



# Nematic Liquid Crystal Phase in a System of Interacting Dimers and Monomers

Ian Jauslin<sup>1</sup>, Elliott H. Lieb<sup>2</sup>

<sup>1</sup> School of Mathematics, Institute for Advanced Study, Princeton, New Jersey, USA.  
E-mail: jauslin@ias.edu

<sup>2</sup> Departments of Mathematics and Physics, Princeton University, Princeton, New Jersey, USA.  
E-mail: lieb@princeton.edu

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**Abstract:** We consider a monomer-dimer system with a strong attractive dimer-dimer interaction that favors alignment. In 1979, Heilmann and Lieb conjectured that this model should exhibit a nematic liquid crystal phase, in which the dimers are mostly aligned, but do not manifest any translational order. We prove this conjecture for large dimer activity and strong interactions. The proof follows a Pirogov-Sinai scheme, in which we map the dimer model to a system of hard-core polymers whose partition function is computed using a convergent cluster expansion.

## 1. Introduction

In a 1979 paper, Heilmann and one of us [HL79] attempted to construct a simple statistical mechanical lattice model of a liquid crystal phase transition. Such a model would have to have the property that the constituent ‘molecules’ would have to show no long-range order at high temperature and, at low temperature, have a transition to a phase in which there is long-range rotational order of the molecules, but *no* long-range translational order. In other words, the molecules are nearly parallel, but their centers show no long-range correlations. Such a model had not been constructed before then, although there was the 1949 heuristic ultra-thin, ultra-long molecule model of Onsager [On49].

In the model considered in [HL79], the molecules are represented by interacting dimers or trimers on a square or cubic lattice. It was shown, by reflection positivity and chessboard estimates, that, for several different models, the system exhibits long-range orientational order at low temperature. Thus, if we specify the orientation of one dimer somewhere in the lattice, any other dimer is oriented in the same way with large probability. It was not proved, however, that this rotational order is *not* accompanied by translational order. That is, it was not proved that fixing a dimer somewhere on the lattice does not induce correlations in the position of distant dimers, even though it does induce a preference for their orientation.

Since then, there have been many new developments in the field, though a complete proof of the lack of translational order for any of the models considered in [HL79] was, until now, still lacking. In [AH80], a new three-dimensional model was added to the list by extending one of the two-dimensional models in [HL79]. In [AZ82, Za96], the result was extended to a model of elongated molecules on a lattice admitting *continuous* orientations, with short- (in three dimensions) and long- (in two dimensions) range attractive interactions. A liquid crystalline (also called *nematic*) phase was later proved to exist [BKL84] (that is, both orientational order and a lack of translational order are shown) in a model of infinitely thin long molecules in two dimensions admitting a *finite* number of orientations (although the discussion in [BKL84] is limited to a remark in the concluding section of the paper). This behavior was also shown to occur in an integrable lattice model of rods admitting two orientations and of varying length [IVZ06] or of a fixed, long length [DG13]. Finally, in [ACM14], a mean-field interacting dimer model was introduced and solved.

There has also been some progress towards proving the conjecture in [HL79]. Most efforts have focused on one of the models in [HL79], model I (see Fig. 1), which is two-dimensional, and involves an interaction between collinear, neighboring dimers. In [AI16], D. Alberici tweaked this model by making the activity of horizontal and vertical dimers different, thus favoring one orientation, and showed the emergence of a liquid crystalline phase. There have also been numerical results [PCF14] supporting the conjecture.

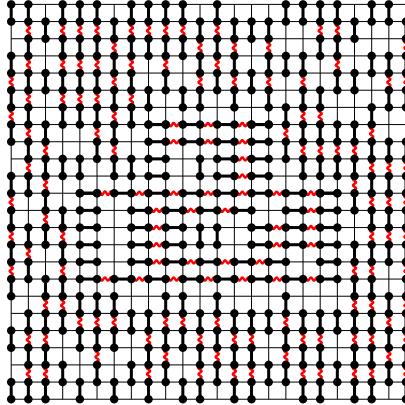
In this paper, we shall prove the conjecture in [HL79] that there is no long-range translational order in model I. There is little doubt that similar proofs could be devised for the other models and other dimensions for which orientational order was proved in [HL79].

Let us describe the model in more precise terms. It is a monomer-dimer system on the square lattice, in which a dimer is an object that covers exactly two neighboring vertices, and a monomer covers a single vertex. No two objects are permitted to cover the same vertex. Monomers are to be thought of, in this context, as empty sites, whereas dimers represent molecules. The dimer-activity  $z$  is large, which favors dimers heavily, but the presence of monomers is crucial. In addition we introduce a strong attractive force that favors alignment. Without this interaction, as was shown in [HL72], the monomer-dimer model would not have phase transitions at positive temperature, and thus, would exhibit no liquid crystalline ordering.

The attractive interaction assigns a negative energy  $-J$  to every pair of dimers that are adjacent and aligned, that is, that are on the same row or the same column, see Fig. 1. We offer two interpretations of this model. One is of polar molecules of length 1, represented by individual dimers; the other is of molecules of varying length, modeled by chains of adjacent and aligned dimers.

We choose boundary conditions that favors vertical dimers, and focus on the parameter regime  $J \gg z \gg 1$ . We first prove that horizontal dimers are unlikely in the bulk, in accordance with the result of [HL79]. The method of proof is completely different from that in [HL79]; in particular we do not use reflection positivity. We further show that the probability of finding a vertical dimer on a given edge is, in the thermodynamic limit, independent of the position of the edge. Furthermore, the joint probability of finding a dimer at an edge  $e$  and another at  $e'$ , up to a constant, decays exponentially in the distance between  $e$  and  $e'$  with a rate  $\gtrsim e^{-\frac{3}{2}J} z^{-\frac{1}{2}}$ . This proves the absence of translational order.

The proof follows a Pirogov-Sinai [PS75] scheme, which is an extension of the Peierls argument. The main idea is to map the interacting dimer model to a system of hard-core



**Fig. 1.** An example of a dimer configuration. Interacting dimers are depicted as connected by a red (color online) wavy line

polymers, and show that the effective activity of these polymers decays *sufficiently* fast in their size. We then use a cluster expansion to compute the partition function of the model in terms of an absolutely convergent series, and estimate the one- and two-point correlation functions.

This paper is organized as follows. In Sect. 2, we define the model in precise terms, state our main theorem and provide a detailed sketch of the proof. Section 3 describes the solution to an ancillary model in which only one dimer orientation is allowed, which plays an important role in the rest of the proof. In Sect. 4, we map the interacting dimer model to the polymer model. In Sect. 5 we prove bounds on the polymer activity and entropy, and compute the partition function of the polymer model in terms of an absolutely convergent cluster expansion. Finally, the proof of the main theorem is concluded in Sect. 6.

## 2. The Model

**2.1. Definition of the model.** A dimer configuration is a collection of non-overlapping edges of  $\mathbb{Z}^2$ . In order to define these formally, we denote the set of edges of a subset  $\Lambda \subset \mathbb{Z}^2$  by

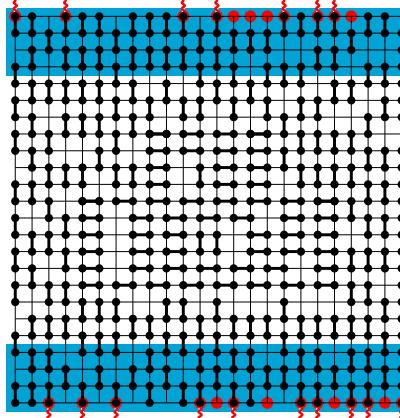
$$\mathcal{E}(\Lambda) := \{\{x, x'\}, x, x' \in \Lambda, \|x - x'\| = 1\} \quad (1)$$

in which  $\|\cdot\|$  denotes the Euclidean distance. The edges of  $\Lambda$  are either horizontal (h-edges) or vertical (v-edges), and given a set of edges  $E \subset \mathcal{E}(\Lambda)$  and  $q \in \{h, v\}$ , we denote the set of  $q$ -edges in  $E$  by  $\mathbb{D}_q(E)$ . We then define the set of **dimer configurations** in  $\Lambda$  as

$$\mathcal{Q}(\Lambda) := \{E \subset \mathcal{E}(\Lambda), \forall e \neq e' \in E, e \cap e' = \emptyset\} \quad (2)$$

(see Fig. 1 for an example).

**1 - Interaction.** We introduce a strong interaction between dimers, that favors configurations in which dimers are aligned, collinear and neighbors (see Fig. 1). Every such pair of dimers contributes  $-J$  to the energy of the configuration, and  $J$  will be taken to be large.



**Fig. 2.** The boundary condition. Here, we have taken  $q = v$  and  $\ell_0 = 4$ . There can be no horizontal dimers in the cyan (color online) region. Some of the vertices on the boundary, depicted as large red (color online) discs, are *magnetized*: they contribute  $-J$  to the energy when they are occupied

**2 - Boundary condition.** We choose the boundary condition in such a way that either vertical or horizontal dimers are favored. To determine which it is, we introduce a variable  $q \in \{v, h\}$  which is set to  $v$  if vertical dimers are favored and  $h$  if horizontal ones are. In addition, we define  $-q$  as the *opposite* of  $q$ , that is, if  $q = h$ , then  $-q = v$  and vice-versa. The boundary condition consists of two forces: first,  $-q$ -dimers are not allowed to be too close to the boundary, and  $q$ -dimers may be attracted by certain parts of the boundary.

Note that we could consider different boundary conditions, as long as they favor horizontal or vertical dimers. It would require an extra computation, which we have chosen not to carry out, as we have achieved our goal of showing that there are two extremal Gibbs states in which the rotational symmetry is broken, but the translational one is not.

In order to define the boundary condition precisely, let us first define the **boundary** of a bounded subset  $\Lambda \subset \mathbb{Z}^2$ , denoted by  $\partial\Lambda$ , as the set of edges  $\{x, x'\} \in \mathcal{E}(\mathbb{Z}^2)$  with  $x \in \Lambda$  and  $x' \in \mathbb{Z}^2 \setminus \Lambda$ . In addition, we define the  **$q$ -distance** between two points  $x, x' \in \mathbb{Z}^2$ , denoted by  $\mathfrak{d}_q(x, x') \in \mathbb{R} \cup \{\infty\}$ , in the following way. If  $x$  and  $x'$  are in the same  $q$ -line (a  $v$ -line is a vertical line and an  $h$ -line is a horizontal one), then  $\mathfrak{d}_q(x, x') = \|x - x'\|$ , and if they are not, then  $\mathfrak{d}_q(x, x') = \infty$ . We can now define the boundary condition, which, we recall, consists of two forces (see Fig. 2 for an example).

- We fix a length scale  $\ell_0 > 1$  and require that every  $-q$ -dimer in  $\Lambda$  be separated from the boundary of  $\Lambda$  by a  $q$ -distance of at least  $\ell_0$ . We denote the set of dimer configurations satisfying this condition by

$$\Omega_{q, \ell_0}(\Lambda) := \{\underline{\delta} \in \Omega(\Lambda), \quad \mathfrak{d}_q(\mathbb{D}_{-q}(\underline{\delta}), \partial\Lambda) \geq \ell_0\}. \quad (3)$$

(In this paper, we will use the convention that the distance between two sets is the smallest distance between the elements of the set. Furthermore, we will use this convention recursively to define the distance between sets of sets, and so forth...)

- In addition to this condition, we will allow part or all of  $\partial\Lambda$  to be *magnetized*, by which we mean that parts of the boundary may attract  $q$ -dimers, as if there were  $q$ -dimers right outside it. Formally, we introduce a subset  $\varrho \subset \partial\Lambda$  of edges on the boundary

which are magnetized, and, given a dimer configuration  $\underline{\delta} \in \Omega_{q,\ell_0}(\Lambda)$ , we define the set of dimers that are bound to the boundary as

$$\mathbb{B}_q(\underline{\delta}, \varrho) := \{d \in \underline{\delta}, \quad \mathfrak{d}_q(d, \varrho) = 0\}. \quad (4)$$

Every dimer in  $\mathbb{B}_q(\underline{\delta}, \varrho)$  contributes  $-J$  to the interaction, as if every such dimer interacted with a phantom dimer outside  $\Lambda$ .

The boundary condition is thus specified by the triplet  $(q, \varrho, \ell_0) \equiv \mathbf{q}$ , and we will use the shorthand  $\Omega_{\mathbf{q}} \equiv \Omega_{q,\ell_0}$  and  $\mathbb{B}_{\mathbf{q}}(\underline{\delta}) \equiv \mathbb{B}_q(\underline{\delta}, \varrho)$ .

**3 - Observables.** In this paper, we will compute the *grand-canonical partition function* of the system, defined as

$$Z(\Lambda | \mathbf{q}) = \sum_{\underline{\delta} \in \Omega_{\mathbf{q}}(\Lambda)} z^{|\underline{\delta}|} e^{-W_0(\underline{\delta})} \mathfrak{B}_{\mathbf{q}}(\underline{\delta}) \quad (5)$$

with

$$e^{-W_0(\underline{\delta})} := \prod_{d_1, d_2 \in \underline{\delta}} e^{\frac{1}{2} J \mathbb{1}(d_1 \sim d_2)}, \quad \mathfrak{B}_{\mathbf{q}}(\underline{\delta}) := \prod_{d \in \mathbb{B}_{\mathbf{q}}(\underline{\delta})} e^J \quad (6)$$

in which

- $z > 0$  is the *dimer activity*,
- $J > 0$  is the *interaction strength*, (the factor  $\frac{1}{2}$  accounts for the fact that each pair is counted twice)
- $|\underline{\delta}|$  denotes the number of dimers in  $\underline{\delta}$ ,
- $\mathbb{1}(d_1 \sim d_2) \in \{0, 1\}$  identifies which pairs of dimers interact: it is equal to 1 if and only if  $\exists q' \in \{v, h\}$  such that  $d_1$  and  $d_2$  are both  $q'$ -dimers and are at  $q'$ -distance 1 from one another.

In addition, we will compute the  $n$ -point correlation functions, defined as follows. We fix a set of edges  $\Upsilon \equiv \{v_1, \dots, v_n\} \subset \mathcal{E}(\Lambda)$ , and define

$$\langle \mathbb{1}_{v_1} \cdots \mathbb{1}_{v_n} \rangle_{\Lambda, \mathbf{q}} := \frac{1}{Z(\Lambda | \mathbf{q})} \sum_{\substack{\underline{\delta} \in \Omega_{\mathbf{q}}(\Lambda) \\ \underline{\delta} \supset \Upsilon}} z^{|\underline{\delta}|} e^{-W_0(\underline{\delta})} \mathfrak{B}_{\mathbf{q}}(\underline{\delta}). \quad (7)$$

The infinite-volume limit of this correlation function is defined by considering a square  $L \times L$  box  $\Lambda_L$  and taking the  $L \rightarrow \infty$  limit

$$\langle \mathbb{1}_{v_1} \cdots \mathbb{1}_{v_n} \rangle_{\mathbf{q}} := \lim_{L \rightarrow \infty} \langle \mathbb{1}_{v_1} \cdots \mathbb{1}_{v_n} \rangle_{\Lambda_L, \mathbf{q}}. \quad (8)$$

We will assume that the different  $v_i$  are at a distance of at least  $\ell_0$  from each other (this assumption is merely a technical requirement). Note that the partition function of dimers in  $\Lambda$  that contain  $\Upsilon$  can, equivalently, be viewed as the partition function on  $\Lambda \setminus (\bigcup_{v \in \Upsilon} v)$  with a special boundary condition. Namely, the endpoints of  $v$  are *magnetized*, in the sense discussed above, but, unlike the boundary of  $\Lambda$ , which excludes  $-q$ -dimers at a distance  $\ell_0$ , the boundary of  $v$  does not exclude any dimers. Formally, defining

$$\Lambda^{(\Upsilon)} := \Lambda \setminus (\bigcup_{v \in \Upsilon} v) \quad (9)$$

and

$$\Omega_{q,\ell_0}^{(\Upsilon)}(\Lambda) := \{\underline{\delta} \in \Omega_{q,\ell_0}(\Lambda), \quad \forall \delta \in \underline{\delta}, \quad \delta \subset \Lambda^{(\Upsilon)}\} \quad (10)$$

we have (recall that we assume that different  $v_i$ 's are not neighbors (so that sources do not interact directly))

$$\langle \mathbb{1}_{v_1} \cdots \mathbb{1}_{v_n} \rangle_{\Lambda, \mathbf{q}} = \frac{Z^{(\mathcal{Y})}(\Lambda | \mathbf{q})}{Z(\Lambda | \mathbf{q})}, \quad Z^{(\mathcal{Y})}(\Lambda | \mathbf{q}) := z^n \sum_{\underline{\delta} \in \mathcal{Q}_{\mathbf{q}}^{(\mathcal{Y})}(\Lambda)} z^{|\underline{\delta}|} e^{-W_0(\underline{\delta})} \mathfrak{B}_{\mathbf{q}}^{(\mathcal{Y})}(\underline{\delta}) \quad (11)$$

in which  $\mathfrak{B}_{\mathbf{q}}^{(\mathcal{Y})}(\underline{\delta})$  includes the interactions with the sources:

$$\mathfrak{B}_{\mathbf{q}}^{(\mathcal{Y})}(\underline{\delta}) := \mathfrak{B}_{\mathbf{q}}(\underline{\delta}) \left( \prod_{v \in \mathbb{D}_q(\mathcal{Y})} \mathfrak{B}_{(q, \partial_q v, \ell_0)}(\underline{\delta}) \right) \left( \prod_{v \in \mathbb{D}_{-q}(\mathcal{Y})} \mathfrak{B}_{(-q, \partial_{-q} v, \ell_0)}(\underline{\delta}) \right) \quad (12)$$

in which  $\partial_q \equiv \mathbb{D}_q(\partial)$  (by which we mean that for any set  $X$ ,  $\partial_q X \equiv \mathbb{D}_q(\partial X)$ ) (note that the index  $\ell_0$  is redundant; we have kept it in in order not to have to introduce yet another notation for the boundary condition  $\mathbf{q} \equiv (q, \varrho, \ell_0)$ ).

**4 - Oriented dimer model.** As was shown in [HL79], when the interaction strength is sufficiently large, the probability of horizontal and vertical dimers coexisting is low. In fact, the main idea is to compute how much the partition function of the model with  $q$ -boundary conditions differs from that of a similar model in which there are *only*  $q$ -dimers and monomers, and to show that, in a sense to be made precise, this difference is small. We first formally define the *oriented dimer model*, in which only one of the two dimer orientations is allowed: let  $\Theta_q(\Lambda) \subset \mathcal{Q}(\Lambda)$  denote the set of  $q$ -dimer configurations on  $\Lambda$ :

$$\Theta_q(\Lambda) := \{\underline{\delta} \in \mathcal{Q}_q(\Lambda), \delta \in \mathbb{D}_q(\mathcal{E}(\Lambda))\} \quad (13)$$

in terms of which the partition function of the  $q$ -dimer model is

$$\mathfrak{Z}_{\mathbf{q}}^{(\mathcal{Y})}(\Lambda) = z^n \sum_{\underline{\delta} \in \Theta_q(\Lambda^{(\mathcal{Y})})} z^{|\underline{\delta}|} e^{-W_0(\underline{\delta})} \mathfrak{B}_{\mathbf{q}}^{(\mathcal{Y})}(\underline{\delta}). \quad (14)$$

In order to compare  $Z^{(\mathcal{Y})}(\Lambda | \mathbf{q})$  and  $\mathfrak{Z}_{\mathbf{q}}^{(\mathcal{Y})}(\Lambda)$ , we will compute the ratio

$$\frac{Z^{(\mathcal{Y})}(\Lambda | \mathbf{q})}{\mathfrak{Z}_{\mathbf{q}}^{(\mathcal{Y})}(\Lambda)}. \quad (15)$$

Note that, in the oriented dimer model, since different columns of vertical dimers and different rows of horizontal dimers do not interact, in order to compute  $\mathfrak{Z}_{\mathbf{q}}^{(\mathcal{Y})}(\Lambda)$ , it suffices to compute the partition function of dimers on a one-dimensional chain.

**2.2. Result.** Our main result is that, at large activities and yet larger interaction strengths, this model exhibits *nematic* order, that is, it exhibits long-range orientational order, yet no long-range translational order. This is stated precisely in the following theorem.

**Theorem 1** (Nematic phase). *Let  $\mathbf{v} \equiv (v, \emptyset, \ell_0)$ , which corresponds to open boundary conditions coupled with the condition that no horizontal dimers come within a distance  $\ell_0$  of the boundary. There exist large constants (which, in principle, can be worked out)  $C_1, C_2 > 0$  such that, if*

$$J > C_1 z \quad \text{and} \quad z > C_2 \quad (16)$$

*then, taking  $\ell_0 = C_3 e^{\frac{3}{2}J} \sqrt{z}$  for some constant  $C_3 > 0$  ( $\ell_0$  is of the order of the correlation length of the oriented dimer model), the following statements hold.*

– Let  $e_v \in \mathbb{D}_v(\mathcal{E}(\mathbb{Z}^2))$  be a vertical edge,  $\langle \mathbb{1}_{e_v} \rangle_v$  is independent of the position of  $e_v$ , and

$$\langle \mathbb{1}_{e_v} \rangle_v = \frac{1}{2} \left( 1 + O\left(\frac{1}{\sqrt{ze^J}}\right) \right). \quad (17)$$

In other words, the probability of finding a dimer at a given edge is independent of the position of that edge, and most vertices are occupied by a vertical dimer (if the lattice were fully packed, then half the edges are occupied).

– Let  $e_h \in \mathbb{D}_h(\mathcal{E}(\mathbb{Z}^2))$  be a horizontal edge,

$$\langle \mathbb{1}_{e_h} \rangle_v = O(e^{-3J}). \quad (18)$$

Thus, horizontal dimers are unlikely. This implies orientational order (in particular, this implies that  $\langle \mathbb{1}_{e_h} \mathbb{1}_{e_v} \rangle_v = O(e^{-3J})$ ).

– For any pair of edges  $e, e' \in \mathcal{E}(\mathbb{Z}^2)$  which are at a distance of at least  $\ell_0$ ,

$$\langle \mathbb{1}_e \mathbb{1}_{e'} \rangle_v - \langle \mathbb{1}_e \rangle_v \langle \mathbb{1}_{e'} \rangle_v = O(e^{-C_4 \text{dist}_{\text{HL}}(e, e')}) \quad (19)$$

for some constant  $C_4 > 0$ , in which the distance  $\text{dist}_{\text{HL}}$  is that induced by the norm

$$\|(x, y)\|_{\text{HL}} := J|x| + \ell_0^{-1}|y|. \quad (20)$$

This means that the probability of placing two dimers at  $e$  and  $e'$  is equal to a term that does not depend on the position of the edges plus a term that decays exponentially with the distance between them. There is, thus, no long-range translational order. The decay rate is of order  $J \gg 1$  in the horizontal direction and  $e^{-\frac{3}{2}J}z^{-\frac{1}{2}} \ll 1$  in the vertical.

**2.3. Sketch of the proof.** Before discussing the proof that is carried out in this paper, let us mention two simpler approaches we have tried which have failed.

In [HL79], orientational order was proved using reflection positivity and chessboard estimates. The main difficulty with extending this method to prove the lack of translational order is that, as can be seen from Theorem 1, the correlation length of the system is very large:  $\ell_0 \approx e^{\frac{3}{2}J} \sqrt{z}$ , and the lack of order is only visible on that scale, and seems difficult to see using only reflection positivity.

Another natural approach to the problem is to integrate out the vertical dimers and manipulate the resulting effective horizontal dimer model. The idea being that, if vertical dimers are favored on the boundary, then they should dominate, so the horizontal dimer model would be a rarefied gas, which could be treated by standard cluster expansion methods. However, since horizontal dimers are subjected to a surface tension, they tend to bunch together into large *swarms*. In order for this approach to be successful, the swarms would have to pay an energetic price proportional to their volume, in order to counterbalance their entropy. Unfortunately, they do not do so. Note, however, that if we made the activity of horizontal dimers slightly smaller than that of vertical ones, as in [A116], then the horizontal swarms would have a sufficiently large volume cost, and this approach would be successful.

Instead, we opted for a Pirogov-Sinai argument.

The main idea of the proof is to estimate how much the partition function of the full dimer model differs from that of the oriented dimer model, which is integrable, and to show that the dominant contribution to the observables in theorem 1 come from the

oriented dimer model. The oriented dimer model is integrable, and one easily shows that the local dimer density is invariant under translations and satisfies (17). In addition, pair correlations decay in the vertical direction with a rate  $\ell_0^{-1} \approx e^{-\frac{3}{2}J} z^{-\frac{1}{2}}$ , and are identically zero in the horizontal direction. Therefore, (19) holds in the oriented dimer model, with the improvement that the decay rate in the horizontal direction is infinite, rather than of order  $J$ . The full model does have horizontal correlations, mediated by horizontal dimers. In order to bound the difference in the partition functions of the oriented dimer model and the full one, we will compute the ratio of the dimer partition function to the oriented dimer partition function (15) in terms of absolutely convergent series.

Obviously, the difference between the full and the oriented dimer models is that there are both horizontal and vertical dimers in the former. With that in mind, we consider dimer configurations in terms of *horizontal* and *vertical phases* and *defects* (see Fig.-7). A vertical phase is a region of  $\mathbb{Z}^2$  that is occupied only by vertical dimers (and monomers); similarly, a horizontal phase is occupied by horizontal dimers. The interface between a vertical and a horizontal phase is a *defect*. This point of view is similar to the *Peierls argument* for the ferromagnetic Ising model, in which one can consider a spin configuration as a collection of contours which delineate regions containing only + or - spins. Unlike the Ising model, the configuration in a uniform phase is not unique (because they can contain monomers), but, since the oriented dimer model is integrable, we can compute the partition function in these regions (this is reminiscent of the models considered in [BKL84, BKL85]). In addition, given that we are computing the ratio (15), the partition functions in uniform phases appearing in the numerator are *approximately* canceled out by the oriented dimer partition function in the denominator, leaving an effective weight for the defects.

The dominant contribution to the weight of a defect comes from the fact that *most* dimers in the denominator of (15) interact with a neighboring dimer (because the dimer activity is large), which means that almost every other vertical edge (we choose  $q = v$ ) contributes a factor  $e^J$ . We can keep track of these factors by assigning a weight  $e^{\frac{J}{2}}$  to each endpoint of a dimer. On the other hand, the dimers on either side of a defect have different orientations, and, therefore, do not interact. By cutting these interactions, a defect of length  $|l|$  contributes a factor  $\approx e^{-\frac{J}{2}|l|}$  (see Fig. 7). This is encouraging: in the language of Pirogov-Sinai theory [PS75], this would indicate that the system satisfies the *Peierls condition* with a *large decay rate*  $\frac{J}{2}$ , which is a sufficient condition for general Pirogov-Sinai constructions [KP84, BKL84] to apply.

There is, however, one important complication. As was mentioned earlier, the partition functions in the uniform phases only approximately cancel. Indeed, in the numerator, one has a product of oriented partition functions over a partition of  $\Lambda$ , whereas, in the denominator, there is only one oriented partition function over all of  $\Lambda$ . However, the partition function in a region depends on its geometry. In addition, while correlations in the oriented dimer model decay exponentially, they have a large correlation length  $\ell_0 \approx e^{\frac{3}{2}J} \sqrt{z}$ . There are, therefore, two length scales at play in this system: the microscopic size of a dimer, and the mesoscopic correlation length of the oriented dimer model. Therefore, the dependence of the oriented dimer partition function on the geometry of the region which it describes is strong when the diameter of the region does not exceed  $\ell_0$ . As a consequence, defects interact with each other, with an exponentially decaying interaction that has a very small decay rate  $\ell_0^{-1}$ . In order to deal with this interaction, we use the Mayer trick [Ur27, Ma37] (that is, we write the pair interaction

$e^{-W}$  as  $(e^{-W} - 1) + 1$  and expand) and split defect configurations into isolated bunches of interacting defects, called *polymers*, which interact only via a hard-core repulsion. We represent polymers graphically as a collection of defects connected to each other by lines representing the interaction (see Fig. 10). The effective activity of the polymers can then be shown to be  $\approx e^{-\frac{J}{2}|l| - \ell_0^{-1}|\sigma|}$  where  $|l|$  is the total length of the defects and  $|\sigma|$  is the total length of the interaction lines. This looks much worse than  $e^{-\frac{J}{2}|l|}$ : the decay rate is now  $\approx \ell_0^{-1}$  which is extremely small and may not, a priori, suffice to control the entropy of the polymers: in a model of arbitrary polymers with activity  $e^{-\ell_0^{-1}|l|}$ , it would be likely to find polymers, whereas we need them to be rare.

The key ingredient to overcome this difficulty is that the interaction is one-dimensional: it comes from the oriented dimer model, and takes place over vertical or horizontal lines, so the contribution to the entropy of a polymer from its interactions is only a one-dimensional sum. In addition, interaction lines are always connected to a defect, which has a very small weight. In fact, the smallest possible defect is of length 6, so the largest possible weight for a defect is  $e^{-3J}$ . On the other hand, the sum over the length of the interaction lengths yields  $\sum_\ell e^{-\ell_0^{-1}\ell} \approx \ell_0$ . Now, since every new interaction line must connect to a new defect, the overall contribution of the interaction line along with the defect to which it is connected, is, at most,  $\ell_0 e^{-3J} \ll 1$ . This allows us to control the entropy of the polymers, even though the decay rate of the interaction lines is small. Having done so, we use a cluster expansion [Ru99, GBG04, KP86, BZ00] to compute the partition function of the polymer model and (15).

There are some more technical complications that arise in the proof. One of these is standard in Pirogov-Sinai theory: unlike the Ising model, the partition function of the oriented dimer model may take different values for vertical and horizontal boundary conditions, which prevents us from using a straight Peierls argument. In order to avoid long-range interactions in the defect model, we must *flip* the boundary condition inside each defect back to the vertical, and, in doing so, introduce an extra factor in the activity of the defect that depends on the partition function of the full dimer model inside the defect with both boundary conditions. We then show that this term is, at most, exponentially large in the size of the defect with a rate that is much smaller than  $\frac{J}{2}$ , and thereby causes no trouble. To do so, we must bound the partition function inside the defect from above and below, which we do by induction, and is the main reason why we compute the ratio (15) instead of merely bounding it.

In addition, we have found it necessary to avoid interaction lines of length  $< \ell_0$ . This is due to the fact that the polymer model we have constructed contains *trivial polymers*, which do not contain any defect and consist of a single interaction line going all the way through  $\Lambda$ . Whenever such lines are of length  $< \ell_0$  (which may occur since, in order to carry out the inductive argument mentioned above, we cannot restrict our attention to  $\Lambda$ 's of large volume), their activity can be close to  $\pm 1$ . This causes a number of issues, which we have opted to remedy by ensuring that no short trivial polymers may arise. This can be accomplished by grouping defects that are closer than  $\ell_0$  from each other into bunches, called *contours* (see Fig. 8).

Finally, the introduction of sources to compute correlation functions comes with its share of pesky complications, which we will not comment on here. In fact, readers who are not interested in the fine details of the proof are invited to consider only the case  $\Upsilon = \emptyset$ , and skip the source-specific paragraphs on a first reading.

### 3. Solution of the One-Dimensional Problem

In this section, we compute the partition function of the oriented dimer model on a finite, connected chain  $\{1, \dots, \ell\} \subset \mathbb{Z}$ , with various boundary conditions.

In order to specify the boundary condition, we introduce the following notation. We introduce a real vector space,

$$\Delta := \text{span}\{|\text{r}\rangle, |\text{l}\rangle, |\times\rangle\} \quad (21)$$

Given a pair of vectors  $\omega \equiv (\omega_1, \omega_\ell) \in \Delta^2$ , we define the partition function  $\Psi^{(\omega)}(\ell)$  in the following way.

- If  $\omega_1 = |\text{r}\rangle, |\text{l}\rangle$  or  $|\times\rangle$ , then the first vertex must be covered by, respectively, a half-dimer pointing right, a half-dimer pointing left or a monomer;
- For symmetry reasons,  $\omega_\ell$  is defined the other way around (this notation may seem slightly counter-intuitive, but it will be useful in the following): if  $\omega_\ell = |\text{r}\rangle, |\text{l}\rangle$  or  $|\times\rangle$ , then the last vertex must be covered by, respectively, a half-dimer pointing left, a half-dimer pointing right or a monomer.
- $\Psi^{(\omega)}(\ell)$  is bilinear in  $\omega$ .

For example, the partition function with open boundary conditions is obtained by taking  $\omega = (|\text{r}\rangle + |\times\rangle, |\text{r}\rangle + |\times\rangle)$ .

**Lemma 1.** *For every  $\ell \geq 1$ , we have, for  $\omega \equiv (\omega_1, \omega_\ell) \in \Delta^2$ ,*

$$\Psi^{(\omega)}(\ell) = v_+(\omega_1)v_+(\omega_\ell)b_+\lambda_+^\ell + v_-(\omega_1)v_-(\omega_\ell)b_-\lambda_-^\ell + v_0(\omega_1)v_0(\omega_\ell)b_0\lambda_0^\ell \quad (22)$$

with

$$\begin{aligned} \lambda_+ &= \left( \sqrt{ze^J} + \frac{1}{2}e^{-J} \right) (1 + O(e^{-J}\epsilon^2)), \\ \lambda_- &= \left( -\sqrt{ze^J} + \frac{1}{2}e^{-J} \right) (1 + O(e^{-J}\epsilon^2)), \\ \lambda_0 &= 1 - e^{-J}(1 + \epsilon^2) + O(e^{-2J}\epsilon^2) \end{aligned} \quad (23)$$

in which

$$\epsilon := \frac{1}{\sqrt{ze^J}} \quad (24)$$

and, for  $i \in \{+, -, 0\}$ ,

$$b_i := \frac{z}{(2\lambda_i(\lambda_i - 1)^2 + z)\lambda_i} \quad (25)$$

$v_i(\omega_j)$  is linear in  $\omega_j$ , and

$$v_i(|\text{r}\rangle) = \lambda_i - 1, \quad v_i(|\text{l}\rangle) = \frac{\lambda_i}{z}(\lambda_i - 1), \quad v_i(|\times\rangle) = 1. \quad (26)$$

*Proof.* We will use a *transfer matrix* approach.

**1 -** Every vertex may be in one of three states: it is either covered by a half-dimer pointing right (r), a half-dimer pointing left (l), or no dimer (x). One easily checks that the partition function can be written as

$$\Psi^{(\omega_1, \omega_\ell)}(\ell) = \omega_1 \cdot T^{\ell-1} \omega_\ell \quad (27)$$

where  $T$  is the *transfer matrix*, whose expression, in the  $(|r\rangle, |l\rangle, |\times\rangle)$  basis, is

$$T := \begin{pmatrix} 0 & z & 0 \\ e^J & 0 & 1 \\ 1 & 0 & 1 \end{pmatrix}. \quad (28)$$

2 - By straightforward computation, we diagonalize  $T$ :

$$T = P \begin{pmatrix} \lambda_+ & 0 & 0 \\ 0 & \lambda_- & 0 \\ 0 & 0 & \lambda_0 \end{pmatrix} P^{-1} \quad (29)$$

where  $\lambda_{\pm}$  and  $\lambda_0$  satisfy (24),

$$P = \begin{pmatrix} \lambda_+ - 1 & \lambda_- - 1 & \lambda_0 - 1 \\ \frac{\lambda_+}{z}(\lambda_+ - 1) & \frac{\lambda_-}{z}(\lambda_- - 1) & \frac{\lambda_0}{z}(\lambda_0 - 1) \\ 1 & 1 & 1 \end{pmatrix} \begin{pmatrix} \frac{1}{N_+} & 0 & 0 \\ 0 & \frac{1}{N_-} & 0 \\ 0 & 0 & \frac{1}{N_0} \end{pmatrix}. \quad (30)$$

and

$$P^{-1} = \begin{pmatrix} \frac{1}{N_+} & 0 & 0 \\ 0 & \frac{1}{N_-} & 0 \\ 0 & 0 & \frac{1}{N_0} \end{pmatrix} \begin{pmatrix} \frac{\lambda_+}{z}(\lambda_+ - 1) & \lambda_+ - 1 & 1 \\ \frac{\lambda_-}{z}(\lambda_- - 1) & \lambda_- - 1 & 1 \\ \frac{\lambda_0}{z}(\lambda_0 - 1) & \lambda_0 - 1 & 1 \end{pmatrix} \quad (31)$$

with, for  $i \in \{+, -, 0\}$ ,

$$N_i := \sqrt{2 \frac{\lambda_i}{z} (\lambda_i - 1)^2 + 1}. \quad (32)$$

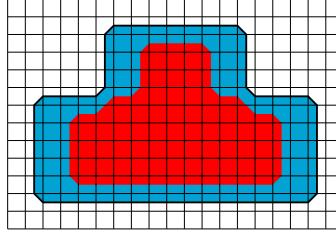
Therefore, the lemma holds with

$$b_i = \frac{1}{N_i^2 \lambda_i}, \quad v_i(\omega) = (\omega^T P)_i. \quad (33)$$

□

#### 4. Dilute Hard-Core Polymer Model

In this section, we will map the high-density dimer model to a dilute model of polymers, which only interact with each other via a hard-core repulsion. We proceed in five steps: we first map the dimer model to a *loop* model, then to an *external* contour model, for which we compute the activity and interaction of external contours, and then map the contour model to a system of *external* polymers, and, finally, introduce the polymer model.



**Fig. 3.** An h-bounding loop. The loop is depicted as a thick line running through the edges that make it up. The core of the loop is colored red and its mantle is colored cyan (color online)

**4.1. Loop model.** First of all, for  $c \in \{v, h\}$ , we define the  **$c$ -support** of  $\underline{\delta}$  as the set of vertices that are covered by  $c$ -dimers:

$$\text{supp}_c(\underline{\delta}) := \bigcup_{\{x, x'\} \in \mathbb{D}_c(\underline{\delta})} \{x, x'\} \quad (34)$$

(recall that  $\mathbb{D}_c(\underline{\delta})$  is the set of  $c$ -dimers in  $\underline{\delta}$ ).

We construct a family  $\mathcal{L}_q(\underline{\delta})$  of *bounding loops* associated to  $\underline{\delta}$ . A **loop** is a set of edges  $l$ , such that there exists a *simply connected* (a simply connected set is a set whose complement is connected) set  $\bar{l} \subset \mathbb{Z}^2$  such that  $l$  is the boundary of  $\bar{l}$ :  $l = \partial \bar{l}$ . To assign a loop to  $\underline{\delta}$ , we will proceed by induction. If  $\underline{\delta}$  only consists of  $q$ -dimers, then  $\mathcal{L}_q(\underline{\delta}) = \emptyset$ . If not, then the boundary of  $\text{supp}_{-q}(\underline{\delta})$  is a non-empty union of disjoint loops, denoted by  $\underline{l} \equiv \{l_1, \dots, l_{|\underline{l}|}\}$ . From these, we extract the *most external* ones  $\underline{l}' \equiv \{l'_1, \dots, l'_{|\underline{l}'|}\} \subset \underline{l}$  by discarding loops that lie inside other loops, that is,  $\bar{l}'_i \cap \bar{l}_j \neq \emptyset$  if and only if  $l'_i = l_j$  (the fact that there is no prime in  $\bar{l}_j$  or  $l_j$  is not a typo: the  $l'_i$  are external to *all* loops).

These loops separate a  $q$ -phase from a  $-q$  phase, which implies some geometric constraints. For one, the inside of each loop is lined with  $-q$ -dimers. To capture these properties, we define the notion of a  **$c$ -bounding loop** for  $c \in \{v, h\}$ . To do so, we split the interior of a loop into a region which *must* be covered by  $c$ -dimers (which we call the *mantle* of the loop), and the rest (the *core*), see Fig. 3. Formally, the **core** is defined as

$$\mathbb{I}_c^{(\gamma)}(l) := \{x \in \bar{l} \setminus (\bigcup_{v \in \gamma} v), \quad \mathfrak{d}_c(x, \mathbb{D}_c(l)) \geq 2, \quad \mathfrak{d}_{-c}(x, \mathbb{D}_{-c}(l)) \geq 1\} \quad (35)$$

(recall that  $\mathfrak{d}_c$  is the  $c$ -distance on  $\mathbb{Z}^2$ ) (remark: we do not count the sources as being part of the interior) and its **mantle** as

$$\mathbb{O}_c(l) := \bar{l} \setminus \mathbb{I}_c^{(\emptyset)}(l). \quad (36)$$

A  **$c$ -bounding loop** is a loop whose mantle is disjoint from the sources  $\bigcup_{v \in \gamma} v$ , and can be completely covered by  $c$ -dimers (see Figs. 3 and 4). We denote the set of  $c$ -bounding loops by  $\mathcal{L}_c^{(\gamma)}(\Lambda)$ , and the set of all bounding loops by  $\mathcal{L}^{(\gamma)}(\Lambda) \equiv \mathcal{L}_v^{(\gamma)}(\Lambda) \cup \mathcal{L}_h^{(\gamma)}(\Lambda)$ . (Note that some loops could be  $v$ -bounding loops as well as  $h$ -bounding loops. The index  $c$  is meant as an extra structure, which is not a function of the geometry of the loop.)

The loops  $l'_i$  are  $-q$ -bounding loops and are disjoint. Finally, denoting the set of dimers that are contained inside a bounding loop  $l'_j$  by  $\underline{\delta} \cap \bar{l}'_j$ , we define, inductively, (see Fig. 4)

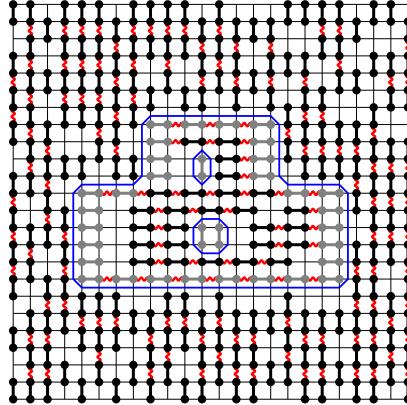


Fig. 4. The loops associated to the dimer configuration in Fig. 1. The dimers in the mantles are gray

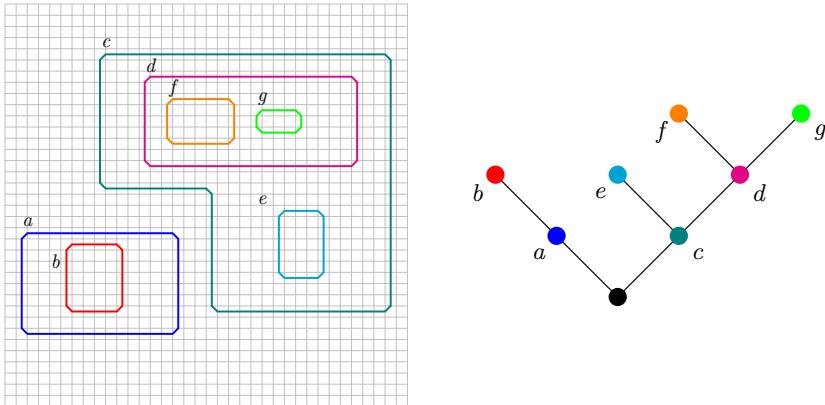


Fig. 5. Nested bounding loops and their corresponding inclusion tree. The root of the tree is drawn in black. The colors (color online) and labels of the nodes in the tree match with those of the loops. The tree is *alternating* because the blue, teal, orange and green (a, c, f and g) loops are h-bounding loops and the red, cyan and magenta (b, e and d) ones are v-bounding loops

$$\mathcal{L}_q(\underline{\delta}) := \underline{l}' \cup \bigcup_{j=1}^{|\underline{l}'|} \mathcal{L}_{-q}(\underline{\delta} \cap \bar{l}'_j). \quad (37)$$

The loops in  $\mathcal{L}_q(\underline{\delta})$  are disjoint, and their mantles are disjoint.

These bounding loops are *alternating*, in the sense that  $q$ -loops may only encircle  $-q$ -loops. It is useful, to define this notion, to introduce *inclusion trees*. The **inclusion tree** associated to  $\mathcal{L}_q(\underline{\delta})$  is a tree  $T(\mathcal{L}_q(\underline{\delta}))$  (see Fig. 5 for an example) that is such that

- $T(\mathcal{L}_q(\underline{\delta}))$  has  $|\mathcal{L}_q(\underline{\delta})| + 1$  nodes. One node corresponds to  $\partial \Lambda$ , and is called the *root* of  $T(\mathcal{L}_q(\underline{\delta}))$ , while the  $|\mathcal{L}_q(\underline{\delta})|$  others each correspond to a loop  $l \in \mathcal{L}_q(\underline{\delta})$ .
- For each  $l \in \mathcal{L}_q(\underline{\delta})$ , the corresponding node has a unique *parent*, chosen in such a way that every loop containing  $l$  is an *ancestor* of  $l$  (a loop  $l'$  contains a loop  $l$  if  $l \subset \mathcal{E}(\bar{l}')$  (recall that  $\mathcal{E}(\bar{l}')$  is the set of edges in  $\bar{l}'$ )).

The inclusion tree is **alternating**, in the sense that the children of a  $c$ -bounding loop are  $-c$ -bounding loops. We define  $c(l)$  as the orientation of the loop  $l$  (that is,  $l$  is a  $c(l)$ -bounding loop), which we will also call the **index** of  $l$ .

Thus, given a dimer configuration, we have constructed a set of bounding loops  $\mathcal{L}_q(\underline{\delta})$ . Conversely, if we fix a family  $\mathcal{L}_q$  of bounding loops that are disjoint and whose mantles are disjoint, whose most external loops are  $q$ -bounding loops, and are such that  $T(\mathcal{L}_q)$  is alternating, then the set of dimer configurations  $\underline{\delta}$  such that  $\mathcal{L}_q(\underline{\delta}) = \mathcal{L}_q$  is equal to the set of configurations satisfying the following: for every loop  $l \in \mathcal{L}_q$ ,

- for every  $-q$ -dimer, there exists  $l' \in \mathcal{L}_q$  such that the dimer is in  $\bar{l}'$ ,
- the *mantle*  $\mathbb{O}_{c(l)}(l)$  of  $l$  is completely covered by  $c(l)$ -dimers.

Therefore, we can rewrite the dimer partition function (15) as

$$\frac{Z^{(\gamma)}(\Lambda|\mathbf{q})}{\mathfrak{Z}_{\mathbf{q}}^{(\gamma)}(\Lambda)} = \sum_{\mathcal{L} \subset \mathcal{L}^{(\gamma)}(\Lambda)} \left( \prod_{l \neq l' \in \mathcal{L}} \varphi(l, l') \right) \mathbb{1}_q(\mathcal{L}) \frac{\mathfrak{Z}_{\mathbf{q}}^{(\gamma)}(\Lambda \setminus \cup_{l \in \mathcal{L}} \bar{l})}{\mathfrak{Z}_{\mathbf{q}}^{(\gamma)}(\Lambda)} \cdot \prod_{l \in \mathcal{L}} \left( \mathfrak{Y}_{c(l)}^{(\gamma)}(\mathbb{O}_{c(l)}(l)) \mathfrak{Z}_{c(l)}^{(\gamma)}(\iota_{c(l)}^{(\gamma)}(l, \mathcal{L})) \right) \quad (38)$$

in which, roughly (these quantities are formally defined below),  $\varphi_{\text{ext}}(l, l')$  is a hard-core pair interaction that keeps the loops or their mantles from intersecting,  $\mathbb{1}_q(\mathcal{L})$  ensures that  $T(\mathcal{L})$  is alternating and that the index of its root is  $q$ ,  $\mathfrak{Z}_{\mathbf{q}}^{(\gamma)}(\Lambda \setminus \cup_{l \in \mathcal{L}} \bar{l})$  is the partition function of  $q$ -dimers outside all loops,  $\mathfrak{Y}_{c(l)}^{(\gamma)}(\mathbb{O}_{c(l)}(l))$  is the weight of the  $c(l)$ -dimers in the mantle of  $l$ , and  $\mathfrak{Z}_{c(l)}^{(\gamma)}(\iota_{c(l)}^{(\gamma)}(l, \mathcal{L}))$  is the partition function of  $c(l)$ -dimers in between loops.

- $\varphi(l, l') \in \{0, 1\}$  is equal to 1 if and only if  $l$  and  $l'$  are disjoint and their mantles are disjoint.
- $\mathbb{1}_q(\mathcal{L}) \in \{0, 1\}$  is equal to 1 if and only if  $T(\mathcal{L})$  is alternating, and the index of its root is  $q$ .
- $\iota_{c(l)}^{(\gamma)}(l, \mathcal{L})$  is the **padding** of  $l$ , and is the space inside the loop  $l$  that is external to all other loops (see Fig. 6):

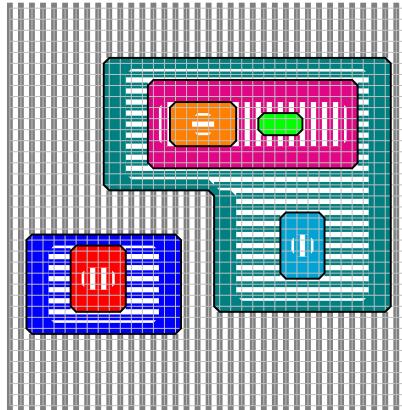
$$\iota_{c(l)}^{(\gamma)}(l, \mathcal{L}) := \mathbb{I}_{c(l)}^{(\gamma)}(l) \setminus \left( \bigcup_{l' \in \mathcal{L} \setminus \{l\}} \bar{l}' \right). \quad (39)$$

(Note that one could restrict the union over  $l'$  to loops *inside*  $l$ , but this is not necessary.)

- $\mathbf{c}(l)$  is the boundary condition of  $\iota_{c(l)}^{(\gamma)}(l, \mathcal{L})$ . Because the  $c(l)$ -dimers that are in  $\mathbb{O}_{c(l)}(l)$  can interact with those in  $\iota_{c(l)}^{(\gamma)}(l, \mathcal{L})$ , the entire boundary is *magnetized* (see Fig. 4): with the notation of Sect. 2,  $\mathbf{c}(l) := (c(l), \partial_{c(l)} \mathbb{I}_{c(l)}^{(\emptyset)}(l), \ell_0)$  (we recall that  $\partial_{c(l)} \equiv \mathbb{D}_{c(l)}(\partial)$ ).
- $\mathfrak{Y}_{c(l)}^{(\gamma)}(\mathbb{O}_{c(l)}(l))$  is the weight of the dimers in  $\mathbb{O}_{c(l)}(l)$ , which is packed with dimers (see Fig. 4):

$$\mathfrak{Y}_c(\mathbb{O}) := z^{|\underline{\delta}_c(\mathbb{O})|} e^{-W_0(\underline{\delta}_c(\mathbb{O}))} \mathfrak{B}_{\mathbf{q}}^{(\gamma)}(\underline{\delta}_c(\mathbb{O})) \quad (40)$$

where  $\underline{\delta}_c(\mathbb{O})$  is the unique closely-packed  $c$ -dimer configuration in  $\mathbb{O}$ , and  $\mathfrak{B}_{\mathbf{q}}^{(\gamma)}$  is the boundary term at the sources (see (6) and (12)).



**Fig. 6.** The segments inside and outside the loops. The sets  $\iota_{c(l)}(l, \mathcal{L})$  are depicted in the same color (color online) as the loop to which they correspond. The mantles of the loops are covered in black

*4.2. External contour model.* These bounding loops *interact* with each other through the dimers in the space between them. Actually, as we will see in the following,  $q$ -bounding loops that are at a  $-q$ -distance of less than  $\ell_0$  interact *strongly*. In order to avoid dealing with distinct objects that interact strongly, we group these loops together to form a larger object, called a *contour*, which is a set of loops that are all at a distance  $< \ell_0$  from each other.

The interaction is one-dimensional, and is either horizontal or vertical. To represent it graphically, it is convenient to introduce the notion of a *segment*. To that end, we define, for  $y \in \mathbb{Z}^2$ , the v- and h-lines going through  $y$ :

$$\chi_y^{(v)} := \{(y, x), x \in \mathbb{Z}\}, \quad \chi_y^{(h)} := \{(x, y), x \in \mathbb{Z}\}. \quad (41)$$

Put simply, a v-line is a vertical line and an h-line is a horizontal one. In addition, we define a map  $\Sigma_c$  that takes a bounded set  $\Lambda' \subset \mathbb{Z}^2$  as an argument, and splits it up into  $c$ -segments. Formally, given  $y \in \mathbb{Z}$ , we denote the set of connected components of  $\Lambda' \cap \chi_y^{(c)}$  by  $\underline{\sigma}_{c,y}(\Lambda')$  (that is,  $\underline{\sigma}_{c,y}(\Lambda')$  is a set whose elements are the connected components of  $\Lambda' \cap \chi_y^{(c)}$ ), and define

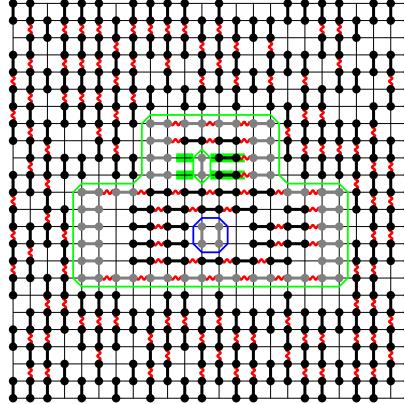
$$\Sigma_c(\Lambda') := \bigcup_{y \in \mathbb{Z}} \underline{\sigma}_{c,y}(\Lambda'). \quad (42)$$

We then define the set of **segments** in between the loops in  $\mathcal{L}_q(\underline{\delta})$  (see Fig. 6) as

$$\mathcal{S}_\Lambda^{(\gamma)}(\mathcal{L}_q(\underline{\delta})) := \left( \Sigma_q \left( \Lambda^{(\gamma)} \setminus \left( \bigcup_{l \in \mathcal{L}} \bar{l} \right) \right) \right) \cup \left( \bigcup_{l \in \mathcal{L}} \Sigma_{c(l)}(\iota_{c(l)}^{(\gamma)}(l, \mathcal{L}_q(\underline{\delta}))) \right). \quad (43)$$

We wish to gather together the loops that are separated by segments of length  $< \ell_0$ . To do so, we define the **support** of  $\mathcal{L}_q(\underline{\delta})$  as the pair

$$\text{supp}_{\Lambda, \ell_0}^{(\gamma)}(\mathcal{L}_q(\underline{\delta})) := (\mathcal{L}_q(\underline{\delta}), \mathcal{S}_{\Lambda, < \ell_0}^{(\gamma)}(\mathcal{L}_q(\underline{\delta}))) \quad (44)$$



**Fig. 7.** The contour configuration associated to the dimer configuration in Fig. 1. Here, we took  $q = v$  and  $\ell_0 = 4$ . There are three bounding loops. The two green (color online) bounding loops form a single contour because the segments that separate the inner loop from the mantle of the outer one (drawn in green in the figure) are of length 1 or 2, which is less than  $\ell_0$ . The blue (color online) bounding loop forms its own contour, but, since it is not external to the green one, it is dropped

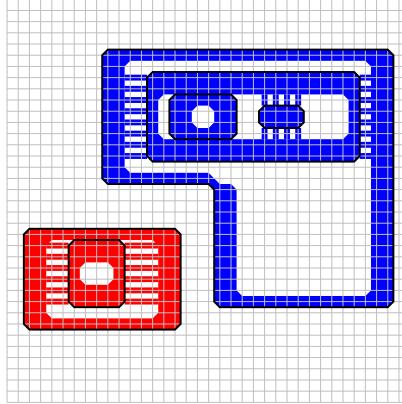
in which  $\mathcal{S}_{\Lambda, < \ell_0}^{(\gamma)}(\mathcal{L}_q(\underline{\delta}))$  is the set of segments of length  $< \ell_0$ :

$$\mathcal{S}_{\Lambda, < \ell_0}^{(\gamma)}(\mathcal{L}_q(\underline{\delta})) := \{\sigma \in \mathcal{S}_{\Lambda}^{(\gamma)}(\mathcal{L}_q(\underline{\delta})), |\sigma| < \ell_0\}. \quad (45)$$

A *contour* is a subset of  $\mathcal{L}_q(\underline{\delta})$  that has a *connected* support, in the following sense. A  $c$ -segment  $s$  is said to be **connected** to a bounding loop  $l$  if  $\partial_c s \cap \partial \mathbb{O}_{c(l)}(l) \neq \emptyset$ . Similarly, two bounding loops  $l, l'$  are said to be connected if  $\partial \mathbb{O}_{c(l)}(l) \cap \partial \mathbb{O}_{c(l')}(l') \neq \emptyset$ . Finally, given a set of segments  $\mathcal{S}$  and a set  $\mathcal{L}$  of disjoint loops,  $(\mathcal{L}, \mathcal{S})$  is said to be **connected** if, for every  $x, y \in \mathcal{S} \cup \mathcal{L}$ , there exists a path from  $x$  to  $y$ , that is, there exists  $\underline{p} \equiv (p_1, \dots, p_{|\underline{p}|})$  with  $p_i \in \mathcal{S} \cup \mathcal{L}$ ,  $p_1 = x$ ,  $p_{|\underline{p}|} = y$ ,  $p_i$  and  $p_{i+1}$  are never both segments, and  $p_i$  and  $p_{i+1}$  are connected.

We then split  $\mathcal{L}_q(\underline{\delta})$  into *connected components* (see Fig. 7),  $\underline{\Gamma}_q(\underline{\delta}) \equiv \{\Gamma_1, \dots, \Gamma_{|\underline{\Gamma}_q(\underline{\delta})|}\}$ , that is,  $\Gamma_i \subset \mathcal{L}_q(\underline{\delta})$ , the support of  $\Gamma_i$  is connected,  $\Gamma_i \cap \Gamma_j = \emptyset$  whenever  $i \neq j$ ,  $\Gamma_1 \cup \dots \cup \Gamma_{|\underline{\Gamma}_q(\underline{\delta})|} = \mathcal{L}_q(\underline{\delta})$  and, finally, the support of  $\Gamma_i \cup \Gamma_j$  is *disconnected* when  $i \neq j$  (when two contours are disconnected from each other, we say they are **compatible**).

The fact that the inclusion tree  $T(\mathcal{L}_q(\underline{\delta}))$  is alternating induces a long-range interaction between contours: a  $q$ -bounding loop must lie inside a  $-q$ -bounding loop, independently of the distance that separates them. In order to avoid this, we will call upon a technique used in Pirogov-Sinai theory [PS75, KP84]. The first step is to focus on the contours that are the most *external*, in the following sense. Two contours  $\Gamma, \Gamma'$  are **external** to each other if every loop  $l \in \Gamma$  and every  $l' \in \Gamma'$  are external to each other:  $\bar{l} \cap \bar{l}' = \emptyset$ . The set of external contours is denoted by  $\underline{\Gamma}'_q(\underline{\delta}) \equiv \{\Gamma'_1, \dots, \Gamma'_{|\underline{\Gamma}'_q(\underline{\delta})|}\}$ , and defined as the subset of  $\underline{\Gamma}_q(\underline{\delta})$  such that for every  $l'_i \in \Gamma'_i$  and  $l_j \in \Gamma_j$ ,  $\bar{l}'_i \cap \bar{l}_j \neq \emptyset$  if and only if  $\Gamma'_i = \Gamma_j$ . (Note that, even though we dropped the contours that are not external, the contours we are left with may still contain several loops, and their inclusion tree is alternating, see Fig. 8.)



**Fig. 8.** Two contours, depicted in different colors (color online). Here, we have taken  $\ell_0 = 4$  and  $q = v$ . The segments in the support are drawn as thick lines, and the mantles have been filled in. These two contours are compatible and external to each other. Comparing with Fig. 5, note that the cyan loop labeled as  $e$  is not depicted here, since it would not be external to the other contours

Grouping the loops in (38) in this way, and dropping the contours that are not the most external, we rewrite the partition function as

$$\frac{Z^{(\Upsilon)}(\Lambda|q)}{\mathfrak{Z}_q^{(\Upsilon)}(\Lambda)} = \sum_{\underline{\Gamma} \subset \mathfrak{C}_q^{(\Upsilon)}(\Lambda)} \left( \prod_{\Gamma \neq \Gamma' \in \underline{\Gamma}} \varphi_{\text{ext}}(\Gamma, \Gamma') \right) \frac{\mathfrak{Z}_q^{(\Upsilon)}(\Lambda \setminus \mathcal{I}(\underline{\Gamma}))}{\mathfrak{Z}_q^{(\Upsilon)}(\Lambda)} \cdot \prod_{\Gamma \in \underline{\Gamma}} \prod_{l \in \Gamma} \left( \mathfrak{Y}_{c(l)}^{(\Upsilon)}(\mathbb{O}_{c(l)}(l)) \mathfrak{Z}_{c(l)}^{(\Upsilon)}(\ell_{c(l), < \ell_0}(l, \Gamma)) Z^{(\Upsilon)}(\ell_{c(l), \geq \ell_0}(l, \Gamma) | \mathbf{c}(l)) \right) \quad (46)$$

in which, roughly (see below for a formal definition of these quantities),  $\varphi_{\text{ext}}(\Gamma, \Gamma')$  is a hard-core pair interaction that ensures that contours are compatible and external to each other,  $\mathfrak{Z}_q^{(\Upsilon)}(\Lambda \setminus \mathcal{I}(\underline{\Gamma}))$  is the partition function of  $q$ -dimers outside the contours,  $\mathfrak{Y}_{c(l)}^{(\Upsilon)}(\mathbb{O}_{c(l)}(l))$  is, as before, the weight of the  $c(l)$ -dimers in the mantle of  $l$ ,  $\mathfrak{Z}_{c(l)}^{(\Upsilon)}(\ell_{c(l), < \ell_0}(l, \Gamma))$  is the partition function of  $c(l)$ -dimers in the segments inside the loop that are of length  $< \ell_0$ , and  $Z^{(\Upsilon)}(\ell_{c(l), \geq \ell_0}^{(\Upsilon)}(l, \Gamma) | \mathbf{c}(l))$  is the partition function of dimers in the remainder of the loops.

- $\mathfrak{C}_q^{(\Upsilon)}(\Lambda)$  is the set of **contours**, which is defined as the set of collections of loops  $\Gamma$  which are pairwise disjoint and have disjoint mantles, have an alternating inclusion tree whose root label is  $q$ , and whose support is connected (see Fig. 8).
- $\varphi_{\text{ext}}(\Gamma, \Gamma') \in \{0, 1\}$  is equal to 1 if and only if  $\Gamma$  and  $\Gamma'$  are compatible (that is, disconnected from each other) and external to each other.
- $\mathcal{I}(\underline{\Gamma})$  is the union of the interiors of the loops:

$$\mathcal{I}(\underline{\Gamma}) := \bigcup_{\Gamma \in \underline{\Gamma}} \bigcup_{l \in \Gamma} \bar{l}. \quad (47)$$

–  $\iota_{c,<\ell_0}^{(\Upsilon)}(l, \Gamma)$  and  $\iota_{c,\geqslant\ell_0}^{(\Upsilon)}(l, \Gamma)$  are the restrictions of  $\iota_c^{(\Upsilon)}(l, \Gamma)$  to the parts of the segments that are of length, respectively,  $< \ell_0$  and  $\geqslant \ell_0$ :

$$\iota_{c,<\ell_0}^{(\Upsilon)}(l, \Gamma) := \bigcup_{\substack{\sigma \in \Sigma_c(\iota_c^{(\Upsilon)}(l, \Gamma)) \\ |\sigma| < \ell_0}} \sigma, \quad \iota_{c,\geqslant\ell_0}^{(\Upsilon)}(l, \Gamma) := \iota_c^{(\Upsilon)}(l, \Gamma) \setminus \iota_{c,<\ell_0}^{(\Upsilon)}(l, \Gamma). \quad (48)$$

–  $\mathbf{c}(l)$  and  $\mathfrak{Y}_{c(l)}^{(\Upsilon)}$  were defined at the end of Sect. 4.1.

**4.3. Effective activity and interaction of the external contour model.** We will now re-organize and re-express the right side of (46). First of all, by inserting trivial identities, we rewrite

$$\begin{aligned} \frac{Z^{(\Upsilon)}(\Lambda|\mathbf{q})}{\mathfrak{Z}_{\mathbf{q}}^{(\Upsilon)}(\Lambda)} &= \sum_{\underline{\Gamma} \subset \mathfrak{C}_q^{(\Upsilon)}(\Lambda)} \left( \prod_{\Gamma \neq \Gamma' \in \underline{\Gamma}} \varphi_{\text{ext}}(\Gamma, \Gamma') \right) A_{\mathbf{q}}^{(\Upsilon)}(\Lambda, \underline{\Gamma}) \cdot \\ &\quad \cdot \prod_{\Gamma \in \underline{\Gamma}} \prod_{l \in \Gamma} \left( B_{\mathbf{q}, \mathbf{c}(l)}^{(\Upsilon)}(\iota_{c(l), \geqslant\ell_0}^{(\Upsilon)}(l, \Gamma)) \frac{\widetilde{Z}^{(\Upsilon)}(\iota_{c(l), \geqslant\ell_0}^{(\Upsilon)}(l, \Gamma)|\mathbf{q})}{\mathfrak{Z}_{\mathbf{q}}^{(\Upsilon)}(\iota_{c(l), \geqslant\ell_0}^{(\Upsilon)}(l, \Gamma))} \right) \end{aligned} \quad (49)$$

in which  $\widetilde{Z} > 0$  will be defined later (see (87)) (since  $\widetilde{Z}$  appears in both the numerator and denominator, it is not crucial, at this stage, to specify what it is, as long as it does not vanish, which it does not),

$$B_{\mathbf{q}, \mathbf{c}}^{(\Upsilon)}(\iota) := \frac{Z^{(\Upsilon)}(\iota|\mathbf{c})}{\mathfrak{Z}_{\mathbf{c}}^{(\Upsilon)}(\iota)} \frac{\mathfrak{Z}_{\mathbf{q}}^{(\Upsilon)}(\iota)}{\widetilde{Z}^{(\Upsilon)}(\iota|\mathbf{q})} \quad (50)$$

and

$$A_{\mathbf{q}}^{(\Upsilon)}(\Lambda, \underline{\Gamma}) := \frac{1}{\mathfrak{Z}_{\mathbf{q}}^{(\Upsilon)}(\Lambda)} \mathfrak{Z}_{\mathbf{q}}^{(\Upsilon)}(\Lambda \setminus \mathcal{I}(\underline{\Gamma})) \prod_{\Gamma \in \underline{\Gamma}} \prod_{l \in \Gamma} \left( \mathfrak{Y}_{c(l)}^{(\Upsilon)}(\mathbb{O}_{c(l)}(l)) \mathfrak{Z}_{\mathbf{c}(l)}^{(\Upsilon)}(\iota_{c(l)}^{(\Upsilon)}(l, \Gamma)) \right) \quad (51)$$

for which we used the fact that

$$\mathfrak{Z}_{\mathbf{c}(l)}^{(\Upsilon)}(\iota_{c(l), <\ell_0}^{(\Upsilon)}(l, \Gamma)) \mathfrak{Z}_{\mathbf{c}(l)}^{(\Upsilon)}(\iota_{c(l), \geqslant\ell_0}^{(\Upsilon)}(l, \Gamma)) = \mathfrak{Z}_{\mathbf{c}(l)}^{(\Upsilon)}(\iota_{c(l)}^{(\Upsilon)}(l, \Gamma)). \quad (52)$$

Recall that  $\mathbf{q} \equiv (q, \varrho, \ell_0)$  and  $\varrho$  is a subset of  $\partial \Lambda$ , not of  $\partial \iota_{c(l), \geqslant\ell_0}^{(\Upsilon)}(l, \Gamma)$ . When  $\mathbf{q}$  appears in  $\mathfrak{Z}_{\mathbf{q}}^{(\Upsilon)}(\iota)$  and the like, it is to be understood in the sense that the boundary of  $\iota$  is *not* magnetized, that is,  $\varrho$  could be replaced with  $\emptyset$  (which is consistent with the notation since the elements of  $\varrho$  never come in contact with the dimers inside  $\iota$ ).

The factors  $B_{\mathbf{q}, \mathbf{c}(l)}^{(\Upsilon)}(\iota_{c(l), \geqslant\ell_0}^{(\Upsilon)}(l, \Gamma))$  contribute to the *activity* of the contour, whereas  $A_{\mathbf{q}}^{(\Upsilon)}(\Lambda, \underline{\Gamma})$  contributes to both the activity and the interaction. In order to separate these contributions, we compute  $A_{\mathbf{q}}^{(\Upsilon)}(\Lambda, \underline{\Gamma})$  more explicitly.

**1 -** Let us first compute the partition function  $\mathfrak{Z}_{\mathbf{c}}^{(\Upsilon)}(\Lambda')$  of the oriented dimer model, for any bounded subset  $\Lambda' \subset \mathbb{Z}^2$  and any boundary condition  $\mathbf{c} \equiv (c, \varrho, \ell_0)$ . Different  $c$ -lines are independent, so  $\mathfrak{Z}_{\mathbf{c}}^{(\Upsilon)}(\Lambda')$  can be expressed as a product over  $c$ -lines. The

boundary conditions of each line depends on  $\varrho$  and  $\Upsilon$ . To specify them, we first introduce two 1-dimensional boundary conditions:  $\omega_0$  and  $\omega_1$  which correspond, respectively, to *open* and *magnetized* boundary conditions:

$$\omega_0 := |\mathbf{r}\rangle + |\times\rangle, \quad \omega_1 := e^J |\mathbf{r}\rangle + |\times\rangle. \quad (53)$$

We then define the boundary condition of a  $c$ -line  $\sigma$ ,  $\varpi_{\varrho, \Upsilon}(\sigma) \equiv (\varpi_{\varrho, \Upsilon}^{(0)}(\sigma), \varpi_{\varrho, \Upsilon}^{(1)}(\sigma))$ , as follows. Let  $x_0(\sigma)$  denote the lower-left-most vertex of  $\sigma$ , and  $x_1(\sigma)$  the upper-right-most. For  $j \in \{0, 1\}$ ,

$$\varpi_{\varrho, \Upsilon}^{(j)}(\sigma) := \begin{cases} \omega_1 & \text{if } \exists e \in \varrho \cup \partial^{(\text{mag})}\Upsilon, e \ni x_j(\sigma) \\ \omega_0 & \text{if not} \end{cases} \quad (54)$$

in which  $\partial^{(\text{mag})}\Upsilon$  is the *magnetized* portion of the boundaries of the sources:

$$\partial^{(\text{mag})}\Upsilon := \bigcup_{v \in \mathbb{D}_q(\Upsilon)} \mathbb{D}_q(\partial v) \cup \bigcup_{v \in \mathbb{D}_{-q}(\Upsilon)} \mathbb{D}_{-q}(\partial v). \quad (55)$$

We now reexpress  $\mathfrak{Z}_c^{(\Upsilon)}(\Lambda')$ :

$$\mathfrak{Z}_c^{(\Upsilon)}(\Lambda') = z^{|\Upsilon \cap \Lambda'|} \prod_{\sigma \in \Sigma_c(\Lambda'^{(\Upsilon)})} \Psi^{(\varpi_{\varrho, \Upsilon}(\sigma))}(|\sigma|) \quad (56)$$

in which  $\Sigma_c$  was defined in (42), and  $\Psi$  is the partition function of the one-dimensional dimer model, computed in Sect. 3. Now, by (22),

$$\mathfrak{Z}_c^{(\Upsilon)}(\Lambda') = z^{|\Upsilon \cap \Lambda'|} \prod_{\sigma \in \Sigma_c(\Lambda')} \left( v_+(\varpi_{\varrho, \Upsilon}^{(0)}(\sigma)) v_+(\varpi_{\varrho, \Upsilon}^{(1)}(\sigma)) b_+ \lambda_+^{|\sigma|} e^{-W_{\varpi_{\varrho, \Upsilon}(\sigma)}(|\sigma|)} \right) \quad (57)$$

where, for  $\omega \equiv (\omega^{(0)}, \omega^{(1)}) \in \Delta^2$ ,

$$e^{-W_\omega(|\sigma|)} := 1 + \frac{v_-(\omega^{(0)}) v_-(\omega^{(1)}) b_-}{v_+(\omega^{(0)}) v_+(\omega^{(1)}) b_+} \left( \frac{\lambda_-}{\lambda_+} \right)^{|\sigma|} + \frac{v_0(\omega^{(0)}) v_0(\omega^{(1)}) b_0}{v_+(\omega^{(0)}) v_+(\omega^{(1)}) b_+} \left( \frac{\lambda_0}{\lambda_+} \right)^{|\sigma|} \quad (58)$$

and  $\lambda$  and  $v$  were defined in Lemma 1.

**2 -** In addition,  $\mathfrak{Y}_c^{(\Upsilon)}(\mathbb{O}_c)$ , which, we recall, is the partition function of close-packed  $c$ -dimers in  $\mathbb{O}_c$  (see (40)), is equal to

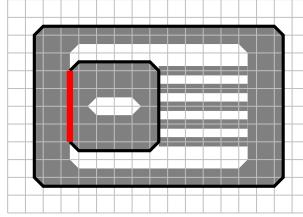
$$\mathfrak{Y}_c^{(\Upsilon)}(\mathbb{O}_c) = (ze^J)^{\frac{1}{2}|\mathbb{O}_c|} e^{-\frac{1}{2}J|\partial_c \mathbb{O}_c|} \exp(J|\{e \in \partial_c \mathbb{O}_c, e \cap \partial_c \mathbb{D}_c(\Upsilon) \neq \emptyset\}|) \quad (59)$$

with  $\partial_c \mathbb{D}_c(\Upsilon) := \bigcup_{v \in \mathbb{D}_c(\Upsilon)} \partial_c v$ .

**3 -** We now plug (57) and (59) into (51) to compute  $A_q(\Lambda, \underline{\Gamma})$ . We split the resulting terms into three contributions as follows.

**3-1 -** First, we focus on the terms involving  $\lambda_+$ . By definition, for any bounded  $\Lambda' \subset \mathbb{Z}^2$  and  $c \in \{v, h\}$ ,

$$\sum_{\sigma \in \Sigma_c(\Lambda')} |\sigma| = |\Lambda'| \quad (60)$$



**Fig. 9.** In this example, one of the loops in the contour touches the mantle of the loop that contains it. The set of edges at which this occurs is denoted by  $\mathbb{X}(\Gamma)$ . In the figure,  $\mathbb{X}(\Gamma)$  is rendered as a thick red (color online) line

so (recall (39) and (47))

$$\begin{aligned} & \frac{\prod_{\sigma \in \Sigma_q(\Lambda^{(Y)} \setminus \mathcal{I}(\underline{\Gamma}))} \lambda_+^{|\sigma|}}{\prod_{\sigma \in \Sigma_q(\Lambda^{(Y)})} \lambda_+^{|\sigma|}} \left( \prod_{\Gamma \in \underline{\Gamma}} \prod_{l \in \Gamma} \left( (ze^J)^{\frac{1}{2}|\mathbb{O}_{c(l)}(l)|} \prod_{\sigma \in \Sigma_{c(l)}(l^{(Y)}_{c(l)}(l, \Gamma))} \lambda_+^{|\sigma|} \right) \right) \\ & = \left( \frac{\sqrt{ze^J}}{\lambda_+} \right)^{\sum_{\Gamma \in \underline{\Gamma}} \sum_{l \in \Gamma} |\mathbb{O}_{c(l)}(l)|}. \end{aligned} \quad (61)$$

**3-2 -** We turn, now, to the terms involving  $b_+$  and  $v_+$ . For the moment, we will ignore the sources. The factors in

$$\begin{aligned} \mathfrak{N}(\underline{\Gamma}, \Lambda) := & \frac{\prod_{\sigma \in \Sigma_q(\Lambda \setminus \mathcal{I}(\underline{\Gamma}))} \left( v_+(\varpi_{\varrho, \emptyset}^{(0)}(\sigma)) v_+(\varpi_{\varrho, \emptyset}^{(1)}(\sigma)) b_+ \right)}{\prod_{\sigma \in \Sigma_q(\Lambda)} \left( v_+(\varpi_{\varrho, \emptyset}^{(0)}(\sigma)) v_+(\varpi_{\varrho, \emptyset}^{(1)}(\sigma)) b_+ \right)} \\ & \cdot \prod_{\Gamma \in \underline{\Gamma}} \prod_{l \in \Gamma} \left( e^{-\frac{1}{2}J|\partial_{c(l)}\mathbb{O}_{c(l)}(l)|} \prod_{\sigma \in \Sigma_{c(l)}(l^{(\emptyset)}_{c(l)}(l, \Gamma))} \left( v_+(\varpi_{\varrho(l), \emptyset}^{(0)}(\sigma)) v_+(\varpi_{\varrho(l), \emptyset}^{(1)}(\sigma)) b_+ \right) \right) \end{aligned} \quad (62)$$

with  $\varrho(l) := \partial_{c(l)}\mathbb{I}_{c(l)}^{(\emptyset)}(l)$  (the inside of the loops is entirely magnetized), regard the boundaries of the lines  $\sigma$  (see Fig. 6). They fall in one of the following categories.

- The terms that are attached to  $\partial_q \Lambda$  appear in the numerator and the denominator and cancel each other out.
- In addition, there are factors attached to the contours. These come in two flavors: those coming from *outside* the contour, which have open boundary conditions, and those coming from *inside*, which have magnetized boundary conditions. Thus, there is a factor  $\sqrt{b_+}v_+(\omega_0)$  associated with each edge of the outer boundary of the mantle of a loop, *provided* that edge comes in contact with a segment. And there is a factor  $\sqrt{b_+}v_+(\omega_1)$  associated with each edge of the inner boundary of the mantle, again, *provided* that edge comes in contact with a segment. The latter caveat is not innocuous: there are cases (see Fig. 9) in which a loop in the contour comes in contact with the inner boundary of the mantle of another loop, in which case there are no such terms. To keep track of these events, we introduce the set

$$\mathbb{X}(\Gamma) := \bigcup_{l \neq l' \in \Gamma} l' \cap \partial_{c(l)}\mathbb{O}_{c(l)}(l). \quad (63)$$

All in all,

$$\mathfrak{N}(\underline{\Gamma}, \Lambda) = \prod_{\Gamma \in \underline{\Gamma}} \prod_{l \in \Gamma} \left( e^{-\frac{1}{2}J|l|} \left( \sqrt{b_+} v_+(\omega_0) e^{\frac{1}{2}J} \right)^{|\mathbb{D}_{-c(l)}(l \setminus \mathbb{X}(\Gamma))|} \cdot \left( \sqrt{b_+} v_+(\omega_1) e^{-\frac{1}{2}J} \right)^{|\partial_{c(l)} \mathbb{I}_{c(l)}^{(\emptyset)}(l \setminus \mathbb{X}(\Gamma))|} \right) \quad (64)$$

in which we used the identities

$$\partial_{c(l)} \mathbb{O}_{c(l)}(l) = \mathbb{D}_{c(l)}(l) \cup \partial_{c(l)} \mathbb{I}_{c(l)}^{(\emptyset)}(l), \quad l = \mathbb{D}_{c(l)}(l) \cup \mathbb{D}_{-c(l)}(l) \quad (65)$$

(and both unions are disjoint unions) and (since the edges in  $\mathbb{X}$  appear on the inside of the mantle of a loop and the outside of another loop ( $\mathbb{X}$  is the interface between a mantle and a loop that touches the mantle), see Fig. 9)

$$\sum_{l \in \Gamma} |\mathbb{D}_{-c(l)}(l \cap \mathbb{X}(\Gamma))| = \sum_{l \in \Gamma} |\partial_{c(l)} \mathbb{I}_{c(l)}^{(\emptyset)}(l) \cap \mathbb{X}(\Gamma)| \quad (66)$$

which means that the  $e^{\frac{1}{2}J}$  and  $e^{-\frac{1}{2}J}$  in (64) that come from edges in  $\mathbb{X}$  cancel each other out.

**3-3 -** Let us now take the sources into account. Sources break up the segments, and, in doing so, contribute their own boundary terms. The main contribution comes from the sources  $v$  that are surrounded by an odd number of loops: indeed, in this case,  $v$  contributes to the denominator  $\mathfrak{Z}_q^{(\gamma)}(\Lambda)$  through the boundary terms on its  $q$ -boundary  $\partial_q v$ , whereas, in the numerator, if it is surrounded by an odd number of loops, then it contributes boundary terms on its  $-q$ -boundary  $\partial_{-q} v$ . In addition, in cases where sources come in contact with contours, the boundary of the contour may be erased at the source. Finally, the sources may interact directly with the dimers in the mantle of a contour. We denote the product of all of these factors by  $u_q^{(\gamma)}(\Gamma)$ . To define  $u$  formally, we will introduce the notion of *contact points*. Given a source  $v \in \gamma$  and a contour  $\Gamma \in \mathfrak{C}_q(\Lambda)$  the **contact points** of  $v$  and  $\Gamma$  is the set of edges  $e$  that both intersect  $v$  and neighbor the mantle of a loop of  $\Gamma$ . Formally, given  $c \in \{v, h\}$ , we define the set of interior and exterior  $c$ -contact points of  $v$  and  $\Gamma$  as

$$\mathfrak{K}_c^{(\text{int})}(v, \Gamma) := \bigcup_{\substack{l \in \Gamma \\ v \in l}} \partial_c v \cap \partial_c \mathbb{O}_{c(l)}(l), \quad \mathfrak{K}_c^{(\text{ext})}(v, \Gamma) := \bigcup_{\substack{l \in \Gamma \\ v \notin l}} \partial_c v \cap \partial_c \mathbb{O}_{c(l)}(l) \quad (67)$$

and the set of  $c$ -contact points as

$$\mathfrak{K}_c(v, \Gamma) := \mathfrak{K}_c^{(\text{int})}(v, \Gamma) \cup \mathfrak{K}_c^{(\text{ext})}(v, \Gamma). \quad (68)$$

In addition, the **background**  $\mathfrak{g}(v, \Gamma) \in \{v, h\}$  of a source  $v$  is defined in the following way: for each  $l \in \Gamma$ , if  $v \in \mathcal{E}(\mathbb{I}_{c(l)}^{(\emptyset)}(l, \Gamma))$ , then  $\mathfrak{g}(v, \Gamma)$  is set to  $c(l)$ . If there is no such loop, then  $\mathfrak{g}(v, \Gamma)$  is set to  $q$ . Correspondingly, we split the set of sources into sources in a vertical and horizontal background:

$$\gamma = \gamma_v \cup \gamma_h, \quad \gamma_c := \{v \in \gamma, \mathfrak{g}(v, \Gamma) = c\}. \quad (69)$$

This allows us to express  $u$ , following the description given above:

$$\begin{aligned}
u_q^{(\Upsilon)}(\Gamma) &= \left( \frac{v_+^4(\omega_0)b_+^2}{v_+^2(\omega_1)b_+} \right)^{|\mathbb{D}_q(\Upsilon_{-q})| - |\mathbb{D}_{-q}(\Upsilon_{-q})|} \left( \prod_{v \in \Upsilon} b_+^{-|\mathfrak{K}_{\mathbf{g}(v, \Gamma)}(v, \Gamma)|} \right) \cdot \\
&\cdot \left( \prod_{v \in \mathbb{D}_q(\Upsilon_q)} v_+(\omega_1)^{-|\mathfrak{K}_q(v, \Gamma)|} v_+(\omega_0)^{-|\mathfrak{K}_q^{(\text{ext})}(v, \Gamma)|} v_+(\omega_1)^{-|\mathfrak{K}_q^{(\text{int})}(v, \Gamma)|} \right) \cdot \\
&\cdot \left( \prod_{v \in \mathbb{D}_{-q}(\Upsilon_{-q})} v_+(\omega_0)^{-|\mathfrak{K}_{-q}(v, \Gamma)|} v_+(\omega_0)^{-|\mathfrak{K}_{-q}^{(\text{ext})}(v, \Gamma)|} v_+(\omega_1)^{-|\mathfrak{K}_{-q}^{(\text{int})}(v, \Gamma)|} \right) \cdot \\
&\cdot \left( \prod_{v \in \mathbb{D}_{-q}(\Upsilon_q)} v_+(\omega_0)^{-|\mathfrak{K}_q(v, \Gamma)|} v_+(\omega_0)^{-|\mathfrak{K}_q^{(\text{ext})}(v, \Gamma)|} v_+(\omega_1)^{-|\mathfrak{K}_q^{(\text{int})}(v, \Gamma)|} \right) \cdot \\
&\cdot \left( \prod_{v \in \mathbb{D}_{-q}(\Upsilon_{-q})} v_+(\omega_1)^{-|\mathfrak{K}_{-q}(v, \Gamma)|} v_+(\omega_0)^{-|\mathfrak{K}_{-q}^{(\text{ext})}(v, \Gamma)|} v_+(\omega_1)^{-|\mathfrak{K}_{-q}^{(\text{int})}(v, \Gamma)|} \right) \cdot \\
&\cdot \exp \left( J \sum_{c \in \{\mathbf{h}, \mathbf{v}\}} \left( \sum_{v \in \mathbb{D}_c(\Upsilon_q)} |\mathfrak{K}_c^{(\text{int})}(v, \Gamma)| + \sum_{v \in \mathbb{D}_c(\Upsilon_{-q})} |\mathfrak{K}_c^{(\text{ext})}(v, \Gamma)| \right) \right). \quad (70)
\end{aligned}$$

Thus, the actual contribution of the terms involving  $b_+$  and  $v_+$  is

$$\mathfrak{N}^{(\Upsilon)}(\underline{\Gamma}, \Lambda) = \mathfrak{N}(\underline{\Gamma}, \Lambda) \prod_{\Gamma \in \underline{\Gamma}} u_q^{(\Upsilon)}(\Gamma). \quad (71)$$

**3-4 -** We now put things together: by plugging (61), (64) and (71) into (51), keeping track of the  $e^{-W}$  terms, and noting that the  $z^{|\Upsilon \cap \Lambda'|}$  factors in (57) cancel out, we find

$$A_{\mathbf{q}}(\Lambda, \underline{\Gamma}) = \mathfrak{y}_{\mathbf{q}}^{(\Upsilon)}(\Lambda) e^{-\mathfrak{W}_{\mathbf{q}}^{(\Upsilon)}(\underline{\Gamma})} \prod_{\Gamma \in \underline{\Gamma}} \left( \mathfrak{x}_{\mathbf{q}}^{(\Upsilon)}(\Gamma) u_q^{(\Upsilon)}(\Gamma) \prod_{l \in \Gamma} \frac{e^{-\frac{1}{2}J|l|}}{\mathfrak{y}_{\mathbf{c}(l)}^{(\Upsilon)}(l_{c(l)}, \geq \ell_0(l, \Gamma))} \right) \quad (72)$$

where

$$\mathfrak{y}_{\mathbf{q}}^{(\Upsilon)}(\Lambda) := \prod_{\sigma \in \Sigma_{\mathbf{q}}(\Lambda)} \frac{1}{e^{-W^{(\varpi_{\bar{\varrho}, \Upsilon(\sigma)})}(|\sigma|)}} \quad (73)$$

(recall (58)), and, if  $\bar{\varrho} \equiv \varrho \cup \bigcup_{l \in \Gamma} \partial \mathbb{I}_{c(l)}^{(\emptyset)}(l)$ , (recall that  $\mathbf{q} \equiv (q, \varrho, \ell_0)$  is the boundary condition) then

$$\begin{aligned}
\mathfrak{x}_{\mathbf{q}}^{(\Upsilon)}(\Gamma) &:= \left( \prod_{\sigma \in \mathcal{S}_{\Lambda, < \ell_0}^{(\Upsilon)}(\Gamma)} e^{-W^{(\varpi_{\bar{\varrho}, \Upsilon(\sigma)})}(|\sigma|)} \right) \left( \prod_{l \in \Gamma} \left( \frac{\sqrt{ze^J}}{\lambda_+} \right)^{|\mathbb{O}_{c(l)}(l)|} \right) \cdot \\
&\cdot \prod_{l \in \Gamma} \left( \left( \sqrt{b_+} v_+(\omega_0) e^{\frac{1}{2}J} \right)^{|\mathbb{D}_{-c(l)}(l \setminus \mathbb{X}(\Gamma))|} \left( \sqrt{b_+} v_+(\omega_1) e^{-\frac{1}{2}J} \right)^{|\mathbb{D}_{c(l)} \mathbb{I}_{c(l)}^{(\emptyset)}(l \setminus \mathbb{X}(\Gamma))|} \right) \quad (74)
\end{aligned}$$

and  $\mathfrak{W}_{\mathbf{q}}^{(\Upsilon)}$  is the effective interaction:

$$e^{-\mathfrak{W}_{\mathbf{q}}^{(\Upsilon)}(\underline{\Gamma})} := \prod_{\substack{\sigma \in \Sigma_q(\Lambda^{(\Upsilon)} \setminus \mathcal{I}(\underline{\Gamma})) \\ |\sigma| \geq \ell_0}} e^{-W^{(\varpi_{\varrho}, \Upsilon(\sigma))}(|\sigma|)}. \quad (75)$$

**4 -** Finally, we are in a position to write the contour model in terms of an effective activity and interaction: by inserting (72) into (49), and multiplying and dividing  $Z/\mathfrak{Z}_{\mathbf{q}}$  by  $\mathfrak{h}_{\mathbf{q}}^{(\Upsilon)}$ , we find

$$\begin{aligned} \frac{Z^{(\Upsilon)}(\Lambda|\mathbf{q})}{\tilde{\mathfrak{Z}}_{\mathbf{q}}^{(\Upsilon)}(\Lambda)} &= \sum_{\underline{\Gamma} \subset \mathfrak{C}_q^{(\Upsilon)}(\Lambda)} \left( \prod_{\Gamma \neq \Gamma' \in \underline{\Gamma}} \varphi_{\text{ext}}(\Gamma, \Gamma') \right) e^{-\mathfrak{W}_{\mathbf{q}}^{(\Upsilon)}(\underline{\Gamma})}. \\ &\cdot \prod_{\Gamma \in \underline{\Gamma}} \left( \eta_{\mathbf{q}}^{(\Upsilon)}(\Gamma) \prod_{l \in \Gamma} \frac{\tilde{Z}^{(\Upsilon)}(\iota_{c(l), \geq \ell_0}^{(\Upsilon)}(l, \Gamma) | \mathbf{q})}{\tilde{\mathfrak{Z}}_{\mathbf{q}}^{(\Upsilon)}(\iota_{c(l), \geq \ell_0}^{(\Upsilon)}(l, \Gamma))} \right) \end{aligned} \quad (76)$$

in which

$$\tilde{\mathfrak{Z}}_{\mathbf{q}}^{(\Upsilon)}(\Lambda') := \mathfrak{Z}_{\mathbf{q}}^{(\Upsilon)}(\Lambda') \mathfrak{h}_{\mathbf{q}}^{(\Upsilon)}(\Lambda') \quad (77)$$

and

$$\eta_{\mathbf{q}}^{(\Upsilon)}(\Gamma) := \mathfrak{x}_{\mathbf{q}}^{(\Upsilon)}(\Gamma) \mathfrak{u}_q^{(\Upsilon)}(\Gamma) \prod_{l \in \Gamma} \left( K_{\mathbf{q}, \mathbf{c}(l)}^{(\Upsilon)}(\iota_{c(l), \geq \ell_0}^{(\Upsilon)}(l, \Gamma)) e^{-\frac{J}{2}|l|} \right) \quad (78)$$

with

$$K_{\mathbf{q}, \mathbf{c}}^{(\Upsilon)}(\Lambda') := \frac{Z^{(\Upsilon)}(\Lambda'|\mathbf{c})}{\tilde{\mathfrak{Z}}_{\mathbf{c}}^{(\Upsilon)}(\Lambda')} \frac{\tilde{\mathfrak{Z}}_{\mathbf{q}}^{(\Upsilon)}(\Lambda')}{\tilde{Z}^{(\Upsilon)}(\Lambda'|\mathbf{q})}. \quad (79)$$

The factor  $\eta_{\mathbf{q}}^{(\Upsilon)}(\Gamma)$  is the *effective activity* of  $\Gamma$ ,  $\varphi_{\text{ext}}$  is a hard-core pair interaction between the contours, and  $\mathfrak{W}_{\mathbf{q}}^{(\Upsilon)}(\underline{\Gamma})$  is a many-body, short-range *effective interaction*, arising from the 1-dimensional partition functions of the dimer configurations separating them.

**4.4. External polymer model.** We have mapped the dimer model to a contour model with hard-core and short-range (exponentially decaying) interactions. The next step is to dispense with the short-range interactions. To that end, we re-sum the interaction  $e^{-\mathfrak{W}_q^{(\Upsilon)}}$  by inserting trivial identities into (75):

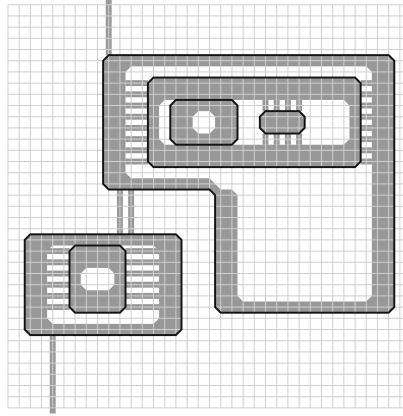
$$e^{-\mathfrak{W}_{\mathbf{q}}^{(\Upsilon)}(\underline{\Gamma})} = \prod_{\substack{\sigma \in \Sigma_q(\Lambda^{(\Upsilon)} \setminus \mathcal{I}(\underline{\Gamma})) \\ |\sigma| \geq \ell_0}} (w_{\varpi_{\varrho}, \Upsilon(\sigma)}(|\sigma|) + 1) \quad (80)$$

where

$$w_{\varpi_{\varrho}, \Upsilon(\sigma)}(|\sigma|) := e^{-W^{(\varpi_{\varrho}, \Upsilon(\sigma))}(|\sigma|)} - 1. \quad (81)$$

and expand  $e^{-\mathfrak{W}}$ :

$$e^{-\mathfrak{W}_{\mathbf{q}}^{(\Upsilon)}(\underline{\Gamma})} := \sum_{S \subset \{\sigma \in \Sigma_q(\Lambda^{(\Upsilon)} \setminus \mathcal{I}(\underline{\Gamma})), |\sigma| \geq \ell_0\}} \prod_{\sigma \in S} w_{\varpi_{\varrho}, \Upsilon(\sigma)}(|\sigma|). \quad (82)$$



**Fig. 10.** An external polymer. It is obtained by joining the two contours in Fig. 8 by segments of length  $\geq \ell_0$ . This polymer is connected to  $\partial \Lambda$

Each term in the sum over  $S$  gives rise to a new object, called an *external polymer*, which consists of contours joined together by segments in  $S$  (see Fig. 10). Formally, an **external polymer** is a couple  $\xi \equiv (\underline{\Gamma}(\xi), \underline{\sigma}(\xi))$  with

- $\underline{\Gamma}(\xi) \equiv \{\Gamma_1, \dots, \Gamma_{|\underline{\Gamma}(\xi)|}\}$  is a (possibly empty) set of contours  $\Gamma_i \in \mathfrak{C}_q^{(\Upsilon)}(\Lambda)$ , that are pairwise *compatible* and *external* to each other,
- $\underline{\sigma}(\xi) \equiv \{\sigma_1(\xi), \dots, \sigma_{|\underline{\sigma}(\xi)|}\}$  is a (possibly empty) set of  $q$ -segments

$$\sigma_i \in \left\{ \sigma \in \Sigma_q \left( \Lambda^{(\Upsilon)} \setminus \left( \bigcup_{\Gamma \in \underline{\Gamma}(\xi)} \Gamma \cup \bigcup_{l \in \Gamma} l \right) \right), \quad |\sigma| \geq \ell_0 \right\} \quad (83)$$

satisfying the following conditions.

- $\underline{\Gamma}(\xi)$  and  $\underline{\sigma}(\xi)$  cannot both be empty.
- We define the **support** of  $\xi$  as

$$\begin{aligned} \text{supp}_{\Lambda, \ell_0}^{(\Upsilon)}(\xi) &:= \left( \bigcup_{\Gamma \in \underline{\Gamma}(\xi)} \Gamma, \mathcal{S}_{\Lambda}^{(\Upsilon)}(\xi) \right), \\ \mathcal{S}_{\Lambda}^{(\Upsilon)}(\xi) &:= \underline{\sigma}(\xi) \cup \left( \bigcup_{\Gamma \in \underline{\Gamma}(\xi)} \mathcal{S}_{\Lambda, < \ell_0}^{(\Upsilon)}(\Gamma) \right) \end{aligned} \quad (84)$$

The support of  $\xi$  is required to be *connected*.

Denoting the set of external polymers in  $\Lambda$  by  $\mathfrak{X}_q^{(\Upsilon)}(\Lambda)$ , we rewrite (76) as

$$\begin{aligned} \frac{Z^{(\Upsilon)}(\Lambda | \mathbf{q})}{\tilde{\mathfrak{Z}}_{\mathbf{q}}^{(\Upsilon)}(\Lambda)} &= \sum_{\xi \subset \mathfrak{X}_q^{(\Upsilon)}(\Lambda)} \left( \prod_{\xi \neq \xi' \in \xi} \varphi_{\text{ext}}(\xi, \xi') \right) \cdot \\ &\cdot \prod_{\xi \in \xi} \left( \zeta_{\mathbf{q}}^{(\Upsilon)}(\xi) \prod_{\Gamma \in \underline{\Gamma}(\xi)} \prod_{l \in \Gamma} \frac{\tilde{Z}^{(\Upsilon)}(\ell_{c(l)}, \geq \ell_0(l, \Gamma) | \mathbf{q})}{\tilde{\mathfrak{Z}}_{\mathbf{q}, (\ell_{c(l)}, \geq \ell_0(l, \Gamma))}^{(\Upsilon)}} \right) \end{aligned} \quad (85)$$

in which

- $\varphi_{\text{ext}}(\xi, \xi') \in \{0, 1\}$  is equal to 1 if and only if  $\xi$  and  $\xi'$  are **compatible**, by which we mean that the support of  $\xi_1 \cup \xi_2 \equiv (\underline{\Gamma}(\xi_1) \cup \underline{\Gamma}(\xi_2), \underline{\sigma}(\xi_1) \cup \underline{\sigma}(\xi_2))$  is *disconnected*,
- $\zeta_{\mathbf{q}}^{(\Upsilon)}(\xi)$  is the *activity* of  $\xi$ :

$$\zeta_{\mathbf{q}}^{(\Upsilon)}(\xi) := \left( \prod_{\Gamma \in \underline{\Gamma}(\xi)} \eta_{\mathbf{q}}^{(\Upsilon)}(\Gamma) \right) \left( \prod_{\sigma \in \underline{\sigma}(\xi)} w_{\varpi_{\mathbf{q}, \Upsilon}(\sigma)}(|\sigma|) \right). \quad (86)$$

We will now define  $\tilde{Z}^{(\Upsilon)}(\iota_{c(l), \geq \ell_0}^{(\Upsilon)}(l, \Gamma) | \mathbf{q})$ , as the partition function of *non-trivial* external polymers. Trivial polymers are  $q$ -segments that go all the way through  $\iota_{c(l), \geq \ell_0}^{(\Upsilon)}(l, \Gamma)$ . By construction, every  $c(l)$ -segment in  $\iota_{c(l), \geq \ell_0}^{(\Upsilon)}(l, \Gamma)$  is of length  $\geq \ell_0$ , but this is not necessarily true of the  $q$ -segments. Since short segments give a poor gain, we wish to avoid them, and, simply, define  $\tilde{Z}^{(\Upsilon)}(\iota_{c(l), \geq \ell_0}^{(\Upsilon)}(l, \Gamma) | \mathbf{q})$  without them: for any finite  $\Lambda' \subset \mathbb{Z}^2$ ,

$$\frac{\tilde{Z}^{(\Upsilon)}(\Lambda' | \mathbf{q})}{\tilde{\mathfrak{Z}}_{\mathbf{q}}^{(\Upsilon)}(\Lambda')} := \sum_{\underline{\xi} \subset \tilde{\mathfrak{X}}_q^{(\Upsilon)}(\Lambda')} \left( \prod_{\underline{\xi} \neq \underline{\xi}' \in \underline{\xi}} \varphi_{\text{ext}}(\xi, \xi') \right) \cdot \prod_{\xi \in \underline{\xi}} \left( \zeta_{\mathbf{q}}^{(\Upsilon)}(\xi) \prod_{\Gamma \in \underline{\Gamma}(\xi)} \prod_{l \in \Gamma} \frac{\tilde{Z}^{(\Upsilon)}(\iota_{c(l), \geq \ell_0}^{(\Upsilon)}(l, \Gamma) | \mathbf{q})}{\tilde{\mathfrak{Z}}_{\mathbf{q}, \Gamma}^{(\Upsilon)}(\iota_{c(l), \geq \ell_0}^{(\Upsilon)}(l, \Gamma))} \right) \quad (87)$$

obtained from (85) by replacing  $\mathfrak{X}_q^{(\Upsilon)}(\Lambda')$  with  $\tilde{\mathfrak{X}}_q^{(\Upsilon)}(\Lambda')$ , which is the set of **non-trivial polymers**:

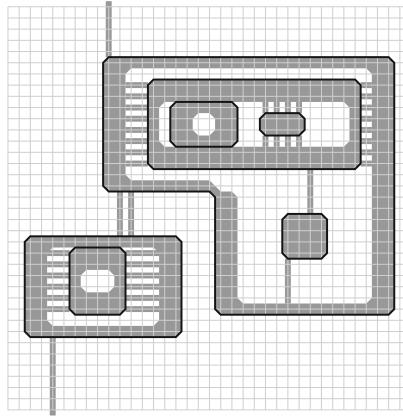
$$\tilde{\mathfrak{X}}_q^{(\Upsilon)}(\Lambda) := \{\xi \in \mathfrak{X}_q^{(\Upsilon)}(\Lambda), \underline{\Gamma}(\xi) \neq \emptyset\}. \quad (88)$$

*Remark.* The reason that we can drop the trivial polymers comes from the inductive structure of the construction: we have  $\tilde{Z}(\iota | \mathbf{q})$  instead of  $Z(\iota | \mathbf{c})$  in (85) because we have multiplied and divided by  $\tilde{Z}(\iota | \mathbf{q})$  and incorporated  $Z(\iota | \mathbf{c})$  into the flipping term  $K_{\mathbf{q}, \mathbf{c}}$ . Therefore, at this stage,  $\tilde{Z}(\iota | \mathbf{q})$  could be, essentially, anything. Later on, we will need the fact that  $K_{\mathbf{q}, \mathbf{c}}$  is, at most, exponentially large in the size of the boundary. This imposes constraints on  $\tilde{Z}(\iota | \mathbf{q})$ , which must not differ too much from  $Z(\iota | \mathbf{c})$ . In this context, ‘not too much’ means that they only differ by boundary terms. Trivial polymers, which go all the way through  $\Lambda$ , are boundary terms, which is why they can be dropped. This will be proved in Lemma 7.

**4.5. Polymer model.** In order to move from external polymers to polymers, we proceed recursively, by placing external polymers inside external polymers. Before defining the set of polymers, let us first introduce a few more definition: an external polymer  $\xi$  is said to be **connected to**  $\partial \Lambda$  if the support of  $(\partial \Lambda \cup \underline{\Gamma}(\xi), \underline{\sigma}(\xi))$  is connected. In addition, the **core** of  $\xi$  is defined as

$$\mathbb{I}_q^{(\Upsilon)}(\xi) := \bigcup_{\Gamma \in \underline{\Gamma}(\xi)} \bigcup_{l \in \Gamma} \mathbb{I}_{c(l)}^{(\Upsilon)}(l). \quad (89)$$

Now, the set of **polymers**  $\mathfrak{P}_q^{(\Upsilon)}(\Lambda)$  is defined recursively. Roughly (see below for a formal definition), a polymer consists of an external polymer with polymers inside it



**Fig. 11.** A polymer. It consists of an external polymer, which is the one drawn in Fig. 10, and a smaller polymer inside it. The smaller polymer consists of a single loop, and is connected to the external polymer by two vertical lines, whose length are  $\geq \ell_0 \equiv 4$ . Note that this loop is an h-bounding loop, even though it is directly inside an h-bounding loop. Similarly, the smaller polymer is connected to the external one by vertical lines, instead of horizontal ones. This comes from the recursive structure of the construction of polymers

(here, the word ‘external’ refers to the definition in Sect. 4.4: the external polymer is, obviously, not external to the polymers inside it). The polymers inside the external polymer are connected to it by segments.

- An external polymer is, itself, a polymer:  $\mathfrak{P}_q^{(\gamma)}(\Lambda) \supset \mathfrak{X}_q^{(\gamma)}(\Lambda)$ .
- A polymer  $\gamma \in \mathfrak{P}_q^{(\gamma)}(\Lambda) \setminus \mathfrak{X}_q^{(\gamma)}(\Lambda)$  that is not external, is the union of an external polymer  $\xi(\gamma) \in \mathfrak{X}_q^{(\gamma)}(\Lambda)$  and of polymers  $g_1, \dots, g_n \in \mathfrak{P}_q^{(\gamma)}(\mathbb{I}_q^{(\emptyset)}(\xi(\gamma)))$  that are all  $q$ -connected to  $\partial_q \mathbb{I}_q^{(\emptyset)}(\xi(\gamma))$ , and compatible with  $\xi(\gamma)$  and with each other. In this case, we define  $\underline{g}(\gamma) := \{g_1, \dots, g_n\}$ .
- A polymer  $\gamma \in \mathfrak{P}_q^{(\gamma)}(\Lambda)$  is said to be **connected to**  $\partial \Lambda$  if  $\xi(\gamma)$  is connected to  $\partial \Lambda$ .
- Two polymers  $\gamma, \gamma' \in \mathfrak{P}_q^{(\gamma)}(\Lambda)$  are said to be **compatible** if, for any  $g \in \{\xi(\gamma)\} \cup \underline{g}(\gamma)$  and any  $g' \in \{\xi(\gamma')\} \cup \underline{g}(\gamma')$ ,  $g$  and  $g'$  are compatible.

The **activity** of a polymer  $\gamma$  is defined as

$$\xi_q^{(\gamma)}(\gamma) := \xi_q^{(\gamma)}(\xi(\gamma)) \prod_{g \in \underline{g}(\gamma)} \xi_q^{(\gamma)}(g). \quad (90)$$

We are now ready to state the main result of this section, namely the mapping to the polymer model, stated in the following lemma.

**Lemma 2** (Polymer model). *Consider a bounded subset  $\Lambda \subset \mathbb{Z}^2$  such that  $\mathbb{Z}^2 \setminus \Lambda$  is connected, and the boundary  $\partial \Lambda$  is a bounding loop. In addition, let  $\mathbf{q}$  be a boundary condition. If every  $q$ -segment of  $\Lambda$  is of length  $\geq \ell_0$ , that is, for every  $\sigma \in \Sigma_q(\Lambda)$ ,  $|\sigma| \geq \ell_0$ , then*

$$\frac{Z^{(\gamma)}(\Lambda | \mathbf{q})}{\mathfrak{Z}_q^{(\gamma)}(\Lambda)} = \sum_{\gamma \subset \mathfrak{P}_q(\Lambda)} \left( \prod_{\gamma \neq \gamma' \in \underline{\gamma}} \Phi(\gamma, \gamma') \right) \prod_{\gamma \in \underline{\gamma}} \xi_q^{(\gamma)}(\gamma) \quad (91)$$

in which  $\Phi(\gamma, \gamma') \in \{0, 1\}$  is equal to 1 if and only if  $\gamma$  and  $\gamma'$  are compatible.

*Proof.* We will actually prove that (87) can be rewritten as

$$\frac{\tilde{Z}^{(\gamma)}(\Lambda'|\mathbf{q})}{\tilde{\mathcal{Z}}_{\mathbf{q}}^{(\gamma)}(\Lambda')} = \sum_{\underline{\gamma} \subset \tilde{\mathcal{P}}_q^{(\gamma)}(\Lambda')} \left( \prod_{\gamma \neq \gamma' \in \underline{\gamma}} \Phi(\gamma, \gamma') \right) \prod_{\gamma \in \underline{\gamma}} \xi_{\mathbf{q}}^{(\gamma)}(\gamma) \quad (92)$$

in which  $\tilde{\mathcal{P}}_q^{(\gamma)}(\Lambda)$  is the set of *non-trivial* polymers, which we define as follows. The set of **trivial** polymers is the set of polymers that consist of a single trivial external polymer. The set of **non-trivial** polymers is the complement of the set of trivial polymers. By (85), this implies (91).

Equation (87) states that we can deduce the expression of the right side of (92) from the same expression for smaller sets  $\Lambda' \subsetneq \Lambda$ . It follows from the principle of mathematical induction, that if we know (92) for the smallest possible sets, then we can compute the left side of (92) for sets of any size. If  $\Lambda$  is so small that it cannot contain a contour, then (92) follows immediately from (87) (both sides of the equation are equal to 1). We now assume that (91) holds for every strict subset of  $\Lambda$ . By inserting (92) into (87), we find

$$\begin{aligned} \frac{\tilde{Z}^{(\gamma)}(\Lambda|\mathbf{q})}{\tilde{\mathcal{Z}}_{\mathbf{q}}^{(\gamma)}(\Lambda)} &= \sum_{\underline{\gamma} \subset \tilde{\mathcal{P}}_q^{(\gamma)}(\Lambda)} \left( \prod_{\gamma \neq \gamma' \in \underline{\gamma}} \varphi_{\text{ext}}(\gamma, \gamma') \right) \left( \prod_{\gamma \in \underline{\gamma}} \xi_{\mathbf{q}}^{(\gamma)}(\gamma) \right) \cdot \\ &\quad \cdot \prod_{\gamma \in \underline{\gamma}} \prod_{l \in \text{supp}_{\mathcal{L}}(\gamma)} \left( \sum_{\underline{\gamma}_l \subset \tilde{\mathcal{P}}_{q, \geq \ell_0}^{(\gamma)}(l_{c(l)}^{(\gamma)}(l, \Gamma))} \left( \prod_{\gamma_l \neq \gamma'_l \in \underline{\gamma}_l} \Phi(\gamma_l, \gamma'_l) \right) \prod_{\gamma_l \in \underline{\gamma}_l} \xi_{\mathbf{q}}^{(\gamma)}(\gamma_l) \right). \end{aligned} \quad (93)$$

Following the recursive structure of the definition of the set of polymers, we group the unions of external polymers  $\gamma$  and polymers  $\gamma_l$  into a set of connected polymers that are pairwise compatible. We thus conclude the proof of (92) for  $\Lambda$  from (92) for strict subsets of  $\Lambda$ .  $\square$

## 5. Cluster Expansion of the Polymer Model

In this section, we will express the partition function of the polymer model (91) as an absolutely convergent *cluster expansion*. To prove the convergence of the expansion, we will proceed by induction: assuming that the cluster expansion is absolutely convergent for strict subsets of  $\Lambda$ , we will prove that it converges for  $\Lambda$ . We split this result into three lemmas (see Lemmas 4, 5 and 7). In the first, we prove a bound for the effective activity of polymers, in the second, we bound the entropy of the polymers, and in the third, prove the convergence of the cluster expansion.

**5.1. Cluster expansion.** The cluster expansion allows us to compute the logarithm of the partition function (91) in terms of an absolutely convergent series. This is a rather standard step in Pirogov-Sinai theory, and has been written about extensively. In this work, we will use a result of Bovier and Zahradník [BZ00, Theorem 1], which, using our notations, is summed up in the following lemma.

**Lemma 3** (Cluster expansion). *If there exist two functions  $a, d$  that map polymers  $\mathfrak{P}_q^{(\gamma)}(\Lambda)$  to  $[0, \infty)$  and a number  $\delta \geq 0$ , such that  $\forall \gamma \in \mathfrak{P}_q^{(\gamma)}(\Lambda)$ ,*

$$|\zeta_{\mathbf{q}}^{(\gamma)}(\gamma)|e^{a(\gamma)+d(\gamma)} \leq \delta < 1, \quad \sum_{\substack{\gamma' \in \mathfrak{P}_q^{(\gamma)}(\Lambda) \\ \gamma' \not\sim \gamma}} |\zeta_{\mathbf{q}}^{(\gamma)}(\gamma')|e^{a(\gamma')+d(\gamma')} \leq \frac{\delta}{|\log(1-\delta)|} a(\gamma) \quad (94)$$

in which  $\gamma' \not\sim \gamma$  means that  $\gamma'$  and  $\gamma$  are not compatible, then

$$\log \left( \frac{Z(\Lambda|\mathbf{q})}{\tilde{Z}_{\mathbf{q}}(\Lambda)} \right) = \sum_{\underline{\gamma} \sqsubset \mathfrak{P}_q^{(\gamma)}(\Lambda)} \Phi^T(\underline{\gamma}) \prod_{\gamma \in \underline{\gamma}} \zeta_{\mathbf{q}}^{(\gamma)}(\gamma) \quad (95)$$

in which  $\underline{\gamma} \sqsubset \mathfrak{P}_q^{(\gamma)}(\Lambda)$  (the symbol  $\sqsubset$  is used instead of  $\subset$ ) means that  $\underline{\gamma}$  is a multiset (a multiset is similar to a set except for the fact that an element may appear several times in a multiset, in other words, a multiset is an unordered tuple) with elements in  $\mathfrak{P}_q^{(\gamma)}(\Lambda)$ , and  $\Phi^T$  is the Ursell function, defined as (see [Ru99, (4.9)])

$$\Phi^T(\emptyset) = 0, \quad \Phi^T(\{\gamma\}) = 1,$$

$$\Phi^T(\{\gamma_1, \dots, \gamma_n\}) := \frac{1}{N_{\underline{\gamma}}!} \sum_{\mathfrak{g} \in \mathcal{G}^T(n)} \prod_{\{j, j'\} \in \mathcal{E}(\mathfrak{g})} (\Phi(\gamma_j, \gamma_{j'}) - 1) \quad (96)$$

in which  $\mathcal{G}^T(n)$  is the set of connected graphs on  $n$  vertices and  $\mathcal{E}(\mathfrak{g})$  is the set of edges of  $\mathfrak{g}$ . In addition, for every  $\gamma \in \mathfrak{P}_q^{(\gamma)}(\Lambda)$ , and, if  $n_{\gamma_i}$  is the multiplicity of  $\gamma_i$  in  $(\gamma_1, \dots, \gamma_n)$ , then  $N_{\underline{\gamma}}! \equiv \prod_{j=1}^n (n_{\gamma_j}!)^{\frac{1}{n_{\gamma_j}}}$ . In addition, for every  $\gamma \in \mathfrak{C}_v(\Lambda)$ ,

$$\sum_{\underline{\gamma} \sqsubset \mathfrak{P}_q^{(\gamma)}(\Lambda)} \left| \Phi^T(\{\gamma\} \sqcup \underline{\gamma}) \prod_{\gamma' \in \underline{\gamma}} \left( \zeta_{\mathbf{q}}^{(\gamma)}(\gamma') e^{d(\gamma')} \right) \right| \leq e^{a(\gamma)} \quad (97)$$

where  $\sqcup$  denotes the union operation in the sense of multisets.

**5.2. Bound on the polymer activity.** We will now prove a bound on the activity  $\zeta_{\mathbf{q}}^{(\gamma)}$  of a polymer. We will prove this bound under the assumption that  $K_{\mathbf{q}, \mathbf{c}}^{(\gamma)}(\Lambda')$  is, at most, exponentially large in  $|\partial \Lambda'|$ , a fact which we will prove in Lemma 7. (That proof is based on a cluster expansion of  $Z$  and  $\tilde{Z}$ , in which the only clusters that contribute are those which interact with the boundary.)

**Lemma 4** (Bound on the polymer activity). *If*

$$e^J \gg z \gg 1, \quad (98)$$

and,  $\exists C_5 > 0$  such that, for every  $\Lambda' \subsetneq \Lambda$ ,

$$\left| K_{\mathbf{q}, \mathbf{c}}^{(\gamma)}(\Lambda') \right| \leq e^{C_5(|\partial_c \Lambda'| + \ell_0 e^{-3J} |\partial_q \Lambda'|)}. \quad (99)$$

then,  $\forall \gamma \in \mathfrak{P}_q^{(\Upsilon)}(\Lambda)$ ,

$$\left| \zeta_{\mathbf{q}}^{(\Upsilon)}(\gamma) \right| \leq e^{-\Xi^{(\Upsilon)}(\gamma)} \quad (100)$$

where

$$\begin{aligned} \Xi^{(\Upsilon)}(\gamma) := & \frac{1}{2} \bar{J} \mathfrak{l}(\gamma) + \bar{\kappa} \mathfrak{s}(\gamma) - (J + \log \bar{\epsilon}) \mathfrak{m}^{(\Upsilon)}(\gamma) - C_5 \ell_0^2 e^{-3J} \mathfrak{b}(\gamma) \\ & - J \mathfrak{v}_1^{(\Upsilon)}(\gamma) - J \mathfrak{v}_2^{(\Upsilon)}(\gamma) - C_6 |\Upsilon| \end{aligned} \quad (101)$$

for some constant  $C_6 > 0$ , with

$$\bar{J} := J - C_7, \quad \bar{\kappa} := \kappa C_8, \quad \bar{\epsilon} := \epsilon C_9 \quad (102)$$

in which  $C_7 > 2C_5$  and  $C_8, C_9 > 0$  are constants,

$$\epsilon := \frac{1}{\sqrt{ze^J}} \ll 1, \quad \kappa := \frac{1}{e^J \sqrt{ze^J}} \ll 1. \quad (103)$$

and (the following definitions are recursive)

$$\mathfrak{l}(\gamma) := \sum_{\Gamma \in \underline{\Gamma}(\xi(\gamma))} \sum_{l \in \Gamma} |l| + \sum_{g \in \underline{g}(\gamma)} \mathfrak{l}(g), \quad \mathfrak{s}(\gamma) := \sum_{\sigma \in \underline{\sigma}(\xi(\gamma))} |\sigma| + \sum_{g \in \underline{g}(\gamma)} \mathfrak{s}(g) \quad (104)$$

$$\begin{aligned} \mathfrak{m}^{(\Upsilon)}(\gamma) := & \sum_{\Gamma \in \underline{\Gamma}(\xi(\gamma))} \sum_{\sigma \in \mathcal{S}_{A, < \ell_0}^{(\Upsilon)}(\Gamma)} 1 + \sum_{\substack{g \in \underline{g}(\gamma) \\ |\sigma|=1}} \mathfrak{m}^{(\Upsilon)}(g), \\ \mathfrak{b}(\gamma) := & \mathbb{1}_{\mathfrak{l}(\gamma) - \mathfrak{m}^{(\Upsilon)}(\gamma) \geq \ell_0} (\mathfrak{l}(\gamma) - \mathfrak{m}^{(\Upsilon)}(\gamma)). \end{aligned} \quad (105)$$

$$\mathfrak{v}_1^{(\Upsilon)}(\gamma) := \sum_{\Gamma \in \underline{\Gamma}(\xi(\gamma))} 3(|\mathbb{D}_{-q}(\Upsilon_{-q})| - |\mathbb{D}_q(\Upsilon_{-q})|) + \sum_{g \in \underline{g}(\gamma)} \mathfrak{v}_1^{(\Upsilon)}(g) \quad (106)$$

$$\mathfrak{v}_2^{(\Upsilon)}(\gamma) := \sum_{\Gamma \in \underline{\Gamma}(\xi(\gamma))} \sum_{v \in \mathbb{D}_q(\Upsilon_{-q}) \cup \mathbb{D}_{-q}(\Upsilon_q)} |\mathfrak{K}^{(\Upsilon, \text{ext})}(v, \Gamma)| + \sum_{g \in \underline{g}(\gamma)} \mathfrak{v}_2^{(\Upsilon)}(g) \quad (107)$$

in which  $\mathfrak{K}^{(\Upsilon, \text{ext})}(v, \Gamma) := \mathfrak{K}_v^{(\Upsilon, \text{ext})}(v, \Gamma) \cup \mathfrak{K}_h^{(\Upsilon, \text{ext})}(v, \Gamma)$  (see (67)).

*Proof.* We recall that  $\zeta_{\mathbf{q}}^{(\Upsilon)}(\gamma)$  was defined in (90). We proceed by first bounding  $\zeta_{\mathbf{q}}^{(\Upsilon)}(\gamma)$  for  $\gamma \equiv \xi \in \mathfrak{X}_q^{(\Upsilon)}(\Lambda)$ , and conclude the proof by induction. To that end we bound the terms appearing in (86) one by one.

**1 -** First of all, by (24),

$$\lambda_+ = \sqrt{ze^J} (1 + \kappa + O(\epsilon^2 \kappa)), \quad \frac{\lambda_-}{\lambda_+} = -(1 - \kappa) + O(\kappa \epsilon), \quad \frac{\lambda_0}{\lambda_+} = \frac{1}{\sqrt{ze^J}} (1 + O(e^{-J})) \quad (108)$$

by (25),

$$b_+ = \frac{1}{2ze^{2J}} (1 + O(\epsilon)), \quad \frac{b_-}{b_+} = 1 - 4\epsilon + 8\epsilon^2 + 5\kappa + O(\epsilon^3), \quad \frac{b_0}{b_+} = 2ze^{2J} (1 + O(\epsilon)) \quad (109)$$

and by (26) and (53),

$$v_+(\omega_0) = \lambda_+, \quad \frac{v_-(\omega_0)}{v_+(\omega_0)} = \frac{\lambda_-}{\lambda_+}, \quad \frac{v_0(\omega_0)}{v_+(\omega_0)} = \frac{\lambda_0}{\lambda_+} \quad (110)$$

and

$$v_+(\omega_1) = e^J \sqrt{ze^J} (1 + O(\epsilon)), \quad \frac{v_-(\omega_1)}{v_+(\omega_1)} = -(1 + 2\epsilon + 2\epsilon^2 - 3\kappa) + O(\epsilon^3), \quad (111)$$

$$\frac{v_0(\omega_1)}{v_+(\omega_1)} = -\frac{e^{-\frac{5}{2}J}}{z^{\frac{3}{2}}} (1 + O(\epsilon)).$$

**2 -** We bound  $\mathfrak{x}_c^{(\gamma)}(\Gamma)$ , which was defined in (74): by (108) through (111),

$$\sqrt{b_+} v_+(\omega_0) e^{\frac{1}{2}J} = \frac{1}{\sqrt{2}} (1 + O(\epsilon)), \quad \sqrt{b_+} v_+(\omega_1) e^{-\frac{1}{2}J} = \frac{1}{\sqrt{2}} (1 + O(\epsilon)), \quad (112)$$

$$\frac{\sqrt{ze^J}}{\lambda_+} < 1.$$

In addition, for any  $\omega \equiv (\omega^{(0)}, \omega^{(1)}) \in \{\omega_0, \omega_1\}^2$ ,

$$\frac{v_-(\omega^{(0)}) v_-(\omega^{(1)}) b_-}{v_+(\omega^{(0)}) v_+(\omega^{(1)}) b_+} < 1, \quad \frac{v_0(\omega^{(0)}) v_0(\omega^{(1)}) b_0}{v_+(\omega^{(0)}) v_+(\omega^{(1)}) b_+} = O(e^J) \quad (113)$$

so, by (58),

$$|e^{-W^{(\omega)}(|\sigma|)}| < 1 + O(e^J \epsilon^{|\sigma|}). \quad (114)$$

If  $|\sigma| \geq 2$ , then  $O(e^J \epsilon^{|\sigma|}) < 1$ , whereas if  $|\sigma| = 1$ , it is of order  $O(e^{J+\log \epsilon})$  and may be large. Therefore,

$$|\mathfrak{x}_q^{(\gamma)}(\Gamma)| < \left( O(e^{J+\log \epsilon}) \right)^{m^{(\gamma)}(\Gamma)} \quad (115)$$

in which  $m^{(\gamma)}$  was defined in (105) (Each segment of length  $\geq 2$  and  $< \ell_0$  contributes, at most,  $1 + O(e^J \epsilon^2)$ , which we absorb into the  $\frac{1}{\sqrt{2}}$  factors in (112), which we can do since we can bound the number of segments by the lengths of the loops (or rather, by the number of  $-c(l)$ -edges in the outer loop and the number of  $c(l)$ -edges on the boundary of its core).)

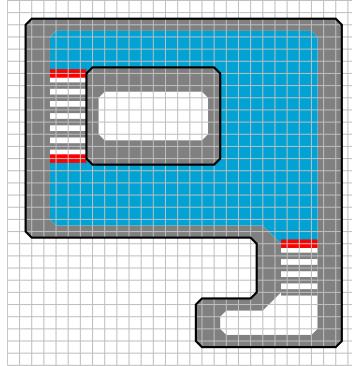
**3 -** We now turn to  $\mathfrak{u}_q^{(\gamma)}(\Gamma)$ , defined in (70). By (112),

$$|\mathfrak{u}_q^{(\gamma)}(\Gamma)| < O(1)^{|\gamma|} e^{J(\mathfrak{v}_1^{(\gamma)}(\Gamma) + \mathfrak{v}_2^{(\gamma)}(\Gamma))} \quad (116)$$

in which  $\mathfrak{v}_i$  was defined in (106) and (107).

**4 -** We will now bound  $K_{\mathbf{q}, \mathbf{c}(l)}^{(\gamma)}$ , which was defined in (79). By (99),

$$\sum_{l \in \Gamma} \log K_{\mathbf{q}, \mathbf{c}(l)}^{(\gamma)}(\iota_{c(l), \geq \ell_0}^{(\gamma)}(l, \Gamma)) \leq C_5 \sum_{l \in \Gamma} |\partial_{c(l)} \iota_{c(l), \geq \ell_0}^{(\gamma)}(l, \Gamma)| + C_5 \ell_0 e^{-3J} \sum_{l \in \Gamma} |\partial_q \iota_{c(l), \geq \ell_0}^{(\gamma)}(l, \Gamma)|. \quad (117)$$



**Fig. 12.** A polymer, in which we have highlighted one of the connected components of the padding (in cyan, color online). Whereas most of the boundary of the padding is part of a loop (or of the boundary of the core of a loop), some parts are segments of length  $< \ell_0 \equiv 4$ . These are depicted as thicker red (color online) lines

(If  $q = c(l)$ , then this inequality is slightly suboptimal: only the first term is needed. However, this bound is good enough for our purposes.) In addition, since every edge in  $\partial_{c(l)} l_{c(l), \geq \ell_0}^{(\Upsilon)}(l, \Gamma)$  is necessarily part of a loop (or, rather, of the boundary of the core of a loop, but this distinction does not matter much since there exists an injective map from the boundary of the core to its loop) or a source (see Fig. 10),

$$\sum_{l \in \Gamma} |\partial_{c(l)} l_{c(l), \geq \ell_0}^{(\Upsilon)}(l, \Gamma)| \leq l(\Gamma) + 4|\Upsilon|. \quad (118)$$

On the other hand, the edges in  $\partial_q l_{c(l), \geq \ell_0}^{(\Upsilon)}(l, \Gamma)$  are not necessarily in a loop or a source: if  $q \neq c(l)$ , then portions of the  $q$ -boundary of  $l_{c(l), \geq \ell_0}^{(\Upsilon)}(l, \Gamma)$  can consist of segments of length  $< \ell_0$  (see Fig. 12). However, we can bound the number of times this may happen, as follows. Consider a connected component  $I$  of  $\partial_{c(l)} l_{c(l), m \geq \ell_0}^{(\Upsilon)}(l, \Gamma)$  and a loop  $l'$  in  $\partial I$ . We go through the edges in  $l'$  in, say, clockwise order, which gives us an ordered list of edges. The edges that intersect a segment of length  $< \ell_0$  are called ‘bad’. We then group consecutive bad edges together, and the edge immediately following a bad group is called ‘good’. By construction, there are at least as many good edges as there are groups of bad ones. In addition, since bad edges touch segments of length  $< \ell_0$ , each group of consecutive bad edges contains  $< \ell_0$  elements. Therefore,

$$\#\{\text{bad edges}\} < \ell_0 \times \#\{\text{good edges}\} \quad (119)$$

We then construct an injective map from the set of good edges to the set of edges in  $l$  that are not connected to a segment of length 1. The map is defined as follows: given a good edge  $e$

- if  $e$  is already part of a loop, then the map returns  $e$  itself, which may not be connected to a segment of length 1 (otherwise, it would not be part of the boundary of  $l_{c(l), \geq \ell_0}^{(\Upsilon)}(l, \Gamma)$ ),
- if  $e$  is on the boundary of the mantle of a loop, then we use the mapping alluded to earlier to map the boundary of the mantle of a loop to the loop itself (we have not defined it formally, a task which we leave to the reader),

- if  $e$  is on the boundary of a source, then that source must, itself, neighbor a loop or its mantle (if it did not, then the loop  $l'$  would simply go around the source and there would be no bad edges), in which case, the map returns the edge at which the source is connected to the loop (if that edge is on the boundary of a mantle, then we use the map mentioned above).

All in all, this implies that

$$\#\{\text{bad edges}\} < \ell_0(\mathbf{l}(\Gamma) - \mathbf{m}^{(\Upsilon)}(\Gamma)). \quad (120)$$

Finally, this may only occur if  $\Gamma$  is large enough to contain a non-empty  $\iota_{c(l), \geq \ell_0}^{(\Upsilon)}$ , that is, if

$$\mathbf{l}(\Gamma) - \mathbf{m}^{(\Upsilon)}(\Gamma) \geq \ell_0. \quad (121)$$

Therefore,

$$\sum_{l \in \Gamma} |\partial_q \iota_{c(l), \geq \ell_0}^{(\Upsilon)}(l, \Gamma)| \leq \mathbf{l}(\Gamma) + 4|\Upsilon| + \mathbb{1}_{\mathbf{l}(\Gamma) - \mathbf{m}^{(\Upsilon)}(\Gamma) \geq \ell_0} \ell_0(\mathbf{l}(\Gamma) - \mathbf{m}^{(\Upsilon)}(\Gamma)). \quad (122)$$

Thus, inserting (118) and (122) into (117), we find

$$\prod_{l \in \Gamma} K_{\mathbf{q}, \mathbf{c}(l)}^{(\Upsilon)}(\iota_{c(l), \geq \ell_0}^{(\Upsilon)}(l, \Gamma)) \leq e^{C_5(2\mathbf{l}(\Gamma) + \mathbb{1}_{\mathbf{l}(\Upsilon) - \mathbf{m}^{(\Upsilon)}(\Upsilon) \geq \ell_0} \ell_0^2 e^{-3J} (\mathbf{l}(\Gamma) - \mathbf{m}^{(\Upsilon)}(\Gamma)))} O(1)^{|\Upsilon|}. \quad (123)$$

**5 -** We turn, now, to  $w$ , which was defined in (81). By (113), since  $|\sigma| \geq \ell_0$ ,

$$|w_{\varpi}(|\sigma|)| = O(e^{-\kappa(1+O(\epsilon))|\sigma|}). \quad (124)$$

**6 -** By injecting (115), (124) and (99) into (86), we find

$$\left| \zeta_{\mathbf{q}}^{(\Upsilon)}(\xi(\Upsilon)) \right| \leq O(e^{J+\log \epsilon})^{\mathbf{m}^{(\Upsilon)}(\xi(\Upsilon))} e^{-\frac{1}{2} J \mathbf{l}(\xi(\Upsilon)) + C_5(2\mathbf{l}(\xi(\Upsilon)) + \ell_0^2 e^{-3J} \mathbf{b}(\xi(\Upsilon)))}.$$

$$\cdot e^{J(\mathbf{v}_1^{(\Upsilon)}(\xi(\Upsilon)) + \mathbf{v}_2^{(\Upsilon)}(\xi(\Upsilon)))} O(1)^{|\Upsilon|} \left( \prod_{\sigma \in \underline{\sigma}(\xi)} O(e^{-\kappa(1+O(\epsilon))|\sigma|}) \right). \quad (125)$$

We conclude the proof by induction.  $\square$

*5.3. Bound on the polymer entropy.* We now bound the number of possible polymers, weighted by their activity.

**Lemma 5** (Bound on the polymer entropy). *For  $0 < \alpha, \beta < 1$  and  $\gamma \in \mathfrak{P}_q^{(\Upsilon)}(\Lambda)$ , let*

$$a(\gamma) := \alpha \mathcal{E}^{(\Upsilon)}(\gamma), \quad d(\gamma) := \beta \mathcal{E}^{(\Upsilon)}(\gamma) \quad (126)$$

*which are both positive. If*

$$J \gg z \gg 1 \quad (127)$$

*and*

$$\ell_0 = C_3 \kappa^{-1} \quad (128)$$

for some constant  $C_3 > \max\{1, (\alpha C_8)^{-1}\}$  (in which  $C_8$  is the constant appearing in (102)), and

$$\theta := 1 - \alpha - \beta, \quad \frac{1}{2}\mathbf{t} \leq \theta \leq 1, \quad \mathbf{t} = 1 + O(zJ^{-1}) \quad (129)$$

then, for every  $\gamma \in \mathfrak{P}_q^{(\gamma)}(\Lambda)$ ,

$$\sum_{\substack{\gamma' \in \mathfrak{P}_q^{(\gamma)}(\Lambda) \\ \gamma' \not\sim \gamma}} e^{-\theta \mathcal{E}^{(\gamma)}(\gamma')} \leq a(\gamma) \quad (130)$$

in which  $\gamma' \not\sim \gamma$  means that  $\gamma'$  and  $\gamma$  are incompatible, which implies that (94), and, consequently, Lemma 3 hold.

*Proof.* We will first focus on the sum over non-trivial polymers, and then turn to the trivial ones.

**1 -** Let us, for the moment, neglect the sources, and discuss their role later on. We will show that for every edge  $e \in \mathbb{Z}^2$ ,

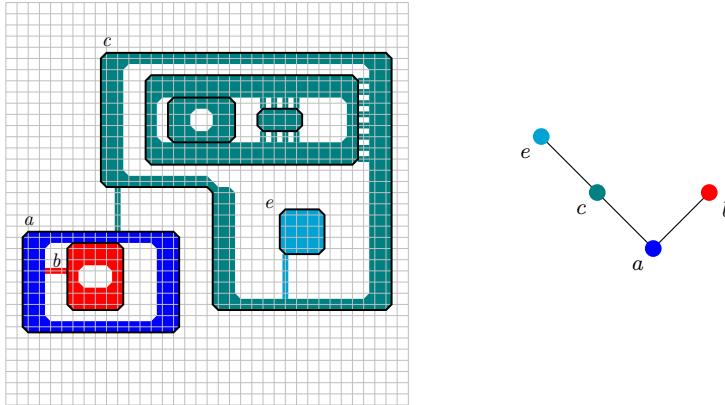
$$\sum_{\substack{\gamma' \in \tilde{\mathfrak{P}}_q^{(\emptyset)}(\Lambda) \\ \partial\xi(\gamma') \ni e}} e^{-\theta \mathcal{E}^{(\emptyset)}(\gamma')} \leq e^{-3\theta\bar{J}} \quad (131)$$

in which  $\partial\xi(\gamma') := \bigcup_{\Gamma \in \underline{\Gamma}(\xi(\gamma'))} \Gamma$ .

**1-1 -** A polymer  $\gamma' \in \tilde{\mathfrak{P}}_q^{(\emptyset)}(\Lambda)$  consists of *loops* and *segments* which are either of length  $< \ell_0$ , in which case they connect two loops in the same contour, or their length is  $\geq \ell_0$ . By Lemma 4, loops come with a gain factor  $e^{-\frac{1}{2}\bar{J}|\mathcal{U}|}$ , and segments which are  $\geq \ell_0$  come with a gain factor  $e^{-\bar{K}|\sigma|}$ . Shorter segments do not have such a gain, and segments of length 1 actually come with a loss factor  $e^{J+\log\bar{\epsilon}}$ . This loss is less dramatic than might seem at first glance: segments of length 1 necessarily connect two loops (since loops are at a distance  $\geq \ell_0$  from the boundary), and, when taking the  $e^{-\frac{1}{2}J}$  factors coming from the endpoints of the segment, one finds that length 1 segments actually contribute  $O(\epsilon)$ , which is small. Nevertheless, this gain factor is much smaller than for longer segments, which is a fact we will have to deal with. The trick is to consider loops that are at distance 1 from each other as a single object, and introduce the notion of a **head**, which is a contour  $\Gamma \in \mathfrak{C}_q^{(\emptyset)}(\Lambda)$  whose segments (if any) are all of length 1:  $\forall \sigma \in \mathcal{S}_{\Lambda, < \ell_0}^{(\emptyset)}(\Gamma)$ ,  $|\sigma| = 1$ . Thus, a polymer  $\gamma' \in \tilde{\mathfrak{P}}_q^{(\emptyset)}(\Lambda)$  consists of *heads* and *segments* (see Fig. 13). For simplicity of exposition, we will consider loops that are not separated by a segment (see Fig. 13) as belonging to the same head.

We then define the set of *backbones*  $\bar{\gamma}'$  of  $\gamma'$  as the set of polymers obtained from  $\gamma'$  by removing segments of length  $> 1$  in such a way that, while the support of  $\bar{\gamma}'$  is still connected, it would not be if we removed any more segments of length  $> 1$ . Among the backbones in  $\bar{\gamma}'$ , we pick one arbitrarily, denote it by  $\bar{\gamma}'$  and call it *the backbone* of  $\gamma'$  (see Fig. 13). The main idea is to bound the entropy of the backbone, after which we bound the entropy of the full polymer.

**1-2 -** The backbone  $\bar{\gamma}'$  consists of  $L \geq 1$  heads and  $n_s$  segments of length  $> 1$ , and has a natural tree structure (see Fig. 13): if we associate a node to each head and a branch to every pair of nodes that corresponds to heads that are connected by a segment, then the resulting graph is a tree (a tree is a graph with no loops), denoted by  $\mathbb{T}(\bar{\gamma}')$ . We



**Fig. 13.** A backbone of the polymer in Fig. 10. The blue ( $a$ ) loop is the root. The lollipops are depicted in different colors and labeled by different letters. Each consists of a stem and a head, which consists of loops which are at distance  $\leq 1$  from each other. The tree structure of the backbone is drawn as well

call the head containing the edge  $e$  the **root** of the tree. Every other head has a unique **parent**, which is defined as the unique neighbor of the head that is closest to the root (using the natural graph distance on the tree). A head together with the segment that connects it to its parent is called a **lollipop**, and the segment is called the **stem** of the lollipop. The backbone is completely determined by the tree  $\mathbb{T}(\bar{\gamma}')$ , the shape of the root head and the lollipops, and the points on the heads to which the lollipops are attached.

- The number of rooted trees with  $L$  nodes is bounded by

$$\# \text{ trees with } L \text{ nodes } \leq 4^{L-1} \quad (132)$$

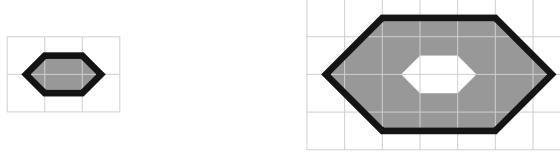
(which can be proved rather easily by mapping the set of trees to 1-dimensional walks with  $2(L-1)$  steps, see [GM01, lemma A.1]).

- The number of possible shapes of a lollipop is estimated as follows. Let us focus on the  $i$ -th lollipop  $\Gamma_i$ . It consists of a stem of length  $\ell_i$ , and a head, which is a union of bounding loops. The head is *connected* to the stem at an edge  $e_i$  (in the sense discussed in Sect. 4.2). By definition, every segment in  $\mathcal{S}_{\Lambda, < \ell_0}^{(\theta)}(\Gamma_i)$  is of length 1, and every such segment is connected to exactly 2 edges of the head. Conversely, every edge in the head may be connected to 0 or 1 length-1 segments. We denote the number of edges in the head that are connected to a length-1 segment by  $n_{z,i}$ , and fix the number of remaining edges to  $n_{l,i}$ . Then, we estimate the number of possible heads with  $n_{z,i}$  and  $n_{l,i}$  fixed. A head can be seen as a connected subgraph of a finite-degree graph: for instance, consider the graph  $\mathfrak{G}$  whose vertices correspond to the edges of  $\mathbb{Z}^2$  and whose edges correspond to every pair of edges of  $\mathbb{Z}^2$  that are at distance  $\leq 2$  from each other. A head is a connected subgraph of this graph and has  $n_{l,i} + n_{z,i}$  vertices. Therefore, the number of possible head shapes is bounded by

$$\# \text{ lollipop head shapes } \leq c_1^{n_{l,i} + n_{z,i}} \quad (133)$$

for some constant  $c_1 > 0$  (which depends only on the degree of the graph  $\mathfrak{G}$ ).

- Once the tree structure and the shapes of the lollipops are fixed, we are left with positioning the lollipops. Given a lollipop  $\Gamma_i$ , the tree structure tells us to which other



**Fig. 14.** Left: the smallest possible head has 6 edges. Right: the smallest h-loop that can contain an h-dimer has 18 edges

lollipop  $\Gamma_j$  its stem is connected. Therefore, it suffices to bound the number ways  $\Gamma_i$  can be connected to  $\Gamma_j$  by  $2^{n_{1,j}}$ . Thus,

$$\# \text{ lollipop positions} \leq \prod_{i=1}^L 2^{n_{1,i}}. \quad (134)$$

We now express the weight  $e^{-\theta \mathcal{E}^{(\emptyset)}(\bar{\gamma}')}$  (see (101)) of the backbone in terms of lollipops. We fix the lengths of the stems  $\ell_2, \dots, \ell_L \geq 2$  (we take the convention that the first head is the root, which does not have a stem), as well as the numbers  $n_{1,1}, \dots, n_{1,L}$  of edges in each head that are not connected to a length-1 segment, and the numbers  $n_{z,1}, \dots, n_{z,L}$  of edges in each head that are connected to a length-1 segment. We have

$$\mathfrak{l}(\gamma') = \sum_{i=1}^L (n_{1,i} + n_{z,i}), \quad \mathfrak{s}(\bar{\gamma}') = \sum_{i=2}^L \ell_i, \quad \mathfrak{m}^{(\emptyset)}(\bar{\gamma}') = \frac{1}{2} \sum_{i=1}^L n_{z,i}. \quad (135)$$

Therefore, by (101)

$$\mathcal{E}^{(\emptyset)}(\bar{\gamma}') = \sum_{i=1}^L \left( \frac{1}{2} \bar{J} n_{1,i} - \frac{1}{2} \log \bar{\epsilon} n_{z,i} \right) + \bar{\kappa} \sum_{i=2}^L \ell_i - \chi(n_1) \quad (136)$$

with

$$\chi(n_1) := C_5 \ell_0^2 e^{-3J} \mathbb{1}_{\sum_i n_{1,i} \geq \ell_0} \sum_{i=1}^L n_{1,i}. \quad (137)$$

Note that, since  $z \ll J$ , this shows that  $\mathcal{E}^{(\emptyset)}(\gamma') \geq 0$ .

In addition, by simple geometric considerations, for every  $i \in \{1, \dots, L\}$ ,

$$n_{1,i} \geq 6 \quad (138)$$

Indeed there are at least 6 edges in every head that can not be connected to a length-1 segment (see Fig. 14). Those edges are the  $q$ -edges with the largest  $q$ -component, the  $q$ -edges with the smallest  $q$ -component, and the  $-q$ -edges with the largest  $q$ -component. There are at least 2 of each, which adds up to 6.

**1-3 -** Given a backbone, we can construct a family of polymers by adding segments to it. The weight of an additional segment of length  $\ell$  is  $\leq 1$ . In addition, the number of ways in which one can add segments is bounded by

$$\# \text{ polymers compatible with backbone} \leq \prod_{i=1}^L 2^{n_{1,i}} \quad (139)$$

(the estimate corresponds to allowing for segments to be added to any point of a loop, which is an over-counting).

**1-3-1 -** Thus, we bound, by (132) through (139),

$$\sum_{\substack{\gamma' \in \tilde{\mathfrak{P}}_q^{(\emptyset)}(\Lambda) \\ \partial \gamma' \ni e}} e^{-\theta \mathcal{E}^{(\emptyset)}(\gamma')} \leq \sum_{L=1}^{\infty} 4^{L-1} \sum_{n_{1,1}, \dots, n_{1,L} \geq 0} \sum_{n_{z,1}, \dots, n_{z,L} \geq 0} \sum_{\ell_2, \dots, \ell_L \geq 2} \cdot \\ \cdot e^{\theta \chi(n_1)} \left( \prod_{i=1}^L \left( c_2 e^{-\theta \frac{1}{2} \bar{J}} \right)^{n_{1,i}} \right) \left( \prod_{i=1}^L \left( c_2 \bar{\epsilon}^{\frac{1}{2} \theta} \right)^{n_{z,i}} \right) \prod_{i=2}^L F(\ell_i) \quad (140)$$

where  $c_2 > 0$  is a constant, and

$$F(\ell) := \begin{cases} e^{-\theta \bar{\kappa} \ell} & \text{if } \ell \geq \ell_0 \\ 1 & \text{otherwise.} \end{cases} \quad (141)$$

**1-3-2 -** Let us now get rid of the  $e^{\theta \chi}$  factor. If  $\sum_i n_{1,i} \geq \ell_0$ , then the first factor can be rewritten as

$$e^{\theta \chi(n_1)} \prod_{i=1}^L \left( e^{-\theta \frac{1}{2} \bar{J}} \right)^{n_{1,i}} = \prod_{i=1}^L \left( e^{-\theta \frac{1}{2} \bar{J} + \theta C_5 \ell_0^2 e^{3J}} \right)^{n_{1,i}} \quad (142)$$

and, since  $\ell_0 = C_3 \kappa^{-1}$  and  $\kappa^{-1} = e^{\frac{3}{2} J} \sqrt{z}$ ,

$$C_5 \ell_0^2 e^{-3J} = C_5 C_3^2 z. \quad (143)$$

Therefore, we can get rid of  $\chi$  by replacing  $\theta$  with  $\theta(1 - 2C_5 C_3^2 z \bar{J}^{-1})$ :

$$e^{\theta \chi(n_1)} \prod_{i=1}^L \left( e^{-\theta \frac{1}{2} \bar{J}} \right)^{n_{1,i}} = \prod_{i=1}^L \left( e^{-\theta(1+O(zJ^{-1})) \frac{1}{2} \bar{J}} \right)^{n_{1,i}}. \quad (144)$$

**1-3-3 -** Therefore, if  $z$  and  $J$  are large enough, since  $\ell_0 = C_3 \kappa^{-1}$ ,  $\sum_{\ell} F(\ell) = O(\theta \bar{\kappa})^{-1}$  and

$$\sum_{\substack{\gamma' \in \tilde{\mathfrak{P}}_q^{(\emptyset)}(\Lambda) \\ \partial \gamma' \ni e}} e^{-\theta \mathcal{E}^{(\emptyset)}(\gamma')} \leq e^{-3\theta \bar{J}} \sum_{L=1}^{\infty} c_3^L \left( e^{-3\bar{\theta} \bar{J}} (\theta \bar{\kappa})^{-1} \right)^{L-1} \quad (145)$$

for some constant  $c_3 > 0$ , and with  $\bar{\theta} = \theta(1 + O(zJ^{-1}))$ . Note that the first factor is  $e^{-3\theta \bar{J}}$  instead of  $e^{-3\theta(1+O(zJ^{-1})) \bar{J}}$ . This follows from the fact that,

- if  $L = 1$ , then the  $(1 + O(zJ^{-1}))$  correction only arises if  $n_{1,1} \geq \ell_0$ , in which case the factor would be  $e^{-\frac{1}{2}\theta(1+O(zJ^{-1})) \bar{J} \ell_0} \gg e^{-3\theta \bar{J}}$
- if  $L > 1$ , then the correction can be absorbed in  $e^{-3\theta \bar{J}(1+O(zJ^{-1}))(L-1)}$ .

In addition, provided  $z \ll e^{(6\bar{\theta}-3)J}$ , which is true if  $\bar{\theta} = \theta(1 + O(zJ^{-1})) > \frac{1}{2}$ , we have  $\bar{\kappa} \ll e^{3\bar{\theta}\bar{J}}$ , from which (131) follows.

**2 -** We now take the sources into account, and show that

$$\sum_{\substack{\gamma' \in \tilde{\mathfrak{P}}_q^{(\Upsilon)}(\Lambda) \\ \partial \xi(\gamma') \ni e}} e^{-\theta \mathcal{E}^{(\Upsilon)}(\gamma')} \leq \begin{cases} e^{-3\theta \bar{J}} & \text{if } \text{dist}_1(e, \Upsilon) > 3 \\ e^{-\theta \bar{J}} & \text{otherwise} \end{cases} \quad (146)$$

in which  $\text{dist}_1$  is the distance induced by the 1-norm:  $|(x, y)|_1 = |x| + |y|$ . For simplicity of exposition, we will only consider the case in which there are two sources. This is enough for the purpose of computing two-point correlations, and the argument can easily be generalized to an arbitrary number of sources.

**2-1 -** We will first deal with  $\mathfrak{v}_1$  (see (106)). The key observation is that, in order for  $|\mathbb{D}_{-q}(\Upsilon_{-q})| > 0$ , the size of one of the heads of  $\Gamma$  must be large enough:

$$\mathfrak{l}(\Gamma) - 2\mathfrak{m}^{(\emptyset)}(\Gamma) \geq 18|\mathbb{D}_{-q}(\Upsilon_{-q})|. \quad (147)$$

Indeed,  $\Upsilon_{-q}$  is empty unless it is contained inside at least one  $-q$ -loop, and the smallest  $-q$ -loop that can contain a  $-q$ -dimer is of length 18 (see Fig. 14). Furthermore, since distinct sources are at distance  $\geq \ell_0$  from each other, if a loop contains two sources, then it is much larger than  $2 \times 18$ . Now, since  $\mathfrak{l}(\Gamma) - 2\mathfrak{m}^{(\emptyset)}(\Gamma) \geq 6$  (which is the size of the smallest possible loop, see the discussion above), we have

$$\frac{1}{2}\mathfrak{l}(\Gamma) - \mathfrak{m}^{(\emptyset)}(\Gamma) - \mathfrak{v}_1^{(\Upsilon)}(\Gamma) \geq \frac{1}{2} \left( \frac{1}{2}\mathfrak{l}(\Gamma) - \mathfrak{m}^{(\emptyset)}(\Gamma) \right) + \frac{3}{2}. \quad (148)$$

We can thus absorb  $\mathfrak{v}_1$  by replacing  $e^{-\theta \frac{1}{2} \bar{J} n_{1,i}}$  in (140) by  $e^{-\theta \frac{1}{4} \bar{J} n_{1,i} - \theta \frac{3}{2} \bar{J}}$ .

**2-2 -** After having thus absorbed  $\mathfrak{v}_1$ , the remaining contribution of sources comes from loops that are in contact with a source (through  $\mathfrak{v}_2$ , see (107)), or at distance 1 from a source (from the  $\Upsilon$ -dependence of  $\mathfrak{m}^{(\Upsilon)}$ , see (105)). When such an event occurs,  $\mathfrak{m}^{(\Upsilon)}$  and  $\mathfrak{v}_2$  give rise to a large factor, which is counter-balanced by the gain in the entropy coming from the constraint that the loop in question is pinned down by the source.

The large factor produced by  $\mathfrak{v}_2$  is  $e^{J|\mathfrak{R}^{(\text{ext})}|}(\nu, \Gamma)$  (see (67)), that is, it is exponentially large in the number of external contact points of each head. Consider a head which is in contact with exactly one source, and does not encircle another source. Denoting its length by  $n_1$  (using the notation introduced above), we note that it can have, at most,  $\min(\frac{1}{2}n_1 - 1, 6)$  external contact points (since the head has to wind around the source in order to have many contact points). A similar argument holds for the factor produced by  $\mathfrak{m}^{(\Upsilon)}$ , which implies that the overall contribution of a head neighboring a source is bounded by  $e^{\frac{1}{2}J \min(n_1 - 2, 12)}$ . Since the head does not encircle another source, there is no need to absorb the contribution of  $\mathfrak{v}_1$  as we did above, and  $\frac{1}{2}\mathfrak{l}(\Gamma) - \mathfrak{m}^{(\emptyset)}(\Gamma)$  will contribute  $\frac{1}{2}n_1$ , instead of  $\frac{1}{4}n_1 + \frac{3}{2}$  as per (148). Thus, the overall contribution of this loop to (140) is

$$e^{\theta \bar{J} \min(-1, -\frac{1}{2}n_1 + 6)}. \quad (149)$$

Now, consider a head that either is in contact with both sources, or touches one and encircles the other. Such a head must, therefore, be quite large: since sources are

separated by at least  $\ell_0$ ,  $n_l \geq 2\ell_0$ . This time, we must absorb  $\mathfrak{v}_1$  as explained above, and, by (148), find that the head will contribute

$$e^{\theta \bar{J}(-\frac{1}{4}n_l + \frac{9}{2})} \mathbb{1}_{n_l \geq 2\ell_0}. \quad (150)$$

When a head that is not the root (recall that, when counting the number of possible backbones, we identified the head containing the edge  $e$  as the *root*) is in contact with a source, then there is no need to sum over the length of its stem. Equivalently, since the sum over the length of a lollipop stem produces a factor proportional to  $\kappa^{-1}$ , we can sum over the length of the stem, and correct the weight of the loop by a factor proportional to  $\kappa$ .

Finally, we turn to the root head. The number of points at which it comes within a distance 1 of a source  $v$  is bounded by  $\frac{1}{2}n_l + 1 - \text{dist}_1(e, v)$ . Therefore, provided it only comes in contact with a single source, and does not encircle another, it contributes

$$e^{\theta \bar{J} \min(1 - \text{dist}_1(e, v), -1, -\frac{1}{2}n_l + 6)}. \quad (151)$$

Note that, by a very similar argument, one checks that  $\mathcal{E}^{(\gamma)}(\gamma') \geq 0$  even in the presence of sources.

**2-3 -** All in all, in the presence of sources, (140) becomes

$$\begin{aligned} \sum_{\substack{\gamma' \in \tilde{\mathfrak{P}}_q^{(\gamma)}(\Lambda) \\ \delta \gamma' \ni e}} e^{-\theta \mathcal{E}^{(\gamma)}(\gamma')} &\leq c_4 \sum_{L=1}^{\infty} 4^{L-1} \sum_{n_{l,1}, \dots, n_{l,L} \geq 6} \sum_{n_{z,1}, \dots, n_{z,L} \geq 0} \sum_{\ell_2, \dots, \ell_L \geq 2} \cdot \\ &\cdot e^{\theta \chi(n_l)} \left( \prod_{i=1}^L \left( c_2 \bar{\epsilon}^{\frac{1}{2}\theta} \right)^{n_{z,i}} \right) \left( \prod_{i=2}^L F(\ell_i) \right) c_2^{n_{l,1}}. \quad (152) \\ &\cdot \left( e^{-\theta \frac{1}{4} \bar{J} n_{l,1} - \theta \frac{3}{2} \bar{J}} \left( 1 + \mathbb{1}_{n_{l,1} \geq 2\ell_0} e^{\theta 6\bar{J}} \right) + e^{-\theta \frac{1}{2} \bar{J} \max(2\text{dist}_1(e, \gamma) - 2, 2, n_{l,1} - 12)} \right) \cdot \\ &\cdot \left( \prod_{i=2}^L \left( c_2^{n_{l,i}} \left( e^{-\theta \frac{1}{4} \bar{J} n_{l,i} - \theta \frac{3}{2} \bar{J}} \left( 1 + \kappa \mathbb{1}_{n_{l,i} \geq 2\ell_0} e^{\theta 6\bar{J}} \right) + \kappa e^{-\theta \frac{1}{2} \bar{J} \max(2, n_{l,i} - 12)} \right) \right) \right) \end{aligned}$$

for some constant  $c_4 > 0$ . The rest of the computation is identical to the case without sources, and yields (146).

**3 -** We can now estimate the sum over non-trivial  $\gamma'$  that intersect a given  $\gamma$  by summing over the position of  $e$  in such a way that  $\gamma'$  is incompatible with  $\gamma$ . Such an incompatibility arises only if a loop of  $\gamma'$  is at a distance  $< \ell_0$  from a loop of  $\gamma$ , or if a segment or loop of  $\gamma'$  intersects a segment or loop of  $\gamma$ . This yields a factor  $O(\ell_0 \mathfrak{l}(\gamma) + \mathfrak{s}(\gamma))$ . However, if  $\gamma'$  is at a 1-distance that is  $\leq 3$  from a source, then the sum over its position yields a constant rather than  $O(\ell_0 \mathfrak{l}(\gamma) + \mathfrak{s}(\gamma))$ . Therefore,

$$\sum_{\substack{\gamma' \in \tilde{\mathfrak{P}}_q^{(\gamma)}(\Lambda) \\ \gamma' \not\sim \gamma}} e^{-\theta \mathcal{E}^{(\gamma)}(\gamma')} \leq \kappa^{-1} e^{-3\theta \bar{J}} O(\mathfrak{l}(\gamma) + \kappa \mathfrak{s}(\gamma)) + O(e^{-\theta \bar{J}}) \ll a(\gamma). \quad (153)$$

**4 -** Let us now turn to the contribution of trivial polymers  $\gamma' \in \mathfrak{P}_q^{(\gamma)}(\Lambda) \setminus \tilde{\mathfrak{P}}_q^{(\gamma)}(\Lambda)$ . The activity of such polymers is bounded by

$$e^{-\theta \mathcal{E}^{(\gamma)}(\gamma')} \leq e^{-\theta \bar{\kappa} \ell_0} = e^{-\theta C_3 C_8} \quad (154)$$

where  $C_8$  was introduced in (102), and  $C_3$  in (128).

**4-1 -** If  $\gamma$  is non-trivial, then  $\gamma' \not\sim \gamma$  only if  $\gamma'$  intersects a loop of  $\gamma$ . Indeed,  $\gamma'$  is a  $q$ -segment, and, in order for it to intersect a  $q$ -segment of  $\gamma$ , it will have to intersect the loops at its endpoints, and  $-q$ -segments of  $\gamma$  must lie *inside* a loop of  $\gamma$ . Therefore,

$$\sum_{\substack{\gamma' \in \mathfrak{P}_q^{(\gamma)}(\Lambda) \setminus \tilde{\mathfrak{P}}_q^{(\gamma)}(\Lambda) \\ \gamma' \not\sim \gamma}} e^{-\theta \mathcal{E}^{(\gamma)}(\gamma')} = O(I(\gamma)) \ll a(\gamma) \quad (155)$$

**4-2 -** If  $\gamma$  is trivial, then there is only one position for  $\gamma'$  that will intersect  $\gamma$ , and

$$\sum_{\substack{\gamma' \in \mathfrak{P}_q^{(\gamma)}(\Lambda) \setminus \tilde{\mathfrak{P}}_q^{(\gamma)}(\Lambda) \\ \gamma' \not\sim \gamma}} e^{-\theta \mathcal{E}^{(\gamma)}(\gamma')} \leq e^{-\theta C_3 C_8} \quad (156)$$

whereas

$$a(\gamma) \geq \alpha \bar{\kappa} \ell_0 = \alpha C_3 C_8 > 1. \quad (157)$$

□

**5.4. Polymer-source interaction.** In this section we introduce the notion of a polymer *interacting* with a source, which will be useful in the following to compute observables from the cluster expansion.

**Definition 1** (Polymer-source interaction). First of all, we generalize the definition of the polymer activity  $\zeta_q^{(\gamma)}(\gamma)$  in (90), which, so far, has only been defined for polymers  $\gamma \in \mathfrak{P}_q^{(\gamma)}(\Lambda)$ . We extend this definition to polymers with a different family of sources  $\gamma'$ : if  $\gamma \in \mathfrak{P}_q^{(\gamma)}(\Lambda) \setminus \tilde{\mathfrak{P}}_q^{(\gamma)}(\Lambda)$ , then we set  $\zeta_q^{(\gamma)}(\gamma) \equiv 0$ .

Given a polymer  $\gamma \in \mathfrak{P}_q^{(\gamma)}(\Lambda)$  and a source  $E \in \mathcal{E}(\Lambda)$ , we say that  $\gamma$  **interacts** with  $E$  if  $\zeta_q^{(E)}(\gamma) \neq \zeta_q^{(\emptyset)}(\gamma)$ . In this case, we write  $\gamma \& E$ . Note that, in order for  $\gamma$  to interact with  $E$ , it must either come within a distance  $\ell_0$  of it or encircle it.

**Lemma 6** (Entropy of a polymer interacting with a source). *There exists a constant  $C_{10} > 0$  such that, for any  $E \in \mathcal{E}(\Lambda)$  and a family of sources  $\gamma'$ ,*

$$\sum_{\substack{\gamma \in \mathfrak{P}_q^{(\gamma')}(\Lambda) \\ \gamma \& E}} |\zeta_q^{(\gamma)}(\gamma)| e^{\alpha \mathcal{E}^{(\gamma)}(\gamma)} \leq C_{10} e^{-(1-\alpha) \bar{J}} + 4 \max_{\substack{\sigma \in \Sigma_q(\Lambda^{(\gamma')}) \\ \mathfrak{d}_q(\sigma, E) \leq 1}} e^{-(1-\alpha) \bar{\kappa} |\sigma|} \quad (158)$$

for  $\alpha < \frac{1}{2} \mathfrak{t}$  (see (129)).

*Proof.* As noted in definition 1,  $\gamma$  may only interact with  $E$  if it surrounds it, or comes within a distance  $\ell_0$  of it. If  $\gamma$  is trivial, then its activity is bounded by

$$\max_{\substack{\sigma \in \Sigma_q(\Lambda^{(\gamma')}) \\ \mathfrak{d}_q(\sigma, E) \leq 1}} e^{-(1-\alpha) \bar{\kappa} |\sigma|} \quad (159)$$

and there are fewer than 4 trivial polymers that interact with  $E$ . If  $\gamma$  is non-trivial, let us fix one edge  $e \in \partial\xi(\gamma)$  of one of its external loops. Furthermore, if  $\text{dist}_1(e, \Upsilon) > 3$ , then, since  $\gamma \& E$ ,  $\mathbb{I}(\gamma) + \mathfrak{s}(\gamma) \geq \text{dist}_1(e, E) - \ell_0$ , which implies that,

$$\mathcal{E}^{(\Upsilon)}(\gamma) \geq \bar{\kappa}(\text{dist}_1(e, E) - \ell_0). \quad (160)$$

Therefore, for  $\beta \in (0, \frac{1}{2}\mathfrak{t} - \alpha)$  (see (129)),

$$\begin{aligned} \sum_{\substack{\gamma \in \mathfrak{P}_q^{(\Upsilon')}(\Lambda) \\ \gamma \& E}} |\zeta_{\mathbf{q}}^{(\Upsilon)}(\gamma)| e^{a(\gamma)} &\leq \sum_{\substack{e \in \mathcal{E}(\Lambda) \\ \text{dist}_1(e, \Upsilon) \leq 3}} \sum_{\substack{\gamma \in \tilde{\mathfrak{P}}_q^{(\Upsilon)}(\Lambda) \\ \partial\xi(\gamma) \ni e}} e^{-(1-\alpha)\mathcal{E}^{(\Upsilon)}(\gamma)} \\ &+ \sum_{\substack{e \in \mathcal{E}(\Lambda) \\ \text{dist}_1(e, \Upsilon) > 3}} e^{-\beta\bar{\kappa}(\text{dist}_1(e, E) - \ell_0)} \sum_{\substack{\gamma \in \tilde{\mathfrak{P}}_q^{(\Upsilon)}(\Lambda) \\ \partial\xi(\gamma) \ni e}} e^{-(1-\alpha-\beta)\mathcal{E}^{(\Upsilon)}(\gamma)} \quad (161) \\ &+ 4 \max_{\substack{\sigma \in \Sigma_q(\Lambda^{(\Upsilon')}) \\ \mathfrak{d}_q(\sigma, E) \leq 1}} e^{-(1-\alpha)\bar{\kappa}|\sigma|}. \end{aligned}$$

Thus, by (146),

$$\begin{aligned} \sum_{\substack{\gamma \in \mathfrak{P}_q^{(\Upsilon')}(\Lambda) \\ \gamma \& E}} |\zeta_{\mathbf{q}}^{(\Upsilon)}(\gamma)| e^{a(\gamma)} &\leq e^{-(1-\alpha-\beta)3\bar{J}} \sum_{\substack{e \in \mathcal{E}(\Lambda) \\ \text{dist}_1(e, \Upsilon) > 3}} e^{-\beta\bar{\kappa}(\text{dist}_1(e, E) - \ell_0)} \\ &+ e^{-(1-\alpha)\bar{J}} \sum_{\substack{e \in \mathcal{E}(\Lambda) \\ \text{dist}_1(e, \Upsilon) \leq 3}} 1 + 4 \max_{\substack{\sigma \in \Sigma_q(\Lambda^{(\Upsilon')}) \\ \mathfrak{d}_q(\sigma, E) \leq 1}} e^{-(1-\alpha)\bar{\kappa}|\sigma|}. \quad (162) \end{aligned}$$

Finally, using (128),

$$\sum_{\substack{e \in \mathcal{E}(\Lambda) \\ \text{dist}_1(e, \Upsilon) > 3}} e^{-\beta\bar{\kappa}(\text{dist}_1(e, E) - \ell_0)} = O(\beta\kappa)^{-1} \quad (163)$$

from which (158) follows, using  $\kappa < e^{2J}$  and taking  $3\beta \leq 1 - 2\alpha$ .  $\square$

**5.5. Flipping term.** We will now conclude the proof of the convergence of the cluster expansion, by proving (99).

**Lemma 7** (Bound on the flipping term). *There exists a constant  $C_5 > 0$  such that, for every boundary condition  $\mathbf{c} \equiv (c, \varsigma, \ell_0)$ ,*

$$\left| K_{\mathbf{q}, \mathbf{c}}^{(\Upsilon)}(\Lambda) \right| \leq e^{C_5(|\partial_c \Lambda| + \ell_0 e^{-3J} |\partial_q \Lambda|)}. \quad (164)$$

*Proof.* The main idea of the proof is to compute  $Z^{(\Upsilon)}(\Lambda' | c) / \tilde{\mathfrak{Z}}_{\mathbf{c}}^{(\Upsilon)}(\Lambda')$  and  $\tilde{Z}^{(\Upsilon)}(\Lambda' | q) / \tilde{\mathfrak{Z}}_{\mathbf{q}}^{(\Upsilon)}(\Lambda')$  using the cluster expansion presented in Lemma 3 whose convergence is ensured by Lemmas 4 and 5. We then isolate the *bulk* terms, which cancel out, and

the *boundary* terms, which yield (164). As we will see, it suffices to consider only the first term in (95) and bound the remainder according to (97).

**1 - Sources.** The first step is to eliminate the sources. We will focus on  $Z^{(\Upsilon)}(\Lambda'|c)/\tilde{\mathfrak{Z}}_c^{(\Upsilon)}(\Lambda')$ , the argument for the other ratio is very similar. Let, for  $t \in [0, 1]$ ,

$$\tilde{\xi}_c^{(\Upsilon, \emptyset)}(\gamma|t) := t \xi_c^{(\Upsilon)}(\gamma) + (1-t) \xi_c^{(\emptyset)}(\gamma) \quad (165)$$

and

$$\bar{\mathfrak{P}}_c^{(\Upsilon, \emptyset)}(\Lambda) := \mathfrak{P}_c^{(\Upsilon)}(\Lambda) \cup \mathfrak{P}_c^{(\emptyset)}(\Lambda) \quad (166)$$

in terms of which

$$\begin{aligned} \log \left( \frac{Z^{(\Upsilon)}(\Lambda|c)}{\tilde{\mathfrak{Z}}_c^{(\Upsilon)}(\Lambda)} \right) - \log \left( \frac{Z^{(\emptyset)}(\Lambda|c)}{\tilde{\mathfrak{Z}}_c^{(\emptyset)}(\Lambda)} \right) &= \int_0^1 dt \sum_{m=1}^{\infty} \sum_{\gamma \in \bar{\mathfrak{P}}_c^{(\Upsilon, \emptyset)}(\Lambda)} \sum_{\underline{\gamma} \sqsubset \bar{\mathfrak{P}}_c^{(\Upsilon, \emptyset)}(\Lambda) \setminus \{\gamma\}} \\ \Phi^T(\{\gamma\}^m \sqcup \underline{\gamma}) \left( \xi_c^{(\Upsilon)}(\gamma) - \xi_c^{(\emptyset)}(\gamma) \right) m(\tilde{\xi}_c^{(\Upsilon, \emptyset)}(\gamma|t))^{m-1} \prod_{\gamma' \in \underline{\gamma}} \tilde{\xi}_c^{(\Upsilon, \emptyset)}(\gamma'|t) \end{aligned} \quad (167)$$

where  $\{\gamma\}^m$  is the multiset with  $m$  elements, all of which are  $\gamma$ . Furthermore, by (100),

$$m|\tilde{\xi}_c^{(\Upsilon, \emptyset)}(\gamma|t)|^{m-1} \leq 1 \quad (168)$$

so, by (97),

$$\left| \log \left( \frac{Z^{(\Upsilon)}(\Lambda|c)}{\tilde{\mathfrak{Z}}_c^{(\Upsilon)}(\Lambda)} \right) - \log \left( \frac{Z^{(\emptyset)}(\Lambda|c)}{\tilde{\mathfrak{Z}}_c^{(\emptyset)}(\Lambda)} \right) \right| \leq \sum_{\gamma \in \bar{\mathfrak{P}}_c^{(\emptyset)}(\Lambda)} \left| \xi_c^{(\Upsilon)}(\gamma) - \xi_c^{(\emptyset)}(\gamma) \right| e^{a(\gamma)}. \quad (169)$$

Furthermore,  $\xi_c^{(\Upsilon)}(\gamma) - \xi_c^{(\emptyset)}(\gamma)$  differs from 0 only if  $\gamma$  interacts with at least one source in  $\Upsilon$  (see Definition 1). Therefore, by Lemma 6,

$$\left| \log \left( \frac{Z^{(\Upsilon)}(\Lambda|c)}{\tilde{\mathfrak{Z}}_c^{(\Upsilon)}(\Lambda)} \right) - \log \left( \frac{Z^{(\emptyset)}(\Lambda|c)}{\tilde{\mathfrak{Z}}_c^{(\emptyset)}(\Lambda)} \right) \right| = O(1). \quad (170)$$

The same bound holds for  $\log(\tilde{Z}^{(\Upsilon)}(\Lambda'|q)/\tilde{\mathfrak{Z}}_q^{(\Upsilon)}(\Lambda'))$ . We are thus left with estimating  $K_{q,c}^{(\emptyset)}(\Lambda)$ .

**2 - Bulk terms.** Some of the terms in the cluster expansion (95) are independent of the boundary, and cannot contribute to  $K_{q,c}^{(\emptyset)}(\Lambda)$  since it only involves boundary terms (since the two ratios in (79) only differ through their boundary conditions). Let us now make this idea more precise.

**2-1 -** Among the polymers, some are connected to the boundary, which we call *boundary polymers*, while the others are not, and are called *bulk polymers*. Boundary polymers depend on the boundary, since they are connected to it, so we partition

$$\mathfrak{P}_c^{(\emptyset)}(\Lambda) = \mathfrak{P}_c^{(\partial)}(\Lambda) \cup \mathfrak{P}_c^{(\circ)}(\Lambda), \quad \tilde{\mathfrak{P}}_q^{(\emptyset)}(\Lambda) = \tilde{\mathfrak{P}}_q^{(\partial)}(\Lambda) \cup \mathfrak{P}_q^{(\circ)}(\Lambda) \quad (171)$$

in which  $\mathfrak{P}_c^{(\partial)}(\Lambda)$  and  $\tilde{\mathfrak{P}}_q^{(\partial)}(\Lambda)$  are sets of boundary polymers, and rewrite (95) as

$$\log \left( \frac{Z^{(\emptyset)}(\Lambda|c)}{\tilde{\mathfrak{Z}}_c^{(\emptyset)}(\Lambda)} \right) = \mathfrak{B}_c(\mathfrak{P}_c^{(\circ)}(\Lambda)) + \bar{\mathfrak{B}}_c(\mathfrak{P}_c^{(\partial)}(\Lambda), \mathfrak{P}_c^{(\circ)}(\Lambda)) \quad (172)$$

and

$$\log \left( \frac{\tilde{Z}^{(\emptyset)}(\Lambda | \mathbf{q})}{\mathfrak{Z}_{\mathbf{q}}^{(\emptyset)}(\Lambda)} \right) = \mathfrak{B}_{\mathbf{q}}(\mathfrak{P}_q^{(\circ)}(\Lambda)) + \overline{\mathfrak{B}}_{\mathbf{q}}(\tilde{\mathfrak{P}}_q^{(\emptyset)}(\Lambda), \mathfrak{P}_q^{(\circ)}(\Lambda)) \quad (173)$$

where  $\mathfrak{B}$  is the contribution of clusters involving only bulk polymers, and  $\overline{\mathfrak{B}}$  is the contribution of clusters that contain at least one boundary polymer:

$$\mathfrak{B}_{\mathbf{q}}(\mathfrak{P}) := \sum_{\underline{\gamma} \sqsubset \mathfrak{P}} \Phi^T(\underline{\gamma}) \prod_{\gamma' \in \underline{\gamma}} \zeta_{\mathbf{q}}^{(\emptyset)}(\gamma'). \quad (174)$$

and

$$\overline{\mathfrak{B}}_{\mathbf{q}}(\mathfrak{P}, \mathfrak{Q}) := \sum_{m=1}^{\infty} \sum_{\gamma \in \mathfrak{P}} \sum_{\underline{\gamma} \sqsubset (\mathfrak{P} \cup \mathfrak{Q}) \setminus \{\gamma\}} \Phi^T(\{\gamma\}^m \sqcup \underline{\gamma}) (\zeta_{\mathbf{q}}^{(\emptyset)}(\gamma))^m \prod_{\gamma' \in \underline{\gamma}} \zeta_{\mathbf{q}}^{(\emptyset)}(\gamma'). \quad (175)$$

*Remark.* When defining  $\overline{\mathfrak{B}}$ , we separate out one of the polymers,  $\gamma$ , and ask that it be a boundary polymer. When doing so, we must sum over the multiplicity of  $\gamma$  *separately* (the sum over  $m$  in (175)). If we did not do so, we would be overcounting polymer configurations (this can easily be seen on a simple example where the set of polymers consists of only two objects). We do this because we are writing identities, but, in the following, we will want bounds, for which we do not need to split the sum over the multiplicity from the sum over  $\underline{\gamma}$ .

**2-2** - Bulk polymers still depend on the boundary, because it restricts the polymers to be inside  $\Lambda$ . To remove this dependence, we introduce the set of *infinite-volume* polymers:  $\mathfrak{P}_q^{(\circ)}(\infty)$  is the set of finite polymers, defined as in Sect. 4.5, except that, instead of requiring that the polymers be inside  $\Lambda$ , they are merely required to be finite. In addition, given a vertex  $\mathbf{v} \in \mathbb{Z}^2$ , we define  $\mathfrak{P}_q^{(\circ)}(\infty | \mathbf{v})$  as the set of polymers whose upper-leftmost vertex is  $\mathbf{v}$ . We then rewrite (95) as the sum over clusters of infinite-volume polymers minus the sum over clusters of infinite-volume polymers which are not contained within  $\Lambda$ :

$$\mathfrak{B}_{\mathbf{q}}(\mathfrak{P}_q^{(\circ)}(\Lambda)) = \mathfrak{F}_q(|\Lambda|) - \mathfrak{G}_q(\Lambda), \quad \mathfrak{B}_{\mathbf{c}}(\mathfrak{P}_c^{(\circ)}(\Lambda)) = \mathfrak{F}_c(|\Lambda|) - \mathfrak{G}_c(\Lambda) \quad (176)$$

with

$$\mathfrak{F}_q(|\Lambda|) := \sum_{\mathbf{v} \in \Lambda} \sum_{m=1}^{\infty} \sum_{\gamma \in \mathfrak{P}_q^{(\circ)}(\infty | \mathbf{v})} (\zeta_{\mathbf{q}}^{(\emptyset)}(\gamma))^m \sum_{\underline{\gamma} \sqsubset \mathfrak{P}_q^{(\circ)}(\infty) \setminus \{\gamma\}} \Phi^T(\{\gamma\}^m \sqcup \underline{\gamma}) \prod_{\gamma' \in \underline{\gamma}} \zeta_{\mathbf{q}}^{(\emptyset)}(\gamma') \quad (177)$$

which, by translation invariance, only depends on  $\Lambda$  through  $|\Lambda|$ , and

$$\mathfrak{G}_q(\Lambda) := \sum_{\mathbf{v} \in \Lambda} \sum_{m=1}^{\infty} \sum_{\gamma \in \mathfrak{P}_q^{(\circ)}(\infty | \mathbf{v})} (\zeta_{\mathbf{q}}^{(\emptyset)}(\gamma))^m \sum_{\substack{\underline{\gamma} \sqsubset \mathfrak{P}_q^{(\circ)}(\infty) \setminus \{\gamma\} \\ \{\gamma\}^m \sqcup \underline{\gamma} \not\sqsubset \mathfrak{P}_q^{(\circ)}(\Lambda)}} \Phi^T(\{\gamma\}^m \sqcup \underline{\gamma}) \prod_{\gamma' \in \underline{\gamma}} \zeta_{\mathbf{q}}^{(\emptyset)}(\gamma'). \quad (178)$$

Thus,

$$\begin{aligned} \log K_{\mathbf{q}, \mathbf{c}}^{(\emptyset)}(\Lambda) &= \left( \overline{\mathfrak{B}}_{\mathbf{c}}(\mathfrak{P}_c^{(\emptyset)}(\Lambda), \mathfrak{P}_c^{(\circ)}(\Lambda)) - \overline{\mathfrak{B}}_{\mathbf{q}}(\widetilde{\mathfrak{P}}_q^{(\emptyset)}(\Lambda), \mathfrak{P}_q^{(\circ)}(\Lambda)) \right) \\ &\quad + (\mathfrak{F}_c(|\Lambda|) - \mathfrak{F}_q(|\Lambda|)) + (\mathfrak{G}_c(\Lambda) - \mathfrak{G}_q(\Lambda)). \end{aligned} \quad (179)$$

**2-3 -** The cancellation of the bulk terms follows from the observation that

$$\mathfrak{F}_q(|\Lambda|) = \mathfrak{F}_c(|\Lambda|) \quad (180)$$

which is obvious if  $c = q$ , and follows from the invariance of the system under  $\frac{\pi}{2}$  rotations if  $c \neq q$ .

**3 - Boundary terms.** We now bound  $\overline{\mathfrak{B}}(\mathfrak{P}^{(\emptyset)}, \mathfrak{P}^{(\circ)})$  and  $\mathfrak{G}$ .

**3-1 -** We isolate the dominant term in (175):

$$\begin{aligned} \overline{\mathfrak{B}}_{\mathbf{q}}(\mathfrak{P}, \mathfrak{Q}) &= \sum_{\gamma \in \mathfrak{P}} \zeta_{\mathbf{q}}^{(\emptyset)}(\gamma) \\ &\quad + \sum_{m=1}^{\infty} \sum_{\gamma \in \mathfrak{P}} (\zeta_{\mathbf{q}}^{(\emptyset)}(\gamma))^m \sum_{\underline{\gamma} \sqsubset (\mathfrak{P} \cup \mathfrak{Q}) \setminus \{\gamma\}} \Phi^T(\{\gamma\}^m \sqcup \underline{\gamma}) \prod_{\gamma' \in \underline{\gamma}} \left( e^{-d(\gamma') + d(\gamma')} \zeta_{\mathbf{q}}^{(\emptyset)}(\gamma') \right) \end{aligned} \quad (181)$$

in which we set  $d(\gamma) := \beta \mathcal{E}^{(\gamma)}(\gamma)$  as in lemma 5.

**3-1-1 -** Let us consider  $\overline{\mathfrak{B}}_{\mathbf{q}}(\widetilde{\mathfrak{P}}_q^{(\emptyset)}(\Lambda), \mathfrak{P}_q^{(\circ)}(\Lambda))$ . By lemma 5 and, more precisely, (131), for any  $\theta \in (\frac{1}{2}t, 1]$ ,

$$\sum_{\gamma \in \widetilde{\mathfrak{P}}_q^{(\emptyset)}(\Lambda)} e^{-\theta \mathcal{E}^{(\gamma)}(\gamma)} \leq \sum_{e \in \Lambda} \sum_{\substack{\gamma \in \widetilde{\mathfrak{P}}_q^{(\emptyset)}(\Lambda) \\ \xi(\gamma) \ni e}} F(\mathfrak{d}_q(e, \partial_q \Lambda)) e^{-\theta \mathcal{E}^{(\gamma)}(\gamma)} \leq c_5 \kappa^{-1} e^{-3\theta \bar{J}} |\partial_q \Lambda| \quad (182)$$

for some constant  $c_5 > 0$ , where  $F$  was defined in (141). In addition using the fact that, by (101), for  $\gamma \in \widetilde{\mathfrak{P}}_q^{(\emptyset)}(\Lambda)$ ,

$$d(\gamma) \equiv \beta \mathcal{E}^{(\gamma)}(\gamma) \geq 3\beta \bar{J} \quad (183)$$

so that, by (97),

$$\sum_{\gamma \in \widetilde{\mathfrak{P}}_q^{(\emptyset)}(\Lambda)} \left| \Phi^T(\{\gamma\} \sqcup \underline{\gamma}) \prod_{\gamma' \in \underline{\gamma}} \left( e^{-d(\gamma') + d(\gamma')} \zeta_{\mathbf{q}}^{(\emptyset)}(\gamma') \right) \right| \leq e^{a(\gamma)} e^{-3\beta \bar{J}} \quad (184)$$

in which we recall that  $a(\gamma) = \alpha \mathcal{E}^{(\gamma)}(\gamma)$  (since we are interested in an upper bound, we can reabsorb the sum over  $m$  into the sum over  $\underline{\gamma}$ ), and, by (182),

$$\sum_{\gamma \in \widetilde{\mathfrak{P}}_q^{(\emptyset)}(\Lambda)} e^{a(\gamma)} |\zeta_{\mathbf{q}}^{(\emptyset)}(\gamma)| \leq c_5 \kappa^{-1} e^{-3(1-\alpha) \bar{J}} |\partial_q \Lambda|. \quad (185)$$

Thus

$$|\overline{\mathfrak{B}}_q(\widetilde{\mathfrak{P}}_q^{(\emptyset)}(\Lambda), \mathfrak{P}_q^{(\circ)}(\Lambda))| \leq c_5 \kappa^{-1} e^{-3\bar{J}} |\partial_q \Lambda| \left( 1 + e^{-3(\beta-\alpha) \bar{J}} \right). \quad (186)$$

**3-1-2 -** We now turn to  $\overline{\mathfrak{B}}_c(\mathfrak{P}_c^{(\partial)}(\Lambda), \mathfrak{P}_c^{(o)}(\Lambda))$ . By lemma 5,

$$\begin{aligned} \sum_{\gamma \in \mathfrak{P}_c^{(\partial)}(\Lambda)} e^{-\theta \mathcal{E}^{(\gamma)}(\gamma)} &= \sum_{\gamma \in \widetilde{\mathfrak{P}}_c^{(\partial)}(\Lambda)} e^{-\theta \mathcal{E}^{(\gamma)}(\gamma)} + \sum_{\gamma \in \mathfrak{P}_c^{(\partial)}(\Lambda) \setminus \widetilde{\mathfrak{P}}_c^{(\partial)}(\Lambda)} e^{-\theta \mathcal{E}^{(\gamma)}(\gamma)} \\ &\leq c_5 \kappa^{-1} e^{-3\theta \bar{J}} |\partial_c \Lambda| + e^{-\theta \bar{\kappa} \ell_0} |\partial_c \Lambda| \end{aligned} \quad (187)$$

so, by a similar reasoning as in the previous paragraph,

$$\begin{aligned} |\overline{\mathfrak{B}}_c(\mathfrak{P}_c^{(\partial)}(\Lambda), \mathfrak{P}_c^{(o)}(\Lambda))| &\leq |\partial_c \Lambda| \left( \left( e^{-\bar{\kappa} \ell_0} + c_5 \kappa^{-1} e^{-3\bar{J}} \right) + e^{-\beta \bar{\kappa} \ell_0} \left( e^{-\bar{\kappa} \ell_0(1-\alpha)} + c_5 \kappa^{-1} e^{-3\bar{J}(1-\alpha)} \right) \right). \end{aligned} \quad (188)$$

**3-2 -** We now turn to  $\mathfrak{G}$ . We fix a vertex  $\mathbf{v} \in \Lambda$ . If  $\mathfrak{d}_q(\mathbf{v}, \partial_q \Lambda) < \ell_0$ , then, proceeding in the same way as for  $\mathfrak{B}$ , we bound

$$\sum_{\gamma \in \mathfrak{P}_q^{(o)}(\infty | \mathbf{v})} |\zeta_{\mathbf{q}}^{(\emptyset)}(\gamma)| \sum_{\underline{\gamma} \sqsubset \mathfrak{P}_q^{(o)}(\infty)} \Phi^T(\{\gamma\} \sqcup \underline{\gamma}) \prod_{\gamma' \in \underline{\gamma}} |\zeta_{\mathbf{q}}^{(\emptyset)}(\gamma')| \leq e^{-3\bar{J}} \left( 1 + e^{-3(1-\alpha)\bar{J}} \right). \quad (189)$$

If  $\mathfrak{d}_q(\mathbf{v}, \partial_q \Lambda) \geq \ell_0$ , then the condition  $\gamma \sqcup \underline{\gamma} \not\sqsubset (\mathfrak{P}_q^{(o)}(\Lambda))$  implies that

$$\sum_{\gamma' \in \gamma \cup \underline{\gamma}} (\mathfrak{l}(\gamma') + \mathfrak{s}(\gamma')) \geq \mathfrak{d}_q(\mathbf{v}, \partial_q \Lambda) - \ell_0. \quad (190)$$

In addition,  $\mathfrak{l}(\gamma) \geq 6$ . Therefore

$$\sum_{\gamma' \in \underline{\gamma}} \mathcal{E}^{(\gamma')}(\gamma) \geq 3\bar{J} |\underline{\gamma}| + \bar{\kappa}(\mathfrak{d}_q(\mathbf{v}, \partial_q \Lambda) - \ell_0 - 4). \quad (191)$$

Therefore, proceeding as for  $\mathfrak{B}$ ,

$$\begin{aligned} \sum_{\gamma \in \mathfrak{P}_q^{(o)}(\infty | \mathbf{v})} |\zeta_{\mathbf{q}}^{(\emptyset)}(\gamma)| \sum_{\substack{\underline{\gamma} \sqsubset \mathfrak{P}_q^{(o)}(\infty) \\ \{\gamma\} \sqcup \underline{\gamma} \not\sqsubset \mathfrak{P}_q^{(o)}(\Lambda)}} \Phi^T(\{\gamma\} \sqcup \underline{\gamma}) \prod_{\gamma' \in \underline{\gamma}} |\zeta_{\mathbf{q}}^{(\emptyset)}(\gamma')| \\ \leq e^{-\beta \bar{\kappa}(\mathfrak{d}_q(\mathbf{v}, \partial_q \Lambda) - \ell_0)} e^{-3\bar{J}} \left( 1 + e^{-3(1-\alpha)\bar{J}} \right). \end{aligned} \quad (192)$$

Therefore,

$$|\mathfrak{G}_q(\Lambda)| \leq c_6 \ell_0 e^{-3\bar{J}} |\partial_q \Lambda| \left( 1 + e^{-3(1-\alpha)\bar{J}} \right) \quad (193)$$

for some constant  $c_6 > 0$ .

**4 -** We conclude the proof by injecting (180), (186), (188) and (193) into (179), and using the fact that  $\kappa^{-1} < \ell_0$  (128).  $\square$

## 6. Nematic Phase

We are now ready to prove theorem 1. Let  $\Lambda_L$  by a square box of side-length  $L$ .

1 - Given an edge  $e \in \mathcal{E}(\Lambda_L)$ , we have

$$\langle \mathbb{1}_e \rangle_{\Lambda_L, \mathbf{q}} = \frac{\tilde{\mathfrak{Z}}_{\mathbf{q}}^{(\{e\})}(\Lambda_L)}{\tilde{\mathfrak{Z}}_{\mathbf{q}}^{(\emptyset)}(\Lambda_L)} \exp \left( \log \left( \frac{Z^{(\{e\})}(\Lambda_L | \mathbf{q})}{\tilde{\mathfrak{Z}}_{\mathbf{q}}^{(\{e\})}(\Lambda_L)} \right) - \log \left( \frac{Z^{(\emptyset)}(\Lambda_L | \mathbf{q})}{\tilde{\mathfrak{Z}}_{\mathbf{q}}^{(\emptyset)}(\Lambda_L)} \right) \right) \quad (194)$$

1-1 - Let us first bound the exponent in the thermodynamic limit. As in Lemma 7, we define

$$\bar{\zeta}_{\mathbf{c}}^{(\{e\}, \emptyset)}(\gamma | t) := t \zeta_{\mathbf{c}}^{(\{e\})}(\gamma_0) + (1 - t) \zeta_{\mathbf{c}}^{(\emptyset)}(\gamma_0) \quad (195)$$

and

$$\bar{\mathfrak{P}}_q^{(\{e\}, \emptyset)}(\Lambda_L) := \mathfrak{P}_q^{(\{e\})}(\Lambda_L) \cup \mathfrak{P}_q^{(\emptyset)}(\Lambda_L) \quad (196)$$

and, using the cluster expansion (95), we write

$$\begin{aligned} & \log \left( \frac{Z^{(\{e\})}(\Lambda_L | \mathbf{q})}{\tilde{\mathfrak{Z}}_{\mathbf{q}}^{(\{e\})}(\Lambda_L)} \right) - \log \left( \frac{Z^{(\emptyset)}(\Lambda_L | \mathbf{q})}{\tilde{\mathfrak{Z}}_{\mathbf{q}}^{(\emptyset)}(\Lambda_L)} \right) \\ &= \int_0^1 dt \sum_{m=1}^{\infty} \sum_{\gamma \in \bar{\mathfrak{P}}_q^{(\{e\}, \emptyset)}(\Lambda_L)} \sum_{\underline{\gamma} \sqsubset \bar{\mathfrak{P}}_q^{(\{e\}, \emptyset)}(\Lambda_L) \setminus \{\gamma\}} \Phi^T(\{\gamma\}^m \sqcup \underline{\gamma}) \cdot \\ & \quad \cdot \left( \zeta_{\mathbf{q}}^{(\{e\})}(\gamma) - \zeta_{\mathbf{q}}^{(\emptyset)}(\gamma) \right) m(\bar{\zeta}_{\mathbf{q}}^{(\{e\}, \emptyset)}(\gamma | t))^{m-1} \prod_{\gamma' \in \underline{\gamma}} \bar{\zeta}_{\mathbf{q}}^{(\{e\}, \emptyset)}(\gamma' | t). \end{aligned} \quad (197)$$

We split the sum into a *bulk* and a *boundary* contribution, similarly to Lemma 7. We define  $\bar{\mathfrak{P}}_q^{(\{e\}, \emptyset)}(\infty)$  as the set of polymers  $\gamma$  for which  $\exists L$  such that  $\gamma$  is in  $\bar{\mathfrak{P}}_q^{(\{e\}, \emptyset)}(\Lambda_L)$  while not being connected to the boundary. We then split

$$\log \left( \frac{Z^{(\{e\})}(\Lambda_L | \mathbf{q})}{\tilde{\mathfrak{Z}}_{\mathbf{q}}^{(\{e\})}(\Lambda_L)} \right) - \log \left( \frac{Z^{(\emptyset)}(\Lambda_L | \mathbf{q})}{\tilde{\mathfrak{Z}}_{\mathbf{q}}^{(\emptyset)}(\Lambda_L)} \right) = I^{(e)} - R_1^{(e)}(\Lambda_L) + R_2^{(e)}(\Lambda_L) \quad (198)$$

where

$$\begin{aligned} I^{(e)} &:= \int_0^1 dt \sum_{m=1}^{\infty} \sum_{\gamma \in \bar{\mathfrak{P}}_q^{(\{e\}, \emptyset)}(\infty)} \sum_{\underline{\gamma} \sqsubset \bar{\mathfrak{P}}_q^{(\{e\}, \emptyset)}(\infty) \setminus \{\gamma\}} \Phi^T(\{\gamma\}^m \sqcup \underline{\gamma}) \cdot \\ & \quad \cdot \left( \zeta_{\mathbf{q}}^{(\{e\})}(\gamma) - \zeta_{\mathbf{q}}^{(\emptyset)}(\gamma) \right) m(\bar{\zeta}_{\mathbf{q}}^{(\{e\}, \emptyset)}(\gamma | t))^{m-1} \prod_{\gamma' \in \underline{\gamma}} \bar{\zeta}_{\mathbf{q}}^{(\{e\}, \emptyset)}(\gamma' | t) \end{aligned} \quad (199)$$

$$\begin{aligned}
R_1^{(e)}(\Lambda_L) &:= \int_0^1 dt \sum_{m=1}^{\infty} \sum_{\gamma \in \overline{\mathfrak{P}}_q^{(\{e\}, \emptyset)}(\infty)} \sum_{\substack{\underline{\gamma} \sqsubset \overline{\mathfrak{P}}_q^{(\{e\}, \emptyset)}(\infty) \setminus \{\gamma\} \\ \{\gamma\}^m \sqcup \underline{\gamma} \not\sqsubset \overline{\mathfrak{P}}_q^{(\{e\}, \emptyset)}(\Lambda_L)}} \Phi^T(\{\gamma\}^m \sqcup \underline{\gamma}) \cdot \\
&\quad \cdot \left( \xi_{\mathbf{q}}^{(\{e\})}(\gamma) - \xi_{\mathbf{q}}^{(\emptyset)}(\gamma) \right) m(\bar{\xi}_{\mathbf{q}}^{(\{e\}, \emptyset)}(\gamma|t))^{m-1} \prod_{\gamma' \in \underline{\gamma}} \bar{\xi}_{\mathbf{q}}^{(\{e\}, \emptyset)}(\gamma'|t)
\end{aligned} \tag{200}$$

and

$$\begin{aligned}
R_2^{(e)}(\Lambda_L) &:= \int_0^1 dt \sum_{m=1}^{\infty} \sum_{\gamma \in \overline{\mathfrak{P}}_q^{(\{e\}, \emptyset)}(\Lambda_L)} \sum_{\substack{\underline{\gamma} \sqsubset \overline{\mathfrak{P}}_q^{(\{e\}, \emptyset)}(\Lambda_L) \setminus \{\gamma\} \\ \{\gamma\}^m \sqcup \underline{\gamma} \not\sqsubset \overline{\mathfrak{P}}_q^{(\{e\}, \emptyset)}(\infty)}} \Phi^T(\{\gamma\}^m \sqcup \underline{\gamma}) \cdot \\
&\quad \cdot \left( \xi_{\mathbf{q}}^{(\{e\})}(\gamma) - \xi_{\mathbf{q}}^{(\emptyset)}(\gamma) \right) m(\bar{\xi}_{\mathbf{q}}^{(\{e\}, \emptyset)}(\gamma|t))^{m-1} \prod_{\gamma' \in \underline{\gamma}} \bar{\xi}_{\mathbf{q}}^{(\{e\}, \emptyset)}(\gamma'|t).
\end{aligned} \tag{201}$$

By (100),

$$m|\bar{\xi}_{\mathbf{q}}^{(\{e\}, \emptyset)}(\gamma|t)|^{m-1} \leq 1 \tag{202}$$

so, by (97),

$$|I^{(e)}| \leq \sum_{\gamma \in \overline{\mathfrak{P}}_q^{(\{e\}, \emptyset)}(\infty)} \left| \xi_{\mathbf{q}}^{(\{e\})}(\gamma) - \xi_{\mathbf{q}}^{(\emptyset)}(\gamma) \right| e^{a(\gamma)}. \tag{203}$$

Furthermore,  $\xi_{\mathbf{q}}^{(\{e\})}(\gamma) - \xi_{\mathbf{q}}^{(\emptyset)}(\gamma)$  differs from 0 only if  $\gamma$  interacts with  $e$  (see Definition-1), so, by Lemma 6,

$$|I^{(e)}| \leq 2C_{10}e^{-(1-\alpha)\bar{J}}. \tag{204}$$

In addition, the clusters  $\gamma_0, \dots, \gamma_n$  that contribute to  $R_1$  or  $R_2$  interact with  $e$  as well as with the boundary of  $\Lambda_L$ . Therefore, for such clusters,

$$\sum_{i=0}^n \mathcal{E}(\gamma_i) \geq \bar{\kappa} (\text{dist}_1(e, \partial \Lambda) - 2\ell_0) \tag{205}$$

which goes to  $\infty$  as  $L \rightarrow \infty$ . Therefore,

$$\prod_{j=0}^n |\xi_{\mathbf{q}}^{(\gamma_j)}(\gamma_j)| \leq e^{-\beta\bar{\kappa}(\text{dist}_1(e, \partial \Lambda) - 2\ell_0)} \prod_{j=0}^n e^{-(1-\beta)\mathcal{E}(\gamma_j)} \tag{206}$$

for  $\beta \in (0, \frac{1}{2}\bar{t} - \alpha)$ . We then use Lemmas 3 and 6 to bound

$$|R_i^{(e)}(\Lambda_L)| = O\left(e^{-\beta\bar{\kappa}(\text{dist}_1(e, \partial \Lambda) - 2\ell_0)}\right) \tag{207}$$

so

$$|R_1^{(e)}(\Lambda_L)| + |R_2^{(e)}(\Lambda_L)| \xrightarrow{L \rightarrow \infty} 0. \tag{208}$$

Therefore,

$$\log \left( \frac{Z^{(\{e\})}(\Lambda_L | \mathbf{q})}{\tilde{\mathfrak{Z}}_{\mathbf{q}}^{(\{e\})}(\Lambda_L)} \right) - \log \left( \frac{Z^{(\emptyset)}(\Lambda_L | \mathbf{q})}{\tilde{\mathfrak{Z}}_{\mathbf{q}}^{(\emptyset)}(\Lambda_L)} \right) \xrightarrow{L \rightarrow \infty} I^{(e)} \quad (209)$$

which is independent of the position of  $e$ , and is bounded as per (204).

**1-2 -** We turn, now, to the ratio of  $\tilde{\mathfrak{Z}}$ 's, which can be computed explicitly from (77) and (57): if  $e$  is vertical and  $L$  is large enough, then

$$\frac{\tilde{\mathfrak{Z}}_{\mathbf{v}}^{(\{e\})}(\Lambda_L)}{\tilde{\mathfrak{Z}}_{\mathbf{v}}^{(\emptyset)}(\Lambda_L)} = z v_+^2(\omega_1) b_+ \lambda_+^{-2} = \frac{1}{2} (1 + O(\epsilon)) \quad (210)$$

in which we used (108) through (111). If  $e$  is horizontal, then

$$\frac{\tilde{\mathfrak{Z}}_{\mathbf{v}}^{(\{e\})}(\Lambda_L)}{\tilde{\mathfrak{Z}}_{\mathbf{v}}^{(\emptyset)}(\Lambda_L)} = z v_+^4(\omega_0) b_+^2 \lambda_+^{-2} = O(e^{-3J}). \quad (211)$$

This proves that  $\langle \mathbb{1}_e \rangle_{\mathbf{v}}$  is independent of the position of  $e$ , as well as (17) and (18).

**2 -** We now consider two edges  $e, e' \in \mathcal{E}(\Lambda_L)$  which are at a distance of at least  $\ell_0$ . We have

$$\begin{aligned} & \langle \mathbb{1}_e \mathbb{1}_{e'} \rangle_{\Lambda_L, \mathbf{q}} - \langle \mathbb{1}_e \rangle_{\Lambda_L, \mathbf{q}} \langle \mathbb{1}_{e'} \rangle_{\Lambda_L, \mathbf{q}} \\ &= \langle \mathbb{1}_e \rangle_{\Lambda_L, \mathbf{q}} \langle \mathbb{1}_{e'} \rangle_{\Lambda_L, \mathbf{q}} \left( \frac{\tilde{\mathfrak{Z}}_{\mathbf{q}}^{(\{e, e'\})}(\Lambda_L) \tilde{\mathfrak{Z}}_{\mathbf{q}}^{(\emptyset)}(\Lambda_L)}{\tilde{\mathfrak{Z}}_{\mathbf{q}}^{(\{e\})}(\Lambda_L) \tilde{\mathfrak{Z}}_{\mathbf{q}}^{(\{e'\})}(\Lambda_L)} e^{\mathbb{K}_{\mathbf{q}}^{(e, e')}(\Lambda_L)} - 1 \right) \end{aligned} \quad (212)$$

with

$$\begin{aligned} \mathbb{K}_{\mathbf{q}}^{(e, e')}(\Lambda_L) &:= \log \left( \frac{Z^{(\{e, e'\})}(\Lambda_L | \mathbf{q})}{\tilde{\mathfrak{Z}}_{\mathbf{q}}^{(\{e, e'\})}(\Lambda_L)} \right) + \log \left( \frac{Z^{(\emptyset)}(\Lambda_L | \mathbf{q})}{\tilde{\mathfrak{Z}}_{\mathbf{q}}^{(\emptyset)}(\Lambda_L)} \right) \\ &\quad - \log \left( \frac{Z^{(\{e\})}(\Lambda_L | \mathbf{q})}{\tilde{\mathfrak{Z}}_{\mathbf{q}}^{(\{e\})}(\Lambda_L)} \right) - \log \left( \frac{Z^{(\{e'\})}(\Lambda_L | \mathbf{q})}{\tilde{\mathfrak{Z}}_{\mathbf{q}}^{(\{e'\})}(\Lambda_L)} \right). \end{aligned} \quad (213)$$

First of all, by (77) and (57),

$$\frac{\tilde{\mathfrak{Z}}_{\mathbf{q}}^{(\{e, e'\})}(\Lambda_L) \tilde{\mathfrak{Z}}_{\mathbf{q}}^{(\emptyset)}(\Lambda_L)}{\tilde{\mathfrak{Z}}_{\mathbf{q}}^{(\{e\})}(\Lambda_L) \tilde{\mathfrak{Z}}_{\mathbf{q}}^{(\{e'\})}(\Lambda_L)} = 1. \quad (214)$$

We then write  $\mathbb{K}_{\mathbf{q}}^{(e, e')}(\Lambda_L)$  using the cluster expansion (95), and note that the only clusters  $\gamma_0, \dots, \gamma_n$  that contribute are those that interact with *both*  $e$  and  $e'$ . For such clusters, denoting the vertical and horizontal distances by  $\text{dist}_v$  and  $\text{dist}_h$  (these are the induced by the semi-norms  $\|(x, y)\|_v := |y|$  and  $\|(x, y)\|_h := |x|$ ),

$$\sum_{i=0}^n \mathcal{E}(\gamma_i) \geq \bar{\kappa} (\text{dist}_v(e, e') - 2\ell_0) + \bar{J} \text{dist}_h(e, e') \quad (215)$$

so, by once again estimating

$$\prod_{j=0}^n |\zeta_{\mathbf{q}}^{(\gamma)}(\gamma_j)| \leq e^{-\beta(\bar{\kappa}(\text{dist}_v(e, e') - 2\ell_0) + \bar{J} \text{dist}_h(e, e'))} \prod_{j=0}^n e^{-(1-\beta)\mathcal{E}^{(\gamma)}(\gamma_j)} \quad (216)$$

for  $\beta \in (0, \frac{1}{2}t - \alpha)$ , and using Lemmas 3 and 6 to bound

$$|\mathbb{K}_{\mathbf{q}}^{(e, e')}(A_L)| = O\left(e^{-\beta(\bar{\kappa}(\text{dist}_v(e, e') - 2\ell_0) + \bar{J}\text{dist}_h(e, e'))}\right) \quad (217)$$

from which (19) follows.  $\square$

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