)}} \right\}. \quad (6)$$

One can find more references on test for complete independence in Mao [9] or Chang and Qi [4].

In practice, the assumption of normality for distributions may be violated. Now we assume $X = (X_1, \cdots, X_p)$ is a random vector and $X_1, \cdots, X_p$ are identically distributed with distribution function F. Given a random sample of size n, $\mathbf{x}_1, \mathbf{x}_2, \cdots, \mathbf{x}_n$, where $\mathbf{x}_i = (x_{i1}, x_{i2}, \cdots, x_{ip})$ for $1 \leq i \leq n$, are drawn from the distribution of $X = (X_1, \cdots, X_p)$, and define Pearson's correlation coefficients r_{ij}'s as in (1). By using the Stein method, Chen and Shao [5] show that (3) holds under some moment conditions of F if p_n/n is bounded.

In this paper, we propose to apply empirical likelihood method to the testing problem (2). The empirical likelihood is a nonparametric statistical method proposed by Owen [12,13], which is originally used to test the mean vector of a population based on a set of independent and identically distributed (i.i.d.) random variables. Empirical likelihood does not require to specify the family of distributions for the data and it possesses some good properties of the likelihood methods.

The rest of the paper is organized as follows. In Section 2, we first introduce a one-sided empirical likelihood method for the mean of a set of random variables with a common mean and then establish the connection between the test of complete independence and the one-sided empirical likelihood method. Our main result concerning the limiting distribution of the one-sided empirical likelihood ratio statistic is also given in Section 2. In Section 3, we carry out a simulation study to compare the performance of the empirical likelihood method and normal approximation based on Schott's test statistic and chi-square approximation based on Chang and Qi's adjusted test statistic. In our simulation study, we also apply these methods to some other distributions such as the exponential distributions and mixture of the exponential and normal distributions so as to compare their adaptability to non-normality. The proofs of the main results are given in Section 4.

2. Main Results

In this section, we apply the empirical likelihood method to the test of complete independence. First, we assume $X = (X_1, \cdots, X_p)$ is a random vector from a p-dimensional multivariate normal distribution. Under the null hypothesis of (2), $\{r_{ij}^2, 1 \leq i < j \leq p\}$ are random variables from an identical distribution with mean $\frac{1}{n-1}$. As a matter of fact, it follows from Corollary 5.1.2 in Muirhead [11] that r_{ij}^2 has the same distribution as $T^2/(n-2+T^2)$ under the null hypothesis of (2), where T is a random variable having t-distribution with $n-2$ degrees of freedom. $\{r_{ij}^2, 1 \leq i < j \leq p\}$ are asymptotically independent if the sample size n is large. We will develop a one-sided empirical likelihood test statistic and apply it to the data set $\{(n-1)r_{ij}^2, 1 \leq i < j \leq p\}$, where $p = p_n$ is a sequence of positive integers such that $p_n \to \infty$ as $n \to \infty$. As an extension, we then consider the case when $X = (X_1, \cdots, X_p)$ is a random vector with an identical marginal distribution function F which is not necessarily Gaussian. When the p components of X are independent, we demonstrate that the empirical likelihood method we develop under normality works for general distribution F as well if some additional conditions are satisfied.

2.1. One-sided empirical likelihood test

Consider a random sample of size N, namely $y_1, \cdots, y_N$. Assume the sample comes from a population with mean μ and variance σ^2. The empirical likelihood function for the mean μ is defined as

$$L(\mu) = \sup \left\{ \prod_{i=1}^{N} \omega_i \,\Big|\, \sum_{i=1}^{N} \omega_i y_i = \mu, \omega_i \geq 0, \sum_{i=1}^{N} \omega_i = 1 \right\}. \tag{7}$$

The function $L(\mu)$ is well defined if μ belongs to the convex hull given by

$$H := \left\{ \sum_{i=1}^{N} \omega_i y_i \,\Big|\, \sum_{i=1}^{N} \omega_i = 1, \omega_i > 0, \ i = 1, \cdots, N \right\};$$

otherwise, set $L(\mu) = 0$. We see that $H = (\min_{1 \leq i \leq N} y_i, \max_{1 \leq i \leq N} y_i)$.

Assume $\mu \in H$. By the standard Lagrange multiplier technique, the supremum on the

6

right-hand side of (7) is achieved at

$$\omega_i = \frac{1}{N(1 + \lambda(y_i - \mu))}, \quad i = 1, \cdots, N, \tag{8}$$

where λ is the solution to equation $g(\lambda) = 0$, with $g(\lambda)$ defined as follows

$$g(\lambda) := \sum_{i=1}^{N} \frac{y_i - \mu}{1 + \lambda(y_i - \mu)}. \tag{9}$$

Assume $\min\limits_{1 \leq i \leq N} y_i < \max\limits_{1 \leq i \leq N} y_i$. When $\mu \in H$, then the function $g(\lambda)$ defined in (9) is strictly increasing for $\lambda \in (-(\max\limits_{1 \leq i \leq N} y_n - \mu)^{-1}, (\mu - \min\limits_{1 \leq i \leq N} y_i)^{-1})$. A solution to $g(\lambda) = 0$ in this range exists and the solution must be unique.

Proposition 2.1. *Assume $y_1, \cdots, y_N$ are N observations with $y_i \neq y_j$ for some i and j. Then $\log L(\mu)$ is strictly concave in H, and $L(\bar{y}) = \sup\limits_{\mu} L(\mu) = N^{-N}$, where $\bar{y} = \frac{1}{N} \sum\limits_{i=1}^{N} y_i$.*

Remark. The results in Proposition 2.1 are well-known among the researchers in the area of empirical likelihood methods. A short proof will be given in Section 4 for completeness.

Consider the following two-sided test problem

$$H_0 : \mu = \mu_0 \quad vs \quad H_a : \mu \neq \mu_0.$$

The empirical likelihood ratio is given by

$$\frac{L(\mu_0)}{\sup\limits_{\mu \in R} L(\mu)} = \frac{L(\mu_0)}{N^{-N}} = \prod_{i=1}^{N} (1 + \lambda(y_i - \mu_0))^{-1},$$

where λ is the solution to the following equation

$$\frac{1}{N} \sum_{i=1}^{N} \frac{y_i - \mu_0}{1 + \lambda(y_i - \mu_0)} = 0.$$

Therefore, the log-empirical likelihood test statistic is given by

$$\ell(\mu_0) := -2\log \frac{L(\mu_0)}{\sup\limits_{\mu \in R} L(\mu)} = 2\sum_{i=1}^{N} \log(1 + \lambda(y_i - \mu_0)). \tag{10}$$

It is proved in Owen [14] that $\ell(\mu_0)$ converges in distribution to a chi-square distribution with one degree of freedom if $y_1, \cdots, y_N$ are i.i.d. random variables with mean μ_0 and a finite second moment.

Our interest here is to consider a one-sided test

$$H_0 : \mu = \mu_0 \quad vs \quad H_a : \mu > \mu_0. \tag{11}$$

According to Proposition 2.1, $L(\mu)$ is increasing in $(-\infty, \bar{y})$ and decreasing in $(\bar{y}, \infty)$, which implies $\sup\limits_{\mu \geq \mu_0} L(\mu) = L(\mu_0)I(\bar{y} < \mu_0) + N^{-N}I(\bar{y} \geq \mu_0)$. Therefore, the empirical likelihood ratio corresponding to test (11) is

$$\frac{L(\mu_0)}{\sup\limits_{\mu \geq \mu_0} L(\mu)} = \begin{cases} \frac{L(\mu_0)}{N^{-N}}, & \text{if } \bar{y} \geq \mu_0; \\ 1, & \text{if } \bar{y} < \mu_0. \end{cases}$$

Then the log-empirical likelihood test statistic for test (11) is

$$\ell_n(\mu_0) := -2\log \frac{L(\mu_0)}{\sup\limits_{\mu \geq \mu_0} L(\mu)} = \ell(\mu_0)I(\bar{y} \geq \mu_0), \tag{12}$$

where $\ell(\mu_0)$ is defined in (10).

2.2. Empirical likelihood method for testing complete independence

Let r denote the sample Pearson correlation coefficient based on a random sample of size n from a bivariate normal distribution with correlation coefficient ρ. From Muirhead [11], page

8

156,

$$E(r^2) = 1 - \frac{n-2}{n-1}(1 - \rho^2)_2 F_1(1, 1; \frac{1}{2}n + 1; \rho^2),$$

where

$$_2F_1(a, b; c; z) = 1 + \frac{ab}{1!c}z + \frac{a(a+1)b(b+1)}{2!c(c+1)}z^2 + \cdots = 1 + \sum_{k=1}^{\infty} \frac{(a)_k(b)_k}{(c)_k}\frac{z^k}{k!}$$

is the hypergeometric function, $(a)_k = \Gamma(a + k)/\Gamma(a)$, and $\Gamma(x) = \int_0^\infty t^{x-1}e^{-t}dt$ is the gamma function. It is easy to check when $\rho = 0$, $_2F_1(1, 1; \frac{1}{2}n + 1; \rho^2) = 1$, and $E(r^2) = 1 - \frac{n-2}{n-1} = \frac{1}{n-1}$; when $\rho \neq 0$, $_2F_1(1, 1; \frac{1}{2}n + 1; \rho^2) < 1 + \sum_{k=1}^{\infty} \rho^{2k} = \frac{1}{1-\rho^2}$, and thus, $E(r^2) > 1 - \frac{n-2}{n-1} = \frac{1}{n-1}$.

First, we assume $X = (X_1, \cdots, X_p)$ is a random vector from a p-dimensional multivariate normal distribution $N_p(\boldsymbol{\mu}, \boldsymbol{\Sigma})$. Review the sample correlation coefficients r_{ij} defined in (1). Denote the correlation matrix of $\boldsymbol{\Sigma}$ by $\boldsymbol{\Gamma} = (\gamma_{ij})$. From the above discussion, we have that under the null hypothesis of (2), $E(r_{ij}^2) = \frac{1}{n-1}$ for all $1 \leq i < j \leq p$, $E(r_{ij}^2) \geq \frac{1}{n-1}$ under the alternative of (2) and at least one of the inequalities is strict. We see that test (2) is equivalent to the following one-tailed test

$$H_0 : E(\bar{r}_{ij}) = 1, \; 1 \leq i < j \leq p \; vs \; H_a : E(\bar{r}_{ij}) > 1 \; \text{for some } 1 \leq 1 < j \leq p,$$

where $\bar{r}_{ij} = (n - 1)r_{ij}^2$. Under the null hypothesis of (2), $\{(n - 1)r_{ij}^2, \; 1 \leq i < j \leq p\}$ are identically distributed with mean 1 and variance $\frac{2(n-2)}{(n+1)}$. We also notice from Chang and Qi [4] that $\{(n - 1)r_{ij}^2, \; 1 \leq i < j \leq p\}$ behave as if they were independent and identically distributed. For these reasons, we propose a one-sided empirical likelihood ratio test as follows.

Rewrite $\{(n - 1)r_{ij}^2, \; 1 \leq i < j \leq p\}$ as $y_1, \cdots, y_N$, where $N = p(p - 1)/2$. Then $y_1, \cdots, y_N$ are asymptotically i.i.d with mean 1. Define the one-sided log-empirical likeli-

9

hood ratio test statistics as in (12) with $\mu_0 = 1$, or equivalently

$$\ell_n := \ell_n(1) = 2I(\bar{r} \geq 1) \sum_{1 \leq i < j \leq p} \log \left(1 + \lambda\big((n-1)r_{ij}^2 - 1\big) \right), \tag{13}$$

where λ is the solution to the equation

$$\sum_{1 \leq i < j \leq p} \frac{(n-1)r_{ij}^2 - 1}{1 + \lambda\big((n-1)r_{ij}^2 - 1\big)} = 0,$$

and $\bar{r} = \bar{y} = \frac{n-1}{N} \sum_{1 \leq i < j \leq p} r_{ij}^2$.

Our first result on empirical likelihood method for testing the complete independence under normality in the paper is as follows.

Theorem 1. *Assume $p = p_n \to \infty$ as $n \to \infty$. Then $\ell_n \overset{d}{\to} Z^2 I(Z > 0)$ as $n \to \infty$ under the null hypothesis of (2), where Z is a standard normal random variable.*

Let Φ denote the cumulative distribution function of the standard normal distribution, i.e,

$$\Phi(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{x} e^{-t^2/2} dt, \quad x \in (-\infty, \infty).$$

Let G denote the cumulative distribution function of $Z^2 I(Z > 0)$. Then

$$G(x) = \begin{cases} 0, & x < 0; \\ \Phi(\sqrt{x}), & x \geq 0. \end{cases}$$

Therefore, for any $\alpha \in (0, \frac{1}{2})$, an α-level critical value of G is given by z_α^2, where z_α is an α-level critical value for the standard normal distribution. Based on Theorem 1, a level α rejection region for test on (11) is

$$\mathcal{R}_e(\alpha) = \left\{ \ell_n \geq z_\alpha^2 \right\}. \tag{14}$$

Here we only consider $\alpha < \frac{1}{2}$ because $Z^2 I(Z > 0)$ is nonnegative, $P(Z^2 I(Z > 0) >$

10

$c|H_0) < \frac{1}{2}$ if $c > 0$, and $P(Z^2 I(Z > 0) > c|H_0) = 1$ if $c \leq 0$.

Now we consider the general case when $X = (X_1, \cdots, X_p)$ is a random vector with independent and identically distributed components. The one-sided empirical likelihood test statistic ℓ_n based on $\{(n-1)r_{ij}^2, \ 1 \leq i < j \leq p\}$ is defined as in (13). The limiting distribution for ℓ_n is the same as that under normality.

Theorem 2. *Assume $X_1, \cdots, X_p$ are independent and identically distributed and $E(X_1^{24}) < \infty$. If $p = p_n \to \infty$ as $n \to \infty$ and p_n/n is bounded, then $\ell_n \xrightarrow{d} Z^2 I(Z > 0)$ as $n \to \infty$.*

Compared with Theorem 1, p_n in Theorem 2 is restricted in a smaller range and it can be of the same order as n.

To demonstrate the performance of empirical likelihood method and two other test statistics, we have a numerical study. Our simulation study indicates that the empirical likelihood test (14) based om ℓ_n maintains a very stable size or type I error. In terms of size, ℓ_n is more accurate t_{np}^c and t_{np}^*. Most of the time, t_{np}^c and t_{np}^* have slightly larger sizes than 0.05 when the nominal level α is 0.05, and their powers are also slightly larger than that of ℓ_n in our simulation study. For simplicity purpose, the simulation result on ℓ_n is not shown in this paper.

In order to balance the size and power for the empirical likelihood method, We introduce a rescaled empirical likelihood statistic, $\bar{\ell}_n$, defined as follows

$$\bar{\ell}_n = \frac{2(n-1)(n+1)}{3(p-1)(p+4)} \ell_n \sum_{1 \leq i < j \leq p} r_{ij}^4. \tag{15}$$

Under conditions of Theorems 1 or 2, $\bar{\ell}_n$ and ℓ_n have the same limiting distribution, that is,

$$\bar{\ell}_n \xrightarrow{d} Z^2 I(Z > 0) \quad \text{as} \quad n \to \infty \tag{16}$$

provided that

$$\frac{2(n-1)(n+1)}{3(p-1)(p+4)} \sum_{1 \leq i < j \leq p} r_{ij}^4 \xrightarrow{p} 1. \tag{17}$$

11

This equation will be verified in Section 4. Based on (16), a level α test rejects the complete independence if $\bar{\ell}_n$ falls into the rejection region

$$\bar{\mathcal{R}}_e(\alpha) = \left\{\bar{\ell}_n \geq z_\alpha^2\right\}. \tag{18}$$

3. Simulation

In this section, we will consider the following three test statistics for testing complete independence (2), including Schott's test statistic t_{np}^* given in (3), Chang and Qi's adjusted test statistic t_{np}^c defined in (5), and the rescaled empirical likelihood test statistic $\bar{\ell}_n$ given in (15). The corresponding rejection regions are given in (4), (6), and (18), respectively. All simulations are implemented by the software R.

For sample size $n = 20, 50, 100$ and dimension $p = 10, 20, 50, 100$, we apply the three test statistics to each of five distributions for 10000 iterations to obtain the empirical sizes and the empirical powers of the tests. We set the nominal type I error $\alpha = 0.05$. The five distributions include the normal, the uniform over $[-1, 1]$, the exponential, the mixture of the normal and exponential distributions, and the sum of normal and exponential distributions.

To control the dependence structure, we introduce a covariance matrix $\mathbf{\Gamma}_\rho$ defined by

$$\mathbf{\Gamma}_\rho = \left(\gamma_{ij}\right)_{p \times p}, \text{ with } \gamma_{ii} = 1, \text{ and } \gamma_{ij} = \rho \text{ if } i \neq j, \tag{19}$$

which is also a correlation matrix. In our simulation study, we generate random samples from the distribution of a random vector $X = (X_1, \cdots, X_p)$ with covariance matrix $\mathbf{\Gamma}_\rho$ or correlation matrix $\mathbf{\Gamma}_\rho$. For details, see the five distributions described below. For all distributions we consider, the observations have independent components when $\rho = 0$ and positively dependent components when $\rho > 0$. We choose very small values for ρ such as $\rho = 0.02$ and 0.05. When the value of ρ is large, the resulting powers for all three methods will be too close to 1, and the comparison is meaningless. Therefore, based on $10,000$ replicates, the sizes for three test statistics are estimated when $\rho = 0$, and their powers are estimated when $\rho = 0.02$ and 0.05. All results are reported in Tables 1 to 5.

12

a. Normal Distribution

The observations are drawn from a multivariate normal random vector $X = (X_1, \cdots, X_p)$ with mean $\boldsymbol{\mu} = (0, \cdots, 0)$ and variance matrix $\boldsymbol{\Gamma}_\rho$ specified in (19). The results on the empirical sizes and powers are given in Table 1.

b. Uniform Distribution

We first generate $p + 1$ i.i.d. random variables $Y_0, Y_1, \cdots, Y_p$ from Uniform $(-1, 1)$ distribution, then set $X_i = \frac{\sqrt{\rho}}{\sqrt{1-\rho}} Y_0 + Y_i, i = 1, 2, \cdots, p$. It is easy to verify that random vector $X = (X_1, \cdots, X_p)$ has mean $\boldsymbol{\mu} = (0, \cdots, 0)$ and correlation matrix $\boldsymbol{\Gamma}_\rho$ as defined in (19). The results on the empirical sizes and powers are given in Table 2.

c. Exponential Distribution

We generate $p+1$ i.i.d. random variables $Y_0, Y_1, \cdots, Y_p$ from the unit exponential distribution, then define $X_i = \frac{\sqrt{\rho}}{\sqrt{1-\rho}} Y_0 + Y_i, i = 1, 2, \cdots, p$. The random vector $X = (X_1, \cdots, X_p)$ has a correlation matrix $\boldsymbol{\Gamma}_\rho$ as defined in (19) for $\rho \in [0, 1)$. The results on the empirical sizes and powers are given in Table 3.

d. Mixture of Normal and Exponential Distributions

The random vector $X = (X_1, \cdots, X_p)$ is sampled from a mixture of the normal and exponential distributions which is with 90% probability from the multivariate normal with mean $\boldsymbol{\mu} = (1, \cdots, 1)$ and covariance matrix $\boldsymbol{\Gamma}_\rho$ given in (19) and with 10% probability from a random vector $(Y_1, \cdots, Y_p)$ where $Y_1, \cdots, Y_p$ are i.i.d. unit exponential random variables. The results on the empirical sizes and powers are given in Table 4.

e. Sum of Normal and Exponential Distribution

The random vector $X = (X_1, \cdots, X_p)$ is a weighted sum of two independent random vectors, U and V, $X = U + 0.01V$, where U is from a multivariate normal distribution with mean $\boldsymbol{\mu} = (0, \cdots, 0)$ and covariance matrix $\boldsymbol{\Gamma}_\rho$ defined in (19), and $V = (Y_1, \cdots, Y_p)$ with Y_i's being i.i.d. unit exponential random variables. The results on the empirical sizes and powers are given in Table 5.

From the simulation results, the empirical sizes for all three tests are close to 0.05 which is the nominal type I error we set in the simulation, especially when both n and p are large. Test

statistic t_{np}^c has the smallest size in most cases, and it is a little bit conservative sometimes. The size of $\bar{\ell}_n$ is between that of t_{np}^c and t_{np}^* and both $\bar{\ell}_n$ and t_{np}^* are comparable for most combinations of n and p.

As we expect, the powers of all three test statistics become higher as p grows larger. The increase in n also brings about an increase in power, but not as much as the increase in p does, because $\frac{p(p-1)}{2}$ is the number of r_{ij}^2's involved in the test. All test statistics achieve high power when $\rho = 0.05$. Three test statistics result in comparable powers in general, although the power of Chang and Qi's test statistic is occasionally a little bit less than the other two test statistics. These differences may be due to the fact that Chang and Qi's test statistic maintain a lower type I error.

In summary, in this paper, we have developed the one-sided empirical likelihood method and proposed the rescaled empirical likelihood test statistic for testing the complete independence for high dimensional random vectors. The rescaled empirical likelihood test statistic performs very well in terms of the size and power and can serve as a good alternative to the existent test statistics in the literature.

4. Proofs

Proof of Proposition 2.1. To prove the strict concavity of $L(\mu)$, we need to show that for $\mu_1, \mu_2 \in H$, $\mu_1 \neq \mu_2$,

$$\log L(t\mu_1 + (1-t)\mu_2) > t \log L(\mu_1) + (1-t) \log L(\mu_2), \ \ t \in (0,1). \tag{20}$$

Since $\mu_j \in H$ for $j = 1, 2$, we have $\log L(\mu_j) = \log \prod_{i=1}^{N} \omega_{ji} = \sum_{i=1}^{N} \log \omega_{ji}$, where $\omega_{ji} > 0$, $i = 1, \cdots, N$ are determined by (8) and (9) with μ being replaced by μ_j, $\sum_{i=1}^{N} \omega_{ji} = 1$, $\sum_{i=1}^{N} \omega_{ji} y_i = \mu_j$ for $j = 1, 2$.

For every $t \in (0,1)$, set $\omega_{ti} = t\omega_{1i} + (1-t)\omega_{2i}$, $i = 1, \cdots, N$. Then $\omega_{ti} > 0$, $\sum_{i=1}^{N} \omega_{ti} = 1$,

14

$\sum_{i=1}^{N} \omega_{ti} y_i = t\mu_1 + (1-t)\mu_2 \in H$. Since $\log x$ is strictly concave in $(0, \infty)$, we have

$$\log(\omega_{ti}) = \log\left(t\omega_{1i} + (1-t)\omega_{2i}\right) \geq t\log\omega_{1i} + (1-t)\log\omega_{2i} \quad i = 1, \cdots, N,$$

and at least one of the inequalities is strict, i.e, $\log(\omega_{ti}) > t\log\omega_{1i} + (1-t)\log\omega_{2i}$ for some i, since $\mu_1 \neq \mu_2$ implies $(\omega_{11}, \omega_{12}, \cdots, \omega_{1N}) \neq (\omega_{21}, \omega_{22}, \cdots, \omega_{2N})$. Therefore, we get

$$\sum_{i=1}^{N} \log(\omega_{ti}) > \sum_{i=1}^{N} \left(t\log\omega_{1i} + (1-t)\log\omega_{2i}\right) = t\log L(\mu_1) + (1-t)\log L(\mu_2),$$

which implies

$$\log L(t\mu_1 + (1-t)\mu_2) \geq \log \prod_{i=1}^{N} \omega_{ti} = \sum_{i=1}^{N} \log(\omega_{ti}) > t\log L(\mu_1) + (1-t)\log L(\mu_2),$$

proving (20).

When $\mu = \bar{y}$, an obvious solution to (9) is $\lambda = 0$. Since the solution to (9) is unique, we see that $\omega_i = N^{-1}$, and thus, $L(\bar{y}) = N^{-N}$. We also notice that

$$\sup_{\mu} L(\mu) = \sup_{\mu \in H} L(\mu) \leq \sup\left\{\prod_{i=1}^{N} \omega_i \Big| \omega_i \geq 0, \sum_{i=1}^{N} \omega_i = 1\right\} = N^{-N}.$$

The last step is obtained by using the Lagrange multipliers. We omit the details here. Therefore, we conclude that $L(\bar{y}) = \sup_{\mu} L(\mu) = N^{-N}$. $\qquad\square$

Proof of Theorem 1. We assume the null hypothesis in (2) is true in the proof.

Define $\sigma_n^2 = \frac{2(n-2)}{n+1}$ and $S_n^2 = \frac{1}{N} \sum_{1 \leq i < j \leq p} \left((n-1)r_{ij}^2 - 1\right)^2$. Review that $N = p(p-1)/2$. We have $\sigma_{np}^2 = N\sigma_n^2/(n-1)^2$. Since the distribution of y_j's depends on n, $\{y_j, 1 \leq j \leq N\}$ forms an array of random variables.

If the following three conditions are satisfied: (i). $\frac{1}{\sigma_n} \max_{1 \leq j \leq N} |y_j - 1| = o_p(N^{1/2})$ as $n \to \infty$;

15

(ii). $\frac{1}{N\sigma_n^2} \sum\limits_{j=1}^{N} (y_j - 1)^2 \xrightarrow{p} 1$ as $n \to \infty$; (iii). $\dfrac{\sum\limits_{j=1}^{N} y_j - N}{\sqrt{N\sigma_n^2}} \xrightarrow{d} N(0,1)$ as $n \to \infty$, equivalently, in term of r_{ij}^2's,

(C1). $\dfrac{1}{\sigma_n} \max\limits_{1 \leq i < j \leq p} |(n-1)r_{ij}^2 - 1| = o_p(N^{1/2})$ as $n \to \infty$;

(C2). $\dfrac{1}{\sigma_n^2} S_n^2 \xrightarrow{p} 1$ as $n \to \infty$;

(C3). $z_n := \dfrac{\sum\limits_{1 \leq i < j \leq p}(n-1)r_{ij}^2 - N}{\sqrt{N\sigma_n^2}} \xrightarrow{d} N(0,1)$ as $n \to \infty$,

we can follow the same procedure as in Owen [14] or use Theorem 6.1 in Peng and Schick [15] to conclude that

$$\ell(1) = \left(\frac{\sum\limits_{1 \leq i < j \leq p}(n-1)r_{ij}^2 - N}{\sqrt{N\sigma_n^2}} \right)^2 (1 + o_p(1)) + o_p(1) = z_n^2(1 + o_p(1)) + o_p(1)$$

where $\ell(1)$ is defined in (10) with $\mu_0 = 1$. Again, by using condition (C3), we have

$$\ell_n = z_n^2(1 + o_p(1))I(\bar{y} > 0) + o_p(1) = z_n^2(1 + o_p(1))I(z_n > 0) + o_p(1) \xrightarrow{d} Z^2 I(Z > 0)$$

as $n \to \infty$, where ℓ_n is defined in (13), proving Theorem 1.

Now we will verify conditions (C1), (C2) and (C3). (C3) has been proved by Chang and Qi [4] as we indicate below equation (3).

Assume (i, j) is a pair of integers with for $1 \leq i < j \leq p$. It is proved in Schott [17] that

$$E(r_{ij}^2) = \frac{1}{n-1}, \quad \text{Var}(r_{ij}^2) = \frac{2(n-2)}{(n+1)(n-1)^2} = \frac{\sigma_n^2}{(n-1)^2}, \tag{21}$$

From Chang and Qi [4], we have

$$E(r_{ij}^4) = \frac{3}{(n-1)(n+1)}, \quad E(r_{ij}^6) = \frac{15}{(n-1)(n+1)(n+3)},$$
$$E(r_{ij}^8) = \frac{105}{(n-1)(n+1)(n+3)(n+5)}. \tag{22}$$

16

By using binomial expansion, we also have

$$m_4 := E\left((r_{ij}^2 - \frac{1}{n-1})^4\right) = E(r_{ij}^8) - 4\frac{E(r_{ij}^6)}{n-1} + 6\frac{E(r_{ij}^4)}{(n-1)^2} - 4\frac{E(r_{ij}^2)}{(n-1)^3} + \frac{1}{(n-1)^4}. \quad (23)$$

Now we can verify condition (C1). By use of Chebyshev's inequality, equations (21) and (22)

$$
\begin{aligned}
P\left(\frac{\max\limits_{1\leq i<j\leq p}(n-1)r_{ij}^2}{\sigma_n} > \delta N^{1/2}\right) &\leq \sum_{1\leq i<j\leq p} P\left(\frac{r_{ij}^2}{\sigma_n} > \frac{\delta N^{1/2}}{n-1}\right) \\
&\leq \frac{N(n-1)^3}{\delta^4 N^{3/2}\sigma_n^3} E(r_{12}^6) \\
&= O\left(\frac{1}{N^{1/2}}\right) \to 0
\end{aligned}
$$

as $n \to \infty$ for every $\delta > 0$. This implies $\frac{1}{\sigma_n}\max\limits_{1\leq i<j\leq p}(n-1)r_{ij}^2 = o(N^{1/2})$. Hence, we have

$$\frac{1}{\sigma_n}\max_{1\leq i<j\leq p}|(n-1)r_{ij}^2 - 1| = \frac{1}{\sigma_n}\max_{1\leq i<j\leq p}(n-1)r_{ij}^2 + O(1) = o_p(N^{1/2}),$$

proving condition (C1).

Below we will use (i,j) and (s,t) to denote two pair of integers with $1 \leq i < j \leq p$ and $1 \leq s < t \leq p$. It follows from Theorem 2 in Veleval and Ignatov [18] that $\{r_{ij}, 1 \leq i < j \leq p\}$ are pairwise independent, that is, If $(i,j) \neq (s,t)$, then r_{ij} and r_{st} are independent, thus we have

$$E\left(\left((n-1)r_{ij}^2-1\right)^2\left((n-1)r_{st}^2-1\right)^2\right) = E\left(\left((n-1)r_{ij}^2-1\right)^2\right)E\left(\left((n-1)r_{st}^2-1\right)^2\right) = \sigma_n^4.$$

Since $E(S_n^2) = \sigma_n^2$, we have

$$
\begin{aligned}
E\left(S_n^2 - \sigma_n^2\right)^2 &= E(S_n^4) - \sigma_n^4 \\
&= \frac{1}{N^2} \sum_{1\leq i<j\leq p}\sum_{1\leq s<t\leq p} E\left(\left((n-1)r_{ij}^2-1\right)^2\left((n-1)r_{st}^2-1\right)^2\right) - \sigma_n^4.
\end{aligned}
$$

17

We can classify the summands within the double summation above into two classes: $N(N-1)$ terms in class 1 when $(i,j) \neq (s,t)$ and N terms in class 2 when $(i,j) = (s,t)$. We see that

$$E\Big(\big((n-1)r_{ij}^2 - 1\big)^2\big((n-1)r_{st}^2 - 1\big)^2\Big) = \sigma_n^4$$

if $(i,j) \neq (s,t)$ by using the independence, and

$$E\Big(\big((n-1)r_{ij}^2 - 1\big)^2\big((n-1)r_{st}^2 - 1\big)^2\Big) = m_4(n-1)^4$$

if $(i,j) = (s,t)$, where m_4 is given by (23). Therefore, we have

$$E\Big(S_n^2 - \sigma_n^2\Big)^2 = \frac{1}{N^2}\big(N(N-1)\sigma_n^4 + Nm_4(n-1)^4\big) - \sigma_n^4 = \frac{m_4(n-1)^4 - \sigma_n^4}{N}.$$

In view of (21), (22) and (23), some tedious calculation shows that

$$E\Big(S_n^2 - \sigma_n^2\Big)^2 = \frac{16(n-2)(7n^3 - 30n^2 + 11n + 60)}{p(p-1)(n+1)^2(n+3)(n+5)} = \frac{4(7n^3 - 30n^2 + 11n + 60)\sigma_n^4}{p(p-1)(n-2)(n+3)(n+5)},$$

which implies

$$E\Big(\frac{S_n^2}{\sigma_n^2} - 1\Big)^2 = \frac{4(7n^3 - 30n^2 + 11n + 60)}{p(p-1)(n-2)(n+3)(n+5)} = O\big(\frac{1}{p_n^2}\big) \to 0,$$

as $n \to \infty$, and thus Condition (C2) holds. The proof of Theorem 1 is complete. $\qquad\square$

Proof of Theorem 2. Theorem 2 can be proved by using similar arguments in the proof of Theorem 1. We will continue to use the notation defined in the proof of Theorem 1.

Under the conditions in Theorem 2, Chen and Shao [5] have obtained the following results:

$$\begin{aligned}
E(r_{ij}^2) &= \frac{1}{n-1}, \quad E(r_{ij}^4) = \frac{3}{n^2} + O\big(\frac{1}{n^3}\big), \quad E(r_{ij}^8) = O\big(\frac{1}{n^4}\big); \\
E(r_{ij_1}^2 r_{ij_2}^2) &= \frac{1}{(n-1)^2}, \quad E(r_{ij_1}^4 r_{ij_2}^4) = \frac{9}{n^4} + O\big(\frac{1}{n^5}\big) \quad \text{if } j_1 \neq j_2,
\end{aligned}$$

$\hfill(24)$

where $1 \leq i \neq j \leq p$, $1 \leq i \neq j_1 \leq p$ and $1 \leq i \neq j_2 \leq p$. It follows from the C_r inequality

18

that

$$E\big((n-1)r_{ij}^2-1\big)^4 \le 2^3\big((n-1)^4 r_{ij}^8+1\big) = O(1). \tag{25}$$

We need to verify conditions (C1), (C2), and (C3) as given in the proof of theorem 1. Condition (C1) can be verified similarly by using estimates of moments in (24), and condition (C3) follows from the central limit theorem (3), which is true under the condition of the theorem in virtue of Theorem 2.2 in Chen and Shao [5].

Now we proceed to verify condition (C2). First, we have

$$\bar{\sigma}_n^2 := E(S_n^2) = E\Big(\big((n-1)r_{12}^2-1\big)^2\Big) = (n-1)^2 E\big(r_{12}^4\big)-1 = 2+O\Big(\frac{1}{n}\Big) = \sigma_n^2\Big(1+O\Big(\frac{1}{n}\Big)\Big) \tag{26}$$

from (24). Then

$$
\begin{aligned}
E\Big(S_n^2-\bar{\sigma}_n^2\Big)^2 &= E(S_n^4)-\bar{\sigma}_n^4 \\
&= \frac{1}{N^2}\sum_{1\le i<j\le p}\sum_{1\le s<t\le p} E\Big(\big((n-1)r_{ij}^2-1\big)^2\big((n-1)r_{st}^2-1\big)^2\Big)-\bar{\sigma}_n^4.
\end{aligned}
$$

Considering the summands within the double summation above, we see that there are $N(p-2)(p-3)/2$ pairs of sets $\{i,j\}$ and $\{s,t\}$ which are disjoint. For these pairs,

$$E\Big(\big((n-1)r_{ij}^2-1\big)^2\big((n-1)r_{st}^2-1\big)^2\Big) = \bar{\sigma}_n^4.$$

because r_{ij} and r_{st} are independent. For all other $N^2-N(p-2)(p-3)/2$ pairs, corresponding summands are dominated by

$$
\begin{aligned}
E\Big(\big((n-1)r_{ij}^2-1\big)^2\big((n-1)r_{st}^2-1\big)^2\Big) &\le \sqrt{E\big((n-1)r_{ij}^2-1\big)^4 E\big((n-1)r_{st}^2-1\big)^4} \\
&= E\big((n-1)r_{ij}^2-1\big)^4 \\
&= O(1)
\end{aligned}
$$

from the Cauchy-Schwarz inequality and equation (25). Therefore, we have

$$
\begin{aligned}
E\left(S_n^2 - \bar{\sigma}_n^2\right)^2 &= \frac{1}{N^2}\left(\frac{1}{2}N(p-2)(p-3)\bar{\sigma}_n^4 + O\left(N^2 - \frac{1}{2}N(p-2)(p-3)\right)\right) - \bar{\sigma}_n^4 \\
&= O\left(\frac{1}{N^2}\left(N^2 - \frac{1}{2}N(p-2)(p-3)\right)\right) \\
&= O(\frac{1}{p}) \to 0
\end{aligned}
$$

as $n \to \infty$. In the estimation above, we also use the fact that $\bar{\sigma}_n^2 \sim \sigma_n^2 \to 2$ from (26). Therefore, we have

$$
E\left(\frac{S_n^2}{\bar{\sigma}_n^2} - 1\right)^2 \to 0
$$

as $n \to \infty$. This yields $\dfrac{S_n^2}{\bar{\sigma}_n^2} \overset{p}{\to} 1$, which together with (26) implies condition (C2). This completes the proof of the theorem. $\square$

Proof of Equation (17). In the proofs of Theorems 2 and 2, we have obtained that $S_n/\sigma_n^2 \overset{p}{\to} 1$, which implies $S_n \overset{p}{\to} 2$ since $\sigma_n^2 \to 2$. From condition (C3), we have

$$
\frac{1}{N}\left(\sum_{1 \le i < j \le p}(n-1)^2 r_{ij}^2 - N\right) \overset{p}{\to} 0.
$$

Therefore, we get

$$
\begin{aligned}
\frac{2(n-1)(n+1)}{3(p-1)(p+4)}\sum_{1 \le i < j \le p} r_{ij}^4 &= \frac{1+o(1)}{3N}\sum_{1 \le i < j \le p}(n-1)^2 r_{ij}^4 \\
&= \frac{1+o(1)}{3}\left(S_n^2 + 1 + \frac{2}{N}\left(\sum_{1 \le i < j \le p}(n-1)^2 r_{ij}^2 - N\right)\right) \\
&\overset{p}{\to} 1
\end{aligned}
$$

as $n \to \infty$, proving (17). $\square$

Disclosure statement

No potential conflict of interest was reported by the authors.

References

[1] Anderson, T. W.(1984). An Introduction to Multivariate Statistical Analysis. 2nd Ed *John Wiley & Sons.*.

[2] Bai, Z., Jiang, D., Yao, J. and Zheng, S. (2009). Corrections to LRT on large dimensional covariance matrix by RMT. *Annals of Statistics, 37*(6B), 3822-3840.

[3] Bartlett, M. S. (1954). A note on multiplying factors for various chi-squared approximations. *J. Royal Stat. Soc.*, Ser. B 16, 296-298.

[4] Chang, S. and Qi, Y. (2018). On Schott's and Mao's test statistics for independence of normal random vectors. *Statistics and Probability Letters* 140, 132-141.

[5] Chen, Y. and Shao, Q.M. (2012). Berry-Esseen inequality for unbounded exchangeable pairs. In Probability Approximations and Beyond, A.D. Barbour et al. (eds.), Lecture Notes in Statistics 205, pp. 13–30, Springer, New York..

[6] Jiang, D., Jiang, T. and Yang, F. (2012). Likelihood ratio tests for covariance matrices of high-dimensional normal distributions. *Journal of Statistical Planning and Inference, 142*(8), 2241-2256.

[7] Jiang, T. and Qi, Y. (2015). Likelihood ratio tests for high-dimensional normal distributions. *Scandinavian Journal of Statistics, 42*(4), 988-1009.

[8] Jiang, T. and Yang, F. (2013). Central limit theorems for classical likelihood ratio tests for high-dimensional normal distributions. *Annals of Statistics, 41*(4), 2029-2074.

[9] Mao, G. (2014). A new test of independence for high-dimensional data. *Statist. Probab. Lett.* 93, 14-18.

[10] Morrison, D. F. (2005). Multivariate Statistical Methods. *Duxbury Press*, 4th Ed.

[11] Muirhead, R. J. (1982). *Aspects of Multivariate Statistical Theory.* Wiley, New York.

[12] Owen, A.B. (1988). Empirical likelihood ratio confidence intervals for a single functional. *Biometrika* 75, 237-249.

[13] Owen, A.B. (1990). Empirical likelihood ratio confidence regions. *Ann. Statist.* 18, 90-120.

[14] Owen, A.B. (2001). Empirical Likelihood. *New York: Chapman and Hall/CRC*.

[15] Peng, H. and Schick, A. (2013). An empirical likelihood approach to goodness of fit testing. *Bernoulli* 19, 954-981.

[16] Qi, Y., Wang, F. and Zhang, L. (2019). Likelihood ratio test of independence of components for high-dimensional normal vectors. *Annals of the Institute of Statistical Mathematics 71*, 911-946.

[17] Schott, J. R. (2005). Testing for complete independence in high dimensions. *Biometrika, 92*(4), 951-956.

[18] Veleva1, E. I. and Ignatov, T. G. (2006). Distributions of joint sample correlation coefficients of independent normally distributed random variables. *Advanced Studies in Contemporary Mathematics* 12, 247-254.

Table 1. Sizes and Powers of tests, Normal distribution

n	p	size ($\rho = 0$)			power ($\rho = 0.02$)			power ($\rho = 0.05$)		
		$\bar{\ell}_n$	t_{np}^c	t_{np}^*	$\bar{\ell}_n$	t_{np}^c	t_{np}^*	$\bar{\ell}_n$	t_{np}^c	t_{np}^*
20	10	0.0593	0.0486	0.0598	0.0646	0.0551	0.0657	0.0974	0.0863	0.0987
	20	0.0609	0.0565	0.0618	0.0670	0.0626	0.0675	0.1357	0.1281	0.1366
	50	0.0594	0.0584	0.0596	0.0836	0.0810	0.0842	0.3079	0.3030	0.3096
	100	0.0524	0.0519	0.0525	0.1156	0.1142	0.1159	0.5584	0.5571	0.5588
50	10	0.0596	0.0484	0.0603	0.0750	0.064	0.0744	0.1660	0.1478	0.1656
	20	0.0533	0.0481	0.0525	0.0855	0.0790	0.0859	0.3257	0.3098	0.3244
	50	0.0508	0.0489	0.0508	0.1457	0.1410	0.1461	0.7372	0.7320	0.7362
	100	0.0536	0.0519	0.0535	0.2684	0.2660	0.2684	0.9609	0.9604	0.9608
100	10	0.0608	0.0504	0.0607	0.0894	0.0743	0.0882	0.3305	0.3025	0.3285
	20	0.0581	0.0511	0.0573	0.1265	0.1167	0.1260	0.6520	0.6345	0.6495
	50	0.0523	0.0494	0.0519	0.2735	0.2675	0.2723	0.9816	0.9809	0.9816
	100	0.0516	0.0506	0.0515	0.5711	0.5669	0.5703	0.9997	0.9997	0.9997

Table 2. Sizes and Powers of tests, Uniform distribution

n	p	size ($\rho = 0$)			power ($\rho = 0.02$)			power ($\rho = 0.05$)		
		$\bar{\ell}_n$	t_{np}^c	t_{np}^*	$\bar{\ell}_n$	t_{np}^c	t_{np}^*	$\bar{\ell}_n$	t_{np}^c	t_{np}^*
20	10	0.0610	0.0529	0.0615	0.0607	0.0525	0.0626	0.0913	0.0802	0.0934
	20	0.0600	0.0571	0.0615	0.0668	0.0621	0.0686	0.1230	0.1147	0.1243
	50	0.0595	0.0582	0.0607	0.0806	0.0785	0.0812	0.2680	0.2647	0.2693
	100	0.0585	0.0582	0.0598	0.1099	0.1093	0.1109	0.5183	0.5171	0.5202
50	10	0.0572	0.0489	0.0575	0.0773	0.0655	0.0774	0.1744	0.1536	0.1749
	20	0.0606	0.0548	0.0600	0.0809	0.0734	0.0816	0.3061	0.2920	0.3062
	50	0.0559	0.0530	0.0562	0.1358	0.1319	0.1359	0.7449	0.7387	0.7449
	100	0.0526	0.0518	0.0527	0.2417	0.2378	0.2422	0.9741	0.9736	0.9741
100	10	0.0596	0.0497	0.0583	0.0842	0.0710	0.0848	0.3225	0.2934	0.3200
	20	0.0589	0.0528	0.0580	0.1269	0.1152	0.1271	0.6430	0.6265	0.6397
	50	0.0516	0.0489	0.0522	0.2650	0.2587	0.2646	0.9889	0.9880	0.9887
	100	0.0504	0.0494	0.0504	0.5567	0.5525	0.5566	1.0000	1.0000	1.0000

Table 3. Sizes and Powers of tests, Exponential distribution

n	p	size ($\rho = 0$)			power ($\rho = 0.02$)			power ($\rho = 0.05$)		
		$\bar{\ell}_n$	t_{np}^c	t_{np}^*	$\bar{\ell}_n$	t_{np}^c	t_{np}^*	$\bar{\ell}_n$	t_{np}^c	t_{np}^*
20	10	0.0718	0.0636	0.0745	0.0799	0.0734	0.0845	0.1419	0.1338	0.1486
	20	0.0638	0.0610	0.0670	0.0867	0.0859	0.0924	0.2135	0.2116	0.2221
	50	0.0581	0.0586	0.0612	0.1224	0.1266	0.1301	0.3982	0.4040	0.4080
	100	0.0602	0.0622	0.0632	0.1901	0.1955	0.1975	0.5774	0.5848	0.5871
50	10	0.0734	0.0620	0.0751	0.0950	0.0840	0.0975	0.2288	0.2156	0.2335
	20	0.0650	0.0606	0.0660	0.1158	0.1114	0.1205	0.3834	0.3800	0.3898
	50	0.0612	0.0608	0.0643	0.1877	0.1912	0.1953	0.7102	0.7140	0.7175
	100	0.0589	0.0607	0.0618	0.3230	0.3289	0.3310	0.9079	0.9107	0.9111
100	10	0.0701	0.0604	0.0698	0.1048	0.0936	0.1088	0.3636	0.3428	0.3689
	20	0.0646	0.0595	0.0647	0.1555	0.1492	0.1578	0.6350	0.6281	0.6400
	50	0.0636	0.0622	0.0653	0.3108	0.3125	0.3187	0.9481	0.9486	0.9495
	100	0.0574	0.0583	0.0600	0.5823	0.5878	0.5899	0.9965	0.9966	0.9966

Table 4. Sizes and Powers of tests, Mixture distribution

n	p	size ($\rho = 0$)			power ($\rho = 0.02$)			power ($\rho = 0.05$)		
		$\bar{\ell}_n$	t_{np}^c	t_{np}^*	$\bar{\ell}_n$	t_{np}^c	t_{np}^*	$\bar{\ell}_n$	t_{np}^c	t_{np}^*
20	10	0.0650	0.0547	0.0664	0.0682	0.0594	0.0680	0.1042	0.0921	0.1052
	20	0.0618	0.0565	0.0630	0.0704	0.0650	0.0706	0.1561	0.1477	0.1572
	50	0.0605	0.0584	0.0608	0.0974	0.0955	0.0983	0.3381	0.3353	0.3394
	100	0.0688	0.0677	0.0692	0.1395	0.1381	0.1407	0.5979	0.5974	0.5998
50	10	0.0624	0.0520	0.0616	0.0778	0.0657	0.0781	0.1991	0.1790	0.1988
	20	0.0602	0.0548	0.0606	0.0958	0.0869	0.0952	0.3832	0.3713	0.3828
	50	0.0587	0.0559	0.0578	0.1643	0.1595	0.1638	0.7827	0.7792	0.7824
	100	0.0569	0.0561	0.057	0.3086	0.3063	0.3097	0.9669	0.9666	0.9672
100	10	0.0610	0.0510	0.0615	0.0990	0.0845	0.0961	0.3777	0.3484	0.3745
	20	0.0599	0.0520	0.0592	0.1395	0.1299	0.1380	0.7200	0.7055	0.7183
	50	0.0556	0.0522	0.0555	0.3137	0.3080	0.3137	0.9888	0.9887	0.9888
	100	0.0561	0.0545	0.0559	0.6389	0.6359	0.6400	1.0000	1.0000	1.0000

Table 5. Sizes and Powers of tests, Sum

n	p	size ($\rho = 0$)			power ($\rho = 0.02$)			power ($\rho = 0.05$)		
		$\bar{\ell}_n$	t_{np}^c	t_{np}^*	$\bar{\ell}_n$	t_{np}^c	t_{np}^*	$\bar{\ell}_n$	t_{np}^c	t_{np}^*
20	10	0.0601	0.0520	0.0607	0.0625	0.0552	0.0634	0.0997	0.0884	0.1019
	20	0.0556	0.0499	0.0558	0.0635	0.0584	0.0637	0.1402	0.1303	0.1401
	50	0.0520	0.0499	0.0520	0.0804	0.0773	0.0807	0.3066	0.3031	0.3072
	100	0.0592	0.0583	0.0593	0.1172	0.1153	0.1173	0.5509	0.5484	0.5515
50	10	0.0587	0.0493	0.0586	0.0723	0.0610	0.0723	0.1702	0.1501	0.1684
	20	0.0558	0.0497	0.0557	0.0843	0.0762	0.0838	0.3315	0.3144	0.3290
	50	0.0537	0.0507	0.0541	0.1429	0.1385	0.1429	0.7446	0.7377	0.7433
	100	0.0557	0.0540	0.0555	0.2637	0.2614	0.2642	0.9550	0.9545	0.9550
100	10	0.0635	0.0552	0.0640	0.0897	0.0758	0.0892	0.3237	0.2955	0.3187
	20	0.0593	0.0527	0.0586	0.1211	0.1097	0.1197	0.6540	0.6360	0.6513
	50	0.0480	0.0459	0.0479	0.2716	0.2644	0.2706	0.9803	0.9791	0.9801
	100	0.0551	0.0537	0.0549	0.5719	0.5690	0.5713	1.0000	1.0000	1.0000