SELF-REPLICATING 3-MANIFOLDS

RYAN BLAIR AND RICKY LEE

ABSTRACT. In this paper we explore the topological properties of self-replicating, 3-dimensional manifolds, which are modeled by idempotents in the (2+1)-cobordism category. We give a classification theorem for all such idempotents. Additionally, we characterize biologically interesting ways in which self-replicating 3-manifolds can embed in $\mathbb{R}^3$.

1. INTRODUCTION

The motivation of this paper is the study of self-replicating 3-manifolds. Intuitively, a 3-manifold M is self-replicating if it contains a surface F such that cutting along F yields two components, each homeomorphic to M. Kauffman previously suggested that the natural model for general self-replication is that of idempotents in appropriately topological categories [6]. Previous classification of idempotents in topological categories include the Temperley-Lieb category [1] and the tangle category [2]. In this paper we model self-replicating 3-manifolds as idempotents in the $(2+1)$-cobordism category and classify all such morphisms. We also explore questions of the embedability of self-replicating 3-manifolds in $\mathbb{R}^3$.

An *idempotent* of a category is a morphism that is idempotent with respect to composition, i.e. a morphism M such that $M = M \circ M$. Idempotents have applications to the theory of quantum observation [8], self-replication in biology [6], and algebraic structures associated to quantum theory [4]. An idempotent M *splits* if there are morphisms P and Q such that $M = P \circ Q$ and $Q \circ P$ is an identity morphism. Any morphism M that splits is idempotent (if $Q \circ P$ is an identity, then $(P \circ Q) \circ (P \circ Q) = P \circ (Q \circ P) \circ Q = P \circ Q$). However, in many categories, not all idempotents split. Let $\mathcal{C}$ denote the $(2+1)$-cobordism category. In this paper, the objects in $\mathcal{C}$ are compact, orientable surfaces. The set of morphisms of $\mathcal{C}$ from F_1 to F_2 is denoted $Mor(F_1, F_2)$ and consists of compact, orientable 3-manifolds M for which $F_1 \cup F_2$ naturally embeds in ∂M.

Theorem 1.1. *If $M \in Mor(G, G)$ is an idempotent such that M and G are connected as manifolds, then M splits.*

The above theorem establishes the first goal of this paper, a classification of self-replicating 3-manifolds. Idempotents that split are particularly simple since they are in one-to-one correspondence with decompositions of the identity morphisms. It was previously shown that all idempotents in the category of unoriented tangles up to isotopy split [2]. Our proof of the above theorem is inspired by the strategy employed in that paper. However, there are several key technical differences.

The second goal of this paper is to explore the concept of realizability of self-replicating 3-manifolds in $\mathbb{R}^3$. As we will show in Proposition 4.1, all idempotents embed in $\mathbb{R}^3$. However, when we view self-replicating 3-manifolds as a model for biological life some examples of embeddings of idempotents in $\mathbb{R}^3$ seem to violate the key biological property of *dispersal*, the process by which individuals move from the immediate environment of their genetic relatives to establish in an area more or less distant from them. For example, each copy of $D^2 \times I$ in the bottom right of Figure 1 can "move" away from the other by, for example, translating the top copy of $D^2 \times I$ upward. Thus, this embedding as a model for a type of biological life seems to have a mechanism for biological dispersal. Alternatively, the two genus 2 handlebodies depicted in the upper right of Figure 2 and on the right of Figure 3 are linked together and thus cannot be "moved" apart. So, this embedding seems to have no mechanism for biological dispersal.

Although there is a rich history of analytically modeling dispersal in a living population, here we are interested in a topological definition that captures the idea that two subsets of $\mathbb{R}^3$ can "move" away from each other. The natural choice seems to be the notion of unlinked subspaces. Given two compact, connected, disjoint subspaces X and Y in $\mathbb{R}^3$ we say that X and Y are *unlinked* if there exists an embedded 2-sphere S in $\mathbb{R}^3$ that separates X from Y. On the left of Figure 3 we give an example of two unlinked handlebodies. On the right of Figure 3 we give an example of two handlebodies that are linked. To our knowledge, this is the first time that the concept of biological dispersal has been modeled topologically.

We adapt our topological model of biological dispersal to self-replicating 3-manifolds in the following way. Given a decomposition of a morphism M of $\mathcal{C}$ as $M = M_1 \circ M_2$, the *decomposing surface* F is the properly embedded surface in M corresponding to the codomain of M_2 and the domain of M_1. An idempotent 3-manifold $M \in Mor(F, F)$ has an *effective* embedding into $\mathbb{R}^3$ if the image of M in $\mathbb{R}^3$ can be surgered along the decomposing surface F corresponding to $M = M \circ M$ to produce two embeddings of M, denoted M_1 and M_2, such that M_1 and

M_2 are unlinked. Figure 3 gives an example of an effective embedding and an example of an embedding that is not effective.

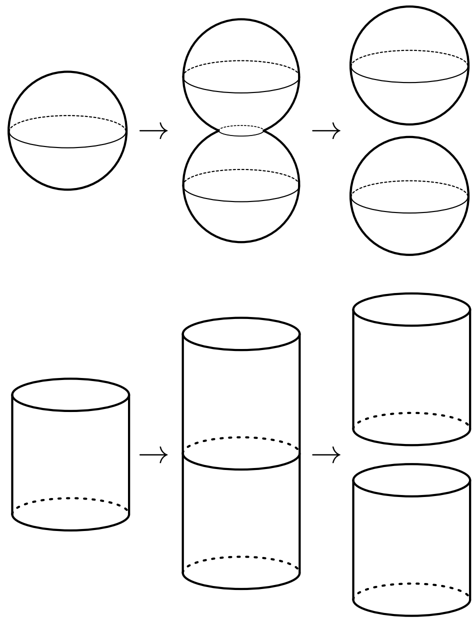

FIGURE 1. Above: A model of cellular division as a 3-ball self-replicating by splitting into two 3-balls. Below: $D^2 \times I$ splitting into two copies of $D^2 \times I$ as an idempotent in the (2+1)-cobordism category. Note that since $D^2 \times I$ is homeomorphic to a 3-ball, we can think of these two models of self-replication as being topologically equivalent.

We say an idempotent in $\mathcal{C}$ is *trivial* if it is the identity morphism on some compact surface F. In particular, an identity morphism is homeomorphic to $F \times I$. We call a surface *planar*, if it embeds in S^2. All identity morphisms on planar surfaces have effective embeddings into $\mathbb{R}^3$. However, the identity morphism on the torus does not. See Figure 2 for an example of a trivial idempotent with an effective embedding and an example of a non-trivial idempotent realized by an embedding that is not effective.

The following theorem gives a characterization of self-replicating 3-manifolds with effective embeddings into $\mathbb{R}^3$.

Theorem 1.2. *Suppose G is a connected compact orientable surface and M is a connected idempotent with $M \in Mor(G, G)$. The 3-manifold, M, effectively embeds into $\mathbb{R}^3$ if and only if M is an identity morphism and G is planar.*

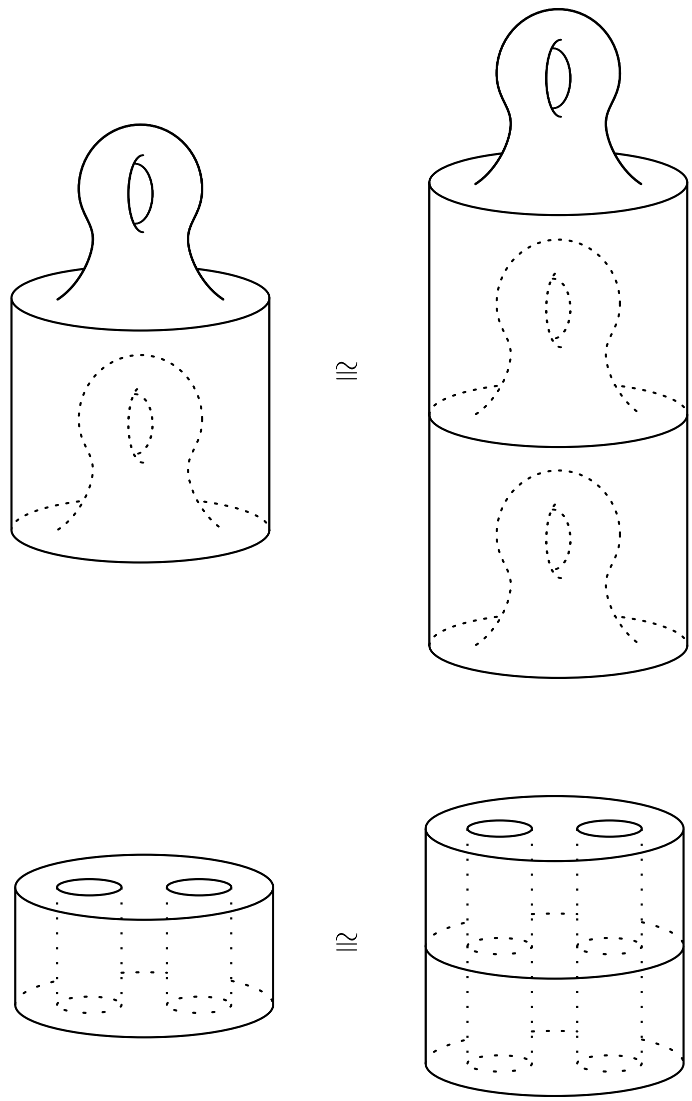

Figure 2. Two representations of the genus 2 handlebody as an idempotent in the (2+1)-cobordism category. The top representation is not an effective embedding. The bottom representation is a trivial idempotent and an effective embedding.

The paper is organized as follows. In Section 2 we give a rigorous definition of the (2+1)-cobordism category and establish the 3-manifold machinery needed for the subsequent proofs. In Section 3 we prove Theorem 1.1. In Section 4, we prove Theorem 1.2.

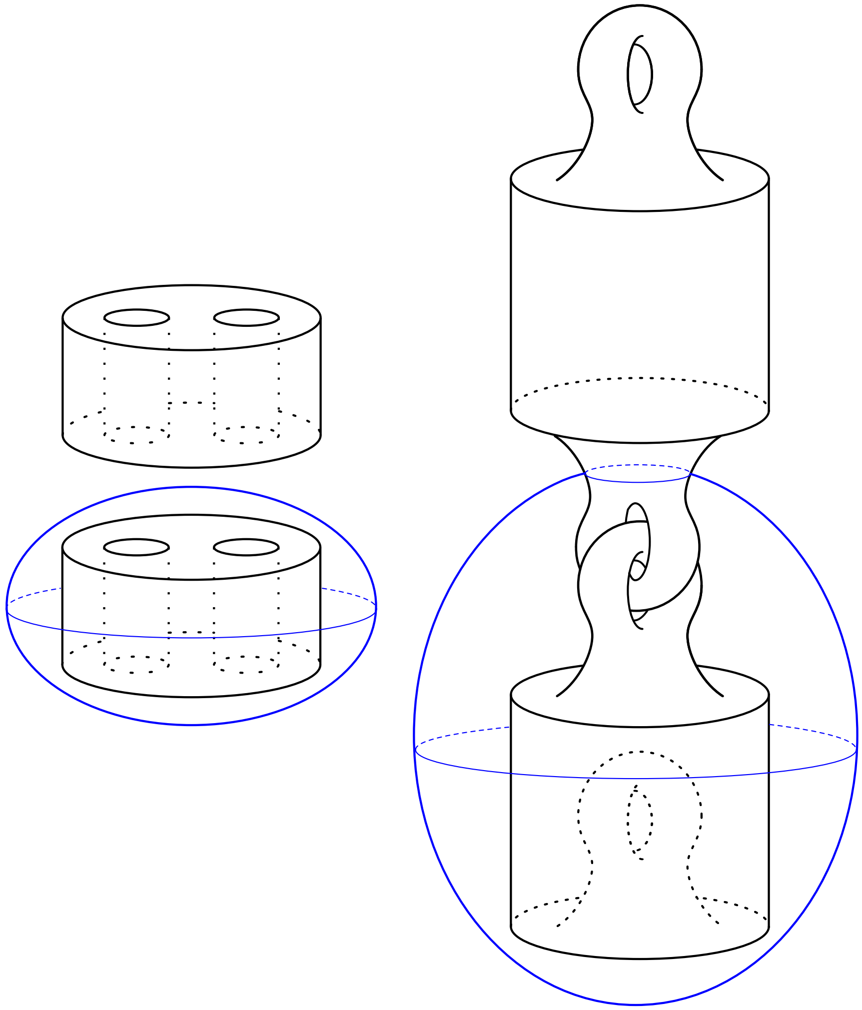

FIGURE 3. Left: Two unlinked genus 2 handlebodies in $\mathbb{R}^3$ representing an effective embedding of a trivial idempotent in the $(2+1)$-cobordism category. Right: Two linked genus 2 handlebodies in $\mathbb{R}^3$ representing an embedding of an idempotent in the $(2+1)$-cobordism category that is not effective. Note that we have isotoped the top-right handlebody slightly so that we can better see that it is linked to the bottom-right handlebody.

2. Preliminaries

2.1. **Idempotents in the $(1+1)$-Cobordism Category.** We begin with a discussion of the $(1+1)$-cobordism category as a means of developing intuition for our results in the $(2+1)$-cobordism category.

Here we take a somewhat informal perspective. However, we carefully build the definition of the (2+1)-cobordism category in the next section.

The (1+1)-cobordism category has objects consisting of disjoint collections of finitely many circles. A morphism in this category is a compact surface with boundary components partitioned into two groups corresponding to the domain and range of the morphism. We can visualize composition of these morphisms as "stacking" one on top of another.

Suppose F is an orientable connected idempotent in the $(1+1)$-cobordism category. Since F is homeomorphic to $F \circ F$, an Euler characteristic computation tells us that $\chi(F) = 2\chi(F)$ and $\chi(F) = 0$. By the classification of surfaces, it follows that F is an annulus, representing the identity morphism on S^1.

As we will see, the classification of idempotents in the $(2+1)$-cobordism category is much more involved. In particular, although there is only one orientable connected idempotent in the (1+1)-cobordism category, there is a rich infinite family of orientable connected idempotents in the $(2+1)$-cobordism category.

2.2. **The $(2+1)$-Cobordism Category.** We begin this section with a description of the $(2+1)$-cobordism category. For additional details on this category see [3] and [7]. A *triad* is a triple (M, F, G) where M is a smooth compact 3-manifold and F and G are smooth, compact, orientable 2-manifolds such that $\partial M = (F \amalg G) \cup X$ with $X \cong (\partial F) \times I$. When ∂F is empty, $\partial M = F \amalg G$. Otherwise, M is a smooth manifold with corners along ∂F and ∂G. However, we will often suppress the manifold with corners structure on M since we will often be content to classify M up to homeomorphism. Given fixed surfaces F and G, a cobordism from F to G is a triple (M, j_F, j_G), where M is a compact smooth 3-manifold and $j_F : F \to \partial M$, $j_G : G \to \partial M$ are embeddings such that $(M, j_F(F), j_G(G))$ is a triad. When appropriate, we will sometimes suppress the additional structure and refer to the cobordism (M, j_F, j_G) as M.

Two cobordisms (M_1, j_{F_1}, j_{G_1}) and (M_2, j_{F_2}, j_{G_2}) are equivalent if there is a diffeomorphism $h : M_1 \to M_2$ such that $h \circ j_{F_1} = j_{F_2}$ and $h \circ j_{G_1} = j_{G_2}$. The set of equivalence classes of cobordisms from F to G is denoted by $Mor(F, G)$. In this more precise definition the trivial cobordism form F to G is $(F \times I, j_F^0, j_F^1)$ where $j_F^i : F \to F \times I$ is the standard inclusion map such that $j_F^i(F) = F \times \{i\}$.

Let $M \in Mor(F, G)$ and $M' \in Mor(G, H)$ be represented by cobordisms (M, j_F, j_G) and (M', j'_G, j'_H). The topological manifold $M \cup_{j'_G \circ (j_G)^{-1}}$

M' admits a smooth structure compatible with those of M and M', giving rise to a smooth manifold $M \circ M'$, and $(M \circ M', j_F, j'_H)$ represents a well-defined class $M \circ M' \in Mor(F, H)$. Then, an idempotent is an equivalence class of cobordism $M \in Mor(F, F)$ such that $M = M \circ M$.

2.3. **Decomposing Surfaces.** *L*et $M \in Mor(F, G)$ be represented by the cobordism (M, j_F, j_G). A surface $H \subset M$ is a *decomposing surface* for M if:

(1) H is a smooth, compact, orientable 2-manifold.
(2) $j_H : H \to M$ is a proper smooth embedding.
(3) If $\partial F \neq \emptyset$, then for each annular component A_i of $\partial M \setminus (j_F(F) \amalg j_G(G))$ there exists a unique connected component α_i in ∂H such that α_i is an essential loop in A_i.
(4) The exterior of H in M is $A \coprod B$, where A and B are 3-manifolds such that $F \subset \partial A$ and $G \subset \partial B$. Moreover, $A \in Mor(F, H)$, $B \in Mor(H, G)$, and $B \circ A$ is equivalent to M as an element of $Mor(F, G)$.

Note that a consequence of this definition is that for any decomposing surface H for $M \in Mor(F, G)$, each of ∂H, ∂F and ∂G contain the same number of components.

A decomposing surface H for $M \in Mor(F, G)$ is *minimal* if there is no decomposing surface H' for M with $-\chi(H') < -\chi(H)$.

2.4. **Essential Surfaces.** A surface F properly embedded in a 3-manifold M is *boundary-parallel* if there is an isotopy of F in M which fixes ∂F and takes F to a subsurface contained in ∂M. Otherwise, we say F is *non-boundary parallel.* A loop γ embedded in a surface F is *essential* if it does not bound a disk in F. A surface F is *compressible* in M if F is a 2-sphere bounding a 3-ball or if there exists a disk D embedded in M such that $D \cap F = \partial D$ and ∂D is essential in F. Such a disk is called a *compressing disk.* Otherwise, we say F is *incompressible.* A surface in M is *essential* if it is incompressible and non-boundary parallel.

Given a properly embedded surface F in a 3-manifold M, we can *compress* F along a compressing disk D to form a new properly embedded surface F^*. Let $D^2 \times I$ be a small fibered neighborhood of D in M such that $D = D^2 \times \{\frac{1}{2}\}$ and $\partial(D^2) \times I$ is an embedded annulus in F. Then we define F^* to be the surface isotopic to $(F \setminus (\partial(D^2) \times I)) \cup (D^2 \times \{0, 1\})$.

The number of non-isotopic, disjoint, essential surfaces properly embedded in a compact 3-manifold is bounded due to the following classical result.

Theorem 2.1. *[Page 49 of [5]] Let M be a compact 3-manifold. If $\{F_1, \ldots, F_n\}$ is a collection of pairwise disjoint, incompressible surfaces in M so that for some integer χ_0, $\chi(F_i) > \chi_0$, $1 \leq i \leq n$, then there is an integer $N_0(M, \chi_0)$ such that either $n < N_0(M, \chi_0)$, some F_i is an annulus, or a disk parallel into ∂M, or for some $i \neq j$, F_i is parallel to F_j in M.*

Note that the conclusion of F_i being parallel to F_j in the above theorem implies that $F_i \cong F_j \cong F$ and that $F_i \cup F_j$ bounds a 3-manifold homeomorphic to $F \times I$ in M such that $F_i = F \times \{0\}$ and $F_j = F \times \{1\}$.

3. Idempotents Split

Lemma 3.1. *Let $M \in Mor(G, G)$ be a nontrivial idempotent such that M and G are connected as manifolds. If F is a minimal decomposing surface for M, then F is essential.*

Proof. Suppose F is compressible. Then we can compress F once to obtain a surface F^*. The surface F^* may have more connected components than F. However, $\partial F = \partial F^*$ and $\chi(F^*) = \chi(F) + 2 > \chi(F)$. Moreover, F^* continues to be a separating surface in M with $\partial_+ M$ to one side and $\partial_- M$ to the other. Hence, F^* is a decomposing surface for M with $-\chi(F^*) < -\chi(F)$, contradicting the minimality of F. Thus, F is incompressible.

Now we show F cannot be boundary parallel. We begin by proving the following claim:

Claim. Suppose $M \in Mor(G, G)$ is a nontrivial idempotent. Then either $\partial_+ M$ or $\partial_- M$ is compressible in M.

Proof. Suppose for contradiction both $\partial_+ M$ and $\partial_- M$ are incompressible. By Theorem 2.1, there exists integers N_0 and χ_0 such that if $F_1, \ldots, F_k$ is a collection of pairwise disjoint incompressible surfaces with $k > N_0$ and $\chi(F_i) > \chi_0$, for all $1 \leq i \leq k$, then some F_i is a boundary parallel annulus, boundary parallel disk, or there exists $i \neq j$ such that F_i is parallel to F_j. Fix such integers N_0 and χ_0 with $\chi_0 < \chi(G)$.

Since M is an idempotent, $M \cong M \circ M \circ M \ldots M$ where we have composed M with itself $N_0 + 2$ times. Therefore, we can find $N_0 + 1$ disjoint decomposing surfaces, $G_1, \ldots, G_{N_0+1}$ in M, each of which is homeomorphic to G. See Figure 4 for the case when $N_0 = 1$. Since each G_i is homeomorphic to G, then $\chi_0 < \chi(G_i)$ for all $1 \leq i \leq N_0+1$. Since both $\partial_+ M$ and $\partial_- M$ are incompressible, then each G_i is incompressible.

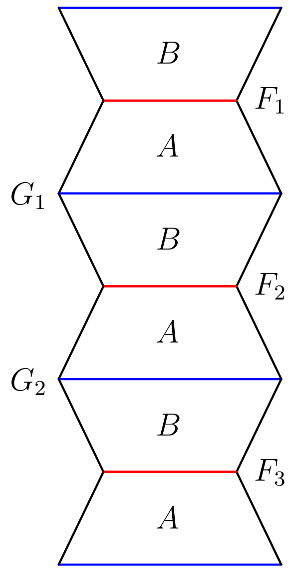

FIGURE 4. A schematic decomposition of an idempotent M as $M \circ M \circ M$ where $M = B \circ A$.

Since the collection of surfaces, $G_1, \ldots, G_{N_0+1}$ meets the hypothesis for Theorem 2.1, then one of the following hold: some G_i is a boundary parallel annulus, some G_i is a boundary parallel disk, or there exists $i \neq j$ such that G_i is parallel to G_j. In each of these cases M is of the form $surface \times I$. But M was assumed to by a nontrivial idempotent, so we get our desired contradiction. □

Suppose F is boundary parallel. Then $F \cong G$, and $\chi(F) = \chi(G)$. By our claim, we can assume one of $\partial_- M$ and $\partial_+ M$ is compressible. Without loss of generality, assume $\partial_+ M$ is compressible. Let $G \times I$ denote a collar neighborhood of $\partial_+ M$ with $\partial_+ M = G \times \{1\}$. Set $N := G \times \{0\}$. Then N is a decomposing surface such that $\chi(N) = \chi(\partial_+ M) = \chi(F)$. Since F is a minimal decomposing surface, then N is also a minimal decomposing surface. Since $\partial_+ M$ is compressible, then N is also compressible. But minimal decomposing surfaces were shown to be incompressible. This gives our desired contradiction. Hence, F is essential. □

The following is the proof of Theorem 1.1.

Proof. Let F be a minimal decomposing surface for M. By Lemma 3.1, F is essential. The surface F decomposes M into cobordisms $A \in Mor(G, F)$, $B \in Mor(F, G)$, such that $M \cong B \circ A$.

By Theorem 2.1, there exists integers N_0 and χ_0 such that if $F_1, \ldots, F_k$ is a collection of pairwise disjoint incompressible surfaces with $N_0 < k$

and $\chi_0 < \chi(F_i)$ for $1 \leq i \leq k$, then some F_i is a boundary parallel annulus, boundary parallel disk, or there exists $i \neq j$ such that F_i is parallel to F_j. Fix such integers N_0 and χ_0.

Since M is an idempotent, $M \cong M \circ M \circ M \dots M$ where we have composed M with itself $N_0 + 1$ times. Since F is an essential minimal decomposing surface, we can find $N_0 + 1$ disjoint essential minimal decomposing surfaces $F_1, \dots, F_{N_0+1}$ for M, each representing the copy of F in each copy of M. The collection of surfaces $F_1, \dots, F_{N_0+1}$ decompose M into one copy of A, one copy of B, and N_0 copies of $A \circ B$. See Figure 4 for the case when $N_0 = 2$.

Note that each surface F_i in the collection $F_1, \dots, F_{N_0+1}$ is a minimal decomposing surface for M and essential by Lemma 3.1. In particular, each surface F_i is not a boundary parallel annulus and not a boundary parallel disk. By Theorem 2.1, there exist two surfaces F_i and F_j in our collection of essential minimal decomposing surfaces that are parallel. This shows that the cobordism bounded by F_i and F_j in M, which is equivalent to $(A \circ B)^l$ for some $1 \leq l$, must be equivalent to the trivial cobordism $F \times I$. Thus $A \circ B$ is the identity morphism in $Mor(F, F)$. □

4. Effective Embeddings

Proposition 4.1. *If $M \in Mor(G, G)$ is an idempotent in $\mathcal{C}$ such that such that M and G are connected as manifolds, then M embeds in $\mathbb{R}^3$*

Proof. Let $M \in Mor(G, G)$ be an idempotent. By Theorem 1.1, there exists a decomposing surface F for M and morphisms $A \in Mor(G, F)$, $B \in Mor(F, G)$ such that $A \circ B \cong F \times I \in Mor(F, F)$ and $B \circ A \cong M$. Since F is a compact orientable surface, both F and $F \times I$ embed in $\mathbb{R}^3$. Hence, M embeds as $B \circ A$ in $A \circ B \circ A \circ B$. In turn, $A \circ B \circ A \circ B$ can be identified with any embedding of $F \times I$ in $\mathbb{R}^3$. So, M embeds in $\mathbb{R}^3$. □

In light of Proposition 4.1, we instead focus on which idempotents embed effectively.

Definition 4.2. *An idempotent 3-manifold $M \in Mor(F, F)$ has an* effective *embedding into $\mathbb{R}^3$ if the image of M in $\mathbb{R}^3$ can be surgered along the decomposing surface F corresponding to $M = M \circ M$ to produce two embeddings of M, denoted M_1 and M_2, such that there is an embedded 2-sphere in $\mathbb{R}^3$ separating M_1 from M_2.*

The following is the proof of Theorem 1.2.

Proof. If G is planar and $M \in Mor(G,G)$ is trivial, then an effective embedding of M in $\mathbb{R}^3$ is given by the product of an embedding of G in $\mathbb{R}^2$ with an embedding of $[0,1]$ in $\mathbb{R}$.

Suppose a connected idempotent $M \in Mor(G,G)$ has an effective embedding $f : M \to \mathbb{R}^3$. Let G' be a decomposing surface in M which realizes $M = M \circ M$. Let $N \cong G \times I$ be a fibered neighborhood of G' in M such that $G' = G \times \{\frac{1}{2}\}$ and $N \cap \partial M = \partial(G) \times I$ is a collection of annuli. Then we can surger M along G' to construct a new 3-manifold $M \setminus [G \times (0,1)]$, which has two connected components. We denote the image of these connected components in $\mathbb{R}^3$ by M_1 and M_2. Since f is an effective embedding of the idempotent M, each of M_1 and M_2 are homeomorphic to M, and there exists an embedded 2-sphere S in $\mathbb{R}^3$ such that S separates M_1 from M_2.

Over all spheres in $\mathbb{R}^3$ that separate M_1 from M_2, choose S to minimize the number of curves of intersection with $\partial f(N)$. If $S \cap \partial f(N)$ is empty, then $f^{-1}(S)$ is a properly embedded sphere in $N \cong G \times I$ that separates $G \times \{0\}$ from $G \times \{1\}$, which is impossible unless $G \cong S^2$. Since every smoothly embedded 2-sphere in $\mathbb{R}^3$ bounds a 3-ball to one side and $\partial f(M)$ is the disjoint union of two 2-spheres, then $f(M)$ is homeomorphic to $S^2 \times I$ and M is an identity morphism. Hence, we can assume $S \cap \partial f(N)$ is a non-empty collection of curves.

Suppose $S \cap f(N)$ is compressible in $f(N)$ with compressing disk D. Since S is a sphere, then $S \cap f(N)$ is planar and ∂D separates boundary components in $S \cap f(N)$. Surgering S along D produces two embedded 2-spheres in $\mathbb{R}^3$, each intersecting $\partial f(N)$ in strictly fewer curves than S. At least one of these two 2-spheres also separates M_1 from M_2, contradicting the minimality of $S \cap f(N)$. Hence, $S \cap f(N)$ is an incompressible surface properly embedded in a 3-manifold homeomorphic to $G \times I$ which separates $G \times \{0\}$ from $G \times \{1\}$. Since all incompressible surfaces in $(surface) \times I$ are vertical or horizontal, then $S \cap f(N)$ is horizontal and, hence, every component of $S \cap f(N)$ is properly isotopic to $f(G \times \{0\})$ in $f(N)$. Thus, G is a planar surface.

In the following paragraph we identify $\mathbb{R}^3$ with its one point compactification, S^3 so that the sphere S bounds a 3-ball to each side. Since every component of $S \cap f(N)$ is properly isotopic to $f(G \times \{0\})$ in $f(N)$, then there is an isotopy of S after which M_1 is embedded in a 3-ball B bounded by S and $M_1 \cap S = \partial_+ M_1 \cong G$. Suppose $\partial_+ M_1$ is compressible in M_1 with compressing disk D. Since ∂D is essential in $\partial_+ M_1$ and $\partial_+ M_1$ is planar, then ∂D separates the boundary components of $\partial_+ M_1$. Let E_1 and E_2 be the two disks in S bounded by ∂D. The disk D cuts B into two 3-balls B_1 and B_2 such that $\partial B_1 = E_1 \cup D$ and $\partial B_2 = E_2 \cup D$. Since $\partial_- M_1$ is connected, then we can assume $\partial_- M_1$

is embedded in B_1, up to relabeling. Let α be a boundary component of $\partial_+ M_1$ contained in E_2. The curve α together with some boundary component β of $\partial_- M_1$ cobound an annulus $A \subset \partial M_1$. See Figure 5. However, $int(B_1)$ and $int(B_2)$ induce a separation on A, contradicting the fact that the annulus is connected. Hence, $\partial_+ M_1$ is incompressible in M_1. By isotoping S so that M_2 is embedded in the other 3-ball bounded by S and $M_2 \cap S = \partial_- M_2 \cong G$ and repeating the above argument, we can show $\partial_- M_2$ is incompressible in M_2. Taken together this implies that both $\partial_+ M$ and $\partial_- M$ are incompressible in M. By the claim in the proof of Lemma 3.1, M is a identity morphism.

□

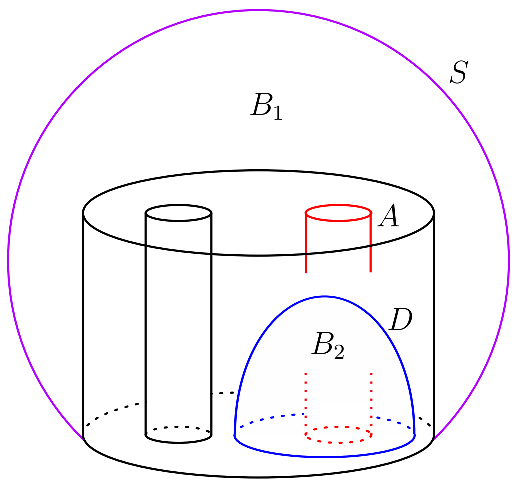

FIGURE 5. M_1 is embedded in the 3-ball B with the compressing disk D for $\partial_+ M_1$ in blue and the annulus A in red.

5. ACKNOWLEDGEMENTS

The authors would like to thank the referee for comments that materially improved the paper. The first author was partially supported by NSF Grant DMS–1821254.

6. DATA AVAILABILITY

Data sharing not applicable to this article as no datasets were generated or analysed during the current study.

References

[1] Samson Abramsky, *Temperley-lieb algebra: from knot theory to logic and computation via quantum mechanics*, Mathematics of quantum computation and quantum technology, 2007, pp. 515–558.

[2] Ryan Blair and Joshua Sack, *Idempotents in tangle categories split*, J. Knot Theory Ramifications **28** (2019), no. 5, 1950025, 9. MR3943699

[3] Nils Carqueville and Ingo Runkel, *Introductory lectures on topological quantum field theory*, Advanced school on topological quantum field theory, 2018, pp. 9–47. MR3838090

[4] Rukhsan Ul Haq and Louis H. Kauffman, *Iterants, idempotents and clifford algebra in quantum theory*, CoRR **abs/1705.06600** (2017).

[5] W.H. Jaco and C.B.M. Sciences, *Lectures on three-manifold topology*, Conference Board of the mathematical science, Conference Board of the Mathematical Sciences, 1980.

[6] Louis H. Kauffman, *Biologic. II*, Woods Hole mathematics, 2004, pp. 94–132. MR2123368

[7] John Milnor, *Lectures on the h-cobordism theorem*, Princeton University Press, Princeton, N.J., 1965. Notes by L. Siebenmann and J. Sondow. MR0190942

[8] Peter Selinger, *Idempotents in dagger categories*, Electronic Notes in Theoretical Computer Science **210** (2008), 107 –122. Proceedings of the 4th International Workshop on Quantum Programming Languages (QPL 2006).

E-mail address: `Corresponding Author: ryan.blair@csulb.edu`