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Action principles and conservation laws for Chew–Goldberger–Low anisotropic plasmas

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The ideal Chew–Goldberger–Low (CGL) plasma equations, including the double adiabatic conservation laws for the parallel ($p_\parallel$) and perpendicular pressure ($p_\perp$), are investigated using a Lagrangian variational principle. An Euler–Poincaré variational principle is developed and the non-canonical Poisson bracket is obtained, in which the non-canonical variables consist of the mass flux $\boldsymbol{M}$, the density ρ, the entropy variable $\sigma = \rho S$ and the magnetic induction $\boldsymbol{B}$. Conservation laws of the CGL plasma equations are derived via Noether's theorem. The Galilean group leads to conservation of energy, momentum, centre of mass and angular momentum. Cross-helicity conservation arises from a fluid relabelling symmetry, and is local or non-local depending on whether the gradient of S is perpendicular to $\boldsymbol{B}$ or otherwise. The point Lie symmetries of the CGL system are shown to comprise the Galilean transformations and scalings.

Key words: fusion plasma, plasma nonlinear phenomena, space plasma physics

1. Introduction

In magnetohydrodynamics (MHD), the ideal Chew–Goldberger–Low (CGL) equations (Chew, Goldberger & Low 1956) describe plasmas in which there is not enough scattering of the particles to have an isotropic pressure and only lowest-order gyro-radius effects terms are taken into account in the particle transport (i.e. finite Larmor-radius terms are neglected). These equations can be viewed as the small gyro-radius limit of the Vlasov fluid moment equations, where the pressure tensor has the anisotropic (gyrotropic) form $\mathsf{p} = p_\perp \mathsf{I} + (p_\parallel - p_\perp)\boldsymbol{\tau}\boldsymbol{\tau}$. Here $\boldsymbol{\tau} = |\boldsymbol{B}|^{-1}\boldsymbol{B}$ is the unit vector along the magnetic field and I is the identity tensor. The pressure component $p_\parallel - p_\perp$ controls the anisotropy, i.e. the pressure tensor is isotropic if $p_\parallel = p_\perp$.

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Higher-order gyro-radius effects and gyro-viscosity lead to the extended anisotropic MHD equations (see, e.g., Macmahon 1965; Ramos 2005*a*,*b*; Sulem & Passot 2015; Hunana *et al.* 2019*a*). Braginskii (1965), Schnack (2009) and Devlen & Pekünlü (2010) used an isotropic pressure decomposition, which is not an anisotropic CGL decomposition. Hunana *et al.* (2022) considered various generalisations of the Braginskii (1965) model using the Landau collisional operator for the case of multi-species plasmas. A recent Hamiltonian version of extended gyro-viscous MHD has been developed by Lingam, Morrison & Wurm (2020).

In the present work, we develop a variational formulation of the ideal CGL plasma equations starting from the Lagrangian formulation of Newcomb (1962). Our main goals are to derive and discuss:

(1) an Euler–Poincaré (EP) action principle;
(2) a non-canonical Poisson bracket and its Casimirs;
(3) conservation laws from application of Noether's theorem to the Galilean group of Lie point symmetries;
(4) a cross-helicity conservation law from particle relabelling symmetries and its non-local nature when the entropy gradients are non-orthogonal to the magnetic field.

Previous work on the CGL equations can be found in Abraham-Shrauner (1967), Hazeltine, Mahajan & Morrison (2013), Holm & Kupershmidt (1986), Ramos (2005*a*,*b*), Hunana *et al.* (2019*a*,*b*) and Du *et al.* (2020).

Cheviakov & Bogoyavlenskij (2004) and Cheviakov & Anco (2008) derived exact, anisotropic MHD equilibria solutions of the CGL equations, with a modified equation of state for incompressible fluid flows. Cheviakov & Bogoyavlenskij (2004) obtained an infinite group of Lie symmetries of the anisotropic plasma equilibrium equations for steady flow configurations. An infinite-dimensional family of transformations between the isotropic MHD equilibrium equations and solutions of the anisotropic CGL equations were obtained. Ilgisonis (1996) studied the stability of steady CGL plasma equilibria in a tokamak for a generalised Grad–Shafranov equation, taking into account the fluid relabelling symmetry, and covers previous stability criteria for ideal CGL plasma and ideal MHD cases.

Our methods are adapted from MHD and ideal fluid mechanics. It is well known how to use a Lagrangian map to relate Eulerian and Lagrangian fluid quantities (Newcomb 1962), and this underpins the EP action principle which is a Lagrangian counterpart of a Hamiltonian formulation (Holm, Marsden & Ratiu 1998; Webb 2018; Webb & Anco 2019). This action principle is based on the Lagrangian map and utilises an associated Lie algebra.

Recent work by Dewar *et al.* (2020) and Dewar & Qu (2022) developed variational principles to describe time- dependent relaxed MHD using a global cross-helicity constraint with a phase space Lagrangian action principle (PSL) as opposed to a configuration space Lagrangian (CSL) action. The theory has been used to describe multi-region, extended MHD (RxMHD) in fusion plasma devices. In ideal MHD current sheets can develop. One of the main ideas in this work is to provide a global fitting together of ideal MHD sub-regions, subject to global cross-helicity and magnetic helicity constraints by using Lagrange multipliers. This is a rather complicated theory that lies beyond the scope of the present paper.

Noether's theorem has been used to obtain conservation laws for ideal MHD (Webb & Anco 2019); see also Padhye & Morrison (1996*a*,*b*) and Padhye (1998). A different approach to deriving conservation laws is Lie dragging of differential forms, vector fields and tensors, as developed by Moiseev *et al.* (1982), Sagdeev, Tur & Yanovskii (1990), Tur & Yanovsky (1993), Webb *et al.* (2014*a*,*b*), Gilbert & Vanneste (2020), Besse & Frisch (2017) and Anco & Webb (2020).

Section 2 summarises the ideal CGL plasma equations. The anisotropic pressure components $p_\parallel$ and $p_\perp$ satisfy double adiabatic conservation laws. The total energy equation for the system arises from combining the internal energy equation, the kinetic energy equation and the electromagnetic energy equation (i.e. Poynting's theorem). The cross-helicity and Galilean conservation laws are obtained for barotropic and non-barotropic gas equations of state. The magnetic helicity transport equation and the conservation of magnetic helicity is described. The thermodynamics of CGL plasmas are discussed. This leads to an internal energy density of the form $e = e(\rho, S, B)$ for the plasma (e.g. Holm & Kupershmidt 1986; Hazeltine *et al.* 2013), where $B = |\boldsymbol{B}|$ is the magnitude of the magnetic field. It applies to the situation when reversible energy changes occur, with the temperature of the plasma being given by $T = e_S$, and it yields an equation of state which incorporates the double adiabatic conservation laws. A comparison is given with a more restrictive different approach that applies the ideal gas law to the plasma (e.g. Du *et al.* 2020), with an equation of state being derived as a consequence.

Section 3 describes the Lagrangian map between the Eulerian fluid particle position $\boldsymbol{x}$ and the Lagrangian particle position $\boldsymbol{x}_0$. This map is obtained from integrating the system of ordinary differential equations $\mathrm{d}\boldsymbol{x}/\mathrm{d}t = \boldsymbol{u}(\boldsymbol{x}, t)$ with the fluid velocity $\boldsymbol{u}(\boldsymbol{x}, t)$ assumed to be a known function of $\boldsymbol{x}$ and t and $\boldsymbol{x} = \boldsymbol{x}_0$ at time $t = 0$. The Lagrangian map $\boldsymbol{x}(\boldsymbol{x}_0, t)$ is used to write the fluid equations as a variational principle (Newcomb 1962; Holm *et al.* 1998). In this description the canonical coordinates are $\boldsymbol{q} = \boldsymbol{x}(\boldsymbol{x}_0, t)$, and the canonical momenta are defined by the Legendre transformation $\boldsymbol{p} = \partial L_0/\partial \dot{\boldsymbol{x}}$ where L_0 is the Lagrangian density in a frame moving with the fluid.

Section 4 develops the EP action principle for the CGL plasma equations by using the general method of Holm *et al.* (1998) and the results of Newcomb (1962).

In § 5, the Poisson bracket $\{F, G\}$ of functionals F and G is described. We obtain the non-canonical Poisson bracket of Holm & Kupershmidt (1986) but give more details. By converting the variational derivatives for functionals from the canonical coordinates $(\boldsymbol{q}, \boldsymbol{p})$ to physically motivated non-canonical coordinates, the Poisson bracket is converted to its non-canonical form in the new variables. The symplectic (i.e. Hamiltonian) form of the non-canonical Poisson bracket is determined, which is useful in proving the Jacobi identity and for writing down the Casimir determining equations. The Casimirs of the Poisson bracket commute with the Hamiltonian functional characterising the system and are obtained using the approach of Hameiri (2004) (see also Morrison 1982; Holm *et al.* 1985; Marsden & Ratiu 1994; Padhye & Morrison 1996*a*,*b*; Padhye 1998).

Section 6 describes Noether's theorem for the CGL variational principle using the Lagrangian variables. Noether's theorem produces conservation laws in terms of these variables from variational symmetries. The Eulerian form of the conservation laws is obtained through the Lagrangian map by use of a result of Padhye (1998). The variational symmetries include the Galilean group, in particular: (i) time translation invariance yields energy conservation; (ii) space translation invariance yields momentum conservation; (iii) invariance under Galilean boosts yields centre of mass conservation; (iv) invariance of under rotations yields angular momentum conservation.

In addition, fluid relabelling symmetries are shown to be variational symmetries which yield the generalised cross-helicity conservation law for CGL plasmas. The conditions under which this conservation law is local or non-local are delineated. A local conservation law is shown to arise when the parallel and perpendicular entropies, $S_\parallel$ and $S_\perp$, have

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zero gradient along the magnetic field. Alternatively, for an internal energy density $e = e(\rho, S, B)$, a local conservation law occurs when $\boldsymbol{B} \cdot \nabla S = 0$. When these conditions fail to hold in the plasma, the cross-helicity conservation law is non-local and depends on the integration of the temperatures back along the Lagrangian fluid particle paths.

The present paper can be thought of as an extension of the work of Holm & Kupershmidt (1986). We restrict our analysis to non-relativistic flows, whereas Holm & Kupershmidt (1986) study both relativistic and non-relativistic flow versions of the CGL equations. Our analysis is more complete than Holm & Kupershmidt (1986) in the following ways. (i) Our analysis takes into account more recent developments in anisotropic moment equations for plasmas with an anisotropic pressure tensor (e.g. Hazeltine *et al.* 2013; Hunana *et al.* 2019*a*). In particular, the formulations of the equation of state and the first law of thermodynamics used by Holm & Kupershmidt (1986) were not clear because of the brevity of the exposition. (ii) We show that a slightly more general, non-separable equation of state can be used than that of Hazeltine *et al.* (2013) (see (2.30) and what follows). (iii) We provide a more complete treatment of the conservation laws for the CGL system, and identify the symmetries of the Lagrangian which give rise to the conservation laws in Noether's theorem. Our analysis shows that the cross-helicity conservation law is a consequence of a fluid relabelling symmetry, which is, in general, a non-local conservation law unless $\boldsymbol{B}$ is normal to the entropy gradients of $S_\parallel$, $S_\perp$ and S, in which case the conservation law is local. (iv) We describe the CGL Poisson bracket of Holm & Kupershmidt (1986) which uses the non-canonical variables ρ, $\boldsymbol{M} = \rho\boldsymbol{u}$, $\sigma = \rho S$ and $\boldsymbol{B}$. The entropy variable used in the non-canonical Poisson bracket is $\sigma = \rho S$ where S is the entropy. The entropy S can, in turn, be decomposed into the form $S = S_\parallel + S_\perp$. The non-canonical Poisson bracket derivation in Appendix D is more rigorous than that used by most authors as it takes into account the variations in the basis vectors used to define vectors and tensors. However, these changes do not modify the net Poisson bracket, because they only lead at most to changes in the boundary terms in the bracket which are discarded (see, e.g., Holm, Kupershmidt & Levermore (1983) for the more rigorous general derivation of the non-canonical Poisson bracket in MHD). (v) We provide a modern version of Noether's theorem which only uses the evolutionary symmetry operator. We describe the classical form of Noether's theorem obtained by Bluman & Kumei (1989), in Appendix H. In § 6.1 we derive CGL conservation laws using the more modern form of Noether's theorem. Section 6.2 gives the classical version of Noether's theorem to derive CGL plasma conservation laws.

Section 7 concludes with a summary and discussion.

The various technical derivations used in the main results, and the Lie point symmetries of the CGL system, are summarised in Appendices A to G.

2. CGL equations and conservation laws

In this section, we first summarise the basic CGL plasma model equations, and then we discuss the thermodynamics of CGL plasmas, followed by the key conservation laws in Eulerian form: total energy; cross-helicity and non-local conserved cross-helicity; magnetic helicity. These conservation laws, in later sections, will be related to the variational symmetries of the action principle for the CGL equations and expressed in Lagrangian form. Some general remarks about Eulerian conservation law equations, which will be used in the discussions, are summarised in § 6.

2.1. *CGL model*

The physical variables which describe CGL plasmas are the fluid velocity $\boldsymbol{u}$, fluid density ρ, magnetic field induction $\boldsymbol{B}$, the anisotropic pressure components $p_\parallel$ and $p_\perp$ and the

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entropy S. The ideal CGL plasma equations are similar to the MHD equations and consist of: the mass continuity equation

$$\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \boldsymbol{u}) = 0; \tag{2.1}$$

the momentum equation (in semi-conservative form)

$$\frac{\partial}{\partial t}(\rho \boldsymbol{u}) + \nabla \cdot (\rho \boldsymbol{u}\boldsymbol{u} + \mathsf{p} + \mathsf{M}_B) = -\rho \nabla \Phi, \tag{2.2}$$

in which p is the gyrotropic pressure tensor (which replaces the isotropic gas pressure $p\mathsf{I}$ used in MHD) given by

$$\mathsf{p} = p_\perp \mathsf{I} + (p_\parallel - p_\perp)\boldsymbol{\tau}\boldsymbol{\tau}, \quad \boldsymbol{\tau} = \frac{\boldsymbol{B}}{B}, \quad B = |\boldsymbol{B}|, \tag{2.3a–c}$$

and M_B is the magnetic pressure tensor

$$\mathsf{M}_B = \frac{1}{\mu_0}\left(\frac{B^2}{2}\mathsf{I} - \boldsymbol{B}\boldsymbol{B}\right) = \frac{B^2}{\mu_0}\left(\frac{1}{2}\mathsf{I} - \boldsymbol{\tau}\boldsymbol{\tau}\right); \tag{2.4}$$

the entropy transport equation

$$\frac{\partial S}{\partial t} + \boldsymbol{u} \cdot \nabla S = 0; \tag{2.5}$$

Faraday's equation and Gauss's law

$$\frac{\partial \boldsymbol{B}}{\partial t} - \nabla \times (\boldsymbol{u} \times \boldsymbol{B}) = 0, \tag{2.6}$$

$$\nabla \cdot \boldsymbol{B} = 0; \tag{2.7}$$

along with the pressure equations

$$\frac{\partial p_\parallel}{\partial t} + \nabla \cdot \left(p_\parallel \boldsymbol{u}\right) + 2\mathsf{p}^\parallel : \nabla \boldsymbol{u} = 0, \tag{2.8}$$

$$\frac{\partial p_\perp}{\partial t} + \nabla \cdot (p_\perp \boldsymbol{u}) + \mathsf{p}^\perp : \nabla \boldsymbol{u} = 0, \tag{2.9}$$

where $\mathsf{p}^\parallel$ and $\mathsf{p}^\perp$ are the terms comprising the gyrotropic pressure tensor

$$\mathsf{p}^\parallel = p_\parallel \boldsymbol{\tau}\boldsymbol{\tau}, \quad \mathsf{p}^\perp = p_\perp (\mathsf{I} - \boldsymbol{\tau}\boldsymbol{\tau}), \quad \mathsf{p} = \mathsf{p}^\parallel + \mathsf{p}^\perp. \tag{2.10a–c}$$

Through Faraday's equation (2.6) and the mass continuity equation (2.1), the pressure equations (2.8)–(2.9) can be expressed as the double adiabatic equations

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{p_\parallel B^2}{\rho^3}\right) = 0, \quad \frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{p_\perp}{\rho B}\right) = 0, \tag{2.11a,b}$$

which represent conservation of the particle magnetic moment and the second longitudinal adiabatic moment of the particles, where $\mathrm{d}/\mathrm{d}t = \partial/\partial t + \boldsymbol{u} \cdot \nabla$ is the Lagrangian time derivative following the flow.

In the momentum equation (2.2), Φ is the potential of an external source of gravity (for example, in the case of the solar wind, it could represent the gravitational potential field of the Sun).

If one counts the number of evolution equations in (2.1)–(2.11*a*,*b*) there are 10 equations for the 10 variables $\boldsymbol{u}$, $\boldsymbol{B}$, ρ, S, $p_\parallel$ and $p_\perp$. If one does not need to know S, then there are nine equations for nine unknowns.

The non-canonical Poisson bracket formulation of MHD in Morrison & Greene (1982) uses Faraday's law (2.6) in the form $\partial \boldsymbol{B}/\partial t - \nabla \times (\boldsymbol{u} \times \boldsymbol{B}) + \boldsymbol{u}\nabla \cdot \boldsymbol{B} = 0$ for the mathematical case in which $\nabla \cdot \boldsymbol{B} \neq 0$ (see also Webb 2018). In fact, the possibility of $\nabla \cdot \boldsymbol{B} \neq 0$ arises in numerical MHD due to numerical errors in the Gauss' law (2.7).

To determine the relationship between the entropy S, the internal energy density ε and the pressure components $p_\parallel$ and $p_\perp$ in (2.1)–(2.11*a*,*b*) we first note that ε is given by

$$\varepsilon = \frac{p_\parallel + 2p_\perp}{2}. \tag{2.12}$$

A different, related approach comes from the entropy law for ideal gases. We consider and compare both approaches.

One link between the approaches is the observation that integration of the double adiabatic equations (2.11*a*,*b*) yields

$$p_\parallel = \exp\left(\bar{S}_\parallel\right) \frac{\rho^3}{B^2}, \quad p_\perp = \exp\left(\bar{S}_\perp\right) \rho B, \tag{2.13a,b}$$

where the quantities $\bar{S}_\parallel$ and $\bar{S}_\perp$ are dimensionless forms of entropy integration constants arising from integrating the double adiabatic equations (2.11*a*,*b*). In principle, $\bar{S}_\parallel$ and $\bar{S}_\perp$ must be scalars advected with the flow. In particular, these quantities could be functions of the entropy S satisfying (2.5). They could also be functions of other advected invariants of the flow, such as $\boldsymbol{B} \cdot \nabla S/\rho$ (e.g. Tur & Yanovsky 1993; Webb *et al.* 2014*a*). For the sake of simplicity, we assume that $S_\parallel$ and $S_\perp$ depend only on S. The overbars on $\bar{S}$, $\bar{S}_\parallel$ and $\bar{S}_\perp$ denote dimensionless versions of these quantities.

If these $S_\parallel$ and $S_\perp$ are functions solely of the entropy S, then the internal energy density ε will have the functional form $\varepsilon(\rho, S, B)$. This form also arises from the first law of thermodynamics coming from the transport equation for ε when reversible thermodynamic processes are considered. Strictly speaking, we should use normalised or dimensionless variables in (2.13*a*,*b*), i.e. we should have used the variables

$$\bar{p}_\parallel = \frac{p_\parallel}{p_{\parallel 0}}, \quad \bar{p}_\perp = \frac{p_\perp}{p_{\perp 0}}, \quad \bar{\boldsymbol{B}} = \frac{\boldsymbol{B}}{B_0}, \quad \bar{\rho} = \frac{\rho}{\rho_0}, \tag{2.14a–d}$$

where the subscript zero quantities are dimensional constants. In a convenient abuse of notation, we have dropped the overbar superscripts in (2.13*a*,*b*).

2.2. *Thermodynamic formulation*

A physical description of $p_\parallel$ and $p_\perp$ and their relation to the internal energy density ε was developed first by Holm & Kupershmidt (1986), which in part uses the work of Volkov (1966), and also later by Hazeltine *et al.* (2013), which discusses both reversible and irreversible thermodynamics. Here we concentrate on the ideal reversible dynamics case.

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An analysis of reversible work done on the plasma reveals that the internal energy per unit mass of the CGL plasma, $e = \varepsilon/\rho$, obeys the Pfaffian equation

$$\mathrm{d}e = T\,\mathrm{d}S + \frac{p_\parallel}{\rho^2}\mathrm{d}\rho - \frac{p_\Delta}{\rho B}\mathrm{d}B. \tag{2.15}$$

where T is the adiabatic temperature in the plasma. The expression (2.12) for the internal energy density gives the relations

$$e = \frac{\varepsilon}{\rho} = \frac{(p_\parallel + 2p_\perp)}{2\rho}, \quad p_\Delta = p_\parallel - p_\perp, \tag{2.16a,b}$$

(see also Holm & Kupershmidt 1986) and

$$p \equiv \frac{(p_\parallel + 2p_\perp)}{3} = \frac{2\varepsilon}{3}, \tag{2.17}$$

which is the gas pressure defined by one-third of the trace of the CGL pressure tensor (2.3*a–c*).

The Pfaffian equation (2.15) constitutes the first law of thermodynamics for a CGL plasma. As outlined in Hazeltine *et al.* (2013), it can be derived from the transport equation for ε under thermodynamic processes that involve reversible work done on the plasma described by the gyrotropic part of the Vlasov distribution function. The derivation can be generalised to include irreversible work comprising a dissipative term and a further term due to gyro-viscosity, which is a non-dissipative term obtained in the limit of no scattering. Equation (2.15) is equivalent to the non-dissipative internal energy equation (2.51) in which a source term $\rho T\,\mathrm{d}S/\mathrm{d}t$ has been added to the right hand-side.

The above interpretation of (2.15) implies $e = e(\rho, S, B)$ along with the thermodynamic relations

$$T = e_S = \varepsilon_S/\rho, \quad p_\parallel = \rho^2 e_\rho = \rho\varepsilon_\rho - \varepsilon, \quad p_\Delta = -\rho B e_B = -B\varepsilon_B. \tag{2.18a–c}$$

Thus, if $e(\rho, S, B)$ is known, i.e. an equation of state has been specified, then the relations (2.18*a–c*) give $p_\parallel$ and $p_\perp$ as functions of ρ, S, B. Consistency must hold with the double adiabatic equations (2.11*a,b*), which constrains the possible expressions for $e(\rho, S, B)$.

To derive the constraints, we first substitute the relations (2.18*a–c*) into the internal energy (2.17), giving

$$e = \frac{1}{\rho}\left(\frac{3p_\parallel}{2} - p_\Delta\right) = \frac{3}{2}\rho e_\rho + B e_B. \tag{2.19}$$

This is a first-order partial differential equation for $e(\rho, S, B)$ which is easily solved by using the method of characteristics (e.g. Sneddon 1957):

$$\frac{\mathrm{d}\rho}{(3\rho/2)} = \frac{\mathrm{d}B}{B} = \frac{\mathrm{d}e}{e}. \tag{2.20}$$

Integrating these differential equations yields the general solution

$$e = \rho^{2/3} f(\xi, S), \quad \xi = B\rho^{-2/3}, \tag{2.21a,b}$$

where $f(\xi, S)$ is an arbitrary function of S and the similarity variable ξ. Now, the relations (2.18*a–c*) imply that

$$T = \rho^{2/3} f_S, \tag{2.22}$$

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and

$$p_{\|}=\tfrac{2}{3}\rho^{5/3}\left(f-\xi f_{\xi}\right), \quad p_{\perp}=\tfrac{1}{3}\rho^{5/3}\left(2f+\xi f_{\xi}\right). \tag{2.23a,b}$$

Substitution of expressions (2.23*a*,*b*) into the double adiabatic equations (2.11*a*,*b*) followed by use of the relation $B=\rho^{2/3}\xi$ then yields

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{2}{3}\xi^2 f-\frac{2}{3}\xi^3 f_{\xi}\right)=0, \quad \frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{2}{3\xi}f+\frac{1}{3}f_{\xi}\right)=0, \tag{2.24a,b}$$

with $\mathrm{d}f/\mathrm{d}t=f_{\xi}\,\mathrm{d}\xi/\mathrm{d}t$ for any process in which S is adiabatic (cf. the entropy equation (2.5)). However, from Faraday's equation (2.6) and the mass continuity equation (2.1), we find that ξ obeys the transport equation

$$\frac{\mathrm{d}}{\mathrm{d}t}\xi=\left(-\frac{1}{3}\nabla\cdot\boldsymbol{u}+\boldsymbol{\tau}\boldsymbol{\tau}:\nabla\boldsymbol{u}\right)\xi. \tag{2.25}$$

Thus, ξ is not an advected quantity, which implies that the adiabatic equations (2.24*a*,*b*) for $f(\xi,S)$ reduce to

$$\frac{2}{3}\xi^2 f-\frac{2}{3}\xi^3 f_{\xi}=c_{\|}(S), \quad \frac{2}{3\xi}f+\frac{1}{3}f_{\xi}=c_{\perp}(S). \tag{2.26a,b}$$

The general solution of this pair of equations is given by

$$f(\xi,S)=c_{\perp}(S)\xi+c_{\|}(S)\frac{1}{2\xi^2}. \tag{2.27}$$

Consequently, expressions (2.23*a*,*b*) yield the equation of state

$$p_{\|}=c_{\|}(S)\rho^3/B^2, \quad p_{\perp}=c_{\perp}(S)\rho B, \tag{2.28a,b}$$

along with the relations

$$\bar{S}_{\|}=\ln c_{\|}(S), \quad \bar{S}_{\perp}=\ln c_{\perp}(S) \tag{2.29a,b}$$

from the double adiabatic integrals (2.13*a*,*b*). The corresponding internal energy (2.21*a*,*b*) and pressure (2.17) have the explicit form

$$e=c_{\perp}(S)B+c_{\|}(S)\frac{\rho^2}{2B^2}=\exp(\bar{S}_{\perp})B+\exp(\bar{S}_{\|})\frac{\rho^2}{2B^2}, \tag{2.30}$$

and

$$p=c_{\perp}(S)\frac{2\rho B}{3}+c_{\|}(S)\frac{\rho^3}{3B^2}=\exp(\bar{S}_{\perp})\frac{2\rho B}{3}+\exp(\bar{S}_{\|})\frac{\rho^3}{3B^2}. \tag{2.31}$$

Note that the expressions for e and p in terms of the double adiabatic integrals $\bar{S}_{\|}$ and $\bar{S}_{\perp}$ hold independently of any form of equation of state. Finally, either (2.22) or (2.18*a*–*c*), both of which rely on $e=e(\rho,S,B)$, yield the plasma temperature:

$$T=c_{\perp}'(S)B+c_{\|}'(S)\frac{\rho^2}{2B^2}=\frac{\mathrm{d}\bar{S}_{\perp}}{\mathrm{d}S}\exp(\bar{S}_{\perp})B+\frac{\mathrm{d}\bar{S}_{\|}}{\mathrm{d}S}\exp(\bar{S}_{\|})\frac{\rho^2}{2B^2}. \tag{2.32}$$

Hazeltine *et al.* (2013) arrived at a similar but less-general result under the assumption that f is separable in S and ξ. This leads to $c_{\|}(S)=Q(S)d_{\|}$ and $c_{\perp}(S)=Q(S)d_{\perp}$, where

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$d_\perp$ and $d_\parallel$ are constants. The assumption of separability includes, for example, the case where the gas pressure p satisfies the ideal gas law:

$$p = \rho R T \quad \text{or} \quad T = \frac{p}{\rho R}, \tag{2.33}$$

where R is the gas constant. In this case, from the relation (2.17), it follows that f must satisfy the equation $\partial f/\partial S = (2/3R)f$. Then it further follows that $Q(S) = \exp(2S/(3R))$.

Conditions on the derivatives of the internal energy e for thermodynamic stability are discussed in Hazeltine *et al.* (2013), but these considerations lie beyond the scope of the present work.

2.2.1. *Ideal gas law and entropy*

The double adiabatic equations (2.8) for a CGL plasma can be combined in the form

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{p_\parallel p_\perp^2}{\rho^5}\right) = 0 \tag{2.34}$$

which suggests the advected quantity $p_\parallel p_\perp^2/\rho^5$ may be viewed as a function of the entropy S. A specific functional relation can be motivated by considering the MHD limit for a non-relativistic gas, where the gas entropy is given by the standard formula

$$S = C_v \ln\left(\frac{p}{\rho^\gamma}\right), \quad \gamma = \frac{C_p}{C_v} = \frac{5}{3}, \tag{2.35a,b}$$

with C_v and C_p being the specific heats at constant volume and pressure, respectively. In a CGL plasma, the gas pressure p is given by expression (2.17) in terms of the pressure components $p_\parallel$ and $p_\perp$ which are equal in the MHD limit, because the pressure tensor (2.3*a–c*) must become isotropic. This results in the relation $p_\parallel = p_\perp = p$, and as a consequence $p_\parallel p_\perp^2/\rho^5 = p^3/\rho^5 = \exp(3S/C_v)$ by assuming the formula (2.35*a*,*b*). Generalising this relation away from the MHD limit then suggests the formula

$$S = C_v \ln\left(\frac{p_\parallel^{1/3} p_\perp^{2/3}}{\rho^{5/3}}\right) \tag{2.36}$$

for the entropy of a CGL plasma (see, e.g., Abraham-Shrauner 1967; Du *et al.* 2020).

Furthermore, because the double adiabatic integrals (2.13*a*,*b*) give $p_\parallel p_\perp^2/\rho^5 = \exp(\bar{S}_\parallel + 2\bar{S}_\perp)$ where $\bar{S}_\parallel$ and $\bar{S}_\perp$ are each advected, the entropy formula (2.36) can be expressed as $S = (C_v/3)(\bar{S}_\parallel + 2\bar{S}_\perp)$. This relation now suggests that the entropy is a sum of components

$$S = S_\parallel + S_\perp \quad \text{where } S_\parallel \equiv C_{v\parallel}\bar{S}_\parallel \quad \text{and} \quad S_\perp \equiv C_{v\perp}\bar{S}_\perp, \tag{2.37}$$

with

$$C_{v_\parallel} \equiv \tfrac{1}{3}C_v, \quad C_{v_\perp} \equiv \tfrac{2}{3}C_v. \tag{2.38a,b}$$

Correspondingly, the internal energy density (2.16*a*,*b*) of the plasma can be split into a sum of densities

$$\varepsilon = \frac{p_\parallel + 2p_\perp}{2} = \frac{p_\parallel}{(\gamma_\parallel - 1)} + \frac{p_\perp}{(\gamma_\perp - 1)} \equiv \varepsilon_\parallel + \varepsilon_\perp, \tag{2.39}$$

with effective adiabatic indices:

$$\gamma_\parallel = 3, \quad \gamma_\perp = 2. \tag{2.40a,b}$$

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Further note that

$$\varepsilon = \frac{p}{\gamma - 1}, \quad p = \frac{1}{3}(p_\parallel + 2p_\perp), \quad \gamma = \frac{5}{3} \tag{2.41a–c}$$

in terms of the gas pressure p.

In addition, by applying the standard ideal gas law (2.33) to each of the pressure components $p_\parallel$ and $p_\perp$, we can define corresponding temperatures:

$$T_\parallel \equiv p_\parallel/(\rho R) \quad \text{and} \quad T_\perp \equiv p_\perp/(\rho R). \tag{2.42a,b}$$

Then the internal energy density (2.39) can be expressed as

$$\varepsilon = R\rho \left(\tfrac{1}{2}T_\parallel + T_\perp\right), \tag{2.43}$$

which yields the relations

$$C_{v_\parallel} = e_{T_\parallel} = \tfrac{1}{2}R, \quad C_{v_\perp} = e_{T_\perp} = R, \quad C_v = \tfrac{3}{2}R \tag{2.44a–c}$$

from the specific heats (2.38*a,b*). Finally, by substituting the double adiabatic integrals (2.13*a,b*) into the quantities (2.42*a,b*) and using expressions (2.37) and (2.44*a–c*), we obtain

$$\frac{\partial T_\parallel}{\partial S_\parallel} = 2T_\parallel/R, \quad \frac{\partial T_\perp}{\partial S_\perp} = T_\perp/R, \tag{2.45a,b}$$

which implies the expected thermodynamic relations:

$$e_{S_\parallel} = T_\parallel, \quad e_{S_\perp} = T_\perp. \tag{2.46a,b}$$

The preceding approach to the entropy of CGL plasmas has been adopted by Du *et al.* (2020). It is equivalent to the thermodynamic approach specialised to the case of an ideal gas in which the gas law is assumed to hold for all three pressures p, $p_\parallel$ and $p_\perp$.

The net upshot of this discussion is that we take $e = e(\rho, S, B)$ as the equation of state for the gas in the rest of the paper. Note, however, that $S_\parallel = S_\parallel(S)$ and $S_\perp(S) = S - S_\parallel(S)$ are functions of S. Here $e = e(\rho, S, B)$ is the form of the equation of state used by Holm & Kupershmidt (1986). The CGL Poisson bracket of Holm & Kupershmidt (1986) uses the entropy variable $\sigma = \rho S$ to describe the complicated thermodynamics of the CGL plasma.

2.3. *Total mass and energy conservation laws*

The mass continuity equation (2.1) can be expressed in the familiar co-moving form

$$\frac{\partial \rho}{\partial t} + \boldsymbol{u}\cdot\nabla\rho = -\rho\nabla\cdot\boldsymbol{u}. \tag{2.47}$$

On a volume $V(t)$ moving with the fluid, the corresponding mass integral is conserved:

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_{V(t)} \rho\, \mathrm{d}^3x = 0. \tag{2.48}$$

A co-moving equation for the internal energy density (2.12) is obtained by combining the parallel and perpendicular pressure equations (2.8)–(2.9), which yields

$$\frac{\partial \varepsilon}{\partial t} + \nabla\cdot(\varepsilon\boldsymbol{u}) + (p_\parallel - p_\perp)\boldsymbol{\tau}\boldsymbol{\tau} : \nabla\boldsymbol{u} + p_\perp\nabla\cdot\boldsymbol{u} = 0. \tag{2.49}$$

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This equation can be written in terms of the gyrotropic pressure tensor p by noting from expression (2.10*a–c*) that

$$\mathsf{p}\cdot\nabla\boldsymbol{u} = p_\perp\nabla\cdot\boldsymbol{u} + (p_\parallel - p_\perp)\boldsymbol{\tau}\boldsymbol{\tau} : \nabla\boldsymbol{u}. \tag{2.50}$$

Then the co-moving equation (2.49) takes the form

$$\frac{\partial\varepsilon}{\partial t} + \nabla\cdot(\varepsilon\boldsymbol{u} + \mathsf{p}\cdot\boldsymbol{u}) = \boldsymbol{u}\cdot(\nabla\cdot\mathsf{p}), \tag{2.51}$$

which is analogous to the internal energy density equation in fluid dynamics and MHD.

The total kinetic energy equation for the plasma is obtained by taking the scalar product of the momentum equation (2.2) with $\boldsymbol{u}$, which yields

$$\frac{\partial}{\partial t}\left(\frac{1}{2}\rho u^2 + \rho\Phi\right) + \nabla\cdot\left(\boldsymbol{u}\left(\frac{1}{2}\rho u^2 + \rho\Phi\right)\right) = -\boldsymbol{u}\cdot\nabla\cdot\mathsf{p} + \boldsymbol{J}\cdot\boldsymbol{E} + \boldsymbol{u}\cdot\boldsymbol{B}\frac{\nabla\cdot\boldsymbol{B}}{\mu_0}, \tag{2.52}$$

where $u = |\boldsymbol{u}|$, and

$$\boldsymbol{E} = -\boldsymbol{u}\times\boldsymbol{B} \quad\text{and}\quad \boldsymbol{J} = \frac{\nabla\times\boldsymbol{B}}{\mu_0}, \tag{2.53a,b}$$

are the electric field $\boldsymbol{E}$ and the electric current $\boldsymbol{J}$.

Using Maxwell's equations (2.6)–(2.7), we obtain Poynting's theorem (the electromagnetic energy equation) in the form:

$$\frac{\partial}{\partial t}\left(\frac{1}{2\mu_0}B^2\right) + \nabla\cdot\left(\frac{1}{\mu_0}\boldsymbol{E}\times\boldsymbol{B}\right) = -\boldsymbol{J}\cdot\boldsymbol{E} - \frac{1}{\mu_0}(\nabla\cdot\boldsymbol{B})(\boldsymbol{u}\cdot\boldsymbol{B}). \tag{2.54}$$

The equation for the total energy in conserved form is obtained by adding the internal energy equation (2.51), the kinetic energy equation (2.52) and the electromagnetic energy equation (2.54), which gives

$$\frac{\partial}{\partial t}\left(\frac{1}{2}\rho u^2 + \rho\Phi + \varepsilon + \frac{1}{2\mu_0}B^2\right) + \nabla\cdot\left(\left(\frac{1}{2}\rho u^2 + \rho\Phi + \varepsilon\right)\boldsymbol{u} + \mathsf{p}\cdot\boldsymbol{u} + \frac{1}{\mu_0}\boldsymbol{E}\times\boldsymbol{B}\right) = 0. \tag{2.55}$$

It is useful here to note that the Poynting electromagnetic energy flux is

$$\boldsymbol{F}_{\mathrm{em}} = \frac{1}{\mu_0}\boldsymbol{E}\times\boldsymbol{B} = \frac{B^2}{2\mu_0}\boldsymbol{u} + \boldsymbol{u}\cdot\mathsf{M}_B. \tag{2.56}$$

The resulting energy balance equation on a volume $V(t)$ moving with the fluid is given by

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_{V(t)}\left(\frac{1}{2}\rho u^2 + \rho\Phi + \varepsilon + \frac{1}{2\mu_0}B^2\right)\mathrm{d}^3x = -\oint_{\partial V(t)}\boldsymbol{u}\cdot(\mathsf{p} + \mathsf{M}_B)\cdot\hat{\boldsymbol{n}}\,\mathrm{d}A \tag{2.57}$$

in terms of the magnetic pressure tensor (2.4). The flux terms in (2.57) represent the rate at which the total pressure tensor $\mathsf{p} + \mathsf{M}_B$ does work on the fluid. Note that there is no contribution due to advection of the total energy density. Further discussion is provided in § 6.1.1 (see also Padhye 1998; Anco & Dar 2009, 2010). In addition, note that if the total pressure tensor has no perpendicular component at the moving boundary, i.e. $(\mathsf{p} + \mathsf{M}_B)\cdot\hat{\boldsymbol{n}} = 0$, then the moving energy integral will be conserved (i.e. a constant of the motion).

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2.4. *Cross-helicity*

The cross-helicity transport equation for a CGL plasma is obtained by taking the scalar product of $\boldsymbol{u}$ with Faraday's equation (2.6) plus the scalar product of $\boldsymbol{B}$ with the momentum equation (2.2), in the following form. By a standard cross-product identity, Faraday's equation can be written

$$\frac{\partial}{\partial t}\boldsymbol{B} - \boldsymbol{B}\cdot\nabla\boldsymbol{u} + \boldsymbol{u}\cdot\nabla\boldsymbol{B} + (\nabla\cdot\boldsymbol{u})\boldsymbol{B} = 0. \tag{2.58}$$

The momentum equation minus $\boldsymbol{u}$ times the mass continuity equation (2.1) yields the velocity equation:

$$\frac{\mathrm{d}}{\mathrm{d}t}\boldsymbol{u} = -\frac{1}{\rho}\nabla\cdot(\mathsf{p} + \mathsf{M}_B) - \nabla\Phi, \tag{2.59}$$

where $-(1/\rho)\nabla\cdot\mathsf{p}$ is the acceleration due to the anisotropic pressure. Forming and adding the respective scalar products with $\boldsymbol{u}$ and $\boldsymbol{B}$ then gives the equation for cross-helicity density $\boldsymbol{u}\cdot\boldsymbol{B}$ as

$$\frac{\partial}{\partial t}(\boldsymbol{u}\cdot\boldsymbol{B}) + \nabla\cdot\left[(\boldsymbol{u}\cdot\boldsymbol{B})\boldsymbol{u} + \left(\Phi - \frac{1}{2}u^2\right)\boldsymbol{B}\right] = -\boldsymbol{B}\cdot\left(\frac{1}{\rho}\nabla\cdot\mathsf{p}\right), \tag{2.60}$$

where $u = |\boldsymbol{u}|$.

The thermodynamic form of this transport equation (2.60) is obtained by using the equation

$$-\nabla\cdot\mathsf{p} = \boldsymbol{B}\times(\nabla\times\boldsymbol{\Omega}) - \boldsymbol{\Omega}(\nabla\cdot\boldsymbol{B}) + \rho(T\nabla S - \nabla h), \tag{2.61}$$

which relies on the Pfaffian equation (first law of thermodynamics) (2.15) for a CGL plasma with an internal energy $e(\rho, B, S)$, where, from the thermodynamic relations (2.18*a–c*), $T = e_S$ is the gas temperature and $h = \varepsilon_\rho = e + \rho e_\rho$ is the enthalpy of the fluid:

$$h = \frac{3p_\parallel + 2p_\perp}{2\rho}. \tag{2.62}$$

Here

$$\boldsymbol{\Omega} = \frac{p_\Delta}{B}\boldsymbol{\tau}. \tag{2.63}$$

Note that the case $\boldsymbol{\Omega} = 0$ corresponds to the MHD limit in which the anisotropy vanishes, $p_\Delta = p_\parallel - p_\perp = 0$. A derivation of the pressure divergence equation (2.61) is given in Appendix B. An alternative derivation is provided within the EP formulation of the CGL equations in § 4.1 and Appendix E. The scalar product of this equation with $(1/\rho)\boldsymbol{B}$ reduces to

$$-\boldsymbol{B}\cdot\left(\frac{1}{\rho}\nabla\cdot\mathsf{p}\right) = \boldsymbol{B}\cdot(T\nabla S) - \nabla\cdot(h\boldsymbol{B}) \tag{2.64}$$

using Gauss' law (2.7). As a result, the cross-helicity density transport equation in thermodynamic form is simply

$$\frac{\partial}{\partial t}(\boldsymbol{u}\cdot\boldsymbol{B}) + \nabla\cdot\left[(\boldsymbol{u}\cdot\boldsymbol{B})\boldsymbol{u} + \left(\Phi + h - \frac{1}{2}u^2\right)\boldsymbol{B}\right] = \boldsymbol{B}\cdot(T\nabla S). \tag{2.65}$$

The term on the right-hand side of the cross-helicity transport equation (2.65) can be written in a conserved form through use of the equation

$$\frac{\partial}{\partial t}(\phi \nabla S \cdot \boldsymbol{B}) + \nabla \cdot [(\phi \nabla S \cdot \boldsymbol{B})\,\boldsymbol{u}] = (\nabla S \cdot \boldsymbol{B})\,\frac{\mathrm{d}}{\mathrm{d}t}\phi, \tag{2.66}$$

which holds for any scalar variable ϕ and follows from Faraday's equation (2.6) and the entropy equation (2.5). This leads to the non-local cross-helicity density conservation law

$$\frac{\partial}{\partial t}(\boldsymbol{w} \cdot \boldsymbol{B}) + \nabla \cdot \left[(\boldsymbol{w} \cdot \boldsymbol{B})\boldsymbol{u} + \left(\Phi + h - \frac{1}{2}u^2\right)\boldsymbol{B}\right] = 0, \tag{2.67}$$

in terms of

$$\boldsymbol{w} = \boldsymbol{u} + r\nabla S, \tag{2.68}$$

where r is a non-local variable obtained by integrating the temperature T back along the path of the Lagrangian fluid element. Specifically,

$$\frac{\mathrm{d}r}{\mathrm{d}t} = -T, \quad r = -\int^t T\,\mathrm{d}t, \tag{2.69a,b}$$

where $\mathrm{d}/\mathrm{d}t = \partial/\partial t + \boldsymbol{u} \cdot \nabla$ is the Lagrangian time derivative.

The non-local conservation law (2.67) yields a moving cross-helicity balance equation (cf. Appendix C):

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_{V(t)} \boldsymbol{w} \cdot \boldsymbol{B}\,\mathrm{d}^3x = -\oint_{\partial V(t)} \left(\Phi + h - \frac{1}{2}u^2\right)\boldsymbol{B} \cdot \hat{\boldsymbol{n}}\,\mathrm{d}A \tag{2.70}$$

on a volume $V(t)$ moving with the fluid. Note that if $\boldsymbol{B}$ is perpendicular to the boundary, i.e. $\boldsymbol{B} \cdot \hat{\boldsymbol{n}} = 0$, then the cross-helicity integral $\int_{V(t)} \boldsymbol{w} \cdot \boldsymbol{B}\,\mathrm{d}^3\boldsymbol{x}$ is conserved in the flow.

An alternative form of the non-local conservation law (2.67) is obtained by directly taking the divergence of the pressure tensor p using the gyrotropic expression (2.10a–c), and combining it with the gradient of the enthalpy (2.62). As shown in Appendix B, this gives

$$\begin{aligned}\boldsymbol{B} \cdot \left(\frac{1}{\rho}\nabla \cdot \mathsf{p} - \nabla h\right) &= -\left(\frac{p_\parallel}{2\rho}\boldsymbol{B} \cdot \nabla \ln c_\parallel(S) + \frac{p_\perp}{\rho}\boldsymbol{B} \cdot \nabla \ln c_\perp(S)\right) \\ &= -\left(\frac{p_\parallel}{2\rho}\boldsymbol{B} \cdot \nabla \bar{S}_\parallel + \frac{p_\perp}{\rho}\boldsymbol{B} \cdot \nabla \bar{S}_\perp\right)\end{aligned} \tag{2.71}$$

in terms of the adiabatic integrals (2.13a,b) and (2.37). Expression (2.71) is equivalent to $-\boldsymbol{B} \cdot (T\nabla S)$ through the temperature expression (2.32). Instead, if we use the gas law temperatures (2.49) and the specific heats (2.51) associated with $p_\parallel$ and $p_\perp$, then we can write

$$\boldsymbol{B} \cdot \left(\frac{1}{\rho}\nabla \cdot \mathsf{p} - \nabla h\right) = -\left(T_\parallel \boldsymbol{B} \cdot \nabla S_\parallel + T_\perp \boldsymbol{B} \cdot \nabla S_\perp\right) \tag{2.72}$$

in term of the gas law entropies (2.37). Substituting expression (2.72) into the cross-helicity density transport equation (2.65) yields

$$\frac{\partial}{\partial t}(\boldsymbol{u} \cdot \boldsymbol{B}) + \nabla \cdot \left[(\boldsymbol{u} \cdot \boldsymbol{B})\boldsymbol{u} + \boldsymbol{B}\left(\Phi + h - \frac{1}{2}u^2\right)\right] = \boldsymbol{B} \cdot (T_\parallel \nabla S_\parallel + T_\perp \nabla S_\perp). \tag{2.73}$$

This result (2.73) can be also derived using the Newcomb (1962) action principle for CGL plasmas shown in § 4.

The corresponding non-local cross-helicity density conservation law is given by

$$\frac{\partial}{\partial t}(\tilde{\boldsymbol{w}}\boldsymbol{\cdot}\boldsymbol{B})+\nabla\boldsymbol{\cdot}\left[(\tilde{\boldsymbol{w}}\boldsymbol{\cdot}\boldsymbol{B})\boldsymbol{u}+\boldsymbol{B}\left(\Phi+h-\frac{1}{2}u^2\right)\right]=0, \tag{2.74}$$

where

$$\tilde{\boldsymbol{w}}=\boldsymbol{u}+r_\parallel\nabla S_\parallel+r_\perp\nabla S_\perp, \tag{2.75}$$

with $r_\parallel$ and $r_\perp$ being non-local variables defined by the equations:

$$\frac{\mathrm{d}r_\parallel}{\mathrm{d}t}=-T_\parallel,\quad \frac{\mathrm{d}r_\perp}{\mathrm{d}t}=-T_\perp. \tag{2.76a,b}$$

The non-local cross-helicity *density* conservation law for MHD analogous to equation (2.67) was developed in Webb *et al.* (2014*a*,*b*), Webb (2018), Webb & Anco (2019), Yahalom (2017*a*,*b*) and Yahalom & Qin (2021). A topological interpretation of the generalised cross-helicity conservation law has been found in terms of an MHD Aharonov–Bohm effect in Yahalom (2017*a*,*b*). Yahalom (2013) discussed a topological interpretation of magnetic helicity as an Aharonov Bohm effect in MHD.

If $\boldsymbol{B}\boldsymbol{\cdot}\nabla S=0$, then the non-local conservation law (2.67) reduces to a local cross-helicity *density* conservation law given by $r=0$ and $\boldsymbol{w}=\boldsymbol{u}$. This result is analogous to the local cross-helicity conservation law in MHD (e.g. Webb 2018; Webb & Anco 2019).

It is interesting to note that the velocity equation (2.59) can be written in the suggestive form:

$$\frac{\mathrm{d}\boldsymbol{u}}{\mathrm{d}t}=T\nabla S-\nabla h+\frac{\tilde{\boldsymbol{J}}\times\boldsymbol{B}}{\rho}-\nabla\Phi, \tag{2.77}$$

where

$$\tilde{\boldsymbol{J}}\equiv\boldsymbol{J}-\nabla\times\boldsymbol{\Omega}=\frac{\nabla\times\tilde{\boldsymbol{B}}}{\mu_0},\quad \tilde{\boldsymbol{B}}\equiv\boldsymbol{B}\left(1-\frac{\mu_0 p_\Delta}{B^2}\right). \tag{2.78a,b}$$

This (2.77) turns out to arise directly from the non-canonical Hamiltonian formulation presented in § 5.1.

2.5. *Magnetic helicity*

In ideal MHD and in ideal CGL plasmas, Faraday's equation (2.6) written in terms of the electric field

$$\frac{\partial\boldsymbol{B}}{\partial t}+\nabla\times\boldsymbol{E}=0 \tag{2.79}$$

can be uncurled to give $\boldsymbol{E}=-\boldsymbol{u}\times\boldsymbol{B}$ in the form

$$\boldsymbol{E}=-\nabla\phi_E-\frac{\partial\boldsymbol{A}}{\partial t}=-(\boldsymbol{u}\times\boldsymbol{B})\quad\text{where }\boldsymbol{B}=\nabla\times\boldsymbol{A}. \tag{2.80}$$

The uncurled form of Faraday's equation (2.80) implies

$$\frac{\partial\boldsymbol{A}}{\partial t}+\boldsymbol{E}+\nabla\phi_E=0. \tag{2.81}$$

Combining the scalar product of Faraday's equation (2.79) with $\boldsymbol{A}$ plus the scalar product of the uncurled equation (2.81) with $\boldsymbol{B}$ yields the magnetic helicity transport equation

$$\frac{\partial}{\partial t}(\boldsymbol{A}\boldsymbol{\cdot}\boldsymbol{B})+\nabla\boldsymbol{\cdot}(\boldsymbol{E}\times\boldsymbol{A}+\phi_E\boldsymbol{B})=0, \tag{2.82}$$

which can also be written in the form

$$\frac{\partial}{\partial t}(\boldsymbol{A}\cdot\boldsymbol{B})+\nabla\cdot[(\boldsymbol{A}\cdot\boldsymbol{B})\boldsymbol{u}+(\phi_E-\boldsymbol{u}\cdot\boldsymbol{A})\boldsymbol{B}]=0. \tag{2.83}$$

The total magnetic helicity for a volume $V(t)$ moving with the fluid is given by

$$H^M=\int_{V(t)}\boldsymbol{A}\cdot\boldsymbol{B}\,\mathrm{d}^3x. \tag{2.84}$$

It satisfies the moving balance equation (cf. Appendix C)

$$\frac{\mathrm{d}H^M}{\mathrm{d}t}=-\oint_{\partial V(t)}(\phi_E-\boldsymbol{A}\cdot\boldsymbol{u})\boldsymbol{B}\cdot\hat{\boldsymbol{n}}\,\mathrm{d}A, \tag{2.85}$$

where $\hat{\boldsymbol{n}}$ is the outward unit normal of the moving boundary $\partial V(t)$. Thus, if $\boldsymbol{B}$ is perpendicular to the boundary, i.e. $\boldsymbol{B}\cdot\hat{\boldsymbol{n}}=0$, then H^M is conserved in the flow. This discussion applies both to ideal MHD and also to ideal CGL plasmas. (e.g. Kruskal & Kulsrud 1958; Woltjer 1958; Moffatt 1969; Berger & Field 1984; Moffatt & Ricca 1992; Arnold & Khesin 1998).

It is beyond the scope of the present exposition to discuss the issues of how to define the relative magnetic helicity for volumes V for which $\boldsymbol{B}\cdot\boldsymbol{n}\neq 0$ on ∂V (e.g. Berger & Field 1984; Finn & Antonsen 1985, 1988; Webb *et al.* 2010). Similar considerations apply to field line magnetic helicity (e.g. Prior & Yeates 2014) and absolute magnetic helicity (e.g. Low 2006, 2011; Berger & Hornig 2018).

It is of interest to note that, under gauge transformations (i.e. $\boldsymbol{A}\to\boldsymbol{A}+\nabla\chi$ and $\phi_E\to\phi_E-\partial\chi/\partial t$ for an arbitrary function $\chi(t,\boldsymbol{x})$), the moving balance equation is gauge invariant, but the magnetic helicity integral changes by addition of a boundary integral: $H^M\to H^M+\oint_{\partial V(t)}\chi\boldsymbol{B}\cdot\hat{\boldsymbol{n}}\,\mathrm{d}A$. However, if the electric field potential ϕ_E satisfies the gauge

$$\phi_E=\boldsymbol{A}\cdot\boldsymbol{u}, \tag{2.86}$$

then the moving flux of the magnetic helicity vanishes and the resulting (gauge-dependent) magnetic helicity H^M is conserved in the flow. This gauge turns out to hold precisely when $\boldsymbol{A}$ is advected by the flow, which is shown in the following subsection.

2.6. *Lie dragged (advected) quantities*

A quantity, a, is Lie dragged by the fluid flow if its advective Lie derivative vanishes:

$$\left(\frac{\partial}{\partial t}+\mathcal{L}_{\boldsymbol{u}}\right)a=0, \tag{2.87}$$

where $\mathcal{L}_{\boldsymbol{u}}$ is the ordinary Lie derivative with respect to the vector field $\boldsymbol{u}$ (e.g. Tur & Yanovsky 1993; Webb *et al.* 2014*a*; Anco & Webb 2020). For a scalar quantity, its advective Lie derivative reduces to its material derivative

$$\frac{\mathrm{d}}{\mathrm{d}t}=\frac{\partial}{\partial t}+\boldsymbol{u}\cdot\nabla. \tag{2.88}$$

For quantities that are vector fields or differential forms, the advective Lie derivative also contains a rotation-shear term which involves $\nabla\boldsymbol{u}$.

In CGL plasmas, S is an advected scalar, as are the double adiabatic integrals (2.13*a*,*b*). There are two basic advected non-scalar quantities: the differential form $\boldsymbol{\alpha}=\nabla S\cdot\mathrm{d}\boldsymbol{x}\equiv$

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$\nabla_i S\, \mathrm{d}x^i$ and the vector field $\boldsymbol{b} = (1/\rho)\boldsymbol{B} \equiv b^i\, \nabla_i$ (note the vector field $\boldsymbol{b}$ is a directional derivative operator). The contraction of the vector field $\boldsymbol{b}$ with the co-vector or differential form $\boldsymbol{\alpha}$ is a scalar (note that $\nabla_i \rfloor \mathrm{d}x^j = \delta^i_j$, and $\boldsymbol{b}\rfloor\boldsymbol{\alpha} = b^i\alpha_i \equiv b^i\nabla_i S$). Note that $\boldsymbol{b}$ is Lie dragged with the flow, (this statement is equivalent to Faraday's equation, when one takes into account the mass continuity equation: i.e. $(\partial_t + \mathcal{L}_{\boldsymbol{u}})\boldsymbol{b} = 0$ which implies $\partial_t\boldsymbol{b} + [\boldsymbol{u}, \boldsymbol{b}] = 0$). Here $\boldsymbol{\alpha}$ is a one-form or co-vector, that is Lie dragged with the flow. The inner product of the vector field $\boldsymbol{b}$ with the one-form $\boldsymbol{\alpha}$ is a scalar invariant which is advected with the flow (e.g. Tur & Yanovsky 1993). Thus, the quantities

$$\frac{\boldsymbol{B}\boldsymbol{\cdot}\nabla S}{\rho}, \quad \frac{\boldsymbol{B}\boldsymbol{\cdot}\nabla \bar{S}_\parallel}{\rho}, \quad \frac{\boldsymbol{B}\boldsymbol{\cdot}\nabla \bar{S}_\perp}{\rho} \tag{2.89a–c}$$

are advected scalars in CGL plasmas.

The gauge (2.86) in which the magnetic helicity integral is conserved is equivalent to Lie dragging the one-form $\alpha = \boldsymbol{A}\boldsymbol{\cdot}\mathrm{d}\boldsymbol{x}$ with the fluid flow:

$$\left(\frac{\partial}{\partial t} + \mathcal{L}_{\boldsymbol{u}}\right)(\boldsymbol{A}\boldsymbol{\cdot}\mathrm{d}\boldsymbol{x}) = \left(\frac{\partial \boldsymbol{A}}{\partial t} - \boldsymbol{u}\times\boldsymbol{B} + \nabla(\boldsymbol{A}\boldsymbol{\cdot}\boldsymbol{u})\right)\boldsymbol{\cdot}\mathrm{d}\boldsymbol{x} = 0, \tag{2.90}$$

which vanishes due to the uncurled form of Faraday's equation (2.80) combined with equation (2.86). Holm & Kupershmidt (1983a,b) used this advected-$\boldsymbol{A}$ gauge in the formulation of non-canonical Poisson brackets for MHD and for multi-fluid plasmas (see also Gordin & Petviashvili 1987, 1989; Padhye & Morrison 1996a,b; Padhye 1998).

3. The Lagrangian map

In a Lagrangian formulation of MHD and CGL plasmas, fluid elements are given labels that are constant in the fluid flow. The simplest labelling consists of initial values $\boldsymbol{x}_0 = \boldsymbol{x}(0)$ for integrating the flow equations of a fluid element

$$\frac{\mathrm{d}\boldsymbol{x}(t)}{\mathrm{d}t} = \boldsymbol{u}(\boldsymbol{x}(t), t), \tag{3.1}$$

in which the fluid velocity $\boldsymbol{u}$ is assumed to be a known function of $\boldsymbol{x}$ and t. In general, fluid labels are given by functions $\boldsymbol{a} = \boldsymbol{a}(\boldsymbol{x}_0)$.

A Lagrangian map is an invertible mapping from the Lagrangian fluid labels $\boldsymbol{a}$ to the Eulerian position coordinates $\boldsymbol{x}(t) = \boldsymbol{X}(\boldsymbol{a}, t)$ for the motion of a fluid element. For simplicity, we take $\boldsymbol{x}$ and $\boldsymbol{a} = \boldsymbol{x}_0$ to be expressed in terms of Cartesian coordinates $\boldsymbol{x} = (x, y, z)$ and $\boldsymbol{x}_0 = (x_0, y_0, z_0)$. Then, the Lagrangian map takes the form

$$\boldsymbol{x} = \boldsymbol{X}(\boldsymbol{x}_0) \tag{3.2}$$

with the t dependence being suppressed in the notation. Invertibility implies that the Jacobian of this map is non-degenerate:

$$J = \det(X_{ij}) \neq 0 \quad \text{where } X_{ij} \equiv \frac{\partial X^i(\boldsymbol{x}_0)}{\partial x_0^j}. \tag{3.3}$$

We write Y_{ij} to denote the inverse of the matrix X_{ij}, whereby

$$Y_{ik}X_{kj} = X_{ik}Y_{kj} = \delta_{ij} \tag{3.4}$$

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with δ_{ij} denoting the components of the identity matrix. Recall the standard formulae:

$$Y_{ij} = J^{-1}A_{ji}, \quad X_{ji}A_{ki} = A_{ik}X_{ij} = J\delta_{jk}, \quad \frac{\partial J}{\partial X_{ij}} = A_{ij}, \tag{3.5a–c}$$

where $A_{ij} = \mathrm{cofac}(X_{ij})$ is the cofactor matrix of X_{ij}. In addition, note that

$$\dot{J} = \frac{\partial J}{\partial X_{ij}}\dot{X}_{ij} = A_{ij}\frac{\partial \dot{X}^i}{\partial x_0^j}. \tag{3.6}$$

3.1. *Map formulae*

We now formulate the maps between the CGL plasma variables ρ, S, $\boldsymbol{B}$, $p_\parallel$ and $p_\perp$ and their Lagrangian counterparts $\rho_0(\boldsymbol{x}_0)$, $S_0(\boldsymbol{x}_0)$, $\boldsymbol{B}_0(\boldsymbol{x}_0)$, $p_\parallel(\boldsymbol{x}_0)$ and $p_\perp(\boldsymbol{x}_0)$, respectively.

The density ρ_0 of a fluid element with label $\boldsymbol{x}_0$ is related to the Eulerian density by the mass conservation equation

$$\rho\, \mathrm{d}^3x = \rho_0\, \mathrm{d}^3x_0, \tag{3.7}$$

where $\mathrm{d}^3x = J\, \mathrm{d}^3x_0$. This implies

$$\rho = \frac{\rho_0}{J}. \tag{3.8}$$

From Newcomb (1962) the Cartesian components of B^i in the Eulerian frame are related to the Lagrangian magnetic field component B_0^k by the equation:

$$B^i = \frac{X_{ik}B_0^k}{J}. \tag{3.9}$$

The derivation of (3.9) follows by noting

$$B^i\, \mathrm{d}\sigma_i = B_0^k\, \mathrm{d}\sigma_{0k}, \tag{3.10}$$

$$\mathrm{d}^3x = \mathrm{d}x^i\, \mathrm{d}\sigma_i = J\, \mathrm{d}^3x_0 = J\, \mathrm{d}x_0^k\, \mathrm{d}\sigma_{ok} \equiv JY_{ki}\, \mathrm{d}x^i\, \mathrm{d}\sigma_{0k}. \tag{3.11}$$

Equation (3.10) describes the conservation of magnetic flux, where $\mathrm{d}\sigma_i$ is the flux tube area normal to the x^i coordinate surface, and $\mathrm{d}\sigma_{0k}$ is the flux tube area normal to the x_0^k surface in the Lagrangian frame. Equation (3.11) relates the volume elements d^3x to d^3x_0. From (3.11) one obtains

$$\mathrm{d}\sigma_i = A_{ik}\, \mathrm{d}\sigma_0^k, \tag{3.12}$$

for the transformation between the area elements $\mathrm{d}\sigma_i$ and $\mathrm{d}\sigma_0^k$. The magnetic flux conservation equation (3.10) now gives the transformation $B_0^k = A_{ik}B^i$ which, in turn, implies the transformation (3.9).

An alternative form of the relation (3.9) arises from the Lie dragged or frozen-in vector field

$$\boldsymbol{b} = \frac{B^i}{\rho}\frac{\partial}{\partial x^i} = \frac{B_0^k}{\rho_0}\frac{\partial}{\partial x_0^k} \equiv \frac{B_0^k}{\rho_0}X_{ik}\frac{\partial}{\partial x^i}, \tag{3.13}$$

which implies the transformation (3.9) between B^i and B_0^k. In (3.13) we use the modern differential geometry notion of a vector field as a directional derivative operator (e.g. Misner, Thorne & Wheeler 1973).

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From the relation (3.9), we have

$$B^2 = \frac{\zeta^2 B_0^2}{J^2} \quad \text{where } \zeta^2 = X_{ij}\tau_0^j X_{ik}\tau_0^k \tag{3.14}$$

in terms of the unit vector $\tau_0 = B_0^{-1}\boldsymbol{B}_0$ along the magnetic field $\boldsymbol{B}_0$ (cf. (2.3*a–c*)), with $B_0 = |\boldsymbol{B}_0|$ being the magnetic field strength at $\boldsymbol{x}_0$. Hence, we obtain

$$B = \frac{\zeta\, B_0}{J}, \quad \tau^i = \frac{B^i}{B} = \frac{X_{ij}\tau_0^j}{\zeta}. \tag{3.15a,b}$$

The latter relation can be inverted to obtain $\boldsymbol{\tau}_0$ in terms of $\boldsymbol{\tau}$ through the formulae (3.5*a–c*):

$$\tau_0^i = \frac{\zeta}{J} A_{ji}\tau^j. \tag{3.16}$$

Next, from the double adiabatic conservation laws (2.11*a*,*b*), we have the frozen-in quantities

$$\frac{p_\perp}{\rho B} = \frac{p_{\perp 0}}{\rho_0 B_0}, \quad \frac{p_\parallel B^2}{\rho^3} = \frac{p_{\parallel 0} B_0^2}{\rho_0^3}. \tag{3.17a,b}$$

By combining these equations with (3.15*a*,*b*) and (3.8), we obtain

$$p_\parallel = \frac{p_{\parallel 0}}{J\zeta^2}, \quad p_\perp = \frac{p_{\perp 0}\zeta}{J^2}, \tag{3.18a,b}$$

which gives the respective relations between the pressures $p_{\parallel 0}$ and $p_{\perp 0}$ at a fluid element with label $\boldsymbol{x}_0$ and the Eulerian pressures $p_\parallel$ and $p_\perp$.

Finally, we note that the Eulerian entropy S is equal to the entropy S_0 at a fluid element with label $\boldsymbol{x}_0$, since S is frozen-in by the transport equation (2.5).

3.2. *Lagrangian variational principle and Euler–Lagrange equations*

The Lagrangian variational principle for MHD and for CGL plasmas was obtained in Newcomb (1962):

$$\mathcal{A} = \int_{t_0}^{t_1}\int_V L\, \mathrm{d}^3x\, \mathrm{d}t = \int_{t_0}^{t_1}\int_{V_0} L_0\, \mathrm{d}^3x_0\, \mathrm{d}t, \tag{3.19}$$

where L is the Lagrangian in Eulerian variables and L_0 is its counterpart arising through the Lagrangian map. Here V is a fixed spatial domain and $[t_0, t_1]$ is a fixed time interval. In general, L will be given by the kinetic energy density minus the potential energy density for a fluid element. The kinetic energy density for both MHD and CGL plasmas is simply $\frac{1}{2}\rho u^2$, with $u = |\boldsymbol{u}|$. Subtracting this expression from the total energy density for a CGL plasma (cf. the moving energy balance equation (2.57))

$$H \equiv \frac{1}{2}\rho u^2 + \rho\Phi + \varepsilon + \frac{1}{2\mu_0}B^2 \tag{3.20}$$

yields the potential energy density $\rho\Phi + \varepsilon + (1/2\mu_0)B^2$, where ε is the internal energy density (2.12). Thus, a CGL plasma has the Lagrangian

$$L = \frac{1}{2}\rho u^2 - \rho\Phi - \varepsilon - \frac{1}{2\mu_0}B^2, \tag{3.21}$$

where, for the sake of completeness, we have included the energy contributed by an external gravitational potential $\Phi(\boldsymbol{x})$ (e.g. which would arise from the gravitational field of the Sun in the case of solar and interplanetary physics). The corresponding Lagrangian L_0 is obtained from the relation $L\,\mathrm{d}^3x = L_0\,\mathrm{d}^3x_0 = LJ\,\mathrm{d}^3x_0$, which gives

$$L_0 = JL. \tag{3.22}$$

Substituting the Eulerian expressions (3.21), (3.18*a*,*b*), (3.14), (3.8) and (3.2) into L_0, and then using expression (2.12) for the internal energy density in terms of the pressures $p_\parallel$ and $p_\perp$, we obtain

$$L_0 = \rho_0(\boldsymbol{x}_0)\left(\frac{1}{2}|\dot{\boldsymbol{X}}|^2 - \Phi(\boldsymbol{X})\right) - \left(\frac{p_{\parallel 0}(\boldsymbol{x}_0)}{2\zeta^2} + \frac{p_{\perp 0}(\boldsymbol{x}_0)\zeta}{J}\right) - \frac{\zeta^2 B_0(\boldsymbol{x}_0)^2}{2\mu_0 J}. \tag{3.23}$$

The Euler–Lagrange equations of the resulting action principle (3.19) for a CGL plasma are obtained by variation of the Cartesian components X^i of the Lagrangian map (3.2), modulo boundary terms: $\delta\mathcal{A}/\delta X^i = 0$. In essence, we work in a reference frame moving with the fluid flow, where the dynamics is described in terms of how the Eulerian position $\boldsymbol{x}$ of the frame varies with the fluid labels $\boldsymbol{x}_0$ and the time t through the Lagrangian map (3.2). Equivalently, the Euler–Lagrange equations are given by applying to L_0 the Euler operator (variational derivative) E_{X^i} in the calculus of variations, where $X^i(x_0^j, t)$ is a function of x_0^j and t. Thus,

$$\frac{\delta\mathcal{A}}{\delta X^i} \equiv E_{X^i}(L_0) = \frac{\partial L_0}{\partial X^i} - \frac{\partial}{\partial t}\left(\frac{\partial L_0}{\partial \dot{X}^i}\right) - \frac{\partial}{\partial x_0^j}\left(\frac{\partial L_0}{\partial X_{ij}}\right) = 0, \tag{3.24}$$

where ∂/∂_t and $\partial/\partial_{x_0^j}$ act as total derivatives. (See § 6 for more details of variational calculus.) In Appendix C, we show that when (3.24) is simplified and expressed in terms of Eulerian variables, it reduces to the Eulerian momentum equation (2.2).

3.3. *Hamilton's equations*

We now derive the corresponding Hamilton formulation of the Euler–Lagrange equation (3.24) for a CGL plasma. Here we write $X^i = x^i$ for simplicity.

First, the generalised momentum is defined as

$$\pi_i = \frac{\partial L_0}{\partial \dot{x}^i} = \rho_0\dot{x}^i. \tag{3.25}$$

Next, the Hamiltonian density is defined by the Legendre transformation:

$$H_0 = \pi_k\dot{x}^k - L_0. \tag{3.26}$$

After substitution of the CGL Lagrangian (3.23) along with $\dot{x}^i$ in terms of the momentum (3.25), we then obtain the expression

$$H_0 = \frac{\pi_i\pi_i}{2\rho_0} + \rho_0\Phi(\boldsymbol{x}) + \frac{p_{\parallel 0}}{2\zeta^2} + \frac{p_{\perp 0}\zeta}{J} + \frac{X_{ij}X_{ik}B_0^jB_0^k}{2\mu_0 J}. \tag{3.27}$$

In Eulerian variables, this expression is given by

$$H_0 = JH, \tag{3.28}$$

where H is the Eulerian total energy density (3.20). The resulting Hamiltonian equations of motion consist of

$$\dot{x}^i = \frac{\delta\mathcal{H}}{\delta\pi^i}, \quad \dot{\pi}^i = -\frac{\delta\mathcal{H}}{\delta x^i}, \tag{3.29a,b}$$

which yield the momentum relation (3.25) and the Euler–Lagrange equation (3.24), where the Hamiltonian function is defined as

$$\mathcal{H} = \int_V H_0 \,\mathrm{d}^3 x_0 = \int_V H \,\mathrm{d}^3 x. \tag{3.30}$$

As in ordinary Hamiltonian mechanics, with the use of the canonical variables $q^i = x^i$ and $p^i = \pi^i$, Hamilton's equations (3.29*a*,*b*) can be written in a canonical Poisson bracket formulation:

$$\dot{\mathcal{F}} = \{\mathcal{F}, \mathcal{H}\} = \int \left(\mathcal{F}_{q^i}\mathcal{H}_{p^i} - \mathcal{F}_{q^i}\mathcal{H}_{p^i}\right) \mathrm{d}^3 x_0. \tag{3.31}$$

Here $\mathcal{F}$ is a general functional of q^i and p^i (see Morrison 1982). There is an equivalent, non-canonical Poisson bracket that employs only Eulerian variables, which we present in § 5.

4. EP action principle

In this section, we formulate the EP action principle for the CGL plasma equations. We follow the developments in Holm *et al.* (1998), Webb *et al.* (2014*b*), which provide the EP formulation for MHD. The main difference for a CGL plasma compared with MHD is that the internal energy density $\varepsilon = \rho e$ (cf. (2.16*a*,*b*)) depends on ρ, S and B, whereas for MHD it depends only on ρ and S.

The Lagrangian and Hamiltonian action principles in § 3 are based on a reference frame that is attached to fluid elements, whereby the Eulerian position $\boldsymbol{x}$ of the frame is a function of fluid element labels $\boldsymbol{x}_0$ and time t, as given by the Lagrangian map (3.2). In contrast, the EP action principle is based on the use of a fixed Eulerian reference frame, in which the motion of a fluid element is given by inverting the Lagrangian map so that the fluid element label $\boldsymbol{x}_0$ becomes a function of $\boldsymbol{x}$ and t. To set up this formulation, it is convenient to introduce the notation

$$g\boldsymbol{x}_0 = \boldsymbol{x} \tag{4.1}$$

with g at any fixed time t representing an element in the group of diffeomorphisms on Euclidean space in Cartesian coordinates $\boldsymbol{x}$ and $\boldsymbol{x}_0$. Note that g has an inverse g^{-1} defined by $g^{-1}\boldsymbol{x} = \boldsymbol{x}_0$, whereas $g|_{t=0}$ is just the identity map. In (4.1) and the sequel, the t dependence of g and $\boldsymbol{x}_0$ is suppressed for simplicity of notation; an overdot stands for $\dot{} = \partial/\partial t$. The diffeomorphism group will be denoted $G \equiv \mathrm{Diff}(\mathbb{R}^3)$.

The coordinate components of a fluid element label, x_0^i, represent advected quantities:

$$\left(\frac{\partial}{\partial t} + \boldsymbol{u}\boldsymbol{\cdot}\nabla\right) x_0^i = 0. \tag{4.2}$$

From (4.1) and (4.2), we see that

$$\dot{x}_0^i = ((\dot{g^{-1}})\boldsymbol{x})^i = -(g^{-1}\dot{g}g^{-1}\boldsymbol{x})^i = -(g^{-1}\dot{g}\boldsymbol{x}_0)^i = -\boldsymbol{u}\boldsymbol{\cdot}\nabla x_0^i. \tag{4.3}$$

Hence, we obtain

$$g^{-1}\dot{g} = \boldsymbol{u}\boldsymbol{\cdot}\partial_{\boldsymbol{x}}, \tag{4.4}$$

which can be viewed as both a left-invariant vector field in the tangent space of G and the directional derivative along the fluid flow in Euclidean space. The property of left-invariance means that, for any fixed element h in G, $g \to hg$ implies $\eta \to (hg)^{-1}(\dot{hg}) = g^{-1}h^{-1}h\dot{g} = g^{-1}\dot{g} = \eta$, with $\dot{h} \equiv 0$.

Compared with the Lagrangian formulation, whose dynamical variable is $\boldsymbol{x}(\boldsymbol{x}_0, t)$, the EP formulation uses the dynamical variables ρ, S, B and $\boldsymbol{u}$, which are functions of $\boldsymbol{x}$ and t. Variations of the EP variables are defined through variation of the inverse Lagrangian map, g, in the following way.

We consider a general variation δg and define

$$\eta \equiv g^{-1}\delta g \tag{4.5}$$

which represents a left-invariant vector field on the group G. Similarly to the identification (4.4), we can write

$$\eta = \boldsymbol{\eta} \cdot \partial_x \tag{4.6}$$

viewed as the directional derivative associated with δg, where $\boldsymbol{\eta}$ is an Eulerian vector field. (In general, $\cdot \partial_x$ identifies Eulerian vector fields in Euclidean space with left-invariant vector fields on the diffeomorphism group G.)

Now, the variation δg is intended to leave the fixed Eulerian reference frame unchanged, whereby $\delta \boldsymbol{x} \equiv 0$. Taking the corresponding variation of the inverse Lagrangian map (4.1) yields $\delta x_0^i = \delta(g^{-1}x)^i = -(g^{-1}(\delta g)g^{-1}x)^i = -\eta x_0^i$. Thus, because x_0^i is a function of x^j and t, we obtain

$$\delta x_0^i = -\boldsymbol{\eta} \cdot \nabla x_0^i = -\mathcal{L}_\eta x_0^i, \tag{4.7}$$

where $\mathcal{L}_\eta$ is the Lie derivative with respect to the Eulerian vector field $\boldsymbol{\eta}$.

More generally, the same Lie derivative operation is used to define the variation of any advected quantity (a scalar, a vector or a differential form), a:

$$\delta a = -\mathcal{L}_\eta a, \tag{4.8}$$

where a satisfies the advection equation (2.87). Expressions for the induced variations of ρ, S, $\boldsymbol{B}$ can then be deduced by considering advected quantities in terms of those variables (cf. §§ 2.6 and 3.1). The basic advected quantities in a CGL plasma are

$$S, \quad (1/\rho)\boldsymbol{B}, \quad \rho\, \mathrm{d}^3x, \quad \boldsymbol{B} \cdot \hat{\boldsymbol{n}}\, \mathrm{d}A, \tag{4.9a–d}$$

where $\mathrm{d}A$ is the area element on a surface moving with the fluid. (These quantities are also referred to as a Cauchy invariant.) As S is advected, its variation is given directly by (4.8). Advection of $\rho\, \mathrm{d}^3x$ combined with $\delta x^i = 0$ implies that $(\delta\rho)\, \mathrm{d}^3x = \delta(\rho\, \mathrm{d}^3x) = -\mathcal{L}_\eta(\rho\, \mathrm{d}^3x) = -(\mathcal{L}_\eta\rho + \rho\nabla \cdot \boldsymbol{\eta})\, \mathrm{d}^3x$ due to the well-known expansion/contraction formula $\mathcal{L}_v(\mathrm{d}^3x) = \nabla \cdot \boldsymbol{v}\, \mathrm{d}^3x$ holding for any vector field $\boldsymbol{v}$. Thus, we have

$$\delta S = -\boldsymbol{\eta} \cdot \nabla S, \tag{4.10}$$

$$\delta\rho = -\nabla \cdot (\rho\boldsymbol{\eta}). \tag{4.11}$$

Next, because $(1/\rho)\boldsymbol{B}$ is advected, this gives

$$\begin{aligned} \delta\left(\frac{\boldsymbol{B}}{\rho}\right) &= -\mathcal{L}_\eta\left(\frac{\boldsymbol{B}}{\rho}\right) = \frac{(\mathcal{L}_\eta\rho)\boldsymbol{B}}{\rho^2} - \frac{\mathcal{L}_\eta\boldsymbol{B}}{\rho} \\ &= -\frac{(\delta\rho)\boldsymbol{B}}{\rho^2} + \frac{\delta\boldsymbol{B}}{\rho}. \end{aligned} \tag{4.12}$$

Thus, after substituting $\delta\rho$ from (4.11), we obtain

$$\delta\boldsymbol{B} = -(\nabla \cdot \boldsymbol{\eta})\boldsymbol{B} - \mathcal{L}_\eta\boldsymbol{B} = \boldsymbol{B} \cdot \nabla\boldsymbol{\eta} - \nabla \cdot (\boldsymbol{\eta}\boldsymbol{B}) = \nabla \times (\boldsymbol{\eta} \times \boldsymbol{B}), \tag{4.13}$$

where $\mathcal{L}_\eta\boldsymbol{B} = [\boldsymbol{\eta}, \boldsymbol{B}]$ is the commutator of vector fields.

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Finally, the induced variation of $\boldsymbol{u}$ is given by combining the equations

$$(\delta\boldsymbol{u})\cdot\partial_x=\delta(g^{-1}\dot{g})=-g^{-1}(\delta g)g^{-1}\dot{g}+g^{-1}(\dot{\delta g})=-\boldsymbol{\eta}\cdot\partial_x\otimes\boldsymbol{u}\cdot\partial_x+g^{-1}(\dot{\delta g}) \quad (4.14)$$

and

$$(\dot{\delta g})=(\dot{g\eta})=\dot{g}\eta+g\dot{\eta}=g(\boldsymbol{u}\cdot\partial_x\otimes\boldsymbol{\eta}\cdot\partial_x+\dot{\boldsymbol{\eta}}\cdot\partial_x). \quad (4.15)$$

This yields

$$\delta\boldsymbol{u}\cdot\partial_x=\dot{\boldsymbol{\eta}}\cdot\partial_x-[\boldsymbol{\eta}\cdot\partial_x,\boldsymbol{u}\cdot\partial_x]=\left(\dot{\boldsymbol{\eta}}-\mathcal{L}_\eta\boldsymbol{u}\right)\cdot\partial_x, \quad (4.16)$$

where $[\cdot,\cdot]$ is the commutator of vector fields which coincides with the Lie bracket. Thus, we obtain the variation:

$$\delta\boldsymbol{u}=\dot{\boldsymbol{\eta}}-\mathcal{L}_\eta\boldsymbol{u}=(\partial/\partial t+\mathcal{L}_u)\boldsymbol{\eta}=\frac{\mathrm{d}}{\mathrm{d}t}\boldsymbol{\eta}\equiv\frac{\partial\boldsymbol{\eta}}{\partial t}+[\boldsymbol{u},\boldsymbol{\eta}]. \quad (4.17)$$

4.1. *The EP equation*

For CGL plasmas, the EP action principle is given by

$$\mathcal{J}=\int_{t_0}^{t_1}\int_V L\,\mathrm{d}^3x\,\mathrm{d}t, \quad (4.18)$$

where L is the Lagrangian density (3.21). Here V is a fixed spatial domain and $[t_0,t_1]$ is a fixed time interval. The stationary points of this action principle, $\delta\mathcal{J}=0$, under the variations (4.17), (4.10), (4.11) and (4.13) of the respective variables $\boldsymbol{u}$, S, ρ and $\boldsymbol{B}$, turn out to yield the Eulerian momentum equation (2.2), as we now show.

The derivation of the stationary points is non-trivial because of the form of the variations in terms of η given by (4.5) and (4.6). Specifically, the variation of $\mathcal{J}$ needs to be put into the form

$$\delta\mathcal{J}=\int_{t_0}^{t_1}\langle\eta,F\rangle\,\mathrm{d}t=\int_{t_0}^{t_1}\int_V\boldsymbol{\eta}\cdot\boldsymbol{F}\,\mathrm{d}^3x\,\mathrm{d}t \quad (4.19)$$

modulo boundary terms, where $F=\boldsymbol{F}\cdot\partial_x$ is both a left-invariant vector field in the tangent space of G and the directional derivative along a vector field $\boldsymbol{F}$ in Euclidean space. Then, because $\boldsymbol{\eta}$ is an arbitrary vector field (corresponding to an arbitrary variation δg), the equation yielding the stationary points of $\mathcal{J}$ is $\boldsymbol{F}=0$.

We now proceed to find $\delta\mathcal{J}$ and $\boldsymbol{F}$. A general expression is available for $\boldsymbol{F}$ in Holm *et al.* (1998), which involves the diamond operator, $\diamond$, defined by property

$$\left\langle\frac{\delta L}{\delta a}\diamond a,\eta\right\rangle=-\left\langle\frac{\delta L}{\delta a},\mathcal{L}_\eta(a)\right\rangle, \quad (4.20)$$

where, as earlier, a is an advected quantity. It will be instructive to instead show how to obtain $\boldsymbol{F}$ from the variation (4.17), (4.10), (4.11) and (4.13) directly using the Eulerian variables.

From the Lagrangian density (3.21), we obtain

$$\left.\begin{aligned}\frac{\delta L}{\delta\boldsymbol{u}}=\rho\boldsymbol{u},\quad\frac{\delta L}{\delta S}&=-\varepsilon_S=-\rho T,\quad\frac{\delta L}{\delta\rho}=\frac{1}{2}u^2-\varepsilon_\rho-\varPhi=\frac{1}{2}u^2-h-\varPhi,\\ \frac{\delta L}{\delta\boldsymbol{B}}&=-\varepsilon_B\boldsymbol{\tau}-\frac{\boldsymbol{B}}{\mu_0}=\left(\frac{p_\Delta}{B^2}-\frac{1}{\mu_0}\right)\boldsymbol{B}\equiv-\frac{1}{\mu_0}\tilde{\boldsymbol{B}},\end{aligned}\right\} \quad (4.21)$$

using the thermodynamic relations (2.18*a–c*), where T is the temperature and h is the enthalpy (2.62) and $p_\Delta=p_\parallel-p_\perp$ is the pressure anisotropy. Here we have also used the

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identity $\delta B = \boldsymbol{\tau}\cdot\delta\boldsymbol{B}$, with $\boldsymbol{\tau} = \boldsymbol{B}/B$ being the unit vector along $\boldsymbol{B}$. Note that $\tilde{\boldsymbol{B}}$ is related $\boldsymbol{B}$ by (2.78*a*,*b*). The main difference in these variational derivative expressions (4.21) compared with the MHD case is the addition of the anisotropy term in $\delta L/\delta\boldsymbol{B}$, i.e. $\boldsymbol{B}$ is replaced by $\tilde{\boldsymbol{B}}$.

The total variation of L is given by combining the products of the variational derivatives (4.21) and the corresponding variations (4.17), (4.10), (4.11) and (4.13). This yields

$$\begin{aligned}\delta L &= \frac{\delta L}{\delta \boldsymbol{u}}\cdot\delta\boldsymbol{u} + \frac{\delta L}{\delta S}\delta S + \frac{\delta L}{\delta\rho}\delta\rho + \frac{\delta L}{\delta\boldsymbol{B}}\cdot\delta\boldsymbol{B}\\ &= \rho\boldsymbol{u}\cdot\left(\dot{\boldsymbol{\eta}} - \mathcal{L}_{\eta}\boldsymbol{u}\right) - \rho T\left(-\boldsymbol{\eta}\cdot\nabla S\right) + \left(\frac{1}{2}u^2 - h - \varPhi\right)\left(-\nabla\cdot(\rho\boldsymbol{\eta})\right)\\ &\quad - \frac{1}{\mu_0}\tilde{\boldsymbol{B}}\cdot\left(\nabla\times(\boldsymbol{\eta}\times\boldsymbol{B})\right).\end{aligned} \tag{4.22}$$

The next step is to bring each term into the form (4.19), modulo a total time derivative and a total divergence, using integration by parts. The first term in expression (4.22) expands out to give

$$\begin{aligned}\rho\boldsymbol{u}\cdot\dot{\boldsymbol{\eta}} + \rho\boldsymbol{u}\boldsymbol{u}:\nabla\boldsymbol{\eta} - \rho(\boldsymbol{\eta}\cdot\nabla\boldsymbol{u})\cdot\boldsymbol{u} &= \boldsymbol{\eta}\cdot\left(-\frac{\partial(\rho\boldsymbol{u})}{\partial t} - \frac{1}{2}\rho\nabla(u^2) - \nabla\cdot(\rho\boldsymbol{u}\boldsymbol{u})\right)\\ &\quad + \frac{\partial(\rho\boldsymbol{u}\cdot\boldsymbol{\eta})}{\partial t} + \nabla\cdot\left(\rho(\boldsymbol{u}\cdot\boldsymbol{\eta})\boldsymbol{u}\right).\end{aligned} \tag{4.23}$$

Similarly, the third term in (4.22) yields

$$\boldsymbol{\eta}\cdot\left(\rho\nabla\left(\tfrac{1}{2}u^2 - h - \varPhi\right)\right) + \nabla\cdot\left(-\left(\tfrac{1}{2}u^2 - h - \varPhi\right)\rho\boldsymbol{\eta}\right). \tag{4.24}$$

The second term in (4.22) simply gives

$$\boldsymbol{\eta}\cdot(\rho T\nabla S). \tag{4.25}$$

The last term in (4.22) can be rearranged by cross-product identities:

$$(\boldsymbol{B}\times\boldsymbol{\eta})\cdot\left(\frac{1}{\mu_0}\nabla\times\tilde{\boldsymbol{B}}\right) + \nabla\cdot\left(\frac{1}{\mu_0}\tilde{\boldsymbol{B}}\times(\boldsymbol{\eta}\times\boldsymbol{B})\right), \tag{4.26}$$

and

$$(\boldsymbol{B}\times\boldsymbol{\eta})\cdot\left(\frac{1}{\mu_0}\nabla\times\tilde{\boldsymbol{B}}\right) = \boldsymbol{\eta}\cdot\left(-\frac{1}{\mu_0}\boldsymbol{B}\times\left(\nabla\times\tilde{\boldsymbol{B}}\right)\right). \tag{4.27}$$

Now, combining the four terms (4.27), (4.25), (4.24) and (4.23) and discarding the total derivatives, we obtain

$$\delta L = \boldsymbol{\eta}\cdot\left(-\frac{\partial(\rho\boldsymbol{u})}{\partial t} - \nabla\cdot(\rho\boldsymbol{u}\boldsymbol{u}) + \rho(T\nabla S - \nabla h - \nabla\varPhi) - \frac{1}{\mu_0}\boldsymbol{B}\times\left(\nabla\times\tilde{\boldsymbol{B}}\right)\right) \tag{4.28}$$

modulo total time derivatives and total divergences. This yields the desired relation

$$\int_V \delta L\,\mathrm{d}^3x = \int_V \boldsymbol{\eta}\cdot\boldsymbol{F}\,\mathrm{d}^3x = \langle\eta, F\rangle, \tag{4.29}$$

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with

$$\boldsymbol{F} = -\frac{\partial(\rho \boldsymbol{u})}{\partial t} - \nabla \cdot (\rho \boldsymbol{u}\boldsymbol{u}) + \rho(T\nabla S - \nabla h - \nabla \Phi) - \frac{1}{\mu_0}\boldsymbol{B} \times \left(\nabla \times \tilde{\boldsymbol{B}}\right). \tag{4.30}$$

The equation $\boldsymbol{F} = 0$ resulting from the variational principle (4.19) is called the EP equation. It is equivalent to the Eulerian momentum equation (2.2), after the pressure divergence identity (2.61) is used. A direct derivation of this identity is provided in Appendix B. Alternatively, (4.30) can be viewed as showing how the pressure divergence identity arises from the variational principle (4.19). Appendix E derives the identity (2.61) and the EP equation $\boldsymbol{F} = 0$ using the approach of Holm *et al.* (1998).

Holm & Kupershmidt (1986) studied a corresponding Hamiltonian form of the CGL plasma equations for both relativistic and non-relativistic flows; however, details of the EP formulation were not covered.

5. Non-canonical Poisson bracket

Morrison & Greene (1980, 1982) and Holm & Kupershmidt (1983*a*,*b*) obtained the non-canonical Poisson bracket for ideal MHD, which involves the basic variables ρ, $\sigma \equiv \rho S$, $\boldsymbol{M} \equiv \rho \boldsymbol{u}$ and $\boldsymbol{B}$:

$$\begin{aligned}\{\mathcal{F}, \mathcal{G}\}^{\text{MHD}} = -\int_V \Big\{&\rho\left(\mathcal{F}_{\boldsymbol{M}} \cdot \nabla \mathcal{G}_\rho - \mathcal{G}_{\boldsymbol{M}} \cdot \nabla \mathcal{F}_\rho\right) + \sigma\left(\mathcal{F}_{\boldsymbol{M}} \cdot \nabla \mathcal{G}_\sigma - \mathcal{G}_{\boldsymbol{M}} \cdot \nabla \mathcal{F}_\sigma\right) \\ &+ \boldsymbol{M} \cdot (\mathcal{F}_{\boldsymbol{M}} \cdot \nabla \mathcal{G}_{\boldsymbol{M}} - \mathcal{G}_{\boldsymbol{M}} \cdot \nabla \mathcal{F}_{\boldsymbol{M}}) + \boldsymbol{B} \cdot (\mathcal{F}_{\boldsymbol{M}} \cdot \nabla \mathcal{G}_{\boldsymbol{B}} - \mathcal{G}_{\boldsymbol{M}} \cdot \nabla \mathcal{F}_{\boldsymbol{B}}) \\ &+ \boldsymbol{B} \cdot ((\nabla \mathcal{F}_{\boldsymbol{M}}) \cdot \mathcal{G}_{\boldsymbol{B}} - (\nabla \mathcal{G}_{\boldsymbol{M}}) \cdot \mathcal{F}_{\boldsymbol{B}})\Big\}\, \mathrm{d}^3 x,\end{aligned} \tag{5.1}$$

where $\mathcal{F}$ and $\mathcal{G}$ are arbitrary functionals, and subscripts denote a variational derivative. This bracket is bilinear, antisymmetric and obeys the Jacobi identity.

It turns out the CGL plasma Poisson bracket obtained by Holm & Kupershmidt (1986), $\{\mathcal{F}, \mathcal{G}\}^{\text{CGL}}$ has the same form as the MHD plasma Poisson bracket except that the thermodynamics and internal energy $e(\rho, S, B)$ are completely different in the two cases. Holm & Kupershmidt (1986) obtained the bracket:

$$\begin{aligned}\{\mathcal{F}, \mathcal{G}\}^{\text{CGL}} = -\int_V \Big\{&\rho\left(\mathcal{F}_{\boldsymbol{M}} \cdot \nabla \mathcal{G}_\rho - \mathcal{G}_{\boldsymbol{M}} \cdot \nabla \mathcal{F}_\rho\right) + \sigma\left(\mathcal{F}_{\boldsymbol{M}} \cdot \nabla \mathcal{G}_\sigma - \mathcal{G}_{\boldsymbol{M}} \cdot \nabla \mathcal{F}_\sigma\right) \\ &+ \boldsymbol{M} \cdot (\mathcal{F}_{\boldsymbol{M}} \cdot \nabla \mathcal{G}_{\boldsymbol{M}} - \mathcal{G}_{\boldsymbol{M}} \cdot \nabla \mathcal{F}_{\boldsymbol{M}}) + \boldsymbol{B} \cdot (\mathcal{F}_{\boldsymbol{M}} \cdot \nabla \mathcal{G}_{\boldsymbol{B}} - \mathcal{G}_{\boldsymbol{M}} \cdot \nabla \mathcal{F}_{\boldsymbol{B}}) \\ &+ \boldsymbol{B} \cdot ((\nabla \mathcal{F}_{\boldsymbol{M}}) \cdot \mathcal{G}_{\boldsymbol{B}} - (\nabla \mathcal{G}_{\boldsymbol{M}}) \cdot \mathcal{F}_{\boldsymbol{B}})\Big\}\, \mathrm{d}^3 x.\end{aligned} \tag{5.2}$$

An overview of Hamiltonian systems is given in Morrison (1998). Banerjee & Kumar (2016) provides a Dirac bracket approach to the MHD Poisson bracket. The property that the MHD bracket is linear in the dynamical variables ρ, σ, $\boldsymbol{M}$ and $\boldsymbol{B}$ has an important mathematical relationship to semi-direct product Lie algebras, which is explained in Holm *et al.* (1983, 1998) for general fluid systems.

The variables appearing in the CGL non-canonical Poisson bracket consist of

$$\rho, \quad \sigma, \quad \boldsymbol{M}, \quad \boldsymbol{B}. \tag{5.3a–d}$$

The analysis in Appendix D starts from the canonical bracket

$$\{\mathcal{F}, \mathcal{G}\} = \int_V \left(\mathcal{F}_{\boldsymbol{q}} \cdot \mathcal{G}_{\boldsymbol{p}} - \mathcal{F}_{\boldsymbol{p}} \cdot \mathcal{G}_{\boldsymbol{q}}\right) \mathrm{d}^3 x_0 \tag{5.4}$$

involving canonical Hamiltonian variables $(\boldsymbol{q}, \boldsymbol{p}) \equiv (\boldsymbol{x}(\boldsymbol{x}_0, t), \boldsymbol{\pi}(\boldsymbol{x}_0, t))$ where $\boldsymbol{\pi} = \rho_0 \dot{\boldsymbol{x}}$ is the canonical momentum (cf. § 3.3). The main steps consist of using the Lagrangian

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map (3.2) to obtain a transformation to the non-canonical variables (5.3*a–d*), followed by applying a variational version of the chain rule to the variational derivatives with respect to $(\boldsymbol{q}, \boldsymbol{p})$ (see e.g. Zakharov & Kuznetsov 1997). These steps are carried out by working in a fixed Lagrangian frame, while the corresponding Eulerian frame given by the Lagrangian map undergoes a variation, which includes varying Cartesian basis vectors associated to the components of $\boldsymbol{x}$. This general approach is described in a short communication by Holm *et al.* (1983).

In the following, we give an alternative more succinct derivation of the CGL Poisson bracket (5.2). It leads to the same Poisson bracket as that obtained in Appendix D. Note that the canonical bracket (5.4) can be written in the form

$$\{\mathcal{F}, \mathcal{G}\} = \int \left(\mathcal{F}_x \cdot \mathcal{G}_p - \mathcal{F}_p \cdot \mathcal{G}_x\right) \frac{\mathrm{d}^3\boldsymbol{x}}{J}. \tag{5.5}$$

Here $\boldsymbol{q} \equiv \boldsymbol{x}$. The transformation of variational derivatives between the canonical variables and the new variables (5.3*a–d*) is effected by noting that

$$\begin{aligned} \delta\mathcal{F} &= \int \frac{1}{J}\left(\mathcal{F}_x \cdot \Delta\boldsymbol{x} + \mathcal{F}_p \cdot \Delta\boldsymbol{p}\right) \mathrm{d}^3\boldsymbol{x} \\ &= \int \left(\hat{\mathcal{F}}_\rho \delta\rho + \hat{\mathcal{F}}_M \cdot \delta\boldsymbol{M} + \hat{\mathcal{F}}_\sigma \delta\sigma + \hat{\mathcal{F}}_B \cdot \delta\boldsymbol{B}\right) \mathrm{d}^3\boldsymbol{x}, \end{aligned} \tag{5.6}$$

where $\hat{\mathcal{F}}(\rho, \boldsymbol{M}, \sigma, \boldsymbol{B}) \equiv \mathcal{F}(\boldsymbol{q}, \boldsymbol{p})$ is the functional $\mathcal{F}(\boldsymbol{q}, \boldsymbol{p})$ expressed in terms of the new variables $(\rho, \boldsymbol{M}^T, \sigma, \boldsymbol{B}^T)^{\mathrm{T}}$. Here $\delta\psi$ denotes the Eulerian variation of ψ and $\Delta\psi$ denotes the Lagrangian variation of ψ.

Using the transformations

$$\left.\begin{array}{c} \Delta J = J\nabla \cdot \Delta\boldsymbol{x}, \quad \Delta\boldsymbol{M} = \dfrac{\Delta\boldsymbol{p}}{J} - \boldsymbol{M}(\nabla \cdot \Delta\boldsymbol{x}), \\ \delta\boldsymbol{M} = \dfrac{\Delta\boldsymbol{p}}{J} - \nabla_j\left(\Delta x^j \boldsymbol{M}\right), \quad \delta\rho = -\nabla \cdot (\rho\Delta\boldsymbol{x}), \quad \delta\sigma = -\nabla \cdot (\sigma\Delta\boldsymbol{x}), \\ \delta\boldsymbol{B} = \nabla \times (\Delta\boldsymbol{x} \times \boldsymbol{B}) - \Delta\boldsymbol{x}(\nabla \cdot \boldsymbol{B}), x \end{array}\right\} \tag{5.7}$$

in (5.6) gives the formulae

$$\left.\begin{array}{c} \mathcal{F}_x = J\left[(\nabla\hat{\mathcal{F}}_M) \cdot \boldsymbol{M} + \sigma\nabla\hat{\mathcal{F}}_\sigma + \rho\nabla\hat{\mathcal{F}}_\rho + (\nabla\hat{\mathcal{F}}_B) \cdot \boldsymbol{B} - (\boldsymbol{B} \cdot \nabla)\hat{\mathcal{F}}_B - (\nabla \cdot \boldsymbol{B})\hat{\mathcal{F}}_B\right], \\ \mathcal{F}_p = \hat{\mathcal{F}}_M, \end{array}\right\} \tag{5.8}$$

for the transformation of variational derivatives from the old variables $(\boldsymbol{x}, \boldsymbol{p})$ to the new variables $\boldsymbol{M}$, σ, ρ and $\boldsymbol{B}$. Using the expressions (5.8) for $\mathcal{F}_x$ and $\mathcal{F}_p$ in (5.5) and dropping the hat accents, the Poisson bracket (5.5) reduces to

$$\{\mathcal{F}, \mathcal{G}\} = \{\mathcal{F}, \mathcal{G}\}^{\mathrm{CGL}} - \int_V \nabla \cdot [\boldsymbol{B}\,(\mathcal{G}_M \cdot \mathcal{F}_B - \mathcal{G}_B \cdot \mathcal{F}_M)]\,\mathrm{d}^3\boldsymbol{x}. \tag{5.9}$$

Dropping the last pure divergence term in (5.9) (which converts to a surface integral over the boundary by Gauss's theorem), (5.9) reduces to the CGL Poisson bracket (5.2), which applies in the general case where $\nabla \cdot \boldsymbol{B} \neq 0$ (i.e. the Jacobi identity applies for the bracket (5.2)), which is the analogue of the Morrison & Greene (1982) bracket in MHD.

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The CGL bracket (5.2) shares the same main features as the MHD bracket: it is bilinear, antisymmetric and obeys the Jacobi identity. A verification of the Jacobi identity can be given using functional multi-vectors (Olver 1993) in the same way as for the MHD bracket (e.g. Webb 2018, Chapter 8). An alternative verification of the Jacobi identity for the MHD bracket is given in Morrison (1982); see also Chandre (2013), Chandre *et al.* (2013), as well as Holm & Kupershmidt (1983*a*,*b*) who used the magnetic vector potential $\boldsymbol{A}$ in the advected gauge (2.90).

In general, any Poisson bracket can be expressed in a cosymplectic form which defines a corresponding Hamiltonian (cosymplectic) operator. Using non-canonical Eulerian variables Z, the cosymplectic form is given by

$$\{\mathcal{F}, \mathcal{G}\} = \int_V \mathcal{F}_{Z^T} \boldsymbol{\mathcal{D}} \mathcal{G}_Z \, \mathrm{d}^3x, \tag{5.10}$$

in which $\boldsymbol{\mathcal{D}}$ is the Hamiltonian (matrix) operator, where T denotes the transpose. Note that, in the case $Z = (\boldsymbol{q}, \boldsymbol{p})$, this operator reduces to the skew matrix $\boldsymbol{\mathcal{D}} = \left(\begin{smallmatrix} 0 & I \\ -I & 0 \end{smallmatrix}\right)$. Antisymmetry of the bracket (5.10), $\{\mathcal{F}, \mathcal{G}\} + \{\mathcal{G}, \mathcal{F}\} = 0$, corresponds to $\boldsymbol{\mathcal{D}}$ being skew-adjoint; the Jacobi identity, $\{\{\mathcal{F}, \mathcal{G}\}, \mathcal{H}\} + \{\{\mathcal{H}, \mathcal{F}\}, \mathcal{G}\} + \{\{\mathcal{G}, \mathcal{H}\}, \mathcal{F}\} = 0$, corresponds to $\boldsymbol{\mathcal{D}}$ having a vanishing Schouten bracket with itself (Olver 1993).

Taking

$$Z = \left(\rho, \sigma, M^i, B^i\right), \tag{5.11}$$

with M^i and B^i denoting the components of $\boldsymbol{M}$ and $\boldsymbol{B}$, respectively, in Cartesian coordinates x^i, we see that the cosymplectic form (5.10) of the CGL bracket (5.2) after integration by parts is given by the Hamiltonian operator:

$$\boldsymbol{\mathcal{D}} = -\begin{pmatrix} 0 & 0 & \nabla^j \circ \rho & 0 \\ 0 & 0 & \nabla^j \circ \sigma & 0 \\ \rho\nabla^i & \sigma\nabla^i & M^j\nabla^i + \nabla^j \circ M^i & B^j\nabla^i - \delta^{ij}B^k\nabla_k \\ 0 & 0 & \nabla^j \circ B^i - \delta^{ij}\nabla_k \circ B^k & 0 \end{pmatrix}, \tag{5.12}$$

where $\circ$ denotes operator composition (i.e. $\nabla \circ a = (\nabla a) + a\nabla$). Here ∇^k acts as the total derivative with respect to x^k. Note that the minus sign in (5.12) is due to the overall minus sign in the CGL Poisson bracket (5.2) which follows the sign convention used in the MHD bracket (5.1) in Morrison & Greene (1980, 1982).

We can convert (5.12) into vector notation by identifying $\nabla^i F = \mathrm{grad}\, F = \nabla F$, $\nabla^j F^j = \mathrm{div}\, \boldsymbol{F} = \nabla \boldsymbol{\cdot} \boldsymbol{F}$ and $\nabla^j F^k - \nabla^k F^j = \mathrm{curl}\, \boldsymbol{F} = \nabla \times \boldsymbol{F}$.

5.1. *Non-canonical Hamiltonian equations*

A non-canonical Poisson bracket (5.10) provides a Hamiltonian formulation once a Hamiltonian functional $\mathcal{H}$ is chosen. The formulation involves expressing the dynamical variables $Z(\boldsymbol{x}, t)$ formally as functionals

$$\mathcal{Z}(\boldsymbol{x}', t) \equiv \int_V Z(\boldsymbol{x}', t)\delta(\boldsymbol{x} - \boldsymbol{x}') \, \mathrm{d}^3x \tag{5.13}$$

and writing Hamilton's equations in the form

$$\frac{\partial \mathcal{Z}}{\partial t} = \{\mathcal{Z}, \mathcal{H}\}. \tag{5.14}$$

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Using the explicit expression (5.10) for the bracket yields

$$\frac{\partial Z}{\partial t} = \mathcal{D}\frac{\delta \mathcal{H}}{\delta Z} \tag{5.15}$$

in terms of the Hamiltonian operator $\mathcal{D}$.

The appropriate Hamiltonian for describing CGL plasmas is given by the conserved total energy (cf. § 2.3):

$$\mathcal{H} = \int_V H\, \mathrm{d}^3 x, \tag{5.16}$$

where H is the Eulerian total energy density (3.20). Substituting this Hamiltonian into the non-canonical Hamilton's equations (5.15) can be shown to yield the CGL plasma equations (2.1)–(2.6) and (2.8)–(2.9).

5.2. *Casimirs*

In a non-canonical Hamiltonian system, a functional $\mathcal{C} = \int_V C\, \mathrm{d}^3 x$ satisfying the equation

$$\{\mathcal{C}, \mathcal{F}\} \equiv 0 \quad \text{for all functionals } \mathcal{F} \tag{5.17}$$

is called a Casimir. Existence of a non-trivial Casimir $\mathcal{C} \not\equiv 0$ indicates that the Poisson bracket is degenerate. Correspondingly, the Hamiltonian operator $\mathcal{D}$ in the cosymplectic form of the bracket (5.10) will have a non-trivial kernel:

$$\mathcal{D}\mathcal{C}_Z \equiv 0, \tag{5.18}$$

where $\mathcal{C}_Z \equiv \delta\mathcal{C}/\delta Z$. Note that a Casimir is a conserved integral, because the time evolution of any functional is given by the Hamiltonian equations (5.14):

$$\frac{\mathrm{d}}{\mathrm{d}t}\mathcal{C} = \frac{\partial \mathcal{C}}{\partial t} = \{\mathcal{C}, \mathcal{H}\} = 0. \tag{5.19}$$

Casimirs are useful in stability analysis of steady flows and plasma equilibria (e.g. Holm *et al.* 1985; Hameiri 2004). The conservation (5.19) holds modulo boundary integrals, and an investigation of boundary conditions is needed for a Casimir to be a strictly conserved integral (i.e. a constant of motion).

All Casimirs can be determined by solving (5.17), or alternatively (5.18) (see, e.g., Hameiri (2004) and Padhye & Morrison (1996*a*,*b*) for the MHD case). For CGL plasmas, we obtain the Casimir determining equations:

$$\left.\begin{aligned}
\nabla^j(\rho\mathcal{C}_{M^j}) = \nabla\cdot(\rho\mathcal{C}_{\boldsymbol{M}}) = 0, \quad \nabla^j(\sigma\mathcal{C}_{M^j}) = \nabla\cdot(\sigma\mathcal{C}_{\boldsymbol{M}}) = 0,\\
\rho\nabla^i\mathcal{C}_\rho + \sigma\nabla^i\mathcal{C}_\sigma + M^j\nabla^i\mathcal{C}_{M^j} + \nabla^j(M^i\mathcal{C}_{M^j}) + B^j\nabla^i\mathcal{C}_{B^j} - B^j\nabla_j\mathcal{C}_{B^i}\\
\equiv \rho\nabla\mathcal{C}_\rho + \sigma\nabla\mathcal{C}_\sigma + (\nabla\mathcal{C}_{\boldsymbol{M}})\cdot\boldsymbol{M} + \nabla\cdot(\mathcal{C}_{\boldsymbol{M}}\boldsymbol{M}) + (\nabla\mathcal{C}_{\boldsymbol{B}})\cdot\boldsymbol{B} - \boldsymbol{B}\cdot\nabla\mathcal{C}_{\boldsymbol{B}} = 0,\\
\nabla^j(B^i\mathcal{C}_{M^j}) - \nabla_j(B^j\mathcal{C}_{M^i}) = \nabla\times(\boldsymbol{B}\times\mathcal{C}_{\boldsymbol{M}}) = 0,
\end{aligned}\right\} \tag{5.20}$$

where ∇ (∇^k) acts as the total derivative with respect to $\boldsymbol{x}$ (x^k). This is an overdetermined system of linear partial differential equations for

$$\mathcal{C} = \int_V C(t, \boldsymbol{x}, Z, \nabla Z, \ldots, \nabla^l Z)\, \mathrm{d}^3 x, \tag{5.21}$$

with Z denoting the dynamical variables (5.11). The system can, in principle, be integrated to find $\mathcal{C}$ explicitly, once a differential order l for the dependence of $\mathcal{C}$ on derivatives of the

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variables is chosen. Note that solutions of the divergence form $C = \nabla\cdot \boldsymbol{F}$ lead to $\mathcal{C}$ being a boundary integral which can be assumed to be trivial if suitable boundary conditions are imposed. This classification problem is beyond the scope of the present work.

5.3. *Mass, cross-helicity and magnetic helicity Casimirs*

The well-known Casimirs for ideal barotropic MHD are the mass integral, the cross-helicity integral and the magnetic helicity integral. CGL plasmas with an isentropic equation of state $e = e(\rho, B)$ possess these same Casimirs:

$$\mathcal{C}_1 = \int_V \rho \, \mathrm{d}^3x, \quad \mathcal{C}_2 = \int_V \boldsymbol{u}\cdot\boldsymbol{B}\, \mathrm{d}^3x, \quad \mathcal{C}_3 = \int_V \boldsymbol{A}\cdot\boldsymbol{B}\, \mathrm{d}^3x. \tag{5.22a–c}$$

In the more physically realistic case with an equation of state $e = e(\rho, S, B)$, $\mathcal{C}_1$ and $\mathcal{C}_3$ still hold as Casimirs, but $\mathcal{C}_2$ turns out to be a Casimir only in the case $\boldsymbol{B}\cdot\nabla S = 0$, as we show shortly.

Physically, the mass integral $\mathcal{C}_1$ is the total mass of the plasma; the cross-helicity integral $\mathcal{C}_2$ describes the linking of the fluid vorticity and magnetic field flux tubes; and the magnetic helicity integral $\mathcal{C}_3$ describes the knotting, linking and twist and writhe of the magnetic flux tubes (see,e.g., Berger & Field 1984; Moffatt & Ricca 1992; Hameiri 2004). (Also see, e.g., Yoshida (2016), for interesting applications). In a fixed volume V with a boundary ∂V, $\mathcal{C}_1$ is conserved, $\partial\mathcal{C}_1/\partial t = 0$, if $\boldsymbol{u}$ has no normal component at the boundary (cf. § 2.3). Conservation of $\mathcal{C}_3$, $\partial\mathcal{C}_3/\partial t = 0$, holds if, in addition, $\boldsymbol{B}$ has no normal component at the boundary, whereas the cross-helicity integral $\mathcal{C}_2$ is conserved, $\partial\mathcal{C}_2/\partial t = 0$, only with the further condition $\boldsymbol{B}\cdot\nabla S = 0$ (cf. §§ 2.4 and 2.5).

5.4. *Advected Casimirs*

In general, Casimirs can be sought by looking among advected scalars, θ, because the corresponding scalar integral $\int_{V(t)} \rho\theta \, \mathrm{d}^3x$ will be conserved on volumes $V(t)$ moving with the flow. This implies that, on a fixed volume V, the integral $\int_V \rho\theta\, \mathrm{d}^3x$ will be conserved up to a flux integral $-\oint_{\partial V} \rho\theta\boldsymbol{u}\cdot\hat{\boldsymbol{n}}\, \mathrm{d}A$ that vanishes if $\boldsymbol{u}$ has no normal component at the boundary ∂V and, therefore, $\int_V \rho\theta\, \mathrm{d}^3x$ will satisfy the Casimir property $\partial\mathcal{C}/\partial t = 0$. Note that this property is necessary but not sufficient for $\int_V \rho\theta\, \mathrm{d}^3x$ to be a Casimir, because there are conserved integrals such as the total energy and angular momentum that do not belong to the kernel of the Poisson bracket.

For CGL plasmas, the basic advected scalars are S, $\bar{S}_\parallel$, $\bar{S}_\perp$. Additional advected scalars are provided by Ertel's theorem: if θ is an advected scalar, then so is $\boldsymbol{B}\cdot\nabla\theta/\rho$. This yields the advected quantities (2.89*a–c*) shown in § 2.6.

One can show that

$$\mathcal{C}_4 = \int_V \rho f(S, \theta)\, \mathrm{d}^3\boldsymbol{x} \quad \text{where} \quad \theta = \frac{\boldsymbol{B}\cdot\nabla S}{\rho}, \tag{5.23}$$

is an advected Casimir, for any function $f(S, \theta)$. We omit the proof.

6. Noether's theorem and conservation laws

Conservation laws of the CGL plasma equations (2.1) to (2.10*a–c*) can be derived from Noether's theorem applied to the Lie point symmetries of the Lagrangian variational principle (3.19). A brief discussion of the MHD case was outlined in Webb & Anco (2019). The CGL plasma conservation laws differ in comparison with the MHD conservation laws mainly in the form of pressure tensor: in particular, the MHD isotropic gas pressure tensor $\mathsf{p} = p\,\mathsf{I}$ is replaced by the non-isotropic CGL plasma pressure tensor $\mathsf{p} = p_\perp \mathsf{I} + (p_\parallel - p_\perp)\boldsymbol{\tau}\boldsymbol{\tau}$.

Appendix H gives a description of Noether's first theorem, for a differential equation system described by an action principle as developed by Bluman & Kumei (1989). The analysis uses canonical Lie symmetry operators X in which both the dependent and independent variables change in the Lie transformation, and also the evolutionary form of the symmetry operator denoted by $\hat{X}$ in which the independent variables do not change, but the dependent variables and the derivatives of the dependent variables change. The prolongation operators $\mathrm{pr}\,X$ and $\mathrm{pr}\,\hat{X}$ are described, and used to derive Noether's first theorem similar to Bluman & Kumei (1989). More recent derivations of Noether's theorem using $\mathrm{pr}\,\hat{X}$, are given by Bluman & Anco (2002) and Olver (1993).

In § 6.1 we use a recent form of Noether's theorem to derive conservation laws using the evolutionary form of the prolonged symmetry operator $\mathrm{pr}\,\hat{X}$. In § 6.2 we obtain the same results from the classical form of Noether's theorem given in Appendix H. Although the recent form of Noether's theorem, is conceptually more appealing, it is not any simpler than the classical form of Noether's theorem. The classical form of Noether's theorem is perhaps easier to understand, as it relates directly back to the invariance of the action integral under Lie and divergence transformations.

6.1. *Noether's theorem and evolutionary symmetries*

To use Noether's theorem, we need to obtain the Lie point symmetries of the Lagrangian variational principle (3.19). As this variational principle employs the variables $x^i = X^i(x_0^j, t)$ given by the Cartesian components of the Lagrangian map (3.2), a Lie point symmetry acting on the coordinate space (t, x_0^j, x^i) has the form

$$t \rightarrow t + \epsilon\xi^t + O(\epsilon^2), \quad x^i \rightarrow x^i + \epsilon\xi^i + O(\epsilon^2), \quad x_0^i \rightarrow x_0^i + \epsilon\xi_0^i + O(\epsilon^2) \qquad (6.1a\text{–}c)$$

with ϵ denoting the parameter in the point symmetry transformation, where ξ^i, ξ_0^i, ξ^t are functions of t, x_0^j, x^i. The infinitesimal transformation corresponds to the generator

$$X = \xi^i \frac{\partial}{\partial x^i} + \xi_0^i \frac{\partial}{\partial x_0^i} + \xi^t \frac{\partial}{\partial t} \qquad (6.2)$$

whereas the finite transformation (6.1*a–c*) is given by exponentiation of the generator, $(t, x^i, x_0^i) \rightarrow \exp(\epsilon X)(t, x^i, x_0^i)$.

To be a symmetry, a generator (6.2) must leave the variational principle invariant modulo boundary terms. This is equivalent to the condition that the change in the Lagrangian must satisfy (Ovsjannikov 1978; Ibragimov 1985)

$$\mathrm{pr}\,XL_0 = \xi^t D_t L_0 + \xi_0^i D_{x_0^i} L_0 + D_t \Lambda_0^t + D_{x_0^i} \Lambda_0^i \qquad (6.3)$$

where the operators $D_t = \partial/\partial_t + \dot{x}^i \partial/\partial_{x^i} + \cdots$ and $D_{x_0^i} = \partial/\partial_{x_0^i} + x^{ji}\partial/\partial_{x^j} + \cdots$ denote total derivative operators with respect to the independent variables t and x_0^i and D_{x^i} denotes the partial derivative with respect to x^i keeping $\boldsymbol{x}_0$ and t constant. Here Λ_0^t and Λ_0^i are arbitrary potentials, that arise in Noether's theorem, because the variational derivative of a perfect derivative term has zero variational derivative. The first two terms on the right-hand side of (6.3) represent the Lie derivative $\mathcal{L}_X L_0$ using the chain rule for differentiation. The terms involving total derivatives of L_0 on the right-hand side of (6.3) can be understood to arise from the change in the spatial domain V_0 and the time interval $[t_0, t_1]$ in the variational principle (3.19) under the action of an infinitesimal

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point transformation (6.1*a–c*). On the left-hand side of (6.3), pr denotes prolongation to the extended coordinate space $(t, x_0^j, x^i, \dot{x}^i, x^{ij})$ (i.e. jet space) in which $\dot{x}^i$ and x^{ij} are coordinates that correspond to $\dot{X}^i(x_0^j, t)$ and $\partial X^i(x_0^j, t)/\partial x_0^j$ on solutions $x^i = X^i(x_0^j, t)$ of the Euler–Lagrange equations (3.24) of the variational principle. An explicit formula for the components of $\mathrm{pr}\,X$ will not be needed if we express the invariance condition (6.3) by using the characteristic form of the generator in which only the dependent variables x^i undergo a transformation

$$\hat{X} = \hat{\xi}^i \frac{\partial}{\partial x^i}, \quad \hat{\xi}^i = \xi^i - \xi^t \dot{x}^i - \xi_0^j x^{ij}, \tag{6.4a,b}$$

which arises from how X acts on solutions $x^i = X^i(x_0^j, t)$. The relationship between the two forms (6.4*a*,*b*) and (6.2) is that $\mathrm{pr}\,X = \mathrm{pr}\,\hat{X} + \xi^t D_t + \xi_0^i D_{x_0^i}$. Hence, (6.3) becomes (Olver 1993; Bluman & Anco 2002)

$$\mathrm{pr}\,\hat{X}(\hat{L}_0) = D_t \Lambda_0^t + D_{x_0^i} \Lambda_0^i, \tag{6.5}$$

where

$$\hat{L}_0 = L_0|_{X=x} \tag{6.6}$$

is a function of $t, x_0^j, x^i, \dot{x}^i, x^{ij}$. The prolongation $\mathrm{pr}\,\hat{X}$ is given by simply extending $\hat{X}$ to act on $\dot{x}^i$ and x^{ij} through the total derivative relations $\mathrm{pr}\,\hat{X}(\dot{x}^i) = D_t(\hat{X}x^i) = D_t\xi^t$ and $\mathrm{pr}\,\hat{X}(x^{ij}) = D_{x_0^j}(\hat{X}x^i) = D_{x_0^j}\xi^i$.

In turn, this form (6.5) of the invariance condition can be expressed succinctly as

$$E_{x^i}(\mathrm{pr}\,\hat{X}(\hat{L}_0)) = 0 \tag{6.7}$$

using the Euler–Lagrange operator E_{x^i} which has the property that it annihilates a function iff the function is given by a total divergence with respect to t and x_0^j. The formulation (6.7) can be used as a determining equation to find all Lie point symmetries of the variational principle (3.19). (See Olver (1993) and Bluman & Anco (2002) for a general discussion.)

Each Lie point symmetry (6.1*a–c*) gives rise to a conservation law through combining the condition (6.5) and Noether's identity

$$\mathrm{pr}\,\hat{X}(\hat{L}_0) = \hat{\xi}^i E_{x^i}(L_0) + D_t W^t + D_{x_0^i} W^i, \tag{6.8}$$

where

$$W^t = \hat{\xi}^j \frac{\partial \hat{L}_0}{\partial \dot{x}^j}, \quad W^i = \hat{\xi}^j \frac{\partial \hat{L}_0}{\partial x^{ji}}. \tag{6.9a,b}$$

Thus, we obtain the following statement of Noether's theorem.

PROPOSITION 6.1. *If the Lagrangian variational principle (3.19) is invariant up to boundary terms under an infinitesimal point transformation (6.1a–c), then the Euler–Lagrange equations (3.24) of the variational principle possess a conservation law*

$$D_t \mathcal{I}_0^t + D_{x_0^i} \mathcal{I}_0^i = 0 \tag{6.10}$$

in which the conserved density and the spatial flux are given by

$$\mathcal{I}_0^t = \left(\Lambda_0^t - W^t\right)\Big|_{x=X}, \quad \mathcal{I}_0^i = \left(\Lambda_0^i - W^i\right)\Big|_{x=X}, \tag{6.11a,b}$$

using (6.5) and (6.9a,b).

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Explicit expressions for $\partial \hat{L}_0/\partial \dot{x}^j = (\partial L_0/\partial \dot{X}^j)|_{X=x}$ and $\partial \hat{L}_0/\partial x^{ji} = (\partial L_0/\partial X^{ji})|_{X=x}$ are provided by (C1) in Appendix C. Note that the conservation law is a local continuity equation which holds when $x^i = X^i(x_0^j, t)$ satisfies the Euler–Lagrange equations (3.24). On solutions, $D_t|_{x=X} = \partial/\partial t$ and $D_{x_0^i}|_{x=X} = \partial/\partial x_0^i$ (acting as total derivatives).

We remark that (6.11*a,b*) can be derived alternatively using the canonical form of a symmetry generator (6.2), which requires computing the prolongation. See Webb & Zank (2007) and Webb & Anco (2019) for the MHD case. The canonical symmetry approach to conservation laws for the CGL system is also described in Appendix H.

6.1.1. *Eulerian form of a Lagrangian conservation law*

A Lagrangian conservation law (6.11*a,b*) can be expressed equivalently as an Eulerian conservation law (Padhye 1998)

$$\partial_t \Psi^t + \nabla_i \Psi^i = 0 \tag{6.12}$$

whose conserved density and spatial flux are given by

$$\Psi^t = \frac{1}{J}\,\mathcal{I}_0^t\Big|_{x=X}, \quad \Psi^i = \frac{1}{J}\,\left(u^i\mathcal{I}_0^t + X_{ik}\mathcal{I}_0^k\right)\Big|_{x=X} \tag{6.13a,b}$$

on solutions $x^i = X^i(x_0^j, t)$ of the Euler–Lagrange equations (3.24). This form (6.12)–(6.13*a,b*) of a conservation law is appropriate for considering conserved integrals

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_V \Psi^t\,\mathrm{d}^3x = -\oint_{\partial V} \boldsymbol{\Psi}\cdot\hat{\boldsymbol{n}}\,\mathrm{d}A \tag{6.14}$$

on a fixed spatial domain V in the CGL plasma.

In general, an Eulerian conservation law is a continuity equation of the form (6.12) holding on the solution space of the CGL plasma equations (2.1)–(2.10*a–c*). The corresponding conserved integral (6.14) has the physical content that the rate of change of the integral quantity $\int_V \Phi^t\,\mathrm{d}^3x$ in V is balanced by the net flux leaving the boundary of V. It is often physically useful to consider instead a spatial domain $V(t)$ that moves with the plasma. The form of the conservation law for moving domains is given by

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_{V(t)} \Psi^t\,\mathrm{d}^3x = -\oint_{\partial V(t)} \boldsymbol{\Gamma}\cdot\hat{\boldsymbol{n}}\,\mathrm{d}A \tag{6.15}$$

in terms of the moving flux

$$\boldsymbol{\Gamma} = \boldsymbol{\Psi} - \boldsymbol{u}\Psi^t = \frac{1}{J}\boldsymbol{X}\cdot\boldsymbol{\mathcal{I}}_0. \tag{6.16}$$

Note that the moving integral quantity $\int_{V(t)} \Psi^t\,\mathrm{d}^3x$ will be an invariant (i.e. a constant of motion) when the net moving flux vanishes on the boundary $\partial V(t)$. (See Anco & Webb (2020) for a discussion of moving domain conservation laws and invariants in fluid mechanics.) The equivalence between the conservation laws (6.14) and (6.15) can be derived by writing the continuity equation (6.12) in terms of the material (co-moving) derivative (2.88): $\mathrm{d}\Phi^t/\mathrm{d}t = -(\nabla\cdot\boldsymbol{u})\Phi^t - \nabla\cdot(\boldsymbol{\Phi} - \Phi^t\boldsymbol{u})$, where $(\nabla\cdot\boldsymbol{u})$ represents the expansion or contraction of an infinitesimal volume d^3x moving with the fluid (see, e.g., Anco & Dar 2009, 2010).

Now, rather than seeking to find all Lie point symmetries, we consider two main classes: kinematic and fluid relabelling. Kinematic symmetries are characterised by the generator

(6.1*a–c*) having $\xi_0^i=0$, with ξ^t and ξ^i being functions only of t, x^i. Fluid relabelling symmetries have $\xi^i=\xi^t=0$ in the generator (6.1*a–c*), with ξ_0^i being a function only of t, x_0^i.

A useful general remark is that any Lie point symmetry of a variational principle corresponds to a Lie point symmetry of the Euler–Lagrange equations, because invariance of a variational principle means that its extremals are preserved. Thus, one way to find all Lie point symmetries of a variational principle is by first obtaining the Lie point symmetries of the Euler–Lagrange equations and, second, checking which of those symmetries leaves invariant the variational principle.

Before deriving conservation laws using the above analysis, it is useful to note that there are three basic steps in the analysis.

(i) First it is necessary to determine for a given Lie symmetry, whether the Lie invariance condition (6.5) for the action can be satisfied by choosing the potentials Λ_0^t and Λ_0^i. Here the left-hand side of (6.5) for the action $\mathrm{pr}\,\hat{X}(L_0)$ is evaluated for the symmetry operator X.

(ii) Determine the surface vector components W^t and W^i that occur in the Noether identity (6.8), where W^t and W^i are given by (6.9*a,b*) (recall $\dot{x}^j=\partial x^j(\boldsymbol{x}_0,t)/\partial t=u^j$ and $x^{ji}=\partial x^j/\partial x_0^i$). Then using Proposition 6.1 one can obtain the Lagrangian conservation law (6.10) with conserved density $\mathcal{I}_0^t$ and flux $\mathcal{I}_0^i$ given in (6.11*a,b*).

(iii) Determine the Eulerian form of the conservation law using the results of Padhye (1998) described by (6.12) and (6.13*a,b*).

6.1.2. *Lie invariance condition*

The Lie invariance condition for the action in (6.5) may be written in the form

$$\begin{aligned}
&J\left\{\nabla\cdot(\rho\hat{\boldsymbol{\xi}})\left[\Phi+h-\frac{1}{2}u^2\right]+\rho\boldsymbol{u}\cdot\left(\frac{\mathrm{d}\hat{\boldsymbol{\xi}}}{\mathrm{d}t}-\hat{\boldsymbol{\xi}}\cdot\nabla\boldsymbol{u}\right)+\rho T\hat{\boldsymbol{\xi}}\cdot\nabla S\right.\\
&\quad-\frac{\tilde{\boldsymbol{B}}}{\mu_0}\cdot\left[\nabla\times(\hat{\boldsymbol{\xi}}\times\boldsymbol{B})-\hat{\boldsymbol{\xi}}\nabla\cdot\boldsymbol{B}\right]\\
&\quad\left.+\nabla\cdot\left\{\rho\hat{\boldsymbol{\xi}}\left[\frac{1}{2}u^2-(h+\Phi)\right]+\hat{\boldsymbol{\xi}}\cdot(\mathsf{p}+\mathsf{M}_B)+\frac{1}{\mu_0}\left(\hat{\boldsymbol{\xi}}\times\boldsymbol{B}\right)\times\tilde{\boldsymbol{B}}\right\}\right\}\\
&=D_t\Lambda_0^t+D_{x_0^j}\left(\Lambda_0^j\right).
\end{aligned}\tag{6.17}$$

Here we use the notation $\hat{\boldsymbol{\xi}}\equiv\hat{\boldsymbol{\xi}}_0$ in the fluid re-labelling symmetry case. The Lie invariance condition (6.17) also applies for the general Lie symmetry case, including the Lie point Galilean symmetry cases, the fluid relabelling symmetry cases and other more general cases. The derivation of (6.17) from (6.5) is outlined in Appendix G. In the general case $\hat{\xi}^i=\xi^i-(\xi^tD_t+\xi_0^sD_{x_0^s})x^i$. The Lie invariance condition (6.17) is similar to the Eulerian Lie invariance condition for the action used by Webb & Anco (2019) for the case of MHD. Here the evolutionary form of the symmetry operator X is used rather than the canonical symmetry operator used by Webb & Anco (2019).

6.1.3. *Galilean symmetries*

The Eulerian form of the CGL plasma equations (2.1)–(2.10*a–c*) clearly indicates that they possess the Galilean group of Lie point symmetries when $\Phi=0$, which are generated

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by (Rogers & Ames 1989; Fuchs 1991)

$$P_0 = \frac{\partial}{\partial t}, \quad P_i = \frac{\partial}{\partial x^i}, \quad K_i = t\frac{\partial}{\partial x^i} + \frac{\partial}{\partial u^i}, \quad J_i = \epsilon_{ijk}\left(x^j\frac{\partial}{\partial x^k} + u^j\frac{\partial}{\partial u^k} + B^j\frac{\partial}{\partial B^k}\right). \tag{6.18a–d}$$

These generators describe a time translation (P_0), space translations (P_i, $i = 1, 2, 3$), Galilean boosts (K_i, $i = 1, 2, 3$) and rotations (J_i, $i = 1, 2, 3$), respectively, about the x, y and z axes. As shown in appendix F, the only additional Lie point symmetries admitted by (2.1) to (2.10*a–c*) consist of scalings. Hence, Galilean symmetries and scaling symmetries comprise all kinematic Lie point symmetries of the CGL plasma equations.

The Galilean symmetries (6.18*a–d*) have a corresponding Lagrangian form

$$X_{P_0} = \frac{\partial}{\partial t}, \quad X_{P_i} = \frac{\partial}{\partial x^i}, \quad X_{K_i} = t\frac{\partial}{\partial x^i}, \quad X_{J_i} = \epsilon_{ijk}x^j\frac{\partial}{\partial x^k}. \tag{6.19a–d}$$

Note that the prolongation of these Lagrangian generators to $\dot{x}^i = \dot{X}^i = u^i$ through the Lagrangian relation (3.1) for the fluid velocity yields the corresponding Eulerian generators (6.18*a–d*) acting on the variables (t, x^i, u^i). As $\xi^i_0 = 0$ for all of the generators (6.19*a–d*), they are of kinematic type.

In the case when the gravitational potential Φ is non-zero, then the preceding generators must satisfy the condition $X\Phi(x^i) = 0$ to be admitted as symmetries. For example, if Φ is invariant under z-translation, then X_{P_3} and X_{K_3} are symmetries; if Φ is invariant under z-rotation, then X_{J_3} is a symmetry; and if Φ is spherically symmetric, then X_{J_i}, $i = 1, 2, 3$, are symmetries.

6.1.4. *Galilean conservation laws*

It is straightforward to show that each Galilean generator (6.19*a–d*) satisfies the condition (6.7) for invariance of the Lagrangian variational principle (3.19) when $\Phi = 0$ and thereby yields a conservation law (6.10).

Specifically, time translation (X_{P_0}) has $\hat{\xi}^i = -\dot{x}^i$ and, thus, $\mathrm{pr}\,\hat{X}_{P_0}L_0 = -\ddot{x}^i\partial L_0/\partial_{\dot{x}^i} - \dot{x}^{ij}\partial L_0/\partial_{x^{ij}} = -D_tL_0$ due to $\partial L_0/\partial_t = 0$. This gives $\Lambda^t_0 = -L_0$, $\Lambda^i_0 = 0$, which yields conservation of energy

$$\mathcal{I}^t_0 = \dot{X}^j\frac{\partial L_0}{\partial \dot{X}^j} - L_0 = H_0, \quad \mathcal{I}^i_0 = \dot{X}^j\frac{\partial L_0}{\partial X^{ji}}, \tag{6.20a,b}$$

with H_0 being the Hamiltonian (3.27).

Space translations (X_{P_j}) have $\hat{\xi}^i = \delta^i{}_j$ and, thus, $\mathrm{pr}\,\hat{X}_{P_j}L_0 = 0$ due to $\partial L_0/\partial_{x^i} = 0$ (where we neglect gravity) combined with $D_t\hat{\xi}^i = 0$ and $D_{x^j_0}\hat{\xi}^i = 0$. Hence, $\Lambda^t_0 = 0$, $\Lambda^i_0 = 0$, which yields conservation of momentum

$$\mathcal{I}^t_0 = -\frac{\partial L_0}{\partial \dot{X}^j}, \quad \mathcal{I}^i_0 = -\frac{\partial L_0}{\partial X^{ji}}. \tag{6.21a,b}$$

Galilean boosts (X_{K_j}) have $\hat{\xi}^i = t\delta^i{}_j$ and, thus, $\mathrm{pr}\,\hat{X}_{K_j}L_0 = \rho_0\dot{x}^j = D_t(\rho_0x^j)$. This gives $\Lambda^t_0 = \rho_0x^j$, $\Lambda^i_0 = 0$, yielding conservation of Galilean momentum (centre of mass)

$$\mathcal{I}^t_0 = \int \frac{\partial L_0}{\partial \dot{X}^j}\mathrm{d}t - t\frac{\partial L_0}{\partial \dot{X}^j}, \quad \mathcal{I}^i_0 = -t\frac{\partial L_0}{\partial X^{ji}}. \tag{6.22a,b}$$

Rotations (X_{J_j}) have $\hat{\xi}^i = \epsilon^i{}_{kj}x^k$. This leads to $\mathrm{pr}\,\hat{X}_{J_j}L_0 = 0$, because L_0 depends on x^i only through the scalars $|\dot{x}^i|$ and $|X_{ij}\tau^j_0|$ which are rotationally invariant. Thus $\Lambda^t_0 = 0$,

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$\Lambda_0^i = 0$, which yields conservation of axial angular momentum

$$\mathcal{I}_0^t = \epsilon_{jkl} x^k \frac{\partial L_0}{\partial \dot{X}^l}, \quad \mathcal{I}_0^i = \epsilon_{jkl} x^k \frac{\partial L_0}{\partial X^{li}}. \tag{6.23a,b}$$

The set of Galilean conservation laws (6.20*a*,*b*) to (6.23*a*,*b*) can be written in a unified form through use of the observation that

$$\Lambda_0^t = \dot{\xi}^j \int \frac{\partial L_0}{\partial \dot{X}^j} \mathrm{d}t - \xi^t L_0, \quad \Lambda_0^i = 0, \tag{6.24a,b}$$

with

$$\xi^t = a_0, \quad \xi^i = a_1^i + a_2^i t + a_3^k \epsilon_{ijk} x^j, \tag{6.25a,b}$$

where a_0, a_1^i, a_2^i and a_3^i $(i = 1, 2, 3)$ are arbitrary constants (parameterising the respective Galilean symmetry generators (6.19*a*–*d*)). Hence, we have

$$\mathcal{I}_0^t = \dot{\xi}^j \int \frac{\partial L_0}{\partial \dot{X}^j} \mathrm{d}t - \xi^t L_0 - (\xi^j - \xi^t \dot{X}^j) \frac{\partial L_0}{\partial \dot{X}^j}, \quad \mathcal{I}_0^i = -(\xi^j - \xi^t \dot{X}^j) \frac{\partial L_0}{\partial X^{ji}}. \tag{6.26a,b}$$

The resulting set of Eulerian conservation laws (6.12)–(6.13*a*,*b*) takes the form

$$\Psi^t = H\xi^t - \rho u^j \xi^j + \rho x^j \dot{\xi}^j, \tag{6.27}$$

$$\Psi^i = u^i \Psi^t - (\xi^j - \xi^t u^j) \left(p^{ij} + M_B^{ij} \right), \tag{6.28}$$

where we have used

$$\frac{\partial L_0}{\partial \dot{X}^j} = \rho_0 u^j, \quad \int \frac{\partial L_0}{\partial \dot{X}^j} \mathrm{d}t = \rho_0 x^j, \quad \frac{\partial L_0}{\partial X^{jk}} = \left(p^{jl} + M_B^{jl} \right) A_{lk}, \tag{6.29a–c}$$

which follow from expressions (C1) and (C6) in Appendix C, along with the formulae (3.5*a*–*c*) and (3.8). Here p^{ij} and M_B^{ij} represents the components of the CGL non-isotropic pressure tensor (2.3*a*–*c*) and the magnetic pressure tensor (2.4), respectively. Note that $H = \rho|\boldsymbol{u}|^2 - L$ is the Hamiltonian (3.20).

In the MHD case, p^{ij} is replaced by the components of the isotropic MHD gas pressure tensor $p\,\delta^{ij}$.

The preceding derivations may be summarised as follows.

(i) Time translation symmetry (a_0) yields energy conservation, for which

$$\left.\begin{aligned} \Psi^t &= H = \rho \left(\frac{1}{2}|\boldsymbol{u}|^2 + \Phi(\boldsymbol{x}) \right) + \varepsilon + \frac{B^2}{2\mu_0}, \\ \boldsymbol{\Psi} &= \left(\frac{1}{2}\rho|\boldsymbol{u}|^2 + \varepsilon + \rho\Phi \right) \boldsymbol{u} + \mathsf{p} \cdot \boldsymbol{u} + \frac{1}{\mu_0} \boldsymbol{E} \times \boldsymbol{B}, \\ \boldsymbol{\Gamma} &= \boldsymbol{\Psi} - \boldsymbol{u} H = (\mathsf{p} + \mathsf{M}_B) \cdot \boldsymbol{u}, \end{aligned}\right\} \tag{6.30}$$

where $\boldsymbol{E} = -\boldsymbol{u} \times \boldsymbol{B}$ is the electric field strength, and $\boldsymbol{E} \times \boldsymbol{B}/\mu_0$ is the Poynting flux.

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(ii) Space translation symmetry (a_1^i, $i = 1, 2, 3$) yields momentum conservation with

$$\left.\begin{aligned} \Psi^t &= -\boldsymbol{M} = -\rho\boldsymbol{u}, \quad \boldsymbol{\Psi} = -\mathsf{T}, \\ \boldsymbol{\Gamma} &= \boldsymbol{\Psi} - \boldsymbol{u}\boldsymbol{\Psi}^t = -(\mathsf{p} + \mathsf{M}_B), \end{aligned}\right\} \tag{6.31}$$

where

$$\mathsf{T} = \rho\boldsymbol{u}\boldsymbol{u} + \mathsf{p} + \mathsf{M}_B \tag{6.32}$$

is the CGL plasma stress tensor.

(iii) Galilean boost symmetry yields centre of mass conservation, with

$$\left.\begin{aligned} \Psi^t &= \rho\boldsymbol{x} - t\boldsymbol{M} = \rho(\boldsymbol{x} - t\boldsymbol{u}), \quad \boldsymbol{\Psi} = \rho\boldsymbol{u}\boldsymbol{x} - t\mathsf{T}, \\ \boldsymbol{\Gamma} &= \boldsymbol{\Psi} - \boldsymbol{u}(\rho\boldsymbol{x} - t\boldsymbol{M}) = -t\,(\mathsf{p} + \mathsf{M}_B). \end{aligned}\right\} \tag{6.33}$$

(iv) Rotational symmetry yields angular momentum conservation, with

$$\left.\begin{aligned} \Psi^t &= \boldsymbol{x}\times\boldsymbol{M} = \rho\boldsymbol{x}\times\boldsymbol{u}, \quad \boldsymbol{\Psi} = \boldsymbol{x}\times\mathsf{T}, \\ \boldsymbol{\Gamma} &= \boldsymbol{\Psi} - \boldsymbol{u}(\boldsymbol{x}\times\boldsymbol{M}) = \boldsymbol{x}\times(\mathsf{p} + \mathsf{M}_B), \end{aligned}\right\} \tag{6.34}$$

where $\boldsymbol{M} = \rho\boldsymbol{u}$ is the momentum density vector.

6.1.5. *Fluid relabelling symmetries*

Fluid relabelling symmetries correspond to transformations (6.1*a–c*) that change the Lagrangian fluid labels $\boldsymbol{x}_0$, but leave the Eulerian variables and $\boldsymbol{x}$ and t invariant: $x_0^i \to x_0^i + \epsilon\xi_0^i + O(\epsilon^2)$, where $\xi_0^i = \xi_0^i(t, x_0^j)$. The symmetry generator has the form

$$X = \xi_0^i \frac{\partial}{\partial x_0^i}. \tag{6.35}$$

For evaluating the condition for invariance of the Lagrangian variational principle (3.19), we need the characteristic form of the generator:

$$\hat{X} = \hat{\xi}_0^i \partial_{x^i}, \quad \hat{\xi}_0^i = -\xi_0^j x^{ij}. \tag{6.36a,b}$$

To proceed we use the formulation (6.5) instead of (6.7), which allows for consideration of functions Λ_0^t, Λ_0^i that can involve non-localpotentials. This generality is necessary to derive the cross-helicity conservation law (2.67)–(2.68) which involves the temperature potential (2.69*a*,*b*).

6.1.6. *Noether's theorem for fluid-relabelling conservation laws*

The Lie invariance condition (6.17) in the fluid relabelling symmetry cases, may be written in the form

$$\begin{aligned} &J\left\{\nabla\cdot(\rho\hat{\boldsymbol{\xi}})\left[\Phi + h - \frac{1}{2}u^2\right] + \rho\boldsymbol{u}\cdot\left(\frac{\mathrm{d}\hat{\boldsymbol{\xi}}}{\mathrm{d}t} - \hat{\boldsymbol{\xi}}\cdot\nabla\boldsymbol{u}\right) - \frac{\tilde{\boldsymbol{B}}}{\mu_0}\cdot\left[\nabla\times(\hat{\boldsymbol{\xi}}\times\boldsymbol{B}) - \hat{\boldsymbol{\xi}}\nabla\cdot\boldsymbol{B}\right]\right\} \\ &\quad = D_t\left(\Lambda_0^t - \rho_0 r\boldsymbol{\xi}_0\cdot\nabla_0 S\right) + D_{x_0^i}\left(\Lambda_0^i - A_{ki}G^k\right), \end{aligned} \tag{6.37}$$

where

$$\boldsymbol{G} = \rho\hat{\boldsymbol{\xi}}\left[\frac{1}{2}u^2 - (h + \Phi)\right] + \hat{\boldsymbol{\xi}}\cdot(\mathsf{p} + \mathsf{M}_B) + \frac{1}{\mu_0}\left(\hat{\boldsymbol{\xi}}\times\boldsymbol{B}\right)\times\tilde{\boldsymbol{B}}. \tag{6.38}$$

For cross-helicity conservation, $\hat{\boldsymbol{\xi}} = -\boldsymbol{B}/\rho$ and $\boldsymbol{\xi}_0 = \boldsymbol{B}_0/\rho_0$. Setting $\nabla\cdot\boldsymbol{B} = 0$ (Gauss's law), the left-hand side of (6.37) vanishes. The condition that the right-hand side of (6.37)

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vanish is satisfied by the choices

$$\Lambda_0^t = \rho_0 r \boldsymbol{\xi}_0 \cdot \nabla S, \quad \Lambda_0^i = A_{ki} G^k, \tag{6.39a,b}$$

where $\boldsymbol{G}$ is given by (6.38). Evaluating formulae (6.38) and (6.39*a*,*b*) gives the formulae

$$\boldsymbol{G} = \boldsymbol{B}\left[e + \Phi - \frac{1}{2}u^2 + \frac{B^2}{2\mu_0\rho}\right], \tag{6.40}$$

$$\Lambda_0^t = r\boldsymbol{B}_0 \cdot \nabla_0 S \equiv r(J\boldsymbol{B} \cdot \nabla S), \tag{6.41}$$

$$\Lambda_0^i = B_0^i\left[h + \Phi - \frac{1}{2}u^2 + \frac{B^2}{2\mu_0\rho} - \frac{p_\parallel}{\rho}\right], \tag{6.42}$$

where $h = e + p_\parallel/\rho$ is the enthalpy of the CGL plasma. Here $\hat{\boldsymbol{\xi}} = -\boldsymbol{B}/\rho$ and $\boldsymbol{\xi}_0 = \boldsymbol{B}_0/\rho_0$, which gives rise to the cross-helicity conservation laws described by (6.45*a*,*b*) and (6.46).

Noether's theorem given by Proposition 6.1 now gives the following main result.

PROPOSITION 6.2. *The Lagrangian variational principle (3.19) is invariant up to boundary terms under the infinitesimal fluid relabelling transformation*

$$x_0^i \to x_0^i + \epsilon B_0^i/\rho_0 + O(\epsilon^2). \tag{6.43}$$

The resulting conservation law (6.10)–(6.11a,b) of the Euler–Lagrange equations (3.24)) is obtained by using (6.41)–(6.42) for Λ_0^t and Λ_0^i and using the results

$$W_0^t = \hat{\xi}_0^j \rho_0 \dot{x}^j = -J\boldsymbol{B}\cdot\boldsymbol{u}, \quad W_0^i = \hat{\xi}_0^j A_{ki}\left(p^{jk} + M_B^{jk}\right) = -\frac{1}{\rho}\left(p_\parallel - \frac{B^2}{2\mu_0}\right)B_0^i. \tag{6.44a,b}$$

Here W_0^t and W_0^i are given by (6.9a,b), in which the derivatives of L_0 are given by (G2a–c). This yields the conserved density and the flux

$$\mathcal{I}_0^t = J\boldsymbol{B}\cdot(\boldsymbol{u} + r\nabla S), \quad \mathcal{I}_0^i = B_0^i\left(h + \Phi - \tfrac{1}{2}u^2\right). \tag{6.45a,b}$$

The corresponding Eulerian conservation law (6.12)–(6.13*a*,*b*) is given by

$$\left.\begin{aligned} \Psi^t = \boldsymbol{B}\cdot(\boldsymbol{u} + r\nabla S), \quad \boldsymbol{\Psi} &= \left(h + \Phi - \tfrac{1}{2}u^2\right)\boldsymbol{B} + (\boldsymbol{B}\cdot(\boldsymbol{u} + r\nabla S))\,\boldsymbol{u}, \\ \boldsymbol{\Gamma} &= \left(h + \Phi - \tfrac{1}{2}u^2\right)\boldsymbol{B}. \end{aligned}\right\} \tag{6.46}$$

This is the cross-helicity density conservation law (2.67)–(2.69*a*,*b*).

6.2. *Classical Noether approach*

In this subsection we use a classical version of Noether's theorem (see, e.g., Bluman & Kumei (1989) and Appendix H) to derive conservation laws for the CGL plasma action (3.19) that uses the canonical Lie symmetry operator $\tilde{X} = \mathrm{pr}\, X$ rather than the evolutionary form $\mathrm{pr}\,\hat{X}$. The canonical symmetry operator form of Noether's theorem was used by Webb & Zank (2007) and Webb & Anco (2019) for the MHD fluid case. In this approach one searches for Lie transformations and divergence transformations that leave the action (6.1*a*–*c*) invariant, where

$$L_0' = L_0 + \epsilon D_\alpha \bar{\Lambda}_0^\alpha + O(\epsilon^2), \tag{6.47}$$

is the divergence transformation. Here $D_0 = \partial/\partial t$ and $D_i = \partial/\partial x_0^i$ are total partial derivatives with respect to t and x_0^i. Note we use $\bar{\Lambda}_0^\alpha$ and $\bar{\Lambda}^\alpha$ to denote the potentials,

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in order to distinguish them from the potentials used for the evolutionary potentials in § 6.1.

The condition for the action to remain invariant under (6.1*a–c*) and (6.47) may be written in the form (cf. Bluman & Kumei 1989):

$$\mathrm{pr}\, XL_0 + L_0\left[D_t\xi^t + D_{x_0^j}\left(\xi_0^j\right)\right] + D_t\bar{\Lambda}_0^0 + D_{x_0^j}\bar{\Lambda}_0^j = 0, \tag{6.48}$$

where

$$\mathrm{pr}\, XL_0 = \xi^t\frac{\partial}{\partial t} + \xi_0^s\frac{\partial}{\partial x_0^s} + \xi^k\frac{\partial}{\partial x^k} + \xi^{x_t^k}\frac{\partial}{\partial x_t^k} + \xi^{x_{kj}}\frac{\partial}{\partial x^{kj}} + \cdots, \tag{6.49}$$

is the prolonged, canonical Lie symmetry operator.

Here $\mathrm{pr}\, X$ is related to $\mathrm{pr}\,\hat{X}$ by the equations (Ovsjannikov 1978; Ibragimov 1985; Bluman & Kumei 1989)

$$\left.\begin{aligned}\mathrm{pr}\, X &= \mathrm{pr}\,\hat{X} + \xi_0^\alpha D_\alpha,\\ \mathrm{pr}\,\hat{X} &= \hat{\xi}^k\frac{\partial}{\partial x^k} + D_\alpha\left(\hat{\xi}^k\right)\frac{\partial}{\partial x_\alpha^k} + D_\alpha D_\beta\left(\hat{\xi}^k\right)\frac{\partial}{\partial x_{\alpha\beta}^k} + \cdots,\end{aligned}\right\} \tag{6.50}$$

where the evolutionary symmetry generator $\hat{\xi}^i$ is given by (6.4*a*,*b*) and $\xi^t \equiv \xi_0^0$.

Noether's theorem follows from Noether's identity:

$$\mathrm{pr}\, XL_0 + L_0 D_\alpha\xi_0^\alpha + D_\alpha\bar{\Lambda}_0^\alpha = \hat{\xi}^i E_{x^i}\left(L_0\right) + D_\alpha\left(W^\alpha + L_0\xi_0^\alpha\right) + D_\alpha\left(\bar{\Lambda}_0^\alpha\right), \tag{6.51}$$

where $E_{x^i}(L_0) \equiv \delta\mathcal{J}/\delta x^i$ is the variational derivative of the action $\mathcal{J}$ with respect to x^i. For the case of CGL plasmas, the surface terms W^α are given by

$$W^t \equiv W^0 = \hat{\xi}^j\frac{\partial L_0}{\partial x_t^j}, \quad W^i = \hat{\xi}^j\frac{\partial L_0}{\partial x^{ji}} \tag{6.52a,b}$$

(see Bluman & Kumei (1989) and Ibragimov (1985) for more general cases).

If the Lie invariance condition (6.48) is satisfied, then the left-hand side of (6.51) vanishes and, consequently, the right-hand side of (6.51) must vanish. If, in addition, the Euler–Lagrange equations $E_{x^i}(L_0) = 0$ are satisfied, then (6.51) implies

$$D_\alpha\left(W^\alpha + L_0\xi_0^\alpha + \bar{\Lambda}_0^\alpha\right) = 0, \tag{6.53}$$

which is the conservation law of Noether's first theorem, which applies if the Euler–Lagrange equations $E_{x^i}(L_0) = 0$ are independent, which is the case for a finite Lie algebra of Lie point symmetries.

In the more general case where the symmetries depend on continuous functions $\{\phi^k(\boldsymbol{x}_0, t) : 1 \leqslant k \leqslant N\}$, then the Lie pseudo-algebra of symmetries is infinite dimensional. In this case, Noether's second theorem implies that the Euler–Lagrange equations are not all independent, and that there exists differential relations between the Euler–Lagrange equations (see, e.g., Olver (1993) and Hydon & Mansfield (2011) for details). Noether's second theorem, in some cases results in mathematically trivial conservation laws. Charron & Zadra (2018) discuss Ertel's theorem and Noether's second theorem.

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For the CGL plasma model, the W^α from (6.52*a,b*) are given by

$$W^t = \hat{\xi}^j \rho_0 u^j, \quad W^i = \hat{\xi}^j \left(p^{js} + M_B^{js} \right) A_{si}. \tag{6.54a,b}$$

Substitution of (6.54*a,b*) for the W^α into Noether's theorem (6.53) gives the Lagrangian conservation law

$$\frac{\partial \mathcal{I}^0}{\partial t} + \frac{\partial \mathcal{I}^i}{\partial x_0^i} = 0, \tag{6.55}$$

where

$$\left.\begin{aligned} \mathcal{I}^0 &= \rho_0 u^j \hat{\xi}^j + \xi^t L_0 + \bar{\Lambda}_0^t, \\ \mathcal{I}^i &= \hat{\xi}^j \left(p^{js} + M_B^{js} \right) A_{si} + \xi_0^i L_0 + \bar{\Lambda}_0^i, \end{aligned}\right\} \tag{6.56}$$

are the conserved density and flux.

The Lagrangian conservation law (6.55) corresponds to an Eulerian conservation law of the form (6.12) (e.g. Padhye 1998) with conserved density Ψ^t, and flux Ψ^j given by

$$\left.\begin{aligned} \Psi^t &= \rho u^k \hat{\xi}^k + \xi^t L + \bar{\Lambda}^0, \\ \Psi^j &= \hat{\xi}^k \left(T^{jk} - L\delta^{jk} \right) + \xi^j L + \bar{\Lambda}^j, \end{aligned}\right\} \tag{6.57}$$

where

$$\left.\begin{aligned} T^{jk} &= \rho u^j u^k + p^{jk} + M_B^{jk}, \\ \bar{\Lambda}^0 &= \bar{\Lambda}_0^0 / J, \quad \bar{\Lambda}^j = \left(u^j \bar{\Lambda}_0^0 + x^{js} \bar{\Lambda}_0^s \right) / J. \end{aligned}\right\} \tag{6.58}$$

The Lagrangian and Eulerian conservation laws for the Galilean group (§§ 6.1.2–6.1.3) and the cross-helicity conservation law associated with the fluid relabelling symmetry $\hat{\boldsymbol{\xi}} = -\boldsymbol{B}/\rho$ (§§ 6.1.4–6.1.5) now follow from (6.56)–(6.58) for appropriate choices of the potentials $\bar{\Lambda}_0^t$ and $\bar{\Lambda}_0^j$ and of the symmetry generators ξ^i, ξ_0^i and ξ^t. Note that the evolutionary symmetry potentials used in § 6.1 are different than those used in this subsection (§ 6.2).

EXAMPLE 6.3. *The time translation symmetry of the Lagrangian action (3.19), satisfies the Lie invariance condition (6.48) by choosing*

$$\xi^t = 1, \quad \xi^i = 0, \quad \xi_0^s = 0, \quad \hat{\xi}^i = -u^i, \quad \bar{\Lambda}_0^\alpha = 0, \tag{6.59a–e}$$

where $i, s = 1, 2, 3$, *and* $\alpha = 0, 1, 2, 3$ *is a variational symmetry of the action (3.19). The corresponding conservation law using Noether's theorem results (6.57) is the energy conservation law:*

$$\frac{\partial}{\partial t}\left[\frac{1}{2}\rho |\boldsymbol{u}|^2 + \varepsilon + \frac{B^2}{2\mu_0} + \rho \Phi(\boldsymbol{x}) \right] + \nabla \boldsymbol{\cdot} \left[\rho \boldsymbol{u} \left(\frac{1}{2}|\boldsymbol{u}|^2 + \Phi(\boldsymbol{x}) \right) + \frac{\boldsymbol{E} \times \boldsymbol{B}}{\mu_0} + \varepsilon \boldsymbol{u} + \mathsf{p} \boldsymbol{\cdot} \boldsymbol{u} \right] = 0, \tag{6.60}$$

where p *is the CGL pressure tensor (2.3a–c),* $\boldsymbol{E} = -\boldsymbol{u} \times \boldsymbol{B}$ *is the electric field strength and* $\boldsymbol{E} \times \boldsymbol{B}/\mu_0$ *is the Poynting flux.*

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EXAMPLE 6.4. *If the gravitational potential $\Phi(\boldsymbol{x})$ is independent of x^{j_1} say, then the Lie invariance condition (6.48) for a divergence symmetry of the action is satisfied, by the choice*

$$\xi^i = \delta^{ij_1}, \quad \xi_0^s = \xi^t = 0, \quad \hat{\xi}^i = \delta^{ij_1}, \quad \bar{\Lambda}_0^0 = \bar{\Lambda}_0^i = 0, \tag{6.61a–d}$$

and condition (6.48) reduces to the equation

$$-\rho_0 \frac{\partial \Phi}{\partial x^{j_1}} = 0. \tag{6.62}$$

Then using (6.47) we obtain the momentum conservation equation in the x^{j_1} direction in the form

$$\left\{ \frac{\partial}{\partial t}(\rho \boldsymbol{u}) + \nabla \cdot \left(\rho \boldsymbol{u} \otimes \boldsymbol{u} + \mathsf{p} + \frac{B^2}{2\mu_0} \mathsf{I} - \frac{\boldsymbol{B} \otimes \boldsymbol{B}}{\mu_0} \right) \right\}^{j_1} = 0. \tag{6.63}$$

In the case where gravity can be neglected (i.e. $\Phi = 0$) the superscript j_1 can be dropped in (6.63). Technically, the conservation of momentum law appears more complicated in non-Cartesian coordinates (e.g. in spherical geometry), where the metric tensor, the covariant derivative and the affine connection play an important role.

EXAMPLE 6.5. *The Galilean boost symmetry, with infinitesimal generators*

$$\xi^i = \Omega^i t, \quad \xi_0^s = 0, \quad \xi^t = 0, \quad \bar{\Lambda}_0^0 = -\rho_0(\boldsymbol{x}_0)\boldsymbol{\Omega} \cdot \boldsymbol{x}, \quad \bar{\Lambda}_0^i = 0, \tag{6.64a–e}$$

$(i, s = 1, 2, 3)$ and (6.57) gives rise to the centre of mass conservation law

$$\frac{\partial}{\partial t}[\boldsymbol{\Omega} \cdot \rho(\boldsymbol{u}t - \boldsymbol{x})] + \nabla \cdot \left[\boldsymbol{\Omega} \cdot \left\{ \rho(\boldsymbol{u}t - \boldsymbol{x}) \otimes \boldsymbol{u} + t \left[\mathsf{p} + \frac{B^2}{2\mu_0} \mathsf{I} - \frac{\boldsymbol{B} \otimes \boldsymbol{B}}{\mu_0} \right] \right\} \right] = 0, \tag{6.65}$$

provided

$$\boldsymbol{\Omega} t \cdot \nabla \Phi \equiv \boldsymbol{\xi} \cdot \nabla \Phi = 0. \tag{6.66}$$

Thus, one obtains the centre of mass conservation law for Galilean boosts perpendicular to the external gravitational field. Thus, for a spherically symmetric gravitational potential (e.g. for the Sun), a Galilean boost conservation law exists for a boost $\boldsymbol{\xi} = \boldsymbol{\Omega} t$ perpendicular to the radial direction.

EXAMPLE 6.6. *The Lie transformation generators*

$$\xi^i = \epsilon_{ijk} \Omega^j x^k, \quad \xi_0^s = \xi^t = 0, \quad \hat{\xi}^i = \xi^{x^i}, \quad \bar{\Lambda}_0^\alpha = 0, \tag{6.67a–d}$$

give rise to the angular momentum conservation equation

$$\frac{\partial}{\partial t}[\boldsymbol{\Omega} \cdot (\boldsymbol{x} \times \boldsymbol{M})] + \nabla [\boldsymbol{\Omega} \cdot (\boldsymbol{x} \times \mathsf{T}] = 0, \tag{6.68}$$

where

$$\boldsymbol{M} = \rho \boldsymbol{u} \quad \text{and} \quad (\boldsymbol{x} \times \mathsf{T})^{pj} = \epsilon_{pqk} x^q T^{kj}, \tag{6.69a,b}$$

define the mass flux or momentum density $\boldsymbol{M}$ and $\boldsymbol{x} \times \mathsf{T}$ respectively. The invariance condition (6.48) for a divergence symmetry of the action, requires

$$\begin{aligned} \tilde{X} L_0 &= -\rho_0(\boldsymbol{\Omega} \times \boldsymbol{x}) \cdot \nabla \Phi + \epsilon_{kps} \Omega^p \left[p^{sk} + \frac{B^2}{2\mu_0} \delta^{sk} - \frac{B^s B^k}{\mu_0} \right] \\ &\equiv -\rho_0(\boldsymbol{\Omega} \times \boldsymbol{x}) \cdot \nabla \Phi = 0. \end{aligned} \tag{6.70}$$

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Note that the second term in the first line of (6.70) vanishes because the term in large square brackets is symmetric in s and k.

For the case where $\boldsymbol{\Omega} = \Omega \boldsymbol{e}_z$ the condition (6.70) reduces to the equation

$$\tilde{X}L_0 = -\rho_0 \Omega r \sin\theta \frac{\partial \Phi}{\partial \phi} = 0, \tag{6.71}$$

where (r, θ, ϕ) are spherical polar coordinates. Condition (6.71) is satisfied for $\Phi = \Phi(r, \theta)$.

EXAMPLE 6.7. *The fluid relabelling symmetry transformations*

$$\xi^i = 0, \quad \xi^t = 0, \quad \xi_0^s = -\frac{B_0^s}{\rho_0}, \quad \hat{\xi}^i = \frac{B^i}{\rho} \equiv b^i, \tag{6.72a–d}$$

and the choices

$$\bar{\Lambda}^0 = r\left(\boldsymbol{B} \boldsymbol{\cdot} \nabla S\right), \quad \bar{\Lambda}^i = u^i \bar{\Lambda}^0, \quad \frac{\mathrm{d}r}{\mathrm{d}t} = -T, \tag{6.73a–c}$$

for the gauge potentials $\bar{\Lambda}^\alpha$ $(\alpha = 0, 1, 2, 3)$, leave the action invariant (i.e. (6.48) is satisfied). Using (6.57) to calculate Ψ^t and Ψ^j gives the results (6.46) for the cross-helicity conservation law (2.67)–(2.69a,b).

The above examples illustrate the use of the classical version of Noether's first theorem in obtaining conservation laws of the CGL equations.

7. Summary and concluding remarks

In this paper an investigation has been carried out of the ideal CGL plasma equations, based in part on the Lagrangian variational formulation of Newcomb (1962), in which there is an anisotropic pressure tensor, with pressure components $p_\parallel$ and $p_\perp$ parallel and perpendicular to the magnetic field $\boldsymbol{B}$, which satisfy the so-called double adiabatic equations which, in turn, can be described by using entropy components: $S_\parallel$ and $S_\perp$ parallel and perpendicular to $\boldsymbol{B}$ (e.g. Du *et al.* 2020).

The total energy conservation law, and the cross-helicity and magnetic helicity conservation laws were obtained (§ 2). The total energy equation was decomposed into the sum of three energy equations, namely the internal energy equation or co-moving energy equation, the total kinetic energy equation and Poynting's theorem (the electromagnetic energy equation). The cross-helicity transport equation involves the effective enthalpy $h = (p_\parallel + \varepsilon)/\rho$ of the gas associated with pressure work terms parallel to $\boldsymbol{B}$. If the internal energy density of the plasma per unit mass has the form $e = e(\rho, S, B) \equiv \varepsilon/\rho$ (Holm & Kupershmidt 1986; Hazeltine *et al.* 2013) one obtains a cross-helicity transport equation of the form

$$\frac{\partial}{\partial t}(\boldsymbol{u} \boldsymbol{\cdot} \boldsymbol{B}) + \left[(\boldsymbol{u} \boldsymbol{\cdot} \boldsymbol{B})\boldsymbol{u} + \boldsymbol{B}\left(\Phi + h - \frac{1}{2}u^2\right)\right] = T\boldsymbol{B} \boldsymbol{\cdot} \nabla S, \tag{7.1}$$

where $T = e_S$ is the temperature of the gas. If $\boldsymbol{B} \boldsymbol{\cdot} \nabla S = 0$, then (7.1) is a local conservation law. More generally, if the source term $Q = T\boldsymbol{B} \boldsymbol{\cdot} \nabla S \neq 0$, (7.1) can be reduced to a non-local conservation law of the form

$$\frac{\partial}{\partial t}(\boldsymbol{w}\cdot\boldsymbol{B})+\left[(\boldsymbol{w}\cdot\boldsymbol{B})\boldsymbol{u}+\boldsymbol{B}\left(\Phi+h-\frac{1}{2}u^2\right)\right]=0, \tag{7.2}$$

where

$$\boldsymbol{w}=\boldsymbol{u}+r\nabla S \quad \text{and} \quad \frac{\mathrm{d}r}{\mathrm{d}t}=-T. \tag{7.3a,b}$$

Here $\mathrm{d}/\mathrm{d}t=\partial/\partial t+\boldsymbol{u}\cdot\nabla$ is the advective time derivative following the flow. The CGL plasma conservation law (7.2) generalises the non-local, MHD cross-helicity conservation law of Webb *et al.* (2014*a*,*b*) and Yahalom (2017*a*,*b*). Here $\Phi(\boldsymbol{x})$ is the potential for an external gravitational field. Similar transport equations for cross-helicity to (7.1) and (7.3*a*,*b*) were also developed in terms of the parallel and perpendicular temperatures $T_\parallel$ and $T_\perp$ defined as $T_\parallel=p_\parallel/(\rho R)$ and $T_\perp=p_\perp/(\rho R)$.

There are two different, but equivalent forms of the momentum (or force) equation for the CGL system. The form of the momentum equation obtained by Newcomb (1962) Lagrangian action principle reduces to the form

$$\frac{\mathrm{d}\boldsymbol{u}}{\mathrm{d}t}=-\frac{1}{\rho}\nabla\cdot\mathsf{p}+\frac{\boldsymbol{J}\times\boldsymbol{B}}{\rho}-\nabla\Phi, \tag{7.4}$$

where $\boldsymbol{J}=\nabla\times\boldsymbol{B}/\mu_0$ is the current. This equation is essentially the same as the MHD momentum equation, except that the isotropic gas pressure tensor $p\mathsf{I}$ is replaced by the anisotropic CGL pressure tensor $\mathsf{p}=p_\parallel\boldsymbol{\tau}\boldsymbol{\tau}+p_\perp(\mathsf{I}-\boldsymbol{\tau}\boldsymbol{\tau})$. However,

$$\nabla\cdot\mathsf{p}=-\left[\boldsymbol{B}\times(\nabla\times\boldsymbol{\Omega})+\rho(T\nabla S-\nabla h)\right] \tag{7.5}$$

(e.g. (2.61), (B6) and (E12)) where

$$\boldsymbol{\Omega}=\frac{p_\Delta}{B^2}\boldsymbol{B}, \quad p_\Delta=p_\parallel-p_\perp, \quad h=\frac{\varepsilon+p_\parallel}{\rho}. \tag{7.6a–c}$$

Using $\nabla\cdot\mathsf{p}$ from (7.5) and (7.6*a*–*c*) in (7.4) results in the CGL momentum equation in the form

$$\frac{\mathrm{d}\boldsymbol{u}}{\mathrm{d}t}=T\nabla S-\nabla h+\frac{\tilde{\boldsymbol{J}}\times\boldsymbol{B}}{\rho}-\nabla\Phi, \tag{7.7}$$

where

$$\tilde{\boldsymbol{J}}=\boldsymbol{J}-\nabla\times\boldsymbol{\Omega}=\frac{\nabla\times\tilde{\boldsymbol{B}}}{\mu_0}, \tag{7.8}$$

$$\tilde{\boldsymbol{B}}=\boldsymbol{B}\left[1-\frac{\mu_0 p_\Delta}{B^2}\right]. \tag{7.9}$$

In the form (7.7) the anisotropic pressure force term $-\nabla\cdot\mathsf{p}/\rho$ has been partly transformed into the modified force $\tilde{\boldsymbol{J}}\times\boldsymbol{B}/\rho$ where $\tilde{\boldsymbol{J}}=(\nabla\times\tilde{\boldsymbol{B}})/\mu_0$ in which $\tilde{\boldsymbol{B}}$ is the modified magnetic induction $\tilde{\boldsymbol{B}}$. The form of $\tilde{\boldsymbol{J}}$ suggests that $-\nabla\times\boldsymbol{\Omega}$ could be interpreted as a magnetisation current.

It is interesting to note that $\tilde{B}=B(1-\mu_0 p_\Delta/B^2)<0$ if $p_\Delta=(p_\parallel-p_\perp)>B^2/\mu_0$, which corresponds to the firehose instability threshold (e.g. Stix 1992; Hunana *et al.* 2016; Hunana & Zank 2017). The mirror instability threshold for the CGL plasma model

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does not correspond to plasma kinetic theory. For the CGL plasma model with zero electron pressure (i.e. cold electrons) the mirror instability occurs if $p_\perp - p_\parallel > (5/6)p_\perp + p_\parallel p_B/p_\perp$, whereas kinetic theory gives the threshold for the instability as $p_\perp - p_\parallel > p_\parallel p_B/p_\perp$ where $p_B = B^2/(2\mu)$ is the magnetic pressure. Note the firehose instability occurs if the parallel pressure dominates the perpendicular pressure. Similarly if $p_\perp$ dominates $p_\parallel$ one obtains the mirror instability.

In the approaches of Hazeltine *et al.* (2013) and Holm & Kupershmidt (1986), the internal energy density per unit mass, e satisfies the first law of thermodynamics, in the form

$$T\,\mathrm{d}S = \mathrm{d}e + p_\parallel\,\mathrm{d}\tau + \left(\frac{p_\Delta}{\rho B}\right)\mathrm{d}B. \tag{7.10}$$

The variational formulation of Newcomb (1962) does not explicitly require (7.10) to apply. The Lagrangian action principle ofNewcomb (1962) leads to the correct momentum equation for the CGL plasma, and to a Hamiltonian formulation of the equations in Lagrangian variables.

The Lagrangian variational principle for ideal CGL plasmas obtained by Newcomb (1962) was used in § 3 to obtain a canonical Hamiltonian formulation of the equations based on the canonical coordinates $\boldsymbol{q} = \boldsymbol{x}(\boldsymbol{x}_0, t)$ and the canonical momentum $\boldsymbol{\pi} = \rho_0\dot{\boldsymbol{x}}(\boldsymbol{x}_0, t)$, and also to establish that stationary variations of the action give the CGL momentum equation in both its Lagrangian and Eulerian forms.

Section 4 provides an EP action principle derivation of the CGL momentum equation (see also Appendix E, which uses the approach of Holm *et al.* 1998).

By transforming the canonical Poisson bracket for the CGL system of § 3, to non-canonical physical variables leads to the non-canonical Poisson bracket for the CGL system of § 5. The detailed transformation formulae from the canonical coordinates to the physical variables $\psi = (\rho, \sigma, \boldsymbol{M}, \boldsymbol{B})^{\mathrm{T}}$ is described in Appendix D. By writing the non-canonical Poisson bracket in cosymplectic form, leads to a system of equations for the Casimirs C, as solutions of the Poisson bracket equation: $\{C, K\} = 0$ where K is an arbitrary functional of the physical variables. The Casimirs satisfy $\mathcal{C}_t = \{C, K\} = 0$. Hamiltonian dynamics of the system takes place on the symplectic leaves $C =$ const. of the system.

The classical Casimirs for ideal fluids and plasmas are the mass conservation integral, the cross-helicity integral for barotropic flows and the magnetic helicity. For the CGL plasma case, there is effectively one entropy function that is Lie dragged with the flow, because $S_\parallel$ and $S_\perp$ are assumed to be functions only of S. In principle, a more complicated bracket would arise, if one used the possibility that the adiabatic integrals for $p_\perp$ and $p_\parallel$ depended on other scalar invariants that are advected with the flow. For example, the integrals (2.13*a*,*b*) could also depend on the scalar invariant $\boldsymbol{B}\boldsymbol{\cdot}\nabla S/\rho$. This possibility was not explored in the present paper.

The links between Noether's theorem and conservation laws for CGL plasmas are developed (§ 6). The evolutionary symmetry form of the Lie invariance condition for the action is used. This approach differs from the canonical symmetry operator form of the invariance condition (e.g. Webb & Anco (2019), see also Appendix H). In the evolutionary form of the symmetry operator, the independent variables are frozen and all the Lie transformation changes are restricted to changes in the dependent variables and their derivatives (see,e.g., Ibragimov 1985; Olver 1993; Bluman & Anco 2002). In the canonical symmetry approach, both the dependent and independent variables and their derivatives change.

The CGL plasma equations admit the Galilean Lie point symmetry group, and three extra scaling symmetries (Appendix F). The Galilean group leads to: (i) the energy conservation law due to time translation invariance of the action; (ii) the momentum conservation equations due to space translation invariances; (iii) the conservation of angular momentum due to rotational invariance about some given rotation axis; and (iv) the uniform centre of mass conservation law which is due to invariance under Galilean boosts. These conservation laws are derived using the evolutionary form of Noether's theorem.

The non-local cross-helicity conservation law (6.45*a*,*b*)–(6.46) (see also (2.67)) arises from a fluid relabelling symmetry with generators $\hat{\boldsymbol{\xi}} = -\boldsymbol{B}/\rho$ and $\boldsymbol{\xi}_0 = \boldsymbol{B}_0/\rho_0$ and with non-trivial potentials Λ_0^t and Λ_0^i. It is a non-local conservation law that depends on the Lagrangian time integral of the temperature back along the fluid flow trajectory. The cross-helicity conservation law for MHD, is a local conservation law for the case of a barotropic gas (i.e. $p = p(\rho)$), but is a non-local conservation law for the non-barotropic case where $p = p(\rho, S)$ (see also Yahalom (2017*a*,*b*); Yahalom & Qin (2021) for a topological interpretation). The CGL entropies $S_\parallel$ and $S_\perp$ are not constants, but are non-trivial scalars that are advected with the background flow.

Lingam *et al.* (2020) studied extended variational principles of MHD (and CGL) type including gyro-viscous effects (i.e. higher-order finite Larmor radius effects in the collisionless limit). A single gyro-viscous term is added to the usual action. The gyro-viscous term alters the total momentum density, but it does not alter the divergence of the mass flux. The total momentum density has the form: $\boldsymbol{M}^c = \boldsymbol{M} + \boldsymbol{M}^*$, where $\boldsymbol{M} = \rho \boldsymbol{u}$ is the mass flux and $\boldsymbol{M}^*$ has the form $\boldsymbol{M}^* = \nabla \times \boldsymbol{L}^*$, where $\boldsymbol{L}^*$ is the internal angular momentum of the particle ($\boldsymbol{L}^* = (2m/e)\boldsymbol{\mu}$ and $\boldsymbol{\mu} = \mu\boldsymbol{\tau}$ is the vector form of the particle adiabatic moment. These ideas are related to papers by Newcomb (1972, 1973, 1983), Morrison, Lingam & Acevedo (2014) and others. Our analysis can be extended in principle to include gyro-viscosity in the collisionless limit. However, to what extent the action principle approach, reproduces the kinetic plasma and fluid approaches to gyro-viscosity requires further investigation.

Analysis of the CGL plasma equations using Clebsch potentials (e.g. Zakharov & Kuznetsov 1997; Yahalom 2017*a*, *b*; Webb 2018, Ch. 8; Yahalom & Qin 2021) may yield further insights. Similarly, further study of the Lie symmetries the CGL plasma equations would be useful. Investigation of conservation laws for the CGL equations by using Lie dragging remains open for further investigation.

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Competing interests

The authors have no competing interests to declare.

Appendix A

In this appendix, we briefly discuss the derivation of the CGL plasma equations. These equations were originally derived by Chew *et al.* (1956) and later by many authors (e.g. Kulsrud 1983; Ramos 2005*a*,*b*). Some of the derivations use the adiabatic drift approximation, but others simply involve taking moments of the collisionless Vlasov equation or Boltzmann equation over the particle momenta. We use the latter approach.

Following Hunana *et al.* (2019*a*,*b*), we introduce the velocity phase space distribution function $f(\boldsymbol{x}, \boldsymbol{v}, t)$ of the particles where $\mathrm{d}N = f(\boldsymbol{x}, \boldsymbol{v}, t)\, \mathrm{d}^3 v\, \mathrm{d}^3\boldsymbol{x}$ is the number of particles in a volume of phase space at the point $(\boldsymbol{x}, \boldsymbol{v})$ at time t, with velocity volume element $d^3 v$ and position volume element $\mathrm{d}^3\boldsymbol{x}$. The lower order moments of the velocity distribution function averaged over all velocities $\boldsymbol{v}$ are defined as

$$\left.\begin{aligned} n &= \int f\, \mathrm{d}^3\boldsymbol{v}, \\ n\boldsymbol{u} &= \int f\boldsymbol{v}\, \mathrm{d}^3\boldsymbol{v}, \\ p_{ij} &= \int f(v^i - u^i)(v^j - u^j)\, \mathrm{d}^3\boldsymbol{v}, \\ q_{ijk} &= \int f(v^i - u^i)(v^j - u^j)(v^k - u^k)\, \mathrm{d}^3\boldsymbol{v}, \end{aligned}\right\} \tag{A1}$$

where n is the particle number density, $\boldsymbol{u}$ is the fluid velocity, and in Cartesian coordinates, p_{ij} represents the components of the pressure tensor p and q_{ijk} represents the components of the heat flux tensor q. Note that both of these tensors are symmetric.

From adiabatic motion of guiding centre theory, the pressure tensor p at lowest order can be written in the form

$$\mathsf{p} = p_\parallel \boldsymbol{\tau\tau} + p_\perp (\mathsf{I} - \boldsymbol{\tau\tau}) + \Pi, \tag{A2}$$

where $\boldsymbol{\tau} = \boldsymbol{B}/B$ is the unit vector along the magnetic field, $p_\parallel$ and $p_\perp$ are the gyrotropic components of p parallel and perpendicular to the magnetic field $\boldsymbol{B}$ and Π represents the non-gyrotropic components of p. In the limit of a strong background magnetic field $\boldsymbol{B}$ in which the particle gyro-radius is $r_g \ll L$ where L is the scale length for variation of $\boldsymbol{B}$, and for times $T \gg T_\Omega$ where T_Ω is the gyro-period, the particle distribution is approximately gyrotropic while the non-gyrotropic pressure Π can be neglected to first order in r_g/L and T_Ω/T.

Hereafter, $\mathsf{A}^S = \mathsf{A} + \mathsf{A}^{\mathrm{T}}$ denotes the symmetrised form of A, where T denotes the transpose.

Taking the first moment of the Vlasov equation with respect to $\boldsymbol{v}$ gives the pressure tensor equation

$$\frac{\partial \mathsf{p}}{\partial t} + \nabla \boldsymbol{\cdot} (\boldsymbol{u} \otimes \mathsf{p} + \mathsf{q}) + (\mathsf{p} \boldsymbol{\cdot} \nabla \boldsymbol{u})^S + \frac{q}{mc} (\boldsymbol{B} \times \mathsf{p})^S = 0, \tag{A3}$$

where

$$(\boldsymbol{B} \times \mathsf{p})_{ik} = \epsilon_{ijl} B_j p_{lk}. \tag{A4}$$

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The equations for the parallel pressure $p_\parallel$ and perpendicular pressure $p_\perp$ obtained from (A3) respectively reduce to

$$\frac{\partial p_\parallel}{\partial t} + \nabla\cdot\left(p_\parallel \boldsymbol{u}\right) + 2p_\parallel \boldsymbol{\tau\tau} : \nabla\boldsymbol{u} + \boldsymbol{\tau\tau} : (\nabla\cdot q) - \Pi : \frac{\mathrm{d}}{\mathrm{d}t}(\boldsymbol{\tau\tau}) + (\Pi\cdot\nabla\boldsymbol{u})^S : \boldsymbol{\tau\tau} = 0, \tag{A5}$$

and

$$\frac{\partial p_\perp}{\partial t} + \nabla\cdot(p_\perp\boldsymbol{u}) + p_\perp\nabla\cdot\boldsymbol{u} - p_\perp\boldsymbol{\tau\tau} : \nabla\boldsymbol{u} + \frac{1}{2}\left(Tr(\nabla\cdot q) - \boldsymbol{\tau\tau} : (\nabla\cdot q)\right) + \frac{1}{2}\left(Tr(\Pi\cdot\nabla\boldsymbol{u})^S + \Pi : \frac{\mathrm{d}}{\mathrm{d}t}(\boldsymbol{\tau\tau}) - (\Pi\cdot\nabla\boldsymbol{u})^S : \boldsymbol{\tau\tau}\right) = 0. \tag{A6}$$

Neglecting the non-gyrotropic component Π of the pressure tensor, and neglecting the heat flux tensor components q gives the simplified CGL plasma equations for $p_\parallel$ and $p_\perp$ as

$$\frac{\partial p_\parallel}{\partial t} + \nabla\cdot\left(p_\parallel\boldsymbol{u}\right) + 2p_\parallel\boldsymbol{\tau\tau} : \nabla\boldsymbol{u} = 0, \tag{A7}$$

$$\frac{\partial p_\perp}{\partial t} + \nabla\cdot(p_\perp\boldsymbol{u}) + p_\perp\nabla\cdot\boldsymbol{u} - p_\perp\boldsymbol{\tau\tau} : \nabla\boldsymbol{u} = 0. \tag{A8}$$

The double adiabatic equations (2.8) for $p_\parallel$ and $p_\perp$ follow by combining these transport equations (A7) and (A8) with the mass continuity equation (2.1) in the form

$$\frac{\mathrm{d}}{\mathrm{d}t}\rho = -\rho\nabla\cdot\boldsymbol{u}, \tag{A9}$$

and the magnetic field strength equation

$$\frac{\mathrm{d}}{\mathrm{d}t}B = -(\nabla\boldsymbol{u}) : (I - \boldsymbol{\tau\tau})\,B \tag{A10}$$

which comes from Faraday's equation (2.6).

Appendix B

In this appendix, we derive the two equivalent forms of the pressure divergence equation (2.61) and (2.71) used in the derivation of the corresponding forms of the cross-helicityconservation law (2.67) and (2.74). Throughout we take $\nabla\cdot\boldsymbol{B} = 0$.

We start from the non-zero terms on the right-hand side of (2.61):

$$\boldsymbol{B}\times(\nabla\times\boldsymbol{\Omega}) + \rho(T\nabla S - \nabla h). \tag{B1}$$

By applying a standard cross-product identity on the first term in (2.61), we expand

$$\begin{aligned}\boldsymbol{B}\times(\nabla\times\boldsymbol{\Omega}) &= \left[\nabla\left(\frac{p_\Delta}{B}\boldsymbol{\tau}\right)\right]\cdot\boldsymbol{\tau}B - B\boldsymbol{\tau}\cdot\nabla\left(\frac{p_\Delta}{B}\boldsymbol{\tau}\right)\\ &= \nabla p_\Delta - (\boldsymbol{\tau}\cdot\nabla p_\Delta)\,\boldsymbol{\tau}\\ &\quad + p_\Delta\left[-\boldsymbol{\tau}\cdot\nabla\boldsymbol{\tau} + (\nabla\boldsymbol{\tau})\cdot\boldsymbol{\tau} - \nabla\ln B + (\boldsymbol{\tau}\cdot\nabla\ln B)\,\boldsymbol{\tau}\right].\end{aligned} \tag{B2}$$

In this expression, the second and third terms combine into $-\boldsymbol{\tau}\cdot\nabla(p_\Delta\boldsymbol{\tau})$; the fourth term vanishes $(\nabla\boldsymbol{\tau})\cdot\boldsymbol{\tau} = \nabla(\frac{1}{2}\boldsymbol{\tau}\cdot\boldsymbol{\tau}) = 0$ since $\boldsymbol{\tau}$ is a unit vector; and with the Gauss' law

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equation written as $\nabla\cdot\boldsymbol{\tau}=-\boldsymbol{\tau}\cdot\nabla\ln B$, the last term can be expressed as $-p_\Delta(\nabla\cdot\boldsymbol{\tau})\boldsymbol{\tau}$. Thus, (B2) reduces to

$$\boldsymbol{B}\times(\nabla\times\boldsymbol{\Omega})=\nabla p_\Delta-p_\Delta\nabla\ln B-\nabla\cdot(p_\Delta\boldsymbol{\tau}\boldsymbol{\tau}). \tag{B3}$$

Next, we expand the remaining term in (B1) by using the Pfaffian differential equation (2.15) in the gradient form:

$$\nabla e=T\nabla S+\frac{p_\parallel}{\rho}\nabla\ln\rho-\frac{p_\Delta}{\rho}\nabla\ln B. \tag{B4}$$

This yields

$$\begin{aligned}\rho(T\nabla S-\nabla h)&=\rho\left(\nabla(e-h)-\frac{p_\parallel}{\rho}\nabla\ln\rho+\frac{p_\Delta}{\rho}\nabla\ln B\right)\\&=-\nabla p_\parallel+p_\Delta\nabla\ln B,\end{aligned} \tag{B5}$$

by using expressions (2.62) for enthalpy and (2.16*a,b*) for internal energy. Finally, we combine the terms (B5) and (B3), which gives

$$\boldsymbol{B}\times(\nabla\times\boldsymbol{\Omega})+\rho(T\nabla S-\nabla h)=-\nabla p_\perp-\nabla\cdot(p_\Delta\boldsymbol{\tau}\boldsymbol{\tau})=-\nabla\cdot\mathsf{p} \tag{B6}$$

because the pressure tensor (2.10*a–c*) can be written in terms of p_Δ as

$$\mathsf{p}=p_\parallel\boldsymbol{\tau}\boldsymbol{\tau}+p_\perp(\mathsf{I}-\boldsymbol{\tau}\boldsymbol{\tau})=p_\perp\mathsf{I}+p_\Delta\boldsymbol{\tau}\boldsymbol{\tau}. \tag{B7}$$

This yields the first form of the pressure divergence equation (2.61).

We derive the second form of the pressure divergence equation (2.71) by starting from the terms on its left-hand side

$$\nabla\cdot\mathsf{p}-\rho\nabla h. \tag{B8}$$

Expanding the first term in (B8) by use of the gyrotropic expression (B7), we obtain

$$\nabla\cdot\mathsf{p}=\nabla p_\perp+\boldsymbol{\tau}\boldsymbol{\tau}\cdot\nabla p_\Delta+p_\Delta(\boldsymbol{\tau}\nabla\cdot\boldsymbol{\tau}+\boldsymbol{\tau}\cdot\nabla\boldsymbol{\tau}). \tag{B9}$$

The scalar product of this expression with $\boldsymbol{\tau}$ yields

$$\boldsymbol{\tau}\cdot(\nabla\cdot\mathsf{p})=\boldsymbol{\tau}\cdot\left(\nabla p_\parallel-p_\Delta\nabla\ln B\right) \tag{B10}$$

through the Gauss' law equation in the form $\nabla\cdot\boldsymbol{\tau}=-\boldsymbol{\tau}\cdot\nabla\ln B$. The second term in (B8) can be expanded using the enthalpy (2.62). After taking the scalar product with $\boldsymbol{\tau}$, this gives

$$-\rho\boldsymbol{\tau}\cdot\nabla h=-\boldsymbol{\tau}\cdot\nabla\left(\tfrac{3}{2}p_\parallel+p_\perp\right)+\left(\tfrac{3}{2}p_\parallel+p_\perp\right)\boldsymbol{\tau}\cdot\nabla\ln\rho. \tag{B11}$$

The combined terms (B10) and (B11) yield

$$\boldsymbol{\tau}\cdot(\nabla\cdot\mathsf{p}-\rho\nabla h)=-\boldsymbol{\tau}\cdot\nabla\left(\tfrac{1}{2}p_\parallel+p_\perp\right)-p_\Delta\boldsymbol{\tau}\cdot\nabla\ln B+\left(\tfrac{3}{2}p_\parallel+p_\perp\right)\boldsymbol{\tau}\cdot\nabla\ln\rho. \tag{B12}$$

Now, we substitute the gradient of the expressions for $p_\parallel$ and $p_\perp$ in terms of adiabatic integrals (2.37):

$$\left.\begin{aligned}\nabla p_\parallel&=p_\parallel\left(\nabla\ln c_\parallel(S)+3\nabla\ln\rho-2\nabla\ln B\right),\\\nabla p_\perp&=p_\perp\left(\nabla\ln c_\perp(S)-\nabla\ln\rho-\nabla\ln B\right).\end{aligned}\right\} \tag{B13}$$

Then (B12) reduces to the form

$$\boldsymbol{\tau}\cdot(\nabla\cdot\mathsf{p}-\rho\nabla h)=-\tfrac{1}{2}p_\parallel\boldsymbol{\tau}\cdot\nabla\ln c_\parallel(S)-p_\perp\boldsymbol{\tau}\cdot\nabla\ln c_\perp(S), \tag{B14}$$

which gives the pressure divergence equation (2.71) after we use the relations (2.29*a,b*) for the adiabatic integrals in terms of $\bar{S}_\parallel$ and $\bar{S}_\perp$.

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Appendix C

In this appendix, we show how the Eulerian momentum equation (2.2) is obtained from the Euler–Lagrange equation (3.24) of the Lagrangian (3.23). We follow the approach in Newcomb (1962).

From expression (3.23), we have

$$\left.\begin{aligned} \frac{\partial L_0}{\partial X^i} = -\rho_0\frac{\partial \Phi}{\partial X^i}, \quad \frac{\partial L_0}{\partial \dot{X}^i} = \rho_0\dot{X}^i, \\ \frac{\partial L_0}{\partial X_{ij}} = \left(\frac{p_{\parallel 0}}{\zeta^4} - \frac{p_{\perp 0}}{\zeta J}\right) X_{ik}\tau_0^k\tau_0^j + \left(\frac{p_{\perp 0}\zeta}{J^2} + \frac{\zeta^2 B_0^2}{2\mu_0 J^2}\right) A_{ij} - \frac{X_{ik}B_0^k B_0^j}{\mu_0 J}, \end{aligned}\right\} \tag{C1}$$

which uses expression (3.14) for ζ^2, and the derivative expression in (3.5*a–c*) for J. Then, substituting the derivatives (C1) into the Euler–Lagrange equation (3.24), we obtain

$$\begin{aligned} E_{X^i}(L_0) = -\rho_0\frac{\partial \Phi}{\partial X^i} - \frac{\partial}{\partial t}\left(\rho_0\dot{X}^i\right) - \frac{\partial}{\partial x_0^j}\left(\left(\frac{p_{\perp 0}\zeta}{J^2} + \frac{\zeta^2 B_0^2}{2\mu_0 J^2}\right) A_{ij}\right) \\ - \frac{\partial}{\partial x_0^j}\left(\left(\frac{p_{\parallel 0}}{\zeta^4} - \frac{p_{\perp 0}}{\zeta J} - \frac{B_0^2}{\mu_0 J}\right) X_{ik}\tau_0^k\tau_0^j\right). \end{aligned} \tag{C2}$$

To proceed, we need the derivative identity:

$$A_{ij}\frac{\partial}{\partial x_0^j} f = \frac{\partial}{\partial x_0^j}\left(A_{ij} f\right) = J\frac{\partial f}{\partial X^i} = J\, \nabla_i f|_{x=X} \tag{C3}$$

holding for any function f (see Newcomb 1962). This identity arises from the middle property in (3.5*a–c*) as follows. Differentiating with respect to x_0^k gives $\partial J/\partial x_0^j = X_{ij}\partial A_{ik}/\partial x_0^k + A_{ik}\partial X_{ij}/\partial x_0^k$. By using commutativity of partial derivatives $\partial X_{ij}/\partial x_0^k = \partial X_{ik}/\partial x_0^k$ and substituting Jacobi's formula for derivative of a determinant $\partial J/\partial x_0^i = A_{jk}\partial X_{jk}/\partial x_0^i$, we find $\partial A_{ik}/\partial x_0^k = 0$. This leads to the identity (C3), after applying the chain rule and again using the middle property in (3.5*a–c*).

There are two main steps for simplifying (C2). First, through relations (3.14), (3.16) and (3.18*a*,*b*), followed by use of the identity (C3), we can express the two divergence terms in (C2) in the form:

$$\frac{\partial}{\partial x_0^j}\left(\left(p_\perp + \frac{B^2}{2\mu_0}\right) A_{ji}\right) = J\frac{\partial}{\partial X^i}\left(\left(p_\perp + \frac{B^2}{2\mu_0}\right)\right), \tag{C4}$$

$$\begin{aligned} \frac{\partial}{\partial x_0^j}\left(\frac{J}{\zeta^2}\left(p_\Delta - \frac{B^2}{\mu_0}\right)\zeta\,\tau_i\tau_0^j\right) &= \frac{\partial}{\partial x_0^j}\left(\left(p_\Delta - \frac{B^2}{\mu_0}\right)\tau_i\tau^k A_{kj}\right) \\ &= \frac{\partial}{\partial X^k}\left(\left(p_\Delta - \frac{B^2}{\mu_0}\right)\tau_i\tau^k\right). \end{aligned} \tag{C5}$$

Note these terms imply that

$$\frac{\partial L_0}{\partial X_{ij}} = A_{kj}\left(\delta_{ik}\left(p_\perp + \frac{B^2}{2\mu_0}\right) + \left(p_\Delta - \frac{B^2}{\mu_0}\right)\tau_i\tau_k\right). \tag{C6}$$

Second, from the density relation (3.8) and the fluid element flow equation (3.1), we see that in (C2) the first term is simply

$$-J\rho\frac{\partial\Phi}{\partial X^i}, \tag{C7}$$

while the second term can be expressed as

$$\begin{aligned}-\left(\dot{J}\rho u^i+J\frac{\mathrm{d}}{\mathrm{d}t}(\rho u^i)\right)&=-\rho u^i A_{kj}\frac{\partial\dot{X}^k}{\partial x_0^j}-J\left((\rho u^i)_t+\dot{X}^j\nabla_j(\rho u^i)\right)\\&=-J\left((\rho u^i)_t+\nabla_j(\rho u^i u^j)\right)\big|_{x=X}\end{aligned} \tag{C8}$$

by use of relation (3.6).

Thus, the Euler–Lagrange equation (C2) becomes

$$E_{X^i}(L_0)=-J\left(\rho\nabla_i\Phi+(\rho u^i)_t+\nabla_i\left(p_\perp+\frac{B^2}{2\mu_0}\right)+\nabla_j\left(\left(p_\Delta-\frac{B^2}{\mu_0}\right)\tau_i\tau^j\right)\right)\Bigg|_{x=X}. \tag{C9}$$

The stationary points of the action principle are given by the equation $E_{X^i}(L_0)=0$. From expression (C9), the resulting equation coincides with the Eulerian momentum equation (2.2).

Note that, in terms of the Lagrangian variables, the Euler–Lagrange equation (C2) is a nonlinear wave system for $X^i(x_0^j,t)$, where $X_{ij}=\partial X^i/\partial x_0^j$ and $A_{ij}=J^{-1}(\partial X^i/\partial x_0^j)^{-1}$. (See also Golovin (2011), Webb *et al.* (2005*b*) and Webb & Anco (2019) for the MHD case).

Appendix D

Eckart (1963) used the Lagrangian map and Jacobians to describe fluids. Lundgren (1963) used Eulerian and Lagrangian variations in MHD and in plasma physics (see also Newcomb 1962). In Lundgren (1963) a small parameter ϵ is used to describe the variations, where the physical quantity ψ has the functional form: $\psi=\psi(\boldsymbol{x},\boldsymbol{x}_0,\epsilon)$ in which $\boldsymbol{x}=\boldsymbol{X}(\boldsymbol{x}_0,t)$ is the Lagrangian map. Eulerian ($\delta\psi$) and Lagrangian ($\Delta\psi$) variations of ψ are defined as

$$\delta\psi=\lim_{\epsilon\to0}\left(\frac{\partial\psi}{\partial\epsilon}\right)_{\boldsymbol{x}},\quad \Delta\psi=\lim_{\epsilon\to0}\left(\frac{\partial\psi}{\partial\epsilon}\right)_{\boldsymbol{x}_0}. \tag{D1*a*,*b*}$$

Thus, for an Eulerian variation, $\delta\psi$ is evaluated with $\boldsymbol{x}$ held constant, whereas for a Lagrangian variation $\Delta\psi$, $\boldsymbol{x}_0$ is held constant. Thus, $\delta\boldsymbol{x}=0$ and $\Delta\boldsymbol{x}_0=0$.

Using the chain rule for differentiation, it follows that

$$\left.\begin{aligned}\delta\psi&=\Delta\psi+\delta\boldsymbol{x}_0\boldsymbol{\cdot}\nabla_0\psi,\\ \Delta\psi&=\delta\psi+\Delta\boldsymbol{x}\boldsymbol{\cdot}\nabla\psi.\end{aligned}\right\} \tag{D2}$$

Dewar (1970) used a variational principle for linear Wentzel–Kramers–Brillouin (WKB) MHD waves in a non-uniform background plasma flow. Webb *et al.* (2005*a*) used a variational principle to describe non-WKB waves in a non-uniform background flow. Webb *et al.* (2005*b*) used similar ideas to describe nonlinear waves in a non-uniform flow by variational methods.

In this appendix, we describe the use of Eulerian and Lagrangian variations in defining the Poisson bracket for CGL plasmas.

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We derive the CGL plasma non-canonical bracket (5.2) starting from the canonical Poisson bracket (5.4). This canonical bracket is properly defined in a Lagrangian frame, i.e. a physical reference frame moving with the fluid flow, where all quantities are functions of the fluid element labels x_0^i and time t. Non-scalar quantities (e.g. vectors, differential forms, tensors) are expressed in terms of their components with respect to the Cartesian basis vectors of this frame (corresponding to the coordinates x_0^i, $i = 1, 2, 3$). These components are designated by a subscript 0.

The canonical bracket (5.4) has the component form

$$\{\mathcal{F}, \mathcal{G}\} = \int \left(\mathcal{F}_{q^i}\mathcal{G}_{p^i} - \mathcal{F}_{p^i}\mathcal{G}_{q^i}\right) \mathrm{d}^3x_0, \tag{D3}$$

where $q^i = x^i(x_0, t)$ and $p^i = \pi^i(x_0, t)$ comprise the canonical coordinates and momenta, with $\pi^i = \rho_0\dot{x}^i$, as used in formulating Hamilton's equations (3.29*a*,*b*). Note that the fluid element motion is expressed through the relation

$$\dot{x}^i(x_0, t) = u^i(x(x_0, t), t). \tag{D4}$$

Here $\mathcal{F}$ and $\mathcal{G}$ are functionals which depend on q^i, p^i, as well as a set of advected quantities $\mathsf{a}(x_0^j, t)$ which are used in describing the dynamics.

For a CGL plasma, the basic advected quantities are listed in (4.9*a–d*). We take

$$\mathsf{a} = (S_0, \boldsymbol{B}_0/\rho_0, \rho_0\, \mathrm{d}^3x_0). \tag{D5}$$

Note we assume $S_\parallel$ and $S_\perp$ are functionals of s. Accordingly, functionals will be expressed as

$$\mathcal{F} = \int F_0(\mathsf{Z}_0)\, \mathrm{d}^3x_0 \tag{D6}$$

in terms of the Lagrangian variables

$$\mathsf{Z}_0 = (\boldsymbol{q}, \boldsymbol{p}, \mathsf{a}). \tag{D7}$$

The non-canonical Poisson bracket (5.2) employs the Eulerian variables (5.11). The transformation from Lagrangian variables (D7) to these Eulerian variables is effected in the following four steps.

Firstly, the Eulerian form of a functional (D6) is given by

$$\mathcal{F} = \int F(\mathsf{Z})\, \mathrm{d}^3x \tag{D8}$$

with $\mathsf{Z} = (\rho, \sigma, \boldsymbol{B}, \boldsymbol{M})$. The vector variables here will be expanded in components with respect to the Eulerian basis vectors corresponding to x^i viewed as coordinates in an Eulerian frame. It will be convenient to take the basis vectors to be derivative operators (via the standard correspondence between vectors and directional derivatives, e.g. Schutz (1980)):

$$\boldsymbol{x} = x^i\partial_{x^i}, \quad \boldsymbol{x}_0 = x_0^i\partial_{x_0^i}. \tag{D9*a*,*b*}$$

Second, using the notation in Newcomb (1962), we define Δx^i to represent a variation of $x^i(x_0, t)$ in which x_0^i and t are held fixed: $\Delta x_0^i = 0$ and $\Delta t = 0$. Note that Δx^i will itself

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be some function of x_0^i and t; we write it in the Eulerian form:

$$\Delta x^i(x_0, t) = \epsilon^i(x, t) \tag{D10}$$

where, on the right-hand side, x^i is regarded as a function of x_0^i and t. Likewise, $\Delta\pi^i = \Delta(\rho_0\dot{x}^i) = \rho_0\delta\dot{x}^i$ is a corresponding variation of $\pi^i(x_0, t)$, where ρ_0 is unchanged because it is a function of only x_0^i and t. In Eulerian form:

$$\Delta\pi^i(x_0, t) = J(x)\rho(x)\Delta u^i(x, t), \tag{D11}$$

using the relation (D4) and the advection result (3.8), where J is the determinant of the Jacobian matrix of partial derivatives of $x^i(x_0, t)$ (cf. (3.3)).

The third step is to derive formulae for the variation of the Eulerian variables Z. These formulae depend on the specific tensorial nature of each variable and will be obtained through the variation of the advected variables (D5) in component form. Hereafter we suppress the t dependence in all variables and quantities whenever it is convenient.

We start with the scalar field S_0. Since it is advected (i.e. frozen in), this implies $S(x) = S_0(x_0)$. Applying a variation (D10), we consequently see that

$$\Delta S(x) = \Delta S_0(x_0) = 0. \tag{D12}$$

Next we consider $\rho(x)\,\mathrm{d}^3x$, which is properly viewed as an advected differential 3-form (see, e.g., Schutz 1980). Its advection property is expressed by (3.8). As x_0 and t are held fixed, a variation (D10) yields

$$\Delta(\rho(x)J(x)) = J(x)\Delta\rho(x) + \rho(x)\Delta J(x) = 0. \tag{D13}$$

Now we use the variation of the determinant relation $\mathrm{d}^3x = J(x)\,\mathrm{d}^3x_0$, which is given by $\Delta\mathrm{d}^3x = \partial_{x^i}(\Delta x^i)\,\mathrm{d}^3x = (\Delta J(x))\,\mathrm{d}^3x_0$. This yields the result

$$\Delta J(x) = J(x)\partial_{x^i}\epsilon^i(x). \tag{D14}$$

Substituting this variation into (D13), we obtain

$$\Delta\rho(x) = -\rho(x)\partial_{x^i}\epsilon^i(x). \tag{D15}$$

Last we consider the vector field $\boldsymbol{b}(x) \equiv \boldsymbol{B}(x)/\rho(x)$, which has the advection property (3.13). We use this property in component form: $b^i(x)\partial_{x^i} = b_0^i(x_0)\partial_{x_0^i}$. Again, because all of the quantities on the right-hand side are held fixed in a variation (D10), we see that

$$\Delta(b^i(x)\partial_{x^i}) = (\Delta b^i(x))\partial_{x^i} + b^i(x)\Delta\partial_{x^i}. \tag{D16}$$

To determine $\Delta\partial_{x^i}$, we use the coordinate basis relation $\partial_{x^i}\rfloor\,\mathrm{d}x^j = \delta_i^j$ where δ_i^j is the Kronecker symbol. Taking the variation gives $\Delta\partial_{x^i}\rfloor\,\mathrm{d}x^j = -\partial_{x^i}\rfloor\Delta\mathrm{d}x^j$, where

$$\Delta\mathrm{d}x^j = \mathrm{d}\Delta x^j = \mathrm{d}\epsilon^j(x) = \mathrm{d}x^k\partial_{x^k}\epsilon^j(x). \tag{D17}$$

As $\Delta\partial_{x^i}$ must have the form of a linear transformation, say $\phi_i{}^k$, applied to ∂_{x^k}, we find $\phi_i{}^k\partial_{x^k}\rfloor\,\mathrm{d}x^j = \phi_i{}^j = -\partial_{x^i}\rfloor\,\mathrm{d}x^k\partial_{x^k}\epsilon^j(x) = -\partial_{x^i}\epsilon^j(x)$. Thus, we have

$$\Delta\partial_{x^i} = -\partial_{x^i}\epsilon^j(x)\partial_{x^j}. \tag{D18}$$

Substituting this formula into (D16) yields

$$\Delta b^i(x) = b^j(x)\partial_{x^j}\epsilon^i(x). \tag{D19}$$

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By applying the variations (D12), (D15) and (D19) to the quantities $\sigma(x) = \rho(x)S(x)$ and $B^i(x) = \rho(x)b^i(x)$, we readily obtain

$$\Delta\sigma(x) = -\sigma(x)\partial_{x^i}\epsilon^i(x), \tag{D20}$$

and

$$\Delta B^i(x) = -(\partial_{x^j}\epsilon^j(x))B^i(x) + B^j(x)\partial_{x^j}\epsilon^i(x). \tag{D21}$$

To complete the derivation of δZ, we also need to find the variation of $M^i(x) = \rho(x)u^i(x)$, which can be obtained directly from the relation (D4). Taking the variation of this relation yields

$$\Delta u^i(x) = \partial_t \Delta x^i(x_0) = \mathrm{d}\epsilon^i(x)/\mathrm{d}t. \tag{D22}$$

This result implies that $\Delta M^i(x) = (\Delta\rho(x))u^i(x) + \rho(x)(\Delta u^i(x)) = \rho(x)[\mathrm{d}\epsilon^i(x)/\mathrm{d}t - (\partial_{x^j}\epsilon^j(x))u^i(x)]$, and thus we obtain

$$\Delta M^i(x) = \rho(x)\frac{\mathrm{d}}{\mathrm{d}t}\epsilon^i(x) - M^i(x)\partial_{x^j}\epsilon^j(x) = \rho(x)\partial_t\epsilon^i(x) + M^j(x)\partial_{x^j}\epsilon^i(x) - M^i(x)\partial_{x^j}\epsilon^j(x). \tag{D23}$$

Now, the fourth step consists of transforming the variational derivatives $\mathcal{F}_{q^i} \equiv \delta\mathcal{F}/\delta x^i$ and $\mathcal{F}_{p^i} \equiv \delta\mathcal{F}/\delta\pi^i$ into an equivalent form with respect to the Eulerian variables that comprise Z. Consider a variation of a functional $\mathcal{F}$. From (D6), we obtain

$$\Delta\mathcal{F} = \int \left(F_{0q^i}\Delta x^i + F_{0p^i}\Delta\pi^i\right)\mathrm{d}^3x_0 = \int \left(J^{-1}F_{0q^i}\epsilon^i + F_{0p^i}\rho\frac{\mathrm{d}}{\mathrm{d}t}\epsilon^i\right)\mathrm{d}^3x, \tag{D24}$$

where we have used (D10)–(D11) and (D22), along with $\Delta a = 0$ which holds because the quantities (D5) comprising a are advected. Similarly, from (D8), we obtain

$$\begin{aligned}\Delta\mathcal{F} &= \int \left(\left(F_\rho\Delta\rho + F_\sigma\Delta\sigma + F_{B^i}\Delta B^i + F_{M^i}\Delta M^i\right)J + F\delta J\right)\mathrm{d}^3x_0 \\ &= \int \left(\left(F - F_\rho\rho - F_\sigma\sigma - F_{B^i}B^i - F_{M^i}M^i\right)\partial_{x^j}\epsilon^j + F_{B^i}B^j\partial_{x^j}\epsilon^i + F_{M^i}\rho\frac{\mathrm{d}}{\mathrm{d}t}\epsilon^i\right)\mathrm{d}^3x \\ &= \int \left(\left(\rho\partial_{x^j}F_\rho + \sigma\partial_{x^j}F_\sigma + M^i\partial_{x^j}F_{M^i} + B^i\partial_{x^j}F_{B^i} - \partial_{x^i}(F_{B^j}B^i)\right)\epsilon^j + F_{M^i}\rho\frac{\mathrm{d}}{\mathrm{d}t}\epsilon^i\right)\mathrm{d}^3x,\end{aligned} \tag{D25}$$

where we have substituted (D15), (D20), (D21) and (D23), integrated by parts, and then used the cancellation

$$F_\rho\partial_{x^j}\rho + F_\sigma\partial_{x^j}\sigma + F_{B^i}\partial_{x^j}B^i + F_{M^i}\partial_{x^j}M^i - \partial_{x^j}F = 0 \tag{D26}$$

which holds by the chain rule. Finally, from the two expressions (D24) and (D25), we equate the coefficients of ϵ^i, and likewise the coefficients of $\mathrm{d}\epsilon^i/\mathrm{d}t$, because ϵ^i and $\mathrm{d}\epsilon^i/\mathrm{d}t$ are arbitrary functions of x. This yields the key result

$$\left.\begin{aligned}F_{0q^i} &= \left(\rho\partial_{x^i}F_\rho + \sigma\partial_{x^i}F_\sigma + M^j\partial_{x^i}F_{M^j} + B^j\partial_{x^i}F_{B^j{}_i} - \partial_{x^j}(F_{B^i}B^j)\right)J, \\ F_{0p^i} &= F_{M^i},\end{aligned}\right\} \tag{D27}$$

which are the transformation formulae for the variational derivatives.

Substitution of these formulae (D27) into the canonical bracket (D3) gives

$$
\begin{aligned}
\{\mathcal{F},\mathcal{G}\} &= \int \Big(G_{M^j}\left(\rho\partial_{x^j}F_\rho + \sigma\partial_{x^j}F_\sigma + M^i\partial_{x^j}F_{M^i}\right.\\
&\quad \left.+B^i\partial_{x^j}F_{B^i} - \partial_{x^i}(F_{B^j}B^i)\right) - F_{M^j}\left(\rho\partial_{x^j}G_\rho + \sigma\partial_{x^j}G_\sigma\right.\\
&\quad \left.+M^i\partial_{x^j}G_{M^i} + B^i\partial_{x^j}G_{B^i} - \partial_{x^i}(G_{B^j}B^i)\right)\Big)\,\mathrm{d}^3x\\
&= \int \big(\rho(G_{M^j}\partial_{x^j}F_\rho - F_{M^j}\partial_{x^j}G_\rho) + \sigma(G_{M^j}\partial_{x^j}F_\sigma - F_{M^j}\partial_{x^j}G_\sigma)\\
&\quad + M^i(G_{M_j}\partial_{x^j}F_{M^i} - F_{M_j}\partial_{x^j}G_{M^i}) + B^i(G_{B_j}\partial_{x^j}F_{B^i} - F_{B_j}\partial_{x^j}G_{B^i})\\
&\quad +B^i(F_{B^j}\partial_{x^i}G_{M^j} - G_{B^j}\partial_{x^i}F_{M^j})\big)\,\mathrm{d}^3x
\end{aligned} \tag{D28}
$$

after integration by parts to remove derivatives off of B^i. This completes the derivation of the CGL Poisson bracket, which is the counterpart of the MHD bracket Morrison & Greene (1982). Note that, for using the bracket (D28), we can convert all terms into vector notation as shown in (5.2).

Equations (D4) for $\dot{x}^i(x_0, t)$ and (D10) for the variation $\delta x^i(x_0, t)$ can be set up in an alternative way by means of the Lagrangian map (4.1), using diffeomorphisms $g(t)$ on Euclidean space similarly to the set up for the EP action principle in § 4. However, here the variations are not the same as those used for the EP action principle, because those variations took place in the Eulerian frame and involved fixing x^i, with x_0^i being a function of x and t. To proceed, we view $g(t)$, at any fixed time t, as an element in the group of diffeomorphisms $G \equiv \mathrm{Diff}(\mathbb{R}^3)$ acting on Euclidean space in terms of the Cartesian coordinates x_0^i. The Lagrangian map (4.1) then can be expressed in component form:

$$x^i(x_0, t) = g(t)x_0^i. \tag{D29}$$

(Concretely, g can be thought of as a matrix in the fundamental representation of the group G.) Thus, we can write

$$\epsilon^i(x, t) = \delta(g(t)x_0^i) = (\delta g(t)g^{-1}(t))x^i = (\delta g(t)g^{-1}(t))^i, \tag{D30}$$

$$u^i(x, t) = \partial_t(g(t)x_0^i) = (g_t(t)g^{-1}(t))x^i = (g_t(t)g^{-1}(t))^i, \tag{D31}$$

where

$$\delta g(t)g^{-1}(t) \equiv (\delta g(t)g^{-1}(t))^i\partial_{x^i}, \quad g_t(t)g^{-1}(t) \equiv (g_t(t)g^{-1}(t))^i\partial_{x^i} \tag{D32a,b}$$

represent right-invariant vector fields on the group G, which are identified with Eulerian vector fields (directional derivatives) at the point x^i in Euclidean space. Note the property of right-invariance means that, for any fixed element h in G, $g \to gh$ implies $\delta gg^{-1} \to \delta(gh)(gh)^{-1} = \delta ghh^{-1}g^{-1} = \delta gg^{-1}$ and $g_tg^{-1} \to (gh)_t(gh)^{-1} = g_thh^{-1}g^{-1} = g_tg^{-1}$, due to $\partial_t h = \delta h \equiv 0$. The variation of the Eulerian variables Z can be shown to arise from using the push-forward action of $g(t)$, and the pull-back action of $g(t)^{-1}$, on scalar fields, vector fields, differential forms and tensor fields in the Lagrangian frame.

A closer connection to the variations used for the EP action principle arises if the variations derived in (D12), (D15), (D20), (D21) and (D23), are reformulated in the following way. A variation of any one of these quantities, $\delta Z(x)$, can be expressed as the sum of a contribution from varying only its dependence on x^i, in addition to a

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contribution from varying Z with x^i being unchanged. For example, consider $\delta S(x) = (\delta S)(x) + (\delta x^i)\partial_{x^i} S(x)$. The pointwise contribution is given by

$$(\delta S)(x) = \delta S(x) - \epsilon^i(x)\partial_{x^i} S(x) = -\epsilon^i(x)\partial_{x^i} S(x) \equiv \Delta S(x) \tag{D33}$$

from the total variation (D12). Likewise, $\delta\rho(x) = (\delta\rho)(x) + (\delta x^i)\partial_{x^i}\rho(x)$ leads to

$$(\delta\rho)(x) = -\partial_{x^i}(\rho(x)\epsilon^i(x)) \equiv \Delta\rho(x) \tag{D34}$$

from the total variation (D15). As another example, $\delta B^i(x) = \delta(B^i)(x) + (\delta x^j)\partial_{x^j} B^i(x)$ gives

$$(\delta B^i)(x) = \delta(B^i(x)) - \epsilon^j(x)\partial_{x^j} B^i(x) = -\partial_{x^j}(\epsilon^j(x)B^i(x)) + B^j(x)\partial_{x^j}\epsilon^i(x) \equiv \Delta B^i(x) \tag{D35}$$

using the total variation (D21). When the analogous pointwise variations are considered for the components of the advected quantities (D5), we see that they take the form of a Lie derivative:

$$(\delta S)(x) = -\mathcal{L}_{\boldsymbol{\epsilon}} S(x), \quad (\delta b^i)(x) = -\mathcal{L}_{\boldsymbol{\epsilon}} b^i(x), \quad (\delta\rho e_{ijk})(x) = -\mathcal{L}_{\boldsymbol{\epsilon}}(\rho(x)e_{ijk}). \tag{D36a–c}$$

Note that here we have expressed the volume element as a 3-form: $\mathrm{d}^3x = e_{ijk}\mathrm{d}x^i \wedge \mathrm{d}x^j \wedge \mathrm{d}x^k$, where e_{ijk} is the Levi–Civita symbol. The Lie derivative in the variations (D36*a–c*) denotes the standard Lie derivative formula in component form. These formulae can be written more properly by including the basis vectors: e.g. $(\delta b^i(x))\partial_{x^i} = -\mathcal{L}_{\boldsymbol{\epsilon}}(b^i(x)\partial_{x^i})$, where the Lie derivative then acts in the standard geometrical way. This result coincides with the general formulation in Holm *et al.* (1983). In addition, it shows that the variations of the non-advected Eulerian variables given by (D34) and (D35) can be expressed in terms of Lie derivatives:

$$\Delta\rho(x) = -(\partial_i\epsilon^i)\rho(x) - \mathcal{L}_{\boldsymbol{\epsilon}}\rho(x), \quad \Delta B^i(x) = -(\partial_{x^j}\epsilon^j)B^i(x) - \mathcal{L}_{\boldsymbol{\epsilon}} B^i(x). \tag{D37a,b}$$

These formulae are used in the classical work of Newcomb (1962). They can be used to derive the transformation formula (D27) for the variational derivatives by taking a pointwise variation of a functional:

$$\Delta\mathcal{F} = \int \left(F_\rho\Delta\rho + F_\sigma\Delta\sigma + F_{B^i}\Delta B^i + F_{M^i}\Delta M^i\right)\mathrm{d}^3x, \tag{D38}$$

which differs from (D27) by lacking a contribution from varying the volume element. Nevertheless, this leads to the same expression (D28) for the bracket (cf. the cancellation of terms (D26)).

Appendix E

In this appendix, we outline the general approach of Holm *et al.* (1998) on the EP variational principles, with application to CGL plasmas. The action is of the form:

$$\mathcal{J} = \int L[u, a]\,\mathrm{d}^3x\,\mathrm{d}t. \tag{E1}$$

The stationary points of $\mathcal{A}$ are given by $\delta\mathcal{J} = 0$, where the variables a are advected quantities (2.87) subject to the variations

$$\delta a = -\mathcal{L}_{\boldsymbol{\eta}}(a). \tag{E2}$$

Here $\boldsymbol{\eta}$ is an arbitrary, sufficiently differentiable vector field. Alternatively, $\boldsymbol{\eta}$ arises from a Lagrangian map as discussed in § 4.

For the CGL plasma case, the Lagrangian density $L[u, a]$ is given by (3.21), where the a are the advected quantities S, $\rho\,\mathrm{d}^3x$, $\boldsymbol{B}/\rho \equiv \boldsymbol{b}$ (cf. (4.9*a–d*)). The magnetic flux 2-form $B_x\,\mathrm{d}y \wedge \mathrm{d}z + B_y\,\mathrm{d}z \wedge \mathrm{d}x + B_z\,\mathrm{d}x \wedge \mathrm{d}y$ is a Lie dragged invariant of the flow (Webb *et al.* 2014*a*). (Note $p_\parallel$, $p_\perp$ and e are defined in terms of ρ, S and B via the equation of state for $e(\rho, S, B)$). The expression for the action $\mathcal{J}$ is the same as in (4.18). From (E2) and (4.9*a–d*), the variation δa is obtained:

$$\left.\begin{aligned} \delta S &= -\mathcal{L}_\eta S = -\boldsymbol{\eta}\boldsymbol{\cdot}\nabla S,\\ \delta\rho\,\mathrm{d}^3x &= -\mathcal{L}_\eta(\rho\,\mathrm{d}^3x) = -\nabla\boldsymbol{\cdot}(\rho\boldsymbol{\eta})\,\mathrm{d}^3x,\\ \delta\boldsymbol{B} &= \nabla\times(\boldsymbol{\eta}\times\boldsymbol{B}) - \boldsymbol{\eta}(\nabla\boldsymbol{\cdot}\boldsymbol{B}). \end{aligned}\right\} \tag{E3}$$

The variational equation $\delta\mathcal{J} = 0$ is expressed as:

$$\delta\mathcal{J} = \int \langle\boldsymbol{\eta}, \boldsymbol{F}\rangle\,\mathrm{d}^3x\,\mathrm{d}t = 0, \tag{E4}$$

which can be shown to imply for arbitrary $\boldsymbol{\eta}$ that

$$\boldsymbol{F} = \frac{\partial}{\partial t}\left(\frac{\delta L}{\delta\boldsymbol{u}}\right) + ad^*_{\boldsymbol{u}}\left(\frac{\delta L}{\delta\boldsymbol{u}}\right)_R - \frac{\delta L}{\delta a}\diamond a = 0 \tag{E5}$$

(see Holm *et al.* (1998) for details). This is the general form of the EP equation. The diamond operator $\diamond$ is defined by the property (4.20).

For the CGL plasma action principle, the EP equation (E5) has the form

$$\frac{\partial}{\partial t}(\rho\boldsymbol{u}) + \nabla\boldsymbol{\cdot}(\rho\boldsymbol{u}\otimes\boldsymbol{u}) + \rho\nabla\left(\frac{1}{2}u^2\right) = \frac{\delta L}{\delta a}\diamond a. \tag{E6}$$

The term in (E6) involving the diamond operator can be determined from the variation

$$\begin{aligned} \int \frac{\delta L}{\delta a}\delta a\,\mathrm{d}^3x &= \int\left(\frac{\delta L}{\delta\rho}\delta\rho + \frac{\delta L}{\delta S}\delta S + \frac{\delta L}{\delta\boldsymbol{B}}\boldsymbol{\cdot}\delta\boldsymbol{B}\right)\mathrm{d}^3x\\ &= \int\left(\frac{\delta L}{\delta\rho}[-\nabla\boldsymbol{\cdot}(\rho\boldsymbol{\eta})] + \frac{\delta L}{\delta S}[-\boldsymbol{\eta}\boldsymbol{\cdot}\nabla S]\right.\\ &\qquad\left. + \frac{\delta L}{\delta\boldsymbol{B}}\boldsymbol{\cdot}[\nabla\times(\boldsymbol{\eta}\times\boldsymbol{B}) - \boldsymbol{\eta}(\nabla\boldsymbol{\cdot}\boldsymbol{B})]\right)\mathrm{d}^3x\\ &= \int\boldsymbol{\eta}\boldsymbol{\cdot}\left[\rho\nabla\frac{\delta L}{\delta\rho} - \frac{\delta L}{\delta S}\nabla S + \boldsymbol{B}\times\left[\left(\nabla\times\frac{\delta L}{\delta\boldsymbol{B}}\right) - \frac{\delta L}{\delta\boldsymbol{B}}(\nabla\boldsymbol{\cdot}\boldsymbol{B})\right]\right]\mathrm{d}^3x, \end{aligned} \tag{E7}$$

where boundary integral terms have been discarded. From (E7) and (E4), we identify the diamond operator term $(\delta L/\delta a)\diamond a$ as being the term in the square brackets. Substituting the variational derivatives of L, which are given by (4.21), we obtain

$$\begin{aligned} \frac{\delta L}{\delta a}\diamond a &= \rho\left(T\nabla S - \nabla h + \nabla\left(\frac{1}{2}u^2 - \varPhi\right)\right) + \boldsymbol{J}\times\boldsymbol{B}\\ &\quad + \frac{\boldsymbol{B}}{\mu_0}(\nabla\boldsymbol{\cdot}\boldsymbol{B}) + \boldsymbol{B}\times(\nabla\times\boldsymbol{\varOmega}) - \boldsymbol{\varOmega}(\nabla\boldsymbol{\cdot}\boldsymbol{B}), \end{aligned} \tag{E8}$$

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where

$$\boldsymbol{J} = \frac{\nabla \times \boldsymbol{B}}{\mu_0}, \quad \boldsymbol{\Omega} = \frac{p_\Delta}{B}\boldsymbol{\tau}, \quad \boldsymbol{\tau} = \frac{\boldsymbol{B}}{B}. \tag{E9a–c}$$

The auxiliary calculations

$$\left.\begin{aligned} \boldsymbol{B} \times (\nabla \times \boldsymbol{\Omega}) &= -p_\Delta \boldsymbol{\tau} \cdot \nabla \boldsymbol{\tau} - p_\Delta (\mathsf{I} - \boldsymbol{\tau\tau}) \cdot \nabla (\ln B) + (\mathsf{I} - \boldsymbol{\tau\tau}) \cdot \nabla p_\Delta, \\ \nabla \cdot \mathsf{p} &= \nabla p_\perp + p_\Delta ((\nabla \cdot \boldsymbol{\tau})\boldsymbol{\tau} + \boldsymbol{\tau} \cdot \nabla \boldsymbol{\tau}) + \boldsymbol{\tau\tau} \cdot \nabla p_\Delta, \\ \nabla \cdot \boldsymbol{B} &= B (\nabla \cdot \boldsymbol{\tau} + \boldsymbol{\tau} \cdot \nabla \ln B), \end{aligned}\right\} \tag{E10}$$

and the thermodynamic relation

$$\rho (T\nabla S - \nabla h) = p_{\Delta\nabla} (\ln B) - \nabla p_\parallel, \tag{E11}$$

can be combined to give the identity

$$\boldsymbol{B} \times (\nabla \times \boldsymbol{\Omega}) - \boldsymbol{\Omega}(\nabla \cdot \boldsymbol{B}) + \nabla \cdot \mathsf{p} + \rho (T\nabla S - \nabla h) = 0. \tag{E12}$$

This equation was derived in (B6).

Substituting the identity (E12) into (E8) gives:

$$\frac{\delta L}{\delta a} \diamond a = \rho \nabla \left(\frac{1}{2}u^2\right) - \rho \nabla \Phi + \boldsymbol{J} \times \boldsymbol{B} + \frac{\boldsymbol{B}}{\mu_0} \nabla \cdot \boldsymbol{B} - \nabla \cdot \mathsf{p}, \tag{E13}$$

which, when used in the EP equation (E6) yields the Eulerian momentum equation (2.2).

Appendix F

In this appendix, we summarise the complete Lie point symmetry group for the CGL system (2.1)–(2.9). We restrict attention to the case where there is no gravity (i.e. $\Phi = 0$ in (2.2)). (For the general theory of Lie point symmetries, see Ovsjannikov 1978; Ibragimov 1985; Olver 1993; Ovsjannikov 1994; Bluman & Anco 2002; Golovin 2009; Bluman, Cheviakov & Anco 2010.)

The independent and dependent variables in (2.1)–(2.9) consist of t, x^i, ρ, u^i, B^i, S, $p_\parallel$ and $p_\perp$. Thus, every Lie point symmetry arises from a generator of the form

$$X = \xi^t \partial_t + \xi^i \partial_{x^i} + \xi^\rho \partial_\rho + \xi^{u^i} \partial_{u^i} + \xi^{B^i} \partial_{B^i} + \xi^S \partial_S + \xi^{p_\parallel} \partial_{p_\parallel} + \xi^{p_\perp} \partial_{p_\perp} \tag{F1}$$

under which (2.1)–(2.9) are invariant on the space of solutions. Here

$$\xi^t, \ \xi^i, \ \xi^\rho, \ \xi^{u^i}, \ \xi^{B^i}, \ \xi^S, \xi^{p_\parallel}, \ \xi^{p_\perp} \tag{F2}$$

are functions of the independent and dependent variables. For computational purposes, it is easiest to work with the characteristic form of the generator in which only the dependent variables undergo a transformation:

$$\hat{X} = \hat{\xi}^\rho \partial_\rho + \hat{\xi}^{u^i} \partial_{u^i} + \hat{\xi}^{B^i} \partial_{B^i} + \hat{\xi}^S \partial_S + \hat{\xi}^{p_\parallel} \partial_{p_\parallel} + \hat{\xi}^{p_\perp} \partial_{p_\perp} \tag{F3}$$

where

$$\hat{\xi}^v = \xi^v - \xi^t v_t - \xi^i v_{x^i} \tag{F4}$$

for each dependent variable $v = (\rho, u^i, B^i, S, p_\parallel, p_\perp)$. The generator $\hat{X}$ has the convenient property that it commutes with total derivatives with respect to t, x^i, and so the prolongation of $\hat{X}$ acting on derivatives of v is easy to compute.

The determining equations for Lie point symmetries are given by applying pr$\hat{X}$ to each equation (2.1)–(2.9) and evaluating the resulting system on the solution space of (2.1)–(2.9) which is carried out by putting (2.1)–(2.9) into a solved form with respect to a set of leading derivatives. (See Olver (1993) for a general discussion.) The resulting system of determining equations then splits with respect to all derivative variables that appear in the system. This yields an overdetermined linear system of equations for the functions (F2). The system can be solved straightforwardly by use of computer algebra (e.g. Maple or Mathematica).

The Lie point symmetry generators are found to be given by the 10 Galilean transformation generators (6.18*a–d*) and the following 3 scaling generators

$$\left.\begin{aligned} S_1 &= t\partial_t + x^i\partial_{x^i}, \\ S_2 &= t\partial_t + 2\rho\partial_\rho - u^i\partial_{u^i}, \\ S_3 &= 2\rho\partial_\rho + 2p_\perp\partial_{p_\perp} + 2p_\parallel\partial_{p_\parallel} + B^i\partial_{B^i}. \end{aligned}\right\} \tag{F5}$$

In terms of the quantities $p = (p_\parallel + 2p_\perp)/3$ and $p_\Delta = p_\parallel - p_\perp$, the third scaling symmetry has the equivalent form:

$$S_3 = 2\rho\partial_\rho + 2p\partial_p + 2p_\Delta\partial_{p_\Delta} + B^i\partial_{B^i}. \tag{F6}$$

For comparison, the Lie point symmetries for MHD (Rogers & Ames 1989; Fuchs 1991) are given by the 10 Galilean transformation generators (6.18*a–d*) and the 2 scaling generators S_1 and S_2 plus a scaling generator $S_{3'} = 2\rho\partial_\rho + 2p\partial_p + B^i\partial_{B^i}$ which differs from S_3 by omitting the term involving p_Δ.

A full study of subalgebras of the Galilean Lie algebra was given in Ovsjannikov (1978, 1994, 1999) and Grundland & Lalague (1995).

Appendix G

Here we derive the Lie invariance condition (6.17) for the action, which follows from (6.5). We start from the appropriate expansion for pr $\hat{X}L_0$:

$$\mathrm{pr}\,\hat{X}(L_0) = \hat{\xi}^i\frac{\partial L_0}{\partial x^i} + D_t\left(\hat{\xi}^i\right)\frac{\partial L_0}{\partial \dot{x}^i} + D_{x_0^j}\left(\hat{\xi}^i\right)\frac{\partial L_0}{\partial x_{ij}}. \tag{G1}$$

The derivatives of L_0 in (G1) are given in Appendix C, namely

$$\frac{\partial L_0}{\partial X^i} = -\rho_0\frac{\partial \Phi}{\partial x^i}, \quad \frac{\partial L_0}{\partial \dot{X}^i} = \rho_0 u^i, \quad \frac{\partial L_0}{\partial X^{ij}} = (p + M_B)^{ik}A_{kj}. \tag{G2a–c}$$

Using (G2*a–c*) in (G1) gives

$$\begin{aligned} \mathrm{pr}\,\hat{X}(L_0) &= -\rho_0\boldsymbol{\xi}\boldsymbol{\cdot}\nabla\Phi + D_t\left(\hat{\boldsymbol{\xi}}\right)\boldsymbol{\cdot}\rho_0\boldsymbol{u} + D_{x_0^j}\left(\hat{\xi}^i\right)(p + M_B)^{ik}A_{kj} \\ &\equiv \rho J u^i\left[\frac{\mathrm{d}\hat{\xi}^i}{\mathrm{d}t} - \hat{\boldsymbol{\xi}}\boldsymbol{\cdot}\nabla u^i\right] + R_1, \end{aligned} \tag{G3}$$

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where

$$R_1 = -\rho_0 \hat{\boldsymbol{\xi}} \cdot \nabla \Phi + \rho_0 \hat{\boldsymbol{\xi}} \cdot \nabla \left(\frac{1}{2} u^2 \right) + D_{x_0^j} \left(\hat{\xi}^i \right) (p + M_B)^{ik} A_{kj}$$
$$\equiv J \left\{ \nabla \cdot \left(\rho \hat{\boldsymbol{\xi}} \right) \left(\Phi - \frac{1}{2} u^2 \right) - \nabla \cdot \left[\rho \hat{\boldsymbol{\xi}} \left(\Phi - \frac{1}{2} u^2 \right) \right] \right\} + D_{x_0^j} \left(\hat{\xi}^i \right) (p + M_B)^{ik} A_{kj}. \tag{G4}$$

By noting that

$$\nabla_k \left(p^{ik} \right) = - \left\{ \boldsymbol{B} \times (\nabla \times \boldsymbol{\Omega}) - \boldsymbol{\Omega} \nabla \cdot \boldsymbol{B} + \rho \left(T \nabla S - \nabla h \right) \right\}^i , \tag{G5}$$

and

$$\nabla_k \left(M_B^{ik} \right) = - \left\{ \boldsymbol{J} \times \boldsymbol{B} + \boldsymbol{B} \frac{\nabla \cdot \boldsymbol{B}}{\mu_0} \right\}^i . \tag{G6}$$

We obtain

$$A_{kj} D_{x_0^j} \left(\hat{\xi}^i \right) (p + M_B)^{ik} = J \left\{ D_{x^k} \left[\hat{\xi}^i (p + M_B)^{ik} \right] + \rho \hat{\boldsymbol{\xi}} \cdot (T \nabla S - \nabla h) \right.$$
$$\left. + \hat{\boldsymbol{\xi}} \cdot [\boldsymbol{B} \times (\nabla \times \boldsymbol{\Omega}) - \boldsymbol{\Omega} \nabla \cdot \boldsymbol{B}] + \hat{\boldsymbol{\xi}} \cdot \left[\boldsymbol{J} \times \boldsymbol{B} + \boldsymbol{B} \frac{\nabla \cdot \boldsymbol{B}}{\mu_0} \right] \right\} . \tag{G7}$$

Noting

$$\tilde{\boldsymbol{J}} = \boldsymbol{J} - \nabla \times \boldsymbol{\Omega} = \frac{\nabla \times \tilde{\boldsymbol{B}}}{\mu_0}, \quad \tilde{\boldsymbol{B}} = \boldsymbol{B} \left(1 - \frac{\mu_0 p_\Delta}{B^2} \right), \tag{G8a,b}$$

we obtain the equations

$$\hat{\boldsymbol{\xi}} \cdot [\boldsymbol{B} \times (\nabla \times \boldsymbol{\Omega}) - \boldsymbol{\Omega} \nabla \cdot \boldsymbol{B}] + \hat{\boldsymbol{\xi}} \cdot \left[\boldsymbol{J} \times \boldsymbol{B} + \boldsymbol{B} (\nabla \cdot \boldsymbol{B}) / \mu_0 \right]$$
$$= \tilde{\boldsymbol{J}} \cdot \left(\boldsymbol{B} \times \hat{\boldsymbol{\xi}} \right) + (\hat{\boldsymbol{\xi}} \cdot \tilde{\boldsymbol{B}}) \nabla \cdot \boldsymbol{B} / \mu_0$$
$$= \frac{1}{\mu_0} \left\{ \nabla \times \left[\left(\hat{\boldsymbol{\xi}} \times \boldsymbol{B} \right) \times \tilde{\boldsymbol{B}} \right] - \tilde{\boldsymbol{B}} \cdot \left[\nabla \times (\hat{\boldsymbol{\xi}} \times \boldsymbol{B}) - \hat{\boldsymbol{\xi}} \nabla \cdot \boldsymbol{B} \right] \right\} . \tag{G9}$$

In the derivation of (G9) we used the identity

$$\nabla \cdot (\boldsymbol{A} \times \boldsymbol{C}) = (\nabla \times \boldsymbol{A}) \cdot \boldsymbol{C} - (\nabla \times \boldsymbol{C}) \cdot \boldsymbol{A}, \tag{G10}$$

with $\boldsymbol{A} = \hat{\boldsymbol{\xi}} \times \boldsymbol{B}$ and $\boldsymbol{C} = \tilde{\boldsymbol{B}}$.

Using (G9) in (G7) gives

$$A_{kj} D_{x_0^j} \left(\hat{\xi}^i \right) \left(p^{ik} + M_B^{ik} \right) = J \left\{ D_{x^k} \left[\hat{\xi}^i \left(p^{ik} + M_B^{ik} \right) \right] + \rho \hat{\boldsymbol{\xi}} \cdot [T \nabla S - \nabla h] \right.$$
$$\left. + \nabla \cdot \left[\frac{(\hat{\boldsymbol{\xi}} \times \boldsymbol{B}) \times \tilde{\boldsymbol{B}}}{\mu_0} \right] - \frac{\tilde{\boldsymbol{B}}}{\mu_0} \cdot \left[\nabla \times (\hat{\boldsymbol{\xi}} \times \boldsymbol{B}) - \hat{\boldsymbol{\xi}} \nabla \cdot \boldsymbol{B} \right] \right\} . \tag{G11}$$

Using (G11) in (G4) gives a simplification of R_1, which can in turn be used to obtain $\mathrm{pr}\, \hat{X}(L_0)$ in (G1). Substitution of the resultant $\mathrm{pr}\, \hat{X}(L_0)$ in (6.5) gives (6.17) as the condition for Lie invariance of the action.

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Appendix H

In this appendix, we provide a derivation of Noether's first theorem, using the approach of Bluman & Kumei (1989). This analysis should be useful for readers not acquainted with the classical approach to Noether's first theorem. These ideas are used in § 6.2 to describe Noether's theorem.

Consider a system of differential equations in the dependent variables u^α $(1 \leqslant \alpha \leqslant m)$ and independent variables x^i $(1 \leqslant i \leqslant n)$ of the form

$$R^s(x_i, u^\alpha, u^\alpha_i, u^\alpha_{ij}, \ldots) = 0, \quad 1 \leqslant s \leqslant m, \tag{H1}$$

the subscripts in $u^\alpha_i, u^\alpha_{ij}, \ldots$, denote partial derivatives with the respect to the independent variables x^i $(1 \leqslant i \leqslant n)$, which arise from extremal variations of the action

$$J[u] = \int_\Omega L\left(\boldsymbol{x}, u^\alpha_i, u^\alpha_{ij}, \ldots\right) \mathrm{d}\boldsymbol{x}, \tag{H2}$$

which remain invariant under infinitesimal Lie transformations of the form

$$x'^i = x^i + \epsilon\xi^i, \quad u'^\alpha = u^\alpha + \epsilon\eta^\alpha, \quad L' = L + \nabla_i\Lambda^i. \tag{H3a–c}$$

The variation of $J[\boldsymbol{u}]$ is defined as

$$\delta J = \int_{\Omega'} L'\left(\boldsymbol{x}', \boldsymbol{u}', u'^\alpha_i, u'^\alpha_{ij}, \ldots\right) \mathrm{d}\boldsymbol{x}' - \int_\Omega L\left(\boldsymbol{x}, \boldsymbol{u}, u^\alpha_i, u^\alpha_{ij}, \ldots\right) \mathrm{d}\boldsymbol{x}, \tag{H4}$$

where Ω is the region of integration. The variation δJ to $O(\epsilon)$ in (H4) reduces to

$$\delta J = \epsilon \int_\Omega \left(\mathrm{pr}\, XL + LD_i\xi^i + D_i\Lambda^i\right) \mathrm{d}\boldsymbol{x} + O(\epsilon^2). \tag{H5}$$

The term $LD_i\xi^i$ in (H5) represents changes in the volume element:

$$\mathrm{d}\boldsymbol{x}' = \left[1 + \epsilon D_i\xi^i + O(\epsilon^2)\right] \mathrm{d}\boldsymbol{x}. \tag{H6}$$

The Lie derivative term:

$$\mathrm{pr}\, XL = \left(\xi^i \frac{\partial}{\partial x^i} + \eta^\alpha \frac{\partial}{\partial u^\alpha} + \eta^\alpha_i \frac{\partial}{\partial u^\alpha_i} + \eta^\alpha_{ij} \frac{\partial}{\partial u^\alpha_{ij}} + \cdots\right) L, \tag{H7}$$

describes the changes in $L(\boldsymbol{x}, u^\alpha, u^\alpha_i, u^\alpha_{ij}, \ldots)$ due to the Lie transformations of x^i and u^α in (H3a–c), in which the form of L does not change. The term $D_i\Lambda^i$ describes changes of J due to changes in the form of L due to a divergence transformation (under such a transformation the action remains invariant). From (H5) the action $J[\boldsymbol{u}]$ remains invariant under the Lie transformations (H3*a–c*) to $O(\epsilon)$ if

$$\mathrm{pr}\, XL + LD_i\xi^i + D_i\Lambda^i = 0. \tag{H8}$$

It turns out that there is an equivalent extended Lie symmetry operator, $\mathrm{pr}(\hat{X})$, of the form

$$\mathrm{pr}(\hat{X}) = \hat{\eta}^\alpha \frac{\partial}{\partial u^\alpha} + D_i\left(\hat{\eta}^\alpha\right)\frac{\partial}{\partial u_i^\alpha} + D_iD_j\left(\hat{\eta}^\alpha\right)\frac{\partial}{\partial u_{ij}^\alpha} + \cdots, \tag{H9}$$

called the evolutionary operator (e.g. Olver 1993) which describes Lie transformations:

$$x'^i = x^i, \quad u'^\alpha = u^\alpha + \epsilon\hat{\eta}^\alpha, \tag{H10a,b}$$

where

$$\hat{\eta}^\alpha = \eta^\alpha - \xi^j D_j u^\alpha. \tag{H11}$$

The operator $\mathrm{pr}(\hat{X})$ is related to $\mathrm{pr}(X)$ by the formula

$$\mathrm{pr}(X) = \mathrm{pr}(\hat{X}) + \xi^j D_j, \tag{H12}$$

where $D_j = \mathrm{d}/\mathrm{d}x^j$ is the total, partial derivative with respect to x^j. From (H9), we obtain the formulae

$$u'^\alpha_i = u_i^\alpha + \epsilon D_i\left(\hat{\eta}^\alpha\right), \quad u'^\alpha_{ij} = u_{ij}^\alpha + \epsilon D_iD_j\left(\hat{\eta}^\alpha\right), \tag{H13a,b}$$

for the transformation of partial derivatives under $\mathrm{pr}(\hat{X})$. These transformations are different from the transformations of derivatives formulae under the canonical prolonged symmetry operator $\mathrm{pr}\, X$, namely

$$\eta_i^\alpha \equiv \mathrm{pr}(X)u_i^\alpha = \mathrm{pr}\, \hat{X}u_i^\alpha + \xi^j D_j u_i^\alpha = D_i\left(\hat{\eta}^\alpha\right) + \xi^j u_{ji}^\alpha, \tag{H14}$$

and similarly for transformations of the higher-order derivatives of u^α.

Evaluation of $\delta J[\boldsymbol{u}]$ using $\mathrm{pr}(\hat{X})$ gives the variational equation:

$$\delta J = \epsilon \int \left(\mathrm{pr}\, \hat{X}L\right)\mathrm{d}\boldsymbol{x} = \epsilon \int \left[D_i W^i[\boldsymbol{u}, \hat{\boldsymbol{\eta}}] + \hat{\eta}^\gamma E_\gamma(L)\right]\mathrm{d}\boldsymbol{x}, \tag{H15}$$

from which it follows that

$$\mathrm{pr}(\hat{X})L = D_i W^i[\boldsymbol{u}, \hat{\boldsymbol{\eta}}] + \hat{\eta}^\gamma E_\gamma(L), \tag{H16}$$

where the $W^i[\boldsymbol{u}, \hat{\boldsymbol{\eta}}]$ are surface terms given by

$$W^i[\boldsymbol{u}, \hat{\boldsymbol{\eta}}] = \hat{\eta}^\gamma \frac{\delta L}{\delta u^\gamma} + \hat{\eta}_j^\gamma \frac{\delta L}{\delta u_{ji}^\gamma} + \hat{\eta}_{jk}^\gamma \frac{\delta L}{u_{jki}^\gamma} + \cdots, \tag{H17}$$

and $\delta L/\delta\psi$ is given by

$$\frac{\delta L}{\delta\psi} \equiv E_\psi(L) = \frac{\partial L}{\partial\psi} - D_i\left(\frac{\partial L}{\partial\psi_i}\right) + D_iD_j\left(\frac{\partial L}{\partial\psi_{ij}}\right) - \cdots. \tag{H18}$$

Here E_ψ denotes the Euler operator or variational derivative with respect to ψ used in the Calculus of variations. In particular, the equations

$$E_\gamma(L) = \frac{\partial L}{\partial u^\gamma} - D_i\left(\frac{\partial L}{\partial u_i^\gamma}\right) + D_iD_j\left(\frac{\partial L}{\partial u_{ij}^\gamma}\right) - \cdots = 0, \tag{H19}$$

are the Euler–Lagrange equations for the variational principle $\delta J = 0$.

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Using pr$(X)L$ from (H12), the Lie invariance condition (H8) for the action reduces to

$$\mathrm{pr}(\hat{X})L + D_i\left(L\xi^i + \Lambda^i\right) = 0. \tag{H20}$$

Using (H16) for pr$(\hat{X})L$ in (H20) we obtain the Noether theorem identity:

$$\hat{\eta}^\gamma E_\gamma(L) + D_i\left[W^i[\boldsymbol{u}, \hat{\boldsymbol{\eta}}] + L\xi^i + \Lambda^i\right] = 0. \tag{H21}$$

For the case of a finite point Lie group of symmetries (i.e. for a finite number of point symmetries $\hat{\eta}^\gamma$), for which the Euler–Lagrange equations $E_\gamma(L) = 0$ are satisfied, and (H21) reduces to the conservation law

$$D_i\left[W^i[\boldsymbol{u}, \hat{\boldsymbol{\eta}}] + L\xi^i + \Lambda^i\right] = 0, \tag{H22}$$

associated with Noether's first theorem.

If the symmetries $\hat{\eta}^\gamma$ depend on continuous functions $\{\phi^k(\boldsymbol{x} : 1 \leqslant k \leqslant N\}$, then the Lie pseudo-algebra of symmetries is infinite dimensional. In this case Noether's second theorem implies that the Euler–Lagrange equations are not all independent and that there exists differential relations between the Euler–Lagrange equations (see, e.g., Olver (1993) and Hydon & Mansfield (2011) for details).

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