NOTES ON REAL INTERPOLATION OF OPERATOR L_p-SPACES

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ABSTRACT. Let $\mathcal{M}$ be a semifinite von Neumann algebra. We equip the associated noncommutative L_p-spaces with their natural operator space structure introduced by Pisier via complex interpolation. On the other hand, for $1 < p < \infty$ let

$$L_{p,p}(\mathcal{M}) = \big(L_\infty(\mathcal{M}),\, L_1(\mathcal{M})\big)_{\frac{1}{p},\,p}$$

be equipped with the operator space structure via real interpolation as defined by the second named author (*J. Funct. Anal.* 139 (1996), 500–539). We show that $L_{p,p}(\mathcal{M}) = L_p(\mathcal{M})$ completely isomorphically if and only if $\mathcal{M}$ is finite dimensional. This solves in the negative the three problems left open in the quoted work of the second author.

We also show that for $1 < p < \infty$ and $1 \le q \le \infty$ with $p \ne q$

$$\big(L_\infty(\mathcal{M};\ell_q),\, L_1(\mathcal{M};\ell_q)\big)_{\frac{1}{p},\,p} = L_p(\mathcal{M};\ell_q)$$

with equivalent norms, i.e., at the Banach space level if and only if $\mathcal{M}$ is isomorphic, as a Banach space, to a commutative von Neumann algebra.

Our third result concerns the following inequality:

$$\|(\sum_i x_i^q)^{\frac{1}{q}}\|_{L_p(\mathcal{M})} \le \|(\sum_i x_i^r)^{\frac{1}{r}}\|_{L_p(\mathcal{M})}$$

for any finite sequence $(x_i) \subset L_p^+(\mathcal{M})$, where $0 < r < q < \infty$ and $0 < p \le \infty$. If $\mathcal{M}$ is not isomorphic, as a Banach space, to a commutative von Meumann algebra, then this inequality holds if and only if $p \ge r$.

1. INTRODUCTION

Interpolation of L_p-spaces is a classical subject. Our reference for interpolation theory is [1]. Let (Ω, μ) be a measure space. Let $1 \le p_0, p_1, p \le \infty$ and $0 < \theta < 1$ with $\frac{1}{p} = \frac{1-\theta}{p_0} + \frac{\theta}{p_1}$. The following well known interpolation equalities hold

$$\big(L_{p_0}(\Omega),\, L_{p_1}(\Omega)\big)_\theta = L_p(\Omega) \quad \text{with equal norms,} \tag{1}$$

$$\big(L_{p_0}(\Omega),\, L_{p_1}(\Omega)\big)_{\theta,p} = L_p(\Omega) \quad \text{with equivalent norms.} \tag{2}$$

Here $(\,\cdot\,,\,\cdot\,)_\theta$ and $(\,\cdot\,,\,\cdot\,)_{\theta,p}$ denote, respectively, the complex and real interpolation functors. It is also well known that the above equalities hold for vector-valued L_p-spaces. More precisely, under the same assumption on the parameters (assuming additionally $p < \infty$), we have

$$\big(L_{p_0}(\Omega;E_0),\, L_{p_1}(\Omega;E_1)\big)_\theta = L_p\big(\Omega;(E_0,\, E_1)_\theta\big) \quad \text{with equal norms,} \tag{3}$$

$$\big(L_{p_0}(\Omega;E_0),\, L_{p_1}(\Omega;E_1)\big)_{\theta,p} = L_p\big(\Omega;(E_0,\, E_1)_{\theta,p}\big) \quad \text{with equivalent norms} \tag{4}$$

for any compatible pair $(E_0,\, E_1)$ of Banach spaces.

The present note concerns interpolation theory in the category of operator spaces. We refer to [2, 10] for operator space theory. The complex and real interpolations for operator spaces are developed in [8] and [13], respectively. Unless explicitly stated otherwise, all (commutative and noncommutative) L_p-spaces in the sequel are equipped with their natural operator space structure as defined in [9, 10]. Pisier proved that (1) and (3) remain true in the category of operator spaces, that is, these equalities hold completely isometrically, $(E_0,\, E_1)$ being, of course, assumed to be operator spaces in the case of (3).

It is more natural to work with noncommutative L_p-spaces in the category of operator spaces. Let $\mathcal{M}$ be a semifinite von Neumann algebra equipped with a normal semifinite faithful trace τ.

2000 Mathematics Subject Classification: Primary: Secondary:
Key words: Operator spaces, L_p-spaces, real interpolation, column Hilbertian spaces.

Let $L_p(\mathcal{M})$ denote the associated noncommutative L_p-space (cf. [11]). If $\mathcal{M} = B(\ell_2)$ with the usual trace, $L_p(\mathcal{M})$ is the Schatten p-class S_p. If $\mathcal{M}$ is hyperfinite, Pisier [9] also introduced the vector-valued $L_p(\mathcal{M}; E)$ for an operator space E, and showed that (3) continues to hold in this more general setting:

$$\big(L_{p_0}(\mathcal{M}; E_0),\, L_{p_1}(\mathcal{M}; E_1)\big)_\theta = L_p\big(\mathcal{M}; (E_0,\, E_1)_\theta\big) \ \text{ completely isometrically} \tag{5}$$

for any hyperfinite $\mathcal{M}$ and any compatible pair $(E_0,\, E_1)$ of operator spaces.

However, real interpolation does not behave as smoothly as complex interpolation in the category of operator spaces. The problem whether (5) can be extended to real interpolation was left unsolved in [13], see Problems 6.1, 6.2 and 6.4 there. Let $1 < p < \infty$. Using the Banach space equality

$$L_p(\mathcal{M}) = \big(L_\infty(\mathcal{M}),\, L_1(\mathcal{M})\big)_{\frac{1}{p},p},$$

we can equip $L_p(\mathcal{M})$ with another operator space structure via real interpolation as in [13], the resulting operator space is denoted by $L_{p,p}(\mathcal{M})$. Then Problem 6.1 of [13] asks whether $L_{p,p}(\mathcal{M}) = L_p(\mathcal{M})$ completely isomorphically for $p \neq 2$ (the answer is affirmative for $p = 2$). The following result resolves this problem in the negative.

Theorem 1. *Let $1 < p < \infty$ with $p \neq 2$. Then $L_{p,p}(\mathcal{M}) = L_p(\mathcal{M})$ completely isomorphically if and only if $\mathcal{M}$ is finite dimensional.*

Consequently, the answers to all three Problems 6.1, 6.2 and 6.4 of [13] are negative. In particular, neither (2) nor (4) holds in the category of operator spaces.

The next theorem provides an even worse answer to Problem 6.4. It shows that (5) extends to real interpolation at the Banach space level only in the commutative case. Recall that one can define $L_p(\mathcal{M}; \ell_q)$ for any von Neumann algebra $\mathcal{M}$ (see section 3 for more information). $L_p(\mathcal{M}; \ell_q)$ coincides with Pisier's space when $\mathcal{M}$ is hyperfinite. Note that $L_p(\mathcal{M}; \ell_q)$ is defined only as a Banach space if $\mathcal{M}$ is not hyperfinite.

Theorem 2. *Let $1 < p < \infty$ and $1 \le q \le \infty$ with $p \neq q$. Then*

$$\big(L_\infty(\mathcal{M}; \ell_q),\, L_1(\mathcal{M}; \ell_q)\big)_{\frac{1}{p},p} = L_p(\mathcal{M}; \ell_q) \ \text{ with equivalent norms} \tag{6}$$

if and only if $\mathcal{M}$ is isomorphic, as a Banach space, to $L_\infty(\Omega, \mu)$ for some measure space (Ω, μ).

Our third theorem does not concern the real interpolation of the $L_p(\mathcal{M}; \ell_q)$ spaces but gives a result that is to be compared with the norm of these spaces. In the commutative case, the norm of a sequence (x_i) in $L_p(\Omega; \ell_q)$ is given by

$$\|(x_i)\|_{L_p(\Omega;\ell_q)} = \Big\|\Big(\sum_i |x_i|^q\Big)^{\frac{1}{q}}\Big\|_{L_p(\Omega)}.$$

It is well known that this expression is no longer valid in the noncommutative setting, which is one source of many difficulties in noncommutative analysis. The following theorem shows that another classical property of the norm $\|(x_i)\|_{L_p(\Omega;\ell_q)}$ does not extend to the noncommutative case. The index p is now allowed to go below 1, $L_p^+(\mathcal{M})$ denotes the positive cone of $L_p(\mathcal{M})$.

Theorem 3. *Let $0 < r < q < \infty$ and $0 < p \le \infty$.*

(i) *If $p \ge r$, then*

$$\Big\|\Big(\sum_i x_i^q\Big)^{\frac{1}{q}}\Big\|_{L_p(\mathcal{M})} \le \Big\|\Big(\sum_i x_i^r\Big)^{\frac{1}{r}}\Big\|_{L_p(\mathcal{M})} \tag{7}$$

for any finite sequence $(x_i) \subset L_p^+(\mathcal{M})$.

(ii) *If $p < r$ and $\mathcal{M}$ is not isomorphic, as a Banach space, to $L_\infty(\Omega, \mu)$ for some measure space (Ω, μ), then there exists no constant C such that*

$$\Big\|\Big(\sum_i x_i^q\Big)^{\frac{1}{q}}\Big\|_{L_p(\mathcal{M})} \le C\,\Big\|\Big(\sum_i x_i^r\Big)^{\frac{1}{r}}\Big\|_{L_p(\mathcal{M})} \tag{8}$$

for any finite sequence $(x_i) \subset L_p^+(\mathcal{M})$.

The previous theorems will be respectively proved in the next three sections.

2. Proof of Theorem 1

We will need some preparations on column and row Hilbertian operator spaces. Let C_p (resp. R_p) denote the first column (resp. row) subspace of S_p consisting of matrices whose all entries but those in the first column (resp. row) vanish. We have the following completely isometric identifications:

$$(C_p)^* \cong C_{p'} \cong R_p \quad \text{and} \quad (R_p)^* \cong R_{p'} \cong C_p\,, \quad \forall\, 1 \le p \le \infty, \tag{9}$$

where p' denotes the conjugate index of p. C_p and R_p can be also defined via complex interpolation from $C = C_\infty$ and $R = R_\infty$. We view (C, R) as a compatible pair by identifying both of them with ℓ_2 (at the Banach space level), i.e., by identifying the canonical bases $(e_{k,1})$ of C_p and $(e_{1,k})$ of R_p with (e_k) of ℓ_2. Then we have the following completely isometric equalities

$$C_p = (C,\ R)_{\frac{1}{p}} = (C_\infty,\ C_1)_{\frac{1}{p}} \quad \text{and} \quad R_p = (R,\ C)_{\frac{1}{p}} = (R_\infty,\ R_1)_{\frac{1}{p}}. \tag{10}$$

We refer to [8, 9] for more details.

Let Rad_p be the closed subspace spanned by the Rademacher sequence (ε_n) in $L_p([0,\,1])$. Then the noncommutative Khintchine inequality can be reformulated in terms of column and row spaces (see [6, 9]). To this end, we introduce

$$CR_p = C_p + R_p \text{ for } p \le 2 \quad \text{and} \quad CR_p = C_p \cap R_p \text{ for } p > 2.$$

Lemma 4. *Let $1 < p < \infty$. Then $\mathrm{Rad}_p = CR_p$ completely isomorphically. Moreover, the orthogonal projection from $L_2([0,\,1])$ onto Rad_2 extends to a completely bounded projection from $L_p([0,\,1])$ onto Rad_p. All relevant constants depend only on p.*

If E is an operator space, $C_p(E)$ (resp. $R_p(E)$) denotes the first column (resp. row) subspace of the E-valued Schatten class $S_p(E)$. It is clear that $C_p(E)$ and $R_p(E)$ are completely 1-complemented in $S_p(E)$. Consequently, applying (5) to $\mathcal{M} = B(\ell_2)$, we get

$$\big(C_{p_0}(E_0),\, C_{p_1}(E_1)\big)_\theta = C_p((E_0,\, E_1)_\theta) \tag{11}$$

for any compatible pair $(E_0,\, E_1)$ of operator spaces, where $\frac{1}{p} = \frac{1-\theta}{p_0} + \frac{\theta}{p_1}$.

Note that $C_p(R_p) = S_p$ and $C_p(C_p) = S_2$ isometrically at the Banach space level. Here we represent the elements in $C_p(R_p)$ and $C_p(C_p)$ by infinite matrices in the canonical bases of C_p and R_p. The following elementary fact is known to experts and implicitly contained in the proof of [14, Lemma 5.9]. We include its simple proof for completeness.

Lemma 5. *Let $1 \le p, q \le \infty$. Let r be determined by $\frac{1}{r} = \frac{1}{2p} + \frac{1}{2q'}$. Then*

$$C_p(CR_q) = S_r \quad \textit{with equivalent norms.}$$

Proof. By (5), (11) and (9), we have the following isometric equalities

$$\begin{aligned} C_\infty(C_q) &= \big(C_\infty(C_\infty),\, C_\infty(C_1)\big)_{\frac{1}{q}} \\ &= \big(C_\infty(C_\infty),\, C_\infty(R_\infty)\big)_{\frac{1}{q}} \\ &= \big(S_2,\, S_\infty\big)_{\frac{1}{q}} = S_{2q'}\,. \end{aligned}$$

Similarly, $C_1(C_q) = S_{(2q)'}$ isometrically. Thus

$$C_p(C_q) = \big(C_\infty(C_q),\, C_1(C_q)\big)_{\frac{1}{p}} = \big(S_{2q'},\, S_{(2q)'}\big)_{\frac{1}{p}} = S_r.$$

Combining this with (9), we also have

$$C_p(R_q) = C_p(C_{p'}) = S_t \ \text{ isometrically},$$

where $\frac{1}{t} = \frac{1}{2p} + \frac{1}{2q}$. Hence,

$$C_p(CR_q) = C_p(C_q) + C_p(R_q) = S_r + S_t = S_r$$

for $q \le 2$ and $C_p(CR_q) = S_r \cap S_t = S_r$ for $q > 2$ too. □

Proof of Theorem 1. By the type decomposition of von Neumann algebras, if $\mathcal{M}$ is not finite dimensional, then $\mathcal{M}$ contains an infinite dimensional commutative $L_\infty(\Omega,\mu)$ as subalgebra which is moreover the image of a trace preserving normal conditional expectation. Indeed, if the type I summand of $\mathcal{M}$ is infinite dimensional, then $\mathcal{M}$ contains an infinite dimensional commutative $L_\infty(\Omega,\mu)$. On the other hand, if the type II_∞ summand of $\mathcal{M}$ exists, then $\mathcal{M}$ contains $\mathcal{B}(\ell_2)$, so ℓ_∞ too. Finally, if the type II_1 summand of $\mathcal{M}$ exists, then $\mathcal{M}$ contains $L_\infty([0,1])$. See [12] for the type decomposition of von Neumann algebras.

Note that if $L_\infty(\Omega,\mu)$ is infinite dimensional, $L_\infty(\Omega,\mu)$ contains, as subalgebra, either $L_\infty([0,1])$ or ℓ_∞. On the other hand, if $L_{p,p}(\mathcal{M}) = L_p(\mathcal{M})$ held for $\mathcal{M} = \ell_\infty$, it would do so for $\mathcal{M} = \ell_\infty^n$ uniformly in $n \geq 1$. Then by a standard approximation argument, we see that it would hold for $\mathcal{M} = L_\infty([0,1])$ too.

Thus we are reduced to showing the theorem for the special case $\mathcal{M} = L_\infty([0,1])$. Namely, we must show that $L_{p,p}([0,1]) \neq L_p([0,1])$ completely isomorphically. In the rest of the proof, we will drop the interval $[0,1]$ from $L_p([0,1])$. By [13, Theorem 5.4], we choose $1 < p_0 < p_1 < \infty$ and $0 < \theta < 1$ such that $\frac{1}{p} = \frac{1-\theta}{p_0} + \frac{\theta}{p_1}$ and

$$L_{p,p} = \big(L_{p_0},\, L_{p_1}\big)_{\theta,p}\,.$$

Then by [13, Proposition 6.3] and its proof, we have

$$S_p(L_{p,p}) = \big(S_p(L_{p_0}),\, S_p(L_{p_1})\big)_{\theta,p}\,.$$

Using the complete complementation of $C_p(E)$ in $S_p(E)$, we deduce

$$C_p(L_{p,p}) = \big(C_p(L_{p_0}),\, C_p(L_{p_1})\big)_{\theta,p}\,.$$

On the other hand, by the complete complementation of Rad_{p_i} in L_{p_i} for $i = 0, 1$, we get the following isomorphic embedding:

$$\big(C_p(\mathrm{Rad}_{p_0}),\, C_p(\mathrm{Rad}_{p_1})\big)_{\theta,p} \subset C_p(L_{p,p}).$$

Now by Lemmas 4 and 5,

$$\big(C_p(\mathrm{Rad}_{p_0}),\, C_p(\mathrm{Rad}_{p_1})\big)_{\theta,p} = (S_{r_0},\, S_{r_1})_{\theta,p} = S_{2,p} \ \text{ with equivalent norms,}$$

where $\frac{1}{r_i} = \frac{1}{2p} + \frac{1}{2p_i'}$ for $i = 0, 1$. Thus the closed subspace spanned by the Rademacher functions in $C_p(L_{p,p})$ is isomorphic to $S_{2,p}$. However, that spanned by the same functions in $C_p(L_p)$ is isomorphic to S_2. But $S_{2,p} = S_2$ only if $p = 2$. Thus the theorem is proved. □

3. Proof of Theorem 2

We begin this proof by recalling the definition of the space $L_p(\mathcal{M};\ell_q)$ that is introduced in [3] for $q = 1$ and $q = \infty$ (see also [4]), and in [5] for $1 < q < \infty$. This definition is inspired by Pisier's description of the norm of $L_p(\mathcal{M};\ell_q)$ in the hyperfinite case.

A sequence (x_i) in $L_p(\mathcal{M})$ belongs to $L_p(\mathcal{M};\ell_\infty)$ iff (x_i) admits a factorization $x_i = ay_ib$ with $a, b \in L_{2p}(\mathcal{M})$ and $(y_i) \in \ell_\infty(L_\infty(\mathcal{M}))$. The norm of (x_i) is then defined as

$$\|(x_i)\|_{L_p(\mathcal{M};\ell_\infty)} = \inf_{x_i = ay_ib} \|a\|_{L_{2p}(\mathcal{M})}\, \|(y_i)\|_{\ell_\infty(L_\infty(\mathcal{M}))}\, \|b\|_{L_{2p}(\mathcal{M})}\,. \tag{12}$$

On the other hand, $L_p(\mathcal{M};\ell_1)$ is defined as the space of all sequences $(x_i) \subset L_p(\mathcal{M})$ for which there exist $a_{ij}, b_{ij} \in L_{2p}(\mathcal{M})$ such that

$$x_i = \sum_j a_{ij}^* b_{ij}\,.$$

$L_p(\mathcal{M};\ell_1)$ is equipped with the norm

$$\|(x_i)\|_{L_p(\mathcal{M};\ell_1)} = \inf_{x_i = \sum_j a_{ij}^* b_{ij}} \Big\| \sum_{i,j} a_{ij}^* a_{ij} \Big\|_{L_p(\mathcal{M})}^{\frac{1}{2}} \Big\| \sum_{i,j} b_{ij}^* b_{ij} \Big\|_{L_p(\mathcal{M})}^{\frac{1}{2}}\,.$$

Now for $1 < q < \infty$ we define $L_p(\mathcal{M};\ell_q)$ as a complex interpolation space between $L_p(\mathcal{M};\ell_\infty)$ and $L_p(\mathcal{M};\ell_1)$:

$$L_p(\mathcal{M};\ell_q) = \big(L_p(\mathcal{M};\ell_\infty),\ L_p(\mathcal{M};\ell_1)\big)_{\frac{1}{q}}\,.$$

The following description of the norm of $L_p(\mathcal{M};\ell_q)$ is proved in [9] for hyperfinite $\mathcal{M}$ and in [5] for a general $\mathcal{M}$.

Lemma 6. *Let $1 \le p, q \le \infty$.*

(i) *If $p \le q$,*

$$\|(x_i)\|_{L_p(\mathcal{M};\ell_q)} = \inf_{x_i = a y_i b} \|a\|_{L_{2r}(\mathcal{M})}\, \|(y_i)\|_{\ell_q(L_q(\mathcal{M}))}\, \|b\|_{L_{2r}(\mathcal{M})}$$

for any $(x_i) \in L_p(\mathcal{M};\ell_q)$, where $\frac{1}{r} = \frac{1}{p} - \frac{1}{q}$.

(ii) *If $p \ge q$,*

$$\|(x_i)\|_{L_p(\mathcal{M};\ell_q)} = \sup_{\|\alpha\|_{L_{2s}(\mathcal{M})}\le 1,\ \|\beta\|_{L_{2s}(\mathcal{M})}\le 1} \|(\alpha x_i \beta)\|_{\ell_q(L_q(\mathcal{M}))}$$

for any $(x_i) \in L_p(\mathcal{M};\ell_q)$, where $\frac{1}{s} = \frac{1}{q} - \frac{1}{p}$.

We will again consider the column subspace $C_p(\ell_q)$ of $S_p(\ell_q)$ for the proof of Theorem 2. As in the previous section, a generic element $x \in C_p(\ell_q)$ is viewed as an infinite matrix

$$x = \sum_{i,j=1}^{\infty} x_{ij} e_{i,1} \otimes e_j.$$

Let $D_{p,q}$ denote the diagonal subspace of $C_p(\ell_q)$ consisting of all x with $x_{ij} = 0$ for $i \ne j$.

Lemma 7. *Let $1 \le p, q \le \infty$. Then $D_{p,q}$ is completely 1-complemented in $C_p(\ell_q)$.*

Proof. The proof is very simple. It suffices to note that the canonical bases of C_p and ℓ_q are completely 1-unconditional. A standard average argument then yields the assertion. □

We will identify an element $x = (x_i e_{i,1}) \in D_{p,q}$ with the sequence (x_i).

Lemma 8. *Let $1 \le p, q \le \infty$ and $r_{p,q}$ be determined by $\frac{1}{r_{p,q}} = \frac{1}{2p} + \frac{1}{2q}$. Then $D_{p,q} = \ell_{r_{p,q}}$ with equal norms.*

Proof. First consider the cases $q = \infty$ and $q = 1$. Let $x = (x_i e_{i,1}) \in D_{p,\infty} \subset S_p(\ell_\infty)$. Let a be the diagonal matrix with the x_i's as its diagonal entries. Then we have the following factorization:

$$x_i e_{i,1} = a e_{i,1} e_{11}, \quad i \ge 1.$$

Thus by the definition of the norm of $S_p(\ell_\infty)$, we get

$$\|x\|_{S_p(\ell_\infty)} \le \|a\|_{S_{2p}}\, \|(e_{i,1})\|_{\ell_\infty(B(\ell_2))}\, \|e_{11}\|_{S_{2p}} = \|x\|_{\ell_{2p}}.$$

On the other hand, for $q = 1$ we factorize x as

$$x_i e_{i,1} = [\operatorname{sgn}(x_i)|x_i|^{\alpha} e_{i,1}]\, [|x_i|^{1-\alpha} e_{1,1}] = a_i b_i,$$

where $\alpha = \frac{r_{p,1}}{2p}$. Therefore, by the definition of the norm of $S_p(\ell_1)$

$$\begin{aligned} \|x\|_{S_p(\ell_1)} &\le \Big\| \sum_i a_i a_i^* \Big\|_p^{\frac{1}{2}} \Big\| \sum_i b_i^* b_i \Big\|_p^{\frac{1}{2}} \\ &= \Big(\sum_i |x_i|^{r_{p,1}} \Big)^{\frac{1}{2p}} \Big(\sum_i |x_i|^{r_{p,1}} \Big)^{\frac{1}{2}} = \|x\|_{\ell_{r_{p,1}}}. \end{aligned}$$

Thus we have proved that for any $1 \le p \le \infty$

$$\ell_{r_{p,\infty}} \subset D_{p,\infty} \quad \text{and} \quad \ell_{r_{p,1}} \subset D_{p,1} \ \text{ contractively.}$$

Dualizing these inclusions and using Lemma 7, we deduce the assertion for $q = \infty$ and $q = 1$. The case $1 < q < \infty$ is then completed by complex interpolation via (1.1) with the help of Lemma 7 again. □

Remark 9. The previous lemma can be proved directly by Lemma 6 without passing to complex interpolation.

Proof of Theorem 2. If $\mathcal{M}$ is isomorphic, as Banach space, to some commutative L_∞, then $\mathcal{M}$ is a finite direct sum of algebras of the form $L_\infty(\Omega,\mu)\otimes\mathbb{M}_n$, where $\mathbb{M}_n$ is the $n\times n$ full matrix algebra. Then (6) goes back to (4).

Conversely, suppose that $\mathcal{M}$ is not isomorphic to a commutative von Neumann algebra. Our first step is to reduce the non validity of (6) to the special case where $\mathcal{M}=B(\ell_\infty)$. To this end, we use the type decomposition of $\mathcal{M}$. If the type I summand of $\mathcal{M}$ is infinite dimensional, then $\mathcal{M}$ contains $\mathbb{M}_n$ for infinite many n's. On the other hand, if the type II_∞ summand of $\mathcal{M}$ exists, then $\mathcal{M}$ contains $\mathcal{B}(\ell_2)$. Finally, it is well known that if the type II_1 summand of $\mathcal{M}$ exists, then $\mathcal{M}$ contains the hyperfinite II_1 factor $\mathcal{R}$ (cf. e.g. [7]); $\mathcal{R}$ is the von Neumann tensor of countable many copies of $(\mathbb{M}_2,\,\mathrm{tr})$, where tr is the normalized trace on $\mathbb{M}_2$; so $\mathcal{M}$ again contains $\mathbb{M}_n$ for infinite many n's. Note that in all the three cases, the $\mathbb{M}_n$'s contained in $\mathcal{M}$ are images of trace preserving normal conditional expectations (up to a normalization in the type II_1 case).

In summary, if $\mathcal{M}$ is not isomorphic to a commutative von Neumann algebra, $\mathcal{M}$ contains $\mathbb{M}_n$ for infinite many n's which are images of trace preserving normal conditional expectations. This shows that if (6) held for $\mathcal{M}$, then it would do so for $\mathcal{M}=\mathbb{M}_n$ for infinite many n's; consequently, by approximation, it would further hold for $\mathcal{M}=B(\ell_2)$ too. This finishes the announced reduction.

It remains to show that (6) fails for $\mathcal{M}=B(\ell_2)$. Namely, we must show

$$\big(S_\infty(\ell_q),\,S_1(\ell_q)\big)_{\frac{1}{p},p}\neq S_p(\ell_q)\,.$$

This is an easy consequence of the previous two lemmas. By Lemma 7,

$$\big(D_{\infty,q},\,D_{1,q}\big)_{\frac{1}{p},p}\subset\big(S_\infty(\ell_q),\,S_1(\ell_q)\big)_{\frac{1}{p},p},\quad\text{an isometric embedding.}$$

On the other hand, by Lemma 8,

$$\big(D_{\infty,q},\,D_{1,q}\big)_{\frac{1}{p},p}=\big(\ell_{r_{\infty,q}},\,\ell_{r_{\infty,q}}\big)_{\frac{1}{p},p}=\ell_{r_{p,q},p}\quad\text{with equivalent norms,}$$

where $\ell_{r,p}$ denotes the Lorentz sequence space. On the other hand, by Lemma 8, the corresponding subspace of $S_p(\ell_q)$ is equal to $\ell_{r_{p,q}}$. However, $\ell_{r_{p,q},p}=\ell_{r_{p,q}}$ if and only if $r_{p,q}=p$, i.e., $q=p$. The theorem is thus proved. □

4. Proof of Theorem 3

In the sequel, $\|\ \|_p$ will denote the norm of $L_p(\mathcal{M})$. Fix a finite sequence $(x_i)\subset L_p^+(\mathcal{M})$. We claim that the function $q\mapsto\big\|\big(\sum_i x_i^q\big)^{\frac{1}{q}}\big\|_p^q$ is log-convex. Namely, for any $q_0,q_1\in(0,\,\infty)$ and any $\alpha\in(0,\,1)$

$$\big\|\big(\sum_i x_i^q\big)^{\frac{1}{q}}\big\|_p^q\le\big\|\big(\sum_i x_i^{q_0}\big)^{\frac{1}{q_0}}\big\|_p^{(1-\alpha)q_0}\,\big\|\big(\sum_i x_i^{q_1}\big)^{\frac{1}{q_1}}\big\|_p^{(1-\alpha)q_1}\,,\tag{13}$$

where $q=(1-\alpha)q_0+\alpha q_1$. It suffices to show this inequality for $\alpha=\frac{1}{2}$. Then it immediately follows from the Hölder inequality for column spaces:

$$\big\|\sum_i x_i^q\big\|_{\frac{p}{q}}=\big\|\sum_i x_i^{\frac{q_0}{2}}\,x_i^{\frac{q_1}{2}}\big\|_{\frac{p}{q}}\le\big\|\big(\sum_i x_i^{q_0}\big)^{\frac{1}{q_0}}\big\|_p^{\frac{q_0}{2}}\,\big\|\big(\sum_i x_i^{q_1}\big)^{\frac{1}{q_1}}\big\|_p^{\frac{q_1}{2}}\,.$$

Now let us show (7). Replacing x_i by x_i^r and dividing all indices by r, we are reduced to the case where $r=1<q$ and $p\ge1$. Thus it suffices to show

$$\big\|\big(\sum_i x_i^q\big)^{\frac{1}{q}}\big\|_p\le\big\|\sum_i x_i\big\|_p\,.\tag{14}$$

Set $x=\sum_i x_i$. We first consider the case where $q=2$. If $p\le2$, then

$$\begin{aligned}\big\|\big(\sum_i x_i^2\big)^{\frac{1}{2}}\big\|_p^p&=\tau\big[\big(\sum_i x_i^2\big)^{\frac{p}{2}}\big]\le\sum_i\tau(x_i^p)\\&\le\sum_i\tau(x_i^{\frac{1}{2}}x^{p-1}x_i^{\frac{1}{2}})=\sum_i\tau(x^{p-1}x_i)=\|x\|_p^p\,.\end{aligned}$$

Note that the same argument yields (14) in the case where $p \le \min(q,2)$. Assume $p>2$. Let (ε_n) be a Rademacher sequence, and let $\mathbb{E}$ denote the corresponding expectation. Then by the triangle inequality in $L_{\frac{p}{2}}(\mathcal{M})$,

$$\big\|\big(\sum_i x_i^2\big)^{\frac12}\big\|_p \le \big[\mathbb{E}\big\|\sum_i \varepsilon_i x_i\big\|_p^2\big]^{\frac12} \le \|x\|_p\,,$$

where we have used the fact that $-x \le \sum_i \varepsilon_i x_i \le x$. Thus inequality (14) is proved for $q=2$ and any $p\ge 1$.

Using (13) and the just proved cases, we deduce (14) for any $1<q\le 2$ and any $p\ge 1$.

Next, we show (14) for $q=p$. If $p\le 2$, this was proved previously. The case $p>2$ easily follows by an iteration argument. Indeed, if $2<p\le 4$, by the two cases already proved

$$\begin{aligned}\big\|\big(\sum_i x_i^p\big)^{\frac1p}\big\|_p &= \big\|\big(\sum_i x_i^p\big)^{\frac2p}\big\|_{\frac p2}^{\frac12} \le \big\|\sum_i x_i^2\big\|_{\frac p2}^{\frac12}\\ &= \big\|\big(\sum_n x_n^2\big)^{\frac12}\big\|_p \le \|x\|_p.\end{aligned}$$

Repeating this argument, we obtain (14) for the case $q=p$.

We then deduce (14) for $1\le p\le q$ as follows

$$\big\|\big(\sum_i x_i^q\big)^{\frac1q}\big\|_p \le \big\|\big(\sum_i x_i^p\big)^{\frac1p}\big\|_p \le \|x\|_p\,.$$

Thus it remains to consider the case where $p\ge q\ge 2$. This is treated by an iteration argument as above. Indeed, if $q\le 4$, then

$$\big\|\big(\sum_i x_i^q\big)^{\frac1q}\big\|_p = \big\|\big(\sum_i (x_i^2)^{\frac q2}\big)^{\frac2q}\big\|_{\frac p2}^{\frac12} \le \big\|\big(\sum_i x_i^2\big)^{\frac12}\big\|_p \le \|x\|_p\,.$$

Thus the proof of (14), so that of (7), is complete.

Now we turn to the proof of part (ii) of the theorem. As in the proof of Theorem 2, it suffices to consider the case where $\mathcal{M}=B(\ell_2)$. Suppose that (8) holds with a constant C and some indices p,q,r such that $0<p<r<q<\infty$. Then by dividing all indices by r, we are again reduced to the case where $p<1=r<q$. Thus

$$\big\|\big(\sum_i x_i^q\big)^{\frac1q}\big\|_p \le C\,\big\|\sum_i x_i\big\|_p \tag{15}$$

for all finite sequences $(x_i)\subset L_p^+(\mathcal{M})$. We claim that C must be equal to 1. Indeed, given a positive integer k, applying (15) to the family $(x_{i_1}\otimes\cdots\otimes x_{i_k})$ we get

$$\big\|\big(\sum_{i_1,\dots,i_k} x_{i_1}^q\otimes\cdots\otimes x_{i_k}^q\big)^{\frac1q}\big\|_p \le C\,\big\|\sum_{i_1,\dots,i_k} x_{i_1}\otimes\cdots\otimes x_{i_k}\big\|_p\,.$$

However,

$$\big\|\sum_{i_1,\dots,i_k} x_{i_1}\otimes\cdots\otimes x_{i_k}\big\|_p = \big\|\big(\sum_i x_i\big)^{\otimes k}\big\|_p = \big\|\sum_i x_i\big\|_p^k\,.$$

Similarly,

$$\big\|\big(\sum_{i_1,\dots,i_k} x_{i_1}^q\otimes\cdots\otimes x_{i_k}^q\big)^{\frac1q}\big\|_p = \big\|\big(\sum_i x_i^q\big)^{\frac1q}\big\|_p^k\,.$$

It then follows that

$$\big\|\big(\sum_i x_i^q\big)^{\frac1q}\big\|_p \le C^{\frac1k}\,\big\|\sum_i x_i\big\|_p\,;$$

whence the claim by letting $k\to\infty$.

Now it is easy to construct counterexamples to (15) with $C=1$. Consider $\mathcal{M}=\mathbb{M}_2$ and the following matrices

$$x=\begin{pmatrix}1&1\\1&1\end{pmatrix}\quad\text{and}\quad y=\begin{pmatrix}0&0\\0&t\end{pmatrix}$$

with $t>0$ very small. Then

$$x+y=\begin{pmatrix}1&1\\1&1+t\end{pmatrix}\quad\text{and}\quad x^q+y^q=2^{q-1}\begin{pmatrix}1&1\\1&1+2^{1-q}\,t^q\end{pmatrix}.$$

The two eigenvalues of $x+y$ are

$$\frac{2+t\pm\sqrt{4+t^2}}{2}.$$

Thus (recalling that $p<1$)

$$\begin{aligned}\big\|x+y\big\|_p^p &= \frac{(2+t+\sqrt{4+t^2}\,)^p}{2^p} + \frac{(2+t-\sqrt{4+t^2}\,)^p}{2^p}\\ &= 2^p + 2^{-p}\,t^p + \mathrm{o}(t^p) \quad \text{as} \quad t\to 0.\end{aligned}$$

Similarly,

$$\big\|(x^q+y^q)^{\frac{1}{q}}\big\|_p^p = 2^p + 2^{-\frac{p}{q}}\,t^p + \mathrm{o}(t^p).$$

Hence, by (15) with $C=1$, we get

$$2^p + 2^{-\frac{p}{q}}\,t^p + \mathrm{o}(t^p) \le 2^p + 2^{-p}\,t^p + \mathrm{o}(t^p).$$

It then follows that $2^q \le 2$, which is a contradiction since $q>1$. Thus Theorem 3 is completely proved.

Acknowledgements. The second named author is partially supported by the French ANR project (No. ANR-19-CE40-0002) and the Natural Science Foundation of China (No.12031004).

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