Renormalized Oscillation Theory for Linear Hamiltonian Systems on $[0, 1]$ via the Maslov Index

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Abstract

Working with a general class of regular linear Hamiltonian systems, we show that renormalized oscillation results can be obtained in a natural way through consideration of the Maslov index associated with appropriately chosen paths of Lagrangian subspaces of $\mathbb{C}^{2n}$. We verify that our applicability class includes Dirac and Sturm-Liouville systems, as well as a system arising from differential-algebraic equations for which the spectral parameter appears nonlinearly.

1 Introduction

For values λ in some interval $I \subset \mathbb{R}$, we consider linear Hamiltonian systems

$$Jy' = \mathbb{B}(x; \lambda)y; \quad x \in (0, 1), \quad y(x; \lambda) \in \mathbb{C}^{2n}, \quad n \in \{1, 2, \dots\}, \tag{1.1}$$

where J denotes the standard symplectic matrix

$$J = \begin{pmatrix} 0_n & -I_n \\ I_n & 0_n \end{pmatrix},$$

and an important feature of the analysis is that $\mathbb{B}(x; \lambda)$ is allowed to depend nonlinearly on the spectral parameter λ. We assume throughout that for each $\lambda \in I$, $\mathbb{B}(x; \lambda)$ is a measurable matrix-valued function of x, self-adjoint for a.e. $x \in (0, 1)$, for which there exists a dominating function $b_0 \in L^1((0, 1), \mathbb{R})$ so that for each $\lambda \in I$, $\|\mathbb{B}(x; \lambda)\| \leq b_0(x)$ for a.e. $x \in (0, 1)$. Moreover, we assume $\mathbb{B}$ is differentiable in λ, and that there exists a dominating function $b_1 \in L^1((0, 1), \mathbb{R})$ so that for each $\lambda \in I$, $\|\mathbb{B}_\lambda(x; \lambda)\| \leq b_1(x)$ for a.e. $x \in (0, 1)$. (Here, $\| \cdot \|$ denotes any matrix norm.) For convenient reference, we refer to these basic assumptions as Assumptions **(A)**.

We consider two types of self-adjoint boundary conditions, *separated* and *generalized.*

(BC1). We express separated self-adjoint boundary conditions as

$$\alpha y(0; \lambda) = 0; \quad \beta y(1; \lambda) = 0,$$

where we assume

$$\alpha \in \mathbb{C}^{n\times 2n}, \quad \operatorname{rank} \alpha = n, \quad \alpha J \alpha^* = 0;$$
$$\beta \in \mathbb{C}^{n\times 2n}, \quad \operatorname{rank} \beta = n, \quad \beta J \beta^* = 0.$$

(BC2). We express general self-adjoint boundary conditions as

$$\Theta \begin{pmatrix} y(0;\lambda) \\ y(1;\lambda) \end{pmatrix} = 0; \quad \Theta \in \mathbb{C}^{2n\times 4n}, \quad \operatorname{rank} \Theta = 2n, \quad \Theta \mathcal{J}_{4n} \Theta^* = 0,$$

where

$$\mathcal{J}_{4n} := \begin{pmatrix} -J & 0 \\ 0 & J \end{pmatrix}.$$

Here, and throughout, we use the superscript $*$ to denote adjoint. We emphasize that the matrices α and β in **(BC1)** are taken independent of λ, as is the matrix Θ in **(BC2)**. These restrictions on α, β, and Θ are not necessary for the approach, and are assumed rather to streamline the statements of our main results.

We will refer to a value $\lambda \in I$ as an eigenvalue of (1.1) if there exists a function $y(\cdot;\lambda) \in \mathrm{AC}([0,1], \mathbb{C}^{2n}) \backslash \{0\}$ that solves (1.1) along with prescribed boundary conditions of the form **(BC1)** or **(BC2)**. (Here, $AC(\cdot)$ denotes absolute continuity.) Given any pair of values $\lambda_1, \lambda_2 \in I$, $\lambda_1 < \lambda_2$, our goal is to obtain a count $\mathcal{N}([\lambda_1, \lambda_2))$, including multiplicity, of the number of eigenvalues that (1.1) has on the interval $[\lambda_1, \lambda_2)$. (The choice of a closed endpoint on the left and an open endpoint on the right is taken by convention associated with the general definition that the Morse index of an operator corresponds with a count of the number of eigenvalues the operator has strictly below 0.) Our tool for this analysis will be the Maslov index, and as a starting point for a discussion of this object, we define what we will mean by a *Lagrangian subspace* of $\mathbb{C}^{2n}$.

Definition 1.1. *We say $\ell \subset \mathbb{C}^{2n}$ is a Lagrangian subspace of $\mathbb{C}^{2n}$ if ℓ has dimension n and*

$$(Ju, v) = 0, \tag{1.2}$$

for all $u, v \in \ell$. (Here, $(\cdot,\cdot)$ denotes the standard inner product on $\mathbb{C}^{2n}$.) In addition, we denote by $\Lambda(n)$ the collection of all Lagrangian subspaces of $\mathbb{C}^{2n}$, and we will refer to this as the Lagrangian Grassmannian.

Remark 1.1. *Following the convention of Arnol'd's foundational paper [2], the notation $\Lambda(n)$ is often used to denote the Lagrangian Grassmanian associated with $\mathbb{R}^{2n}$. Our expectation is that it can be used in the current setting of $\mathbb{C}^{2n}$ without confusion. We note that the Lagrangian Grassmannian associated with $\mathbb{C}^{2n}$ has been considered by a number of authors, including (ordered by publication date) Bott [7], Kostrykin and Schrader [23], Arnol'd [3], and Schulz-Baldes [29, 30]. It is shown in all of these references that $\Lambda(n)$ is homeomorphic to the set of $n \times n$ unitary matrices $U(n)$, and in [29, 30] the relationship is shown to be diffeomorphic. It is also shown in [29] that the fundamental group of $\Lambda(n)$ is the integers $\mathbb{Z}$.*

Any Lagrangian subspace of $\mathbb{C}^{2n}$ can be spanned by a choice of n linearly independent vectors in $\mathbb{C}^{2n}$. We will generally find it convenient to collect these n vectors as the columns of a $2n \times n$ matrix $\mathbf{X}$, which we will refer to as a *frame* for ℓ. Moreover, we will often coordinatize our frames as $\mathbf{X} = \binom{X}{Y}$, where X and Y are $n \times n$ matrices. Following [10] (p. 274), we specify a metric on $\Lambda(n)$ in terms of appropriate orthogonal projections. Precisely, let $\mathcal{P}_i$ denote the orthogonal projection matrix onto $\ell_i \in \Lambda(n)$ for $i = 1, 2$. I.e., if $\mathbf{X}_i$ denotes a frame for ℓ_i, then $\mathcal{P}_i = \mathbf{X}_i(\mathbf{X}_i^* \mathbf{X}_i)^{-1}\mathbf{X}_i^*$. We take our metric d on $\Lambda(n)$ to be defined by

$$d(\ell_1, \ell_2) := \|\mathcal{P}_1 - \mathcal{P}_2\|,$$

where $\|\cdot\|$ can denote any matrix norm. We will say that a path of Lagrangian subspaces $\ell : \mathcal{I} \to \Lambda(n)$ is continuous provided it is continuous under the metric d.

Suppose $\ell_1(\cdot), \ell_2(\cdot)$ denote continuous paths of Lagrangian subspaces $\ell_i : \mathcal{I} \to \Lambda(n)$, for some parameter interval $\mathcal{I}$. The Maslov index associated with these paths, which we will denote $\operatorname{Mas}(\ell_1, \ell_2; \mathcal{I})$, is a count of the number of times the paths $\ell_1(\cdot)$ and $\ell_2(\cdot)$ intersect, counted with both multiplicity and direction. (In this setting, if we let t_* denote the point of intersection (often referred to as a *crossing point*), then multiplicity corresponds with the dimension of the intersection $\ell_1(t_*) \cap \ell_2(t_*)$; a precise definition of what we mean in this context by *direction* will be given in Section 2.)

The key ingredient we will need for connecting Maslov index calculations with renormalized oscillation results is monotonicity. We say that the evolution of $\mathcal{L} = (\ell_1, \ell_2)$ is *monotonic* provided all intersections occur with the same direction. If the intersections all correspond with the positive direction, and if the crossing points are all discrete, then we can compute

$$\operatorname{Mas}(\ell_1, \ell_2; \mathcal{I}) = \sum_{t \in \mathcal{I}} \dim(\ell_1(t) \cap \ell_2(t)).$$

Suppose $\mathbf{X}_1(t) = \binom{X_1(t)}{Y_1(t)}$ and $\mathbf{X}_2(t) = \binom{X_2(t)}{Y_2(t)}$ respectively denote frames for Lagrangian subspaces of $\mathbb{C}^{2n}$, $\ell_1(t)$ and $\ell_2(t)$. Then we can express this last relation as

$$\operatorname{Mas}(\ell_1, \ell_2; \mathcal{I}) = \sum_{t \in \mathcal{I}} \dim \ker(\mathbf{X}_1(t)^* J \mathbf{X}_2(t)).$$

(See Lemma 2.2 below.) The right-hand side of this final expression, expressed in terms of the matrix Wronskian $W(t) = \mathbf{X}_1(t)^* J \mathbf{X}_2(t)$, has the form we associate with renormalized oscillation theory (see, e.g., [13]), and we will sometimes adopt the notation of [13] and use the *counting function*

$$N_{\mathcal{I}}(\mathbf{X}_1(\cdot)^* J \mathbf{X}_2(\cdot)) := \sum_{t \in \mathcal{I}} \dim \ker(\mathbf{X}_1(t)^* J \mathbf{X}_2(t)). \tag{1.3}$$

Remark 1.2. *Renormalized oscillation theory was introduced in [12] in the context of single Sturm-Liouville equations, and subsequently was developed in [31, 32] for Jacobi operators and Dirac operators. More recently, Gesztesy and Zinchenko have extended these early results to the setting of (1.1) with $\mathbb{B}(x; \lambda) = \lambda A(x) + B(x)$ (with suitable assumptions on A and B) and for three classes of domain: bounded, half-line, and $\mathbb{R}$ (see [13]). This last reference served as the direct motivation for our analysis.*

In order to formulate our theorem regarding **(BC1)**, we fix any pair $\lambda_1, \lambda_2 \in I$, with $\lambda_1 < \lambda_2$, and let $\mathbf{X}_1(x; \lambda)$ denote a $2n \times n$ matrix solving

$$\begin{aligned} J\mathbf{X}_1' &= \mathbb{B}(x; \lambda)\mathbf{X}_1 \\ \mathbf{X}_1(0; \lambda) &= J\alpha^*. \end{aligned} \tag{1.4}$$

Under our assumptions **(A)** on $\mathbb{B}(x;\lambda)$, we can conclude that for each $\lambda \in I$, $\mathbf{X}_1(\cdot;\lambda) \in AC([0,1],\mathbb{C}^{2n\times n})$, and additionally that $\mathbf{X}_1(x;\lambda)$ is differentiable in λ with $\partial_\lambda \mathbf{X}_1(\cdot;\lambda) \in AC([0,1],\mathbb{C}^{2n\times n})$ and

$$J(\partial_\lambda \mathbf{X}_1(x;\lambda))' = \mathbb{B}_\lambda(x;\lambda)\mathbf{X}_1(\cdot;\lambda) + \mathbb{B}(x;\lambda)\partial_\lambda \mathbf{X}_1(x;\lambda), \tag{1.5}$$

for a.e. $x \in (0,1)$.

Remark 1.3. *Regarding (1.5), the observation is simply that we can justify switching the order of differentiation of $\mathbf{X}_1(x;\lambda)$ with respect to x and λ. Our assumptions allow us to do this by integrating (1.4) to*

$$J\mathbf{X}_1(x;\lambda) = -\alpha^* + \int_0^x \mathbb{B}(\xi;\lambda)\mathbf{X}_1(\xi;\lambda)d\xi,$$

and then differentiating through the integral in λ, followed by differentiation in x. These are straightforward calculations that follow the approach of Chapter 2 in [33].

As shown in [14], for each pair $(x,\lambda) \in [0,1] \times I$, $\mathbf{X}_1(x;\lambda)$ is the frame for a Lagrangian subspace $\ell_1(x;\lambda)$. (In [14], the authors make slightly stronger assumptions on $\mathbb{B}(x;\lambda)$, but their proof carries over immediately into our setting.) Likewise, keeping in mind that λ_2 is fixed, we let $\ell_2(x;\lambda_2)$ denote the map of Lagrangian subspaces associated with frames $\mathbf{X}_2(x;\lambda_2)$ solving

$$\begin{aligned} J\mathbf{X}_2' &= \mathbb{B}(x;\lambda_2)\mathbf{X}_2 \\ \mathbf{X}_2(1;\lambda_2) &= J\beta^*. \end{aligned} \tag{1.6}$$

We emphasize that $\mathbf{X}_2(x;\lambda_2)$ is initialized at $x = 1$.

In addition to Assumptions **(A)**, we make the following *positivity* assumptions:

(B1) For any $\lambda \in I$, the matrix

$$\int_0^1 \mathbf{X}_1(x;\lambda)^* \mathbb{B}_\lambda(x;\lambda)\mathbf{X}_1(x;\lambda)dx \tag{1.7}$$

is positive definite.

(B2) For the fixed values $\lambda_1, \lambda_2 \in I$, $\lambda_1 < \lambda_2$, the matrix $(\mathbb{B}(x;\lambda_2) - \mathbb{B}(x;\lambda_1))$ is non-negative for a.e. $x \in (0,1)$, and moreover there is no interval $[a,b] \subset [0,1]$, $a < b$, so that

$$\dim(\ell_1(x;\lambda_1) \cap \ell_2(x;\lambda_2)) \neq 0$$

for all $x \in [a,b]$.

Remark 1.4. *Assumption* **(B1)** *is standard for ensuring that as λ varies, with x fixed, crossings of $\ell_1(x;\lambda)$ and $\ell_2(x;\lambda_2)$ will all occur in the same direction. The frame $\mathbf{X}(x;\lambda_2)$ does not appear in the assumption, because it does not vary with λ. For Assumption* **(B2)***, the a.e. non-negativity of $(\mathbb{B}(x;\lambda_2) - \mathbb{B}(x;\lambda_1))$ ensures that as x varies, crossings of $\ell_1(x;\lambda_1)$ and $\ell_2(x;\lambda_2)$ all occur in the same direction, while the moreover part ensures that $\ell_1(x;\lambda_1)$ and $\ell_2(x;\lambda_2)$ cannnot get stuck at an intersection.*

We will verify in Section 4 that the moreover part of **(B2)** *is implied by the following form of Atkinson positivity: for any $[a, b] \subset [0, 1]$, $a < b$, and any non-trivial solution $y(\cdot; \lambda_1) \in \mathrm{AC}([0, 1], \mathbb{C}^{2n})$ of $Jy' = \mathbb{B}(x; \lambda_1)y$, we must have*

$$\int_a^b ((\mathbb{B}(x; \lambda_2) - \mathbb{B}(x; \lambda_1))y(x; \lambda_1), y(x; \lambda_1))dx > 0. \tag{1.8}$$

In the case that $\mathbb{B}(x; \lambda)$ is linear in λ (as in Remark 1.2), non-negativity of $(\mathbb{B}(x; \lambda_2) - \mathbb{B}(x; \lambda_1))$ corresponds with non-negativity of the matrix A, and the integral conditions (1.7) and (1.8) are both equivalent to Atkinson positivity (see, e.g., Section 4 in [25] and Section IV.4 in [24]). For a development of renormalized oscillation theory under fewer such restrictions, we refer the reader to [9] and the references therein.

Suppose that for some value $\lambda \in I$ equation (1.1) with specified boundary conditions admits one or more linearly independent solutions. We denote the subspace spanned by these solutions by $\mathbb{E}(\lambda)$, noting that $\dim \mathbb{E}(\lambda) \leq 2n$. Given any two values $\lambda_1, \lambda_2 \in I$, with $\lambda_1 < \lambda_2$, it is shown in [14] that under positivity assumptions **(B1)** the *spectral count*

$$\mathcal{N}([\lambda_1, \lambda_2)) := \sum_{\lambda \in [\lambda_1, \lambda_2)} \dim \mathbb{E}(\lambda), \tag{1.9}$$

is well-defined. It's clear that $\mathcal{N}([\lambda_1, \lambda_2))$ is a count of the eigenvalues of (1.1) on $[\lambda_1, \lambda_2)$, counted with geometric multiplicity. In order to understand the nature of algebraic multiplicity in this setting, as well as the notion of essential spectrum, it's useful to frame our discussion in terms of the operator pencil

$$\mathcal{L}(\lambda) = J\frac{d}{dx} - \mathbb{B}(x; \lambda),$$

specified on the domain (independent of λ)

$$\begin{aligned} \mathcal{D} := \{y \in L^2((0, 1), \mathbb{C}^{2n}) : y \in \mathrm{AC}([0, 1], \mathbb{C}^{2n}), \\ \mathcal{L}y \in L^2((0, 1), \mathbb{C}^{2n}),\ \alpha y(0) = 0,\ \beta y(1) = 0\} \end{aligned}$$

(for boundary conditions **(BC1)**, and with a similar specification for boundary conditions **(BC2)**). Using the methods of [33], we can readily verify that for each $\lambda \in I$, $\mathcal{L}(\lambda)$ is Fredholm and self-adjoint on $\mathcal{D}$, from which we can conclude that $\mathcal{L}$ has no essential spectrum on I. Moreover, under slightly stronger assumptions on $\mathbb{B}$ (in particular, $\mathbb{B}(\cdot; \lambda) \in L^2((0, 1), \mathbb{C}^{2n \times 2n})$ for all $\lambda \in I$), we can verify that $\mathcal{L}$ has no Jordan chains of length greater than one, implying that the algebraic and geometric multiplicities of its eigenvalues agree. (See the appendix for further discussion, and also Section 1.2 of [16], in which the authors consider the same operator pencil under slightly stronger assumptions on $\mathbb{B}(x; \lambda)$.)

We will establish the following theorem.

Theorem 1.1. *Fix $\lambda_1, \lambda_2 \in I$, $\lambda_1 < \lambda_2$. For equation (1.1), let Assumptions* **(A)** *and* **(B1)** *hold, and let $\mathbf{X}_1$ and $\mathbf{X}_2$ respectively denote the Lagrangian frames specified in (1.4) and (1.6). If $\mathcal{N}([\lambda_1, \lambda_2))$ denotes the spectral count for (1.1) with boundary conditions* **(BC1)***, then*

$$\mathcal{N}([\lambda_1, \lambda_2)) = \mathrm{Mas}(\ell_1(\cdot; \lambda_1), \ell_2(\cdot; \lambda_2); [0, 1]). \tag{1.10}$$

Moreover, under the additional assumption **(B2)**, *with* λ_1 *and* λ_2 *as in* **(B2)**, *we have*

$$\mathcal{N}([\lambda_1,\lambda_2)) = \sum_{x\in(0,1]} \dim\ker(\mathbf{X}_1(x;\lambda_1)^* J \mathbf{X}_2(x;\lambda_2)). \tag{1.11}$$

We note that the final relation in Theorem 1.1 could be stated with $[\lambda_1,\lambda_2)$ and $(0,1]$ replaced respectively with any of the following three combinations: (λ_1,λ_2) and $(0,1)$, $[\lambda_1,\lambda_2]$ and $[0,1]$, or $(\lambda_1,\lambda_2]$ and $[0,1)$. For the correspondence between $\lambda=\lambda_2$ and $x=0$ (i.e., the fact that the associated delimiters are either both round or both square), we observe that λ_2 is an eigenvalue of (1.1) if and only if $\ell_1(0;\lambda_2)$ and $\ell_2(0;\lambda_2)$ intersect, and since $\ell_1(0;\lambda_1)=\ell_1(0;\lambda_2)$, we can conclude that λ_2 is an eigenvalue of (1.1) if and only if $\ell_1(0;\lambda_1)$ and $\ell_2(0;\lambda_2)$ intersect. This final intersection is precisely the quantity detected at $x=0$ in the sum. Likewise, for the correspondence between $\lambda=\lambda_1$ and $x=1$, λ_1 is an eigenvalue of (1.1) if and only if $\ell_1(1;\lambda_1)$ and $\ell_2(1;\lambda_1)$ intersect, and $\ell_2(1;\lambda_1)=\ell_2(1;\lambda_2)$. In addition, we observe that by exchanging the pair $(\ell_1(x;\lambda_1),\ell_2(x;\lambda_2))$ in our analysis with the pair $(\ell_1(x;\lambda_2),\ell_2(x;\lambda_1))$, we arrive at the alternative formulation

$$\mathcal{N}([\lambda_1,\lambda_2)) = -\operatorname{Mas}(\ell_1(\cdot;\lambda_2),\ell_2(\cdot;\lambda_1);[0,1]),$$

which, under Assumption **(B2)**, implies

$$\mathcal{N}([\lambda_1,\lambda_2)) = \sum_{x\in[0,1)} \dim\ker(\mathbf{X}_1(x;\lambda_2)^* J \mathbf{X}_2(x;\lambda_1)). \tag{1.12}$$

In particular, as opposed to (1.11), the sum on the right-hand side of (1.12) includes $x=0$ and excludes $x=1$. Moreover, in this case, the pairing of $[\lambda_1,\lambda_2)$ with $[0,1)$ can be replaced by any other pairing in which the delimiters of λ_1 and 0 agree, and the delimiters of λ_2 and 1 agree.

Our Theorem 1.1 is quite similar to Theorem 3.10 of [13], though we observe the following differences: in [13], the authors use Theorem 3.10 as a statement about both the case of bounded intervals and the case of half-lines, and we are comparing with the bounded-interval statement. Keeping this in mind, our theorem is slightly less restrictive, in that (1) it allows for a more general class of matrices $\mathbb{B}(x;\lambda)$; and (2) it allows for the possibility that λ_1 and/or λ_2 is an eigenvalue of (1.1). Regarding Item (1), our Theorem 1.1 allows for $\mathbb{B}(x;\lambda)$ to depend nonlinearly on λ. This happens, for example, when (1.1) arises from consideration of certain differential-algebraic systems; in Section 5 we give one such example. Having drawn this comparison, we should emphasize that the primary goal of [13] (and indeed the primary motivation for the original work of [12]) was to develop a theory that would allow the authors to count discrete eigenvalues between bands of essential spectrum. We do not consider that important case here.

Remark 1.5. *It's instructive to view Theorem 1.1 in relation to the results of [14, 17], for which the authors specify* $\mathbf{X}_1(x;\lambda)$ *precisely as here, but in lieu of* $\mathbf{X}_2(x;\lambda_2)$, *use the fixed frame* $\tilde{\mathbf{X}}_2 = J\beta^*$ *for a target space* $\tilde{\ell}_2$. *Under Assumptions* **(A)** *and* **(B1)**, *and assuming boundary conditions* **(BC1)**, *the methods of [14, 17] can be used to establish the relation*

$$\mathcal{N}([\lambda_1,\lambda_2)) = -\operatorname{Mas}(\ell_1(\cdot;\lambda_1),\tilde{\ell}_2;[0,1]) + \operatorname{Mas}(\ell_1(\cdot;\lambda_2),\tilde{\ell}_2;[0,1]). \tag{1.13}$$

Critically, however, in the setting of [14, 17], intersections between $\ell_1(\cdot;\lambda_i)$ $(i=1,2)$ and $\tilde{\ell}_2$ are not necessarily monotone, and so (1.13) cannot generally be formulated as a simple count of nullities as in (1.11) of Theorem 1.1. One way in which this difference in approaches manifests is in the nature of spectral curves, by which we mean continuous paths of crossing-point pairs (x,λ) (assuming for simplicity of the discussion that these curves are non-intersecting). In the setting of [14, 17], such curves can reverse direction as depicted on the left-hand sides of Figures 2, 3, and 5 in Section 5, but in the current setting these spectral curves are necessarily monotone, as depicted on the right-hand sides of the same figures. This dynamic can be viewed as a graphical interpretation of why renormalized oscillation theory works in the elegant way that it does.

For the case of (1.1) with boundary conditions **(BC2)**, we follow [14] and begin by defining a Lagrangian subspace in terms of the "trace" operator

$$\mathcal{T}_x y := \mathcal{M}\begin{pmatrix} y(0) \\ y(x) \end{pmatrix}, \tag{1.14}$$

where

$$\mathcal{M} = \begin{pmatrix} I_n & 0 & 0 & 0 \\ 0 & 0 & I_n & 0 \\ 0 & -I_n & 0 & 0 \\ 0 & 0 & 0 & I_n \end{pmatrix}.$$

In [14], the authors verify that the subspace

$$\ell_3(x;\lambda) := \{\mathcal{T}_x y : y(\cdot;\lambda) \in \mathrm{AC}([0,1],\mathbb{C}^{2n}),\ Jy' = \mathbb{B}(x;\lambda)y \text{ a.e } x \in (0,1)\} \tag{1.15}$$

is a Lagrangian subspace of $\mathbb{C}^{2n}$ for all $(x,\lambda) \in [0,1]\times I$.

In order to establish notation for the statement of our second theorem, we let $\Phi(x;\lambda)$ denote the $2n\times 2n$ fundamental matrix solution to

$$J\Phi' = \mathbb{B}(x;\lambda)\Phi; \quad \Phi(0;\lambda) = I_{2n}, \tag{1.16}$$

and write

$$\Phi(x;\lambda) = \begin{pmatrix} \Phi_{11}(x;\lambda) & \Phi_{12}(x;\lambda) \\ \Phi_{21}(x;\lambda) & \Phi_{22}(x;\lambda) \end{pmatrix}.$$

With this notation, we can express the frame for $\ell_3(x;\lambda)$ as

$$\mathbf{X}_3(x,\lambda) = \begin{pmatrix} X_3(x,\lambda) \\ Y_3(x,\lambda) \end{pmatrix} = \begin{pmatrix} I_n & 0 \\ \Phi_{11}(x;\lambda) & \Phi_{12}(x;\lambda) \\ 0 & -I_n \\ \Phi_{21}(x;\lambda) & \Phi_{22}(x;\lambda) \end{pmatrix}. \tag{1.17}$$

We see by direct calculation that $\mathbf{X}_3(x;\lambda)$ can be interpreted as the frame associated with a linear Hamiltonian system

$$J_{4n}\mathbf{X}_3' = \mathcal{B}(x;\lambda)\mathbf{X}_3,$$

where

$$\mathcal{B}(x;\lambda) = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & B_{11}(x;\lambda) & 0 & B_{12}(x;\lambda) \\ 0 & 0 & 0 & 0 \\ 0 & B_{21}(x;\lambda) & 0 & B_{22}(x;\lambda) \end{pmatrix}; \quad \text{using } \mathbb{B} = \begin{pmatrix} B_{11} & B_{12} \\ B_{21} & B_{22} \end{pmatrix}, \tag{1.18}$$

and the flow is initialized by

$$\mathbf{X}_3(0;\lambda) = \begin{pmatrix} I_n & 0 \\ I_n & 0 \\ 0 & -I_n \\ 0 & I_n \end{pmatrix}.$$

Here, we use the notation J_{4n} to designate the matrix J with each I_n replaced by I_{2n}.

For our second path of Lagrangian subspaces, we let $\mathbf{X}_4(x;\lambda_2)$ solve

$$\begin{aligned} J_{4n}\mathbf{X}_4' &= \mathcal{B}(x;\lambda_2)\mathbf{X}_4 \\ \mathbf{X}_4(1;\lambda_2) &= \mathcal{M}\mathcal{J}_{4n}\Theta^*. \end{aligned} \tag{1.19}$$

In [14], the authors verify that $\mathcal{M}\mathcal{J}_{4n}\Theta^*$ is the frame for a Lagrangian subspace of $\mathbb{C}^{4n}$, and that intersections of $\ell_3(1;\lambda)$ with this Lagrangian subspace correspond with eigenvalues of (1.1)-**(BC2)**. (Again, strictly speaking, the authors of [14] are working with Lagrangian subspaces of $\mathbb{R}^{4n}$.)

In this case, we make the following positivity assumptions:

(B1)$'$ For any $\lambda \in I$, the matrix

$$\int_0^1 \Phi(x;\lambda)^* \mathbb{B}_\lambda(x;\lambda)\Phi(x;\lambda)dx \tag{1.20}$$

is positive definite.

(B2)$'$ For the fixed values $\lambda_1, \lambda_2 \in I$, $\lambda_1 < \lambda_2$, the matrix $(\mathbb{B}(x;\lambda_2) - \mathbb{B}(x;\lambda_1))$ is non-negative for a.e. $x \in (0,1)$, and moreover there is no interval $[a,b] \subset [0,1]$, $a < b$, so that

$$\dim(\ell_3(x;\lambda_1) \cap \ell_4(x;\lambda_2)) \neq 0$$

for all $x \in [a,b]$.

We are now in a position to state our second theorem.

Theorem 1.2. *Fix $\lambda_1, \lambda_2 \in I$, $\lambda_1 < \lambda_2$. For equation (1.1), let Assumptions* **(A)** *and* **(B1)**$'$ *hold, and let $\mathbf{X}_3$ and $\mathbf{X}_4$ respectively denote the Lagrangian frames specified in (1.17) and (1.19). If $\mathcal{N}([\lambda_1,\lambda_2))$ denotes the spectral count for (1.1) with boundary conditions* **(BC2)**, *then*

$$\mathcal{N}([\lambda_1,\lambda_2)) = \text{Mas}(\ell_3(\cdot;\lambda_1), \ell_4(\cdot;\lambda_2); [0,1]). \tag{1.21}$$

Moreover, under the additional assumption **(B2)**$'$, *with λ_1 and λ_2 as in* **(B2)**$'$, *we have,*

$$\mathcal{N}([\lambda_1,\lambda_2)) = \sum_{x\in(0,1]} \dim\ker(\mathbf{X}_3(x;\lambda_1)^* J\mathbf{X}_4(x;\lambda_2)). \tag{1.22}$$

Remark 1.6. *The considerations discussed in Remark 1.4 carry over to the setting of Theorem 1.2, and in particular, we will verify in Section 4 that the moreover part of* **(B2)**$'$ *is implied by (1.8) precisely as previously stated (i.e., it's not necessary to replace* $\mathbb{B}$ *with* $\mathcal{B}$*).*

Although the proof of Theorem 1.2 is essentially identical to the proof of Theorem 1.1 (at least in our development), we are not aware of a statement along the lines of Theorem 1.2 in the literature.

The remainder of the paper is organized as follows. In Section 2, we develop the Maslov index framework in $\mathbb{C}^{2n}$, and in Section 3 we develop the tools we will need to verify the monotonicity that will be necessary to conclude the second statements in Theorems 1.1 and 1.2. In Section 4, we prove Theorems 1.1 and 1.2, and in Section 5 we conclude by verifying that our assumptions hold for five example cases: Dirac systems, Sturm-Liouville systems, the class of systems analyzed in [13], a system associated with differential-algebraic Sturm-Liouville systems (for which $\mathbb{B}(x;\lambda)$ depends nonlinearly on λ), and a self-adjoint fourth-order equation for which we take periodic boundary conditions. In a short appendix, we discuss the interpretation of $\mathcal{L}(\lambda)$ as an operator pencil.

2 The Maslov Index on $\mathbb{C}^{2n}$

In this section, we verify that the framework developed in [15] for computing the Maslov index for Lagrangian pairs in $\mathbb{R}^{2n}$ extends to the case of Lagrangian pairs in $\mathbb{C}^{2n}$. A similar framework has been developed in [30], and in particular, some of the results in this section correspond with results in Section 2 of that reference. We include details here (1) for completeness; and (2) because we need some additional information that is not developed in [30].

As a starting point, we note the following direct relation between Lagrangian subspaces on $\mathbb{C}^{2n}$ and their associated frames.

Proposition 2.1. *A* $2n \times n$ *matrix* $\mathbf{X} = \binom{X}{Y}$ *is a frame for a Lagrangian subspace of* $\mathbb{C}^{2n}$ *if and only if the columns of* $\mathbf{X}$ *are linearly independent, and additionally*

$$X^*Y - Y^*X = 0.$$

We refer to this relation as the Lagrangian property for frames.

Remark 2.1. *The straightforward proof of Proposition 2.1 is essentially the same as for the case of* $\mathbb{R}^{2n}$ *(see Proposition 2.1 in [15]). We note that the Lagrangian property can also be expressed as* $\mathbf{X}^*J\mathbf{X} = 0$*. According to the Fredholm Alternative,* $\mathbb{C}^{2n} = \operatorname{ran}(\mathbf{X}) \oplus \ker(\mathbf{X}^*)$*, and since* $\dim \operatorname{ran} \mathbf{X} = n$*, we must have* $\dim \ker \mathbf{X}^* = n$*. I.e., Lagrangian subspaces on* $\mathbb{C}^{2n}$ *are maximal; no subspace of* $\mathbb{C}^{2n}$ *with dimension greater than* n *can have the Lagrangian property.*

Proposition 2.2. *If* $\mathbf{X} = \binom{X}{Y}$ *is the frame for a Lagrangian subspace of* $\mathbb{C}^{2n}$*, then the matrices* $X \pm iY$ *are both invertible.*

Proof. First, it's standard that $\mathbf{X}^*\mathbf{X} = X^*X + Y^*Y$ is invertible if and only if the columns of $\mathbf{X}$ are linearly independent. Now suppose $v \in \ker(X + iY)$ so that $(X + iY)v = 0$. We can multiply this equation by $(X^* - iY^*)$ and use the Lagrangian property to see that $(X^*X + Y^*Y)v = 0$. This implies $v = 0$, from which we can conclude that $(X + iY)$ is invertible. The case $(X - iY)$ is similar. □

Proposition 2.3. *If* $\mathbf{X} = \binom{X}{Y}$ *is the frame for a Lagrangian subspace of* $\mathbb{C}^{2n}$*, then*

$$(X \pm iY)^{-1} = M^2(X^* \mp iY^*),$$

where $M := (\mathbf{X}^*\mathbf{X})^{-1/2}$*.*

Proof. Computing directly, we see that

$$\begin{aligned}(X^* - iY^*)(X + iY) &= X^*X + Y^*Y + i(X^*Y - Y^*X)\\ &= (M^2)^{-1},\end{aligned}$$

where we have used the Lagrangian property for frames to see that the imaginary part is 0. The claim now follows upon multiplication on the left by M^2 and on the right by $(X+iY)^{-1}$. The case $(X - iY)^{-1}$ is similar. □

For the next proposition, we set

$$\tilde{W}_D := (X + iY)(X - iY)^{-1},$$

noting that the subscript D indicates (as we will check just below) that $\tilde{W}_D$ detects intersections of $\ell = \text{colspan}(\mathbf{X})$ with the Dirichlet plane $\ell_D = \text{colspan}(\binom{0}{I_n})$.

Proposition 2.4. *If* $\mathbf{X} = \binom{X}{Y}$ *is the frame for a Lagrangian subspace of* $\mathbb{C}^{2n}$*, then* $\tilde{W}_D$ *is unitary.*

Proof. Computing directly, we find

$$\begin{aligned}\tilde{W}_D^*\tilde{W}_D &= (X^* + iY^*)^{-1}(X^* - iY^*)(X + iY)(X - iY)^{-1}\\ &= (X^* + iY^*)^{-1}\Big\{X^*X + Y^*Y\Big\}(X - iY)^{-1}\\ &= (X^* + iY^*)^{-1}(M^2)^{-1}M^2(X^* + iY^*) = I,\end{aligned}$$

where for the second equality we used the Lagrangian property for frames, and for the third we used Proposition 2.3. □

Definition 2.1. *Let* $\mathbf{X}_1 = \binom{X_1}{Y_1}$ *and* $\mathbf{X}_2 = \binom{X_2}{Y_2}$ *denote frames for two Lagrangian subspaces of* $\mathbb{C}^{2n}$*. We define*

$$\tilde{W} := -(X_1 + iY_1)(X_1 - iY_1)^{-1}(X_2 - iY_2)(X_2 + iY_2)^{-1}. \tag{2.1}$$

According to Proposition 2.4, $\tilde{W}$ *is the product of unitary matrices, and so is unitary. Consequently, the eigenvalues of* $\tilde{W}$ *will be confined to the unit circle in the complex plane,* S^1*.*

Lemma 2.1. *Suppose* $\mathbf{X}_1 = \binom{X_1}{Y_1}$ *and* $\mathbf{X}_2 = \binom{X_2}{Y_2}$ *respectively denote frames for Lagrangian subspaces of* $\mathbb{C}^{2n}$*. Then*

$$\dim\ker(\mathbf{X}_1^* J \mathbf{X}_2) = \dim\ker(\tilde{W} + I).$$

More precisely,

$$\ker(\mathbf{X}_1^* J \mathbf{X}_2) = \operatorname{ran}\Big((X_2 + iY_2)^{-1}\Big|_{\ker(\tilde{W}+I)}\Big).$$

Proof. First, suppose

$$\dim\ker \mathbf{X}_1^* J \mathbf{X}_2 = m > 0,$$

and let $\{v_k\}_{k=1}^m$ denote a basis for this kernel. Then, in particular,

$$(-X_1^* Y_2 + Y_1^* X_2) v_k = 0 \tag{2.2}$$

for all $k \in \{1, 2, \dots, m\}$. Set

$$w_k = (X_2 + iY_2) v_k,$$

and notice that since $X_2 + iY_2$ is invertible, $\{v_k\}_{k=1}^m$ comprises a linearly independent set of vectors if and only if $\{w_k\}_{k=1}^m$ comprises a linearly independent set of vectors.

Now, we compute directly,

$$\begin{aligned}
\tilde{W} w_k &= -(X_1 + iY_1)(X_1 - iY_1)^{-1}(X_2 - iY_2) v_k \\
&= -(X_1 + iY_1) M_1^2 (X_1^* + iY_1^*)(X_2 - iY_2) v_k \\
&= -(X_1 + iY_1) M_1^2 \Big\{ X_1^* X_2 + Y_1^* Y_2 + i(Y_1^* X_2 - X_1^* Y_2) \Big\} v_k.
\end{aligned}$$

Using (2.2), we see that

$$\begin{aligned}
\tilde{W} w_k &= -(X_1 + iY_1) M_1^2 (X_1^* - iY_1^*)(X_2 + iY_2) v_k \\
&= -(X_1 + iY_1)(X_1 + iY_1)^{-1} w_k = -w_k,
\end{aligned}$$

and so $w_k \in \ker(\tilde{W} + I)$. We can conclude that

$$\dim\ker \mathbf{X}_1^* J \mathbf{X}_2 \le \dim\ker(\tilde{W} + I).$$

For the second part, our calculation has established

$$\ker \mathbf{X}_1^* J \mathbf{X}_2 \subset \operatorname{ran}\Big((X_2 + iY_2)^{-1}\Big|_{\ker(\tilde{W}+I)}\Big).$$

Turning the argument around, we get the inequality and the associated inclusion in the other direction, so we can conclude equality in both cases. □

Lemma 2.2. *Suppose* $\mathbf{X}_1 = \binom{X_1}{Y_1}$ *and* $\mathbf{X}_2 = \binom{X_2}{Y_2}$ *respectively denote frames for Lagrangian subspaces of* $\mathbb{C}^{2n}$*,* ℓ_1 *and* ℓ_2*. Then*

$$\dim\ker(\mathbf{X}_1^* J \mathbf{X}_2) = \dim(\ell_1 \cap \ell_2).$$

More precisely,

$$\operatorname{ran}\Big(\mathbf{X}_2\Big|_{\ker(\mathbf{X}_1^* J \mathbf{X}_2)}\Big) = \ell_1 \cap \ell_2.$$

Proof. First, suppose

$$\dim \ker(\mathbf{X}_1^* J \mathbf{X}_2) = m > 0,$$

and let $\{v_k\}_{k=1}^m$ denote a basis for this kernel. Then, in particular,

$$\mathbf{X}_1^* J \mathbf{X}_2 v_k = 0 \tag{2.3}$$

for all $k \in \{1, 2, \dots, m\}$.

Set $\zeta_k = \mathbf{X}_2 v_k$. Then $\zeta_k \in \ell_2$ (as a linear combination of the columns of $\mathbf{X}_2$), and since $\mathbf{X}_1^* J \zeta_k = 0$, we must have $\zeta_k \in \ell_1$ by maximality. We see that we can associate with the $\{v_k\}_{k=1}^m$ a linearly independent set $\{\zeta_k\}_{k=1}^m \subset \ell_1 \cap \ell_2$, and so

$$\dim \ker \mathbf{X}_1^* J \mathbf{X}_2 \leq \dim(\ell_1 \cap \ell_2).$$

For the second part, our calculation has established

$$\operatorname{ran} \left(\mathbf{X}_2 \Big|_{\ker(\mathbf{X}_1^* J \mathbf{X}_2)} \right) \subset \ell_1 \cap \ell_2.$$

Turning the argument around, we get the inequality and the associated inclusion in the other direction, so we can conclude equality in both cases. □

Combining Lemmas 2.1 and 2.2, we see that

$$\dim \ker(\tilde{W} + I) = \dim(\ell_1 \cap \ell_2), \tag{2.4}$$

which is the key relation in our computation of the Maslov index. Before properly defining the Maslov index, we note that the point $-1 \in S^1$ is chosen essentially at random, and any other point on S^1 would serve just as well. Indeed, we have the following proposition.

Proposition 2.5. *Suppose* $\mathbf{X}_1 = \binom{X_1}{Y_1}$ *and* $\mathbf{X}_2 = \binom{X_2}{Y_2}$ *respectively denote frames for Lagrangian subspaces of* $\mathbb{C}^{2n}$*,* ℓ_1 *and* ℓ_2*, and let* $\tilde{W}$ *be as in (2.1). Given any value* $\tilde{w} \in S^1$*, set*

$$\mathbf{X}_3 = \mathbf{X}_3(\tilde{w}) := i(1 - \tilde{w})\mathbf{X}_2 - (1 + \tilde{w}) J \mathbf{X}_2. \tag{2.5}$$

Then $\mathbf{X}_3$ *is the frame for a Lagrangian subspace of* $\mathbb{C}^{2n}$*,* $\ell_3 = \ell_3(\tilde{w})$*, and*

$$\dim \ker(\tilde{W} - \tilde{w} I) = \dim \ker(\mathbf{X}_1^* J \mathbf{X}_3) = \dim(\ell_1 \cap \ell_3).$$

Proof. In order to check that $\mathbf{X}_3$ is Lagrangian, we note that by straightforward calculations we find

$$\mathbf{X}_3^* \mathbf{X}_3 = (|1 - \tilde{w}|^2 + |1 + \tilde{w}|^2) \mathbf{X}_2^* \mathbf{X}_2,$$

and likewise $\mathbf{X}_3^* J \mathbf{X}_3 = 0$. It's clear from the first of these relations that $\dim \operatorname{colspan}(\mathbf{X}_3) = n$ and from the second that $\mathbf{X}_3$ satisfies the Lagrangian property.

Next, again by direct calculation, we find that

$$(X_3 - iY_3)(X_3 + iY_3)^{-1} = -\frac{1}{\tilde{w}} (X_2 - iY_2)(X_2 + iY_2)^{-1},$$

from which we see that $\tilde{w}$ is an eigenvalue of

$$\tilde{W} = -(X_1 + iY_1)(X_1 - iY_1)^{-1}(X_2 - iY_2)(X_2 + iY_2)^{-1}$$

if and only if -1 is an eigenvalue of

$$\tilde{\mathcal{W}} = -(X_1 + iY_1)(X_1 - iY_1)^{-1}(X_3 - iY_3)(X_3 + iY_3)^{-1}.$$

In this way,

$$\dim(\ell_1 \cap \ell_3) = \dim \ker(\mathbf{X}_1^* J \mathbf{X}_3) = \dim \ker(\tilde{\mathcal{W}} + I) = \dim \ker(\tilde{W} - \tilde{w} I).$$

□

Remark 2.2. *We can interpret Proposition 2.5 in two useful ways. First, each eigenvalue of* $\tilde{W}$*,* $\tilde{w} \in S^1$*, indicates an intersection between* ℓ_1 *and* $\ell_3(\tilde{w})$*. In this way, we can associate with any Lagrangian subspace* ℓ_1 *a family of up to* n *Lagrangian subspaces (depending on multiplicities) obtained through (2.5) as the Lagrangian subspaces* $\operatorname{colspan}(\mathbf{X}_3(\tilde{w}))$ *for some* $\tilde{w}$ *such that* $\dim \ker(\tilde{W} - \tilde{w} I) \neq 0$*. On the other hand, suppose we would like to move our spectral flow calculation from* -1 *to some other* $\tilde{w} \in S^1$*. We let* $\mathbf{X}_3$ *denote our target (generally denoted* $\mathbf{X}_2$*), and solve (2.5) for* $\mathbf{X}_2$*. Then* $\tilde{w} \in \sigma(\tilde{W})$ *corresponds precisely with intersections between* ℓ_1 *and* ℓ_3*, allowing us to compute the Maslov index as a spectral flow through* $\tilde{w}$ *(using* $\tilde{W}$*).*

Turning now to our definition of the Maslov index, we note that since $\tilde{W}$ is unitary, we can define the Maslov index in the $\mathbb{C}^{2n}$ setting precisely as in the $\mathbb{R}^{2n}$ setting in [15]. For completeness, we sketch the development.

Given two continuous maps $\ell_1(t), \ell_2(t)$ on a parameter interval $\mathcal{I}$, we denote by $\mathcal{L}(t)$ the path

$$\mathcal{L}(t) = (\ell_1(t), \ell_2(t)).$$

In what follows, we will define the Maslov index for the path $\mathcal{L}(t)$, which will be a count, including both multiplicity and direction, of the number of times the Lagrangian paths ℓ_1 and ℓ_2 intersect. In order to be clear about what we mean by multiplicity and direction, we observe that associated with any path $\mathcal{L}(t)$ we will have a path of unitary complex matrices as described in (2.1). We have already noted that the Lagrangian subspaces ℓ_1 and ℓ_2 intersect at a value $t_* \in \mathcal{I}$ if and only if $\tilde{W}(t_*)$ has -1 as an eigenvalue. (Recall that we refer to the value t_* as a *crossing point*.) In the event of such an intersection, we define the multiplicity of the intersection to be the multiplicity of -1 as an eigenvalue of $\tilde{W}(t_*)$ (since $\tilde{W}(t_*)$ is unitary the algebraic and geometric multiplicites are the same). When we talk about the direction of an intersection, we mean the direction the eigenvalues of $\tilde{W}(t_*)$ are moving (as t increases) along the unit circle S^1 when they cross -1 (we take counterclockwise as the positive direction). We note that we will need to take care with what we mean by a crossing in the following sense: we must decide whether to increment the Maslov index upon arrival or upon departure. Indeed, there are several different approaches to defining the Maslov index (see, for example, [8, 28]), and they often disagree on this convention.

Following [5, 10, 26] (and in particular Definition 1.5 from [5]), we proceed by choosing a partition $a = t_0 < t_1 < \cdots < t_n = b$ of $\mathcal{I} = [a, b]$, along with numbers $\{\epsilon_j\}_{j=1}^n \subset (0, \pi)$

so that $\ker\big(\tilde{W}(t) - e^{i(\pi \pm \epsilon_j)} I\big) = \{0\}$ for $t_{j-1} \le t \le t_j$; that is, $e^{i(\pi \pm \epsilon_j)} \in \mathbb{C} \setminus \sigma(\tilde{W}(t))$, for $t_{j-1} \le t \le t_j$ and $j = 1, \dots, n$. Moreover, we notice that for each $j = 1, \dots, n$ and any $t \in [t_{j-1}, t_j]$ there are only finitely many values $\beta \in [0, \epsilon_j)$ for which $e^{i(\pi + \beta)} \in \sigma(\tilde{W}(t))$.

Fix some $j \in \{1, 2, \dots, n\}$ and consider the value

$$k(t, \epsilon_j) := \sum_{0 \le \beta < \epsilon_j} \dim\ker\big(\tilde{W}(t) - e^{i(\pi+\beta)} I\big). \tag{2.6}$$

for $t_{j-1} \le t \le t_j$. This is precisely the sum, along with multiplicity, of the number of eigenvalues of $\tilde{W}(t)$ that lie on the arc

$$A_j := \{e^{it} : t \in [\pi, \pi + \epsilon_j)\}.$$

The stipulation that $e^{i(\pi \pm \epsilon_j)} \in \mathbb{C} \setminus \sigma(\tilde{W}(t))$, for $t_{j-1} \le t \le t_j$ ensures that no eigenvalue can enter A_j in the clockwise direction or exit in the counterclockwise direction during the interval $t_{j-1} \le t \le t_j$. In this way, we see that $k(t_j, \epsilon_j) - k(t_{j-1}, \epsilon_j)$ is a count of the number of eigenvalues that enter A_j in the counterclockwise direction (i.e., through -1) minus the number that leave in the clockwise direction (again, through -1) during the interval $[t_{j-1}, t_j]$.

In dealing with the catenation of paths, it's particularly important to understand the difference $k(t_j, \epsilon_j) - k(t_{j-1}, \epsilon_j)$ if an eigenvalue resides at -1 at either $t = t_{j-1}$ or $t = t_j$ (i.e., if an eigenvalue begins or ends at a crossing). If an eigenvalue moving in the counterclockwise direction arrives at -1 at $t = t_j$, then we increment the difference forward, while if the eigenvalue arrives at -1 from the clockwise direction we do not (because it was already in A_j prior to arrival). On the other hand, suppose an eigenvalue resides at -1 at $t = t_{j-1}$ and moves in the counterclockwise direction. The eigenvalue remains in A_j, and so we do not increment the difference. However, if the eigenvalue leaves in the clockwise direction then we decrement the difference. In summary, the difference increments forward upon arrivals in the counterclockwise direction, but not upon arrivals in the clockwise direction, and it decrements upon departures in the clockwise direction, but not upon departures in the counterclockwise direction.

We are now ready to define the Maslov index.

Definition 2.2. *Let $\mathcal{L}(t) = (\ell_1(t), \ell_2(t))$, where $\ell_1, \ell_2 : \mathcal{I} \to \Lambda(n)$ are continuous paths in the Lagrangian–Grassmannian. The Maslov index* $\operatorname{Mas}(\mathcal{L}; \mathcal{I})$ *is defined by*

$$\operatorname{Mas}(\mathcal{L}; \mathcal{I}) = \sum_{j=1}^{n} (k(t_j, \epsilon_j) - k(t_{j-1}, \epsilon_j)). \tag{2.7}$$

Remark 2.3. *As we did in the introduction, we will typically refer explicitly to the individual paths with the notation* $\operatorname{Mas}(\ell_1, \ell_2; \mathcal{I})$.

Remark 2.4. *As discussed in [5], the Maslov index does not depend on the choices of $\{t_j\}_{j=0}^n$ and $\{\epsilon_j\}_{j=1}^n$, so long as these choices follow the specifications described above. Also, we emphasize that Phillips' specification of the spectral flow allows for an infinite number of crossings. In such cases, all except a finite number are necessarily transient (i.e., an eigenvalue crosses -1, but then crosses back, yielding no net contribution to the Maslov index).*

One of the most important features of the Maslov index is homotopy invariance, for which we need to consider continuously varying families of Lagrangian paths. To set some notation, we denote by $\mathcal{P}(\mathcal{I})$ the collection of all paths $\mathcal{L}(t) = (\ell_1(t), \ell_2(t))$, where $\ell_1, \ell_2 : \mathcal{I} \to \Lambda(n)$ are continuous paths in the Lagrangian–Grassmannian. We say that two paths $\mathcal{L}, \mathcal{M} \in \mathcal{P}(\mathcal{I})$ are homotopic provided there exists a family $\mathcal{H}_s$ so that $\mathcal{H}_0 = \mathcal{L}$, $\mathcal{H}_1 = \mathcal{M}$, and $\mathcal{H}_s(t)$ is continuous as a map from $(t,s) \in \mathcal{I} \times [0,1]$ into $\Lambda(n) \times \Lambda(n)$.

The Maslov index has the following properties.

(P1) (Path Additivity) If $\mathcal{L} \in \mathcal{P}(\mathcal{I})$ and $a, b, c \in \mathcal{I}$, with $a < b < c$, then

$$\mathrm{Mas}(\mathcal{L}; [a,c]) = \mathrm{Mas}(\mathcal{L}; [a,b]) + \mathrm{Mas}(\mathcal{L}; [b,c]).$$

(P2) (Homotopy Invariance) If $\mathcal{I} = [a,b]$, $a < b$, and $\mathcal{L}, \mathcal{M} \in \mathcal{P}(\mathcal{I})$ are homotopic with $\mathcal{L}(a) = \mathcal{M}(a)$ and $\mathcal{L}(b) = \mathcal{M}(b)$ (i.e., if $\mathcal{L}, \mathcal{M}$ are homotopic with fixed endpoints) then

$$\mathrm{Mas}(\mathcal{L}; [a,b]) = \mathrm{Mas}(\mathcal{M}; [a,b]).$$

Straightforward proofs of these properties appear in [15] for Lagrangian subspaces of $\mathbb{R}^{2n}$, and proofs in the current setting of Lagrangian subspaces of $\mathbb{C}^{2n}$ are essentially identical.

3 Direction of Rotation

As noted in the previous section, the direction we associate with a crossing point is determined by the direction in which eigenvalues of $\tilde{W}$ rotate through -1 (counterclockwise is positive, while clockwise is negative). When analyzing the Maslov index, we need a convenient framework for analyzing this direction, and the development of such a framework is the goal of this section.

First, in order to understand monotonicity as the spectral parameter λ evolves, we can use the following lemma from [15]. (In [15], the statement takes the frames to be C^1, but the proof only requires differentiability, as asserted here.)

Lemma 3.1. *Suppose $\ell_1, \ell_2 : \mathcal{I} \to \Lambda(n)$ denote paths of Lagrangian subspaces of $\mathbb{C}^{2n}$ with respective frames $\mathbf{X}_1 = \binom{X_1}{Y_1}$ and $\mathbf{X}_2 = \binom{X_2}{Y_2}$ that are differentiable at $t_0 \in \mathcal{I}$. If the matrices*

$$-\mathbf{X}_1(t_0)^* J \mathbf{X}_1'(t_0) = X_1(t_0)^* Y_1'(t_0) - Y_1(t_0)^* X_1'(t_0)$$

and (noting the sign change)

$$\mathbf{X}_2(t_0)^* J \mathbf{X}_2'(t_0) = -(X_2(t_0)^* Y_2'(t_0) - Y_2(t_0)^* X_2'(t_0))$$

are both non-negative, and at least one is positive definite, then the eigenvalues of $\tilde{W}(t)$ rotate in the counterclockwise direction as t increases through t_0. Likewise, if both of these matrices are non-positive, and at least one is negative definite, then the eigenvalues of $\tilde{W}(t)$ rotate in the clockwise direction as t increases through t_0.

In order to analyze monotonicity as the independent variable x evolves, we require a more detailed analysis, and for this we'll consider a general linear Hamiltonian system

$$Jy' = \mathcal{B}(t)y, \quad t \in (0,1), \quad y(t) \in \mathbb{C}^{2n}, \tag{3.1}$$

for which we assume $\mathcal{B} \in L^1((0,1), \mathbb{C}^{2n})$, and that $\mathcal{B}(t)$ is self-adjoint for a.e. $t \in (0,1)$. Throughout this discussion, we will let $\mathbf{X}(t) = \binom{X(t)}{Y(t)}$ denote a $2n \times n$ matrix solution of (3.1), and we will assume that for some $t_0 \in [0,1]$ $\mathbf{X}(t_0)$ is a frame for a Lagrangian subspace, from which it follows that $\mathbf{X}(t)$ is the frame for a Lagrangian subspace $\ell(t)$ for all $t \in [0,1]$. In addition, we will denote by $\tilde{\ell}$ a fixed Lagrangian target with frame $\tilde{\mathbf{X}} = \binom{\tilde{X}}{\tilde{Y}}$, and consider the Maslov index

$$\operatorname{Mas}(\ell(\cdot), \tilde{\ell}; [0,1]), \tag{3.2}$$

which can be computed as described above with the matrix

$$\tilde{\mathcal{W}}(t) := -(X(t) + iY(t))(X(t) - iY(t))^{-1}(\tilde{X} - i\tilde{Y})(\tilde{X} + i\tilde{Y})^{-1}. \tag{3.3}$$

As discussed in [17], if we fix any $t_0 \in [0,1]$, then we can write

$$\tilde{\mathcal{W}}(t) = \tilde{\mathcal{W}}(t_0) e^{iR(t)}$$

for t sufficiently close to t_0. Here, $iR(t)$ is the logarithm of $\tilde{\mathcal{W}}(t_0)^{-1}\tilde{\mathcal{W}}(t)$, and we clearly have $R(t_0) = 0$. (Since $\tilde{\mathcal{W}}(t_0)^{-1}\tilde{\mathcal{W}}(t_0) = I$, it's clear that $\tilde{\mathcal{W}}(t_0)^{-1}\tilde{\mathcal{W}}(t)$ has a unique logarithm for t sufficiently close to t_0.) For any $t \in [0,1]$, we set $\tilde{\Omega}(t) := -i\tilde{\mathcal{W}}(t)^{-1}\tilde{\mathcal{W}}'(t)$, so that

$$\tilde{\mathcal{W}}'(t) = i\tilde{\mathcal{W}}(t)\tilde{\Omega}(t),$$

for all $t \in [0,1]$. Comparing expressions, we see that $R'(t_0) = \tilde{\Omega}(t_0)$.

As discussed particularly in the proof of Lemma 3.11 in [17], the direction of rotation for eigenvalues of $\tilde{\mathcal{W}}(t)$ as they cross $\tilde{w}_0 \in \sigma(\tilde{\mathcal{W}}(t_0))$ is determined by the nature of $R(t)$ for t near t_0. For the current analysis, we will require an extension of Lemma 3.11 in [17], and in developing this we will repeat part of the argument from [17].

First, following [10][p. 306], we fix any $\theta \in [0, 2\pi)$ so that $e^{i\theta} \notin \ker(\tilde{\mathcal{W}}(t_0))$ and, for t sufficiently close to t_0, define the auxiliary matrix

$$\tilde{A}(t) := i(e^{i\theta}I - \tilde{\mathcal{W}}(t))^{-1}(e^{i\theta}I + \tilde{\mathcal{W}}(t)). \tag{3.4}$$

It is straightforward to check that $\tilde{A}(t)$ is self-adjoint, and this allows us to conclude that its eigenvalues will all be real-valued. If we denote the eigenvalues of $\tilde{\mathcal{W}}(t)$ by $\{\tilde{w}_j(t)\}_{j=1}^n$, and the eigenvalues of $\tilde{A}(t)$ by $\{\tilde{a}_j(t)\}_{j=1}^n$ then by spectral mapping we have the correspondence

$$\tilde{a}_j(t) = i\frac{e^{i\theta} + \tilde{w}_j(t)}{e^{i\theta} - \tilde{w}_j(t)}. \tag{3.5}$$

By a short argument in the proof of Lemma 3.11 in [17], the authors find that if the eigenvalue $\tilde{a}_j(t)$ is increasing as t increases through t_0 then the corresponding eigenvalue $\tilde{w}_j(t)$ of $\tilde{W}(t)$ will rotate in the counterclockwise direction along S^1 as t increases through t_0. This means that the rotation of the eigenvalues of $\tilde{\mathcal{W}}(t)$ can be determined by the linear motion of the eigenvalues of $\tilde{A}(t)$.

Suppose $\tilde{w}_0$ is an eigenvalue of $\tilde{\mathcal{W}}(t_0)$ with multiplicity m. Since $\tilde{\mathcal{W}}(t_0)$ is unitary, the algebraic and geometric multiplicities of its eigenvalues agree, so the eigenspace associated with $\tilde{w}_0$, which we denote $\tilde{V}_0$, has dimension m. From Theorem II.5.4 in [19], we know there exists a corresponding eigenvalue group $\{\tilde{w}_j(t)\}_{j=1}^m \subset \sigma(\tilde{\mathcal{W}}(t))$ so that $\tilde{w}_j(t_0) = \tilde{w}_0$ for $j = 1, 2, ..., m$. By a natural choice of indexing, each such $\tilde{w}_j(t)$ will have a corresponding eigenvalue $\tilde{a}_j(t) \in \sigma(\tilde{A}(t))$, and the eigenspace associated with $\{\tilde{a}_j(t_0)\}_{j=1}^m \subset \sigma(\tilde{A}(t_0))$ will be $\tilde{V}_0$. In particular, it follows from the discussion above that the rotation of the eigenvalues $\{\tilde{w}_j(t)\}_{j=1}^m \subset \sigma(\tilde{\mathcal{W}}(t))$ through $\tilde{w}_0$ will be determined by the linear motion of the eigenvalues $\{\tilde{a}_j(t)\}_{j=1}^m \subset \sigma(\tilde{A}(t))$ through $\tilde{a}_0$.

In order to apply these observations in the current setting, we will use the following lemma, which is based on Theorem II.5.4 from [19].

Lemma 3.2. *Let $A \in AC([0,1], \mathbb{C}^{n\times n})$, with $A(t)$ self-adjoint for each $t \in [0,1]$. Fix $t_0 \in [0,1]$, and suppose there exists $\delta > 0$ sufficiently small so that $A'(t)$ is non-negative (resp. non-positive) for a.e. $t \in (t_0 - \delta, t_0 + \delta) \cap [0,1]$. Then the n eigenvalues of $A(t)$ must be non-decreasing (resp. non-increasing) on $(t_0 - \delta, t_0 + \delta) \cap [0,1]$.*

Proof. Since $A \in AC([0,1], \mathbb{C}^{n\times n})$, we have that A is differentiable a.e. in $(0,1)$. Suppose A is differentiable at a value $\tau \in (0,1)$, and let $a(\tau)$ denote any eigenvalue of $A(\tau)$. If m denotes the multiplicity of $a(\tau)$ as an eigenvalue of $A(\tau)$, then we have from Theorem II.5.4 in [19] that there will correspond an eigenvalue group $\{a_j(t)\}_{j=1}^m \subset \sigma(A(t))$ that can be expressed as

$$a_j(t) = a(\tau) + \alpha_j^\tau (t - \tau) + \mathbf{o}(|t-\tau|), \quad \forall\, j \in \{1, 2, \dots, m\}, \tag{3.6}$$

for t sufficiently close to τ, and where the values $\{\alpha_j^\tau\}_{j=1}^m$ are eigenvalues of $P_\tau A'(\tau) P_\tau$ in the space $P_\tau \mathbb{C}^{2n}$, with P_τ denoting projection onto the eigenspace of $A(\tau)$ associated to $a(\tau)$.

Since $A'(t)$ is non-negative for a.e. $t \in (t_0 - \delta, t_0 + \delta) \cap [0,1]$, we can conclude that the values $\{\alpha_j^\tau\}_{j=1}^m$ must be non-negative for a.e. $\tau \in (t_0 - \delta, t_0 + \delta) \cap [0,1]$. In particular, we see that

$$a_j'(\tau) = \alpha_j^\tau \geq 0; \quad \text{a.e. } \tau \in (t_0 - \delta, t_0 + \delta) \cap [0,1].$$

Upon integrating this last relation on any interval $(t_1, t_2) \subset (t_0 - \delta, t_0 + \delta) \cap [0,1]$, we see that

$$a_j(t_2) \geq a_j(t_1), \quad \forall\, j \in \{1, 2, \dots, m\}.$$

Since this is true for all eigenvalue groups of $A(t)$, the proof is complete for the case in which $A'(t)$ is non-negative for a.e. $t \in (t_0 - \delta, t_0 + \delta) \cap [0,1]$. The case in which $A'(t)$ is non-positive for a.e. $t \in (t_0 - \delta, t_0 + \delta) \cap [0,1]$ can be established similarly. □

In our analysis, monotonicity as the independent vararaible x evolves will be a consequence of the following lemma.

Lemma 3.3. *For $\tilde{\mathcal{W}}(t)$ as specified in (3.3), suppose $\mathcal{B}(t)$ is non-negative (resp. non-positive) for a.e. $t \in (0,1)$. Then no eigenvalue of $\tilde{\mathcal{W}}(t)$ can rotate in the counterclockwise (resp. clockwise) direction on any interval $[a,b] \subset [0,1]$, $a < b$.*

Proof. In Section 4 of [15], the authors show that if $\tilde{A}(t)$ from (3.4) is differentiable at t, then

$$\tilde{A}'(t) = 2\Big((e^{i\theta}I - \tilde{\mathcal{W}}(t))^{-1}\Big)^*\tilde{\Omega}(t)(e^{i\theta}I - \tilde{\mathcal{W}}(t))^{-1}, \tag{3.7}$$

with (under the assumptions of the current lemma)

$$\tilde{\Omega}(t) = -2((X(t) - iY(t))^{-1}\tilde{\mathcal{W}})^*\mathbf{X}(t)^*\mathcal{B}(t)\mathbf{X}(t)((X(t) - iY(t))^{-1}\tilde{\mathcal{W}}), \tag{3.8}$$

where $\tilde{\mathcal{W}}$ denotes the constant matrix

$$\tilde{\mathcal{W}} := (\tilde{X} - i\tilde{Y})(\tilde{X} + i\tilde{Y})^{-1}.$$

(In [15], the authors take $\mathbf{X}(t), \tilde{\mathbf{X}} \in \mathbb{R}^{2n\times n}$, but the calculation in our setting simply replaces transpose with adjoint where appropriate.)

If $\mathcal{B}(t)$ is non-negative for a.e. $t \in (0,1)$, then $\tilde{\Omega}(t)$ is non-positive for a.e. $t \in (0,1)$, and consequently $\tilde{A}'(t)$ is non-positive for a.e. $t \in (0,1)$. It follows from Lemma 3.2 that the eigenvalues of $\tilde{A}(t)$ must be non-increasing on $(0,1)$, and using (3.5) we can conclude that no eigenvalue of $\tilde{\mathcal{W}}(t)$ can rotate in the counterclockwise direction on any interval $[a,b] \subset [0,1]$, $a < b$. □

Since $\mathcal{B}(t)$ need not be strictly positive for all $t \in (0,1)$, Lemma 3.3 leaves open the possibility that there exists an interval $[a,b] \subset [0,1]$, $a < b$, so that

$$\dim(\ell(t) \cap \tilde{\ell}) \neq 0 \tag{3.9}$$

for all $t \in [a,b]$. (E.g., $\mathcal{B}(t) \equiv 0$ would be a trivial example allowing this possibility.) In our applications we will have a slightly stronger condition than (3.9), namely that there exists some $m \in \{1, 2, \dots, n\}$ so that

$$\dim(\ell(t) \cap \tilde{\ell}) = m \tag{3.10}$$

for all $t \in [a,b]$. In particular, we will make use of the following lemma.

Lemma 3.4. *Let* $\mathbf{X}(t) = \binom{X(t)}{Y(t)}$ *be a continuous matrix function* $\mathbf{X} : [0,1] \to \mathbb{C}^{2n\times n}$ *such that for each* $t \in [0,1]$, $\mathbf{X}(t)$ *is the frame for a Lagrangian subspace* $\ell(t)$, *and let* $\tilde{\mathbf{X}} \in \mathbb{C}^{2n\times n}$ *denote any fixed Lagrangian frame. Suppose that as* t *increases from* 0 *to* 1 *any eigenvalue of* $\tilde{\mathcal{W}}(t)$ *(from (3.3), except under the current assumptions on* $\mathbf{X}(t)$ *and* $\tilde{\mathbf{X}}$*) that crosses* -1 *makes the crossing in the counterclockwise direction. If there exists an interval* $[a,b] \subset [0,1]$, $a < b$, *so that* $\dim\ker(\tilde{\mathcal{W}}(t) + I) \neq 0$ *for all* $t \in [a,b]$, *then there exists an integer* $m \in \{1, 2, \dots, n\}$ *and a subinterval* $[c,d] \subset [a,b]$, $c < d$, *so that* $\dim\ker(\tilde{\mathcal{W}}(t) + I) = m$ *for all* $t \in [c,d]$. *The same conclusion holds true under the alternative assumption that as* t *increases from* 0 *to* 1 *any eigenvalue of* $\tilde{\mathcal{W}}(t)$ *that crosses* -1 *makes the crossing in the clockwise direction.*

Proof. Fix any $t_0 \in (a,b)$, and observe that we necessarily have

$$\dim\ker(\tilde{\mathcal{W}}(t_0) + I) = m_0$$

for some $m_0 \in \{1, 2, \dots, n\}$. Using continuity of $\tilde{\mathcal{W}}(t)$, fix $\delta_0 > 0$ sufficiently small so that none of the eigenvalues of $\tilde{\mathcal{W}}(\cdot)$ arrive at -1 during the interval $[t_0, t_0 + \delta_0]$. (Note that, by assumption, arrivals would necessarily occur from the counterclockwise direction, and also that we don't count an eigenvalue that has remained at -1 for the full interval $[t_0, t_0 + \delta_0]$ as arriving at -1.)

As t increases from t_0, one or more eigenvalues residing at -1 could rotate away from -1 in the counterclockwise direction, though the total number of eigenvalues residing at -1 could not be reduced to 0 (by assumption). Since there are only a finite number of eigenvalues, these departure times must be separated by intervals, and so we can take $[c, d]$ to be any interval between departures. □

According to Lemma 2.1, we can re-state (3.10) in the following way: the matrix Wronskian

$$\mathcal{W}(t) := \tilde{\mathbf{X}}^* J \mathbf{X}(t)$$

satisfies $\dim \ker \mathcal{W}(t) = m$ for all $t \in [a, b]$. In particular, 0 is an eigenvalue of $\mathcal{W}(t)$ with multiplicity exactly m for all $t \in [a, b]$. In the terminology of [19], there is an eigenvalue group associated with 0, with each eigenvalue in the group identically 0 for all $t \in [a, b]$, and such that the associated projection $P(t)$ projects onto the m-dimensional space $\ker \mathcal{W}(t)$ for all $t \in [a, b]$. According to a slight extension of Theorem II.5.4 in [19], if $\mathbf{X}(t)$ is absolutely continuous on (a, b) then $P(t)$ will be absolutely continuous on this interval as well.

Lemma 3.5. *For $\tilde{\mathcal{W}}(t)$ as specified in (3.3), suppose either that $\mathcal{B}(t)$ is non-negative for a.e. $t \in (0, 1)$ or that $\mathcal{B}(t)$ is non-positive for a.e. $t \in (0, 1)$. If there exists an interval $[a, b] \subset [0, 1]$ and an integer $m \in \{1, 2, \dots, n\}$ so that $\dim \ker(\tilde{\mathcal{W}}(t) + I) = m$ for all $t \in [a, b]$ then the following hold.*

(i) There exists a function $v \in AC([a, b], \mathbb{C}^n)$ so that $\mathbf{X}(t)v(t) \in \ell(t) \cap \tilde{\ell}$ for all $t \in [a, b]$;

(ii) Given any $v(t)$ such that $\mathbf{X}(t)v(t) \in \ell(t) \cap \tilde{\ell}$ for all $t \in [a, b]$ (not necessarily the choice of $v(t)$ from Item (i)), there exists $w \in AC([a, b], \mathbb{C}^n)$ to that

$$\mathbf{X}(t)v(t) = \tilde{\mathbf{X}} w(t), \quad \forall\, t \in (a, b),$$

and additionally

$$\mathbf{X}(t)v'(t) = \tilde{\mathbf{X}} w'(t), \quad \forall\, t \in (a, b).$$

Proof. For Item (i), we fix some $t_* \in (a, b)$ and observe that, by assumption, $\dim(\ell(t_*) \cap \tilde{\ell}) = m$. It follows that there exists a vector $v_* \in \mathbb{C}^n$ so that

$$\mathbf{X}(t_*)v_* \in \ell(t_*) \cap \tilde{\ell}.$$

Letting $P(t)$ denote orthogonal projection onto $\ker \mathcal{W}(t)$, we set $v(t) := P(t)v_*$. Since $P \in \mathrm{AC}([a, b], \mathbb{C}^n)$, we can conclude that $v \in \mathrm{AC}([a, b], \mathbb{C}^n)$. In addition, $\mathbf{X}(t)v(t)$ is clearly in $\ell(t)$ for all $t \in [a, b]$, and since $\mathcal{W}(t)v(t) = 0$ for all $t \in [a, b]$, we have $\tilde{\mathbf{X}}^* J \mathbf{X}(t)v(t) = 0$ for all $t \in [a, b]$, from which we can conclude by Lagrangian maximality that $\mathbf{X}(t)v(t) \in \tilde{\ell}$ for all $t \in [a, b]$. This complete the proof of Item (i).

Turning to Item (ii), let v denote any vector function $v \in AC([a, b], \mathbb{C}^n)$ so that $\mathbf{X}(t)v(t) \in \ell(t) \cap \tilde{\ell}$ for all $t \in [a, b]$. For each $t \in [a, b]$ there must exist some $w = w(t)$ so that

$$\mathbf{X}(t)v(t) = \tilde{\mathbf{X}} w(t),$$

and we can use the Moore-Penrose pseudoinverse of $\tilde{\mathbf{X}}$, to write

$$w(t) = (\tilde{\mathbf{X}}^* \tilde{\mathbf{X}})^{-1} \tilde{\mathbf{X}}^* \mathbf{X}(t) v(t).$$

We see from this expression that $w(t)$ must also be absolutely continuous on $[a, b]$.

Next, by direct calculation we find

$$\mathcal{W}'(t) = \tilde{\mathbf{X}}^* J \mathbf{X}'(t) = \tilde{\mathbf{X}}^* \mathcal{B}(t) \mathbf{X}(t).$$

We are justified in differentiating the relation $\mathcal{W}(t)v(t) = 0$, and we obtain

$$\mathcal{W}'(t)v(t) + \mathcal{W}(t)v'(t) = 0, \quad \text{a.e. } t \in (a, b),$$

and this relation can be expressed as

$$\tilde{\mathbf{X}}^* \mathcal{B}(t) \mathbf{X}(t) v(t) + \tilde{\mathbf{X}}^* J \mathbf{X}(t) v'(t) = 0, \quad \text{a.e. } t \in (a, b). \tag{3.11}$$

If we take a $\mathbb{C}^n$ inner product of this equation with $w(t)$, we find that

$$(\tilde{\mathbf{X}}^* \mathcal{B}(t) \mathbf{X}(t) v(t), w(t)) + (\tilde{\mathbf{X}}^* J \mathbf{X}(t) v'(t), w(t)) = 0, \quad \text{a.e. } t \in (a, b). \tag{3.12}$$

For the second in this last expression, we see that

$$\Big(\tilde{\mathbf{X}}^* J \mathbf{X}(t) v'(t), w(t)\Big) = -\Big(v'(t), \mathbf{X}(t)^* J \tilde{\mathbf{X}} w(t)\Big) = 0,$$

because $\mathbf{X}(t)^* J \tilde{\mathbf{X}}(t) w(t) = 0$ (since $\tilde{\mathbf{X}}(t) w(t) \in \ell(t)$). This leaves only a single summand in (3.12), and we can write

$$0 = (\tilde{\mathbf{X}}^* \mathcal{B}(t) \mathbf{X}(t) v(t), w(t)) = (\mathcal{B}(t) \mathbf{X}(t) v(t), \tilde{\mathbf{X}} w(t)).$$

Using the relation $\mathbf{X}(t) v(t) = \tilde{\mathbf{X}} w(t)$, we can express this as

$$(\mathcal{B}(t) \mathbf{X}(t) v(t), \mathbf{X}(t) v(t)) = 0, \quad \text{a.e. } t \in (a, b).$$

Since $\mathcal{B}(t)$ is non-negative for a.e. $t \in (a, b)$ (or, alternatively, non-positive for a.e. $t \in (a, b)$), we must have $\mathbf{X}(t) v(t) \in \ker \mathcal{B}(t)$ for a.e. $t \in (a, b)$. This allows us to additionally compute

$$J \mathbf{X}'(t) v(t) = \mathcal{B}(t) \mathbf{X}'(t) v(t) = 0,$$

from which we can conclude that $\mathbf{X}'(t) v(t) = 0$ for a.e. $t \in (a, b)$, and consequently $\mathbf{X}(t) v'(t) = \tilde{\mathbf{X}} w'(t)$ for a.e. $t \in (a, b)$, establishing the final claim of the lemma. $\square$

Lemma 3.6. *For $\tilde{\mathcal{W}}(t)$ as specified in (3.3), suppose either that $\mathcal{B}(t)$ is non-negative for a.e. $t \in (0, 1)$ or that $\mathcal{B}(t)$ is non-positive for a.e. $t \in (0, 1)$. If there exists an interval $[a, b] \subset [0, 1]$ and an integer $m \in \{1, 2, \dots, n\}$ so that $\dim \ker(\tilde{\mathcal{W}}(t) + I) = m$ for all $t \in [a, b]$ then there exists an interval $[c, d] \subset [a, b]$, $c < d$, and a constant vector $v_0 \in \mathbb{C}^n \backslash \{0\}$ so that $\mathbf{X}(t) v_0 \in \ell(t) \cap \tilde{\ell}$ for all $t \in [c, d]$.*

Proof. First, if $m = n$, then we must have $\ell(t) = \tilde{\ell}$ for all $t \in [a,b]$, and so for *any* $v_0 \in \mathbb{C}^n \backslash \{0\}$, we have $\mathbf{X}(t)v_0 \in \ell(t) \cap \tilde{\ell}$ for all $t \in [a,b]$. In particular, in this case, the claim holds for $[c,d] = [a,b]$.

Next, suppose $m \in \{1, 2, \cdots, n-1\}$. If $\dim(\ell(t) \cap \tilde{\ell}) = m$ for all $t \in [a,b]$, then we can fix some $t_* \in (a,b)$ and let $\{v_j^*\}_{j=1}^m$ denote a basis for $\ker \tilde{\mathbf{X}}^* J \mathbf{X}(t_*)$. If, as in the proof of Lemma 3.5, we let $P(t)$ denote projection onto $\ker \tilde{\mathbf{X}}^* J \mathbf{X}(t)$, then the elements $v_j(t) := P(t) v_j^*$, $j = 1, 2, \dots, m$, comprise a collection of vector functions $v_j \in \mathrm{AC}([a,b], \mathbb{C}^n)$ that are linearly independent for t sufficiently close to t_*. We denote this interval of linear independence I_*. It follows immediately that the collection of vector functions $\{\mathbf{X}(t)v_j(t)\}_{j=1}^m$ forms a basis for $\ell(t) \cap \tilde{\ell}$ for each $t \in I_*$. (Here, $\mathbf{X}(t)v_j(t) \in \tilde{\ell}$ because $\tilde{\mathbf{X}}^* J \mathbf{X}(t) v_j(t) = 0$.)

We observe that if we let $V(t)$ denote the $n \times m$ matrix with columns $\{v_j(t)\}_{j=1}^m$, then the columns of $\mathbf{X}(t)V(t)$ are precisely the basis elements for $\ell(t) \cap \tilde{\ell}$ selected above. In the usual way, we can make a change of basis by multiplying $\mathbf{X}(t)V(t)$ on the right by any non-singular $m \times m$ matrix $M(t)$. We claim that by restricting to a smaller interval if necessary, we can choose $M(t)$ so that $\tilde{V}(t) := V(t)M(t)$ is in column reduced echelon form for all t in the smaller interval.

In order to understand this claim, we think as follows. We begin with the first components of the vectors $\{v_j(t)\}_{j=1}^m$, which we will denote $\{v_{1j}(t)\}_{j=1}^m$. If each of these is 0 on the entirety of I_*, then the entire first row of $\tilde{V}(t)$ will be 0. Otherwise, there exists some index $j \in \{1, 2, \dots, m\}$ and some value $t_1 \in I_*$ so that $v_{1j}(t_1) \neq 0$. For notational convenience, let's suppose $j = 1$, so that $v_{11}(t_1) \neq 0$. Then by continuity there exists some interval $I_1 \subset I_*$ so that $v_{11}(t) \neq 0$ for all $t \in I_1$. This allows us to divide each entry in the column $v_1(t)$ by the entry $v_{11}(t)$, and consquently we can perform column operations to eliminate the first component of each column $\{v_j(t)\}_{j=2}^m$.

Following the preceding operations, the second component in the second column of the resulting matrix can be expressed as

$$v_{22}(t) - v_{12}(t) \frac{v_{21}(t)}{v_{11}(t)}. \tag{3.13}$$

If this quantity is 0 for all $t \in I_1$, then this entry will be 0 in $\tilde{V}(t)$, and we move to the third entry in the second column. If (3.13) is not 0 on the entirety of I_1, then by continuity there exists an interval $I_2 \subset I_1$ so that (3.13) is non-zero on the entirety of I_2. In this case, we divide the second column of our matrix by (3.13) and use this pivot to eliminate the remaining entries in the second row of our new matrix. Continuing in this way, we obtain a matrix $\tilde{V}(t)$ on some final (smallest) interval I, whose columns span the same space as the columns of $V(t)$, and which has at least m rows with a single 1 as the only non-zero entry.

Let $\{\tilde{v}_j(t)\}_{j=1}^m$ denote the columns of this new matrix, and set

$$\mathbf{v}(t) := \sum_{j=1}^m \tilde{v}_j(t).$$

Then, in particular, we have $\mathbf{X}(t)\mathbf{v}(t) \in \ell(t) \cap \tilde{\ell}$ for all $t \in I$. Since $\mathcal{B}(t)$ is non-negative for a.e. $t \in I$, we can conclude from Lemma 3.5 that there exists some $w \in AC(I, \mathbb{C}^n)$ so that $\mathbf{X}(t)v(t) = \tilde{\mathbf{X}}w(t)$ and $\mathbf{X}(t)v'(t) = \tilde{\mathbf{X}}w'(t)$ both hold for a.e. $t \in I$. So, in particular,

$\mathbf{X}(t)\mathbf{v}'(t) \in \ell(t) \cap \tilde{\ell}$ for a.e. $t \in I$. If $\mathbf{v}'(t) = 0$ for a.e. $t \in I$, then we are done, because we can take $v_0 := \mathbf{v}(t)$, which is constant in t. (We recall that $\mathbf{v}(t)$ is absolutely continuous.) If $\mathbf{v}'(t)$ is not 0 for a.e. $t \in I$, then it must be linearly independent of the set $\{\tilde{v}_j(t)\}_{j=1}^m$, because each of the $\{\tilde{v}_j(t)\}_{j=1}^m$ will have a non-zero entry in a row where $\mathbf{v}'(t)$ has only zeros. But this means $\ell(t) \cap \tilde{\ell}$ has dimension at least $m+1$, a contradiction. □

Lemma 3.7. *For $\tilde{\mathcal{W}}(t)$ as specified in (3.3), suppose either that $\mathcal{B}(t)$ is non-negative for a.e. $t \in (0,1)$ or that $\mathcal{B}(t)$ is non-positive for a.e. $t \in (0,1)$. If there exists an interval $[a,b] \subset [0,1]$, $a < b$, so that $\mathbf{X}(t)v_0 \in \ell(t) \cap \tilde{\ell}$ for all $t \in [a,b]$, then $\mathbf{X}(t)v_0$ is constant on $[a,b]$.*

Proof. We know from Lemma 3.5 that there exists $w \in \mathrm{AC}([a,b], \mathbb{C}^n)$ so that

$$\mathbf{X}(t)v_0 = \tilde{\mathbf{X}}w(t), \quad \forall\, t \in [a,b],$$

with also

$$0 = \mathbf{X}(t)\frac{d}{dt}v_0 = \tilde{\mathbf{X}}w'(t).$$

We can conclude that $w'(t) = 0$ for a.e. $t \in (a,b)$, and so $w(t) = w_0$ for some fixed $w_0 \in \mathbb{C}^n \backslash \{0\}$. But then

$$\mathbf{X}(t)v_0 = \tilde{\mathbf{X}}w_0, \quad \forall\, t \in [a,b],$$

giving the claim. □

4 Proofs of Theorems 1.1 and 1.2

In this section, we use our Maslov index framework to prove our two main theorems.

4.1 Proof of Theorem 1.1

Fix any pair $\lambda_1, \lambda_2 \in I$, with $\lambda_1 < \lambda_2$, and let $\ell_1(x;\lambda)$ denote the map of Lagrangian subspaces associated with the frames $\mathbf{X}_1(x;\lambda)$ specified in (1.4). Keeping in mind that λ_2 is fixed, let $\ell_2(x;\lambda_2)$ denote the map of Lagrangian subspaces associated with the frames $\mathbf{X}_2(x;\lambda_2)$ specified in (1.6). We emphasize that $\mathbf{X}_2(x;\lambda_2)$ is initialized at $x=1$. Effectively, this means that we are looking sideways at the usual Maslov Box, setting the target as the right shelf $\lambda = \lambda_2$, rather than the top shelf.

By Maslov Box in this setting, we mean the following sequence of contours: (1) fix $x=0$ and let λ increase from λ_1 to λ_2 (the *bottom shelf*); (2) fix $\lambda = \lambda_2$ and let x increase from 0 to 1 (the *right shelf*); (3) fix $x = 1$ and let λ decrease from λ_2 to λ_1 (the *top shelf*); and (4) fix $\lambda = \lambda_1$ and let x decrease from 1 to 0 (the *left shelf*). (See Figure 1.)

Following the general framework discussion in Section 2, we will compute the Maslov index along the contours specified in the Maslov box as a spectral flow of the unitary matrix function

$$\begin{aligned}\tilde{W}(x;\lambda) &:= -(X_1(x;\lambda) + iY_1(x;\lambda))(X_1(x;\lambda) - iY_1(x;\lambda))^{-1} \\ &\quad \times (X_2(x;\lambda_2) - iY_2(x;\lambda_2))(X_2(x;\lambda_2) + iY_2(x;\lambda_2))^{-1}.\end{aligned} \tag{4.1}$$

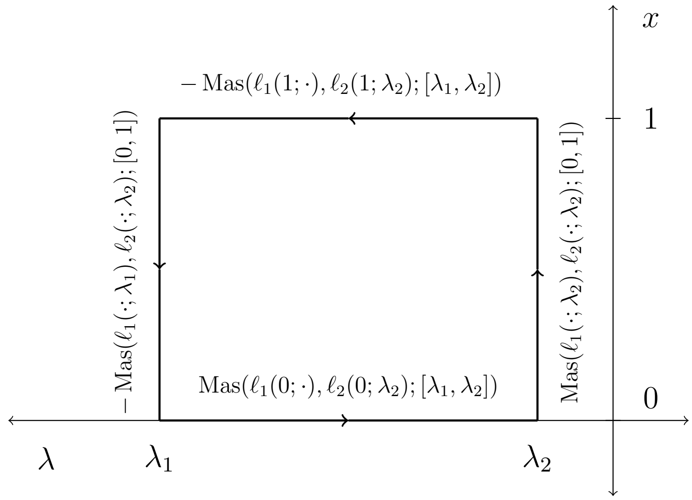

Figure 1: The Maslov Box.

The bottom shelf. We begin our analysis with the bottom shelf. Since $\mathbf{X}_1(0;\lambda) = J\alpha^*$ for all $\lambda \in [\lambda_1, \lambda_2]$ (in particular, is independent of λ), and $\mathbf{X}_2(0;\lambda_2)$ does not vary with λ, we see that in fact the matrix $\tilde{W}(0;\lambda)$ is constant as λ varies from λ_1 to λ_2, and so

$$\operatorname{Mas}(\ell_1(0;\cdot),\ell_2(0;\lambda_2);[\lambda_1,\lambda_2]) = 0. \tag{4.2}$$

This does not necessarily mean that -1 is not an eigenvalue of $\tilde{W}(0;\lambda)$; rather, if -1 is an eigenvalue of $\tilde{W}(0;\lambda)$ with multiplicity m for some $\lambda \in [\lambda_1, \lambda_2]$, then it remains fixed as an eigenvalue of $\tilde{W}(0;\lambda)$ with multiplicity m for all $\lambda \in [\lambda_1, \lambda_2]$.

The right shelf. For the right shelf, λ is fixed at λ_2 for both $\mathbf{X}_1$ and $\mathbf{X}_2$. By construction, $\ell_1(\cdot;\lambda_2)$ will intersect $\ell_2(\cdot;\lambda_2)$ at some $x = x_*$ with dimension m if and only if λ_2 is an eigenvalue of (1.1) with multiplicity m. In the event that λ_2 is not an eigenvalue of (1.1), there will be no crossing points along the right shelf. On the other hand, if λ_2 is an eigenvalue of (1.1) with multiplicity m, then $\tilde{W}(x;\lambda_2)$ will have -1 as an eigenvalue with multiplicity m for all $x \in [0,1]$. In either case,

$$\operatorname{Mas}(\ell_1(\cdot;\lambda_2),\ell_2(\cdot;\lambda_2);[0,1]) = 0. \tag{4.3}$$

The top shelf. For the top shelf, we know from Lemma 3.1 that monotonicity in λ is determined by $-\mathbf{X}_1(1;\lambda)^* J\partial_\lambda \mathbf{X}_1(1;\lambda)$, and we readily compute

$$\begin{aligned}
\frac{\partial}{\partial x}\mathbf{X}_1^*(x;\lambda)J\partial_\lambda\mathbf{X}_1(x;\lambda) &= (\mathbf{X}_1')^* J\partial_\lambda\mathbf{X}_1 + \mathbf{X}_1^* J\partial_\lambda\mathbf{X}_1' \\
&= -(\mathbf{X}_1')^* J^t\partial_\lambda\mathbf{X}_1 + \mathbf{X}_1^*\partial_\lambda J\mathbf{X}_1' \\
&= -\mathbf{X}_1^*\mathbb{B}(x;\lambda)\partial_\lambda\mathbf{X}_1 + \mathbf{X}_1^*\partial_\lambda(\mathbb{B}(x;\lambda)\mathbf{X}_1) = \mathbf{X}_1^*\mathbb{B}_\lambda\mathbf{X}_1.
\end{aligned}$$

Integrating on $[0,1]$, and noting that $\partial_\lambda\mathbf{X}_1(0;\lambda) = 0$, we see that

$$\mathbf{X}_1(1;\lambda)^* J\partial_\lambda\mathbf{X}_1(1;\lambda) = \int_0^1 \mathbf{X}_1(x;\lambda)^*\mathbb{B}_\lambda(x;\lambda)\mathbf{X}_1(x;\lambda)dy.$$

In this way, we see that condition **(B1)** ensures that as λ increases the eigenvalues of $\tilde{W}(1;\lambda)$ will rotate in the clockwise direction. Since each crossing along the top shelf corresponds with an eigenvalue, we can conclude that

$$\mathcal{N}([\lambda_1,\lambda_2)) = -\operatorname{Mas}(\ell_1(1;\cdot),\ell_2(1;\lambda_2);[\lambda_1,\lambda_2]).$$

We note that λ_1 is included in the count, because in the event that $(1,\lambda_1)$ is a crossing point, eigenvalues of $\tilde{W}(1;\lambda)$ will rotate away from -1 in the clockwise direction as λ increases from λ_1 (thus decrementing the Maslov index). Likewise, λ_2 is not included in the count, because in the event that $(1,\lambda_2)$ is a crossing point, eigenvalues of $\tilde{W}(1;\lambda)$ will rotate into -1 in the clockwise direction as λ increases to λ_2 (thus leaving the Maslov index unchanged).

The left shelf. Our analysis so far leaves only the left shelf to consider, and we observe that the Maslov index along the left shelf can be expressed as

$$-\operatorname{Mas}(\ell_1(\cdot;\lambda_1),\ell_2(\cdot;\lambda_2);[0,1]).$$

Using path additivity and homotopy invariance, we can sum the Maslov indices on each shelf of the Maslov Box to arrive at the relation

$$\mathcal{N}([\lambda_1,\lambda_2)) = \operatorname{Mas}(\ell_1(\cdot;\lambda_1),\ell_2(\cdot;\lambda_2);[0,1]), \tag{4.4}$$

which is (1.10) from Theorem 1.1.

In order to get from (4.4) to the second claim in Theorem 1.1, we need to verify that crossings along the left shelf occur monotonically in the counterclockwise direction as x increases. In this case we will have

$$\begin{aligned}\operatorname{Mas}(\ell_1(\cdot;\lambda_1),\ell_2(\cdot;\lambda_2);[0,1]) &= \sum_{0<x\le 1} \dim(\ell_1(x;\lambda_1)\cap\ell_2(x;\lambda_2)) \\ &= \sum_{0<x\le 1} \dim\ker(\mathbf{X}_1(x;\lambda_1)^* J\mathbf{X}_2(x;\lambda_2)).\end{aligned}$$

Here, $x=0$ is not included in the sum, because if $x=0$ is a crossing point, then as x increases from 0, the eigenvalues of $\tilde{W}(x;\lambda_1)$ will rotate away from -1 in the counterclockwise direction, and so will not increment the Maslov index.

In order to address this question of monotonicity, we adapt an idea from Remark 4.2 in [9]. For this, we begin by introducing a $2n\times 2n$ fundamental matrix $\Psi(x;\lambda)$ for (1.1), specified by

$$\begin{aligned} J\Psi' &= \mathbb{B}(x;\lambda_2)\Psi \\ \Psi(1;\lambda_2) &= I_{2n}. \end{aligned} \tag{4.5}$$

With this notation, we can express $\mathbf{X}_2(x;\lambda_2)$ as

$$\mathbf{X}_2(x;\lambda_2) = \Psi(x;\lambda_2)J\beta^*.$$

We now set

$$\mathbf{X}(x;\lambda,\lambda_2) := \Psi(x;\lambda_2)^{-1}\mathbf{X}_1(x;\lambda), \tag{4.6}$$

and note that

$$J\mathbf{X}'(x;\lambda,\lambda_2) = J\{(\Psi(x;\lambda_2)^{-1})'\mathbf{X}_1(x;\lambda) + \Psi(x;\lambda_2)^{-1}\mathbf{X}_1'(x;\lambda)\}.$$

By a straightforward calculation, we can check that

$$\Psi(x;\lambda)^* J \Psi(x;\lambda) = J, \quad \forall\, x \in [0,1], \tag{4.7}$$

from which we see that

$$\Psi(x;\lambda_2)^{-1} = -J\Psi(x;\lambda_2)^* J.$$

This allows us to write

$$\begin{aligned} J\mathbf{X}'(x;\lambda,\lambda_2) &= J\{-J\Psi'(x;\lambda_2)^* J\mathbf{X}_1(x;\lambda) - J\Psi(x;\lambda_2)^* J\mathbf{X}_1'(x;\lambda)\} \\ &= J\{J(J\Psi'(x;\lambda_2))^*\mathbf{X}_1(x;\lambda) - J\Psi(x;\lambda_2)^*\mathbb{B}(x;\lambda)\mathbf{X}_1(x;\lambda)\}. \end{aligned}$$

Next, we use (4.6) to re-write this relation as

$$\begin{aligned} J\mathbf{X}'(x;\lambda,\lambda_2) &= \{-(J\Psi'(x;\lambda_2))^*\Psi(x;\lambda_2)\mathbf{X}(x;\lambda,\lambda_2) + \Psi(x;\lambda_2)^*\mathbb{B}(x;\lambda)\Psi(x;\lambda_2)\mathbf{X}(x;\lambda,\lambda_2)\} \\ &= \{-(\mathbb{B}(x;\lambda_2)\Psi(x;\lambda_2))^*\Psi(x;\lambda_2)\mathbf{X}(x;\lambda,\lambda_2) + \Psi(x;\lambda_2)^*\mathbb{B}(x;\lambda)\Psi(x;\lambda_2)\mathbf{X}(x;\lambda,\lambda_2)\} \\ &= \Psi(x;\lambda_2)^*\{\mathbb{B}(x;\lambda) - \mathbb{B}(x;\lambda_2)\}\Psi(x;\lambda_2)\mathbf{X}(x;\lambda,\lambda_2). \end{aligned}$$

In summary, we can write

$$J\mathbf{X}' = \mathcal{B}(x;\lambda,\lambda_2)\mathbf{X}, \quad \mathcal{B}(x;\lambda,\lambda_2) := \Psi(x;\lambda_2)^*\{\mathbb{B}(x;\lambda) - \mathbb{B}(x;\lambda_2)\}\Psi(x;\lambda_2), \tag{4.8}$$

initialized with

$$\mathbf{X}(1;\lambda,\lambda_2) = \Psi(1;\lambda_2)^{-1}\mathbf{X}_1(1;\lambda) = \mathbf{X}_1(1;\lambda).$$

Since $\mathcal{B}(x;\lambda,\lambda_2)$ is self-adjoint for a.e. $x \in (0,1)$ and $\mathbf{X}_1(1;\lambda)$ is the frame for a Lagrangian subspace, we can conclude that $\mathbf{X}(x;\lambda,\lambda_2)$ is the frame for a Lagrangian subspace for each $(x,\lambda) \in [0,1] \times [\lambda_1,\lambda_2]$. In addition, according to Assumption **(B2)** $\mathcal{B}(x;\lambda_1,\lambda_2)$ is a non-positive matrix for a.e. $x \in (0,1)$.

Lemma 4.1. *For each $x \in [0,1]$,*

$$\mathbf{X}(x;\lambda,\lambda_2)^* J(J\beta^*) = \mathbf{X}_1(x;\lambda)^* J\mathbf{X}_2(x;\lambda_2).$$

Proof. To see this, we simply use (4.7) to write

$$\begin{aligned} \mathbf{X}(x;\lambda,\lambda_2)^* J(J\beta^*) &= (\Phi(x;\lambda_2)^{-1}\mathbf{X}_1(x;\lambda))^* JJ\beta^* = -\mathbf{X}_1(x;\lambda)^*(\Phi(x;\lambda_2)^{-1})^*\beta^* \\ &= -\mathbf{X}_1(x;\lambda)^*(-J\Phi(x;\lambda_2)^* J)^*\beta^* = \mathbf{X}_1(x;\lambda)^* J\Phi(x;\lambda_2)J\beta^* \\ &= \mathbf{X}_1(x;\lambda)^* J\mathbf{X}_2(x;\lambda_2). \end{aligned}$$

□

Lemma 4.1 allows us to detect intersections between $\ell_1(x;\lambda)$ and $\ell_2(x;\lambda_2)$ by instead detecting intersections between $\ell(x;\lambda,\lambda_2) := \operatorname{colspan}(\mathbf{X}(x;\lambda,\lambda_2))$ and the fixed target space $\ell_\beta := \operatorname{colspan}(J\beta^*)$. For these latter intersections, the associated rotation matrix is (with $\mathbf{X}(x;\lambda,\lambda_2) = \binom{X(x;\lambda,\lambda_2)}{Y(x;\lambda,\lambda_2)}$)

$$\begin{aligned} \tilde{\mathcal{W}}(x;\lambda,\lambda_2) := &-(X(x;\lambda,\lambda_2) + iY(x;\lambda,\lambda_2))(X(x;\lambda,\lambda_2) - iY(x;\lambda,\lambda_2))^{-1} \\ &\times (-\beta^* - i\beta_1^*)(-\beta_2^* + i\beta_1^*)^{-1}. \end{aligned} \tag{4.9}$$

At this point, we can proceed with a Maslov-box argument for intersections between $\ell(x;\lambda,\lambda_2)$ and ℓ_β. As before, we get no contributions from the bottom and right shelves, so all that's left to verify is monotonicity on the top and left shelves.

For the top shelf, the rotation is determined as usual by

$$
\begin{aligned}
\mathbf{X}(1;\lambda,\lambda_2)^* J\partial_\lambda \mathbf{X}(1;\lambda,\lambda_2) &= \int_0^1 \mathbf{X}(x;\lambda,\lambda_2)^* \mathcal{B}_\lambda(x;\lambda,\lambda_2)\mathbf{X}(x;\lambda,\lambda_2)dx \\
&= \int_0^1 \mathbf{X}(x;\lambda,\lambda_2)^* \Psi(x;\lambda_2)^* \mathbb{B}_\lambda(x;\lambda)\Psi(x;\lambda_2)\mathbf{X}(x;\lambda,\lambda_2)dx \\
&= \int_0^1 \mathbf{X}_1(x;\lambda)^* \mathbb{B}_\lambda(x;\lambda)\mathbf{X}_1(x;\lambda)dx,
\end{aligned}
$$

and so we obtain positivity as previously from **(B1)**.

For the left shelf, we've seen that $\mathcal{B}(x;\lambda_1,\lambda_2)$ is non-positive for a.e. $x \in (0,1)$, and it follows immediately from Lemma 3.3 that no eigenvalue of $\tilde{\mathcal{W}}(x;\lambda_1,\lambda_2)$ can rotate in the clockwise direction as x increases along $[0,1]$. This still leaves open the possibility that an eigenvalue of $\tilde{\mathcal{W}}(x;\lambda_1,\lambda_2)$ rotates into -1 at $x = x_*$ and remains there for some interval $[x_*, x_* + \delta]$, $\delta > 0$, but this is precisely the event that is ruled out by the second part of Assumption **(B2)**. We can conclude that the Maslov index for the left shelf is a monotonic (positive) count of intersections between $\ell(x;\lambda_1,\lambda_2)$ and ℓ_β, and this can be expressed as

$$
\begin{aligned}
\operatorname{Mas}(\ell(\cdot;\lambda_1,\lambda_2),\ell_\beta;[0,1]) &= \sum_{0<x\leq 1} \dim(\ell(x;\lambda_1,\lambda_2)\cap \ell_\beta) \\
&= \sum_{0<x\leq 1} \dim(\ell_1(x;\lambda_1)\cap \ell_2(x;\lambda_2)) \\
&= \sum_{0<x\leq 1} \dim\ker(\mathbf{X}_1(x;\lambda_1)^* J\mathbf{X}_2(x;\lambda_2)).
\end{aligned}
\tag{4.10}
$$

Using again path additivity and homotopy invariance, we can sum the Maslov indices on each shelf of the Maslov Box to arrive at the relation

$$
\begin{aligned}
\mathcal{N}([\lambda_1,\lambda_2)) &= \operatorname{Mas}(\ell(\cdot;\lambda_1,\lambda_2),\ell_\beta;[0,1]) \\
&= \sum_{0<x\leq 1} \dim(\ell_1(x;\lambda_1)\cap \ell_2(x;\lambda_2)) \\
&= \sum_{0<x\leq 1} \dim\ker(\mathbf{X}_1(x;\lambda_1)^* J\mathbf{X}_2(x;\lambda_2)),
\end{aligned}
\tag{4.11}
$$

which is (1.11) in Theorem 1.1.

Remark 4.1. *Comparing the relations*

$$\mathcal{N}([\lambda_1,\lambda_2)) = \operatorname{Mas}(\ell_1(\cdot;\lambda_1),\ell_2(\cdot;\lambda_2);[0,1]),$$

and

$$\mathcal{N}([\lambda_1,\lambda_2)) = \operatorname{Mas}(\ell(\cdot;\lambda_1,\lambda_2),\ell_\beta;[0,1]),$$

we see that

$$\mathrm{Mas}(\ell_1(\cdot;\lambda_1),\ell_2(\cdot;\lambda_2);[0,1]) = \mathrm{Mas}(\ell(\cdot;\lambda_1,\lambda_2),\ell_\beta;[0,1]). \tag{4.12}$$

Since these two Maslov indices detect precisely the same intersections with the same multiplicities, and the index on the right-hand side is a monotonic count of crossings, it must be the case that the index on the left-hand side is also a monotonic count of crossings. It's interesting to note, however, that while the eigenvalues of $\tilde{\mathcal{W}}(x;\lambda_1,\lambda_2)$ *rotate monotonically as* x *increases along* $(0,1)$*, the eigenvalues of* $\tilde{W}(x;\lambda_1)$ *are only necessarily monotonic at crossing points. In this way, the computation of* $\mathrm{Mas}(\ell(\cdot;\lambda_1,\lambda_2),\ell_\beta;[0,1])$ *is more tractable than the corresponding calculation of* $\mathrm{Mas}(\ell_1(\cdot;\lambda_1),\ell_2(\cdot;\lambda_2);[0,1])$*. Nonetheless, our preference has been to cast our analysis primarily in terms of the latter, because it is in no way tied to the specific application considered here, and so generalizes more readily to cases such as singular Hamiltonian systems.*

The authors are grateful to the referee for pointing out that (4.12) can also be viewed as a consequence of symplectic invariance (Property V in Section 1 of [8]); namely, we have the relations

$$\mathbf{X}(x;\lambda_1,\lambda_2) = \Psi(x;\lambda_2)^{-1}\mathbf{X}_1(x;\lambda_1) \quad \textit{and} \quad J\beta^* = \Psi(x;\lambda_2)^{-1}\mathbf{X}_1(x;\lambda_1),$$

and (4.12) follows immediately by symplectic invariance and the observation that $\Psi(x;\lambda_2)^{-1}$ *is a symplectic matrix for all* $x \in [0,1]$*. Moreover, if we specify the symplectic matrix*

$$\mathfrak{B} := \begin{pmatrix} \beta \\ -(\beta\beta^*)\beta J \end{pmatrix},$$

we see that $\mathfrak{B}J\beta^* = (0 \;\; I)^T$*, so that symplectic invariance additionally implies*

$$\mathrm{Mas}(\ell(\cdot;\lambda_1,\lambda_2),\ell_\beta;[0,1]) = \mathrm{Mas}(\ell_{\mathfrak{B}}(\cdot;\lambda_1,\lambda_2),\ell_D;[0,1]), \tag{4.13}$$

where $\ell_{\mathfrak{B}}(\cdot;\lambda_1,\lambda_2)$ *is the Lagrangian subspace with frame* $\mathfrak{B}\mathbf{X}(x;\lambda_1,\lambda_2)$*, and* ℓ_D *denotes the usual Dirichlet subspace. Restricted to the setting of* $\mathbb{R}^{2n}$*, the right-hand side of (4.13) now fits into the framework of [22], with our crossing points corresponding precisely with the* **focal points** *of Definition 1.1.1(ii) in that reference. In addition, our Assumption* **(B2)** *implies the* **controllability** *property specified in Definition 4.1.1 of [22] (see also Theorem 4.1.3 of [22]).*

We conclude this section by establishing a convenient criterion for verifying Assumption **(B2)**.

Lemma 4.2. *Let Assumptions* **(A)** *hold, and also assume there exist values* $\lambda_1,\lambda_2 \in I$*,* $\lambda_1 < \lambda_2$*, so that* $\mathbb{B}(x;\lambda_2) - \mathbb{B}(x;\lambda_1)$ *is non-negative for a.e.* $x \in (0,1)$*. Suppose that for any* $[a,b] \subset [0,1]$*,* $a < b$*, and any non-trivial solution* $y(x;\lambda_1)$ *of* $Jy' = \mathbb{B}(x;\lambda_1)y$*, we must have*

$$\int_a^b ((\mathbb{B}(x;\lambda_2) - \mathbb{B}(x;\lambda_1))y(x;\lambda_1), y(x;\lambda_1))dx > 0. \tag{4.14}$$

We have the following:

(i) Let Assumption **(B1)** *hold, and let* $\mathbf{X}_1(x;\lambda_1)$ *and* $\mathbf{X}_2(x;\lambda_2)$ *be the Lagrangian frames specified in (1.4) and (1.6), with respective Lagrangian subspaces* $\ell_1(x;\lambda_1)$ *and* $\ell_2(x;\lambda_2)$*. Then there is no interval* $[a,b]\subset[0,1]$*,* $a<b$*, so that*

$$\dim(\ell_1(x;\lambda_1)\cap\ell_2(x;\lambda_2))\neq\{0\}$$

for all $x\in[a,b]$*.*
(ii) Let Assumption $\textbf{(B1)}'$ *hold, and let* $\mathbf{X}_3(x;\lambda_1)$ *and* $\mathbf{X}_4(x;\lambda_2)$ *be the Lagrangian frames specified in (1.17) and (1.19), with respective Lagrangian subspaces* $\ell_3(x;\lambda_1)$ *and* $\ell_4(x;\lambda_2)$*. Then there is no interval* $[a,b]\subset[0,1]$*,* $a<b$*, so that*

$$\dim(\ell_3(x;\lambda_1)\cap\ell_4(x;\lambda_2))\neq\{0\}$$

for all $x\in[a,b]$*.*

Proof. We proceed by contradiction. First, for (i), suppose there exists an interval $[a,b]\subset[0,1]$, $a<b$, so that

$$\ell_1(x;\lambda_1)\cap\ell_2(x;\lambda_2)\neq\{0\}$$

for all $x\in[a,b]$. Then from Lemma 4.1, we have

$$\ell(x;\lambda_1,\lambda_2)\cap\ell_\beta\neq\{0\}$$

for all $x\in[a,b]$. For this latter relation, we've seen that any eigenvalue of $\tilde{\mathcal{W}}(x;\lambda_1,\lambda_2)$ that crosses -1 as x increases through a crossing point x_* must cross in the counterclockwise direction. This allows us to apply Lemma 3.4 to conclude that there exists some interval $[c,d]\subset[a,b]$, $c<d$, along with some integer $m\in\{1,2,\dots,n\}$, so that $\dim\ker(\tilde{\mathcal{W}}(x;\lambda_1,\lambda_2)+I)=m$ for all $x\in[c,d]$. According, then, to Lemma 3.6, we can conclude that there exists $v_0\in\mathbb{C}^n\backslash\{0\}$ so that $\mathbf{X}(x;\lambda_1,\lambda_2)v_0\in\ell(x;\lambda_1,\lambda_2)\cap\tilde{\ell}$ for all x in some possibly smaller interval $[\tilde{a},\tilde{b}]\subset[c,d]$, $\tilde{a}<\tilde{b}$, and subsequently we can conclude from Lemma 3.7 that $\mathbf{X}(x;\lambda_1,\lambda_2)v_0$ is independent of x. It follows from the relation $J\mathbf{X}'=\mathcal{B}(x;\lambda_1,\lambda_2)\mathbf{X}$ that $\mathcal{B}(x;\lambda_1,\lambda_2)\mathbf{X}v_0=0$ for a.e. $x\in[\tilde{a},\tilde{b}]$, and consequently we see that

$$\int_{\tilde{a}}^{\tilde{b}}(\mathcal{B}(x;\lambda_1,\lambda_2)\mathbf{X}(x;\lambda_1,\lambda_2)v_0,\mathbf{X}(x;\lambda_1,\lambda_2)v_0)dx=0. \tag{4.15}$$

Recalling (4.8), we can express (4.15) as

$$\begin{aligned}\int_{\tilde{a}}^{\tilde{b}}&(\Psi(x;\lambda_2)^*\{\mathbb{B}(x;\lambda_1)-\mathbb{B}(x;\lambda_2)\}\Psi(x;\lambda_2)\mathbf{X}(x;\lambda_1,\lambda_2)v_0,\mathbf{X}(x;\lambda_1,\lambda_2)v_0)dx\\ &=\int_{\tilde{a}}^{\tilde{b}}(\{\mathbb{B}(x;\lambda_1)-\mathbb{B}(x;\lambda_2)\}\Psi(x;\lambda_2)\mathbf{X}(x;\lambda_1,\lambda_2)v_0,\Psi(x;\lambda_2)\mathbf{X}(x;\lambda_1,\lambda_2)v_0)dx\\ &=\int_{\tilde{a}}^{\tilde{b}}(\{\mathbb{B}(x;\lambda_1)-\mathbb{B}(x;\lambda_2)\}\mathbf{X}_1(x;\lambda_1)v_0,\mathbf{X}_1(x;\lambda_1)v_0)dx=0.\end{aligned}$$

But $\mathbf{X}_1(x;\lambda_1)v_0$ is a non-trivial solution to $Jy'=\mathbb{B}(x;\lambda_1)y$, so this contradicts (4.14), giving Item (i).

Turning to Item (ii), we now suppose there exists an interval $[a,b] \subset [0,1]$, $a < b$, so that

$$\ell_3(x;\lambda_1) \cap \ell_4(x;\lambda_2) \neq \{0\} \tag{4.16}$$

for all $x \in [a,b]$. Arguing similarly as for Item (i), we find that there exists $v_0 \in \mathbb{C}^{2n} \backslash \{0\}$ so that $z(x;\lambda_1) := \mathbf{X}_3(x;\lambda_1)v_0$ satisfies the relation

$$\int_{\tilde{a}}^{\tilde{b}} (\{\mathcal{B}(x;\lambda_1) - \mathcal{B}(x;\lambda_2)\}z(x;\lambda_1), z(x;\lambda_1))dx = 0,$$

for some interval $[\tilde{a}, \tilde{b}] \subset [0,1]$, not necessarily the same as for Item (i). (Here, we recall that $\mathcal{B}(x;\lambda)$ denotes the matrix specified in (1.18) associated with (1.1)–**(BC2)**.) If we write $z = (z_1, z_2, z_3, z_4)^T$, and set $w := (z_2, z_4)^T$ then we can express this last integral relation as

$$\int_{\tilde{a}}^{\tilde{b}} ((\mathbb{B}(x;\lambda_1) - \mathbb{B}(x;\lambda_2))w(x;\lambda_1), w(x;\lambda_1))dx = 0,$$

which contradicts (4.14) unless $z_2(x;\lambda_1)$ and $z_4(x;\lambda_1)$ are identically 0 on $[\tilde{a}, \tilde{b}]$. In this case, it's clear from the specification of $\mathbf{X}_3(x;\lambda_1)$ in (1.17) that there must exist constant vectors $c_1, c_3 \in \mathbb{C}^n$ so that $z(x;\lambda_1) = (c_1, 0, c_3, 0)^T$ for all $x \in [\tilde{a}, \tilde{b}]$. But $z(x;\lambda_1) \in \ell_3(x;\lambda_1)$ as specified in (1.15), from which we see that $z(x;\lambda_1)$ must have the form $z(x;\lambda_1) = (y_1(0;\lambda_1), y_1(x;\lambda_1), -y_2(0;\lambda_1), y_2(x;\lambda_1))$ for some $y(x;\lambda_1)$ satisfying $Jy' = \mathbb{B}(x;\lambda_1)y$. I.e., we must have $y_1(x;\lambda_1) = 0$ and $y_2(x;\lambda_1) = 0$ for all $x \in [\tilde{a}, \tilde{b}]$, and this can only happen if $y_1(0;\lambda_1) = 0$ and $y_2(0;\lambda_1) = 0$, in which case $z(x;\lambda_1) = 0$ for all $x \in [\tilde{a}, \tilde{b}]$, contradicting the implication of (4.16). □

4.2 Proof of Theorem 1.2

Given the framework developed in the Introduction, the proof of Theorem 1.2 is almost identical to the proof of Theorem 1.1, and we omit the details.

5 Applications

In this section, we verify that our assumptions are satisfied by five example cases: Dirac systems, Sturm-Liouville systems, the family of linear Hamiltonian systems considered in [13], a system associated with differential-algebraic Sturm-Liouville systems, and a fourth-order self-adjoint equation. In the fourth case, $\mathbb{B}(x;\lambda)$ is nonlinear in λ, and in the fifth we will consider periodic boundary conditions. In all cases, we also demonstrate our theory with a numerical calculation, though our numerical calculation for Sturm-Liouville systems serves also as the numerical calculation for the family of systems from [13] (which includes Sturm-Liouville systems). The particular coefficient functions and boundary conditions chosen for these numerical calculations are not taken from physical applications, but rather have been selected to correspond with illustrative spectral curves (see Figures 2, 3, 4, and 5).

5.1 Dirac Systems

The canonical systems in which our assumptions clearly hold are Dirac systems, by which we mean equations of the general form

$$Jy' = (\lambda Q(x) + V(x))y, \tag{5.1}$$

where $Q, V \in L^1((0,1), \mathbb{C}^{2n})$ are self-adjoint matrices and $Q(x)$ is positive definite for a.e. $x \in (0,1)$. For this example, we will assume separated boundary conditions **(BC1)**.

We can think of this system in terms of the operator

$$\mathcal{L}_d := Q(x)^{-1}(J\frac{d}{dx} - V(x)),$$

with which we associate the domain

$$\mathcal{D}(\mathcal{L}_d) = \{y \in L^2((0,1), \mathbb{C}^{2n}) : y \in AC([0,1], \mathbb{C}^{2n}), \mathcal{L}_d y \in L^2((0,1), \mathbb{C}^{2n}); \textbf{(BC1)} \text{ holds}\},$$

and the inner product

$$\langle f, g\rangle_Q := \int_0^1 (Qf, g)dx.$$

With this choice of domain and inner product, $\mathcal{L}_d$ is densely defined, closed, and self-adjoint, (see, e.g., [24]), so $\sigma(\mathcal{L}_d) \subset \mathbb{R}$. I.e., we can take the interval I associated with (1.1) to be $I = \mathbb{R}$.

In this case, $\mathbb{B}(x;\lambda) = \lambda Q(x) + V(x)$, and we see immediately that our Assumptions **(A)** hold. For **(B1)**, $\mathbb{B}_\lambda(x;\lambda) = Q(x)$, so that

$$\int_0^1 \mathbf{X}_1(x;\lambda)^* \mathbb{B}_\lambda(x;\lambda)\mathbf{X}_1(x;\lambda)dx = \int_0^1 \mathbf{X}_1(x;\lambda)^* Q(x)\mathbf{X}_1(x;\lambda)dx,$$

which is positive definite (since $Q(x)$ is positive definite for a.e. $x \in (0,1)$).

For **(B2)**, given any $\lambda_1, \lambda_2 \in \mathbb{R}$, $\lambda_1 < \lambda_2$, we have $\mathbb{B}(x;\lambda_2) - \mathbb{B}(x;\lambda_1) = (\lambda_2 - \lambda_1)Q(x)$, which is clearly non-negative a.e. (in fact, positive definite). For the moreover part, we notice that

$$\mathbb{B}(x;\lambda_2) - \mathbb{B}(x;\lambda_1) = (\lambda_2 - \lambda_1)\mathbb{B}_\lambda(x;\lambda),$$

which allows us to use the argument establishing **(B1)** above to show that the condition in Claim 4.2 is satisfied.

We conclude from Theorem 1.1 that if $\mathcal{N}([\lambda_1, \lambda_2); \mathcal{L}_d)$ denotes the spectral count for $\mathcal{L}_d$, we have

$$\mathcal{N}([\lambda_1, \lambda_2); \mathcal{L}_d) = \mathcal{N}_{(0,1]}(\mathbf{X}_1(\cdot;\lambda_1)^* J\mathbf{X}_2(\cdot;\lambda_2)),$$

where $\mathbf{X}_1(x;\lambda_1)$ and $\mathbf{X}_2(x;\lambda_2)$ denote the frames specified respectively in (1.4) and (1.6), with $\mathbb{B}(x;\lambda)$ as in this section, and the notation $\mathcal{N}_{(0,1]}(\cdot)$ is as in (1.3).

In order to illustrate the difference between the approach taken in [14, 17] and the renormalized approach taken here, we consider a specific example with $Q = I_4$,

$$V(x) := \begin{pmatrix} .13 + \frac{.7*\cos(6\pi x)}{2+\cos(6\pi x)} & \frac{\cos(\pi x)}{2+\cos(4\pi x)} & 0 & 0 \\ \frac{\cos(\pi x)}{2+\cos(4\pi x)} & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \tag{5.2}$$

and Neumann boundary conditions specified by

$$\alpha = \begin{pmatrix} 0_2 & I_2 \end{pmatrix} \quad \beta = \begin{pmatrix} 0_2 & I_2 \end{pmatrix}. \tag{5.3}$$

As noted in Remark 1.5, the authors of [14, 17] specify $\mathbf{X}_1(x;\lambda)$ precisely as here, but in lieu of $\mathbf{X}_2(x;\lambda_2)$, use the fixed target space $\tilde{\mathbf{X}}_2 = J\beta^*$. In particular, in the setting of [14, 17] the Maslov index is computed via the unitary matrix

$$\tilde{\mathcal{W}}_\beta(x;\lambda) := -(X_1(x;\lambda) + iY_1(x;\lambda))(X_1(x;\lambda) - iY_1(x;\lambda))^{-1}(-\beta_2^* - i\beta_1^*)(-\beta_2^* + i\beta_1^*)^{-1}.$$

The spectral curves discussed in Remark 1.5 can be computed in the case of [14, 17] as the pairs (x,λ) for which $\dim\ker(\tilde{\mathcal{W}}_\beta(x;\lambda) + I) \neq 0$, and likewise can be computed in the current setting as the pairs (x,λ) for which $\dim\ker(\tilde{W}(x;\lambda) + I) \neq 0$. Spectral curves for (5.1) with $Q = I_4$, V specified in (5.2), and boundary conditions (5.3) are depicted in Figure 2, with the approach of [14, 17] on the left and the renormalized approach on the right. Several things are worth noting about this comparison of images: (1) the difference between the non-monotonic curve on the left and the monotonic curve on the right is striking and illustrates precisely the main difference in the two approaches; (2) while the spectral curve on the left emerges from the bottom shelf, the spectral curve on the right enters the Maslov box through the left shelf; and (3) since crossings along the top shelf correspond with eigenvalues in both cases, the spectral curves in the left and right sides of Figure 2 both cross the top shelf at the same value of λ. Regarding Item (2), in the setting of [14, 17], spectral curves can enter through any of the three shelves—left, bottom, or right—while in the renormalized setting, spectral curves can only enter through the left shelf.

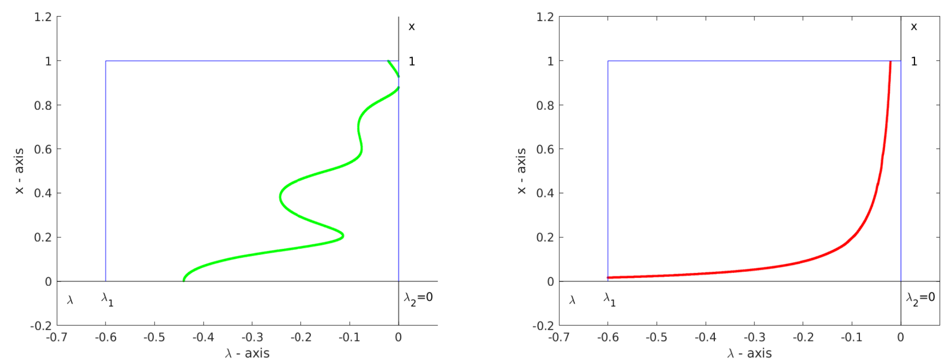

Figure 2: Spectral curves for the Dirac equation example: approach of [14, 17] on left; renormalized approach on right.

5.2 Linear Hamiltonian Systems with Block Matrix Coefficients

In [13], the authors consider linear Hamiltonian systems

$$Jy' = (\lambda Q(x) + V(x))y, \tag{5.4}$$

where

$$Q(x) = \begin{pmatrix} R(x) & 0 \\ 0 & 0 \end{pmatrix},$$

for some $r \times r$ matrix $R(x)$, $1 \le r \le 2n$. The matrices $R(x)$ and $V(x)$ are taken to be self-adjoint for a.e. $x \in (0,1)$, with $R(x)$ additionally positive definite for a.e. $x \in (0,1)$, and in the bounded-interval case, the authors assume $Q, V \in L^1((0,1), \mathbb{C}^{2n \times 2n})$. (In [13], the authors work on a general bounded interval (a,b), but this can always be scaled for convenience to $(0,1)$.)

In order to accommodate the form of Q, the authors of [13] introduce a Hilbert space

$$L^2_R((0,1), \mathbb{C}^r) := \{f : (0,1) \to \mathbb{C}^r \text{ measurable}, \|f\|_{L^2_R((0,1),\mathbb{C}^r)} < \infty\},$$

where $\|\cdot\|_{L^2_R((0,1),\mathbb{C}^r)}$ denotes the weighted norm

$$\|f\|^2_{L^2_R((0,1),\mathbb{C}^r)} := \int_0^1 (R(x)f(x), f(x))dx.$$

In addition, denoting the natural restriction operator $\hat{E}_r : \mathbb{C}^{2n} \to \mathbb{C}^r$, the authors introduce

$$L^2_Q((0,1), \mathbb{C}^{2n}) := \{g : (0,1) \to \mathbb{C}^{2n} \text{ measurable}, \hat{E}_r g \in L^2_R((0,1), \mathbb{C}^r)\},$$

along with the seminorm

$$\|g\|_{L^2_Q((0,1),\mathbb{C}^{2n})} := \|\hat{E}_r g\|_{L^2_R((0,1),\mathbb{C}^r)}.$$

Finally, the authors assume Atkinson's definiteness condition, described as follows: assume that for all $a, b \in (0,1)$ with $a < b$, any nonzero solution $y \in AC([0,1], \mathbb{C}^{2n})$ of (5.4) satisfies

$$\|\chi_{[a,b]} y\|_{L^2_Q((0,1),\mathbb{C}^{2n})} > 0,$$

where $\chi_{[a,b]}$ denotes the usual characteristic function on $[a,b]$.

Under these assumptions, the authors of [13] are able to express (5.4) in terms of the operator

$$\mathcal{L}_b := C(x)(J\frac{d}{dx} - V(x)), \tag{5.5}$$

where

$$C(x) = \begin{pmatrix} R(x)^{-1} & 0 \\ 0 & I_{2n-r} \end{pmatrix},$$

and the domain of $\mathcal{L}_b$ is specified as

$$\mathcal{D}(\mathcal{L}_b) := \{y \in L^2_Q((0,1), \mathbb{C}^{2n}) : y \in AC([0,1], \mathbb{C}^{2n}), \mathcal{L}_b y \in E_r L^2_Q((0,1), \mathbb{C}^{2n}), \textbf{(BC1)} \text{ holds}\},$$

with

$$E_r = \begin{pmatrix} I_r & 0 \\ 0 & 0 \end{pmatrix}.$$

In [13] the authors verify in Section 2 that $\sigma(\mathcal{L}_b) \subset \mathbb{R}$.

We see directly from these specifications that our assumptions **(A)** hold in this case. To check **(B1)**, we compute $\mathbb{B}_\lambda(x;\lambda) = Q(x)$, from which we see that

$$\int_0^1 \mathbf{X}_1(x;\lambda)^* \mathbb{B}_\lambda(x;\lambda) \mathbf{X}_1(x;\lambda) dx = \int_0^1 \mathbf{X}_1(x;\lambda)^* Q(x) \mathbf{X}_1(x;\lambda) dx.$$

Since $Q(x)$ is non-negative for a.e. $x \in (0,1)$, this integral is certainly non-negative, and moreover, it can only be zero if there exists a vector $v \in \mathbb{C}^n$ so that $Q(x)\mathbf{X}_1(x;\lambda)v = 0$ for a.e. $x \in (0,1)$. By definition of $\mathbf{X}_1$, $\psi(x) := \mathbf{X}_1(x;\lambda)v$ solves $J\psi' = \mathbb{B}(x;\lambda)\psi$. If we write $\psi = \binom{\psi_1}{\psi_2}$, with $\psi_1(x;\lambda) \in \mathbb{C}^r$ and $\psi_2(x;\lambda) \in \mathbb{C}^{2n-r}$, we see that since $Q(x)\psi(x;\lambda) = 0$ for a.e. $x \in (0,1)$, we must have $R(x)\psi_1(x;\lambda) = 0$ for a.e. $x \in (0,1)$, and so $\psi_1(x;\lambda) = 0$ for a.e. $x \in (0,1)$. But then

$$\|\chi_{[0,1]}\psi\|_{L^2_Q((0,1),\mathbb{C}^{2n})} = \int_0^1 (R(x)\psi_1(x;\lambda), \psi_1(x;\lambda)) dx = 0,$$

and this contradicts Atkinson's positivity assumption.

For **(B2)**, we fix $\lambda_1, \lambda_2 \in I$, $\lambda_1 < \lambda_2$, and observe that

$$\mathbb{B}(x;\lambda_2) - \mathbb{B}(x;\lambda_1) = (\lambda_2 - \lambda_1)\mathbb{B}_\lambda(x;\lambda).$$

We see immediately that $\mathbb{B}(x;\lambda_2) - \mathbb{B}(x;\lambda_1)$ is non-negative, and in addition, the same argument used to verify **(B1)** shows that the condition assumed in Claim 4.2 holds. Assumption **(B2)** follows.

We conclude from Theorem 1.1 that if $\mathcal{N}([\lambda_1,\lambda_2);\mathcal{L}_b)$ denotes the spectral count for $\mathcal{L}_b$, we have

$$\mathcal{N}([\lambda_1,\lambda_2);\mathcal{L}_b) = \mathcal{N}_{(0,1]}(\mathbf{X}_1(\cdot;\lambda_1)^* J \mathbf{X}_2(\cdot;\lambda_2)),$$

where $\mathbf{X}_1(x;\lambda_1)$ and $\mathbf{X}_2(x;\lambda_2)$ denote the frames (1.4) and (1.6), with $\mathbb{B}(x;\lambda)$ as specified in this section.

5.3 Sturm-Liouville Systems

As an important special case of the general family of systems discussed in Section 5.2, we consider the Sturm-Liouville system

$$-(P(x)\phi')' + V(x)\phi = \lambda Q(x)\phi, \tag{5.6}$$

with boundary conditions

$$\begin{aligned} \alpha_1\phi(0) + \alpha_2 P(0)\phi'(0) = 0 \\ \beta_1\phi(1) + \beta_2 P(1)\phi'(1) = 0. \end{aligned} \tag{5.7}$$

Here, $\phi(x) \in \mathbb{C}^n$, and our notational convention is to take $\alpha = (\alpha_1\ \alpha_2) \in \mathbb{C}^{2n\times n}$ and $\beta = (\beta_1\ \beta_2) \in \mathbb{C}^{2n\times n}$, with α and β satisfying **(BC1)**. We assume $P \in AC([0,1],\mathbb{C}^{n\times n})$, $V, Q \in L^1((0,1),\mathbb{C}^{n\times n})$, and that all three matrices are self-adjoint for a.e. $x \in (0,1)$. In addition, we assume that $P(x)$ is invertible for each $x \in [0,1]$, and that $Q(x)$ is positive definite for a.e. $x \in (0,1)$.

We can think of this system in terms of the operator

$$\mathcal{L}_s\phi := Q(x)^{-1}\{-(P(x)\phi')' + V(x)\phi\},$$

with which we associate the domain

$$\mathcal{D}(\mathcal{L}_s) = \{\phi \in L^2((0,1),\mathbb{C}^n) : \phi, \phi' \in AC([0,1],\mathbb{C}^n), \mathcal{L}_s\phi \in L^2((0,1),\mathbb{C}^n), (5.7)\,\text{holds}\},$$

and the inner product

$$\langle \phi, \psi \rangle_Q := \int_0^1 (Q(x)\phi(x), \psi(x))_{\mathbb{C}^n} dx.$$

With this choice of domain and inner product, $\mathcal{L}_s$ is densely defined, closed, and self-adjoint, so $\sigma(\mathcal{L}_s) \subset \mathbb{R}$. I.e., we can take the interval I associated with (1.1) to be $I = \mathbb{R}$.

For each $x \in [0,1]$, we define a new vector $y(x) \in \mathbb{C}^{2n}$ so that $y(x) = (y_1(x)\ y_2(x))^t$, with $y_1(x) = \phi(x)$ and $y_2(x) = P(x)\phi'(x)$. In this way, we express (5.3) in the form

$$\begin{aligned} &y' = \mathbb{A}(x;\lambda)y; \quad \mathbb{A}(x;\lambda) = \begin{pmatrix} 0 & P(x)^{-1} \\ V(x) - \lambda Q(x) & 0 \end{pmatrix}, \\ &\alpha y(0) = 0; \quad \beta y(1) = 0. \end{aligned}$$

Upon multiplying both sides of this equation by J, we obtain (1.1) with

$$\mathbb{B}(x;\lambda) = \begin{pmatrix} \lambda Q(x) - V(x) & 0 \\ 0 & P(x)^{-1} \end{pmatrix}.$$

It is clear that $\mathbb{B}(x;\lambda)$ satifies our basic assumptions **(A)**. We check that $\mathbb{B}(x;\lambda)$ also satisfies Assumptions **(B1)** and **(B2)**.

First, for **(B1)**, we compute

$$\mathbb{B}_\lambda(x;\lambda) = \begin{pmatrix} Q(x) & 0 \\ 0 & 0 \end{pmatrix},$$

so that

$$\int_0^1 \mathbf{X}_1(x;\lambda)^*\mathbb{B}_\lambda(x;\lambda)\mathbf{X}_1(x;\lambda)dx = \int_0^1 X_1(x;\lambda)^*Q(x)X_1(x;\lambda)dx,$$

which is clearly non-negative (since Q is positive definite), and moreover it cannot have 0 as an eigenvalue, because the associated eigenvector $v \in \mathbb{C}^n$ would necessarily satisfy $X_1(x;\lambda)v = 0$ for all $x \in [0,1]$, and this would contradict linear independence of the columns of $X_1(x;\lambda)$ (as solutions of (5.6)).

For **(B2)**, we fix any $\lambda_1, \lambda_2 \in \mathbb{R}$, $\lambda_1 < \lambda_2$, and notice that

$$\mathbb{B}(x;\lambda_2) - \mathbb{B}(x;\lambda_1) = (\lambda_2 - \lambda_1)\mathbb{B}_\lambda(x;\lambda),$$

which is clearly non-negative. In addition, the same argument used to verify **(B1)** shows that the condition assumed in Claim 4.2 holds. Assumption **(B2)** follows.

We conclude from Theorem 1.1 that if $\mathcal{N}([\lambda_1,\lambda_2);\mathcal{L}_s)$ denotes the spectral count for $\mathcal{L}_s$, we have

$$\mathcal{N}([\lambda_1,\lambda_2);\mathcal{L}_s) = \mathcal{N}_{(0,1]}(\mathbf{X}_1(\cdot;\lambda_1)^*J\mathbf{X}_2(\cdot;\lambda_2)),$$

where $\mathbf{X}_1(x;\lambda_1)$ and $\mathbf{X}_2(x;\lambda_2)$ denote the frames specified in (1.4) and (1.6), with $\mathbb{B}(x;\lambda)$ as specified in this section.

As a specific example in this case, we consider (5.6) with $P = I_2$, $Q = 9I_2$,

$$V(x) = \begin{pmatrix} -2.7 & -18\sin(3x) + .0081x^2 \\ -18\sin(3x) + .0081x^2 & 0 \end{pmatrix}, \tag{5.8}$$

and boundary conditions **(BC1)** specified by $\alpha = (\frac{1}{\sqrt{2}}I_2 \; \frac{1}{3\sqrt{2}}I_2)$ and $\beta = (\frac{1}{\sqrt{2}}I_2 \; \frac{1}{3\sqrt{2}}I_2)$. Spectral curves for this equation are depicted in Figure 3, with the approach of [14, 17] on the left and the renormalized approach on the right. As in our example for Dirac equations, several things are worth noting about this comparison of images: (1) for the figure on the left, we see that the middle spectral curve is non-monotonic, while for the figure on the right, all three spectral curves are monotonic; (2) as in Figure 2, we see that in the setting of [14, 17] spectral curves can emerge from any of the lower three shelves (bottom and right in this case), while in the renormalized setting they can only emerge from the left shelf; and (3) for the figure on the right, we see that spectral curves in the renormalized setting can almost become vertical. Regarding Item (3), we verify in our proof of Theorem 1.1 that the spectral curves cannot be vertical along any interval $\lambda \times (a, b)$ for $0 \le a < b \le 1$, and indeed this is also clear from the analysis of [11] in which the authors show that there can only be a finite number of crossing points along any vertical shelf (see also [13] for the same result in a more general setting).

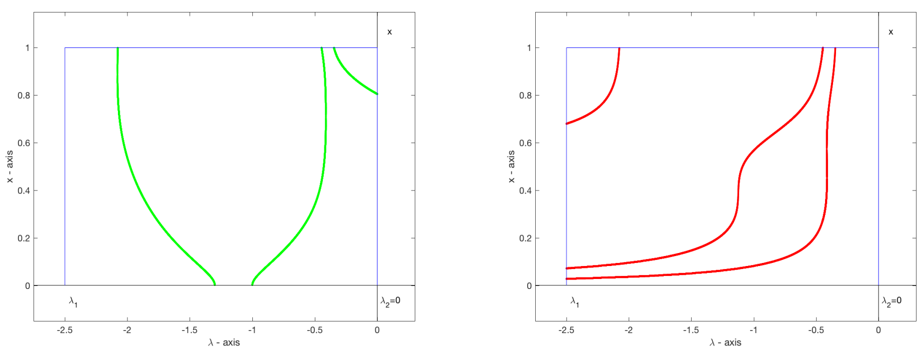

Figure 3: Spectral curves for the Sturm-Liouville system example: approach of [14, 17] on left; renormalized approach on right.

5.4 Differential-Algebraic Sturm-Liouville Systems

We consider systems

$$\mathcal{L}_a\phi = -(P(x)\phi')' + V(x)\phi = \lambda\phi, \tag{5.9}$$

with degenerate matrices

$$P(x) = \begin{pmatrix} P_{11}(x) & 0 \\ 0 & 0 \end{pmatrix}.$$

Here, for some $0 < m < n$, $P_{11} \in AC([0,1], \mathbb{C}^{m \times m})$ is a map into the space of self-adjoint matrices. We assume $P_{11}(x)$ is invertible for all $x \in [0,1]$, and additionally that $V \in C([0,1], \mathbb{C}^{n \times n})$. For notational convenience, we will write

$$V(x) = \begin{pmatrix} V_{11}(x) & V_{12}(x) \\ V_{12}(x)^* & V_{22}(x) \end{pmatrix},$$

where for each $x \in [0,1]$, $V_{11}(x)$ is a self-adjoint $m \times m$ matrix, $V_{12}(x)$ is an $m \times (n-m)$ matrix, and $V_{22}(x)$ is a self-adjoint $(n-m) \times (n-m)$ matrix. We will write

$$\phi = \begin{pmatrix} \phi_1 \\ \phi_2 \end{pmatrix}; \quad \phi_1(x;\lambda) \in \mathbb{C}^m; \quad \phi_2(x;\lambda) \in \mathbb{C}^{n-m},$$

allowing us to express the system as

$$\begin{aligned} -(P_{11}(x)\phi_1')' + V_{11}(x)\phi_1 + V_{12}(x)\phi_2 &= \lambda\phi_1 \\ V_{12}(x)^*\phi_1 + V_{22}(x)\phi_2 &= \lambda\phi_2. \end{aligned}$$

We place separated, self-adjoint boundary conditions on the first m components,

$$\begin{aligned} \alpha_1\phi_1(0) + \alpha_2 P_{11}(0)\phi_1'(0) &= 0 \\ \beta_1\phi_1(1) + \beta_2 P_{11}(1)\phi_1'(1) &= 0, \end{aligned} \tag{5.10}$$

with $\alpha = (\alpha_1 \ \alpha_2)$ and $\beta = (\beta_1 \ \beta_2)$ satisfying the same assumptions as in Section 5.3, except with n replaced by m. We specify the domain

$$\begin{aligned} \mathcal{D}(\mathcal{L}_a) = \{\phi = (\phi_1, \phi_2) \in L^2((0,1), \mathbb{C}^m) \times L^2((0,1), \mathbb{C}^{n-m}) : \phi_1, \phi_1' \in AC([0,1], \mathbb{C}^m), \\ (5.10) \text{ holds}, \ \mathcal{L}_a\phi \in L^2((0,1), \mathbb{C}^m) \times L^2((0,1), \mathbb{C}^{n-m})\}, \end{aligned}$$

and note that with this domain, it is straightfoward to verify that $\mathcal{L}_a$ is densely defined (in $L^2((0,1), \mathbb{C}^m) \times L^2((0,1), \mathbb{C}^{n-m})$), closed, and self-adjoint. In addition, we can apply Theorem 2.2 of [1] to see that the essential spectrum of $\mathcal{L}_a$ is precisely the union of the ranges of the eigenvalues of $V_{22}(x)$ as x ranges over $[0,1]$. More precisely, let $\{\nu_k(x)\}_{k=1}^{n-m}$ denote the eigenvalues of $V_{22}(x)$, and let $\mathcal{R}_k$ denote the range of $\nu_k : [0,1] \to \mathbb{R}$. Then

$$\sigma_{\text{ess}}(\mathcal{L}_a) = \bigcup_{k=1}^{n-m} \mathcal{R}_k.$$

Now, fix any $\lambda_1 < \lambda_2$ so that $[\lambda_1, \lambda_2] \cap \sigma_{\text{ess}}(\mathcal{L}_a) = \emptyset$, and take any $\lambda \in [\lambda_1, \lambda_2]$. Then we can write

$$\phi_2(x;\lambda) = (\lambda I - V_{22}(x))^{-1} V_{12}(x)^* \phi_1(x;\lambda).$$

Upon substitution of ϕ_2 into the equations for ϕ_1, we obtain

$$-(P_{11}(x)\phi_1')' + V_{11}(x)\phi_1 + V_{12}(x)(\lambda I - V_{22}(x))^{-1} V_{12}(x)^* \phi_1 = \lambda\phi_1.$$

We can express this system as a first-order system in the usual way, writing $y_1 = \phi_1$ and $y_2 = P_{11}\phi_1'$, so that

$$y' = \mathbb{A}(x;\lambda)y,$$

with

$$\mathbb{A}(x;\lambda) = \begin{pmatrix} 0 & P_{11}(x)^{-1} \\ \mathbf{V}(x;\lambda) - \lambda I & 0 \end{pmatrix}; \quad \mathbf{V}(x;\lambda) = V_{11} + V_{12}(\lambda I - V_{22})^{-1} V_{12}^*.$$

We multiply by J to obtain the usual Hamiltonian form (1.1) with

$$\mathbb{B}(x;\lambda) = \begin{pmatrix} \lambda I - \mathbf{V}(x;\lambda) & 0 \\ 0 & P_{11}(x)^{-1} \end{pmatrix}.$$

It's clear from our Assumptions on $P(x)$ and $V(x)$ that the first part of Assumptions **(A)** (addressing $\mathbb{B}(x;\lambda)$, not $\mathbb{B}_\lambda(x;\lambda)$) holds in this case for any closed interval I so that $I \cap \sigma_{\text{ess}}(\mathcal{L}_a) = \emptyset$. In order to verify the second part of Assumptions **(A)** and also Assumption **(B1)**, we first compute

$$\mathbb{B}_\lambda(x;\lambda) = \begin{pmatrix} I - \mathbf{V}_\lambda(x;\lambda) & 0 \\ 0 & 0 \end{pmatrix},$$

where

$$\mathbf{V}_\lambda(x;\lambda) = -V_{12}(x)(\lambda I - V_{22}(x))^{-2} V_{12}(x)^*.$$

We see immediately that the second part of Assumptions **(A)** holds for any closed interval I as described just above. In addition, since $(\lambda I - V_{22}(x))^{-1}$ is self-adjoint for all $x \in [0,1]$, we can express $\mathbf{V}_\lambda(x;\lambda)$ as

$$\mathbf{V}_\lambda(x;\lambda) = -((\lambda I - V_{22}(x))^{-1} V_{12}(x)^*)^*((\lambda I - V_{22}(x))^{-1} V_{12}(x)^*),$$

which is negative definite as long as $V_{12}(x)$ has trivial kernel and non-positive in any case. We see that $I - \mathbf{V}_\lambda(x;\lambda)$ is positive definite, and monotonicity (i.e., **(B1)**) now follows in almost precisely the same way as in Section 5.3.

Turning to **(B2)**, we fix any λ_1 and λ_2 as described above, and observe that

$$\mathbb{B}(x;\lambda_2) - \mathbb{B}(x;\lambda_1) = \begin{pmatrix} (\lambda_2 - \lambda_1) I - \mathbf{V}(x;\lambda_2) + \mathbf{V}(x;\lambda_1) & 0 \\ 0 & 0 \end{pmatrix}.$$

Computing directly, we see that

$$\begin{aligned} -\mathbf{V}(x;\lambda_2) + \mathbf{V}(x;\lambda_1) &= V_{12}(x)\{-(\lambda_2 I - V_{22})^{-1} + (\lambda_1 I - V_{22})^{-1}\} V_{12}(x)^* \\ &= (\lambda_2 - \lambda_1) V_{12}(x)\{(\lambda_1 I - V_{22})^{-1}(\lambda_2 I - V_{22})^{-1}\} V_{12}(x)^*. \end{aligned}$$

The matrix in curved brackets is self-adjoint, and by spectral mapping, its eigenvalues are

$$\Big\{ \frac{1}{(\lambda_1 - \nu_k(x))(\lambda_2 - \nu_k(x))} \Big\}_{k=1}^{n-m}. \tag{5.11}$$

By our assumption that $[\lambda_1, \lambda_2] \cap \sigma_{\text{ess}} = \emptyset$, we see that for each $k \in \{1, 2, \dots, n-m\}$ and each $x \in [0,1]$, we either have $\lambda_1 < \lambda_2 < \nu_k(x)$ or we have $\nu_k(x) < \lambda_1 < \lambda_2$. (The idea is simply that $\nu_k(x)$ cannot lie between λ_1 and λ_2.) In either case, the eigenvalues (5.11) are all positive, verifying that the self-adjoint matrix $(\lambda_1 I - V_{22})^{-1}(\lambda_2 I - V_{22})^{-1}$ is positive definite. It follows immediately that $(\lambda_2 - \lambda_1) I - \mathbf{V}(x;\lambda_2) + \mathbf{V}(x;\lambda_1)$ is positive definite, and $\mathbb{B}(x;\lambda_2) - \mathbb{B}(x;\lambda_1)$ is non-negative. In addition, an argument similar to our verification of **(B1)** in this case serves to verify that the assumptions of Claim 4.2 hold in this case, and the moreover part of **(B2)** follows.

We conclude from Theorem 1.1 that if $\mathcal{N}([\lambda_1,\lambda_2);\mathcal{L}_a)$ denotes the spectral count for $\mathcal{L}_a$, we have

$$\mathcal{N}([\lambda_1,\lambda_2);\mathcal{L}_a) = \mathcal{N}_{(0,1]}(\mathbf{X}_1(\cdot;\lambda_1)^* J \mathbf{X}_2(\cdot;\lambda_2)),$$

where $\mathbf{X}_1(x;\lambda_1)$ and $\mathbf{X}_2(x;\lambda_2)$ denote the frames (1.4) and (1.6), with $\mathbb{B}(x;\lambda)$ as specified in this section.

As a specific example in this case, we consider (5.9) with

$$P = \begin{pmatrix} I_2 & 0_2 \\ 0_2 & 0_2 \end{pmatrix},$$

$$V(x) = \begin{pmatrix} -8 - \frac{.7\cos(6\pi x)}{2+\cos(6\pi x)} & -\frac{\cos(\pi x)}{2+\cos(4\pi x)} & 1 & 0 \\ -\frac{\cos(\pi x)}{2+\cos(4\pi x)} & 1 & 0 & 1 \\ 1 & 0 & 1 - .8x\sin(x) & 0 \\ 0 & 1 & 0 & 1 - .8x\sin(x) \end{pmatrix},$$

and Neumann boundary conditions specified by $\alpha = (0_2 \ I_2)$ and $\beta = (0_2 \ I_2)$.

Spectral curves are depicted in Figure 4 for this example, with the approach of [14, 17] on the left and the renormalized approach on the right. In this case,

$$V_{22}(x) = \begin{pmatrix} 1 - .8x\sin(x) & 0 \\ 0 & 1 - 8x\sin(x) \end{pmatrix},$$

so the essential spectrum is confined to the range of $(1 - .8x\sin(x))|_{[0,1]} = [1 - .8\sin(1), 1]$, well to the right of our depicted Maslov boxes.

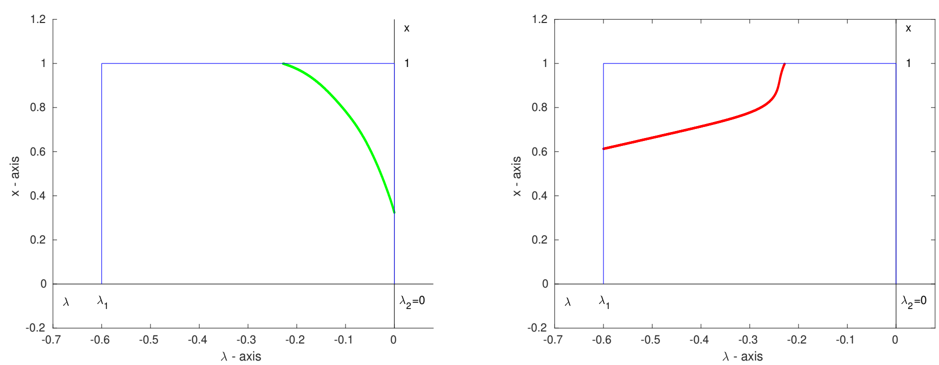

Figure 4: Spectral curves for the differential-algebraic equation example: approach of [14, 17] on left; renormalized approach on right.

5.5 Fourth-Order Equations

We consider fourth-order systems

$$\phi'''' - (V_2(x)\phi')' + V_0(x)\phi = \lambda\phi, \tag{5.12}$$

with periodic boundary conditions

$$\phi^{(k)}(1) = \phi^{(k)}(0), \quad k = 0, 1, 2, 3. \tag{5.13}$$

Here, $\phi(x) \in \mathbb{C}^n$, and we take $V_0 \in L^1((0,1), \mathbb{C}^{n\times n})$ and $V_2 \in AC([0,1], \mathbb{C}^{n\times n})$, with both matrix functions self-adjoint for a.e. $x \in (0,1)$, and $V_2(0) = V_2(1)$. (This final condition isn't necessary, and is used only to retain periodicity of the boundary conditions for the first-order system in the form we'll express it.)

We can think of this system in terms of the operator

$$\mathcal{L}_f\phi := \phi'''' - (V_2(x)\phi')' + V_0(x)\phi,$$

with which we associate the domain

$$\mathcal{D}(\mathcal{L}_f) = \{\phi \in L^2((0,1), \mathbb{C}^n) : \phi^{(k)} \in AC([0,1], \mathbb{C}^n), k = 0, 1, 2, 3,\\ \mathcal{L}_f\phi \in L^2((0,1), \mathbb{C}^n), (5.13) \text{ holds}\}.$$

With this choice of domain and inner product, $\mathcal{L}_f$ is densely defined, closed, and self-adjoint, so $\sigma(\mathcal{L}_f) \subset \mathbb{R}$. I.e., we can take the interval I associated with (1.1) to be $I = \mathbb{R}$. (See, e.g., [33].)

For each $x \in [0,1]$, we define a new vector $y(x) \in \mathbb{C}^{4n}$ so that

$$y(x) = (y_1(x)\ y_2(x)\ y_3(x)\ y_4(x))^T,$$

with $y_1(x) = \phi(x)$, $y_2(x) = \phi''(x)$, $y_3(x) = -\phi'''(x) + V_2(x)\phi'(x)$, and $y_4(x) = -\phi'(x)$. In this way, we express (5.12) in the form

$$y' = \mathbb{A}(x;\lambda)y; \quad \mathbb{A}(x;\lambda) = \begin{pmatrix} 0 & 0 & 0 & -I_n \\ 0 & 0 & -I_n & -V_2(x) \\ V_0(x) - \lambda I_n & 0 & 0 & 0 \\ 0 & -I_n & 0 & 0 \end{pmatrix},$$

$$\Theta \begin{pmatrix} y(0) \\ y(1) \end{pmatrix} = 0, \quad \Theta = (I_{4n} \ \ -I_{4n}).$$

(See [14] for a discussion of the rationale for this choice in defining y.) Upon multiplying both sides of this equation by J, we obtain (1.1) with

$$\mathbb{B}(x;\lambda) = \begin{pmatrix} \lambda I_n - V_0(x) & 0 & 0 & 0 \\ 0 & I_n & 0 & 0 \\ 0 & 0 & 0 & -I_n \\ 0 & 0 & -I_n & -V_2(x) \end{pmatrix}.$$

It is clear that $\mathbb{B}(x;\lambda)$ satifies our basic assumptions **(A)**. We check that $\mathbb{B}(x;\lambda)$ also satisfies Assumptions **(B1)**$'$ and **(B2)**$'$.

First, for **(B1)**$'$, we compute

$$\mathbb{B}_\lambda(x;\lambda) = \begin{pmatrix} I_n & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}.$$

so that

$$\int_0^1 \Phi(x;\lambda)^* \mathbb{B}_\lambda(x;\lambda)\Phi(x;\lambda)dx = \int_0^1 \Phi_1(x;\lambda)^*\Phi_1(x;\lambda)dx,$$

where $\Phi_1(x;\lambda)$ denotes the $n \times 4n$ matrix comprising the first n rows of the $4n \times 4n$ fundamental matrix $\Phi(x;\lambda)$. The matrix $\int_0^1 \Phi_1(x;\lambda)^*\Phi_1(x;\lambda)dx$ is clearly non-negative, and moreover it cannot have 0 as an eigenvalue, because the associated eigenvector $v \in \mathbb{C}^{4n}$ would necessarily satisfy $\Phi_1(x;\lambda)v = 0$ for all $x \in [0,1]$, and this would contradict linear independence of the columns of $\Phi_1(x;\lambda)$ (as solutions of (5.12)).

For **(B2)**$'$, we fix any $\lambda_1, \lambda_2 \in \mathbb{R}$, $\lambda_1 < \lambda_2$, and notice that

$$\mathbb{B}(x;\lambda_2) - \mathbb{B}(x;\lambda_1) = (\lambda_2 - \lambda_1)\mathbb{B}_\lambda(x;\lambda),$$

which is clearly non-negative. In addition, the same argument used to verify **(B1)**$'$ shows that the condition assumed in Claim 4.2 holds. Assumption **(B2)**$'$ follows.

We conclude from Theorem 1.2 that if $\mathcal{N}([\lambda_1,\lambda_2);\mathcal{L}_f)$ denotes the spectral count for $\mathcal{L}_f$, we have

$$\mathcal{N}([\lambda_1,\lambda_2);\mathcal{L}_f) = \mathcal{N}_{(0,1]}(\mathbf{X}_3(\cdot;\lambda_1)^* J \mathbf{X}_4(\cdot;\lambda_2)),$$

where $\mathbf{X}_3(x;\lambda_1)$ and $\mathbf{X}_4(x;\lambda_2)$ denote the frames specified respectively in (1.17) and (1.19), with $\mathbb{B}(x;\lambda)$ as in this section.

As a specific example in this case, we consider (5.12) with $n = 1$ and

$$\begin{aligned} V_0(x) &= -2 + 10\sin(12x) \\ V_2(x) &= 10\cos(2\pi x). \end{aligned}$$

The boundary conditions have the form **(BC2)** with $\Theta = (I_4 \quad -I_4)$. Spectral curves for this equation are depicted in Figure 5, with the approach of [14, 17] on the left and the renormalized approach on the right. For the approach of [14, 17], each point on the bottom shelf is a crossing point, and in addition a spectral curve emerges from the bottom shelf.

Appendix

In this short appendix, we briefly discuss the view of our operator

$$\mathcal{L}(\lambda) := J\frac{d}{dx} - \mathbb{B}(\cdot;\lambda) \tag{5.14}$$

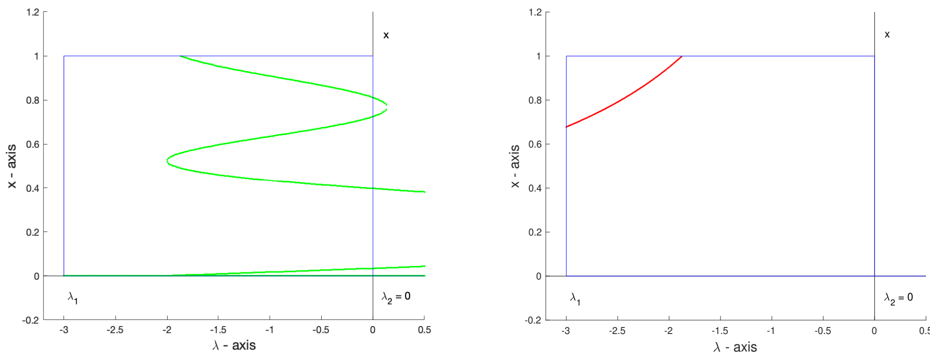

Figure 5: Spectral curves for the Sturm-Liouville system example: approach of [14, 17] on left; renormalized approach on right.

as an operator pencil. In order to keep the discussion brief, we focus on the case of boundary conditions **(BC1)**, for which the domain of $\mathcal{L}(\lambda)$ can be taken to be

$$\begin{aligned}\mathcal{D}(\mathcal{L}(\lambda)) := \{y \in L^2((0,1),\mathbb{C}^{2n}) : y \in \mathrm{AC}([0,1],\mathbb{C}^{2n}),\\ \mathcal{L}y \in L^2((0,1),\mathbb{C}^{2n}),\ \alpha y(0) = 0,\ \beta y(1) = 0\}.\end{aligned}$$

We will confine the discussion in this appendix to the case in which $\mathbb{B}(\cdot;\lambda) \in L^2((0,1),\mathbb{C}^{2n\times 2n})$ for all $\lambda \in I$. Under this additional assumption, $\mathcal{D}(\mathcal{L}(\lambda))$ is independent of λ, and in order to emphasize this independence we will express $\mathcal{D}(\mathcal{L}(\lambda))$ as $\mathcal{D}$. Here, $I \subset \mathbb{R}$ continues to be the interval specified in the introduction containing all values λ under consideration.

Following the development of [6], we specify the resolvent set of $\mathcal{L}$ as

$$\rho(\mathcal{L}) := \{\lambda \in I : \mathcal{L}(\lambda)^{-1} \in \mathcal{B}(L^2((0,1),\mathbb{C}^{2n}))\}, \tag{5.15}$$

where $\mathcal{B}(L^2((0,1),\mathbb{C}^{2n}))$ denotes the linear space of all bounded linear operators mapping $L^2((0,1),\mathbb{C}^{2n})$ to itself, and we specify the spectrum of $\mathcal{L}$ as $\sigma(\mathcal{L}) = I\backslash\rho(\mathcal{L})$. More generally, operator pencils are often defined on open sets of the complex plane $\Omega \subset \mathbb{C}$, but such a specification is not necessary for this brief discussion. In order to be precise about terminology, we define what we mean by the essential spectrum and the point spectrum (adapted from [21]). For this, we assume, as in the current setting, that $\mathcal{D} := \mathrm{dom}(\mathcal{L}(\lambda))$ is independent of λ, and we denote by $\mathbb{L}(L^2((0,1),\mathbb{C}^{2n}))$ the space of all closed linear operators mapping $\mathcal{D} \subset L^2((0,1),\mathbb{C}^{2n})$ to $L^2((0,1),\mathbb{C}^{2n})$.

Definition 5.1. *We define the essential spectrum $\sigma_{\mathrm{ess}}(\mathcal{L})$ of an operator pencil $\mathcal{L} : I \to \mathbb{L}(L^2((0,1),\mathbb{C}^{2n}))$ as the set of $\lambda \in I$ for which either $\mathcal{L}(\lambda)$ is not Fredholm or $\mathcal{L}(\lambda)$ is Fredholm with Fredholm index $\mathrm{ind}(\mathcal{L}(\lambda)) \neq 0$. We define the point spectrum $\sigma_{\mathrm{pt}}(\mathcal{L})$ as the set of $\lambda \in I$ so that $\mathrm{ind}(\mathcal{L}(\lambda)) = 0$, but $\mathcal{L}(\lambda)$ is not invertible.*

With these definitions, we see that the sets $\rho(\mathcal{L})$, $\sigma_{\mathrm{ess}}(\mathcal{L})$, and $\sigma_{\mathrm{pt}}(\mathcal{L})$ are mutually exclusive, and

$$I = \rho(\mathcal{L}) \cup \sigma_{\mathrm{ess}}(\mathcal{L}) \cup \sigma_{\mathrm{pt}}(\mathcal{L}); \quad \sigma(\mathcal{L}) = \sigma_{\mathrm{ess}}(\mathcal{L}) \cup \sigma_{\mathrm{pt}}(\mathcal{L}).$$

Another way to view the definitions is as follows. A value $\lambda_0 \in I$ is categorized as an element of $\rho(\mathcal{L})$, $\sigma_{\text{ess}}(\mathcal{L})$, or $\sigma_{\text{pt}}(\mathcal{L})$ according precisely to the categorization of 0 relative to the operator $\mathcal{L}(\lambda_0)$.

Returning to our particular operator pencil from (5.14), it's a straightforward application of the methods of [33] to verify that under our Assumptions **(A)**, we have the following: for each $\lambda \in I$, $\mathcal{L}(\lambda)$ is Fredholm with index zero, and indeed is self-adjoint. We can conclude that $\sigma(\mathcal{L})$ is comprised entirely of point spectrum, and in particular that for each $\lambda \in \sigma(\mathcal{L})$ there exist a finite number of linearly independent eigenfunctions $\{y_i(x;\lambda)\}_{i=1}^m \subset \mathcal{D}$ so that $\mathcal{L}(\lambda)y_i(\cdot;\lambda) = 0$ for all $i \in \{1, 2, \dots, m\}$. In addition, our Assumption **(B1)** ensures that the eigenvalues are all discrete (i.e., isolated). Our Theorems 1.1 and 1.2 count the number of such discrete eigenvalues, including geometric multiplicity, and it's natural to consider how this relates to the same count using algebraic multiplicity. First, proceeding as in [20], we can define the algebraic multiplicity of an eigenvalue λ_0 of $\mathcal{L}$ in terms of the nature of the Jordan chains associated with it. Readers interested in a complete definition along these lines can find it in Definition 6 of [20], but for our purposes, we only require the following.

Definition 5.2. *Let $\lambda_0 \in I$ be an eigenvalue of an operator pencil $\mathcal{L} : I \to \mathbb{L}(L^2((0,1), \mathbb{C}^{2n}))$ with geometric multiplicity m, and assume $\mathcal{L}'(\lambda_0) \in \mathbb{L}(L^2((0,1), \mathbb{C}^{2n}))$ exists, with additionally* $\operatorname{dom}(\mathcal{L}'(\lambda_0)) = \mathcal{D}$. *Suppose that for any pair (y, ζ) with $y \in \ker \mathcal{L}(\lambda_0)$, and $\zeta \in$* $\operatorname{dom}(\mathcal{L}(\lambda_0))$ *satisfying*

$$\mathcal{L}(\lambda_0)\zeta = \mathcal{L}'(\lambda_0)y, \tag{5.16}$$

we must have $y \equiv 0$. Then λ_0 has algebraic multiplicity m.

We are now in a position to verify that under slightly stronger conditions on $\mathbb{B}(x;\lambda)$ than assumed for Theorems 1.1 and 1.2, the geometric and algebraic multiplicies of eigenvalues of the operator pencil $\mathcal{L} : I \to \mathbb{L}(L^2((0,1), \mathbb{C}^{2n}))$ coincide.

Claim 5.1. *Let Assumptions* **(A)** *and* **(B1)** *hold, and additionally assume that for all $\lambda \in I$, we have $\mathbb{B}(\cdot;\lambda), \mathbb{B}_\lambda(\cdot;\lambda) \in L^2((0,1), \mathbb{C}^{2n\times 2n})$. Then for any eigenvalue λ_0 of the operator pencil $\mathcal{L}$, geometric and algebraic multiplicities agree.*

Proof. In our setting, $\mathcal{L}'(\lambda) = \mathbb{B}_\lambda(x;\lambda)$. Suppose λ_0 is an eigenvalue of $\mathcal{L}$, and that for some $y(\cdot;\lambda_0) \in \ker \mathcal{L}(\lambda_0)$, there is a corresponding $\zeta(\cdot;\lambda_0) \in \mathcal{D}(\mathcal{L})$ so that

$$\mathcal{L}(\lambda_0)\zeta(x;\lambda_0) = \mathbb{B}_\lambda(x;\lambda_0)y(x;\lambda_0), \quad \text{a.e. } x \in (0,1).$$

(Our additional assumption $\mathbb{B}_\lambda(\cdot;\lambda) \in L^2((0,1), \mathbb{C}^{2n\times 2n})$ ensures that $\mathcal{L}'(\lambda_0)$ maps $\mathcal{D}$ to $L^2((0,1), \mathbb{C}^{2n})$, and in particular that $\mathbb{B}_\lambda(x;\lambda_0)y(x;\lambda_0)$ is in the range of $\mathcal{L}(\lambda_0)$.) If we take an L^2 inner product of this equation with y, we obtain the relation

$$\langle \mathcal{L}(\lambda_0)\zeta, y\rangle = \langle \mathbb{B}_\lambda(x;\lambda_0)y, y\rangle.$$

Since $\mathcal{L}(\lambda_0)$ is self-adjoint, the left-hand side can be computed as

$$\langle \mathcal{L}(\lambda_0)\zeta, y\rangle = \langle \zeta, \mathcal{L}(\lambda_0)y\rangle = 0.$$

We see that the right-hand side satisfies $\langle \mathbb{B}_\lambda(\cdot;\lambda_0)y, y\rangle = 0$, and by our positivity condition **(B1)** this means $y = 0$ for a.e. $x \in (0,1)$. According to Definition 5.2, we can conclude that the algebraic multiplicity of λ_0 agrees with the geometric multiplicity of λ_0. □

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