Exponential bound of the integral of Hermite functions product with Gaussian weight

C.E. Wayne [1] and V. Zharnitsky [2]

[1] Department of Mathematics and Statistics, Boston University, Boston, MA 02215
[2] Department of Mathematics, University of Illinois, Urbana, IL 61801
Correspondence to be sent to: vzh@illinois.com

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Abstract

In this paper we derive a bound on the integral of a product of two Hermite-Gaussian functions with a Gaussian weight. We prove that such integrals decay exponentially in the difference of the indices of the Hermite-Gaussian functions. Such integrals arise naturally in mathematical physics and applied mathematics. The estimate is applied to a variational problem related to a Strichartz functional.

1 Introduction

Hermite polynomials are classical orthogonal polynomials, see e.g. [7], arising in many areas of mathematics and physics. There is also an extensive literature on a particular subject of integrals involving Hermite polynomials with some Gaussian weight, see e.g. [5], [8], [9], [10].

The goal of this note is to obtain an exponential bound on the integral of the product of two Hermite-Gaussian functions with a general Gaussian weight and to apply this estimate to a variational problem related to a Strichartz functional [3] .

Recall that Hermite polynomials can be obtained from the generating function as follows:

$$e^{2tx-t^2} = \sum_{n=0}^{\infty} H_n(x)\frac{t^n}{n!} \Rightarrow H_m(x) = \frac{d^m}{dt^m}\Big|_{t=0} e^{2tx-t^2} \ . \tag{1}$$

It is also well known that Hermite polynomials give rise to an orthonormal system of functions:

$$h_n(x) = c_n e^{-x^2/2} H_n(x) = \frac{1}{\pi^{1/4}\sqrt{2^n n!}} e^{-x^2/2} H_n(x)$$

so that

$$\int h_n(x)h_m(x)dx = \delta_{mn}. \tag{2}$$

We wish to consider integrals of products of Hermite-Gaussian functions with different Gaussian weights.

$$I^{\alpha}_{m,n} = \int e^{-(\alpha-1)x^2} h_n(x)h_m(x)dx = c_m c_n \int e^{-\alpha x^2} H_n(x)H_m(x)dx. \tag{3}$$

The case of $\alpha = 1$ just corresponds to the orthogonality relationship (2). If $\alpha = 2$, there is an estimate due to W.-M. Wang [1] proving that these integrals decay exponentially fast as $|n - m|$ grows. The proof relies on the Stirling's formula and logarithmic type inequalities. Using similar methods, Wang also proved an exponential bound for a product of four Hermite functions in [2].

We prove that a similar estimate holds for any $\alpha > 1$ and apply it to a variational problem involving a Strichartz functional. Our approach is based on the Cauchy representation theorem.

Integrals of the form (3) arise in numerous contexts. Among those are:

1. In minimizing the so-called Strichartz functional one needs to estimate sums of the type

$$\sum_{l_1+l_2+l_3=K} I^{\alpha}_{k_1,l_1} I^{\alpha}_{k_2,l_2} I^{\alpha}_{k_3,l_3}, \tag{4}$$

   where l_1, l_2, l_3 run over positive integers and $k_1, k_2, k_3, k_1+k_2+k_3 = K$ are some fixed positive integers. Numerical calculations show that such sums are decreasing with the size of K. With the exponential estimate it is possible to bound such sums for large K and for smaller values of K, the appropriate bound can be verified numerically. This was the source of our original interest in such integrals [3, 4] and we prove such an estimate in section 8.

2. In the analysis of the nonlinear quantum harmonic oscillator with time-periodic perturbation, one naturally encounters integrals of products of several Hermite functions with exponential weight, see [1].

3. Integrals similar to (3) arise in geophysics where approximations for them have been derived in various asymptotic regimes, [6].

4. This integral is equivalent to a hypergeometric function and thus the results below provide an estimate for the asymptotics of a hypergeometric function. A series of papers on the subject of integrals involving Hermite polynomials and their connection to hypergeometric functions appeared in 1940s [5], [8], [9], [10]. The major goal in these works was to find the integrals involving Hermite polynomials which reduce to a single term involving Gamma functions and to obtain the explicit formulas.

Our main result is the following estimate:

Theorem 1 *There exist positive constants $C(\alpha)$ and $\gamma(\alpha)$, independent of m and n, such that*

$$|I^{\alpha}_{m,n}| \le \frac{C(\alpha)}{\sqrt{n+m}} e^{-\gamma(\alpha)\frac{(n-m)^2}{(n+m)}} , \tag{5}$$

where $n \geq 0, m \geq 0$ and $(n,m) \neq (0,0)$. The constant $\gamma(\alpha) \to 0$ as $\alpha \to \infty$.

Remark 1 *It is interesting to note that a faster decaying Gaussian $e^{-\alpha x^2}$ leads to a weaker exponential decay. Of course, for each fixed $\alpha > 1$, the decay is still exponential and we prove in section 7 that the decay rate satisfies $\gamma(\alpha) \leq \frac{1}{2}|\ln(1-\alpha^{-1})|$.*

To prove this estimate, we first rewrite the integral in (3) with the aid of the generating functions for the Hermite polynomials, and then transform them again with the aid of Cauchy's theorem. Note first of all that integrals with n, m of different parity vanish, since the corresponding integrand would be a product of an even and an odd function. In addition, since the integral is symmetric in m and n, we can, without loss of generality, assume that $n \geq m$. Thus, for the remainder of this note we will assume that $n \geq m$ and that $n+m$ is even.

Using (1) we have

$$I^{\alpha}_{m,n} = c_n c_m \int e^{-\alpha x^2} H_n(x) H_m(x) dx = c_n c_m \int e^{-\alpha x^2} D^n_0(e^{2tx-t^2}) D^m_0(e^{2sx-s^2}) dx =$$

(by D^n_0, we denote $n-th$ derivative, evaluated at $t=0$)

$$= c_n c_m D^{n,m}_0 \int e^{-\alpha x^2} e^{2tx-t^2} e^{2sx-s^2} dx,$$

where $D^{n,m}_0$ denotes $n-th$ derivative in t, $m-th$ derivative in s, both evaluated at $t=s=0$.

Next, completing the square we have

$$I^{\alpha}_{m,n} = c_n c_m D^{n,m}_0 \int \exp\left(-\alpha(x - \frac{s+t}{\alpha})^2 - s^2 - t^2 + \frac{(s+t)^2}{\alpha}\right) dx.$$

The integral over x, can be easily evaluated since

$$\int e^{-\alpha x^2} dx = \frac{\sqrt{\pi}}{\sqrt{\alpha}}.$$

Then we have

$$I^{\alpha}_{m,n} = c_n c_m \frac{\sqrt{\pi}}{\sqrt{\alpha}} D^{n,m}_0 \exp\left(-s^2 - t^2 + \frac{(s+t)^2}{\alpha}\right). \tag{6}$$

Now, we need to Taylor expand the exponent and collect the terms with $t^n s^m$ which are the only terms that give a non-zero contribution.

2 Special case (Wang [1]): $\alpha = 2$

Consider the case $\alpha = 2$, then we have

$$I^{2}_{m,n} = c_n c_m \frac{\sqrt{\pi}}{\sqrt{2}} D^{n,m}_0 \exp\left(-\frac{1}{2}(t-s)^2\right).$$

Using the Taylor series expansion $e^y = \sum_{k=0}^{\infty} y^k/k!$, we find that only the term with $k = (m+n)/2$ matters (all other k will correspond to zero terms), and we have:

$$I^2_{m,n} = c_n c_m \frac{\sqrt{\pi}}{\sqrt{2}} D_0^{n,m} \frac{1}{\frac{n+m}{2}!} (-\frac{1}{2})^{(n+m)/2} \binom{n+m}{n} (-1)^m t^n s^m$$

Since $D_0^{n,m} t^n s^m = n!m!$

$$|I^2_{m,n}| = c_n c_m \frac{\sqrt{\pi}}{\sqrt{2}} \frac{1}{\frac{n+m}{2}!} (\frac{1}{2})^{(n+m)/2} \binom{n+m}{n} n!m! =$$

$$= \frac{1}{\sqrt{\pi}} \frac{1}{\sqrt{2^{n+m} n!m!}} \frac{\sqrt{\pi}}{\sqrt{2}} \frac{1}{2^{(n+m)/2}} \frac{(n+m)!}{\frac{n+m}{2}!} = \frac{1}{\sqrt{2}} \frac{1}{2^{(n+m)}} \frac{(n+m)!}{\sqrt{n!m!}\,(\frac{n+m}{2})!}.$$

Using Stirling's formula and a logarithmic type inequality Wang [1] obtained the following exponential bound:

$$|I^2_{m,n}| \le C \frac{1}{\sqrt{n+m}} e^{-\frac{(n-m)^2}{4(n+m)}}. \tag{7}$$

Remark 2 *When we prove the estimates for general values of α below, it will be useful to express various quantities solely in terms of n, rather than both m and n. Recalling that we are assuming that $n \ge m$, we can define $m = \nu n$, for some $\nu \in [0,1]$. Then $n \pm m = (1 \pm \nu)n$ and we can rewrite* (7) *as*

$$|I^2_{m,n}| \le \frac{\tilde{C}}{\sqrt{n}} e^{-\frac{(1-\nu)^2}{4(1+\nu)} n}. \tag{8}$$

In what follows, we will sometimes write our estimates in a form analogous to (8)*, rather than (the equivalent)* (7)*.*

3 Reexpressing $I^\alpha_{m,n}$ with the aid of Cauchy's Theorem

Consider

$$I^\alpha_{m,n} = c_n c_m \frac{\sqrt{\pi}}{\sqrt{\alpha}} D_0^{n,m} \exp\left(-t^2 - s^2 + \frac{(s+t)^2}{\alpha}\right),$$

which is nonzero only if m, n have the same parity. Calculate first

$$D_0^{n,m} \exp\left(-s^2 - t^2 + \frac{(s+t)^2}{\alpha}\right) = D_0^{n,m} \exp\left(\delta st - \beta t^2 - \beta s^2\right) \tag{9}$$

$$= D_0^{n,m} \sum_{q=0}^{\infty} \frac{(\delta st - \beta s^2 - \beta t^2)^q}{q!} \tag{10}$$

$$= \frac{1}{Q!} D_0^{n,m} (\delta st - \beta t^2 - \beta s^2)^Q\ . \tag{11}$$

where $\delta = 2/\alpha$, $\beta = 1 - \alpha^{-1}$ and $Q = (n+m)/2$ since all other terms in the sum will vanish.

When $\alpha = 2$, expanding the expression in s and t and taking the derivative in this last equality results in only a single non-zero term. For other values of α there are many non-zero terms in the derivative, making it harder to derive the desired exponential bound. As a consequence, we look for an alternative way of bounding this expression.

Note that since Q is an integer, $f(z_1, z_2) = (\delta z_1 z_2 - \beta z_1^2 - \beta z_2^2)^Q$ is an analytic function of two complex variables. Thus, we can use the Cauchy integral formula to represent the function and its derivatives:

$$f(z_1, z_2) = \frac{1}{(2\pi i)^2} \int_{C_1(r_1)} \int_{C_2(r_2)} \frac{f(\zeta_1, \zeta_2)}{(\zeta_1 - z_1)(\zeta_2 - z_2)} d\zeta_1 d\zeta_2 \ , \tag{12}$$

where $C_j(r_j)$ is a circle centered at zero of radius r_j and $|z_j| < r_j$, $j = 1, 2$. Then

$$D_0^{n,m} (\delta st - \beta t^2 - \beta s^2)^Q = \frac{n!m!}{(2\pi i)^2} \int_{C_1(r_1)} \int_{C_2(r_2)} \frac{(\delta \zeta_1 \zeta_2 - \beta \zeta_1^2 - \beta \zeta_2^2)^Q}{(\zeta_1)^{(n+1)} (\zeta_2)^{(m+1)}} d\zeta_1 d\zeta_2 \tag{13}$$

and so we have

$$I^{\alpha}_{m,n} = c_n c_m \frac{\sqrt{\pi}}{\sqrt{\alpha}} \frac{n!m!}{Q!} \frac{1}{(2\pi i)^2} \int_{C_1(r_1)} \int_{C_2(r_2)} \frac{(\delta \zeta_1 \zeta_2 - \beta \zeta_1^2 - \beta \zeta_2^2)^Q}{(\zeta_1)^{(n+1)} (\zeta_2)^{(m+1)}} d\zeta_1 d\zeta_2 \ . \tag{14}$$

Now let us choose $\zeta_1 = e^{i\theta_1}$ and $\zeta_2 = \rho e^{i\theta_2}$ and change both variables in the integral, so that

$$I^{\alpha}_{m,n} = c_n c_m \frac{\sqrt{\pi}}{\sqrt{\alpha}} \frac{n!m!}{Q!} \frac{1}{(2\pi i)^2} \int_0^{2\pi} \int_0^{2\pi} \frac{(\delta \rho e^{i(\theta_1+\theta_2)} - \beta e^{i2\theta_1} - \beta e^{2i\theta_2} \rho^2)^Q}{(e^{i\theta_1})^{(n+1)} (\rho e^{i\theta_2})^{(m+1)}} d\, e^{i\theta_1} d\, \rho e^{i\theta_2} \ = \tag{15}$$

$$(-1)^Q c_n c_m \frac{\sqrt{\pi}}{\sqrt{\alpha}} \frac{n!m!}{Q!} \frac{1}{(2\pi)^2} \int_0^{2\pi} \int_0^{2\pi} (\beta + \beta \rho^2 e^{2i(\theta_2 - \theta_1)} - \delta \rho e^{i(\theta_2-\theta_1)} \rho)^Q e^{-im(\theta_2-\theta_1)} \rho^{-m} d\theta_1 d\theta_2. \tag{16}$$

Introducing new second variable $\psi = \theta_2 - \theta_1$, keeping the first variable θ_1 the same and integrating over it, we obtain

$$I^{\alpha}_{m,n} = (-1)^Q c_n c_m \frac{\sqrt{\pi}}{\sqrt{\alpha}} \frac{n!m!}{Q!} \frac{1}{(2\pi)} \int_0^{2\pi} (\beta + \beta \rho^2 e^{2i\psi} - \delta \rho e^{i\psi} \rho)^Q e^{-im\psi} \rho^{-m} d\psi \ . \tag{17}$$

We estimate this expression in the following three sections, beginning with the product of the normalization constants and the factorials.

Remark 3 *In the integral* (14) *we could have parameterized* $C(r_1)$ *as* $\zeta_1 = r_1 e^{i\theta_1}$ *and* $C(r_2)$ *as* $\zeta_2 = r_2 e^{i\theta_2}$*, but it turns out that only the ratio* $|\zeta_1|/|\zeta_2|$ *is relevant for the estimates that follow, so to simplify things we just chose* $r_1 = 1$*.*

4 Stirling's Formula

We use Stirling's approximation in the well known form

$$\sqrt{2\pi n} \left(\frac{n}{e}\right)^n e^{\frac{1}{12n+1}} < n! < \sqrt{2\pi n} \left(\frac{n}{e}\right)^n e^{\frac{1}{12n}}, \tag{18}$$

where $n \geq 1$.

Recall that

$$c_n = \frac{1}{\pi^{1/4} 2^{n/2}} \frac{1}{\sqrt{n!}} \ . \tag{19}$$

Thus,

$$c_n c_m \frac{m!n!}{Q!} = \frac{1}{\sqrt{\pi}} \frac{1}{2^Q} \frac{\sqrt{n!m!}}{Q!} \ . \tag{20}$$

We now use Stirling's Formula to bound the quotient of factorials (first ignoring multiplicative error terms as they are bounded by constants):

$$\begin{aligned} \frac{\sqrt{n!m!}}{Q!} &= \frac{\sqrt{n!m!}}{\left(\frac{m+n}{2}\right)!} \leq \frac{9(nm)^{1/4} n^{n/2} m^{m/2} e^{-(n+m)/2}}{\sqrt{2\pi}\left(\frac{1}{2}(m+n)\right)^{\frac{1}{2}+\frac{m+n}{2}} e^{-(n+m)/2}} &(21)\\ &\leq 2^Q \frac{9}{\sqrt{2\pi}} \frac{n^{\frac{1}{4}+\frac{n}{2}} m^{\frac{1}{4}+\frac{m}{2}}}{(m+n)^{\frac{1}{2}+\frac{m+n}{2}}} = 2^Q \frac{9}{\sqrt{2\pi}} \frac{n^{\frac{1}{4}+\frac{n}{2}} (\nu n)^{\frac{1}{4}+\frac{\nu n}{2}}}{((1+\nu)n)^{\frac{1}{2}+\frac{1}{2}(1+\nu)n}} &(22)\\ &= C(\nu)\ 2^Q \frac{\nu^{\frac{\nu}{2}n}}{(1+\nu)^{\frac{1}{2}(1+\nu)n}} = C(\nu) \left(\frac{2\nu^{\nu/(1+\nu)}}{1+\nu}\right)^Q \ , &(23) \end{aligned}$$

where $C(\nu)$ is a non-negative constant that is uniformly bounded for $\nu \in [0, 1]$

Taking into account multiplicative error terms, one obtains that $C(\nu) \leq e^{1/12}$, including the special case $\nu = 0, n \neq 0$ (corresponding to $m = 0, n \neq 0$), which had to be checked separately.

Inserting this estimate into (20) gives

$$c_n c_m \frac{m!n!}{Q!} \leq \frac{C(\nu)}{\sqrt{\pi}} \left(\frac{\nu^{\nu/(1+\nu)}}{1+\nu}\right)^Q \ . \tag{24}$$

Thus, if we define

$$F(\alpha, \rho, \nu, \psi) = (\beta + \beta\rho^2 e^{2i\psi} - \delta\rho e^{i\psi}) \left(\frac{\nu^{\nu/(1+\nu)}}{1+\nu}\right) \rho^{-\frac{2\nu}{\nu+1}} \ , \tag{25}$$

we see that if $n \geq m$,

$$|I^{\alpha}_{m,n}| \leq C(\alpha) \Big| \int_{\psi=0}^{2\pi} e^{-im\psi} (F(\alpha, \rho, \nu, \psi))^Q d\psi \Big| \ , \tag{26}$$

where we can choose

$$C(\alpha) = \frac{e^{1/12}}{2\pi\sqrt{\alpha}} \, . \tag{27}$$

We now examine the behavior of the integrand in (26). The angle dependence of the magnitude of the integrand occurs solely in the factor

$$G(\alpha, \rho, \psi) = (\beta + \beta\rho^2 e^{2i\psi} - \delta\rho e^{i\psi}) \ . \tag{28}$$

Thus, we first examine how the magnitude of G depends on ψ. Note that

$$\begin{aligned} |G(\alpha,\rho,\psi)|^2 &= (\beta + \beta\rho^2\cos 2\psi - \delta\rho\cos(\psi))^2 + (\beta\rho^2\sin(2\psi) - \delta\rho\sin(\psi))^2 \\ &= ((1-\frac{1}{\alpha}) + (1-\frac{1}{\alpha})\rho^2\cos 2\psi - \frac{2}{\alpha}\rho\cos(\psi))^2 \\ &\qquad +((1-\frac{1}{\alpha})\rho^2\sin(2\psi) - \frac{2}{\alpha}\rho\sin(\psi))^2 \ . \end{aligned} \tag{29}$$

Differentiating this expression with respect to ψ gives

$$\partial_\psi |G(\alpha,\rho,\psi)|^2 = 2\beta\rho\sin(\psi)\left(-4\beta\rho\cos(\psi) + \delta\rho^2 + \delta\right) \ . \tag{30}$$

From this expression we easily have:

Lemma 1 *For fixed $\alpha > 1$ and $\rho > 0$, the modulus of $G(\alpha,\rho,\psi)$ has a critical point at $\psi = 0,\pi$ for all values of α and ρ. There may be two additional critical points that satisfy the equation*

$$\left(-4(1-\frac{1}{\alpha})\rho\cos(\psi) + \frac{2}{\alpha}\rho^2 + \frac{2}{\alpha}\right) = 0 \ , \tag{31}$$

depending on the values of α and ρ.

Differentiating (29) a second time with respect to ψ and setting $\psi = \pi$, one sees that the magnitude of G always has a local maximum at $\psi = \pi$. Evaluating (29) at $\psi = 0$ and $\psi = \pi$, one sees that one always has $|G(\alpha,\rho,0)|^2 < |G(\alpha,\rho,\pi)|^2$. Combining this with the periodicity of this function in ψ one has

Corollary 2 *The modulus of $G(a,\rho,\psi)$ has a global maximum at $\psi = \pi$ for all $\alpha > 1$ and $0 < \rho$.*

Now that we know where the maximum value of the integrand occurs, we can examine how large it is. Reconsider

$$F(\alpha,\rho,\nu,\pi) = (\beta + \beta\rho^2 + \delta\rho)\left(\frac{\nu^{\frac{\nu}{(1+\nu)}}}{1+\nu}\right)\rho^{-\frac{2\nu}{\nu+1}} \tag{32}$$

and recall that we are free to choose $\rho > 0$. Set

$$\rho = \sqrt{\nu} \ . \tag{33}$$

Then

$$F(\alpha,\sqrt{\nu},\nu,\pi) = \frac{\beta + \beta\nu + \delta\sqrt{\nu}}{\nu+1} = \frac{(1-\frac{1}{\alpha})(1+\nu) + \frac{2}{\alpha}\sqrt{\nu}}{\nu+1} \ . \tag{34}$$

Note that with this choice of ρ, we also have

$$F(\alpha,\sqrt{\nu},\nu,\psi) = (\beta + \beta\nu e^{2i\psi} - \delta\sqrt{\nu}e^{i\psi})\frac{1}{\nu+1} \tag{35}$$

We are interested in values of ν between $\nu = 1$ (which corresponds to $m = n$) and $\nu = 0$ (which corresponds to $m = 0$.) Thus, we set $\nu = 1 - \mu$, where μ serves as a measure of how different m and n are. (In particular, $n - m = \mu n$.) Define

$$\mathcal{M}(\alpha,\mu) = F(\alpha, \sqrt{1-\mu}, 1-\mu, \pi) = \frac{(1-\frac{1}{\alpha})(2-\mu) + \frac{2}{\alpha}\sqrt{1-\mu}}{2-\mu} = (1-\frac{1}{\alpha}) + \frac{2\sqrt{1-\mu}}{\alpha(2-\mu)} \ . \tag{36}$$

The following property of $\mathcal{M}$ will be useful at several points in the remainder of the argument.

Lemma 2 *The function $\mathcal{M}(\alpha,\mu)$ is a monotonic non-increasing function of μ and is strictly decreasing for $0 < \mu < 1$.*

Proof:
Differentiating $\mathcal{M}$ with respect to μ, we find:

$$\frac{\partial \mathcal{M}}{\partial \mu} = -\frac{\mu}{\alpha\sqrt{1-\mu}(\mu-2)^2} \leq 0 \ . \tag{37}$$

■

5 The cases when $n \gg m$

Note that up to this point, we have made no assumption about the relative sizes of m and n (other than that $m \leq n$.) We now specialize to consider the case when $n - m$ is $\mathcal{O}(n)$ - i.e. when μ is bounded strictly away from zero.

Using the monotonicity of $\mathcal{M}$ with respect to μ we have:

Lemma 3 *Fix $\mu_0 > 0$. Then for all $\mu_0 \leq \mu \leq 1$, and for all $\alpha > 1$,*

$$\mathcal{M}(\alpha,\mu) \leq \mathcal{M}(\alpha,\mu_0) < 1 \ . \tag{38}$$

Remark 4 *Using the fact that $\mathcal{M}$ is smooth, one can show that*

$$\mathcal{M}(\alpha,\mu_0) = 1 - \frac{\mu_0^2}{8\alpha} + O(\mu_0^4) \tag{39}$$

We will also use the upper bound

Lemma 4 *For $\alpha > 1$ and $\mu \in [0,1]$, $\mathcal{M}(\alpha,\mu) \leq 1 - \frac{\mu^2}{8\alpha}$.*

Proof:

$$\mathcal{M}(\alpha,\mu) = (1-\frac{1}{\alpha}) + \frac{\sqrt{1-\mu}}{\alpha(1-\mu/2)} \leq (1-\frac{1}{\alpha}) + \frac{1}{\alpha}\cdot\frac{1-\frac{\mu}{2}-\frac{\mu^2}{8}}{1-\frac{\mu}{2}} = (1-\frac{1}{\alpha}) + \frac{1}{\alpha}(1-\frac{\mu^2/8}{1-\mu/2}) =$$

$$= 1-\frac{1}{\alpha}\frac{\mu^2/8}{1-\mu/2} \leq 1-\frac{\mu^2}{8\alpha}.$$

We used in the first inequality $\sqrt{1-\mu} \leq 1-\frac{\mu}{2}-\frac{\mu^2}{8}$ for $\mu \in [0,1]$, which is easy to verify.

■

With this estimate we can bound $I^{\alpha}_{m,n}$ for any $m \leq (1-\mu_0)n$. From (26), and the fact that $|F(\alpha,\sqrt{\nu},\nu,\psi)| \leq \mathcal{M}(\alpha,\mu) \leq \mathcal{M}(\alpha,\mu_0)$, we have

$$|I^{\alpha}_{m,n}| \leq C(\alpha)\int_0^{2\pi} |F(\alpha,\sqrt{\nu},\nu,\psi)|^Q d\psi \tag{40}$$

$$\leq 2\pi C(\alpha)(\mathcal{M}(\alpha,\mu_0))^Q = 2\pi C(\alpha)(\mathcal{M}(\alpha,\mu_0))^{\frac{1}{2}\frac{(1+\nu)}{(1-\nu)}|n-m|}. \tag{41}$$

Note that $\frac{(1+\nu)}{(1-\nu)} = \frac{n+m}{n-m}$, so this can be rewritten as

$$|I^{\alpha}_{m,n}| \leq 2\pi C(\alpha)(\mathcal{M}(\alpha,\mu_0))^{\frac{1}{2}(\frac{1+\nu}{1-\nu})^2\frac{(n-m)^2}{(n+m)}} = 2\pi C(\alpha)(\mathcal{M}(\alpha,\mu_0))^{\frac{1}{2}(\frac{2-\mu}{\mu})^2\frac{(n-m)^2}{(n+m)}}. \tag{42}$$

From the form of $\mathcal{M}$, we obtain the bound $\mathcal{M}(\alpha,\mu_0) \leq e^{-\frac{\mu_0^2}{8\alpha}}$ so we have

$$|I^{\alpha}_{m,n}| \leq 2\pi C(\alpha)e^{-\frac{\mu_0^2}{16\alpha}(\frac{2-\mu}{\mu})^2\frac{(n-m)^2}{(n+m)}} \leq 2\pi C(\alpha)e^{-\frac{\mu_0^2}{16\alpha}\frac{(n-m)^2}{(n+m)}}. \tag{43}$$

Thus, since $\mu_0 \leq \mu \leq 1$, we get exponential decay in $(n-m)$, with the exponent of the same form as in Wang's bound for the $\alpha = 2$ case.

Remark 5 *We can rewrite the bound with an extra factor* $\frac{1}{\sqrt{m+n}}$ *by reducing the coefficient in the exponent using the inequality*

$$e^{-\frac{\mu_0^2}{32\alpha}\frac{(n-m)^2}{(n+m)}} \leq D\frac{1}{\sqrt{m+n}}.$$

To find D, we need to bound

$$\sqrt{m+n}\,e^{-\frac{\mu_0^2}{32\alpha}\frac{(n-m)^2}{(n+m)}} = \sqrt{n(2-\mu)}\,e^{-\frac{\mu_0^2}{32\alpha}\frac{n\mu^2}{2-\mu}} \leq \sqrt{2n}\,e^{-\frac{n\mu_0^4}{64\alpha}} \leq \frac{8\sqrt{\alpha}}{\mu_0^2}e^{-1/2} = D.$$

In the last inequality we used the inequality $ye^{-by^2} \leq \frac{1}{\sqrt{2b}}e^{-1/2}$, with $b > 0$, $y \geq 0$. Therefore, we can rewrite the inequality

$$|I^{\alpha}_{m,n}| \leq 2\pi C(\alpha)e^{-\frac{\mu_0^2}{32\alpha}\frac{(n-m)^2}{(n+m)}}\frac{1}{\sqrt{m+n}}\frac{8\sqrt{\alpha}}{\mu_0^2}e^{-1/2}. \tag{44}$$

and recalling that $C(\alpha) = \frac{e^{1/12}}{2\pi\sqrt{\alpha}}$*, we obtain*

$$|I^{\alpha}_{m,n}| \leq \frac{8\sqrt{\alpha}}{\mu_0^2} \frac{1}{\sqrt{m+n}} e^{-\frac{\mu_0^2}{32\alpha}\frac{(n-m)^2}{(n+m)}} . \quad (45)$$

6 The cases when $n \approx m$

In this section we discuss the bounds for $I^{\alpha}_{m,n}$ when $n \approx m$, or equivalently, when μ is small.

Returning to (26), we know that the global max of $F(\alpha, \rho, \nu, \psi)$ occurs when $\psi = \pi$, and that if there is another local max it occurs at $\psi = 0$. As in the previous section, we set $\rho = \sqrt{\nu}$, and $\nu = 1 - \mu$, and consider $F(\alpha, \sqrt{1-\mu}, 1-\mu, \psi)$.

Recall that

$$\mathcal{M}(\alpha, \mu) = F(\alpha, \sqrt{1-\mu}, 1-\mu, \pi) \quad (46)$$

satisfies

- $\mathcal{M}(\alpha, 0) = 1$, and $\mathcal{M}(\alpha, \mu)$ is a strictly decreasing function of μ for $\mu > 0$.
- By Taylor's theorem we have

  $$\mathcal{M}(\alpha, \mu) = 1 - \frac{\mu^2}{8\alpha} + \dots \quad (47)$$

  for μ small (and positive).

We are interested in the bound for $I^{\alpha}_{m,n}$ for small values of μ, In particular, assume that there exist small, positive numbers $\delta_0 = \delta_0(\alpha)$ and μ_0 such that the following hypotheses hold: We first show how these properties imply the desired bounds on $I^{\alpha}_{m,n}$ and we then prove that δ_0 and μ_0 satisfying these properties exist.

(H1) [1] For all $0 \leq \mu \leq \mu_0$,

$$1 - \frac{\mu^2}{4\alpha} \leq \mathcal{M}(\alpha, \mu) \leq 1 - \frac{\mu^2}{10\alpha} . \quad (48)$$

(H2) For all $|\psi - \pi| \leq \delta_0$, and for all $0 \leq \mu \leq \mu_0$,

$$|F(\alpha, \sqrt{1-\mu}, 1-\mu, \psi)| \leq \mathcal{M}(\alpha, \mu) - \frac{(\alpha-1)}{8\alpha}(\pi - \psi)^2 . \quad (49)$$

(H3) For all $|\psi - \pi| > \delta_0$, and for all $0 \leq \mu \leq \mu_0$,

$$|F(\alpha, \sqrt{1-\mu}, 1-\mu, \psi)| \leq \mathcal{M}(\alpha, \mu) - \frac{(\alpha-1)}{8\alpha}\delta_0^2 . \quad (50)$$

[1] By Lemma 4 we have a slightly sharper inequality $\mathcal{M}(\alpha, \mu) \leq 1 - \frac{\mu^2}{8\alpha}$ for any $\mu \in [0, 1]$

Assuming that (H1)-(H3) hold, we now bound $I^\alpha_{m,n}$. Returning to (26), we split the integral into two pieces:

$$|I^\alpha_{m,n}| \leq C(\alpha) \int_{|\pi-\psi|>\delta_0} |F(\alpha, \sqrt{1-\mu}, 1-\mu, \psi)|^Q d\psi + C(\alpha) \int_{|\pi-\psi|\leq\delta_0} |F(\alpha, \sqrt{1-\mu}, 1-\mu, \psi)|^Q d\psi \ . \tag{51}$$

Denote the two terms on the RHS of this inequality as I_A and I_B. The first of these terms is the easiest to bound, so we begin with it. We bound the integrand in this term using (H3), and see that

$$I_A \leq 2\pi C(\alpha)(\mathcal{M}(\alpha,\mu) - \frac{\alpha-1}{8\alpha}\delta_0^2)^Q = 2\pi C(\alpha)(\mathcal{M}(\alpha,\mu))^Q \left(1 - \frac{(\alpha-1)\delta_0^2}{8\alpha\mathcal{M}(\alpha,\mu)}\right)^Q \ . \tag{52}$$

The following Lemma bounds the factor of $\mathcal{M}^Q$:

Lemma 5 *For $0 \leq \delta \leq \delta_0$ and $0 \leq \mu \leq \mu_0$, one has*

$$(\mathcal{M}(\alpha,\mu))^Q \leq \exp\left(-\frac{(2-\mu)^2}{8\alpha}\frac{(n-m)^2}{(n+m)}\right) \ . \tag{53}$$

Proof:
Recalling that $Q = \frac{1}{2}(2-\mu)n$ and $n - m = \mu n$, we can bound

$$\begin{aligned} (\mathcal{M}(\alpha,\mu))^Q &= (\mathcal{M}(\alpha,\mu))^{\frac{1}{2}\frac{(2-\mu)}{\mu}(n-m)} \leq (1 - \frac{\mu^2}{10\alpha})^{\frac{1}{2}\frac{(2-\mu)}{\mu}(n-m)} \\ &\leq \exp(-\frac{\mu(2-\mu)}{20\alpha}(n-m)) = \exp(-\frac{(2-\mu)^2}{20\alpha}\frac{(n-m)^2}{(n+m)}) \ . \end{aligned} \tag{54}$$

The last inequality on the first line of this expression uses (H1), while the first inequality on the second line uses the fact that $(1-x) \leq e^{-x}$, and the last inequality uses $\mu = (2-\mu)\frac{n-m}{n+m}$.

■

We now turn to the expression $\left(1 - \frac{(\alpha-1)\delta_0^2}{8\alpha\mathcal{M}(\alpha,\mu)}\right)^Q$. Using again the fact that $(1-x) \leq e^{-x}$ and the fact that $Q = \frac{1}{2}(2-\mu)n$ gives

$$\left(1 - \frac{(\alpha-1)\delta_0^2}{8\alpha\mathcal{M}(\alpha,\mu)}\right)^Q \leq \exp\left(-\frac{(\alpha-1)\delta_0^2(2-\mu)}{16\alpha\mathcal{M}(\alpha,\mu)}n\right) \leq \exp\left(-\frac{(\alpha-1)\delta_0^2}{16\alpha}n\right) \leq \frac{C(\alpha,\delta_0)}{\sqrt{n+m}} \ . \tag{55}$$

To find an explicit bound on $C(\alpha, \delta_0)$, we estimate

$$\sqrt{n+m}\,\exp\left(-\frac{(\alpha-1)\delta_0^2}{16\alpha}n\right) = \sqrt{2-\mu}\sqrt{n}\exp\left(-\frac{(\alpha-1)\delta_0^2}{16\alpha}n\right) \leq$$

using again the inequality $ye^{-by^2} \leq \frac{1}{\sqrt{2b}} e^{-1/2}$, with $b > 0$, $y \geq 0$

$$\leq \sqrt{2} \frac{e^{-1/2}}{\sqrt{2(\alpha-1)\delta_0^2/(16\alpha)}} = \frac{1}{\delta_0} \frac{4\sqrt{\alpha}}{\sqrt{\alpha-1}} := C(\alpha, \delta_0) .$$

Note that we could, in the inequality (55), bound the expression by any inverse power of $n+m$, but we choose $(n+m)^{-1/2}$ since it will match the contribution of I_B in (51) which gives the leading order contribution to $I^{\alpha}_{m,n}$. Combining (54) with (55) implies

$$I_A \leq \frac{2\pi C(\alpha,\delta_0)}{\sqrt{n+m}} \exp\left(-\frac{(2-\mu)^2}{20\alpha} \frac{(n-m)^2}{(n+m)}\right) . \tag{56}$$

We now turn to bound I_B. In this case, we use $(H2)$ to bound the integrand obtaining

$$\begin{aligned}
I_B &\leq C(\alpha) \int_{\pi-\delta_0}^{\pi+\delta_0} \left(\mathcal{M}(\alpha,\mu) - \frac{(\alpha-1)}{8\alpha} (\pi-\psi)^2 \right)^Q d\psi \qquad (57)\\
&\leq C(\alpha)(\mathcal{M}(\alpha,\mu))^Q \int_{\pi-\delta_0}^{\pi+\delta_0} \left(1 - \frac{(\alpha-1)}{8\alpha \mathcal{M}(\alpha,\mu)} (\pi-\psi)^2 \right)^Q d\psi \\
&\leq C(\alpha)(\mathcal{M}(\alpha,\mu))^Q \int_{\pi-\delta_0}^{\pi+\delta_0} \exp\left(-\frac{(\alpha-1)Q(\pi-\psi)^2}{8\alpha\mathcal{M}(\alpha,\mu)}\right) d\psi \\
&\leq C(\alpha) \sqrt{\frac{8\pi\alpha\mathcal{M}(\alpha,\mu)}{(\alpha-1)Q}} (\mathcal{M}(\alpha,\mu))^Q
\end{aligned}$$

The last inequality in this expression extended the limits of integration from $-\infty$ to ∞ and evaluated the resulting integral. Recalling that $Q = \frac{1}{2}(n+m)$, and using the bound from Lemma 5 we have

$$I_B \leq \frac{C(\alpha,\delta_0)}{\sqrt{n+m}} \exp\left(-\frac{(2-\mu)^2}{8\alpha} \frac{(n-m)^2}{(n+m)}\right) . \tag{58}$$

Combining (56) and (58) we have

Proposition 1 *For $0 \leq \delta \leq \delta_0$ and $0 \leq \mu \leq \mu_0$, one has*

$$|I^{\alpha}_{m,n}| \leq \frac{C(\alpha,\delta_0)}{\sqrt{n+m}} \exp\left(-\frac{(2-\mu)^2}{20\alpha} \frac{(n-m)^2}{(n+m)}\right) . \tag{59}$$

Using the bound for $C(\alpha, \delta_0)$, we can write more explicit bound

$$|I^{\alpha}_{m,n}| \leq \frac{1}{\delta_0} \frac{4\sqrt{\alpha}}{\sqrt{\alpha-1}} \frac{1}{\sqrt{(n+m)}} \exp\left(-\frac{1}{20\alpha} \frac{(n-m)^2}{(n+m)} \right) . \tag{60}$$

All that remains is to verify hypotheses (H1)-(H3).

Remark 6 *It turns out that for $\alpha \in (1,\infty)$ and $m/n \in (0,1)$, there is no uniform lower bound on δ_0 as it can get arbitrarily close to zero. Also, an expression for a lower bound for δ_0 as function of α, μ appears to be rather complicated. Thus, in the section 6.2 we derive a specialized inequality for the case $\alpha_3 = 5/3$, where we can even take $\delta_0 = \pi$.*

6.1 Verifying (H1)-(H3)

Recall that

$$\mathcal{M}(\alpha,\mu) = F(\alpha,\sqrt{1-\mu},1-\mu,\pi) = (1-\frac{1}{\alpha}) + \frac{2\sqrt{1-\mu}}{\alpha(2-\mu)}\,. \tag{61}$$

Note that this is an analytic function of μ in a neighborhood of $\mu = 0$ for any $\alpha > 1$. Furthermore, by Taylor's theorem

$$\mathcal{M}(\alpha,\mu) = 1 - \frac{\mu^2}{8\alpha} + \mathcal{O}(\mu^3)\ . \tag{62}$$

The error in the quadratic Taylor polynomial can be bounded by the maximum of the third derivative times μ^3. Set $\tau_3(\alpha,\mu_1) = \max_{0\le\mu\le\mu_1} |\partial_\mu^3 \mathcal{M}(\alpha,\mu)|$. Then from the error estimates in Taylor's theorem we have

$$1 - \left(\frac{1}{8\alpha} + \tau_3(\alpha,\mu_1)\mu\right)\mu^2 \le \mathcal{M}(\alpha,\mu) \le 1 - \left(\frac{1}{8\alpha} - \tau_3(\alpha,\mu_1)\mu\right)\mu^2\ . \tag{63}$$

By choosing μ_1 sufficiently small we can insure that (H1) holds.

We now turn to (H2). Again, our tool is Taylor's theorem. If we expand $F(\alpha,\sqrt{1-\mu},1-\mu,\psi)$ with respect to ψ, around $\psi = \pi$, we have

$$\begin{aligned} F(\alpha,\sqrt{1-\mu},1-\mu,\psi) &= F(\alpha,\sqrt{1-\mu},1-\mu,\pi) + \frac{1}{2}\partial_\psi^2 F(\alpha,\sqrt{1-\mu},1-\mu,\psi)|_{\psi=\pi}(\pi-\psi)^2 + \mathcal{O}((\pi-\psi)^3) \\ &= \mathcal{M}(\alpha,\mu) + \frac{1}{2}\partial_\psi^2 F(\alpha,\sqrt{1-\mu},1-\mu,\psi)|_{\psi=\pi}(\pi-\psi)^2 + \mathcal{O}((\pi-\psi)^3)\ . \end{aligned} \tag{64}$$

As in the proof of (H1), the error in the second order Taylor approximation is bounded by the third derivative of F. Thus, we define

$$\mathcal{T}_3(\alpha,\mu,\delta_1) = \max_{|\psi-\pi|\le\delta_1} |\partial_\psi^3 F(\alpha,\sqrt{1-\mu},1-\mu,\psi)|\ . \tag{65}$$

Note that $\mathcal{T}_3$ will be a smooth function of both μ and δ_1 for each choice of α.

For ψ near π, we have $\partial_\psi^2 F(\alpha,\sqrt{1-\mu},1-\mu,\psi)|_{\psi=\pi} = -\frac{\alpha-1}{\alpha} + \mathcal{O}(\mu^2)$. Again, using the smoothness F for μ small and ψ close to π, we use the error bounds in Taylor's theorem to choose $\mu_2 > 0$ (depending on α) such that for $0 \le \mu \le \mu_2$,

$$-\frac{9}{8}\frac{\alpha-1}{\alpha} \le \partial_\psi^2 F(\alpha,\sqrt{1-\mu},1-\mu,\psi)|_{\psi=\pi} \le -\frac{6}{8}\frac{\alpha-1}{\alpha}\ . \tag{66}$$

Combining (66), with the error estimate from Taylor's theorem, we see that for $0 \le \mu \le \mu_2$, and $|\psi-\pi| \le \delta_1$, we have

$$F(\alpha,\sqrt{1-\mu},1-\mu,\psi) \le \mathcal{M}(\alpha,\mu) - \frac{3}{8}\frac{\alpha-1}{\alpha}(\psi-\pi)^2 + \frac{\delta_1}{6}\mathcal{T}_3(\alpha,\mu,\delta_1)(\psi-\pi)^2\ . \tag{67}$$

Now choose δ_1 sufficiently small that

$$\sup_{0\le\mu\le\mu_1} \delta_1 \mathcal{T}_3(\alpha,\mu,\delta_1) \le \frac{\alpha-1}{\alpha}\ , \tag{68}$$

(Again, note that δ_1 will depend on α.) Then using this bound in (67) implies the bound in (H2).

Finally, we turn to (H3). First note that from the estimate that we just proved,

$$F(\alpha, \sqrt{1-\mu}, 1-\mu, \pi \pm \delta_2) \leq \mathcal{M}(\alpha, \mu) - \frac{(\alpha-1)}{8\alpha}\delta_2^2 \ , \tag{69}$$

for all $0 \leq \delta_2 \leq \delta_1$. The goal is to find a choice of δ_2 such that this bound holds for all ψ with $|\psi - \pi| \geq \delta_2$. Since (by (H2)),

$$F(\alpha, \sqrt{1-\mu}, 1-\mu, \psi) \leq \mathcal{M}(\alpha, \mu) - \frac{\alpha-1}{8\alpha}(\psi-\pi)^2 \tag{70}$$

for $|\psi - \pi| \leq \delta_2$, if there exists ψ such that $F(\alpha, \sqrt{1-\mu}, 1-\mu, \psi) > \mathcal{M}(\alpha, \mu) - \frac{\alpha-1}{8\alpha}\delta_2^2$, then $F(\alpha, \sqrt{1-\mu}, 1-\mu, \psi)$ must have a local minimum for some ψ. From Lemma 1, we know that there are two possibilities. Either this local minimum occurs at $\psi = 0$, or there is a local minimum at some value of ψ between π and 0, and then $\psi = 0$ is a local maximum. In either case, (70) will hold if the value of

$$\mathcal{M}(\alpha, \mu) - \frac{\alpha-1}{8\alpha}\delta_2^2 > \tilde{F}(\alpha, \sqrt{1-\mu}, 1-\mu, 0) = (1 - \frac{1}{\alpha}) - \frac{2\sqrt{1-\mu}}{\alpha(2-\mu)} \ . \tag{71}$$

Recalling the value of $\mathcal{M}(\alpha, \mu)$, we see that (71) will hold if $\frac{\alpha-1}{8\alpha}\delta_2^2 < \frac{4\sqrt{1-\mu}}{\alpha(2-\mu)}$, i.e if

$$\delta_2^2 < \frac{32\sqrt{1-\mu}}{(2-\mu)(\alpha-1)} \tag{72}$$

then (H3) holds. If $\mu \leq 1/4$, then (72) will hold provided

$$\delta_2 < \frac{2\sqrt{2}}{(\alpha-1)} \ . \tag{73}$$

Choosing $\delta_0 = \min(\delta_1, \delta_2)$ then insures that $(H1) - (H3)$ hold.

6.2 Special case corresponding to dimension 3: $(\alpha_3 = 5/3)$

To obtain explicit and more optimal estimates, in this section we fix the value of $\alpha = \alpha_3 = 5/3$ corresponding to dimension $d = 3$ in the Strichartz functional.

Consider again

$$F(\alpha, \rho, \nu, \psi) = (\beta + \beta\rho^2 e^{2i\psi} - \delta\rho e^{i\psi})\frac{1}{1+\nu} \tag{74}$$

and using $\alpha = \alpha_3 = 5/3, \beta = 1 - \alpha_3^{-1} = 0.4, \delta = 2/\alpha = 1.2$ and the relation $\nu = \rho^2$,

$$F(\alpha_3, \rho, \rho^2, \psi) = (0.4 + 0.4\rho^2 e^{2i\psi} - 1.2\rho e^{i\psi})\frac{1}{1+\rho^2} = 0.4(1 + \rho^2 e^{2i\psi} - 3\rho e^{i\psi})\frac{1}{1+\rho^2}. \tag{75}$$

For convenience we change $\psi = \pi + \xi$, so that with some abuse of notation we have

$$F(\alpha_3, \rho, \rho^2, \xi) = 0.4(1 + \rho^2 e^{2i\xi} + 3\rho e^{i\xi})\frac{1}{1+\rho^2}. \tag{76}$$

Now, we estimate

$$|F(\alpha_3, \rho, \rho^2, \xi)| \le \frac{0.4}{1+\rho^2}(1 + \rho|3 + \rho e^{i\xi}|) = \frac{0.4}{1+\rho^2}(1 + 3\rho|1 + \frac{\rho}{3}e^{i\xi}|) \tag{77}$$

using two lemmas.

Lemma 6 *On the interval $\xi \in [-\pi, \pi]$, and for any $\sigma \in [0,1]$*

$$|1 + \sigma e^{i\xi}| \le 1 + \sigma - \frac{\sigma}{5(\sigma+1)}\xi^2 .$$

Proof:
Rewriting in trigonometric form

$$|1 + \sigma e^{i\xi}| = \sqrt{(1 + \sigma\cos\xi)^2 + \sigma^2 \sin^2\xi} = \sqrt{1 + \sigma^2 + 2\sigma\cos\xi} \le$$

and using the inequality (see the proof below)

$$\cos\xi \le 1 - \frac{\xi^2}{5},$$

on the interval $\xi \in [-\pi, \pi]$ we have

$$\le \sqrt{1 + \sigma^2 + 2\sigma(1 - \frac{\xi^2}{5})} = (1+\sigma)\sqrt{1 - \frac{2\sigma}{5(1+\sigma)^2}\xi^2} \le (1+\sigma)(1 - \frac{\sigma}{5(1+\sigma)^2}) = 1 + \sigma - \frac{\sigma}{5(1+\sigma)}\xi^2,$$

where we used the inequality $\sqrt{1-y} \le 1 - y/2$, for $y \in [0,1]$.

■

Lemma 7 *On the interval $\xi \in [-\pi, \pi]$, one has $\cos\xi \le 1 - \frac{\xi^2}{5}$.*

Proof:
Consider an auxilary function

$$f(\xi) = \cos(\xi) - (1 - \frac{\xi^2}{5}) .$$

Note that $f(0) = 0$, $f(\pi) < 0$, $f(2\pi) > 0$.

Next consider $f'(\xi) = -\sin(\xi) + \frac{2\xi}{5}$ and observe that $f'(\xi)$ has a unique zero on the interval $(0, 2\pi)$ – denote it by ξ_0. It is easy to see that $f'(\xi) < 0$ for $\xi \in (0, \xi_0)$, $f'(\xi) > 0$ for $\xi \in (\xi_0, 2\pi)$. Thus, $f(\xi)$ is decreasing for $\xi \in (0, \xi_0)$ and increasing for $\xi \in (\xi_0, 2\pi)$ and thus has at most one zero in the interval $(0, 2\pi)$. By the intermediate value theorem, $f(\xi)$ must have a zero in the interval $(\pi, 2\pi)$, and hence it has no zero in the interval $(0, \pi)$. Thus, $f(\xi) \le 0$ for all $\xi \in [0, \pi]$, or equivalently,

$$\cos(\xi) \le (1 - \frac{\xi^2}{5}) . \tag{78}$$

■

Applying the above lemmas to equation (77), we obtain

$$|F(\alpha_3, \rho, \rho^2, \xi)| \leq \frac{0.4}{1+\rho^2}(1 + 3\rho[1 + \frac{\rho}{3} - \frac{\rho/3}{5(1+\rho/3)}\xi^2]) = \\ 0.4(1 + \frac{3\rho}{1+\rho^2}) - \frac{0.4\rho^2}{5(1+\rho^2)(1+\rho/3)}\xi^2 \,. \tag{79}$$

Now, recall that $\rho = \sqrt{\nu} = \sqrt{1-\mu}$ and estimate

$$|F(\alpha_3, \sqrt{1-\mu}, 1-\mu, \xi)| \leq 0.4(1 + \frac{3\sqrt{1-\mu}}{2-\mu}) - \frac{0.4(1-\mu)}{5(2-\mu)(1+\sqrt{1-\mu}/3)}\xi^2. \tag{80}$$

By Lemma 4 the first term on the right hand-side is $\mathcal{M}(\alpha_3, \mu)$, bounded by

$$\mathcal{M}(\alpha_3, \mu) = (1 - \frac{1}{\alpha_3}) + \frac{\sqrt{1-\mu}}{\alpha_3(1-\mu/2)} \leq 1 - \frac{\mu^2}{8\alpha_3} = 1 - 0.075\mu^2 \tag{81}$$

and with the assumption $\mu \leq \mu_0 < 1$ we bound the second term from below (since it is negative)

$$\frac{0.4(1-\mu)}{5(2-\mu)(1+\sqrt{1-\mu}/3)} \geq \frac{0.4(1-\mu_0)}{5 \cdot 2(1+1/3)} = 0.024(1-\mu_0)\,. \tag{82}$$

Therefore the final estimate holds with the assumption $\mu \leq \mu_0$ and no restriction on ψ

$$|F(\alpha_3, \sqrt{1-\mu}, 1-\mu, \psi)| \leq 1 - 0.075\mu^2 - 0.024(1-\mu_0)(\psi - \pi)^2\,. \tag{83}$$

Now, we will estimate the second integral I_B in (51). Note that with $\delta_0 = \pi$, we do not have the first integral I_A

$$|I^{\alpha_3}_{m,n}| \leq C(\alpha) \int_{|\pi-\psi|\leq\pi} |F(\alpha, \sqrt{1-\mu}, 1-\mu, \psi)|^Q d\psi \;. \tag{84}$$

In principle, we could directly estimate this integral with the given bound on F, but it is easier to use the estimate from Lemma 5

$$(\mathcal{M}(\alpha_3, \mu))^Q \leq \exp\left(-\frac{(2-\mu)^2}{8\alpha_3}\frac{(n-m)^2}{(n+m)}\right) \;. \tag{85}$$

Then we have

$$
\begin{aligned}
|I^{\alpha_3}_{m,n}| &\leq C(\alpha_3)\int_0^{2\pi}|\mathcal{M}(\alpha_3,\mu)-0.024(1-\mu_0)(\pi-\psi)^2|^Q d\psi \\
&= C(\alpha_3)\mathcal{M}(\alpha_3,\mu)^Q\int_0^{2\pi}|1-\frac{0.024(1-\mu_0)}{\mathcal{M}(\alpha_3,\mu)}(\pi-\psi)^2|^Q d\psi \\
&\leq C(\alpha_3)\mathcal{M}(\alpha_3,\mu)^Q\int_0^{2\pi}\exp\left(-\frac{0.024(1-\mu_0)}{\mathcal{M}(\alpha_3,\mu)}(\pi-\psi)^2 Q\right)d\psi \\
&\leq C(\alpha_3)\mathcal{M}(\alpha_3,\mu)^Q\sqrt{\frac{\pi\mathcal{M}(\alpha_3,\mu)}{0.024(1-\mu_0)Q}} \\
&\leq C(\alpha_3)\sqrt{\frac{\pi\mathcal{M}(\alpha_3,\mu)}{0.024(1-\mu_0)Q}}\exp\left(-\frac{(2-\mu)^2}{8\alpha_3}\frac{(n-m)^2}{(n+m)}\right). \qquad (86)
\end{aligned}
$$

Using that $\mathcal{M}\leq 1$, $8\alpha_3=0.075$, $2-\mu\geq 1$, $Q=\frac{n+m}{2}$ we obtain that for $\mu\leq\mu_0$

$$
|I^{\alpha_3}_{m,n}| \leq C(\alpha_3)\sqrt{\frac{2\pi}{0.024(1-\mu_0)(m+n)}}\exp\left(-0.075\frac{(n-m)^2}{(n+m)}\right). \qquad (87)
$$

The final step is to combine the estimates for $\mu\geq\mu_0$, given by (45) and for $\mu\leq\mu_0$, given by the above equation. We fix $\mu_0=1/2$, and then taking the maximum for the factors multiplying the exponent and taking the minimum of the factors in the exponent ($\frac{\mu_0^2}{32\alpha_3}=0.01875$ and 0.075), we obtain

$$
|I^{\alpha_3}_{m,n}| \leq \frac{42}{\sqrt{m+n}}\exp\left(-0.01875\frac{(n-m)^2}{(n+m)}\right). \qquad (88)
$$

7 Estimating the integral from below for the case $m=0$

In this final section we prove the observation made in Remark 1 that the exponential decay rate in Theorem 1 tends to zero as α tends to infinity.

Setting $m=0$, we have from (6)

$$
\begin{aligned}
I^{\alpha}_{0,n} &= c_n c_0\frac{\sqrt{\pi}}{\sqrt{\alpha}}D_0^{n,0}\exp\left(-t^2+\frac{t^2}{\alpha}\right)= \\
&= \frac{1}{\sqrt{\pi}}\frac{1}{2^{n/2}\sqrt{n!}}\frac{\sqrt{\pi}}{\sqrt{\alpha}}D_0^{n,0}\exp\left((\alpha^{-1}-1)t^2\right)=\frac{1}{2^{n/2}\sqrt{n!}}\frac{1}{\sqrt{\alpha}}D_0^{n,0}\frac{((\alpha^{-1}-1)t^2)^{n/2}}{(n/2)!}= \\
&= \frac{1}{2^{n/2}\sqrt{n!}}\frac{1}{\sqrt{\alpha}}\frac{((\alpha^{-1}-1))^{n/2}n!}{(n/2)!}=\frac{\sqrt{n!}}{2^{n/2}(n/2)!}\frac{(\alpha^{-1}-1)^{n/2}}{\sqrt{\alpha}}.
\end{aligned}
$$

Using Stirling's formula

$$n! = \sqrt{2\pi n}\left(\frac{n}{e}\right)^n e^{\lambda_n}, \frac{1}{12n+1} \leq \lambda_n \leq \frac{1}{12n}, n \geq 1$$

to evaluate $\dfrac{\sqrt{n!}}{2^{n/2}(n/2)!}$, we have

$$|I_{0,n}^\alpha| \geq c\frac{|\alpha^{-1}-1|^{n/2}}{\sqrt{\alpha}\,\sqrt[4]{n}},$$

where $c > 0$ and independent of α, n.

To read off the exponential decay, which comes from $|\alpha^{-1}-1|^{n/2}$, we write

$$|\alpha^{-1}-1|^{n/2} = \exp(n\ln(1-\alpha^{-1})/2).$$

This immediately implies that in the Wang's asymptotics the upper bound for

$$\gamma(\alpha) \leq \frac{|\ln(1-\alpha^{-1})|}{2}. \tag{89}$$

8 Gaussian is a local minimizer in Strichartz functional in $\mathbb{R}^3$

8.1 Computation of the Hessian

In [3], the restricted Hessian of the Strichartz functional in $\mathbb{R}^d$, evaluated at the Gaussian, was calculated in arbitrary dimension $d \geq 1$. It has already been proven using a special structure of the Strichartz functional in low dimensions $d = 1, 2$ that the Gaussian is a global minimizer. The higher dimensional case is still open. The following inequality was derived in [3] which would imply that the Gaussian is a local minimizer in dimension d.

$$\sum_{|l|=|k|,k\neq 0} |I^-(k,l,q)| \leq \frac{2}{q} I^-(0,0,q), \tag{90}$$

where $q = 2 + (4/d)$ or equivalently (using the definition of I^- in [3])

$$\sum_{|l|=|k|,k\neq 0}\prod_{j=1}^d c_{k_j}c_{l_j}\left|\int e^{-qx^2/2}H_{k_j}(x)H_{l_j}(x)dx\right| \leq \frac{2}{q}c_0^{2d}\left(\int e^{-qx^2/2}dx\right)^d, \tag{91}$$

where multi-index $k = (k_1, k_2, ..., k_d)$ is fixed and $k \neq 0$. Some components of k can be zeros, e.g. $k = (1, 0, 0, ...)$.

Recall again (3)

$$I_{mn}^\alpha = c_n c_m \int e^{-\alpha x^2} H_m(x)H_n(x)dx$$

and the exponential estimate written in a slightly modified form

$$|I_{mn}^\alpha| \leq C(\alpha)\frac{1}{\sqrt{\langle m+n\rangle}}e^{-\gamma(\alpha)\frac{(m-n)^2}{(m+n)}}, \tag{92}$$

where $\langle n\rangle = \max(n,1)$. We will use the convention that if $m = n = 0$ in the exponent, then it is equal to 1.

We will estimate the convolution sum (91) for $d = 3$, which corresponds to the case $\alpha_3 = q/2 = 1 + 2/3$

$$\sum_{l_1+l_2+l_3=k_1+k_2+k_3} I_{k_1l_1} I_{k_2l_2} I_{k_3l_3} \tag{93}$$

where $k_1 + k_2 + k_3 = K > 0$ are all fixed and summation is taken over $l_1, l_2, l_3 \geq 0$. The estimate will prove the inequality for sufficiently large K. For the lower values of K, we have checked the inequality by a computer program, see [3], though there remains a gap between the values of K checked in this reference, and the value of K for which we establish (91) - see Subsection 8.3, below.

We prove

Theorem 3 *For the case $d = 3$, we have*

$$\sum_{l_1+l_2+l_3=K} I_{k_1l_1} I_{k_2l_2} I_{k_3l_3} \leq \frac{3^{1/4} C(\alpha_3)^3}{K^{1/4}} \left(1 + \frac{3}{\sqrt{\gamma(\alpha_3)}} + \frac{6}{\gamma(\alpha_3)}\right)\left(1 + \frac{3}{\sqrt{\gamma(\alpha_3)}} + \frac{3}{\gamma(\alpha_3)}\right).$$

The theorem, implies that for sufficiently large $K = k_1+k_2+k_3$, the convolution sum is small enough to satisfy the inequality, (90) - one can then, in principle, check smaller values of k numerically, as described in [3].

8.2 Estimating the triple convolution

In this section we fix $d = 3$ which corresponds to $\alpha = 5/3$ and we omit this superscript below. We also omit writing $\gamma(5/3)$ and $C(5/3)$, writing instead γ, C. We will use the bound from Theorem 1, to obtain a bound on the left hand-side of (91) with $d = 3$, $k = (k_1, k_2, k_3)$

$$\sum_{l_1+l_2+l_3=k_1+k_2+k_3} I_{k_1l_1} I_{k_2l_2} I_{k_3l_3}, \tag{94}$$

where all $k_i, l_i \geq 0$, $k_1 + k_2 + k_3 = K > 0$ are all fixed and the summation is taken over $l_1, l_2, l_3 \geq 0$.

Because of the inequality $\langle m\rangle^{1/4}\langle n\rangle^{1/4} \leq \sqrt{\langle m+n\rangle}$, $m, n \in \mathbb{N}$, it is sufficient to estimate

$$\sum_{l_1+l_2+l_3=K} I_{k_1l_1} I_{k_2l_2} I_{k_3l_3} \leq \sum_{l_1+l_2+l_3=K} \frac{C}{\sqrt[4]{\langle l_1\rangle\langle k_1\rangle}} e^{-\gamma\frac{(k_1-l_1)^2}{(k_1+l_1)}} \cdot \frac{C}{\sqrt[4]{\langle l_2\rangle\langle k_2\rangle}} e^{-\gamma\frac{(k_2-l_2)^2}{(k_2+l_2)}} \cdot \frac{C}{\sqrt[4]{\langle l_3\rangle\langle k_3\rangle}} e^{-\gamma\frac{(k_3-l_3)^2}{(k_3+l_3)}}. \tag{95}$$

We can rewrite this sum as:

$$\frac{C^3}{\sqrt[4]{\langle k_1\rangle\langle k_2\rangle\langle k_3\rangle}} \sum_{l_3=0}^{K} \frac{1}{\sqrt[4]{\langle l_3\rangle}} e^{-\gamma\frac{(k_3-l_3)^2}{(k_3+l_3)}} \sum_{l_1+l_2=K-l_3} \frac{1}{\sqrt[4]{\langle l_1\rangle}} e^{-\gamma\frac{(k_1-l_1)^2}{(k_1+l_1)}} \cdot \frac{1}{\sqrt[4]{\langle l_2\rangle}} e^{-\gamma\frac{(k_2-l_2)^2}{(k_2+l_2)}}. \tag{96}$$

If we bound the second sum by the Cauchy-Schwartz inequality, this quantity is bounded above by:

$$\frac{C^3}{\sqrt[4]{\langle k_1\rangle\langle k_2\rangle\langle k_3\rangle}}\sum_{l_3=0}^{K}\frac{1}{\sqrt[4]{\langle l_3\rangle}}e^{-\gamma\frac{(k_3-l_3)^2}{(k_3+l_3)}}\left[\sum_{l_1=0}^{\infty}\left(\frac{1}{\sqrt[4]{\langle l_1\rangle}}e^{-\gamma\frac{(k_1-l_1)^2}{(k_1+l_1)}}\right)^2\sum_{l_2=0}^{\infty}\left(\frac{1}{\sqrt[4]{\langle l_2\rangle}}e^{-\gamma\frac{(k_2-l_2)^2}{(k_2+l_2)}}\right)^2\right]^{1/2}. \tag{97}$$

Remark 7 *Note that* (95) *is symmetric in* k_1, k_2, *and* k_3. *Thus, possibly by relabeling the indices, we can assume without loss of generality that* $k_3 = \min(k_1, k_2, k_3)$.

These expressions are controlled with the aid of the following lemma.

Lemma 8

$$S_k = \sum_{l=0}^{\infty}\frac{1}{\sqrt{\langle l\rangle}}e^{-2\gamma\frac{(k-l)^2}{k+l}} \le 1 + \frac{3}{\sqrt{\gamma}} + \frac{3}{\gamma}. \tag{98}$$

Proof:
Note that if we change variables to $m = \ell - k$, we have

$$\begin{aligned} S_k = \sum_{m=-k}^{\infty}\frac{1}{\sqrt{\langle m+k\rangle}}e^{-2\gamma\frac{m^2}{m+2k}} &= \sum_{m=-k}^{-1}\frac{1}{\sqrt{\langle m+k\rangle}}e^{-2\gamma\frac{m^2}{m+2k}} \\ &\quad+\frac{1}{\sqrt{\langle k\rangle}} + \sum_{m=1}^{\infty}\frac{1}{\sqrt{\langle m+k\rangle}}e^{-2\gamma\frac{m^2}{m+2k}}. \end{aligned} \tag{99}$$

We first bound the infinite sum by noting that $f(x) = e^{-2\gamma\frac{x^2}{x+2k}}$ is a positive, monotonic, non-increasing function so that

$$\sum_{m=1}^{\infty}\frac{1}{\sqrt{\langle m+k\rangle}}e^{-2\gamma\frac{m^2}{m+2k}} \le \frac{1}{\sqrt{\langle k\rangle}}\sum_{m=1}^{\infty}e^{-2\gamma\frac{m^2}{m+2k}} \le \frac{1}{\sqrt{\langle k\rangle}}\int_0^{\infty}e^{-2\gamma\frac{x^2}{x+2k}}dx \tag{100}$$

$$= \frac{1}{\sqrt{\langle k\rangle}}\int_0^{k}e^{-2\gamma\frac{x^2}{x+2k}}dx + \frac{1}{\sqrt{\langle k\rangle}}\int_k^{\infty}e^{-2\gamma\frac{x^2}{x+2k}}dx. \tag{101}$$

We bound the first of the two integrals in (101) by noting that for $x \in [0, k]$, $\frac{1}{x+2k} \ge \frac{1}{3k}$, so

$$\frac{1}{\sqrt{\langle k\rangle}}\int_0^{k}e^{-2\gamma\frac{x^2}{x+2k}}dx \le \frac{1}{\sqrt{\langle k\rangle}}\int_0^{k}e^{-\frac{2\gamma}{3k}x^2}dx \le \frac{1}{\sqrt{\langle k\rangle}}\sqrt{\frac{3\pi}{8}}\sqrt{\frac{k}{\gamma}} \le \sqrt{\frac{3\pi}{8\gamma}}. \tag{102}$$

The second integral in (101) is bounded by noting that for $x \ge k$, $\frac{x}{x+2k} \ge \frac{1}{3}$, so

$$\frac{1}{\sqrt{\langle k\rangle}}\int_k^{\infty}e^{-2\gamma\frac{x^2}{x+2k}}dx \le \frac{1}{\sqrt{\langle k\rangle}}\int_k^{\infty}e^{-\frac{2\gamma}{3}x}dx \le \frac{3}{2\gamma}. \tag{103}$$

We treat the remaining sum on the right hand side of (99) in a similar fashion. Begin by rewriting and splitting the sum as

$$\begin{aligned}\sum_{m=-k}^{-1} \frac{1}{\sqrt{\langle m+k\rangle}} e^{-2\gamma \frac{m^2}{m+2k}} &= \sum_{n=1}^{k} \frac{1}{\sqrt{\langle k-n\rangle}} e^{-2\gamma \frac{n^2}{2k-n}} \\ &= \sum_{n=1}^{[k/2]} \frac{1}{\sqrt{\langle k-n\rangle}} e^{-2\gamma \frac{n^2}{2k-n}} + \sum_{n=[k/2]+1}^{k} \frac{1}{\sqrt{\langle k-n\rangle}} e^{-2\gamma \frac{n^2}{2k-n}} \ .\end{aligned} \tag{104}$$

Here, $[k/2]$ denotes the integer part of $k/2$ and we note that if $k = 1$ the first sum on the right hand side of this last equation may have no terms, in which case its value is set equal to zero. The first term on the right hand side of (104) is bounded by

$$\begin{aligned}\sum_{n=1}^{[k/2]} \frac{1}{\sqrt{\langle k-n\rangle}} e^{-2\gamma \frac{n^2}{2k-n}} &\leq \frac{1}{\sqrt{\langle k/2\rangle}} \sum_{n=1}^{[k/2]} e^{-\frac{\gamma}{k} n^2} \leq \frac{1}{\sqrt{\langle k/2\rangle}} \int_0^\infty e^{-\frac{\gamma}{k} x^2} dx \\ &= \frac{1}{\sqrt{\langle k/2\rangle}} \sqrt{\frac{k\pi}{4\gamma}} \leq \sqrt{\frac{\pi}{2\gamma}} .\end{aligned} \tag{105}$$

The remaining sum in (104) is bounded by noting that for n between $[k/2] + 1$ and k, $\frac{n}{2k-n} \geq \frac{1}{3}$, so that

$$\sum_{n=[k/2]+1}^{k} \frac{1}{\sqrt{\langle k-n\rangle}} e^{-2\gamma \frac{n^2}{2k-n}} \leq \sum_{n=[k/2]+1}^{k} e^{-\frac{2\gamma}{3} n} \leq \frac{e^{-\frac{2\gamma}{3}}}{1-e^{-\frac{2\gamma}{3}}} = \frac{1}{e^{\frac{2\gamma}{3}}-1} \leq \frac{3}{2\gamma} \ . \tag{106}$$

Combining the estimates in (102), (103), (105), and (106), with (99) we have

$$S_k \leq \frac{1}{\sqrt{\langle k\rangle}} + \frac{3}{2\gamma} + \sqrt{\frac{\pi}{2\gamma}} + \frac{3}{2\gamma} + \sqrt{\frac{3\pi}{8\gamma}} \ , \tag{107}$$

from which the bound in Lemma 8 follows.

■

Using similar means, we now estimate the first sum in (97) .

Lemma 9 *For any $k \geq 0$,*

$$T_k = \sum_{l=0}^{\infty} \frac{1}{\langle l\rangle^{1/4}} e^{-\gamma \frac{(k-l)^2}{k+l}} \leq 1 + \frac{3\langle k\rangle^{1/4}}{\sqrt{\gamma}} + \frac{6}{\gamma}.$$

Proof:

Begin as in the previous lemma by rewriting the sum as

$$T_k = \sum_{m=-k}^{\infty} \frac{1}{\langle m+k\rangle^{1/4}} e^{-\gamma\frac{m^2}{2k+m}} = \sum_{m=-k}^{-1} \frac{1}{\langle m+k\rangle^{1/4}} e^{-\gamma\frac{m^2}{2k+m}} + \frac{1}{\langle k\rangle^{1/4}} + \sum_{m=1}^{\infty} \frac{1}{\langle m+k\rangle^{1/4}} e^{-\gamma\frac{m^2}{2k+m}} . \tag{108}$$

Note that if we first consider the infinite sum,

$$\sum_{m=1}^{\infty} \frac{1}{\langle m+k\rangle^{1/4}} e^{-\gamma\frac{m^2}{2k+m}} \le \frac{1}{\langle k\rangle^{1/4}} \sum_{m=1}^{\infty} e^{-\gamma\frac{m^2}{2k+m}} , \tag{109}$$

the sum that appears on the right hand side of this inequality is exactly the same as the sum in the first inequality in (100), except that 2γ is replaced by γ. Thus, using the same bounds that were derived there (but replacing $\gamma \to \gamma/2$) we find

$$\sum_{m=1}^{\infty} \frac{1}{\langle m+k\rangle^{1/4}} e^{-\gamma\frac{m^2}{2k+m}} \le \frac{1}{\langle k\rangle^{1/4}} \left(\sqrt{\frac{3\pi}{8}} \sqrt{\frac{2k}{\gamma}} + \frac{3}{\gamma} \right) . \tag{110}$$

Similarly, we write

$$\begin{aligned} \sum_{m=-k}^{-1} \frac{1}{\langle m+k\rangle^{1/4}} e^{-\gamma\frac{m^2}{2k+m}} &= \sum_{n=1}^{k} \frac{1}{\langle k-n\rangle^{1/4}} e^{-\gamma\frac{n^2}{2k-n}} && (111) \\ &= \sum_{n=1}^{[k/2]} \frac{1}{\langle k-n\rangle^{1/4}} e^{-\gamma\frac{n^2}{2k-n}} + \sum_{n=[k/2]+1}^{k} \frac{1}{\langle k-n\rangle^{1/4}} e^{-\gamma\frac{n^2}{2k-n}} \\ &\le \frac{1}{\langle k/2\rangle^{1/4}} \sum_{n=1}^{[k/2]} e^{-\gamma\frac{n^2}{2k-n}} + \sum_{n=[k/2]+1}^{k} e^{-\gamma\frac{n^2}{2k-n}} . && (112) \end{aligned}$$

Each of these last two sums are bounded exactly as were the sums in (105) and (106), again with the replacement $\gamma \to \gamma/2$, yielding

$$\frac{1}{\langle k/2\rangle^{1/4}} \sum_{n=1}^{[k/2]} e^{-\gamma\frac{n^2}{2k-m}} \le \frac{1}{\langle k/2\rangle^{1/4}} \sqrt{\frac{k\pi}{2\gamma}} \tag{113}$$

and

$$\sum_{n=[k/2]+1}^{k} e^{-\gamma\frac{n^2}{2k-m}} \le \frac{3}{\gamma} . \tag{114}$$

Combining these estimates we have

$$T_k \le \frac{1}{\langle k\rangle^{1/4}} + \langle k\rangle^{1/4} \sqrt{\frac{3\pi}{8}} \sqrt{\frac{2}{\gamma}} + \frac{3}{\gamma} + \frac{1}{\langle k/2\rangle^{1/4}} \sqrt{\frac{k\pi}{2\gamma}} + \frac{3}{\gamma} \le 1 + \frac{3\langle k\rangle^{1/4}}{\sqrt{\gamma}} + \frac{6}{\gamma} , \tag{115}$$

and the lemma follows.

■

Proof of theorem 3

Combining the estimates from both lemmas in (97), we obtain

$$\sum_{l_1+l_2+l_3=K} I_{k_1l_1}I_{k_2l_2}I_{k_3l_3} \le \frac{C^3}{\sqrt[4]{\langle k_1\rangle\langle k_2\rangle\langle k_3\rangle}}\left(1+\frac{3\langle k_3\rangle^{1/4}}{\sqrt{\gamma}}+\frac{6}{\gamma}\right)\left(1+\frac{3}{\sqrt{\gamma}}+\frac{3}{\gamma}\right). \tag{116}$$

Next, using $1 \le \langle k_3\rangle$, recalling that $k_3 = \min(k_1, k_2, k_3)$, and observing that $\max\{k_1, k_2, k_3\} \ge K/3$, we obtain the stated bound.

■

8.3 Numerical bound on the convolution

Now, we substitute the numbers from the exponential inequality (88) in the triple convolution bound obtained from Theorem 3: $C = 42, \gamma = 0.01875$

$$\sum_{l_1+l_2+l_3=K} I_{k_1l_1}I_{k_2l_2}I_{k_3l_3} \le \frac{3^{1/4}C(\alpha_3)^3}{K^{1/4}}\left(1+\frac{3}{\sqrt{\gamma(\alpha_3)}}+\frac{6}{\gamma(\alpha_3)}\right)\left(1+\frac{3}{\sqrt{\gamma(\alpha_3)}}+\frac{3}{\gamma(\alpha_3)}\right) \tag{117}$$

$$\le \frac{3^{1/4}42^3}{K^{1/4}}\cdot 350\cdot 200 \le \frac{10^{10}}{K^{1/4}}. \tag{118}$$

This shows that the inequality (91) for $K = k_1 + k_2 + k_3$ of the order of 10^{40} and larger.

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