A Lower Bound for the Hanf Number for Joint Embedding

Will Boney
Department of Mathematics
Texas State University
San Marcos, TX 78666 USA
E-mail: wb1011@txstate.edu

Ioannis Souldatos
Department of Mathematics
Aristotle University of Thessaloniki
Thessaloniki, 54124, Greece
E-mail: souldatos@math.auth.gr

January 5, 2022

Abstract

In [13] the authors show that if μ is a strongly compact cardinal, $\boldsymbol{K}$ is an Abstract Elementary Class (AEC) with $LS(\boldsymbol{K}) < \mu$, and $\boldsymbol{K}$ satisfies joint embedding (amalgamation) cofinally below μ, then $\boldsymbol{K}$ satisfies joint embedding (amalgamation) in all cardinals $\geq \mu$. The question was raised if the strongly compact upper bound was optimal.

In this paper we prove the existence of an AEC $\boldsymbol{K}$ that can be axiomatized by an $\mathbb{L}_{\omega_1,\omega}$-sentence in a countable vocabulary, so that if μ is the first measurable cardinal, then

1. $\boldsymbol{K}$ satisfies joint embedding cofinally below μ ;

2. $\boldsymbol{K}$ fails joint embedding cofinally below μ; and

3. $\boldsymbol{K}$ satisfies joint embedding above μ.

Moreover, the example can be generalized to an AEC $\boldsymbol{K}^{\chi}$ axiomatized in $\mathbb{L}_{\chi^+,\omega}$, in a vocabulary of size χ, such that (1)-(3) hold with μ being the first measurable above χ.

This proves that the Hanf number for joint embedding is contained in the interval between the first measurable and the first strongly compact. Since these two cardinals can consistently coincide, the upper bound from [13] is consistently optimal.

This is also the first example of a sentence whose joint embedding spectrum is (consistently) neither an initial nor an eventual interval of cardinals. By Theorem 3.27, for any club C on the first measurable μ, it is consistent that JEP holds exactly on $\lim C$ and everywhere above μ.

1   Background

In [8, Conjecture 9.3], Grossberg made (essentially) the following conjecture.

Conjecture 1.1. *For every κ, there is a cardinal $\mu(\kappa)$ such that for every Abstract Elementary Class (AEC) $\boldsymbol{K}$, if $\boldsymbol{K}$ has the $\mu(LS(\boldsymbol{K}))$-amalgamation property[1] then $\boldsymbol{K}$ has the λ-amalgamation property for all $\lambda \geq \mu(LS(\boldsymbol{K}))$.*

This cardinal $\mu(LS(\boldsymbol{K}))$ (if it exists) is called the Hanf number for amalgamation, and we can define similarly what is means for a cardinal to be the Hanf number for joint embedding.

Baldwin and Boney in [13] proved the existence of a Hanf number for joint embedding (and amalgamation and a few other variants of these two properties) from large cardinals, although their definition of 'Hanf number' is slightly different from Grossberg's: if μ is a strongly compact cardinal, $\boldsymbol{K}$ is an AEC with $LS(\boldsymbol{K}) < \mu$, and $\boldsymbol{K}$ satisfies amalgamation cofinally below μ, then $\boldsymbol{K}$ satisfies amalgamation in all cardinals $\geq \mu$.

The confusion comes from the fact that Hanf number is most often used for the property of model existence. In this case, there are a number of equivalent formulations.

Fact 1.2. *Let $(\boldsymbol{K}, \prec_{\boldsymbol{K}})$ be an AEC, let $P(\lambda)$ be the property that "there exists a model of size λ in $\boldsymbol{K}$"(model-existence), and let κ be the Hanf number of $\boldsymbol{K}$. The following properties are all equivalent:*

1. *$P(\lambda)$ holds in some $\lambda \geq \kappa$;*

2. *$P(\lambda)$ holds in every $\lambda \geq \kappa$;*

---

[1]Throughout the paper we assume the common convention that $LS(\boldsymbol{K}) \geq |\tau(\boldsymbol{K})|$.

3. $P(\lambda)$ holds for cofinally *many $\lambda < \kappa$;*

4. $P(\lambda)$ holds for eventually *many $\lambda < \kappa$; and*

5. $P(\lambda)$ holds in every *cardinality.*

These equivalences heavily use the downward-closed nature of model existence (and the computation of the Hanf number using Morley's Omitting Types Theorem for (3) and (4)). However this is not the case for other properties, such as joint embedding and amalgamation, and the equivalence of (1)-(5) fails in these cases.

More importantly, the computation of the Hanf number does not use any of the specifics of the AEC $\boldsymbol{K}$ except for the $LS(\boldsymbol{K})$. So, two AECs with the same $LS(\boldsymbol{K})$ will yield the same Hanf number. This motivates Definition 1.3, the Hanf number for a general property P.

For the purposes of this paper we take the definition of the Hanf number to be that of Baldwin-Boney (cf. [13]).

Definition 1.3. Fix a property $P(\lambda)$. A function $\mu_P(\kappa)$ from cardinals to cardinals will be called the *Hanf number for P* iff it satisfies the following:

1. If $\boldsymbol{K}$ is an AEC that satisfies property $P(\lambda)$ for cofinally many $\lambda < \mu_P(LS(\boldsymbol{K}))$, then $\boldsymbol{K}$ satisfies $P(\lambda)$ in every $\lambda \geq \mu_P(LS(\boldsymbol{K}))$ and

2. $\mu_P(LS(\boldsymbol{K}))$ is the least such cardinal.

If P is obvious from the context, then we will write $\mu(\kappa)$ instead of $\mu_P(\kappa)$.

In the language of Fact 1.2, Definition 1.3 defines the Hanf number for property P to be the least cardinal such that the implication $(3) \to (2)$ holds true for all AECs with the same LS-number. Because the Hanf number is a function, we investigate certain values of this function. We fix property P and $LS(\boldsymbol{K})$ and we ask the value of $\mu_P(LS(\boldsymbol{K}))$. The example stated in the abstract proves that $\mu_{JEP}(\chi)$ cannot be smaller than the first measurable cardinal above χ.

Properties P of interest are joint embedding, amalgamation, categoricity, existence of maximal models, etc. In this paper, we focus on the properties of joint embedding (JEP) and amalgamation (AP), especially the former.

In a parallel line of inquiry we ask the following: Fix $LS(\boldsymbol{K})$ and let P be the property of joint embedding, or amalgamation, or some other property of interest. For which of the implications between (1)-(5), other

than $(3) \rightarrow (2)$, there exists some κ that makes the implication true? What is the least such κ? Provide counterexamples when no such κ exists.

One of the contributions of this paper is to provide an example that (3) does not imply (4) when P is JEP and κ is the first measurable[2]. In particular, as stated in the abstract, there exists an AEC $\boldsymbol{K}$ that both satisfies and fails JEP cofinally below the first measurable.

This example also relates to Question 4.0.2 in [3], that asks whether there is an AEC, in particular one defined by an $\mathbb{L}_{\omega_1,\omega}$-sentence, whose finite amalgamation spectrum is not an interval (that is, amalgamation restricted to the $\aleph_n$ for $n < \omega$). If we drop the restriction to a finite spectrum and replace AP by JEP, we prove that the answer is positive. Notice that while our example exhibits quite interesting JEP-spectrum, the same is not true for the AP-spectrum. By Theorem 4.4, $\boldsymbol{K}^\chi$ fails amalgamation in every cardinal above 2^{χ^+}. This leaves open the corresponding question (does (3) imply (4)), when P is the amalgamation property and κ is the first measurable cardinal.

If P is either JEP or AP, and κ the first measurable or the first strongly compact cardinal, the status of the implications $(1) \rightarrow (2)$ (as envisioned by Grossberg) and $(2) \rightarrow (3)$ also remains open.

Open Questions 1.4.

1. *Is there an AEC $\boldsymbol{K}$ that fails JEP eventually below κ, κ being the first measurable or the first super compact, but satisfies JEP in all cardinalities above κ?*

2. *Same question as (1), but satisfy JEP in* one *cardinality above κ?*

Although in this paper we will not deal with it, we ought to say that similar considerations about categoricity have long occupied researchers in the area, e.g. the main open problem for AECs (Shelah's Categoricity Conjecture) is whether there exists some κ such that (1) implies (2) for categoricity[3].

A lower bound for $\mu_{AP}(\kappa)$ is given in [11]. For every cardinal κ and every α with $\kappa \leq \alpha < \kappa^+$ there exists an AEC W_α in a vocabulary of size κ, W_α has countable Löwenheim-Skolem number, and it satisfies amalgamation up to $\beth_\alpha$, but fails amalgamation in $\beth_{\kappa^+}$. This proves that $\mu_{AP}(\kappa) \geq \beth_{\kappa^+}$.

---

[2]In this example $LS(\boldsymbol{K}) = \aleph_0$, but the construction works for any $LS(\boldsymbol{K})$ smaller than the first measurable cardinal.

[3]Categoricity means that $P(\lambda)$ is the property that "there exists a unique model (up to isomorphism) of size λ".

On the other hand, results from [13], prove that if $LS(\boldsymbol{K}) = \kappa$, then $\mu_{AP}(\kappa)$ is no larger than the first strongly compact above κ. Combining the results from [11] and [13], $\mu_{AP}(\kappa)$, the Hanf number for AP, is contained in the interval between $\beth_{\kappa^+}$ and the first strongly compact above κ. As noted in [13], the gap between these two cardinals is immense.

The picture is not very promising for JEP either. From [13], we know that the first strongly compact above κ is also an upper bound for $\mu_{JEP}(\kappa)$. A lower bound is given by the examples from [4] and [5]. From Fact 3.37 in [4], for every countable α there exists an AEC $(\boldsymbol{K}_\alpha, \subset)$ defined by a universal $\mathbb{L}_{\omega_1,\omega}$-sentence such that $LS(\boldsymbol{K}_\alpha) = \aleph_0$ and $\boldsymbol{K}_\alpha$ satisfies JEP up to and including $\beth_\alpha$, but it has no larger models. This proves that $\beth_{\omega_1}$ is a lower bound for $\mu_{JEP}(\aleph_0)$, the Hanf number for JEP.

The above example can easily be generalized: for every $\kappa \leq \alpha < \kappa^+$, there exists an AEC $(\boldsymbol{K}_\alpha, \subset)$ defined by a universal $\mathcal{L}_{\kappa^+,\omega}$-sentence such that $LS(\boldsymbol{K}_\alpha) = \kappa$ and $\boldsymbol{K}_\alpha$ satisfies JEP up to and including $\beth_\alpha$, but it has no larger models. This proves that as in the case for AP, $\mu_{JEP}(\kappa)$ is contained in the interval between $\beth_{\kappa^+}$ and the first strongly compact above κ.

Stimulated by early versions of [5], Baldwin and Shelah announced in [1] that (under certain set theoretic hypotheses) there exists a *complete* $\mathbb{L}_{\omega_1,\omega}$-sentence with maximal models in cofinally many cardinalities below the first measurable. They also note that every model of size equal to or larger than the first measurable has a proper $\mathbb{L}_{\omega_1,\omega}$-elementary extension; one can take a countably complete ultrapower. This proves the first measurable cardinal to be equal to $\mu_P(\aleph_0)$, where P is the property of "maximality". More precisely, "maximality" is defined to be the property $P(\lambda)$: the exists a maximal model of size λ.

Notice that this introduces yet another notion of a Hanf number. If μ is the first measurable cardinal, then no $\mathbb{L}_{\omega_1,\omega}$-sentence has a maximal model above μ. In the language of Fact 1.2, letting P be "maximality" and κ be the first measurable, (1) and (2) are equivalent for the trivial reason that they are both always false. The example from [1] proves that (3) does not imply (1) (or (2)) for maximality. So, this notion of a Hanf number does not fit into Definition 1.3, which is our working definition for this paper.

The main result of this paper is to prove a lower bound for the Hanf number for JEP. In particular, we show that $\mu_{JEP}(\aleph_0)$ is bounded below by the first measurable cardinal. Since by results of [12, Theorem 3.1], the first measurable and the first strongly compact can be consistently equal, this

proves that the known bounds are consistently optimal. Of course, the first measurable and the first strongly compact can also be different. So, this leaves open the question whether the lower bound from the current paper or the upper bound from [13] can be improved.

Our example was inspired by [1]. In fact, the idea of the main construction in the current paper appears as Example 3.0.3. in [1]. Our contribution is to compute the JEP- and AP- spectra of this example. Notice that our $\mathbb{L}_{\omega_1,\omega}$-sentence (as well as the sentence from Example 3.0.3 in [1]) is incomplete. This is in contrast to the main construction of [1], where Baldwin and Shelah go in great lengths to create a *complete* sentence. It is an open question whether the results from this paper can extend to the complete example of [1].

In Section 2 we describe the construction. The JEP-spectrum is given in Section 3 and the AP-spectrum in Section 4.

For the main definitions of Abstract Elementary Classes, we refer the reader to Baldwin [2]; however, they are not necessary to understand the main construction and theorems.

2   The main example

Fix an infinite cardinal χ, although this can be taken to be $\aleph_0$. We will define a language τ^χ and an AEC $\boldsymbol{K}^\chi$.

The prototypical elements of $\boldsymbol{K}^\chi$ will be structures of the form

$$M = (\kappa, \mathcal{P}(\kappa), {}^\chi\mathcal{P}(\kappa), \in, \vee, \wedge, \cdot^c, \mathbf{1}, \cap, \pi_\alpha)_{\alpha<\chi}$$

where

1. ${}^\chi\mathcal{P}(\kappa)$ are the χ-length sequences from $\mathcal{P}(\kappa)$ (the power set of κ);

2. $\in$ is the (extensional) 'member of' relation between κ and $\mathcal{P}(\kappa)$;

3. $(\mathcal{P}(\kappa), \vee, \wedge, \cdot^c, \mathbf{1})$ is the standard Boolean Algebra;

4. $\cap : {}^\chi\mathcal{P}(\kappa) \to \mathcal{P}(\kappa)$ returns the intersection of all elements of the sequence; and

5. $\pi_\alpha : {}^\chi\mathcal{P}(\kappa) \to \mathcal{P}(\kappa)$ returns the αth element of the sequence.

However, we don't want the entire theory of these structures and only demand that the members of $\boldsymbol{K}^\chi$ satisfy a particular subset of their $\mathbb{L}_{\chi^+,\omega}$-theory.

Formally, set τ^χ to consist of

- sorts K, P, and Q;

- constant $\mathbf{1}$;

- unary functions $\cdot^c$, $\cap$, and π_α for $\alpha < \chi$;

- binary functions $\wedge$ and $\vee$; and

- a binary relation ε.

Now we define a sentence $\psi^\chi \in \mathbb{L}_{\chi^+,\omega}$ that consists of the conjunction of the following first-order assertions

1. $\varepsilon \subset K \times P$ is extensional;

2. $(P, \vee, \wedge, \cdot^c, \mathbf{1})$ is a Boolean Algebra;

3. ε interacts with the Boolean Algebra operations in the expected way: $\mathbf{1}$ contains every element of K, a complement contains exactly the elements not in the original, etc;

4. P contains all singletons: $\forall x \in K \exists Y \in P \forall z \in K \, (z \in Y \iff z = x)$; and

the conjunction of the following infinitary assertions:

(5) the functions $\pi_\alpha : Q \to P$ are jointly extensional in the sense that

$$\forall A, B \in Q \left(A = B \iff \bigwedge_{\alpha < \chi} \pi_\alpha(A) = \pi_\alpha(B) \right)$$

(6) the function $\cap : Q \to P$ returns the intersection in the sense that

$$\forall A \in Q \forall x \in K \left(x\varepsilon \cap A \iff \bigwedge_{\alpha < \chi} x\varepsilon\pi_\alpha(A) \right)$$

Given any model M of ψ^χ, there is a natural embedding of the Boolean Algebra induced on P^M into the Boolean Algebra on $\mathcal{P}(K^M)$, the powerset of K^M.

Definition 2.1. Let $M \subset N$ model ψ^χ and $X \in P^M$.

1. Define $\widehat{X}^M := \{x \in K^M \mid M \vDash x\varepsilon X\}$.

2. *M and N agree on X* iff $\widehat{X}^M = \widehat{X}^N$.

3. M *and* N *agree on finite subsets* iff they agree on every $X \in P^M$ such that $\widehat{X}^M$ is a finite subset of K^M.

4. Define $\boldsymbol{K}^\chi$ to be the collection of all models of ψ^χ of size $\geq \chi$. For $M, N \in \boldsymbol{K}^\chi$, let $M \prec_{\boldsymbol{K}}^\chi N$ if $M \subset N$ and M and N agree on finite subsets.

   Following standard convention, we will often use $\boldsymbol{K}^\chi$ to refer to the pair $(\boldsymbol{K}^\chi, \prec_{\boldsymbol{K}}^\chi)$.

Crucial to our later analysis, the lack of full elementarity in the $\prec_{\boldsymbol{K}}^\chi$ relation means that M and N do not need to agree on all sets. In particular, M and N agree on $\mathbf{1}$ iff $K^M = K^N$. Note also that $\widehat{X}^M = \widehat{X}^N \cap K^M$.

Proposition 2.2. *For each* χ, $(\boldsymbol{K}^\chi, \prec_{\boldsymbol{K}}^\chi)$ *is an Abstract Elementary Class with Löwenheim-Skolem number* χ.

Proof. This is an easy argument and we leave the details to the reader. One approach is that a definitional expansion (by adding "finite subset" functions) turns $\boldsymbol{K}^\chi$ into a universal class. $\square$

Observation 2.3.

1. *It is immediate from the definition that the size of* P *is bounded by* $2^{|K|}$ *and the size of* Q *by* $|P|^\chi$. *There is no restriction on the size of* K. *Therefore,* ψ^χ *has models in all infinite cardinalities.*

2. *Given any model* $M \vDash \psi^\chi$, *we can find an isomorphic copy* $\widehat{M}$ *of* M *such that*

   (a) $K^{\widehat{M}} = K^M$;

   (b) $P^{\widehat{M}} \subset \mathcal{P}(K^M)$; *and*

   (c) $Q^{\widehat{M}} \subset {}^\chi \mathcal{P}(K^M)$.

   However, this choice is very non-canonical.

Definition 2.4.

1. An AEC $\boldsymbol{K}$ satisfies the κ-*Joint Embedding Property* or JEP(κ) iff $\boldsymbol{K}_\kappa$ is not empty and for all $M_0, M_1 \in \boldsymbol{K}_\kappa$, there are $N \in \boldsymbol{K}$ and $\boldsymbol{K}$-embeddings $f_\ell : M_\ell \to N$. The *joint embedding spectrum of* $\boldsymbol{K}$ or JEP-spectrum is the collection of all cardinals κ such that $\boldsymbol{K}$ satisfies JEP(κ).

2. Similarly, define $AP(\kappa)$ and the AP-spectrum for the amalgamation property.

3. A model M of ψ^χ is of type (λ, κ) with $\lambda \geq \kappa$, if $|M| = \lambda$ and $|K^M| = \kappa$.

4. Given $M \in \mathbf{K}^\chi$, we say that an ultrafilter U on the Boolean Algebra P^M is Q^M-*complete* iff for every $A \in Q^M$, if $\pi_\alpha^M(A) \in U$ for all $\alpha < \chi$, then $\cap^M(A) \in U$.

3 The Joint Embedding Property

In this section we determine the joint embedding spectrum of $\mathbf{K}^\chi$. It turns out (Lemma 3.5) that the question of when two models can be jointly embedded depends on whether they can be extended in a certain way, called K-extendibility.

Definition 3.1. Let $M \in \mathbf{K}^\chi$.

1. M is K-*extendible* iff there is $N \in \mathbf{K}^\chi$ such that $M \prec_{\mathbf{K}}^\chi N$ and $K^M \subsetneq K^N$.

2. M is K-*maximal* iff whenever $N \in \mathbf{K}^\chi$ has $M \prec_{\mathbf{K}}^\chi N$, then $K^M \subsetneq K^N$.

So given a K-maximal model, it is either K-extendible or truly maximal. The next lemma connects K-extendibility to the existence of sufficiently complete ultrafilters.

Lemma 3.2. *Let $M \in \mathbf{K}^\chi$. The following are equivalent:*

1. *M is K-extendible.*

2. *For all cardinals λ, there exists some $N \in \mathbf{K}^\chi$ with $M \prec_{\mathbf{K}}^\chi N$ and $|K^N \setminus K^M| = \lambda$.*

3. *There exists a non-principal ultrafilter U on P^M that is Q^M-complete.*

Proof. The implication (2) $\Rightarrow$ (1) is immediate. We start by proving that (1) $\Rightarrow$ (3). Suppose we have $M, N \in \mathbf{K}^\chi$ that witness the K-extendibility of M. By assumption, there exists some $d \in K^N \setminus K^M$. We then build an ultrafilter $U_d \subset P^M$ as follows: for $X \in P^M$, we set

$$X \in U_d \text{ if and only if } N \vDash d\varepsilon X$$

Then U_d is an ultrafilter by the standard set-theoretic argument; see, e.g. Theorem 5.6 in [10].[4] We only prove U_d is non-principal and Q^M-complete.

Assume that U_d is principal and generated by some $X_0 \in P^M$. By definition, P^M contains all finite subsets of K^M.[5] It follows that $\widehat{X_0}^M$ is a singleton, say $\widehat{X_0}^M = \{x\}$. By definition of $\prec_K$, M, N agree on all finite subsets of K^M. In particular, $\widehat{X_0}^M = \widehat{X_0}^N$. By the definition of U_d, this implies that $d = x \in K^M$. Contradiction.

Let $A \in Q^M$ such that $X_\alpha := \pi_\alpha^M(A) \in U_d$, for all $\alpha < \chi$. Since $N \vDash d\varepsilon X_\alpha$ for every $\alpha < \chi$, we must have $N \vDash d\varepsilon \cap A$. Thus, $\cap^N(A) = \cap^M(A) \in U_d$.

Next we prove that $(3) \Rightarrow (2)$. Suppose that $M \in \boldsymbol{K}^\chi$ and there is a non-principal, Q^M-complete ultrafilter U on P^M. Without loss of generality M satisfies the conclusions of Observation 2.3.(2). Define N to be an $\prec_{\boldsymbol{K}}^\chi$-extension of M as follows: Extend K^M by λ-many new elements to form K^N. Say $K^N = K^M \cup \lambda$. Define ε^N on the new elements: For all $x \in K^N \setminus K^M$ and all $X \in P^M$, stipulate that

$$N \vDash x\varepsilon X \iff X \in U.$$

This implies that for every $X \in P^M$, either $\widehat{X}^N = \widehat{X}^M$, or $\widehat{X}^N = \widehat{X}^M \cup (K^N \setminus K^M)$. Which of the two is the case is determined by membership in U.

Additionally, we extend P so that P^N is the Boolean Algebra that is generated by P^M together with all finite subsets of K^N. In particular, for every F finite subset of K^N which is not a subset of K^M, we introduce a new element X_F in P^N and define ε^N is such a way that $\widehat{X_F}^N = F$.

Moreover, M and N agree on Q, π_α and $\cap$, and on $\varepsilon \restriction_{K^M \times P^M}$, and $\vee, \wedge, ^c$ have the intended interpretations.

We verify that N is in $\boldsymbol{K}^\chi$ and $M \prec_{\boldsymbol{K}}^\chi N$. ε^N is extensional because the same is true for ε^M and by definition of N. Condition (2) is immediate and condition (3) is easy to verify using the fact that U is an ultrafilter. We leave the details to the reader. (4) and (5) are immediate.

We only prove (6) using the Q-completeness of U. If $A \in Q^N = Q^M$, then $Y = \cap^N A$ is in U if and only if for all $\alpha, \pi_\alpha(A) \in U$. The one direction follows from U being an ultrafilter, the other direction follows from U being Q-complete.

---

[4]Another approach is to observe that U_d is the set of all sets whose μ-measure equals to 1, where μ is defined on P^M by $\mu(X) = 1$ when $N \vDash d \in X$, and $\mu(X) = 0$ otherwise. U_d is an ultrafilter because μ is 2-valued.

[5]This means that if K_0 is a finite subset of K^M, there exists some $X \in P^M$ with $\widehat{X}^M = K_0$. See also clause (4) of ψ^χ.

We take cases: If $Y = \cap^N A \in U$, then $\widehat{Y}^N = \widehat{Y}^M \cup (K^N \setminus K^M)$ and $\widehat{\pi_\alpha(A)}^N = \widehat{\pi_\alpha(A)}^M \cup (K^N \setminus K^M)$, for all α. If $Y \notin U$, then $\widehat{Y}^N = \widehat{Y}^M$ and $\widehat{\pi_\alpha(A)}^N = \widehat{\pi_\alpha(A)}^M$, for *some* α (maybe not for all α). In either case, $\widehat{Y}^N = \cap_\alpha \widehat{\pi_\alpha(A)}^N$, which completes the proof that $N \in \boldsymbol{K}^\chi$.

It remains to prove $M \prec^\chi_{\boldsymbol{K}} N$. The fact that $M \subset N$ is immediate. Let X be in P^M so that $\widehat{X}^M$ is a finite set. We need to prove that $\widehat{X}^N = \widehat{X}^M$. Since $\widehat{X}^M$ is finite and U is non-principal, $X \notin U$. Therefore, $\widehat{X}^N = \widehat{X}^M$ as desired. $\qquad\square$

Corollary 3.3. *Assume $M \prec^\chi_{\boldsymbol{K}} N$. If M is not K-extendible, then the same is true for N.*

Proof. By definition of K-extendibility, K^N must equal K^M. Since $M \subset N$, P^M is a sub-Boolean Algebra of P^N. If N were K-extendible, by Lemma 3.2, there would be a non-principal ultrafilter U on P^N which is Q^N-complete. The restriction of U on P^M leads to a contradiction. $\qquad\square$

The key factor for determining whether two $M_0, M_1 \in \boldsymbol{K}^\chi$ can be jointly embedded is the size of K^{M_0} and K^{M_1}. If $|K^{M_0}| = |K^{M_1}|$, then this is possible as seen by the next Lemma 3.4. In fact more is true: For every κ, there is a single structure $M \in \boldsymbol{K}^\chi$ that can embed all other $M' \in \boldsymbol{K}^\chi$ with $|K^{M'}| = \kappa$. This universality property holds true because the Boolean Algebra $(P, \vee, \wedge, {}^c, \boldsymbol{1}, \boldsymbol{0})$ interpreted in M (the universal model) coincides with the Boolean Algebra of the power set of K^M. If $|K^{M_0}| < |K^{M_1}|$, then M_0, M_1 can be jointly embedded if and only if M_0 is K-extendible. This is the content of Lemma 3.5

Lemma 3.4. *Let $\kappa \geq \chi$. There exists a model $M \in \boldsymbol{K}^\chi$ of type $(2^\kappa, \kappa)$ such that for any other $N \in \boldsymbol{K}^\chi$ with $|K^N| = \kappa$, there is an $\prec^\chi_{\boldsymbol{K}}$-embedding from N to M. Moreover, M is K-maximal.*

Proof. Let M be the standard model generated by $K^M = \kappa$, $P^M = \mathcal{P}(\kappa)$, $Q^M = {}^\chi\mathcal{P}(\kappa)$. We claim that M is the desired model.

Let $N \in \boldsymbol{K}^\chi$ with $|K^N| = \kappa$. Define a $\boldsymbol{K}^\chi$-embedding f from N to M as follows: f is a bijection from K^N to κ; this is possible by cardinality assumptions. For each $X \in P^N$, set $f(X) = \{f(x) \mid x \in \widehat{X}^N\} \in \mathcal{P}(\kappa)$. For each $A \in Q^N$, let $f(A)$ be the sequence $\langle f\left(\pi^N_\alpha(A)\right) \mid \alpha < \chi \rangle \in {}^\chi\mathcal{P}(\kappa)$. It is immediate that f is an $\boldsymbol{K}^\chi$-embedding.

For the moreover, the extensionality of $\in$ and the π_α imply that any extension of M cannot grow P or Q without growing K. $\qquad\square$

Lemma 3.5. *Let $M_0, M_1 \in \boldsymbol{K}^\chi$. If $|K^{M_0}| < |K^{M_1}|$, then M_0 and M_1 can be jointly embedded if and only if M_0 is K-extendible.*

Proof. Left-to-right: Suppose that M_0 and M_1 can be jointly embedded into some $M \in \boldsymbol{K}^\chi$. Then $|K^M| \geq |K^{M_1}| > |K^{M_0}|$. By Definition 3.1, M_0 is K-extendible.

Right-to-left: Assume that M_0 is K-extendible. By Lemma 3.2, there exists some $M_0' \in \boldsymbol{K}^\chi$, $M_0 \prec_{\boldsymbol{K}}^\chi M_0'$ and $|K^{M_0'}| = |K^{M_1}|$. Use Lemma 3.4 to joint embed M_0' and M_1 to a common M. Then M serves also as the joint embedding of M_0 and M_1.   □

Recall that there are no countably complete, non-principal ultrafilters on any set of size less than the first measurable cardinal. More generally, fixing χ, there is no χ^+-complete, non-principal ultrafilter on any set of size less than the first measurable larger than χ (if one exists). If there is no measurable above χ, then there are no χ^+-complete, non-principal ultrafilters at all. We utilize these facts to prove the existence of models that are not K-extendible.

Definition 3.6. Given a cardinal κ, set

$$m(\kappa) := \inf\{\lambda \mid \lambda > \kappa \text{ and } \lambda \text{ is measurable}\}$$

If there are no measurable cardinals above κ, then set $m(\kappa) = \infty$ (which is greater than every ordinal by convention).

Lemma 3.7. *Let $\chi \leq \kappa < m(\chi)$. There exists a model $M \in \boldsymbol{K}^\chi$ of type $(2^\kappa, \kappa)$ that is $\prec_{\boldsymbol{K}}^\chi$-maximal. In particular, M is not K-extendible.*

Proof. Let M be the standard model on $(\kappa, \mathcal{P}(\kappa), {}^\chi\mathcal{P}(\kappa))$ as in Lemma 3.4. We know it is K-maximal, and we claim that M is maximal. By the remark following Definition 3.1, it suffices to show that it is not K-extendible.

If it were K-extendible, then Lemma 3.2 would imply that there is a Q^M-complete, non-principal ultrafilter U on the Boolean Algebra P^M. However, since $P^M = \mathcal{P}(\kappa)$, U is an ultrafilter on κ. Moreover, since $Q^M = {}^\chi\mathcal{P}(\kappa)$, the Q^M-completeness of U is a different name for χ^+-completeness (in the normal sense). This would imply that there is a χ^+-complete ultrafilter on κ; however, this contradicts $\kappa < m(\chi)$.   □

Theorem 3.8. *Let $\chi \leq \kappa < m(\chi)$. Then $\boldsymbol{K}^\chi$ fails $JEP(2^\kappa)$.*

Proof. Let M be the model from Lemma 3.7. M is of type $(2^\kappa, \kappa)$. Let N be any model in $\boldsymbol{K}$ of type $(2^\kappa, 2^\kappa)$. Take for instance $K^N = 2^\kappa$, P^N contains all finite and co-finite subsets of K^N and Q^N is empty.

By Lemma 3.5, M and N can be jointly embedded if and only if M is K-extendible. But M is not K-extendible by Lemma 3.7, which proves the theorem. $\square$

Once we are above a measurable, K-extendibility and, therefore, joint embedding become trivial to accomplish.

Lemma 3.9. *If $\kappa \geq m(\chi)$, then every model in $\boldsymbol{K}^\chi_\kappa$ is K-extendible.*

Proof. Let U be a $m(\chi)$-complete, non-principal ultrafilter and $M \in \boldsymbol{K}^\chi_\kappa$. Since measurable cardinals are strong limits, $|K^M| \geq m(\chi)$. By Łoś' Theorem for infinitary logics, the ultrapower $\prod M/U$ is a $\mathbb{L}_{m(\chi),m(\chi)}$-elementary extension of M (up to isomorphism). In particular, $M \prec^\chi_{\boldsymbol{K}} \prod M/U \in \boldsymbol{K}^\chi$. Moreover, $K^{\prod M/U} = \prod K^M/U \supsetneq K^M$. This witnesses that M is K-extendible. $\square$

Theorem 3.10. *If $\kappa \geq m(\chi)$, then $\boldsymbol{K}^\chi$ satisfies $JEP(\kappa)$.*

Proof. As we noted, if $M \in \boldsymbol{K}^\chi_\kappa$, then $|K^M| \geq m(\chi)$. The statement follows from Lemma 3.4 in the case $|K^{M_0}| = |K^{M_1}|$, or otherwise from Lemmas 3.5 and 3.9. $\square$

We can also show that joint embedding holds at strong limit cardinals.

Theorem 3.11. *If $\kappa \geq \chi$ is a strong limit cardinal, then $\boldsymbol{K}^\chi$ satisfies $JEP(\kappa)$.*

Proof. The strong limit assumption implies that if $M \in \boldsymbol{K}^\chi_\kappa$, then $|K^M| = \kappa$ by Observation 2.3. To prove $JEP(\kappa)$, let $M_0, M_1 \in \boldsymbol{K}^\chi$ both of size κ. It follows that $|K^{M_0}| = |K^{M_1}| = \kappa$, so they are both jointly embeddable into the universal model N given by Lemma 3.4. Since $\boldsymbol{K}^\chi$ is an AEC (Proposition 2.2), there is $N' \prec^\chi_{\boldsymbol{K}} N$ of size κ that contains the images of M_0, M_1. This proves $JEP(\kappa)$. $\square$

Notice that Theorem 3.11 holds true even for $\kappa = \chi$.

Suppose that there is a measurable above χ and let $\mu = m(\chi)$. Then Theorem 3.11 yields a cofinal sequence in μ on which joint embedding holds, while Theorem 3.8 yields a cofinal sequence on which JEP fails. Under GCH this gives a complete characterization of the JEP-spectrum of $\boldsymbol{K}$.

Corollary 3.12. *Assume GCH holds.* $\boldsymbol{K}^\chi$ *satisfies* $JEP(\kappa)$ *if and only if* $\kappa \geq \chi$ *is a limit cardinal below* $m(\chi)$ *or* $\kappa \geq m(\chi)$.

Corollary 3.13. *The Hanf number* $\mu_{JEP}(\aleph_0)$ *is at least a measurable cardinal.*

In the rest of this section we determine the joint embedding spectrum even when GCH fails, in particular for cardinals $\chi \leq \kappa < m(\chi)$ such that $\kappa < 2^{<\kappa}$. Our results shows that $JEP(\kappa)$ fails for almost all such cardinals. Depending on cardinal arithmetic there might be some cardinals for which our method does not give an answer. The precise statement is given in Theorem 3.24.

Our abstract tool is the following lemma, which reduces our problem to finding a particular model.

Lemma 3.14. *Suppose that* $\kappa \geq \chi$ *and* $\lambda < \kappa \leq 2^\lambda$. *If* $\boldsymbol{K}^\chi$ *contains a model of type* (κ, λ) *that is not K-extendible, then* $\boldsymbol{K}^\chi$ *fails* $JEP(\kappa)$.

Proof. Let M be a model as in the assumption and let N be a model of type (κ, κ). Then $\|M\| = \|N\| = \kappa$ and $|K^N| > |K^M|$. By Lemma 3.5, M and N cannot be jointly embedded, so $JEP(\kappa)$ fails. $\square$

When $\kappa = \chi$, such a model is easy to build.

Lemma 3.15. *There exists a model* $M \in \boldsymbol{K}^\chi$ *of type* (χ, χ) *which is not K-extendible.*

Proof. Define M by setting $K^M = \chi$; P^M is the Boolean Algebra that is generated by all finite subsets of χ plus all tails of the form $[\alpha, \chi)$, with $\alpha < \chi$; and Q^M contains for every limit ordinal $\beta \leq \chi$ the sequence of tails $\langle [\alpha, \chi) \mid \alpha < \beta \rangle$.

A couple of remarks are at hand before we proceed:

1. We do not impose any completeness requirements on P^M. This means that the size of P^M is equal to χ.

2. If β is a limit ordinal less than χ, then the sequence of tails in Q^M, $\langle [\alpha, \chi) \mid \alpha < \beta \rangle$, has length smaller than χ. Since sequences in Q^M must be of length χ, we tacitly assume that some of the sets in this sequence repeat.

It follows that M has type (χ, χ) and $M \in \boldsymbol{K}^\chi$.

We also claim that M is not K–extendible, or, equivalently by Lemma 3.2, there is no non-principal Q^M-complete ultrafilter on P^M. Assume otherwise and let U be such an ultrafilter. We prove by induction on α that $[\alpha, \chi) \in U$. For the successor stage use the fact that U is non-principal. For the limit stages use the fact that U is Q^M-complete.

Therefore, U contains all the sets of the form $[\alpha, \chi)$, $\alpha < \chi$. Using Q^M-completeness once more, U must contain $\cap_\alpha [\alpha, \chi) = \emptyset$. Contradiction. $\square$

Corollary 3.16. $\boldsymbol{K}^\chi$ *fails* $JEP(\kappa)$ *for all* $\chi^+ \le \kappa \le 2^\chi$.

Proof. Let $\chi^+ \le \kappa \le 2^\chi$ and let M be the model from Lemma 3.15. Define some $N \in \boldsymbol{K}^\chi$ that extends M and has size κ: $K^N = K^M$, P^N is a Boolean Algebra that extends P^M and has size κ, and $Q^N = Q^M$. If N were K-extendible, then by Lemma 3.2, there would be a non-principal Q^N-complete ultrafilter on P^N. The restriction of U on P^M contradicts Lemma 3.15. $\square$

Our next goal is to extend Lemma 3.15 and Corollary 3.16 to higher cardinalities. This will be achieved in Corollary 3.22. We need some preliminary work before we can prove Corollary 3.22.

The proof of the following fact is standard, see, e. g., [7, Lemma 4.2.3].

Fact 3.17. *Let B be a Boolean Algebra on κ and let U be an ultrafilter on B. Then U is λ-complete, for some $\lambda \le \kappa$, if and only if for every $W \subset B$, a partition of B of size $|W| < \lambda$, there exists some $w \in W$ that belongs to U.*

Recall that a cardinal κ is weakly compact if and only if for every κ-complete Boolean Algebra $B \subset P(\kappa)$ generated by κ-many subsets, there is a κ-complete non-principal ultrafilter on B. We will consider a weakening of this large cardinal notion as in [6, Definition 2.1].

Definition 3.18. A cardinal κ is δ-*weakly compact* for $\delta \le \kappa$ iff every κ-complete Boolean Algebra $B \subset P(\kappa)$ generated by κ-many subsets has a δ-complete non-principal ultrafilter on B.

Note that κ-weakly compact is the same as weakly compact.

Lemma 3.19. *Fix an infinite cardinal χ. If $\chi < \kappa \le m(\chi)$ and κ is χ^+-weakly compact, then κ is weakly compact.*

Proof. Assume otherwise. That is there exists some κ-complete Boolean Algebra $B \subset P(\kappa)$ generated by κ-many subsets such that there is no

κ-complete non-principal ultrafilter on B, but there is a χ^+-complete, non-principal ultrafilter U on B. Let μ^+ be the least cardinal so U is not μ^+-complete. Note that we must have $\chi^+ \leq \mu < \kappa$.

Since U is not μ^+-complete, by Fact 3.17, there exists some partition $W = \{w_i \mid i < \mu\}$ of B such that $w_i \notin U$ for all $i < \mu$. Define a function $f : \kappa \to \mu$ by

$$f(x) = i \text{ if } x \in w_i$$

The function f is defined on κ and is onto μ. Use f to define a complete Boolean Algebra $C \subset \mathcal{P}(\mu)$ and some ultrafilter V on C as follows: $Y \in C$ if and only if $f^{-1}(Y) \in B$, and $Y \in V$ if and only if $f^{-1}(Y) \in U$.

It is routine to verify that $C = \mathcal{P}(\mu)$ and that V is a μ-complete, non-principal ultrafilter on C. Thus, μ must be measurable, contradicting the assumption that $\chi < \mu < \kappa \leq m(\chi)$. $\qquad\qquad\square$

We introduce two functions on cardinals, one standard and one not.

Definition 3.20.   1. For any cardinal κ,

- $\beth_0(\kappa) = \kappa$
- $\beth_{\alpha+1}(\kappa) = 2^{\beth_\alpha(\kappa)}$
- $\beth_\lambda(\kappa) = \sup_{\alpha<\lambda} \beth_\alpha(\kappa)$, for λ limit

If no κ is mentioned, we assume that $\kappa = \aleph_0$.

2. For any cardinal κ,

$$A(\kappa) = \begin{cases} \kappa^{<\kappa} & \text{if } \kappa \text{ is regular} \\ 2^\kappa & \text{if } \kappa \text{ is singular} \end{cases}$$

Fact 3.21.

1. *All strong limit cardinals are of the form $\beth_\lambda$ for some limit λ and, given any κ, the least strong limit above κ is $\beth_\omega(\kappa)$.*

2. *If B is a κ-complete Boolean algebra on $\mathcal{P}(\kappa)$ that contains the finite subsets of κ, then B must be of size $A(\kappa)$. Recall that we assume the Boolean algebras we deal with have all finite subsets of κ.*

Corollary 3.22. *Assume $\chi \leq \kappa < m(\chi)$. If κ is not weakly compact, then $JEP(\lambda)$ fails for all λ satisfying*

$$\max\{\kappa^+, A(\kappa)\} \leq \lambda \leq 2^\kappa$$

Note that 2^κ always satisfies this inequality, but it can be the *only* cardinal there if case κ is singular or GCH holds at κ.

Proof. By Lemma 3.19, κ is not χ^+-weakly compact. By definition there is a Boolean Algebra B on κ that admits no non-principal χ^+-complete ultrafilter. The size of B is $A(\kappa)$ by Fact 3.21 (if $\kappa = \chi$ regular, then use Corollary 3.16). Then use B to construct a model $M \in \mathbf{K}^\chi$ of type $(A(\kappa), \kappa)$ that is not K-extendible by defining $K^M = \kappa$, $P^M = B$ and $Q^M = B^\chi$. By Lemma 3.14, if $\kappa < A(\kappa)$, $\mathbf{K}^\chi$ fails $JEP(A(\kappa))$.

For λ any cardinal in the interval $(A(\kappa), 2^\kappa]$ (if any), we work similarly. Construct a model $N \in \mathbf{K}^\chi$ of type (λ, κ) such that $M \prec^\chi_{\mathbf{K}} N$. Define N by letting $K^N = K^M = \kappa$, P^N is a Boolean Algebra extension of $P^M = B$ that has size λ and $Q^N = Q^M = B^\chi$. By Corollary 3.3, N is not K-extendible and by Lemma 3.14 again, $\mathbf{K}^\chi$ fails $JEP(\lambda)$. □

We can strengthen this failure by using induction to build many failures between κ and the first strong limit above κ.

Lemma 3.23. *Let $\chi \leq \kappa < m(\chi)$ such that κ is not weakly compact. Then $JEP(\lambda)$ fails for all λ satisfying*

$$\max\{\kappa^+, A(\kappa)\} \leq \lambda < \beth_\omega(\kappa)$$

except perhaps when there is a singular limit μ such that $2^{<\mu} \leq \lambda < 2^\mu$, in which case our method does not determine whether $JEP(\lambda)$ holds or not.

The cardinal restrictions on λ may seem strange, but are necessary from our methods. Essentially, we will apply Corollary 3.22 to an interval of cardinals to get failures of JEP. When applying to successor λ, the start of the new interval covers the first cardinal missed by the previous intervals. However, at limit cardinals, there can be a gap at the places indicated in the lemma. Notice that under GCH this exception can not happen. Additionally, the JEP is guaranteed to fail on the $\beth_n(\kappa)$'s.

Proof. We apply Corollary 3.22 to all cardinals $\lambda \in [\kappa, \beth_\omega(\kappa))$ to get failure of JEP on each interval

$$I_\lambda = [\max\{\lambda^+, A(\lambda)\}, 2^\lambda].$$

We take the following cases:

- $\lambda = \lambda_0^+$ successor: The start of the interval I_λ is at worst $\left(2^{\lambda_0}\right)^+$, which is the first cardinal above I_{λ_0}. Using that λ is regular and $A(\lambda) = \lambda^{<\lambda}$, we have that
$$\max\{\lambda^+, A(\lambda)\} \leq (2^{<\lambda})^+.$$

- λ limit cardinal: Set $I_{<\lambda} = \cup_{\mu<\lambda}I_\mu$. The first cardinal not in $I_{<\lambda}$ is either $2^{<\lambda}$ or $\left(2^{<\lambda}\right)^+$, depending on whether $I_{<\lambda}$ has supremum or a maximum respectively. We split this case into the following subcases.

- λ is regular and $2^{<\lambda} = 2^\mu$ for some $\mu < \lambda$.
  In this case, the first cardinal above $I_{<\lambda}$ is $(2^{<\lambda})^+$. By regularity, $A(\lambda) = \lambda^{<\lambda}$. Then there is no gap between $I_{<\lambda}$ and I_λ because
$$\max\{\lambda^+, \lambda^{<\lambda}\} \leq (2^{<\lambda})^+$$

- λ is regular and $2^{<\lambda} > 2^\mu$ for all $\mu < \lambda$.
  In this case, the first cardinal above $I_{<\lambda}$ is $2^{<\lambda}$. By regularity, $A(\lambda) = \lambda^{<\lambda} = 2^{<\lambda}$, so the left endpoint of $I_\lambda = \max\{\lambda^+, 2^{<\lambda}\}$. If $2^{<\lambda} = \lambda$, then λ would have been a strong limit. But we noted already that the first strong limit above κ is $\beth_\omega(\kappa) > \lambda$. So, $2^{<\lambda} > \lambda$ and there is no gap between $I_{<\lambda}$ and I_λ.

- λ is singular and $2^{<\lambda} = 2^\mu$ for some $\mu < \lambda$.
  In this case, the continuum function is eventually constant below λ and $2^\lambda = 2^{<\lambda} = 2^\mu$. By singularity, $A(\lambda) = 2^\lambda$. So I_λ begins at (and only contains) 2^λ and I_λ is a subset of $I_{<\lambda}$.

- λ is singular and $2^{<\lambda} > 2^\mu$ for all $\mu < \lambda$.
  In this case, the first cardinal above $I_{<\lambda}$ is $2^{<\lambda}$. By singularity, $A(\lambda) = 2^\lambda$. So I_λ begins at (and only contains) 2^λ, thus the gap.

Putting the above items together, we have the above lemma statement.

$\square$

Summarizing the results of this section, we have shown that the JEP spectrum of $\boldsymbol{K}^\chi$ is as desired.

Theorem 3.24. *Fix an infinite cardinal χ. Then there is an AEC $\boldsymbol{K}^\chi$ with $LS(\boldsymbol{K}^\chi) = \chi$ (that is given by models of a sentence in $\mathbb{L}_{\chi^+,\omega}$) whose JEP spectrum satisfies the following:*

1. *JEP holds cofinally below $m(\chi)$;*

2. *JEP fails cofinally below* $m(\chi)$; *and*

3. *JEP holds everywhere above* $m(\chi)$.

In particular, we have

- *JEP holds at every strong limit and above* $m(\chi)$;

- *if* $\chi \leq \kappa < m(\chi)$, *then JEP fails at* 2^κ; *and*

- *more generally, for regular* κ *with* $\chi \leq \kappa < m(\chi)$, *JEP fails on the interval*

$$\left[\max\{\kappa^+, \kappa^{<\kappa}\}, 2^\kappa\right]$$

Proof. The first bullet point is Theorems 3.11 and 3.10. The last two bullets are Theorem 3.8 and Lemma 3.23. □

Corollary 3.25. *If GCH holds, then JEP fails in* $\boldsymbol{K}^\chi$ *exactly at the* κ *satisfying* $\chi \leq \kappa < m(\chi)$ *that are not strong limit.*

We finish this section by producing models of $ZFC+$ "there is a measurable" where Theorem 3.24 characterizes the JEP spectrum of K^χ. The main tool is the following theorem of Paris and Kunen (see [9, Theorem 21.3]).

Fact 3.26. *Assume GCH and let* κ *be a measurable cardinal. Let D be a normal measure on* κ *and let A be a set of regular cardinals below* κ *such that* $A \notin D$. *Let F be a function on A such that $F(\alpha) < \kappa$ for all $\alpha \in A$, and:*

1. $cf F(\alpha) > \alpha$;

2. $F(\alpha_1) \leq F(\alpha_2)$ *whenever* $\alpha_1 \leq \alpha_2$.

Then there is a generic extension $V[G]$ of V with the same cardinals and cofinalities, such that κ *is measurable in $V[G]$, and for every $\alpha \in A$, $V[G] \vDash 2^\alpha = F(\alpha)$.*

Moreover, the powersets of cardinals not in A have the smallest possible cardinality that satisfies $\kappa < cf(2^\kappa)$ *and that the powerset function is increasing.*

Theorem 3.27. *Assume GCH and fix an infinite cardinal* χ. *Given a club C on* $m(\chi)$, *there is a generic extension $V[G]$ that preserves cardinalities and cofinalities,* $m(\chi)$ *remains a measurable cardinal and* $\boldsymbol{K}^\chi$ *satisfies* $JEP(\lambda)$ *iff* $\lambda \in \lim C$ *or* $\lambda \geq m(\chi)$.

Proof. The goal is to force the cardinal arithmetic of $V[G]$ to make the limit points of C the uncountable strong limit cardinals while preserving the measurability of $m(\chi)$. Since limits of strong limit cardinals are also strong limits, it suffices that we ensure that all cardinals in $\lim C \setminus \lim \lim C$ are strong limits.

Let U be a normal ultrafilter on $m(\chi)$. U contains all clubs, so $\lim C \in U$. Given $\lambda \in \lim C \setminus \lim \lim C$, λ has cofinality ω and the set $\lim C \cap \lambda$ is bounded in λ. Let $\{\kappa_n^\lambda \mid n < \omega\}$ be an increasing sequence of regular cardinals converging to λ and choose κ_0^λ such that it is the least regular cardinal above all cardinals in $\lim C \cap \lambda$. Then define a function F with domain $\{\kappa_n^\lambda \mid n < \omega, \lambda \in \lim C - \lim \lim C\}$ by

$$F(\kappa_n^\lambda) = \kappa_{n+1}^\lambda$$

Using Theorem 3.26, we can force to preserve cofinalities and the measurability of $m(\chi)$ while enforcing

$$V[G] \vDash \text{``} 2^{\kappa_n^\lambda} = \kappa_{n+1}^\lambda \text{''}$$

Thus we have guaranteed that in $V[G]$, $\beth_\omega(\kappa_0^\lambda) = \lambda$. In particular all cardinals in $\lim C$ are strong limit cardinals. We prove that the reverse is true too.

Let μ be a limit cardinal (in $V[G]$) that is not in $\lim C$. We prove that μ is not strong limit. Let λ be the least cardinal in $\lim C$ above μ. If $\kappa_0^\lambda \leq \mu$, then we noted that $\lambda = \beth_\omega(\kappa_0^\lambda)$ is the least strong limit above κ_0^λ. In this case, μ is not strong limit. If $\mu < \kappa_0^\lambda$, then μ must be a singular limit of cardinals in $\lim C$. Given that C is a club, μ must also belong to $\lim C$, which is a contradiction. Therefore, μ can not be a strong limit cardinal.

So, in $V[G]$, $\lim C = \{\beth_\lambda \mid \lambda\text{:limit and } \lambda < m(\chi)\}$. It follows from the moreover clause of Theorem 3.26 that in $V[G]$, $2^{\beth_\lambda} = (\beth_\lambda)^+$, for all limit $\lambda < m(\chi)$, that is GCH holds at strong limit cardinals.

We claim that $JEP(\lambda)$ if and only if $\lambda \in \lim C$ or $\lambda \geq m(\chi)$. The right-to-left direction is from Theorems 3.11 and 3.10, and the fact that the cardinals in $\lim C$ are strong limits.

For the left-to-right direction we take cases. If κ is a strong limit that is not a weakly compact cardinal, then we claim that JEP fails for all λ in the interval $[\kappa^+, \beth_\omega(\kappa))$. We observed already that GCH holds at strong limits. So $2^\kappa = \kappa^+$ and $\max\{\kappa^+, A(\kappa)\} = \kappa^+$. By Lemma 3.23, we know that JEP fails on the interval $[\max\{\kappa^+, A(\kappa)\}, \beth_\omega(\kappa)) = [\kappa^+, \beth_\omega(\kappa))$, except possibly some cardinals mentioned in Lemma 3.23. We argue that there are no such

cardinals, i.e. there is no singular limit μ such that $2^{<\mu} < 2^\mu$ in the interval $[\kappa^+, \beth_\omega(\kappa))$. Let $\lambda = \beth_\omega(\kappa)$. By the way κ_0^λ was defined, it must be $\kappa_0^\lambda = \kappa^+$. Let $\mu \in [\kappa^+, \beth_\omega(\kappa))$ be a limit cardinal and let $n \in \omega$ be the least such that $\kappa_n^\lambda < \mu < \kappa_{n+1}^\lambda$. Since κ_{n+1}^λ is a regular cardinal, by the moreover clause of Theorem 3.26, in $V[G]$ it holds that $2^\mu = \kappa_{n+1}^\lambda$. In addition, in $V[G]$, $2^{\kappa_n^\lambda} \leq 2^{<\mu} \leq 2^\mu = \kappa_{n+1}^\lambda$. Therefore, $2^{<\mu} = 2^\mu$ and there is no gap in the failures of JEP given by Lemma 3.23 in the interval $[\kappa^+, \beth_\omega(\kappa))$.

If κ is a weakly compact cardinal, then κ is a strong limit and again $\kappa^+ = 2^\kappa$. By Theorem 3.8, JEP fails at κ^+. Moreover, apply Lemma 3.23 to κ^+. Since $\max\{\kappa^{++}, (\kappa^+)^{(<\kappa^+)}\} = \kappa^{++}$, JEP fails on the interval $[\kappa^{++}, \beth_\omega(\kappa))$. Combined together, JEP fails on the interval $[\kappa^+, \beth_\omega(\kappa))$. $\qquad\square$

4 Amalgamation

In this section, we investigate the amalgamation spectrum of $\boldsymbol{K}^\chi$ and show that amalgamation will always eventually fail, regardless of large cardinals.

We start by providing a strong condition for when elements can be identified in the amalgam. Then we prove Lemma 4.3, which is an analogue of Lemma 3.2 for disjoint amalgamation.

Recall that by the proof of Lemma 3.2, if $M_0 \prec_{\boldsymbol{K}}^\chi M_1$ and $K^{M_1} \setminus K^{M_0} \neq \emptyset$, then for every $d \in K^{M_1} \setminus K^{M_0}$ we can define a Q^{M_0}-complete, non-principal ultrafilter U_d on P^{M_0} by

$$X \in U_d \text{ if and only if } M_1 \models d\varepsilon X$$

When there is ambiguity, we will refer to this ultrafilter by $U_d^{M_0,M_1}$.

Lemma 4.1. *Assume $M_0, M_1, M_2 \in \boldsymbol{K}^\chi$ and let $d_i \in K^{M_i} \setminus K^{M_0}$, $i = 1, 2$. If there exists an amalgam $N \in \boldsymbol{K}^\chi$ of M_1, M_2 over M_0 where d_1 and d_2 are identified, then $U_{d_1}^{M_1,M_0} = U_{d_2}^{M_2,M_0}$.*

Proof. Suppose that d_1 and d_2 are identified in the amalgam N. Since M_1, M_2 are $\prec_{\boldsymbol{K}}^\chi$-substructures of N, $U_{d_1}^{M_1,M_0} = U_{d_1}^{N,M_0} = U_{d_2}^{N,M_0} = U_{d_2}^{M_2,M_0}$. $\qquad\square$

Definition 4.2. Let $N, M_l \in \boldsymbol{K}^\chi$, $l = 0, 1, 2$ and N is an amalgam of M_1 and M_2 over M_0. Say that the amalgamation is *disjoint on the K-sort* if in N no elements of $K^{M_1} \setminus K^{M_0}$ are identified with any elements of $K^{M_2} \setminus K^{M_0}$.

Lemma 4.3. *Let $M_0, M_1, M_2 \in \boldsymbol{K}^\chi$ with $M_0 \prec_{\boldsymbol{K}}^\chi M_1$ and $M_0 \prec_{\boldsymbol{K}}^\chi M_2$. There is an amalgamation of M_1, M_2 over M_0 that is disjoint on the K-sort if and only if for every $d \in K^{M_i} \setminus K^{M_0}$ the ultrafilter $U_d^{M_i,M_0}$ on P^{M_0}*

defined above can be extended to a non-principal ultrafilter on $P^{M_{3-i}}$ that is $Q^{M_{3-i}}$-complete, for $i = 1, 2$.

Note that $U_d^{M_i, M_0}$ is an ultrafilter on the Boolean Algebra (defined by) P^{M_0}. Since $M_0 \prec_{\boldsymbol{K}}^{\chi} M_{3-i}$, we have that P^{M_0} is a sub-Boolean Algebra of $P^{M_{3-i}}$. Consequently, $U_d^{M_i, M_0}$ is a filter on $P^{M_{3-i}}$. Then Lemma 4.3 says that there is an amalgamation of M_1, M_2 over M_0 which is disjoint on the K-sort exactly when $U_d^{M_i, M_0}$ can be extended to a non-prinicipal and $Q^{M_{3-i}}$-complete ultrafilter on $P^{M_{3-i}}$ and this can be done for every d.

Proof of Lemma 4.3. First, suppose that N is a K-disjoint amalgam of M_1, M_2 over M_0. Consider the case where $d \in K^{M_1} \setminus K^{M_0}$. The case $d \in K^{M_2} \setminus K^{M_0}$ is symmetric and we omit it.

In the amalgam d is an element of $K^N \setminus K^{M_2}$. Define as before the ultrafilter U_d^{N, M_2}. Then U_d^{N, M_2} is a non-principal ultrafilter on P^{M_2} that is Q^{M_2}-complete. We need to prove that U_d^{N, M_2} extends $U_d^{M_1, M_0}$. Let $X \in P^{M_0}$. Then

$$X \in U_d^{N, M_2} \text{ iff } N \models d\varepsilon X$$
$$\text{iff } M_1 \models d\varepsilon X$$
$$\text{iff } X \in U_d^{M_1, M_0}.$$

Second, suppose that we have this extension property and let $V_d^{M_i, M_0}$ denote the $Q^{M_{3-i}}$-complete ultrafilter on $P^{M_{3-i}}$ that extends $U_d^{M_i, M_0}$. Furthermore, suppose that M_1 and M_2 are disjoint except for the common copy of M_0 and that the elements of Q^{M_i} are actually χ-sequences from P^{M_i}.

We define the amalgam N of M_1, M_2 over M_0: K^N equals $K^{M_1} \cup K^{M_2}$.

P^N is the Boolean Algebra generated by $P^{M_1} \cup P^{M_2}$ and all the finite subsets of K^N. We identify two elements $X \in P^{M_1}$ and $Y \in P^{M_2}$, if $\widehat{X}^{M_1}$ and $\widehat{Y}^{M_2}$ is the same subset of K^{M_0}.

Q^N equals $Q^{M_1} \cup Q^{M_2}$, modulo the requirement that if $A \in Q^{M_1}$ and $B \in Q^{M_2}$ are such that for all $\alpha < \chi$, $\pi_\alpha^{M_1}(A) = \pi_\alpha^{M_2}(B)$, then we identify A and B in the amalgam.

All that remains is to define ε^N. It suffices to define ε^N on $K^N \times (P^{M_1} \cup P^{M_2})$, and then extend it to the rest of P^N by the Boolean Algebra rules. We require that ε^N extend $\varepsilon^{M_1} \cup \varepsilon^{M_2}$. Suppose $d \in K^{M_i} \setminus K^{M_0}$ and $X \in P^{M_{3-i}}$. Then we set

$$N \vDash d \in X \iff X \in V_d^{M_i, M_0}$$

Notice that if $X \in P^{M_0}$, then $X \in V_x^{M_i, M_0}$ iff $X \in U_x^{M_i, M_0}$ iff $M_i \models x\varepsilon X$. So, N and M_i agree on ε on their common domain.

The reader can verify that $N \in \boldsymbol{K}^\chi$ and that $M_i \prec_{\boldsymbol{K}}^\chi N$ working as in the proof of Lemma 3.2. In particular, the $Q^{M_{3-i}}$-completeness of $V_d^{M_i, M_0}$ is crucial as in Lemma 3.2. $\qquad\square$

Observe that the proof of the above Lemma does not yield a disjoint amalgam for M_1, M_2. The reason is that some elements of P^{M_1}, P^{M_2} and some elements of Q^{M_1}, Q^{M_2} may be identified. Nevertheless, amalgamation is disjoint on the K-sort.

Using Lemma 4.3, we prove that $\boldsymbol{K}^\chi$ fails amalgamation above 2^{χ^+}. The idea of the proof is due to Spencer Unger.

Theorem 4.4. *Let $\kappa \geq 2^{\chi^+}$. Then $\boldsymbol{K}^\chi$ fails $AP(\kappa)$.*

Proof. First, we will build a filter F on $\mathcal{P}(\kappa)$ generated by $\leq \kappa$-many sets that cannot be extended to a χ^+-complete ultrafilter on all of $\mathcal{P}(\kappa)$; indeed, we will identify a Boolean Algebra P_1 and collection of χ-sequences Q_1 such that F cannot be extended to a Q_1-complete ultrafilter on P_1. To do so, partition κ into $\{A_\alpha \mid \alpha < \chi^+\}$ and define the filter F on κ by, for $X \subset \kappa$,

$$X \in F \text{ if and only if there is } \beta < \chi^+ \text{ such that } \bigcup_{\alpha > \beta} A_\alpha \subset X$$

Note that F is χ^+-complete and contains every cofinite set (and even the co-χ-sized sets). Let $P_0 \subset \mathcal{P}(\kappa)$ be the Boolean Algebra generated by the sets measured by F. Then set $P_1 \supset P_0$ be the Boolean Algebra generated by

$$\left\{ \bigcup_{\alpha \in S} A_\alpha \mid S \subset \chi^+ \right\}$$

Now, define our set of χ-sequences by

$$Q_1 := \left\{ \left\langle \bigcup_{\alpha \in S_i} A_\alpha \mid i < \chi \right\rangle \mid \langle S_i \rangle_{i < \chi} \in {}^\chi \mathcal{P}(\chi^+) \right\}$$

Claim: F cannot be extended to a Q_1-complete filter G that measures all sets in P_1.

Suppose it could. From G, we can define a non-principal ultrafilter U on χ^+ by, for $Y \subset \chi^+$,

$$Y \in U \text{ if and only if } \bigcup_{\alpha \in Y} A_\alpha \in G$$

Each of these sets is in P_1 by construction, so this is a non-principal ultrafilter following the standard argument. Moreover, U is χ^+-complete precisely because G is Q_1-complete. This is a contradiction because there can be no χ^+-complete, non-principal ultrafilter over χ^+. This completes the proof of the claim.

Second, we reverse engineer the proof of Lemma 4.3 to show that this construction forces a failure of amalgamation. We build a triple of models M_0, M_1, M_2. Unless otherwise specified ε is the regular $\in$ ('belongs to') relation and the Boolean Algebra operations are the usual intersection, union and complement. We specify (K, P, Q):

- M_0 is defined by $(\kappa, P_0, \emptyset)$;

- M_1 is defined by $(\kappa \cup \{d\}, P_0, \emptyset)$; d belongs to some $X \in P_0$ if and only if $X \in F$; and

- M_2 is defined by (κ, P_1, Q_1).

Note that F is an ultrafilter on P_0 as P_0 contains precisely the sets that F measures. By construction, $M_0 \prec_{\boldsymbol{K}}^{\chi} M_1, M_2$. Tracing the definition, $U_d^{M_1, M_0} = F$. So by Lemma 4.3, the triple (M_0, M_1, M_2) can be amalgamated iff F can be extended to a Q^{M_2}-complete filter on P^{M_2}. However, this is impossible by the claim.

We finish the proof by observing that all these models have size $\kappa + \left(2^{\chi^+}\right)^{\chi} = \kappa$. So $\boldsymbol{K}_{\kappa}^{\chi}$ fails AP(κ). $\qquad\square$

Acknowledgements

References

[1] John Baldwin and Saharon Shelah. Hanf numbers for extendibility and related phenomena. pre-print.

[2] John T. Baldwin. *Categoricity.*, volume 50. Providence, RI: American Mathematical Society (AMS), 2009.

[3] John T. Baldwin, Martin Koerwien, and Michael C. Laskowski. Disjoint amalgamation in locally finite AEC. *J. Symb. Log.*, 82(1):98–119, 2017.

[4] John T. Baldwin, Martin Koerwien, and Ioannis Souldatos. The joint embedding property and maximal models. *Archive for Mathematical Logic*, 55(3):545–565, 2016.

[5] John T. Baldwin and Ioannis Souldatos. Complete $\mathbb{L}_{\omega_1,\omega}$ with maximal models in multiple cardinalities.

[6] Will Boney and Spencer Unger. Large cardinal axioms from tameness in AECs. *Proc. Am. Math. Soc.*, 145(10):4517–4532, 2017.

[7] C.C. Chang and H.J. Keisler. *Model theory. 3rd rev. ed.* Amsterdam etc.: North-Holland, 3rd rev. ed. edition, 1990.

[8] Rami Grossberg. Classification theory for abstract elementary classes. In *Logic and algebra*, pages 165–204. Providence, RI: American Mathematical Society (AMS), 2002.

[9] Thomas Jech. *Set theory. The third millennium edition, revised and expanded.* Berlin: Springer, the third millennium edition, revised and expanded edition, 2003.

[10] Akihiro Kanamori. *The higher infinite. Large cardinals in set theory from their beginnings. Paperback reprint of the 2003 second edition.* Berlin: Springer, paperback reprint of the 2003 second edition. edition, 2009.

[11] Alexei Kolesnikov and Christopher Lambie-Hanson. The hanf number for amalgamation of coloring classes. *The Journal of Symbolic Logic*, 81(2):570–583, 2016.

[12] Menachem Magidor. How large is the first strongly compact cardinal? or: A study on identity crises. *Ann. Math. Logic*, 10:33–57, 1976.

[13] John Baldwin Will Boney. Hanf numbers and presentation theorems in aecs. In Jose Iovino, editor, *Beyond First Order Model Theory*, pages 81–106. Chapman Hall, 2017.