Reconciling Formal, Multi-Layer, and Hetero-functional Graphs with the Hetero-functional Incidence Tensor

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***Abstract*—The developing consensus across a number of STEM fields is that each of the NAE game-changing goals is is characterized by an "engineering system" that is analyzed and re-synthesized using a meta-problem-solving skill set. Two fields in particular have attempted to traverse this convergence challenge: systems engineering and network science. Systems engineering has developed as a practical and interdisciplinary engineering discipline that enables the successful realization of complex systems from concept, through design, to full implementation based upon graphical modeling languages. In contrast, network science has developed to quantitatively analyze networks that appear in a wide variety of engineering systems but suffers from disparate terminology and a lack of consensus. This paper provides a tensor-based formulation of several of the most important parts of hetero-functional graph theory. More specifically, it discusses the system concept, the hetero-functional adjacency matrix, and introduces the hetero-functional incidence tensor for the first time. The tensor-based formulation described in this work makes a stronger tie between HFGT and its ontological foundations in MBSE. Finally, the tensor-based formulation facilitates an understanding of the relationships between HFGT and multi-layer networks "despite its disparate terminology and lack of consensus". In so doing, this tensor-based treatment is likely to advance Kivela et. al's goal to discern the similarities and differences between these mathematical models in as precise a manner as possible.**

***Index Terms*—Hetero-Functional Graph Theory, Model Based Systems Engineering, sustainable energy transition, American Multi-modal Energy System, Sustainability**

I. Introduction

One defining characteristic of twenty-first century engineering challenges is the breadth of their scope. The National Academy of Engineering (NAE) has identified 14 "game-changing goals" [1]. The developing consensus across a number of STEM (science, technology, engineering, and mathematics) fields is that each of these goals is characterized by an "engineering system" that is analyzed and re-synthesized using a meta-problem-solving skill set.

Definition 1: Engineering system [2]: A class of systems characterized by a high degree of technical complexity, social intricacy, and elaborate processes, aimed at fulfilling important functions in society. ■

Two fields in particular have attempted to traverse this convergence challenge: systems engineering and network science. Systems engineering, and more recently model-based systems engineering (MBSE), has developed as a practical and interdisciplinary engineering discipline that enables the successful realization of complex systems from concept, through design, to full implementation [3]. It equips the engineer with methods and tools to handle systems of ever-greater complexity arising from greater interactions within these systems or from the expanding heterogeneity they demonstrate in their structure and function. Despite its many accomplishments, model-based systems engineering still relies on graphical modeling languages that provide limited quantitative insight (on their own) [4], [5].

In contrast, network science has developed to quantitatively analyze networks that appear in a wide variety of engineering systems. And yet, despite its methodological developments in multi-layer networks, network science has often been unable to address the explicit heterogeneity often encountered in engineering systems [5], [6]. In a recent comprehensive review Kivela et. al [6] write: *"Numerous similar ideas have been developed in parallel, and the literature on multi-layer networks has rapidly become extremely messy. Despite a wealth of antecedent ideas in subjects like sociology and engineering, many aspects of the theory of multi-layer networks remain immature, and the rapid onslaught of papers on various types of multilayer networks necessitates an attempt to unify the various disparate threads and to discern their similarities and differences in as precise a manner as possible. . . . [The multi-layer network community] has produced an equally immense explosion of disparate terminology, and the lack of consensus (or even generally accepted) set of terminology and mathematical framework for studying is extremely problematic."*

Recently, hetero-functional graph theory (HFGT) has developed as a means to mathematically model the structure of large-scale complex, flexible, engineering systems. It does so by fusing concepts from network science and model-based systems engineering (MBSE). For the former, it utilizes multiple graph-based data structures to support a matrix-based quantitative analysis. For the latter, HFGT inherits the heterogeneity of conceptual and ontological constructs found in model-based systems engineering including system form, system function, and system concept.

The ontological strength of hetero-functional graph theory comes from the “systems thinking” foundations in the model-based systems engineering literature [5], [7]. In effect, and very briefly, all systems have a “subject + verb + operand” form where the system form is the subject, the system function is the verb + operand (i.e. predicate) and the system concept is the mapping of the two to each other. The key distinguishing feature of HFGT (relative to multi-layer networks) is its introduction of system function. In that regard, it is more complete than multi-layer networks if system function is accepted as part of an engineering system abstraction. Another key distinguishing feature of HFGT is the differentiation between elements related to transformation and transportation. In that regard, it takes great care to not *overload* mathematical modeling elements and preserve lucidity. These diverse conceptual constructs indicate multi-dimensional rather than two-dimensional relationships.

A. Original Contribution

This paper provides a tensor-based formulation of several of the most important parts of hetero-functional graph theory. More specifically, it discusses the system concept, the hetero-functional adjacency matrix, and introduces the hetero-functional incidence tensor for the first time. Whereas the hetero-functional graph theory text [5] is a comprehensive discussion of the subject, the treatment is based entirely on two-dimensional matrices. The tensor-based formulation described in this work makes a stronger tie between HFGT and its ontological foundations in MBSE. Finally, the tensor-based formulation facilitates an understanding of the relationships between HFGT and multi-layer networks (“despite its disparate terminology and lack of consensus”). In so doing, this tensor-based treatment is likely to advance Kivela et. al’s goal to discern the similarities and differences between these mathematical models in as precise a manner as possible.

B. Paper Outline

The rest of the paper is organized as follows. Section II provides some HFGT preliminaries to support the remainder of the discussion. Section III, then, introduces the hetero-functional incidence tensor which describes the relationships between system capabilities, operands, and physical locations in space (i.e. system buffers as defined later). Section IV provides a mathematical discussion comparing formal, multi-layer, and hetero-functional graphs. Section V brings the work to a close. The interested reader is referred to [8] for deeper explanations of the tensor-based mathematics.

II. Preliminaries

HFGT is predicated on a system concept A_S that describes the allocation of system function to system form as the central question of engineering design. This dichotomy of form and function is repeatedly emphasized in the fields of engineering design and systems engineering [7], [9]–[11]. More specifically, the allocation of system processes to system resources is captured in the “design equation” [11], [12]:

$$P = A_S \odot R \tag{1}$$

where R is set of system resources, P is the set of system processes, A_S is the system concept, and $\odot$ is matrix Boolean multiplication.

Definition 2 – System Resource: [3] An asset or object $r_v \in R$ that is utilized during the execution of a process. ■

Definition 3 – System Process [3], [13]: An activity $p \in P$ that transforms a predefined set of input operands into a predefined set of outputs. ■

Definition 4 – System Operand: [3] An asset or object $l_i \in L$ that is operated on or consumed during the execution of a process. ■

Definition 5 – System Concept [11], [12], [14]: A binary matrix A_S of size $\sigma(P) \times \sigma(R)$ whose element $A_S(w, v) \in \{0, 1\}$ is equal to one when action $e_{wv} \in \mathcal{E}_S$ (in the SysML sense) is available as a system process $p_w \in P$ being executed by a resource $r_v \in R$. ■

In other words, the system concept forms a bipartite graph between the set of system processes and the set of system resources [14].

Hetero-functional graph theory further recognizes that there are inherent differences within the set of resources as well as within the set of processes. $R = M \cup B \cup H$ where M is the set of transformation resources, B is the set of independent buffers, and H is the set of transportation resources. Furthermore, the set of buffers $B_S = M \cup B$ is introduced for later discussion. Similarly, $P = P_\mu \cup P_{\bar{\eta}}$ where P_μ is the set of transformation processes and $P_{\bar{\eta}}$ is the set of refined transportation processes. The latter, in turn, is determined from the Cartesian product ($\bigtimes$) of the set of transportation processes P_η and the set of holding processes P_γ. $P_{\bar{\eta}} = P_\gamma \bigtimes P_\eta$. Every filled element of the system concept indicates a *system capability* of the form: “Resource r_v does process p_w”. The system capabilities are quantified by the structural degrees of freedom.

Definition 6 – Structural Degrees of Freedom [12]: The set of independent actions $\mathcal{E}_S$ that completely defines the instantiated processes in a large flexible engineering system. Their number is given by:

$$DOF_S = \sigma(\mathcal{E}_S) = \sum_{w}^{\sigma(P)} \sum_{v}^{\sigma(R)} A_S(w, v) \tag{2}$$

■

Once the system’s physical capabilities (or structural degrees of freedom have been defined), the hetero-functional adjacency matrix A_ρ is introduced to represent their pair-wise sequences. [14], [15].

Definition 7 – Hetero-functional Adjacency Matrix [14]: A square binary matrix A_ρ of size $\sigma(R)\sigma(P) \times \sigma(R)\sigma(P)$ whose element $A_\rho(\chi_1, \chi_2) \in \{0,1\}$ is equal to one when string $z_{\chi_1,\chi_2} = e_{w_1v_1}e_{w_2v_2} \in \mathcal{Z}$ is available and exists, where index $\chi_i \in [1, \dots, \sigma(R)\sigma(P)]$. ■

In other words, the hetero-functional adjacency matrix corresponds to a hetero-functional graph $G_\rho = \{\mathcal{E}_S, \mathcal{Z}\}$ with structural degrees of freedom (i.e. capabilities) $\mathcal{E}_S$ as nodes and feasible sequences $\mathcal{Z}$ as edges. In contrast, formal graphs, are typically defined as $G_F = \{B_S, H\}$ where transportation resources (e.g. power lines, roads, water pipes) connect independent buffers (e.g. substations, traffic intersections, and water junctions).

Alternatively, the hetero-functional adjacency matrix may be expressed as a fourth order tensor.

Definition 8 – Hetero-functional Adjacency Tensor: A fourth-order tensor $\mathcal{A}_\rho$ of size $\sigma(R) \times \sigma(P) \times \sigma(R) \times \sigma(P)$ whose element $\mathcal{A}_\rho(w_1, v_1, w_2, v_2) \in \{0,1\}$ is equal to one when string $e_{w_1v_1}e_{w_2v_2} \in \mathcal{Z}$ is available and exists.

$$A_\rho = \mathcal{F}_M \left(\mathcal{A}_\rho, [1,2], [3,4]\right) \tag{3}$$

where $\mathcal{F}_M$ is matricization [8]. ■

For systems of substantial size, the size of the hetero-functional adjacency matrix may be challenging to process computationally. However, the matrix is generally very sparse. Therefore, projection operators are used to eliminate the sparsity by projecting the matrix onto a one's vector [15]. This is demonstrated below for A_S^V and A_ρ:

$$\mathbb{P}_S A_S^V = \mathbb{1}^{\sigma(\mathcal{E}_S)} \tag{4}$$

$$\mathbb{P}_S A_\rho \mathbb{P}_S^T = \widetilde{A}_\rho \tag{5}$$

where $()^V$ is shorthand for $vec()$, $\mathbb{P}_S$ is a (non-unique) projection matrix for the vectorized system knowledge base [15].

III. The Hetero-functional Incidence Tensor

To complement the concept of a hetero-functional adjacency matrix A_ρ, the hetero-functional incidence tensor $\widetilde{\mathcal{M}}_\rho$ describes the structural relationships between the physical capabilities (i.e. structural degrees of freedom) $\mathcal{E}_S$, the system operands L, and the system buffers B_S.

$$\widetilde{\mathcal{M}}_\rho = \widetilde{\mathcal{M}}_\rho^+ - \widetilde{\mathcal{M}}_\rho^- \tag{6}$$

Definition 9 – The Negative 3^{rd} Order Hetero-functional Incidence Tensor $\widetilde{\mathcal{M}}_\rho^-$: The negative hetero-functional incidence tensor $\widetilde{\mathcal{M}_\rho}^- \in \{0,1\}^{\sigma(L)\times\sigma(B_S)\times\sigma(\mathcal{E}_S)}$ is a third-order tensor whose element $\widetilde{\mathcal{M}}_\rho^-(i,y,\psi) = 1$ when the system capability $\epsilon_\psi \in \mathcal{E}_S$ pulls operand $l_i \in L$ from buffer $b_{s_y} \in B_S$. ■

Definition 10 – The Positive 3^{rd} Order Hetero-functional Incidence Tensor $\widetilde{\mathcal{M}}_\rho^+$: The positive hetero-functional incidence tensor $\widetilde{\mathcal{M}}_\rho^+ \in \{0,1\}^{\sigma(L)\times\sigma(B_S)\times\sigma(\mathcal{E}_S)}$ is a third-order tensor whose element $\widetilde{\mathcal{M}}_\rho^+(i,y,\psi) = 1$ when the system capability $\epsilon_\psi \in \mathcal{E}_S$ injects operand $l_i \in L$ into buffer $b_{s_y} \in B_S$. ■

The calculation of these two tensors depends on the definition of two more matrices.

Definition 11 – The Negative Process-Operand Incidence Matrix M_{LP}^-: A binary incidence matrix $M_{LP}^- \in \{0,1\}^{\sigma(L)\times\sigma(P)}$ whose element $M_{LP}^-(i,w) = 1$ when the system process $p_w \in P$ pulls operand $l_i \in L$ as an input. To take into account the heterogeneity of processes, it is further decomposed into the negative transformation process-operand incidence matrix $M_{LP_\mu}^-$ and the negative refined transformation process-operand incidence matrix $M_{LP_{\bar{\eta}}}^-$ which by definition is in turn calculated from the negative holding process-operand incidence matrix $M_{LP_\gamma}^-$.

$$M_{LP}^- = \begin{bmatrix} M_{LP_\mu}^- & M_{LP_{\bar{\eta}}}^- \end{bmatrix} = \begin{bmatrix} M_{LP_\mu}^- & M_{LP_\gamma}^- \otimes \mathbb{1}^{\sigma(P_\eta)T} \end{bmatrix} \tag{7}$$

where $\otimes$ is the Kronecker product. ■

Definition 12 – The Positive Process-Operand Incidence Matrix M_{LP}^+: A binary incidence matrix $M_{LP}^+ \in \{0,1\}^{\sigma(L)\times\sigma(P)}$ whose element $M_{LP}^+(i,w) = 1$ when the system process $p_w \in P$ injects operand $l_i \in L$ as an output. To take into account the heterogeneity of processes, it is further decomposed into the positive transformation process-operand incidence matrix $M_{LP_\mu}^+$ and the positive refined transformation process-operand incidence matrix $M_{LP_{\bar{\eta}}}^+$ which, by definition, is, in turn, calculated from the positive holding process-operand incidence matrix $M_{LP_\gamma}^+$

$$M_{LP}^+ = \begin{bmatrix} M_{LP_\mu}^+ & M_{LP_{\bar{\eta}}}^+ \end{bmatrix} = \begin{bmatrix} M_{LP_\mu}^+ & M_{LP_\gamma}^+ \otimes \mathbb{1}^{\sigma(P_\eta)T} \end{bmatrix} \tag{8}$$

■

With the definitions of these incidence matrices in place, the calculation of the negative and positive hetero-functional incidence tensors $\widetilde{\mathcal{M}}_\rho^-$ and $\widetilde{\mathcal{M}}_\rho^+$ follows straightforwardly as a third-order outer product $\circ$. For $\widetilde{\mathcal{M}}_\rho^-$:

$$\widetilde{\mathcal{M}}_\rho^- = \sum_{i=1}^{\sigma(L)} \sum_{y_1=1}^{\sigma(B_S)} e_i^{\sigma(L)} \circ e_{y_1}^{\sigma(B_S)} \circ \mathbb{P}_S \left(\left(X_{iy_1}^-\right)^V \right) \tag{9}$$

where

$$X_{iy_1}^- = \left[\frac{M_{LP_\mu}^{-T} e_i^{\sigma(L)} e_{y_1}^{\sigma(M)T} \quad | \quad \mathbf{0}}{M_{LP_\gamma}^{-T} e_i^{\sigma(L)} \otimes \left(e_{y_1}^{\sigma(B_S)} \otimes \mathbb{1}^{\sigma(B_S)} \right) \otimes \mathbb{1}^{\sigma(R)T}} \right] \tag{10}$$

The $X_{iy_1}^-$ matrix is equivalent in size to the system concept A_S. It has a value of one in all elements where the associated process both withdraws input operand l_i and originates at the buffer $b_{s_{y_1}}$. Consequently, when $X_{iy_1}^-$ is vectorized and then projected with $\mathbb{P}_S$, the result is a vector with a value of one only where the associated system capabilities meet these criteria.

For $\widetilde{\mathcal{M}}_\rho^+$:

$$\widetilde{\mathcal{M}}_\rho^+ = \sum_{i=1}^{\sigma(L)} \sum_{y_2=1}^{\sigma(B_S)} e_i^{\sigma(L)} \circ e_{y_2}^{\sigma(B_S)} \circ \mathbb{P}_S \left(\left(X_{iy_2}^+ \right)^V \right) \tag{11}$$

where

$$X_{iy_2}^+ = \left[\; M_{LP_\gamma}^{+T} e_i^{\sigma(L)} \otimes \left(\mathbb{1}^{\sigma(B_S)} \otimes e_{y_2}^{\sigma(B_S)} \right) \otimes \mathbb{1}^{\sigma(R)T} \; \right] \tag{12}$$

The $X_{iy_2}^+$ matrix is equivalent in size to the system concept A_S. It also has a value of one in all elements where the associated process both injects output operand l_i and terminates at the buffer $b_{s_{y_2}}$. Consequently, when $X_{iy_2}^+$ is vectorized and then projected with $\mathbb{P}_S$, the result is a vector with a value of one only where the associated system capabilities meet these criteria.

It is important to note that the definitions of the 3^{rd} order hetero-functional incidence tensors $\widetilde{\mathcal{M}}_\rho^-$, and $\widetilde{\mathcal{M}}_\rho^+$ are provided in projected form as indicated by the presence of the projection operator $\mathbb{P}_S$ in Equations 9 and 11 respectively. It is often useful to use the un-projected form of these tensors.

$$\mathcal{M}_\rho^- = \sum_{i=1}^{\sigma(L)} \sum_{y_1=1}^{\sigma(B_S)} e_i^{\sigma(L)} \circ e_{y_1}^{\sigma(B_S)} \circ \left(X_{iy_1}^- \right)^V \tag{13}$$

$$\mathcal{M}_\rho^+ = \sum_{i=1}^{\sigma(L)} \sum_{y_2=1}^{\sigma(B_S)} e_i^{\sigma(L)} \circ e_{y_2}^{\sigma(B_S)} \circ \left(X_{iy_2}^+ \right)^V \tag{14}$$

The third dimension of these unprojected 3^{rd} order hetero-functional incidence tensors can then be split into two dimensions to create 4^{th} order hetero-functional incidence tensors.

$$\mathcal{M}_{PR}^+ = vec^{-1} \left(\mathcal{M}_\rho^+, [\sigma(P), \sigma(R)], 3 \right) \tag{15}$$

$$\mathcal{M}_{PR}^- = vec^{-1} \left(\mathcal{M}_\rho^-, [\sigma(P), \sigma(R)], 3 \right) \tag{16}$$

where $vec^{-1}()$ is inverse vectorization [8]. These fourth order tensors describe the structural relationships between the system processes P, the physical resources R that realize them, the system operands L that are consumed and injected in the process, and the system buffers B_S from which these are operands are sent and the system buffers B_S to which these operands are received. They are used in the following section as part of the discussion on layers.

$$\mathcal{M}_{PR} = \mathcal{M}_{PR}^+ - \mathcal{M}_{PR}^- \tag{17}$$

Definition 13 – The Negative 4^{th} Order Hetero-functional Incidence Tensor $\mathcal{M}_{PR}^-$: The negative 4^{th} Order hetero-functional incidence tensor $\mathcal{M}_{PR}^- \in \{0,1\}^{\sigma(L)\times\sigma(B_S)\times\sigma(P)\times\sigma(R)}$ has element $\mathcal{M}_{PR}^-(i,y,w,v) = 1$ when the system process $p_w \in P$ realized by resource $r_v \in R$ pulls operand $l_i \in L$ from buffer $b_{s_y} \in B_S$. ■

Definition 14 – The Positive 4^{th} Order Hetero-functional Incidence Tensor $\mathcal{M}_{PR}^-$: The positive 4^{th} Order hetero-functional incidence tensor $\mathcal{M}_{PR}^- \in \{0,1\}^{\sigma(L)\times\sigma(B_S)\times\sigma(P)\times\sigma(R)}$ has element $\mathcal{M}_{PR}^-(i,y,w,v) = 1$ when the system process $p_w \in P$ realized by resource $r_v \in R$ injects operand $l_i \in L$ into buffer $b_{s_y} \in B_S$. ■

Returning back to the hetero-functional incidence tensor $\widetilde{M_\rho}$, it and its positive and negative components $\widetilde{M}_\rho^+, \widetilde{M}_\rho^-$, can also be easily matricized.

$$M_\rho = \mathcal{F}_M \left(\mathcal{M}_\rho, [1,2], [3] \right) \tag{18}$$

$$M_\rho^- = \mathcal{F}_M \left(\mathcal{M}_\rho^-, [1,2], [3] \right) \tag{19}$$

$$M_\rho^+ = \mathcal{F}_M \left(\mathcal{M}_\rho^+, [1,2], [3] \right) \tag{20}$$

The resulting matrices have a size of $\sigma(L)\sigma(B_S) \times \sigma(\mathcal{E}_\mathcal{S})$ which have a corresponding physical intuition. Each buffer b_{s_y} has $\sigma(L)$ copies to reflect a place (i.e. bin) for each operand at that buffer. Each of these places then forms a bipartite graph with the system's physical capabilities. Consequently, and as expected, the hetero-functional adajacency matrix A_ρ can be calculated as a matrix product of the positive and negative hetero-functional incidence matrices M_ρ^+ and M_ρ^+.

$$A_\rho = M_\rho^{+T} M_\rho^- \tag{21}$$

Such a product systematically enforces all five types of feasibility constraints. I:

1) $P_\mu P_\mu$. Two transformation processes that follow each other must occur at the same transformation resource.
2) $P_\mu P_{\bar{\eta}}$. A refined transportation process that follows a transformation process must have an origin equivalent to the transformation resource at which the transformation process was executed.
3) $P_{\bar{\eta}} P_\mu$. A refined transportation process that precedes a transformation process must have a destination equivalent to the transformation resource at which the transformation process was executed.
4) $P_{\bar{\eta}} P_{\bar{\eta}}$. A refined transportation process that follows another must have an origin equivalent to the destination of the other.
5) PP. The type of operand of one process must be equivalent to the type of output of another process.

IV. Discussion

The introduction of the hetero-functional incidence tensor serves to reconcile the gap in terminology between formal, multi-layer, and hetero-functional graphs.

A. Comparing Formal and Hetero-functional Graphs

At first glance, formal graphs defined as $G_F = \{B_S, H\}$ appear entirely unrelated to hetero-functional graphs defined as $G_\rho = \{\mathcal{E}_S, \mathcal{Z}\}$. Nevertheless, they are related by virtue of the hetero-functional incidence tensor.

$$A_{B_S}(y_1, y_2) = \bigvee_\psi \left(\bigvee_i \mathcal{M}_\rho^-(i, y_1, \psi) \right) \left(\bigvee_i \mathcal{M}_\rho^+(i, y_2, \psi) \right) \tag{22}$$

Note that first OR operation eliminates any possibility of distinguishing between two or more operands flowing from one independent buffer to another. Similarly, the second OR operation eliminates any possibility of distinguishing between two or more capabilities occurring in between a pair of (not-necessarily distinct) buffers. In contrast, this information is not lost in hetero-functional graph adjacency matrix (or tensor) A_ρ because the definition of each capability $\epsilon \in \mathcal{E}_S$ includes the associated origin and destination buffers B_S, the flowing operands L, the associated resources R and the associated process P. Therefore, the hetero-functional adjacency matrix is a more complete model of system structure when there are multiple operands and multiple capabilities between buffers.

B. Comparing Multi-layer and Hetero-functional Graphs

Despite the disparate terminology and lack of consensus in the multi-layer network literature, we adopt the multi-layer adjacency tensor ($\mathcal{A}_{MLN}$) defined by De Dominico et. al. [16]to facilitate the discussion. This fourth order tensor has elements $\mathcal{A}_{MLN}(\alpha_1, \alpha_2, \beta_1, \beta_2)$ where the indices α_1, α_2 denote "vertices" and β_1, β_2 denote "layers". De Dominico et. al write that this multilayer adjacency tensor is a [16]: *"... very general object that can be used to represent a wealth of complicated relationships among nodes."* The challenge in reconciling the multi-layer adjacency tensor $\mathcal{A}_{MLN}$ and the hetero-functional adjacency tensor $\mathcal{A}_\rho$ is an ontological one as the definition of a "layer" is not well-defined in the multi-layer network literature. The closest interpretation to hetero-functional graph theory is if $\mathcal{A}_{MLN}(\alpha_1, \alpha_2, \beta_1, \beta_2) = \mathcal{A}_{B_S}(y_1, y_2, i_1, i_2)$ where the multi-layer network's vertices are equated to the buffers B_S and the layers are equated to the operands L. This interpretation would well describe the departure of an operand l_{i_1} from buffer b_{sy_1} and arriving as l_{i_2} at b_{sy_2}. In such a case, the multi-layer adjacency matrix can be calculated from the hetero-functional incidence tensor.

$$\mathcal{A}_{B_S}(y_1, y_2, i_1, i_2) = \bigvee_{\psi} \mathcal{M}_\rho^-(i, y_1, \psi)\mathcal{M}_\rho^+(i, y_2, \psi) \quad (23)$$

Again, the OR operation eliminates any possibility of distinguishing between two or more capabilities occuring in between a pair of (not-necessarily distinct) buffers. Therefore, the hetero-functional adjacency matrix is a more complete model of system structure when there are multiple operands and multiple capabilities between buffers.

C. Layers in Hetero-functional Graphs

Definition 15 – Layer: A layer $G_\lambda = \{\mathcal{E}_{S\lambda}, Z_{S\lambda}\}$ of a hetero-functional graph $G = \{\mathcal{E}_S, Z_S\}$ is a subset of a hetero-functional graph, $G_\lambda \subseteq G$, for which a predefined layer selection (or classification) criterion applies. A set of layers in a hetero-functional graph adhere to a a classification scheme composed of a number of selection criteria. ■

Note that this definition of a layer is particularly flexible because it depends on the nature of the classification scheme and its associated selection criteria. Nevertheless, and as discussed later, it is important to choose a classification scheme that leads to a set of mutually exclusive layers that are also collectively exhaustive of the hetero-functional graph as a whole.

To select out specific subsets of capabilities (or structural degrees of freedom), HFGT has used the concept of "selector matrices" of various types [5]. Here a layer selector matrix is defined.

Definition 16: Layer Selector Matrix: A binary matrix Λ_λ of size $\sigma(P) \times \sigma(R)$ whose element $\Lambda_\lambda(w, v) = 1$ when the capability $e_{wv} \subset E_{S\lambda}$. ■

From this definition, the calculation of a hetero-functional graph layer follows straightforwardly. First, a layer projection operator $\mathbb{P}_\lambda$ is calculated.

$$\mathbb{P}_\lambda \Lambda_\lambda^V = \mathbb{1}^{\sigma(\mathcal{E}_S)} \quad (24)$$

Next, the associated negative and positive hetero-functional incidence tensors $\widetilde{\mathcal{M}}_{\rho\lambda}^-$ and $\widetilde{\mathcal{M}}_{\rho\lambda}^+$ for a given layer λ are calculated straightforwardly.

$$\widetilde{\mathcal{M}}_{\rho\lambda}^- = \sum_{i=1}^{\sigma(L)} \sum_{y_1=1}^{\sigma(B_S)} e_i^{\sigma(L)} \circ e_{y_1}^{\sigma(B_S)} \circ \mathbb{P}_\lambda \left(\left(X_{iy_1}^- \right)^V \right) \quad (25)$$

$$= \widetilde{\mathcal{M}}_\rho^- \odot_3 \mathbb{P}_\lambda \quad (26)$$

$$\widetilde{\mathcal{M}}_{\rho\lambda}^+ = \sum_{i=1}^{\sigma(L)} \sum_{y_2=1}^{\sigma(B_S)} e_i^{\sigma(L)} \circ e_{y_2}^{\sigma(B_S)} \circ \mathbb{P}_S \left(\left(X_{iy_2}^+ \right)^V \right) \quad (27)$$

$$= \widetilde{\mathcal{M}}_\rho^+ \odot_3 \mathbb{P}_\lambda \quad (28)$$

From there, the positive and negative hetero-functional incidence tensors for a given layer can be matricized and the adjacency matrix of the associated layer $\widetilde{A}_{\rho\lambda}$ follows straightforwardly.

$$\widetilde{M}_{\rho\lambda}^+ = \mathcal{F}_M(\widetilde{\mathcal{M}}_{\rho\lambda}^+, [1,2], [3]) \quad (29)$$

$$\widetilde{M}_{\rho\lambda}^- = \mathcal{F}_M(\widetilde{\mathcal{M}}_{\rho\lambda}^-, [1,2], [3]) \quad (30)$$

$$\widetilde{A}_{\rho\lambda} = \widetilde{M}_{\rho\lambda}^{+T} \widetilde{M}_{\rho\lambda}^- \quad (31)$$

This approach of separating a hetero-functional graph into its constituent layers is quite generic because the layer selector matrix Λ_λ can admit a wide variety of classification schemes including: input operand sets, output operand sets, and process type. The first of these was used in the HFGT text [5] to partition the Trimetrica test case into the multi-layer depiction in Figure 1. One advantage of a classification scheme based on ***sets*** of operands (or processes) is that they lead to the generation of a mutually exclusive and collectively exhaustive set of layers. It is worth noting that a classification scheme based on ***individual*** operands would not yield these properties. For example, a water pump consumes electricity and water as input operands. Consequently, it would have a problematic existence in both the "water layer" as well as the "electricity layer". In contrast, a classification scheme based on operand sets creates an "electricity-water" layer.

V. Conclusion

This paper provides a tensor-based formulation of several of the most important parts of hetero-functional graph theory including the system concept, the hetero-functional adjacency

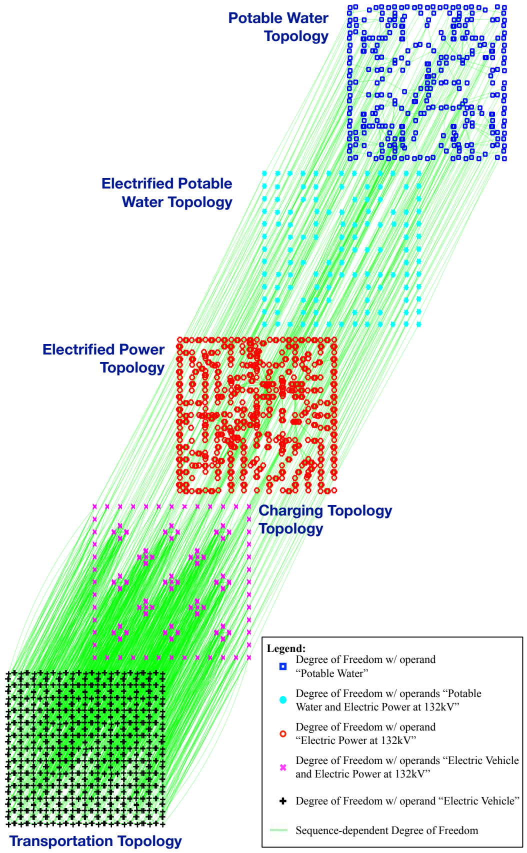

Fig. 1. The Trimetrica Smart City Infrastructure Test Case Visualized as Five Layers Defined by Input Operand Sets: The Potable Water Topology, The Electrified Potable Water Topology, the Electric Power Topology, the Charging Topology, and the Transportation Topology [5].

matrix, and introduces the hetero-functional incidence tensor for the first time. The tensor-based formulation described in this work makes a stronger tie between HFGT and its ontological foundations in MBSE. It also facilitates an understanding of the relationships between HFGT and multi-layer networks ("despite its disparate terminology and lack of consensus"). In so doing, this tensor-based treatment is likely to advance Kivela et. al's goal to discern the similarities and differences between these mathematical models in as precise a manner as possible.

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