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FINITE PERMUTATION GROUPS WITH FEW ORBITS UNDER THE ACTION ON THE POWER SET

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We study the orbits under the natural action of a permutation group $G \leq \mathsf{S}_n$ on the powerset $\mathscr{P}(\{1, \dots, n\})$. The permutation groups having exactly $n+1$ orbits on the powerset can be characterized as set-transitive groups and were fully classified by Beaumont and Peterson in 1955. In this paper, we establish a general method that allows one to classify the permutation groups with $n+r$ set-orbits for a given r, and apply it to integers $2 \leq r \leq 15$ with the help of GAP.

1. Introduction

Throughout the paper, we let $n \geq 2$ denote a positive integer and let $N = \{1, \dots, n\}$.[1] By a *permutation group on n letters* we mean a subgroup G of S_n endowed with the natural action $(g, x) \mapsto gx := g(x) : G \times N \to N$. We call n the *degree* of the permutation group G. The action of G on N induces an action of G on $\mathscr{P}(N)$, given by $(g, X) \mapsto gX = \{gx : x \in X\} : G \times \mathscr{P}(N) \to \mathscr{P}(N)$. In this case, the elements being acted on are subsets of N. Accordingly, we shall call the orbits under this action *set-orbits*. Note that for all $g \in G$ and $X \subseteq N$, $|gX| = |X|$. Since there are $n+1$ distinct sizes of subsets of N, it follows that there are at least $n+1$ distinct set-orbits. Additionally, this shows that all sets in the same set-orbit will have the same cardinality. A set-orbit containing sets of size t will be called a *t-set-orbit*. For a given permutation group G on n letters, the number of distinct t-set-orbits under the action of G on $\mathscr{P}(N)$ will be denoted by $s_t(G)$ and the total number of set-orbits will be denoted by $s(G)$. Clearly $s(G) = \sum_{s=0}^{n} s_t(G)$.

Definition 1.1 (Beaumont and Peterson [2]). Given an integer $0 \leq t \leq n$, a permutation group G on n letters is called *t-set-transitive* if for all t-element subsets $S, T \subseteq N$, there exists $g \in G$ such that $gS = T$.

In terms of the action of G on $\mathscr{P}(N)$, we see that G is t-set-transitive if and only if $s_t(G) = 1$. In other words, a t-set-transitive group is a permutation group with exactly one t-set-orbit. Clearly, all permutation groups are 0-set-transitive and n-set-transitive. A 1-set-transitive permutation group is simply transitive on N.

Definition 1.2 (Beaumont and Peterson [2]). A permutation group G on n letters is called *set-transitive* if G is t-set-transitive for all integers $0 \leq t \leq n$.

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[1] We exclude case $n = 1$ as the only group action on the set $\{1\}$ is the trivial action.

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Set-transitive groups were studied as early as 1944 by Neumann and Morgenstern [9]. In [2], Beaumont and Peterson proved that a set-transitive permutation group on n letters, with $n \notin \{5, 6, 9\}$, always contains the alternating group A_n. There has also been significant research devoted to bounding the number of set-orbits $s(G)$ of a degree n permutation group G. A trivial lower bound is $s(G) \geq 2^n/|G|$. In [5], Cameron proved that if G has order $\exp(o(n^{1/2}))$, then $s(G) = (2^n/|G|)(1 + o(1))$. In [1], Babai and Pyber showed that if G does not contain A_l $(l > t \geq 4)$ as a composition factor, then $\frac{1}{n} \log_2 s(G) \geq \frac{c}{t}$ for some positive constant c. In the same paper, they raised the following question: what is $\inf\left(\frac{1}{n} \log_2 s(G)\right)$ over all solvable degree n permutation groups G? This question was answered by Yang in [12].

On the other hand, there has been relatively little work done on the problem of classifying groups in terms of their number of set-orbits. Beaumont and Peterson [2] successfully classified all set-transitive permutation groups, and Kantor [6] classified 2, 3, 4-set-transitive groups which are not 2, 3, 4-transitive. These classifications lend itself to a natural generalization in the following sense. Viewed in terms on the action of G on $\mathscr{P}(N)$, we see that G is set-transitive if and only if $s(G) = n + 1$. Our paper seeks to completely classify the permutation groups G on n letters satisfying $s(G) = n + r$, for small positive integers r. The paper is laid out in the following manner. In Sections 1 and 2, we give necessary definitions and useful facts. In Section 3, we develop a general method mimicking the strategy in [2]. In Section 4, we exemplify the method by classifying groups with $n+2$, $n+3$, $n+4$, and $n+5$ set-orbits. Then we use GAP [11] to calculate all such groups for $r \leq 15$. Before continuing, we state the following useful facts from [2] and [7].

Given a permutation group G on n letters:

(1) If G contains a t-set-transitive subgroup H, then G is t-set-transitive.

(2) If G is t-transitive, then G is s-set-transitive for all positive integers $s \leq t$.

(3) The symmetric group S_n is set-transitive.

(4) The alternating group A_n is set-transitive for all $n \geq 3$.

(5) If $s_t(G) = 1$ for some $2 \leq t \leq \left\lfloor \frac{n}{2} \right\rfloor$, then G is primitive.[2]

2. Main theorems

Theorem 3 in [2] states that if G is t-set-transitive, then G is $(n-t)$-set-transitive for any integer t such that $1 \leq t \leq n-1$. It is easy to generalize this to the following observation:

Lemma 2.1. *If G is a permutation group on n letters and $0 \leq t \leq n$ is an integer, then $s_t(G) = s_{n-t}(G)$.*

Note that there are $n+1$ distinct sizes of sets in $\mathscr{P}(N)$. Hence, if n is odd, there are an even number of distinct set sizes, and so Lemma 2.1 implies that a permutation group on n letters must have an even number of set-orbits. Therefore, if n is odd and a permutation group G on n letters has $s(G) = n + r$, then r will have to be odd as well. Now consider the situation where n is even and a permutation group G on n letters has an odd number of set-orbits. In this case, since $s_t(G) = s_{n-t}(G)$ for all $0 \leq t \leq n$, it follows that $s_{n/2}(G)$ must be odd. We summarize these results in the next lemma.

[2] Throughout this paper, we let $\lfloor x \rfloor$ be the floor function.

Lemma 2.2. *Let G be a permutation group on n letters and suppose $s(G) = n + r$.*

(1) *If r is even, then n is even.*

(2) *If r is odd and n is even, then $s_{n/2}(G)$ is odd.*

Theorem 2.3 (Livingstone and Wagner [7]). *Given a permutation group G on n letters and an integer $1 \le t \le \frac{n}{2}$, we have $s_{t-1}(G) \le s_t(G)$.*

It follows that if $1 \le t < \lfloor \frac{n}{2} \rfloor - 1$, then $s_t(G) > 1$ if $s_{t-1}(G) > 1$. We will make frequent use of this fact in our classification.

Lemma 2.4. *Let G be $\left(\lfloor \frac{n}{2} \rfloor + k\right)$-set-transitive for a positive integer k. If there exists a prime p such that $\lfloor \frac{n}{2} \rfloor + k < p \le n$, then G is $(n-p+1)$-transitive.*

Proof. By Theorem 7, Corollary 1 in [2], it suffices to show that $p > \max\left(\lfloor \frac{n}{2} \rfloor + k, n - \lfloor \frac{n}{2} \rfloor - k\right) = \lfloor \frac{n}{2} \rfloor + k$. This is true by assumption, so we are done. □

Lemma 2.5. *Let k be a positive integer and let G be a permutation group on n letters that does not contain A_n. If there exists a prime p such that $\lfloor \frac{n}{2} \rfloor + k < p < \frac{2n}{3}$, then G is not $\left(\lfloor \frac{n}{2} \rfloor + k\right)$-set-transitive.*

Proof. Assume for contradiction that G is $\left(\lfloor \frac{n}{2} \rfloor + k\right)$-set-transitive, then by Lemma 2.4 such a group G is $(n-p+1)$-transitive. Since G does not contain A_n, G is at most $\left(\frac{n}{3}+1\right)$-transitive (see [4, p. 152]). Now notice that $n - p + 1 > n - \frac{2n}{3} + 1$ since $p < \frac{2n}{3}$ and since $n - \frac{2n}{3} + 1 = \frac{n}{3} + 1$, we have reached a contradiction and thus are done. □

Note that both Lemmas 2.4 and 2.5 hold for $k = 0$ when n is even. If we find a maximum k_0 for which there exists a prime p such that $\lfloor \frac{n}{2} \rfloor + k_0 < p < \frac{2n}{3}$, then G cannot be $\lfloor \frac{n}{2} \rfloor + k$ set-transitive for any $k \le k_0$. If n is odd and such a k_0 exists, then G is not $\lfloor \frac{n}{2} \rfloor + 1$ set-transitive. Thus it is not $\lfloor \frac{n}{2} \rfloor$ set-transitive either. These results are summarized in the following corollary.

Corollary 2.6. *Let G be a permutation group on n letters not containing A_n, where n is even (odd). Let k_0 be the greatest nonnegative (positive) integer such that there exists a prime p with $\lfloor \frac{n}{2} \rfloor + k_0 < p < \frac{2n}{3}$. Then for all $0 \le k \le k_0$, G is not $\lfloor \frac{n}{2} \rfloor + k$ set-transitive.*

Proof. If G is even, then the existence of such a nonnegative k_0 shows that there exists a prime p such that $\frac{1}{2} + k \le \frac{1}{2}n + k_0 < p < \frac{2n}{3}$. By Lemma 2.5, G is not $\frac{1}{2}n + k$ set-transitive. For the case of G being odd, a positive k_0 shows that there exists a prime p such that $\lfloor \frac{n}{2} \rfloor + k \le \lfloor \frac{n}{2} \rfloor + k_0 < \frac{2n}{3}$. Since this holds at least for $k = 1$, it holds for $k = 0$ since $n - \left(\lfloor \frac{n}{2} \rfloor + 1\right) = \lfloor \frac{n}{2} \rfloor$. □

Theorem 2.7. *A permutation group G on n letters not containing A_n is not $\lfloor \frac{n}{2} \rfloor + k$ set-transitive for any positive integer values of $k \le k_0$ where k_0 is the largest integer such that $48 - \frac{n+1}{2} \le k_0 \le \frac{5}{54}n - \frac{1}{2}$.*

Proof. First note that $48 \le \frac{n+1}{2} + k_0$. Due to a result by Breusch [3], which states that there exists a prime between x and $\frac{9}{8}x$ for $x \ge 48$, there exists a prime p between $\frac{n+1}{2} + k_0$ and $\frac{9}{16}n + \frac{9}{8}k_0 + \frac{9}{16}$. Note that $p > \lfloor \frac{n}{2} \rfloor + k$ for all $k \le k_0$. Since $k_0 \le \frac{5}{54}n - \frac{1}{2}$, we have

$$p < \frac{9}{16}n + \frac{9}{8}k_0 + \frac{9}{16} \le \frac{9}{16}n + \frac{9}{8}\left(\frac{5}{54}n - \frac{1}{2}\right) + \frac{9}{16} = \frac{2}{3}n.$$

Thus by Lemma 2.5, G cannot be $\lfloor \frac{n}{2} \rfloor + k$ set-transitive for any $k \le k_0$. □

Remark. For $k = 0$ the theorem still holds true for even n. For odd n, G is not $\lfloor \frac{n}{2} \rfloor$ set-transitive when a positive k_0 exists because G will not be $\lfloor \frac{n}{2} \rfloor + 1$ set-transitive. This theorem is powerful as it shows that often sets of the same size lie in different orbits. The larger n is, the more set-orbits G will have. The theorem can be applied to make an upper bound on the amount of letters G can permute and have exactly $n + r$ set-orbits. For example, the first value of n for which we get an applicable k_0 is $n = 81$, which gives $k_0 = 7$. This implies that a permutation group G on n letters not containing A_n has at least 14 additional set-orbits, which leads to a corollary.

Corollary 2.8. *If a permutation group G on n letters that does not contain A_n has less than $n + 16$ set-orbits, then $n \leq 81$.*

The following lemmas are in [1].

Lemma 2.9. *If $L \leq G \leq \mathrm{Sym}(\Omega)$, then $s(G) \leq s(L) \leq s(G) \cdot |G : L|$.*

Lemma 2.10. *Assume G is intransitive on Ω and has orbits $\Omega_1, \ldots, \Omega_m$. Let G_i be the restriction of G to Ω_i. Then*

$$s(G) \geq s(G_1) \times \cdots \times s(G_m).$$

Proof. Since $G \leq G_1 \times \cdots \times G_m$ we can apply Lemma 2.9. Clearly

$$s(G_1 \times \cdots \times G_m) = s(G_1) \times \cdots \times s(G_m). \qquad \square$$

Lemma 2.11. *Let G be a transitive permutation group acting on a set Ω where $|\Omega| = n$. Let $(\Omega_1, \ldots, \Omega_m)$ denote a system of imprimitivity of G with maximal block-size b ($1 \leq b < n$; $b = 1$ if and only if G is primitive; $bm = n$). Let N denote the normal subgroup of G stabilizing each of the blocks Ω_i. Let $G_i = \mathrm{Stab}_G(\Omega_i)$, and denote $s = s(G_1)$. Then*

$$s(G) \geq \binom{s + m - 1}{s - 1}.$$

3. Outline of methods

In this section we will outline a step-by-step method on how we fully classify groups with $n + r$ set-orbits for $2 \leq r \leq 5$ which can also be applied to classify groups with $n + r$ set-orbits for even greater r. To outline our method we first reduce the amount of letters n on which G could act, then once we have a reasonable sized list we can test a number of specific permutation groups for the remaining n values. Throughout the whole method we assume that any permutation group we consider does not contain A_n because if it did then $s(G) = n + 1$. We will exemplify how to do some of the steps in the calculation sections of this paper.

Step 0. Choose the r value for which you want to classify all groups with $s(G) = n + r$. Let $k_0 = \lfloor \frac{r-1}{2} \rfloor$ and find the smallest n such that k_0 fits in the bounds specified by Theorem 2.7. The smallest n for which any k_0 appears is $n = 81$ with $k_0 = 7$. This is "Step 0" because for any reasonable r, say $r < 16$, we know that $n \leq 81$.

Step 1. Now that we have an upper bound on n we can start eliminating some of the possible n. Right away, if r is even then we can eliminate all the odd n by Lemma 2.2. Since we need $n + r$ set-orbits, we know that G cannot be s set-transitive for at most $r - 1$ different set sizes s. This is where we can use Corollary 2.6. If n is odd then we need a k_0 value such that there is a prime $\frac{n-1}{2}+k_0 < p < \frac{2n}{3}$ and $2k_0 > r-1$. If n is even then we need a k_0 value such that there is a prime $\frac{1}{2}n+k_0 < p < \frac{2n}{3}$ and $2k_0+1 > r-1$. This is because then we would know that $s_t(G) > 1$ for $2k_0$ different s values in the odd case, and $2k_0+1$ different s values in the even case (since $s_{n/2}(G) = s_{n-n/2}(G)$). If we look at a table of primes and find any such p values for the necessary k_0 for a given n, then we can remove that n from the list of candidates.

Step 2 (Miller's method). Now we look at our remaining n values and apply a theorem of Miller [8], which states that if $n = mp_0 + r$, p_0 is prime, $m \in \mathbb{N}$, $p_0 > m$, $r > m$, then a group G on n symbols, not containing A_n, cannot be more than r-transitive. We decompose n so we have values of m, p_0, and r that fit the conditions and we try to find a sufficiently small r. Using Lemmas 2.4 and 2.5, we see if we can find a small enough p such that $\lfloor \frac{n}{2} \rfloor + k_1 < p \le n$, then $n-p+1 > r$ will contradict that G is $\lfloor \frac{n}{2} \rfloor + k_1$ set-transitive. For this, k_1 is the same as k_0 in the last step but it is not necessary that $\lfloor \frac{n}{2} \rfloor + k_1 < \frac{2n}{3}$. If we reach a contradiction, then we can remove that n from the list.

Step 3. After applying Miller's method we have reduced the number of possible groups on n letters that can have $n+r$ set-orbits. We will now reduce the number of groups even further using the following argument. For a given n and r where $r < n-4$, if $s_2(G) > 1$ then it follows that $s(G) > n+r$, in which case G does not have $n+r$ set-orbits. Thus, we can assume $s_2(G) = 1$ which implies that G is primitive by Theorem 6 in [2]. Similarly, if $r < n-2$, $s_1(G) > 1$ implies that $s(G) > n+r$, in which case G does not have $n+r$ set-orbits. By assuming $s_1(G) = 1$ in this case, we know that G is transitive by Theorem 5 in [2]. In these two cases, we will use [10] to find the structure of the transitive groups. To reduce even further, we will introduce a fact from [2] which follows simply from the orbit stabilizer theorem.

Fact 3.1. *If a permutation group G on n letters is s set-transitive, then $\binom{n}{s}$ divides $|G|$.*

Since we know $s_{k_1}(G) = 1$, then $\binom{n}{k_1}$ must divide $|G|$. Under the above assumption, G is primitive or transitive, in which case one only needs to check a few groups. If the above restriction on r and n does not hold, then we cannot use the fact that $s_t(G) = 1$ for any t. Under this situation, one must check all nontrivial subgroups of S_n. We can use GAP to compute these cases.

Step 4 (computation). At this point, we have a list of possible n for the degree of G, and a list of possible groups for each n. Now we can simply run a GAP program to calculate $s(G)$ for each candidate, and list out the ones that have $s(G) = n+r$ as desired.

Remark. We will do this process by hand for $r = 2, 3, 4, 5$ to exemplify the process and then use a GAP program to go up to $r = 15$. Since we are going in a linear order for $r = 2, 3, 4, 5$, we will often run into the same group twice. An example of this is when we consider all primitive groups on eight letters such that $\binom{8}{3} = 56$ divides $|G|$, and in a later section we consider all primitive groups such that $\binom{8}{2} = 28$ divides $|G|$. We will not consider the same groups twice if we have already calculated $s(G)$, but rather just list the groups we know have $s(G) = n+r$ from previous sections and then only consider the groups whose order is divisible by 28 but not 56. There will be several different instances where we can use previous knowledge to reduce the possible number of groups with $n+r$ set-orbits.

4. Groups with few set-orbits

Groups with $n+2$ set-orbits. Assume a group G on the set $N = \{1, \dots, n\}$ has $n+2$ set-orbits.

Looking at only even n and applying Step 1 with $k_0 = 1$, we are left with the following possibilities:

$$n = 2, 4, 6, 8, 10, 12, 14, 16, 24.$$

To exemplify Step 2, we will show one of the calculations done. Since $24 = 1 \times 19 + 5$, then G is at most 5-transitive. Using $k_1 = 1$ we need to find the smallest prime p such that $13 < p \leq 24$, so $p = 17$. Thus, if G was 13 set-transitive, then it would be $n - p + 1 = 8$-transitive, which would contradict that it is at most 5-transitive. Thus, we know a permutation group on 24 letters cannot have $s(G) = n + 2$.

We spare the reader from having to see any more of these calculations. At the end of this method, we are left with

$$n = 2, 4, 6, 8, 12.$$

Before continuing to Step 3, we take care of the trivial case of $n = 2$. The only permutation groups on two letters are the trivial group and S_2. It happens that the trivial group has $s(G) = 4$, so we must include it. We will no longer consider $n = 2$ for any r.

Now we move on to Step 3. We consider transitive groups on four letters and primitive groups on 6, 8, 12 letters such that 15, 56, 792 divides the group orders, respectively.

Now we move on to Step 4 (computation). We show all of the groups for which $s(G) = n+2$ in the following table. For the GAP ID we let nTr be TransitiveGroup(n, r) and nPr be PrimitiveGroup(n, r). We let nSr be ConjugacyClassesSubgroups(S_n)[r].

n	G	$\lvert G\rvert$	GAP ID
2	1	1	2S1
4	C_4	4	4T1
4	D_8	8	4T3
6	PSL(2, 5)	60	6P1
8	AGL(1, 8)	56	8P1
8	AΓL(1, 8)	168	8P2
8	PGL(2, 7)	336	8P5
8	ASL(3, 2)	1344	8P3
12	M_{12}	95040	12P2

Groups with $n+3$ set-orbits. From the previous computations we know that $C_2 \times C_2$ on four letters and PSL(2, 7) on eight letters have $s(G) = n+3$.

For Step 1, we use a k_0 value of 2 for odd n and 1 for even n. We find primes in the necessary range and are left with

$$n = 3, 4, \dots, 16, 19, 23, 24, 25, 43.$$

Applying Step 2, we are left with

$$n = 3, 4, 5, 6, 7, 8, 9, 11, 12.$$

For $n = 3$, we check all subgroups of S_3. For $n = 4, 5$ we check the transitive groups. For $n = 6, 7, 8, 9, 11, 12$ we check primitive groups with order divisible by 15, 21, 56, 84, 330, 792, respectively.

Note that for the even n we are already done from the previous section. We show the results for all groups with $n+3$ set-orbits in a table, using the same GAP identification key as in the previous section.

n	G	$\lvert G\rvert$	GAP ID
3	C_2	2	3S2
4	$C_2 \times C_2$	4	4T2
5	C_5	5	5T1
5	D_{10}	10	5T2
7	AGL(1, 7)	42	7P4
7	PSL(3, 2)	168	7P5
8	PSL(2, 7)	168	8P4
11	M_{11}	7920	11P6

Groups with $n+4$ set-orbits. We know that the same numbers we were unable to remove in the earlier sections will reappear. Since for even numbers we need $k_0 = 2$, the following n's can no longer be removed using Step 1: $n = 18, 22, 34, 42$. So we must continue with

$$n = 4, 6, 8, 10, 12, 14, 16, 18, 22, 24, 34, 42.$$

After Step 2 we have only

$$n = 4, 6, 8, 10, 12.$$

So we must check all subgroups of S_4, the transitive groups on six letters, and the primitive groups on 8, 10, 12 letters that whose order is divisible by 28, 120, 495. We do not recheck the primitive groups on eight letters divisible by 56.

Below we list all groups with $s(G) = n+4$.

n	G	$\lvert G\rvert$	GAP ID
4	C_3	3	4S4
4	S_3	6	4T8
6	$C_3 \times S_3$	18	6T5
6	S_4	24	6T8
6	$S_3 \times S_3$	36	6T9
6	$(C_3 \times C_3) \rtimes C_4$	36	6T10
6	$C_2 \times S_4$	48	6T11
6	$(C_3 \times C_3) \rtimes D_8$	72	6T13
10	PGL(2, 9)	720	10P4
10	PΓL(2, 9)	1440	10P7

Groups with $n+5$ set-orbits. From previous sections we have found five groups with $n+5$ set-orbits. For even n we continue to use $k_0 = 2$, and for odd n we take $k_0 = 3$. After Steps 1 and 2 we have the same even numbers as in the above section, so $n = 4, 6, 8, 10, 12$. The odd numbers after Step 1 will be the same as in the $n+3$ case, but now we must add $n = 17, 21, 33, 41$. We must apply Step 2 to the odd numbers

$$n = 5, 7, 9, 11, 13, 15, 17, 19, 21, 23, 25, 33, 41, 43.$$

After applying Step 2 we have

$$n = 4, 5, 6, 7, 8, 9, 10, 11, 12.$$

Note that once again, we handled all the necessary computations for the even n in the previous section. So we must check all subgroups of S_5, the transitive subgroups of S_7, and the primitive groups on 9, 11 letters whose order is divisible by 36, 165 but not 84, 330, respectively. Below we list all groups with $n+5$ set-orbits.

n	G	$\lvert G\rvert$	GAP ID
3	1	1	3S1
4	$C_2 \times C_2$	4	4S6
5	A_4	12	5S14
5	S_4	24	5S17
6	$C_2 \times \mathsf{A}_4$	24	6T6
6	S_4	24	6T7
7	$C_7 \rtimes C_3$	21	7P3
9	ASL(2, 3)	216	9P6
9	AGL(2, 3)	432	9P7
10	M_{10}	720	10P6

Remaining computations. We indeed classify all the cases till $r = 15$ using the same method.

The cases of $12 \le r \le 15$ require some additional work. In each of these cases we could check all subgroups of $\mathsf{S}_n (n \le 11)$ using GAP. Since our computers cannot use GAP to compute all subgroups of $\mathsf{S}_n (n \ge 12)$, we will use Lemmas 2.10 and 2.11 to eliminate nontransitive or imprimitive subgroups of S_n. It's important to note that for transitive and primitive subgroups of S_n we can still use GAP since GAP's built in library has all primitive groups of degree less than or equal to 4096. We shall discuss the case when $r = 12$ in detail and then list the results for $13 \le r \le 15$ as the process will be very similar.

In the case $r = 12$, note that we may check the number of set-orbits of all the subgroups of S_n $(n \le 11)$ using GAP, and we can also check all the transitive and primitive subgroups of S_n for n not too large. We will discuss how to handle the case when $n = 12$, and the other possible n values can be checked in a similar way.

Assume the action of group G is not transitive, then the 1-set-orbits will be partitioned into at least two orbits. Assume it is partitioned into at least three orbits, then $s_1 \ge 3$ and by Theorem 2.3, we have $s_i \ge 3$ for $1 \le i \le 11$. Thus $s(G) \ge 3\cdot 11+2 = 35 = 12+23$, and this is impossible. Then assume the 1-set-orbits is partitioned in two, then it can be a subgroup of $\mathsf{S}_1 \times \mathsf{S}_{11}$, $\mathsf{S}_2 \times \mathsf{S}_{10}$, $\mathsf{S}_3 \times \mathsf{S}_9$, $\mathsf{S}_4 \times \mathsf{S}_8$, $\mathsf{S}_5 \times \mathsf{S}_7$, $\mathsf{S}_6 \times \mathsf{S}_6$. By Lemma 2.10 we check that in the corresponding cases, the set orbits will be at least 24, 33, 40, 45, 48, 49. In which case the only possibility is a subgroup of $\mathsf{S}_1 \times \mathsf{S}_{11}$, and in this case, G is either S_{11} or A_{11}.

Assume the action of group G is transitive but not primitive, note that $12 = 2\cdot 6$, $3\cdot 4$, $4\cdot 3$, or $6\cdot 2$, and by applying Lemma 2.11, we see that all the cases will lead to more set-orbits.

Assume the action of group G is transitive and primitive, we use GAP to run through all the possible primitive groups of degree 12 and we see none of them satisfy the requirement.

We list all the results in the next few tables.

Groups with $n+6$ set-orbits

n	G	$\lvert G\rvert$	GAP ID
4	C_2	2	4S2
6	A_4	12	6S31
6	$C_5 \rtimes C_4$	20	6S38
6	A_5	60	6S50
6	S_5	120	6S53

Groups with $n+7$ set-orbits

n	G	$\lvert G\rvert$	GAP ID
5	C_4	4	5S6
5	S_3	6	5S10
5	C_6	6	5S11
5	D_8	8	5S12
5	D_{12}	12	5S15
6	D_{12}	12	6S33
7	S_5	120	7S89
7	A_6	360	7S93
7	S_6	720	7S94
8	$(((C_2 \times C_2 \times C_2 \times C_2) \rtimes C_2) \rtimes C_2) \rtimes C_3$	192	8T38
8	$(((C_2 \times C_2 \times C_2) \rtimes (C_2 \times C_2)) \rtimes C_3) \rtimes C_2$	192	8T40
8	$(((C_2 \times C_2 \times C_2 \times C_2) \rtimes C_3) \rtimes C_2) \rtimes C_3$	288	8T42
8	$((((C_2 \times C_2 \times C_2 \times C_2) \rtimes C_2) \rtimes C_2) \rtimes C_3) \rtimes C_2$	384	8T44
8	$((\mathsf{A}_4 \times \mathsf{A}_4) \rtimes C_2) \rtimes C_2$	576	8T45
8	$(\mathsf{A}_4 \times \mathsf{A}_4) \rtimes C_4$	576	8T46
8	$(\mathsf{S}_4 \times \mathsf{S}_4) \rtimes C_2$	1152	8T47
9	$(C_3 \times C_3) \rtimes C_8$	72	9T15
9	$(C_3 \times C_3) \rtimes QD_{16}$	144	9T19
12	M_{11}	7920	12P1

Groups with $n+8$ set-orbits

n	G	$\lvert G\rvert$	GAP ID
4	C_2	2	4S3
6	C_6	6	6S17
8	$((C_2 \times C_2 \times C_2) \rtimes (C_2 \times C_2)) \rtimes C_3$	96	8S242
8	$((C_2 \times C_2 \times C_2 \times C_2) \rtimes C_2) \rtimes C_3$	96	8S247
8	$(((C_2 \times C_2 \times C_2) \rtimes (C_2 \times C_2)) \rtimes C_3) \rtimes C_2$	192	8S268
8	$(((C_2 \times C_2 \times C_2 \times C_2) \rtimes C_2) \rtimes C_3) \rtimes C_2$	192	8S270
8	A_7	2520	8S293
8	S_7	5040	8S294
12	$\mathrm{PSL}(2, 11) \rtimes \mathrm{C}_2$	1320	12T218

Groups with $n+9$ set-orbits

n	G	$\|G\|$	GAP ID
5	$C_2 \times C_2$	4	5S5
6	$C_2 \times \mathsf{A}_4$	24	6S40
6	S_4	24	6S41
6	$C_2 \times \mathsf{S}_4$	48	6S49
7	A_5	60	7S81
8	$((C_2 \times C_2 \times C_2 \times C_2) \rtimes C_3) \rtimes C_2$	96	8S240
9	$(C_3 \times C_3) \rtimes Q_8$	72	9S370
9	A_8	20160	9S551
9	S_8	40320	9S552
10	S_6	720	10T32

Groups with $n+10$ set-orbits

n	G	$\|G\|$	GAP ID
6	C_5	5	6S14
6	S_3	6	6S16
6	$C_3 \times C_3$	9	6S28
6	D_{10}	10	6S29
6	$(C_3 \times C_3) \rtimes C_2$	18	6S35
6	$C_3 \times \mathsf{S}_3$	18	6S37
6	$\mathsf{S}_3 \times \mathsf{S}_3$	36	6S45
8	GL(2, 3)	48	8S216
10	A_6	360	10S1396
10	PSL(2, 8)	504	10S1448
10	$\mathrm{PSL}(2, 8) \rtimes \mathrm{C}_3$	1512	10S1539
10	A_9	181440	10S1590
10	S_9	362880	10S1591
12	PSL(2, 11)	660	12T179

Groups with $n+11$ set-orbits

n	G	$\|G\|$	GAP ID
5	C_3	3	5S4
5	S_3	6	5S9
7	D_{14}	14	7S48
7	$C_2 \times (C_5 \rtimes C_4)$	40	7S75
7	S_5	120	7S87
7	$C_2 \times \mathsf{A}_5$	120	7S88
7	$C_2 \times \mathsf{S}_5$	240	7S92
8	SL(2, 3)	24	8S154
9	$(C_2 \times C_2 \times C_2) \rtimes C_7$	56	9S355
9	$((C_3 \times C_3 \times C_3) \rtimes C_3) \rtimes C_2$	162	9S457
9	$((C_3 \times C_3 \times C_3) \rtimes C_3) \rtimes C_2$	162	9S458
9	$(C_2 \times C_2 \times C_2) \rtimes (C_7 \rtimes C_3)$	168	9S462
9	$((C_3 \times C_3 \times C_3) \rtimes C_3) \rtimes (C_2 \times C_2)$	324	9S497
9	$\mathrm{PSL}(3, 2) \rtimes \mathrm{C}_2$	336	9S499
9	$(((C_3 \times C_3 \times C_3) \rtimes (C_2 \times C_2)) \rtimes C_3) \rtimes C_2$	648	9S522
9	$(((C_3 \times C_3 \times C_3) \rtimes (C_2 \times C_2)) \rtimes C_3) \rtimes C_2$	648	9S524
9	$((((C_3 \times C_3 \times C_3) \rtimes (C_2 \times C_2)) \rtimes C_3) \rtimes C_2) \rtimes C_2$	1296	9S534
9	$(C_2 \times C_2 \times C_2) \rtimes \mathrm{PSL}(3, 2)$	1344	9S535
10	$(C_5 \times C_5) \rtimes ((C_4 \times C_4) \rtimes C_2)$	800	10S1496
10	$(C_2 \times C_2 \times C_2 \times C_2) \rtimes \mathsf{S}_5$	1920	10S1542
10	$C_2 \times ((C_2 \times C_2 \times C_2 \times C_2) \rtimes \mathsf{A}_5)$	1920	10S1543
10	$C_2 \times ((C_2 \times C_2 \times C_2 \times C_2) \rtimes \mathsf{S}_5)$	3840	10S1561
10	$(\mathsf{A}_5 \times \mathsf{A}_5) \rtimes C_2$	7200	10S1569
10	$(\mathsf{A}_5 \times \mathsf{A}_5) \rtimes (C_2 \times C_2)$	14400	10S1576
10	$(\mathsf{A}_5 \times \mathsf{A}_5) \rtimes C_4$	14400	10S1577
10	$(\mathsf{A}_5 \times \mathsf{A}_5) \rtimes D_8$	28800	10S1584
11	A_{10}	1814400	11S3091
11	S_{10}	3628800	11S3092

Groups with $n+12$ set-orbits

n	G	$\|G\|$	GAP ID
4	1	1	4S1
6	$C_4 \times C_2$	8	6S24
6	D_8	8	6S27
6	$D_8 \times C_2$	16	6S34
8	$C_7 \rtimes C_6$	42	8S196
8	PSL(3, 2)	168	8S264
10	$(C_5 \times C_5) \rtimes C_8$	200	10S1311
10	$(C_5 \times C_5) \rtimes (C_8 \rtimes C_2)$	400	10S1418
10	$(C_2 \times C_2 \times C_2 \times C_2) \rtimes \mathsf{A}_5$	960	10S1504
10	$(C_2 \times C_2 \times C_2 \times C_2) \rtimes \mathsf{S}_5$	1920	10S1541
12	A_{11}	$11!/2$	
12	S_{11}	$11!$	

Groups with $n+13$ set-orbits

n	G	$\|G\|$	GAP ID	n	G	$\|G\|$	GAP ID
5	$C_2 \times C_2$	4	5S7	8	$(C_4 \times C_4) \rtimes C_2$	32	8S181
6	D_8	8	6S26	8	$((C_4 \times C_4) \rtimes C_2) \rtimes C_2$	64	8S226
7	C_7	7	7S23	8	$((C_4 \times C_4 \times C_4) \rtimes C_4) \rtimes C_2$	64	8S227
7	$C_3 \times \mathsf{S}_3$	18	7S51	8	$((C_4 \times C_4 \times C_4) \rtimes C_4) \rtimes C_2$	64	8S228
7	$C_5 \rtimes C_4$	20	7S53	8	$(D_8 \times D_8) \rtimes C_2$	128	8S259
7	S_4	24	7S63	8	$C_2 \times \mathsf{S}_5$	240	8S272
7	$C_3 \times \mathsf{A}_4$	36	7S71	8	$C_2 \times \mathsf{A}_6$	720	8S286
7	$(C_3 \times C_3) \rtimes C_4$	36	7S73	8	S_6	720	8S287
7	$\mathsf{S}_3 \times \mathsf{S}_3$	36	7S74	8	$C_2 \times \mathsf{S}_6$	1440	8S292
7	$C_2 \times \mathsf{S}_4$	48	7S78	9	$C_9 \rtimes C_6$	54	9S354
7	$\mathsf{A}_4 \times \mathsf{S}_3$	72	7S82	9	PSL(3, 2)	168	9S460
7	$(C_3 \times \mathsf{A}_4) \rtimes C_2$	72	7S83	11	PSL(2, 11)	660	11S2754
7	$C_3 \times \mathsf{S}_4$	72	7S84	13	A_{12}	$12!/2$	
7	$(\mathsf{S}_3 \times \mathsf{S}_3) \rtimes C_2$	72	7S85	13	S_{12}	$12!$	
7	$\mathsf{S}_4 \times \mathsf{S}_3$	144	7S90				

5. Closing remarks

Now that a general method is developed for calculating all groups with $n+r$ orbits where r is not too large. The GAP code used for the calculation is available at https://www.math.txstate.edu/research-conferences/summerreu/yang_documents.html. We have successfully classified all the cases for $r \leq 15$. We remark that by far the most computationally taxing step is finding all the subgroups of S_n. If one could come up with a method to circumvent this, the classification could go much farther. Another computationally taxing step is calculating how many set-orbits a large group has. If both of these steps can be improved upon, the classification could go further.

Groups with $n+14$ set-orbits

n	G	$\lvert G\rvert$	GAP ID
6	S_3	6	6S19
6	A_4	12	6S30
6	S_4	24	6S44
8	$C_2\times\mathsf{A}_4$	24	8S157
8	$(C_2\times C_2\times C_2)\rtimes C_4$	32	8S176
8	$C_2\times\mathsf{S}_4$	48	8S214
8	$((C_2\times C_2\times C_2\times C_2)\rtimes C_2)\rtimes C_2$	64	8S229
8	S_5	120	8S257
10	$(C_5\times C_5)\rtimes D_8$	200	10S1305
10	$(C_5\times C_5)\rtimes Q_8$	200	10S1307
10	$(C_5\times C_5)\rtimes(C_4\times C_2)$	200	10S1309
10	$((C_2\times C_2\times C_2\times C_2)\rtimes C_5)\rtimes C_4$	320	10S1386
10	$(C_5\times C_5)\rtimes((C_4\times C_2)\rtimes C_2)$	400	10S1419
10	$C_2\times\big(((C_2\times C_2\times C_2\times C_2)\rtimes C_5)\rtimes C_4\big)$	640	10S1472
14	A_{13}	$13!/2$	
14	S_{13}	$13!$	

Groups with $n+15$ set-orbits

n	G	$\lvert G\rvert$	GAP ID
5	C_2	2	5S3
6	$C_2\times C_2\times C_2$	8	6S22
6	D_8	8	6S23
7	$C_2\times\mathsf{A}_4$	24	7S60
7	S_4	24	7S67
7	S_4	24	7S70
8	S_4	24	8S158
9	$(C_3\times C_3)\rtimes C_6$	54	9S352
9	$(C_3\times C_3\times C_3)\rtimes C_3$	81	9S401
9	$((C_3\times C_3)\rtimes C_3)\rtimes(C_2\times C_2)$	108	9S425
9	$((C_3\times C_3\times C_3)\rtimes C_3)\rtimes C_2$	162	9S459
9	$((C_3\times C_3\times C_3)\rtimes(C_2\times C_2)\rtimes C_3$	324	9S496
9	$(\mathsf{S}_3\times\mathsf{S}_3\times\mathsf{S}_3)\rtimes C_3$	648	9S523
9	$C_2\times\mathsf{A}_7$	5040	9S548
9	S_7	5040	9S549
9	$C_2\times\mathsf{S}_7$	10080	9S550
10	$((C_2\times C_2\times C_2\times C_2)\rtimes C_5)\rtimes C_4$	320	10S1385
13	M_{12}	95040	$\star$
15	A_{14}	$14!/2$	
15	S_{14}	$14!$	

$\star$ Here M_{12} acts transitively on 12 elements (with 14 set-orbits in this action) and trivially on the other element.

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