THE ASTROPHYSICAL JOURNAL, 928:176 (10pp), 2022 April 1 https://doi.org/10.3847/1538-4357/ac596e

On the Conservation of Turbulence Energy in Turbulence Transport Models

B.-B. Wang[1], G. P. Zank[1,2], L. Adhikari[1,2], and L.-L. Zhao[1,2]
[1] Center for Space Plasma and Aeronomic Research (CSPAR), University of Alabama in Huntsville, Huntsville, AL 35899, USA
[2] Department of Space Science, University of Alabama in Huntsville, Huntsville, AL 35899, USA
Received 2021 December 27; revised 2022 February 19; accepted 2022 February 23; published 2022 April 7

Abstract

Zank et al. developed models describing the transport of low-frequency incompressible and nearly incompressible turbulence in inhomogeneous flows. The formalism was based on expressing the fluctuating variables in terms of the Elsässar variables and then taking "moments" subject to various closure hypotheses. The turbulence transport models are different according to whether the plasma beta regime is large, of order unity, or small. Here, we show explicitly that the three sets of turbulence transport models admit a conservation representation that resembles the well-known WKB transport equation for Alfvén wave energy density after introducing appropriate definitions of the "pressure" associated with the turbulent fluctuations. This includes introducing a distinct turbulent pressure tensor for 3D incompressible turbulence (the large plasma beta limit) and pressure tensors for quasi-2D and slab turbulence (the plasma beta order-unity or small regimes) that generalize the form of the WKB pressure tensor. Various limits of the different turbulent pressure tensors are discussed. However, the analogy between the conservation form of the turbulence transport models and the WKB model is not close for multiple reasons, including that the turbulence models express fully nonlinear physical processes unlike the strictly linear WKB description. The analysis presented here both serves as a check on the validity and correctness of the turbulence transport models and also provides greater transparency of the energy dissipation term and the "turbulent pressure" in our models, which is important for many practical applications.

Unified Astronomy Thesaurus concepts: Magnetohydrodynamics (1964); Interplanetary turbulence (830)

1. Introduction

The transport of incompressible MHD turbulence in inhomogeneous flows is a fundamentally important problem for both space physics and astrophysics, particularly in the context of the transport and acceleration of energetic particles such as solar energetic particles and galactic cosmic rays. Historically, the transport of incompressible fluctuations in a large-scale inhomogeneous flow has been modeled on the basis of a linear Alfvén wave description, colloquially known as the WKB model (Parker 1965; Hollweg 1973), and has been popular due to its tractability and simplicity. Being a linearized wave description, the leading-order WKB model describes noninteracting propagating waves and neglects possible mixing or coupling between propagating modes (although see the higher-order corrections discussed by Heinemann & Olbert (1980)). The need to incorporate turbulence effects explicitly was recognized in the 1990s with the development of transport models that departed from the assumption of linearized modes and incorporated mode mixing and nonlinear dissipation via the energy cascade through the inertial range (Zhou & Matthaeus 1989; Marsch & Tu 1990; Zhou & Matthaeus 1990a, 1990b; Matthaeus et al. 1994a; Zank et al. 1996; Breech et al. 2008; Oughton et al. 2011; Zank et al. 2012, 2017). Some discussion was presented by Matthaeus et al. (1994b) about the connection of the earlier turbulence models to the WKB description. However, the connection of the simpler WKB description to the much more elaborate turbulence transport models of Zank et al. (1996), Breech et al. (2008), Oughton et al. (2011), Zank et al. (2012, 2017), and Adhikari et al. (2017) has not been established. This paper addresses the connection between detailed turbulence transport models, their conservation form, and their relation to WKB models.

The application of the turbulence transport models developed by Zank et al. (2012, 2017) to solar wind has shown good agreement with a large variety of observations (Adhikari et al. 2015, 2017; Shiota et al. 2017; Zank et al. 2018b; Adhikari et al. 2020a, 2020b, 2020c; Zhao et al. 2020). The derivation of the turbulence models is based on a two-scale separation method, which is a common approach to obtain turbulence transport models (Zhou & Matthaeus 1989, 1990a, 1990b; Zank et al. 1996; Zank 2014; Zank et al. 2012, 2017). Since the fluctuations are well-separated from the scale associated with the large-scale inhomogeneities, the MHD variables can be decomposed into small-scale rapidly varying fluctuations and large-scale slowly varying mean values. The fluctuations are random variables with zero mean but can have an arbitrarily large amplitude. By applying an ensemble average operator $\langle ... \rangle$ to the MHD equations, i.e., the mass continuity, momentum, energy, and Faraday's induction equations, we can obtain a system of evolution equations for the mean fields that are coupled to the fluctuating fields. On subtracting the operator-averaged equations from the original MHD equations, we obtain the evolution equations for the fluctuating fields. The fluctuating fields can be combined and expressed in terms of the fluctuating Elsässer variables, $\boldsymbol{z}^{\pm} \equiv \boldsymbol{u} \pm \boldsymbol{b}/\sqrt{4\pi\rho}$, and $\boldsymbol{u}$, $\boldsymbol{b}$, and ρ are the fluctuating or turbulent velocity and magnetic field, and the mean plasma mass density, respectively. The dynamical equations for $\partial \boldsymbol{z}^{\pm}/\partial t$ are the basis for constructing a turbulence transport model. By computing the second-order moments of $\boldsymbol{z}^{\pm}$ through the dynamical equations describing the evolution of the Elsässer variables, we can derive systems of equations describing the evolution of the moments $\langle z^{+2} \rangle$, $\langle z^{-2} \rangle$, and $\langle \boldsymbol{z}^{+} \cdot \boldsymbol{z}^{-} \rangle$. Such one-point closure schemes are utilized to derive the dissipation terms and the corresponding evolution equations for the correlation lengths. The nonlinear terms that arise in the evolution equations for both the turbulence quantities and the correlation functions are simplified by using a structural similarity hypothesis. The structural similarity hypothesis is an approximation (essentially a closure hypothesis) for the variance and covariance of the field components as a fraction of the variance of the total, or equivalently relating the off-diagonal elements of the variance or covariance tensor to the corresponding trace. For more details, we refer the reader to Zank et al. (2012, 2017).

Zank et al. (2012, 2017) derived a coupled system of equations that describes the transport of the Elsässer energy in backward-propagating modes $\langle z^{+2}\rangle$, forward-propagating modes $\langle z^{-2}\rangle$, the cross helicity $E_C \equiv (\langle z^{+2}\rangle - \langle z^{-2}\rangle)/2$, the residual energy $E_D \equiv \langle z^{+} \cdot z^{-}\rangle$, and correlation lengths corresponding to backward-propagating modes λ^{+}, forward-propagating modes λ^{-}, and the residual energy λ_D. Under an additional set of simplifying assumptions, the large plasma beta model of Zank et al. (2012) can be further reduced to a single transport equation in the magnetic energy density as derived by Zank et al. (1996) (see also Adhikari et al. 2020c), from which one can recover the well-known WKB model after neglecting the dissipation and mixing terms (Zank et al. 2012).

The focus of this work is to present a conservation form of the three sets of turbulence transport equations that were derived in the beta large or beta order-unity or small regimes. This analysis serves both as a check on the validity and correctness of the transport models and provides greater transparency of the energy dissipation term and the "turbulent pressure" in our models, which is important for many practical applications. The importance of the dissipation of turbulence is of course related to the heating of gas or plasma in numerous space and astrophysical environments, especially in the heating of the solar corona and the acceleration of the solar wind (Matthaeus et al. 1999a; Dmitruk et al. 2001; Oughton et al. 2001; Dmitruk et al. 2002; Cranmer et al. 2007; Chandran & Hollweg 2009; Verdini et al. 2010; Woolsey & Cranmer 2014; van Ballegooijen & Asgari-Targhi 2016; Zank et al. 2018a; Adhikari et al. 2020a, 2020c, 2021), and the heating of the extended heliosphere (Matthaeus et al. 1999b; Smith et al. 2001; Isenberg et al. 2003; Isenberg 2005; Adhikari et al. 2017; Montagud-Camps et al. 2018; Zank et al. 2018b; Adhikari et al. 2020b). Besides the effects of turbulent dissipation in heating the thermal gas, turbulence can contribute to the dynamical behavior of a gas via its contribution to the total pressure. This has been of particular interest in the context of shock waves mediated by cosmic rays, where the pressure contributed by the turbulence excited by cosmic ray streaming decelerates the gas flow, thereby changing the shock profile and modifying the accelerated or energetic particle spectrum (McKenzie & Völk 1982; Jones 1993; Ko 1995; Caprioli et al. 2009).

2. Transport of Turbulent Energy

It is convenient to represent turbulence quantities by a set of one-point moments of the Elsässer variables,

$$E_T \equiv \langle \boldsymbol{u}^2 + \boldsymbol{b}^2/(4\pi\rho)\rangle = \frac{\langle z^{+2}\rangle + \langle z^{-2}\rangle}{2}; \tag{1}$$

$$E_C \equiv 2\langle \boldsymbol{u}\cdot\boldsymbol{b}/(\sqrt{4\pi\rho})\rangle = \frac{\langle z^{+2}\rangle - \langle z^{-2}\rangle}{2}; \tag{2}$$

$$E_D \equiv \langle \boldsymbol{u}^2 - \boldsymbol{b}^2/(4\pi\rho)\rangle = \langle \boldsymbol{z}^{+}\cdot\boldsymbol{z}^{-}\rangle, \tag{3}$$

where E_T is twice the total turbulent kinetic and magnetic energy per unit mass, E_C is the cross helicity measuring the correlation between the fluctuating velocity and magnetic fields, and E_D is the residual energy representing the difference between (twice) the turbulent kinetic and magnetic energies per unit mass. The normalized cross helicity and residual energy are defined as $\sigma_C = E_C/E_T$ and $\sigma_D = E_D/E_T$, respectively. Our focus here is on the turbulence energy density E_w, which is defined as the sum of the turbulence kinetic and magnetic energy densities,

$$E_w \equiv \frac{\rho}{2}\langle \boldsymbol{u}^2\rangle + \frac{\langle \boldsymbol{b}^2\rangle}{8\pi} = \frac{\rho}{2}E_T. \tag{4}$$

2.1. Transport of Incompressible Turbulence in the Plasma Beta Large Regime

Consider first the turbulence transport equations derived from the 3D incompressible MHD equations. As discussed in Zank & Matthaeus (1993), the 3D incompressible MHD equations represent the leading-order description of nearly incompressible MHD in the limit of large plasma beta, and can be derived from the Elsässer variables representation introduced by Zhou & Matthaeus (1989, 1990a, 1990b) and Marsch & Tu (1989). The 3D time-dependent turbulence transport model is then given by Zank et al. (2012),

$$\begin{aligned}&\frac{\partial E_T}{\partial t} + \boldsymbol{U}\cdot\nabla E_T + \nabla\cdot\boldsymbol{V}_A E_C - \boldsymbol{V}_A\cdot\nabla E_C\\&\quad + \nabla\cdot\boldsymbol{U}\left[\frac{E_T}{2} + \left(2a - \frac{1}{2}\right)E_D\right] - 2aE_D\mathbf{nn}\colon\nabla\boldsymbol{U}\\&= -\frac{(E_T + E_C)(E_T - E_C)^{1/2}}{\lambda^{+}}\\&\quad - \frac{(E_T - E_C)(E_T + E_C)^{1/2}}{\lambda^{-}};\end{aligned} \tag{5}$$

$$\begin{aligned}&\frac{\partial E_C}{\partial t} + \boldsymbol{U}\cdot\nabla E_C + \frac{1}{2}\nabla\cdot\boldsymbol{U}E_C - \boldsymbol{V}_A\cdot\nabla E_T\\&\quad + \nabla\cdot\boldsymbol{V}_A E_T - \nabla\cdot\boldsymbol{V}_A E_D - 2bE_D\mathbf{nn}\colon\nabla\boldsymbol{B}/\sqrt{4\pi\rho}\\&= -\frac{(E_T + E_C)(E_T - E_C)^{1/2}}{\lambda^{+}}\\&\quad + \frac{(E_T - E_C)(E_T + E_C)^{1/2}}{\lambda^{-}};\end{aligned} \tag{6}$$

$$\begin{aligned}&\frac{\partial E_D}{\partial t} + \boldsymbol{U}\cdot\nabla E_D + \frac{1}{2}\nabla\cdot\boldsymbol{U}E_D + \left(2a - \frac{1}{2}\right)\nabla\cdot\boldsymbol{U}E_T\\&\quad + \frac{E_C\boldsymbol{V}_A\cdot\nabla E_T - E_T\boldsymbol{V}_A\cdot\nabla E_C}{\sqrt{E_T^2 - E_C^2}}\\&\quad + \nabla\cdot\boldsymbol{V}_A E_C - 2\mathbf{nn}\colon(aE_T\nabla\boldsymbol{U} - bE_C\nabla\boldsymbol{B}/\sqrt{4\pi\rho})\\&= -E_D\left[\frac{(E_T + E_C)^{1/2}}{\lambda^{-}} + \frac{(E_T - E_C)^{1/2}}{\lambda^{+}}\right];\end{aligned} \tag{7}$$

$$\begin{aligned}&\frac{\partial\lambda^{\pm}}{\partial t} + (\boldsymbol{U}\mp\boldsymbol{V}_A)\cdot\nabla\lambda^{\pm} + \frac{E_D}{E_T\pm E_C}\left[\left(a - \frac{1}{4}\right)\nabla\cdot\boldsymbol{U}\right.\\&\quad \mp\frac{1}{2}\nabla\cdot\boldsymbol{V}_A \mp b\mathbf{nn}\colon\nabla\boldsymbol{B}/\sqrt{4\pi\rho}\\&\quad \left. - a\mathbf{nn}\colon\nabla\boldsymbol{U}\right](\lambda_D - 2\lambda^{\pm}) = 2(E_T\mp E_C)^{1/2};\end{aligned} \tag{8}$$

$$\frac{\partial \lambda_D}{\partial t} + \boldsymbol{U} \cdot \nabla \lambda_D + \frac{2E_T}{E_D}\left[\left(a - \frac{1}{4}\right)\nabla \cdot \boldsymbol{U} - a\mathbf{nn}: \nabla \boldsymbol{U}\right]$$
$$\times \left[\frac{(E_T + E_C)\lambda^+ + (E_T - E_C)\lambda^-}{E_T} - \lambda_D\right]$$
$$- \frac{2E_C}{E_D}\left[-\frac{1}{2}\nabla \cdot \boldsymbol{V}_A - b\mathbf{nn}: \nabla \boldsymbol{B}\Big/\sqrt{4\pi\rho}\right]$$
$$\times \left[\frac{(E_T + E_C)\lambda^+ - (E_T - E_C)\lambda^-}{E_C} - \lambda_D\right]$$
$$+ \frac{E_C \boldsymbol{V}_A \cdot \nabla E_T - E_T \boldsymbol{V}_A \cdot \nabla E_C}{E_D\sqrt{E_T^2 - E_C^2}}(2\sqrt{\lambda^+\lambda^-} - \lambda_D)$$
$$+ \frac{\sqrt{E_T^2 - E_C^2}}{E_D}\left[\left(\frac{\lambda^+}{\lambda^-}\right)^{1/2}\boldsymbol{V}_A \cdot \nabla \lambda^-\right.$$
$$\left. - \left(\frac{\lambda^-}{\lambda^+}\right)^{1/2}\boldsymbol{V}_A \cdot \nabla \lambda^+\right]$$
$$= \lambda_D\left[\frac{(E_T + E_C)^{1/2}}{\lambda^-} + \frac{(E_T - E_C)^{1/2}}{\lambda^+}\right], \tag{9}$$

where $\boldsymbol{U}$ is the large-scale fluid velocity, $\boldsymbol{V}_A$ the large-scale Alfvén velocity, $\lambda^{\pm}$ is the correlation length for backward/forward-propagating modes, and $\boldsymbol{n}$ corresponds to a specified direction for axisymmetric turbulence (typically the imposed mean magnetic field direction). The parameters a and b are structural similarity parameters, and their origin in the context of the transport model above is a little subtle (Zank et al. 2012). Specifically, a is a closure that relates the off-diagonal elements of the second-order tensors $\langle z_i z_j \rangle$ (where we deliberately leave the superscripts $\pm$ off to indicate generality) to the trace through a (or b)$\langle z^2 \rangle$. Since $\langle z_i z_j \rangle$ occurs in conjunction with the gradient of either the large-scale flow velocity $\boldsymbol{U}$ or the Alfvén velocity $\boldsymbol{V}_A$, a is associated with gradients in the large-scale flow $\boldsymbol{U}$ whereas the structural similarity parameter b is associated specifically with gradients in $\boldsymbol{V}_A$. The choice of $a = b = 1/2$ or $a = 1/3$ corresponds to either the 2D or the 3D mixing tensor in the Matthaeus et al. (1994a) and Zank et al. (1996) turbulence transport models. For 3D isotropic turbulence, the axisymmetric direction vector $\boldsymbol{n}$ should be a zero vector and disappears together with parameter b.

In deriving the energy-conservation equation, we need some essential vector and tensor relations,

$$\nabla \cdot (\alpha \boldsymbol{A}) = \alpha \nabla \cdot \boldsymbol{A} + \nabla \alpha \cdot \boldsymbol{A}; \tag{10}$$

$$\boldsymbol{T}: \nabla \boldsymbol{A} = \nabla \boldsymbol{A}: \boldsymbol{T}; \tag{11}$$

$$\nabla \cdot \boldsymbol{A} = \nabla \boldsymbol{A}: \boldsymbol{I}; \tag{12}$$

$$\nabla(\alpha \boldsymbol{T}) = \alpha \nabla \cdot \boldsymbol{T} + \nabla \alpha \cdot \boldsymbol{T}; \tag{13}$$

$$\nabla \cdot (\boldsymbol{A} \cdot \boldsymbol{T}) = \boldsymbol{A} \cdot \nabla \cdot \boldsymbol{T} + \boldsymbol{T}: \nabla \boldsymbol{A}, \tag{14}$$

where α is a scalar, $\boldsymbol{A}$ is a vector, $\boldsymbol{T}$ is a tensor, and $\boldsymbol{I}$ is an identity tensor.

On neglecting the dissipation terms in Equation (5), the transport equation for E_T can be written as

$$\frac{\partial E_T}{\partial t} + \boldsymbol{U} \cdot \nabla E_T - \boldsymbol{V}_A \cdot \nabla E_C + \nabla \cdot \boldsymbol{V}_A E_C$$
$$+ \nabla \cdot \boldsymbol{U}\left[\frac{E_T}{2} + \left(2a - \frac{1}{2}\right)E_D\right] - 2aE_D\mathbf{nn}: \nabla \boldsymbol{U} = 0. \tag{15}$$

Using Equation (10) on the second and third terms of (15) and adding the fourth term yields

$$\boldsymbol{U} \cdot \nabla E_T - \boldsymbol{V}_A \cdot \nabla E_C + \nabla \cdot \boldsymbol{V}_A E_C$$
$$= \nabla \cdot (\boldsymbol{U} E_T - \boldsymbol{V}_A E_C) - E_T \nabla \cdot \boldsymbol{U} + 2E_C \nabla \cdot \boldsymbol{V}_A. \tag{16}$$

By using Equations (12) and (11), the fifth and the last terms on the left-hand side of Equation (15) become

$$\nabla \cdot \boldsymbol{U}\left[\frac{E_T}{2} + \left(2a - \frac{1}{2}\right)E_D\right] - 2aE_D\mathbf{nn}: \nabla \boldsymbol{U} = \nabla \boldsymbol{U}: \frac{2}{\rho}\boldsymbol{P}_w,$$

where we have introduced the turbulence pressure tensor $\boldsymbol{P}_w$,

$$\boldsymbol{P}_w \equiv \frac{\rho}{2}\left[\left(\frac{E_T}{2} + \left(2a - \frac{1}{2}\right)E_D\right)\boldsymbol{I} - 2aE_D\boldsymbol{nn}\right]. \tag{17}$$

Equation (15) can therefore be expressed as

$$\frac{\partial E_T}{\partial t} + \nabla \cdot (\boldsymbol{U} E_T - \boldsymbol{V}_A E_C) + \nabla \boldsymbol{U}: \left(\frac{2}{\rho}\boldsymbol{P}_w\right)$$
$$- E_T \nabla \cdot \boldsymbol{U} + 2E_C \nabla \cdot \boldsymbol{V}_A = 0. \tag{18}$$

After substituting $E_C = \sigma_c E_T$, $E_D = \sigma_D E_T$, and $E_T = \rho/2E_w$ into Equation (18), and multiplying by $\rho/2$, we obtain

$$\frac{\partial E_w}{\partial t} + \nabla \cdot [(\boldsymbol{U} - \boldsymbol{V}_A \sigma_C)E_w] + \nabla \boldsymbol{U}: \mathbf{P}_w + \rho E_w \frac{\partial}{\partial t}\left(\frac{1}{\rho}\right)$$
$$+ \rho(\boldsymbol{U} - \boldsymbol{V}_A \sigma_c) \cdot \nabla\left(\frac{1}{\rho}\right) - \frac{E_w}{\rho}\rho \nabla \cdot \boldsymbol{U}$$
$$+ 2\sigma_c E_w \nabla \cdot \boldsymbol{V}_A = 0. \tag{19}$$

Since $\nabla \cdot \boldsymbol{V}_A = -\boldsymbol{V}_A \cdot \nabla\rho/(2\rho)$, the last four terms can be eliminated as follows:

$$\rho E_w \frac{\partial}{\partial t}\left(\frac{1}{\rho}\right) + \rho E_w(\boldsymbol{U} - \boldsymbol{V}_A \sigma_c) \cdot \nabla\left(\frac{1}{\rho}\right)$$
$$- \frac{E_w}{\rho}\rho \nabla \cdot \boldsymbol{U} + 2\sigma_c E_w \nabla \cdot \boldsymbol{V}_A$$
$$= -\frac{E_w}{\rho}\left(\frac{\partial \rho}{\partial t} + \boldsymbol{U} \cdot \nabla \rho - \sigma_c \boldsymbol{V}_A \cdot \nabla \rho\right.$$
$$\left. + \rho \nabla \cdot \boldsymbol{U} + \sigma_c \boldsymbol{V}_A \cdot \nabla \rho\right)$$
$$= -\frac{E_w}{\rho}\left[\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \boldsymbol{U})\right]$$
$$= 0, \tag{20}$$

thanks to conservation of mass. Finally, using Equation (14), we can express the transport equation for the total turbulence energy in a conservative form resembling that of a WKB model,

$$\frac{\partial E_w}{\partial t} + \nabla \cdot [(\boldsymbol{U} - \sigma_C \boldsymbol{V}_A) E_w + \boldsymbol{U} \cdot \boldsymbol{P}_w] = \boldsymbol{U} \cdot \nabla \cdot \boldsymbol{P}_w. \quad (21)$$

The term in square brackets is the energy density flux vector, which is the amount of turbulence energy passing in unit time through a unit area perpendicular to the direction of the velocity (Landau & Lifshitz 1987). Within the square brackets, the first term is the energy transported through the unit surface area in unit time, and the second term is the work done by the turbulent "pressure" force on the plasma within the surface. The right-hand term is the rate of work of the turbulence pressure gradient on the background plasma flow. For the present, we remind the reader that we have neglected the dissipation term in deriving Equation (21)—this term is given below.

The turbulence propagation velocity is the energy-averaged Alfvén velocity,

$$\begin{aligned} -\sigma_C \boldsymbol{V}_A = -\frac{\langle z^{+2}\rangle - \langle z^{-2}\rangle}{\langle z^{+2}\rangle + \langle z^{-2}\rangle}\boldsymbol{V}_A = \frac{\langle z^{-2}\rangle}{\langle z^{+2}\rangle + \langle z^{-2}\rangle}\boldsymbol{V}_A \\ - \frac{\langle z^{+2}\rangle}{\langle z^{+2}\rangle + \langle z^{-2}\rangle}\boldsymbol{V}_A, \end{aligned} \quad (22)$$

which resembles the mean local velocity of Alfvénic turbulence. Since the turbulence consists of structures that move in all directions, the mean local velocity of the Alfvén turbulence is the energy-averaged Alfvén velocity weighted by the ratio of the forward or backward wave energy to the total energy (Bell & Lucek 2001).

We can express the turbulent pressure tensor in terms of the turbulence energy and the fluctuating fields as

$$\begin{aligned} \boldsymbol{P}_w &= \left[\frac{E_w}{2} + \left(2a - \frac{1}{2}\right)\sigma_D E_w\right]\boldsymbol{I} - 2a\sigma_D E_w \boldsymbol{nn} \\ &= \left[a\rho\langle \boldsymbol{u}^2\rangle + (1-2a)\left(\frac{\langle \boldsymbol{b}^2\rangle}{8\pi}\right)\right]\boldsymbol{I} - a\left(\rho\langle \boldsymbol{u}^2\rangle - \frac{\langle \boldsymbol{b}^2\rangle}{4\pi}\right)\mathbf{nn}. \end{aligned} \quad (23)$$

It is worth noting that, if $a > 1/2$, it is possible for the isotropic part of the pressure tensor to be negative. For Alfvén-like turbulence with $E_D = 0 = \sigma_D$, the turbulence pressure is the familiar isotropic Alfvén wave pressure $\boldsymbol{b}^2/(8\pi)\boldsymbol{I}$. For 3D isotropic turbulence, $a = 1/3$ and $\boldsymbol{n} = 0$, the reduced turbulence pressure tensor is

$$\boldsymbol{P}_w = \frac{\rho}{4}\left(E_T + \frac{1}{3}E_D\right) = \frac{\rho\langle \boldsymbol{u}^2\rangle}{3} + \frac{1}{3}\left(\frac{\langle \boldsymbol{b}^2\rangle}{8\pi}\right). \quad (24)$$

Note that the turbulence pressure, including the "ram pressure" (i.e., the kinetic or fluctuating Reynold's pressure) and the fluctuating magnetic stress, is (McKee & Zweibel 1995)

$$\boldsymbol{P} = \rho\langle \boldsymbol{uu}\rangle + \frac{\langle \boldsymbol{b}^2\rangle}{8\pi}\boldsymbol{I} - \frac{\langle \boldsymbol{bb}\rangle}{4\pi}. \quad (25)$$

In the case of 3D isotropic turbulence, the local average fluctuating ram pressure $\rho\langle u_i u_j\rangle = \rho\langle u_i^2\rangle$ is $\rho\langle u^2\rangle/3$ because the average of $u_i u_j = 0$ for $i \neq j$. This is true also for the fluctuating magnetic stress, and is given by $\langle \boldsymbol{b}^2\rangle/(8\pi) - (\langle \boldsymbol{b}^2\rangle/(4\pi))/3 = (\langle \boldsymbol{b}^2\rangle/(8\pi))/3$.

For turbulence that is axisymmetric with respect to the directional vector $\boldsymbol{n}$, and has $a = 1/2$, the turbulence pressure tensor is given by

$$\boldsymbol{P}_w = \frac{\rho\langle u^2\rangle}{2}(\boldsymbol{I} - \boldsymbol{nn}) + \frac{\langle b^2\rangle}{8\pi}\boldsymbol{nn}. \quad (26)$$

This results from our structural similarity assumptions for 2D turbulence (Zank et al. 2012),

$$\langle \boldsymbol{uu}\rangle = \frac{1}{2}\langle \boldsymbol{u}^2\rangle\boldsymbol{I} - \frac{1}{2}\langle \boldsymbol{u}^2\rangle\boldsymbol{nn}; \quad \langle \boldsymbol{bb}\rangle = \frac{1}{2}\langle \boldsymbol{b}^2\rangle\boldsymbol{I} - \frac{1}{2}\langle \boldsymbol{b}^2\rangle\boldsymbol{nn}, \quad (27)$$

which, when substituted into the turbulence pressure Equation (25), yields Equation (26).

The dissipation of turbulence energy $E_{\rm diss}$ is easily found to be given by Zank et al. (2012):

$$\begin{aligned} E_{\rm diss} = -\frac{\rho}{2}\Bigg[&\frac{(E_T + E_C)(E_T - E_C)^{1/2}}{\lambda^+} \\ &- \frac{(E_T - E_C)(E_T + E_C)^{1/2}}{\lambda^-}\Bigg]. \end{aligned} \quad (28)$$

The complete energy transport equation, including the dissipation term, is therefore given by

$$\begin{aligned} &\frac{\partial E_w}{\partial t} + \nabla \cdot [(\boldsymbol{U} - \boldsymbol{V}_A\sigma_C)E_w + \boldsymbol{U} \cdot \boldsymbol{P}_w] \\ &= \boldsymbol{U} \cdot \nabla \cdot \boldsymbol{P}_w - \frac{\rho}{2}\Bigg[\frac{(E_T + E_C)(E_T - E_C)^{1/2}}{\lambda^+} \\ &\quad - \frac{(E_T - E_C)(E_T + E_C)^{1/2}}{\lambda^-}\Bigg]. \end{aligned} \quad (29)$$

2.2. Transport of Nearly Incompressible 2D Turbulence in an Inhomogeneous $\beta \sim 1$ Plasma

From the perspective of nearly incompressible MHD, the incompressible MHD description is valid only for a plasma beta regime much large than unity; for a plasma beta of order unity or less, the turbulence is a superposition of a dominant 2D incompressible component and a minority slab component (Zank & Matthaeus 1992, 1993; Zank et al. 2017, 2020). The equations governing the evolution of 2D incompressible turbulence in the plasma beta order-unity limit are (Zank et al. 2017)

$$\begin{aligned} &\frac{\partial E_T^\infty}{\partial t} + \boldsymbol{U} \cdot \nabla E_T^\infty + \nabla \cdot \boldsymbol{U}\left[\frac{E_T^\infty}{2} + \left(2a - \frac{1}{2}\right)E_D^\infty\right] \\ &= \frac{\boldsymbol{n}_{z\infty} \cdot \nabla\rho}{4\rho}[(E_T^\infty + E_C^\infty)^{3/2} \\ &\quad + (E_T^\infty - E_C^\infty)^{3/2} - E_D^\infty(E_T^\infty + E_C^\infty)^{1/2} \\ &\quad - E_D^\infty(E_T^\infty - E_C^\infty)^{1/2}] \\ &\quad - \frac{(E_T^\infty + E_C^\infty)(E_T^\infty - E_C^\infty)^{1/2}}{\lambda_\perp^+} \\ &\quad - \frac{(E_T^\infty - E_C^\infty)(E_T^\infty + E_C^\infty)^{1/2}}{\lambda_\perp^-}; \end{aligned} \quad (30)$$

$$\frac{\partial E_C^\infty}{\partial t} + \boldsymbol{U}\cdot\nabla E_C^\infty + \frac{\nabla\cdot\boldsymbol{U}}{2}E_C^\infty$$
$$= \frac{\boldsymbol{n}_{z\infty}\cdot\nabla\rho}{4\rho}[(E_T^\infty + E_C^\infty)^{3/2} - (E_T^\infty - E_C^\infty)^{3/2}$$
$$+ E_D^\infty(E_T^\infty - E_C^\infty)^{1/2} - E_D^\infty(E_T^\infty + E_C^\infty)^{1/2}]$$
$$- \frac{(E_T^\infty + E_C^\infty)(E_T^\infty - E_C^\infty)^{1/2}}{\lambda_\perp^+}$$
$$+ \frac{(E_T^\infty - E_C^\infty)(E_T^\infty + E_C^\infty)^{1/2}}{\lambda_\perp^-}; \quad (31)$$

$$\frac{\partial E_D^\infty}{\partial t} + \boldsymbol{U}\cdot\nabla E_D^\infty + \nabla\cdot\boldsymbol{U}\left[\frac{E_D^\infty}{2} + \left(2a - \frac{1}{2}\right)E_T^\infty\right]$$
$$= \frac{\boldsymbol{n}_{z\infty}\cdot\nabla\rho}{4\rho}[E_D^\infty(E_T^\infty + E_C^\infty)^{1/2}$$
$$+ E_D^\infty(E_T^\infty - E_C^\infty)^{1/2} - (E_T^\infty + E_C^\infty)(E_T^\infty - E_C^\infty)^{1/2}$$
$$- (E_T^\infty - E_C^\infty)(E_T^\infty + E_C^\infty)^{1/2}]$$
$$- E_D^\infty\left[\frac{(E_T^\infty - E_C^\infty)^{1/2}}{\lambda_\perp^+} + \frac{(E_T^\infty + E_C^\infty)^{1/2}}{\lambda_\perp^-}\right]; \quad (32)$$

$$\frac{\partial L_\infty^\pm}{\partial t} + \boldsymbol{U}\cdot\nabla L_\infty^\pm + \nabla\cdot\boldsymbol{U}\left[\frac{L_\infty^\pm}{2} + \left(a - \frac{1}{4}\right)L_D^\infty\right]$$
$$+ \frac{\boldsymbol{n}_{z\infty}\cdot\nabla\rho}{4\rho}(E_T^\infty \pm E_C^\infty)^{1/2}(L_D^\infty - 2L_\infty^\pm) = 0; \quad (33)$$

$$\frac{\partial L_D^\infty}{\partial t} + \boldsymbol{U}\cdot\nabla L_D^\infty + \nabla\cdot\boldsymbol{U}\left[\frac{L_D^\infty}{2} + \left(2a - \frac{1}{2}\right)(L_\infty^+ + L_\infty^-)\right]$$
$$- \frac{\boldsymbol{n}_{z\infty}\cdot\nabla\rho}{4\rho}[L_D^\infty(E_T^\infty + E_C^\infty)^{1/2}$$
$$+ L_D^\infty(E_T^\infty - E_C^\infty)^{1/2} - 2L_\infty^+(E_T^\infty - E_C^\infty)^{1/2}$$
$$- 2L_\infty^-(E_T^\infty + E_C^\infty)^{1/2}] = 0, \quad (34)$$

where the subscript "∞" denotes MHD variables that satisfy the incompressible equations, and "$*$" denotes the higher-order corrections. We assume that the structural similarity parameter for the fluctuating velocity and magnetic fields are the same and denoted by a. Note that, in this (and the next) subsection, $\boldsymbol{n}_{z\infty}$ is the weight-averaged direction vector of $z^{\infty\pm}$ modeled in the local coordinate system with the z-axis along the large-scale magnetic field.

Here, $L_\infty^\pm$ and $\lambda_\perp^\pm$ are the correlation function and correlation length corresponding to the backward/forward propagating modes, respectively, and L_D^∞ and λ_D^∞ are the correlation function and correlation length corresponding to the residual energy. The correlation functions and correlation lengths are related by

$$L_\infty^\pm = \int\langle \boldsymbol{z}^{\infty\pm}\cdot\boldsymbol{z}^{\infty\pm'}\rangle dr = \langle z^{\infty\pm 2}\rangle\,\lambda_\perp^\pm; \quad (35)$$

$$L_D^\infty = \int\langle \boldsymbol{z}^{\infty+}\cdot\boldsymbol{z}^{\infty-'} + \boldsymbol{z}^{\infty+'}\cdot\boldsymbol{z}^{\infty-}\rangle dr = E_D^\infty\,\lambda_D^\infty, \quad (36)$$

where $\boldsymbol{z}^{\infty\pm'} \equiv \boldsymbol{z}^{\infty\pm}(\boldsymbol{x} + \boldsymbol{r})$ indicates the lagged Elsässer variable at location $\boldsymbol{r}$ from $\boldsymbol{x}$.

Were we to neglect the terms containing $\boldsymbol{V}_A$ and $\boldsymbol{nn}\colon\nabla\boldsymbol{U}$ on the left-hand side of Equation (5), we would obtain the same as the left-hand side of Equation (30). Thus, the conservation form of the evolution equation for the 2D turbulence energy is given immediately by

$$\frac{\partial E_w^\infty}{\partial t} + \nabla\cdot(\boldsymbol{U}E_w^\infty + \boldsymbol{U}\cdot\boldsymbol{P}_w^\infty) = \boldsymbol{U}\cdot\nabla\cdot\boldsymbol{P}_w^\infty$$
$$+ \frac{\mathbf{n}_{z\infty}\cdot\nabla\rho}{8}[(E_T^\infty + E_C^\infty)^{3/2} + (E_T^\infty - E_C^\infty)^{3/2}$$
$$- E_D^\infty(E_T^\infty + E_C^\infty)^{1/2} - E_D^\infty(E_T^\infty - E_C^\infty)^{1/2}]$$
$$- \frac{\rho}{2}\left[\frac{(E_T^\infty + E_C^\infty)(E_T^\infty - E_C^\infty)^{1/2}}{\lambda_\perp^+}\right.$$
$$\left. + \frac{(E_T^\infty - E_C^\infty)(E_T^\infty + E_C^\infty)^{1/2}}{\lambda_\perp^-}\right]. \quad (37)$$

The turbulence pressure tensor is now defined as

$$\boldsymbol{P}_w^\infty \equiv \left[\frac{E_w^\infty}{2} + \left(2a - \frac{1}{2}\right)\sigma_D E_w^\infty\right]\boldsymbol{I}$$
$$= \left[a\rho\langle \boldsymbol{u}^{\infty 2}\rangle + (1 - 2a)\left(\frac{\langle \boldsymbol{b}^{\infty 2}\rangle}{8\pi}\right)\right]\boldsymbol{I}, \quad (38)$$

where $\boldsymbol{u}^\infty$ and $\boldsymbol{b}^\infty$ are the 2D turbulent velocity and magnetic fields, respectively. This was first noted by Le Roux et al. (2018). The properties are similar to those for the incompressible turbulence case discussed in Section 2.1.

2.3. Transport of Nearly Incompressible Slab Turbulence in an Inhomogeneous $\beta \sim 1$ Plasma

The transport equations that describe the evolution of slab turbulence expressed in terms of the nearly incompressible corrections to the incompressible MHD variables are given by Zank et al. (2017),

$$\frac{\partial E_T^*}{\partial t} + \boldsymbol{U}\cdot\nabla E_T^* + \nabla\cdot\boldsymbol{V}_A E_C^* - \boldsymbol{V}_A\cdot\nabla E_C^*$$
$$+ \nabla\cdot\boldsymbol{U}\left[\frac{E_T^*}{2} + \left(2b - \frac{1}{2}\right)E_D^*\right] - 2bE_D^*\mathbf{ss}\colon\nabla\boldsymbol{U}$$
$$= \frac{\boldsymbol{n}_{z\infty}\cdot\nabla\rho}{4\rho}[(E_T^* - E_C^*)(E_T^\infty - E_C^\infty)^{1/2}$$
$$+ (E_T^* + E_C^*)(E_T^\infty + E_C^\infty)^{1/2}$$
$$- E_D^*(\sqrt{E_T^\infty + E_C^\infty} + \sqrt{E_T^\infty - E_C^\infty})]$$
$$- \frac{(E_T^\infty - E_C^\infty)^{1/2}(E_T^* + E_C^*)}{\lambda_\perp^+}$$
$$- \frac{(E_T^\infty + E_C^\infty)^{1/2}(E_T^* - E_C^*)}{\lambda_\perp^-}; \quad (39)$$

$$\frac{\partial E_C^*}{\partial t} + \boldsymbol{U} \cdot \nabla E_C^* - \boldsymbol{V}_A \cdot \nabla E_T^* + \frac{\nabla \cdot \boldsymbol{U}}{2} E_C^* + \nabla \cdot \boldsymbol{V}_A (E_T^* - E_D^*) - 2bE_D^* \mathbf{ss}\colon \nabla \boldsymbol{B} \big/ \sqrt{4\pi\rho} = \frac{\boldsymbol{n}_{z\infty} \cdot \nabla\rho}{4\rho} [(E_T^* + E_C^*)(E_T^\infty + E_C^\infty)^{1/2} - (E_T^* - E_C^*)(E_T^\infty - E_C^\infty)^{1/2} - E_D^* (\sqrt{E_T^\infty + E_C^\infty} - \sqrt{E_T^\infty - E_C^\infty})] - \frac{(E_T^\infty - E_C^\infty)^{1/2}(E_T^* + E_C^*)}{\lambda_\perp^+} + \frac{(E_T^\infty + E_C^\infty)^{1/2}(E_T^* - E_C^*)}{\lambda_\perp^-}; \tag{40}$$

$$\frac{\partial E_D^*}{\partial t} + \boldsymbol{U} \cdot \nabla E_D^* + \nabla \cdot \boldsymbol{U} \left[\frac{E_D^*}{2} + \left(2b - \frac{1}{2}\right) E_T^* \right] - 2bE_T^* \mathbf{ss}\colon \nabla \boldsymbol{U} + 2bE_C^* \mathbf{ss}\colon \nabla \boldsymbol{B} \big/ \sqrt{4\pi\rho} + \nabla \cdot \boldsymbol{V}_A E_C^* = \frac{\boldsymbol{n}_{z\infty} \cdot \nabla\rho}{4\rho} \times [E_D^* (\sqrt{E_T^\infty + E_C^\infty} + \sqrt{E_T^\infty - E_C^\infty}) - (E_T^* - E_C^*)(E_T^\infty + E_C^\infty)^{1/2} - (E_T^* + E_C^*)(E_T^\infty - E_C^\infty)^{1/2}] - E_D^* \left[\frac{(E_T^\infty - E_C^\infty)^{1/2}}{\lambda_\perp^+} + \frac{(E_T^\infty + E_C^\infty)^{1/2}}{\lambda_\perp^-} \right]; \tag{41}$$

$$\frac{\partial L_*^\pm}{\partial t} + (\boldsymbol{U} \mp \boldsymbol{V}_A) \cdot \nabla L_*^\pm + \frac{\nabla \cdot \boldsymbol{U}}{2} \left(L_*^\pm - \frac{L_D^*}{2} \right) + \nabla \cdot \boldsymbol{V}_A \left(\pm L_*^\pm \mp \frac{L_D^*}{2} \right) + bL_D^* (\nabla \cdot \boldsymbol{U} - \mathbf{ss}\colon \nabla \boldsymbol{U} \mp \mathbf{ss}\colon \nabla \boldsymbol{B} \big/ \sqrt{4\pi\rho}) + \frac{\boldsymbol{n}_{z\infty} \cdot \nabla\rho}{4\rho} (E_T^\infty \pm E_C^\infty)^{1/2} (L_D^* - 2L_*^\pm) = 0; \tag{42}$$

$$\frac{\partial L_D^*}{\partial t} + \boldsymbol{U} \cdot \nabla L_D^* + \frac{\nabla \cdot \boldsymbol{U}}{2} (L_D^* - L_*^+ - L_*^-) + 2b(\nabla \cdot \boldsymbol{U} - \mathbf{ss}\colon \nabla \boldsymbol{U})(L_*^+ + L_*^-) + 2b\mathbf{ss}\colon \nabla \boldsymbol{B} \big/ \sqrt{4\pi\rho} (L_*^+ - L_*^-) + \nabla \cdot \boldsymbol{V}_A (L_*^+ - L_*^-) + \frac{\boldsymbol{n}_{z\infty} \cdot \nabla\rho}{4\rho} [(E_T^\infty - E_C^\infty)^{1/2} (2L_*^+ - L_D^*) + (E_T^\infty + E_C^\infty)^{1/2} (2L_*^- - L_D^*)] = 0, \tag{43}$$

where $\boldsymbol{s}$ denotes the large-scale magnetic field direction. To distinguish the structural similarity parameter a for 2D incompressible turbulence from that of slab turbulence, we introduce the notation b. As for 2D incompressible turbulence (Section 2.1), the correlation lengths for slab turbulence are defined as

$$L_*^\pm = \int \langle \boldsymbol{z}^{*\pm} \cdot \boldsymbol{z}^{*\pm\prime} \rangle dr = \langle \boldsymbol{z}^{*\pm} \cdot \boldsymbol{z}^{*\pm} \rangle \lambda_*^\pm; \tag{44}$$

$$L_D^* = \int \langle \boldsymbol{z}^{*\pm} \cdot \boldsymbol{z}^{*\mp\prime} + \boldsymbol{z}^{*\pm\prime} \cdot \boldsymbol{z}^{*\mp} \rangle dr = E_D^* \lambda_D^*. \tag{45}$$

The transport equation for E_T^* in Equation (39) is similar to Equation (5), and thus Equation (39) can be expressed as

$$\frac{\partial E_w^*}{\partial t} + \nabla \cdot [(\boldsymbol{U} - \boldsymbol{V}_A \sigma_c^*) E_w^* + \boldsymbol{U} \cdot \boldsymbol{P}_w^*)] = \boldsymbol{U} \cdot \nabla \cdot \boldsymbol{P}_w^* + \frac{\boldsymbol{n}_{z\infty} \cdot \nabla\rho}{8} [(E_T^* - E_C^*)(E_T^\infty - E_C^\infty)^{1/2} + (E_T^* + E_C^*)(E_T^\infty + E_C^\infty)^{1/2}] - \frac{\rho}{2} \left[\frac{(E_T^\infty - E_C^\infty)^{1/2}(E_T^* + E_C^*)}{\lambda_\perp^+} + \frac{(E_T^\infty + E_C^\infty)^{1/2}(E_T^* - E_C^*)}{\lambda_\perp^-} \right]. \tag{46}$$

Here, the turbulence pressure tensor is defined as

$$\boldsymbol{P}_w^* \equiv \left[\frac{E_w^*}{2} + \left(2b - \frac{1}{2}\right) \sigma_D^* E_w^* \right] \boldsymbol{I} - 2b\sigma_D^* E_w^* \boldsymbol{nn} = \left[b\rho \langle \boldsymbol{u}_1^2 \rangle + (1 - 2b) \left(\frac{\langle \boldsymbol{b}^{*2} \rangle}{8\pi} \right) \right] \boldsymbol{I} - b \left(\rho \langle \boldsymbol{u}_1^2 \rangle - \frac{\langle \boldsymbol{b}^{*2} \rangle}{4\pi} \right) \boldsymbol{nn}, \tag{47}$$

where $\boldsymbol{u}_1$ and $\boldsymbol{b}^*$ are the fluctuating velocity and magnetic field for slab turbulence, respectively. It is worth noting that, if $\sigma_D \neq 0$ and $b > 1/2$, it is possible for the isotropic part of the turbulence pressure tensor to become negative. To avoid these unphysical situations, it is necessary that care be exercised in choosing the value of the structural similarity parameter.

We illustrate different forms of the turbulence pressure tensor under different assumptions. In the case that $\sigma_D^* = 0$, the slab turbulence pressure tensor, Equation (47), is given by

$$\boldsymbol{P}_w^* = \frac{E_w^*}{2} \boldsymbol{I} = \frac{\langle \boldsymbol{b}^{*2} \rangle}{8\pi} \boldsymbol{I}, \tag{48}$$

which is isotropic and corresponds to the pressure exerted by Alfvén waves. For the case of $b = 1/2$, the slab turbulence pressure tensor can be expressed as

$$\boldsymbol{P}_w^* = \left[\frac{E_w^*}{2} + \sigma_D^* E_w^* \right] \boldsymbol{I} - \sigma_D^* E_w^* \mathbf{nn} = \frac{\rho \langle \boldsymbol{u}_1^2 \rangle}{2} \boldsymbol{I} - \left(\frac{\rho \langle \boldsymbol{u}_1^2 \rangle}{2} - \frac{\langle \boldsymbol{b}^{*2} \rangle}{8\pi} \right) \mathbf{nn}. \tag{49}$$

2.4. Transport of Nearly Incompressible 2D and Slab Turbulence in an Inhomogeneous β ≪ 1 Plasma

The transport equations that describe the evolution of the leading-order 2D turbulence in the $\beta \ll 1$ plasma are the same as the equations (in Section 2.2) for $\beta \sim 1$. However, for the minor slab turbulence, the transport equations are different in the $\beta \ll 1$ and $\beta \sim 1$ plasma. The transport equations for slab turbulence in the $\beta \ll 1$ regimes can be expressed as (see

Appendix for the derivation)

$$\begin{aligned}&\frac{\partial E_T^*}{\partial t} + \boldsymbol{U}\cdot\nabla E_T^* + \nabla\cdot\boldsymbol{V}_A E_C^* - \boldsymbol{V}_A\cdot\nabla E_C^*\\ &+\nabla\cdot\boldsymbol{U}\left[\frac{E_T^*}{2} + (2b - \frac{1}{2})E_D^*\right] - 2bE_D^*\mathbf{ss}:\nabla\boldsymbol{U}\\ &= \frac{\boldsymbol{n}_{u\infty}\cdot\nabla\rho}{2\sqrt{2}\rho}(E_T^\infty + E_D^\infty)^{1/2}(E_T^* - E_D^*)\\ &- \frac{(E_T^\infty - E_C^\infty)^{1/2}(E_T^* + E_C^*)}{\lambda_\perp^+}\\ &- \frac{(E_T^\infty + E_C^\infty)^{1/2}(E_T^* - E_C^*)}{\lambda_\perp^-}\\ &+ \eta\frac{E_C^*}{2}\left[\frac{(E_T^\infty - E_C^\infty)^{1/2}}{\lambda_\perp^+} - \frac{(E_T^\infty + E_C^\infty)^{1/2}}{\lambda_\perp^-}\right];\end{aligned}\tag{50}$$

$$\begin{aligned}&\frac{\partial E_C^*}{\partial t} + \boldsymbol{U}\cdot\nabla E_C^* - \boldsymbol{V}_A\cdot\nabla E_T^* + \frac{\nabla\cdot\boldsymbol{U}}{2}E_C^*\\ &+\nabla\cdot\boldsymbol{V}_A(E_T^* - E_D^*) - 2bE_D^*\mathbf{ss}:\nabla\boldsymbol{B}/\sqrt{4\pi\rho}\\ &= \frac{\boldsymbol{n}_{u\infty}\cdot\nabla\rho}{2\sqrt{2}\rho}(E_T^\infty + E_D^\infty)^{1/2}E_C^*\\ &- \frac{(E_T^\infty - E_C^\infty)^{1/2}(E_T^* + E_C^*)}{\lambda_\perp^+}\\ &+ \frac{(E_T^\infty + E_C^\infty)^{1/2}(E_T^* - E_C^*)}{\lambda_\perp^-}\\ &+ \eta\frac{E_T^* - E_D^*}{2}\left[\frac{(E_T^\infty - E_C^\infty)^{1/2}}{\lambda_\perp^+} - \frac{(E_T^\infty + E_C^\infty)^{1/2}}{\lambda_\perp^-}\right];\end{aligned}\tag{51}$$

$$\begin{aligned}&\frac{\partial E_D^*}{\partial t} + \boldsymbol{U}\cdot\nabla E_D^* + \nabla\cdot\boldsymbol{U}\left[\frac{E_D^*}{2} + \left(2b - \frac{1}{2}\right)E_T^*\right]\\ &- 2bE_T^*\mathbf{ss}:\nabla\boldsymbol{U} + 2bE_C^*\mathbf{ss}:\nabla\boldsymbol{B}/\sqrt{4\pi\rho} + \nabla\cdot\boldsymbol{V}_A E_C^*\\ &= \frac{\boldsymbol{n}_{u\infty}\cdot\nabla\rho}{2\sqrt{2}\rho}(E_T^\infty + E_D^\infty)^{1/2}(E_D^* - E_T^*)\\ &- E_D^*\left[\frac{(E_T^\infty - E_C^\infty)^{1/2}}{\lambda_\perp^+} + \frac{(E_T^\infty + E_C^\infty)^{1/2}}{\lambda_\perp^-}\right]\\ &+ \eta\frac{E_C^*}{2}\left[\frac{(E_T^\infty - E_C^\infty)^{1/2}}{\lambda_\perp^+} - \frac{(E_T^\infty + E_C^\infty)^{1/2}}{\lambda_\perp^-}\right];\end{aligned}\tag{52}$$

$$\begin{aligned}&\frac{\partial L^{*\pm}}{\partial t} + (\boldsymbol{U}\mp\boldsymbol{V}_A)\cdot\nabla L^{*\pm} + \frac{\nabla\cdot\boldsymbol{U}}{2}\left(L^{*\pm} - \frac{L_D^*}{2}\right)\\ &+ \nabla\cdot\boldsymbol{V}_A\left(\pm L^{*\pm} \mp \frac{L_D^*}{2}\right)\\ &+ bL_D^*(\nabla\cdot\boldsymbol{U} - \mathbf{ss}:\nabla\boldsymbol{U} \mp \mathbf{ss}:\nabla\boldsymbol{B}/\sqrt{4\pi\rho})\\ &- \frac{\boldsymbol{n}_{u\infty}\cdot\nabla\rho}{2\sqrt{2}\rho}(E_T^\infty + E_D^\infty)^{1/2}\left(L^{*\pm} - \frac{L_D^*}{2}\right) = 0;\end{aligned}\tag{53}$$

$$\begin{aligned}&\frac{\partial L_D^*}{\partial t} + \boldsymbol{U}\cdot\nabla L_D^* + \frac{\nabla\cdot\boldsymbol{U}}{2}(L_D^* - L^{*+} - L^{*-})\\ &+ 2b(\nabla\cdot\boldsymbol{U} - \mathbf{ss}:\nabla\boldsymbol{U})(L^{*+} + L^{*-})\\ &+ 2b\mathbf{ss}:\nabla\boldsymbol{B}/\sqrt{4\pi\rho}(L^{*+} - L^{*-})\\ &+ \frac{\boldsymbol{n}_{u\infty}\cdot\nabla\rho}{2\sqrt{2}\rho}(E_T^\infty + E_D^\infty)^{1/2}(L^{*+} + L^{*-} - L_D^*)\\ &+ \nabla\cdot\boldsymbol{V}_A(L^{*+} - L^{*-}) = 0,\end{aligned}\tag{54}$$

where $\eta = 1$, and $\boldsymbol{n}_{u\infty}$ indicates the averaged direction of turbulent velocity $\boldsymbol{u}^\infty$ weighted by the associated terms. For an isotropic 2D turbulent velocity $\boldsymbol{u}^\infty$ or a weak dependence between $\boldsymbol{u}^\infty$ and $\boldsymbol{z}^{*\pm}$, the terms associated with $\boldsymbol{n}_{u\infty}\cdot\nabla\rho$ are zero. Similarly, the terms associated with $\boldsymbol{n}_{z\infty}\cdot\nabla\rho$ can also be eliminated from the turbulence transport equations for $\beta\sim 1$. In this case, the only differences between the transport equations for $\beta\sim 1$ and $\beta\ll 1$ result from the additional dissipation terms in the $\beta\ll 1$ equations. On setting $\eta = 0$, we recover the $\beta\sim 1$ turbulence transport equations. To generalize the transport equations for a plasma in which β lies between these two limits, we can parameterize η using β such that $\eta(\beta\sim 1) = 0$ and $\eta(\beta\ll 1) = 1$.

The transport equation for E_T^* in Equation (50) is similar to Equations (39) and (5), thus the conservative form of the transport equation for E_T^* is

$$\begin{aligned}&\frac{\partial E_w^*}{\partial t} + \nabla\cdot[(\boldsymbol{U} - \boldsymbol{V}_A\sigma_c^*)E_w^* + \boldsymbol{U}\cdot\boldsymbol{P}_w^*)]\\ &= \boldsymbol{U}\cdot\nabla\cdot\boldsymbol{P}_w^*\\ &+ \frac{\boldsymbol{n}_{u\infty}\cdot\nabla\rho}{4\sqrt{2}}(E_T^\infty + E_D^\infty)^{1/2}(E_T^* - E_D^*)\\ &- \frac{\rho}{2}\left[\frac{(E_T^\infty - E_C^\infty)^{1/2}(E_T^* + E_C^*)}{\lambda_\perp^+}\right.\\ &\left.+ \frac{(E_T^\infty + E_C^\infty)^{1/2}(E_T^* - E_C^*)}{\lambda_\perp^-}\right]\\ &+ \eta\frac{\rho E_C^*}{4}\left[\frac{(E_T^\infty - E_C^\infty)^{1/2}}{\lambda_\perp^+} - \frac{(E_T^\infty + E_C^\infty)^{1/2}}{\lambda_\perp^-}\right],\end{aligned}\tag{55}$$

where the turbulence pressure tensor P_w^* is identical to Equation (47).

3. Conclusions

In a formal sense, when considering the relationship of the ideal incompressible MHD equations to the ideal compressible MHD equations that results in the theory of nearly incompressible MHD (Zank & Matthaeus 1993), essentially the three limits of large plasma beta, plasma beta of order unity, or small plasma beta are relevant. Based on this ordering, Zank et al. (2012) (large beta), Zank et al. (2017) (beta $\sim$1), and this work (beta $\ll$1) derived three sets of equations describing the evolution of incompressible and nearly incompressible MHD turbulence in inhomogeneous flows as expressed through "moments" of the fluctuating Elsässer variables. The large beta limit yields a transport formalism that is at leading order based on the fully 3D incompressible MHD equations whereas the beta $\sim$1 or $\ll$1 limit yields a superposition of quasi-2D MHD as the leading-order or dominant component and a minority slab component. Each set of transport equations includes the nonlinear interaction between forward and backward modes, introduces the cross helicity and residual energy, and is applicable to sub-Alfvénic as well as super-Alfvénic flows. Each set of equations consists of six coupled equations to model the evolution of turbulence energy, cross helicity, residual energy, and associated correlation lengths. The advanced transport equations are likely more realistic and provide much more information about the evolution of turbulence than earlier turbulence transport models that made a number of quite severe approximations, including the standard WKB description (e.g., Parker 1965; Hollweg 1973; Zank et al. 1996; Breech et al. 2008). Depending on the plasma beta, we should choose an appropriate set of turbulence transport equations. For example, to investigate the evolution of solar wind turbulence, different choices for turbulence transport models must be made carefully for different regions. Neither Zank et al. (2012) nor Zank et al. (2017) derived a conservation form of the transport equations for the energy in the fluctuations. Expressing the fluctuating energy in the form of a conservation law is an important check on the physical and mathematical consistency of the turbulence transport formalism. Of course, the results based on the application of our non-conservation form of the turbulence transport equations to the solar wind should remain unchanged (Adhikari et al. 2021; Zank et al. 2021). Despite the evident complexity of the underlying turbulence transport equations, we show here that, in both limits, the transport equations for the turbulence energy can be expressed in conservative form through the introduction of generalized forms of the pressure tensor for the fluctuating velocity and magnetic field components. The generalized turbulence pressure tensor is quite different from the simple isotropic Alfvén turbulence pressure that is present in the WKB description and is quite unlike the typical concept of a "pressure" derived from the fluctuating magnetic and velocity fields. Under some symmetries or configurations, the turbulence pressure tensor is degenerate and resembles the Alfvén turbulence pressure. It would be of great interest to determine the various forms of the turbulence pressure tensor $\boldsymbol{P}_w$ and turbulence energy density E_w from direct numerical simulations (Lugones et al. 2019).

Our principal results are Equations (29) (beta $\gg$1), (37) (quasi-2D turbulence, beta $\sim$1, or beta $\ll$1), (46) (slab turbulence, beta $\sim$1), and (55) (slab turbulence, beta $\ll$1) together with the respective definitions of the turbulence pressure (23), (38), and (47). The fluctuating energy conservation laws for the beta $\gg$1 and the slab turbulence beta order $\sim$1 or $\ll$1 cases resemble formally the well-known WKB transport equations for the energy density of linear Alfvén waves in an inhomogeneous flow. However, the analogy is not close for multiple reasons: (i) The turbulence transport formalism does not assume that the fluctuations are small-amplitude or linear. (ii) The turbulence pressure tensor $\boldsymbol{P}_w$ is significantly different from the wave pressure tensor of WKB models, containing typically both the energy densities of the fluctuating velocity and magnetic fields. The relevant anisotropies of the underlying turbulence are also contained in the turbulence pressure tensors, as expressed through the structural similarity parameters that represent a closure relation between the trace and the covariance terms in the one-point Elsässer energy tensor terms. (iii) The energy density flux in the turbulence conservation laws is similar to the WKB formalism in that the turbulence form contains the cross helicity, although in the turbulence case, the cross helicity is governed by an independent turbulence transport equation that must be solved in conjunction with the energy transport equation. (iv) Finally, the role of dissipation is properly incorporated in the conservation laws and is based on a Kolmogorov formalism (equally, the dissipation terms can be treated using an Iroshnikov–Kraichnan formalism (Ng et al. 2010) provided one is not modeling quasi-2D turbulence (Zank et al. 2020)). The conservation form of the dominant quasi-2D turbulence case (beta $\sim$1 or beta $\ll$1), Equation (38), also resembles the WKB formalism in some terms, but is quite different in that the Alfvén velocity term is absent entirely. This of course is because the fluctuations are quasi-2D structures such as flux ropes and not Alfvén waves. The differences (i)–(iv) above also apply to the quasi-2D turbulence conservation law. For the beta $\sim$1 or $\ll$1 case, the separation into dominant and minority components means that the quasi-2D turbulence energy density E_w^∞ and pressure $\boldsymbol{P}_w^\infty$ and the slab turbulence energy density E_w^* and pressure $\boldsymbol{P}_w^*$ can be combined to obtain the total turbulence energy density and pressure tensor contribution.

The conservation forms of the turbulence energy density equations cannot be solved in isolation, since they are coupled to the evolution of the cross helicity, the residual energy, and the various correlation lengths. Nonetheless, the conservation form can be used in place of the total energy transport equation formalism used in Zank et al. (2012) and Zank et al. (2017) and applications thereof. If one chooses to impose certain constraints on, e.g., the cross helicity or residual energy (Zank et al. 1996, 2012), the conservation form is particularly useful in deriving simplified forms of the turbulence transport equation, many of which are readily amenable to analytic solution (Zank et al. 1996).

In conclusion, we have derived conservation forms of the turbulence energy transport equations, and shown explicitly the dissipation terms and derived generalized forms of the turbulence pressure tensor. We anticipate that our results will be useful for a range of important and interesting problems in solar, stellar, and other large-scale astrophysical winds, cosmic ray physics, shock waves, and especially the heating of the solar corona and solar wind.

We acknowledge the partial support of an NSF EPSCoR RII-Track-1 Cooperative Agreement OIA-1655280, partial support from a NASA Parker Solar Probe contract SV4-84017, partial support from a NASA LWS grant 80NSSC20K1783, and partial support from a NASA IMAP subaward under NASA contract 80GSFC19C0027. G.P.Z. was partially supported by an NSF CDS&E Award 2009871.

Appendix
Derive the Transport Equations for the Slab Turbulence in the $\beta \ll 1$ Plasma

Following Hunana & Zank (2010), but assuming the ratio of the typical length scale for fluctuations and the typical length scale for the large-scale inhomogeneous background is not as small as the expansion parameter, the evolution equation for the slab turbulent velocity and magnetic field are given by

$$\rho\frac{\partial \boldsymbol{u}_1}{\partial t} + \rho(\boldsymbol{U} + \boldsymbol{u}^\infty)\cdot\nabla\boldsymbol{u}_1 + \rho\boldsymbol{u}_1\cdot\nabla(\boldsymbol{U}+\boldsymbol{u}^\infty) + \rho^\infty\frac{\partial \boldsymbol{u}^\infty}{\partial t} + \rho^\infty(\boldsymbol{U}^\infty + \boldsymbol{u}^\infty)\cdot\nabla\boldsymbol{u}^\infty + \rho^\infty(\boldsymbol{U}+\boldsymbol{u}^\infty)\cdot\nabla\boldsymbol{U} = -\nabla p^* + \frac{(\nabla\times b^*)\times B + (\nabla\times B)\times b^*}{4\pi}; \quad (A1)$$

$$\frac{\partial \boldsymbol{b}^*}{\partial t} + (\boldsymbol{U}+\boldsymbol{u}^\infty)\cdot\nabla\boldsymbol{b}^* - \boldsymbol{b}^*\cdot\nabla\boldsymbol{u}^\infty - \nabla\times(\boldsymbol{u}_1\times\boldsymbol{B}^\infty) = \boldsymbol{B}\cdot\nabla\boldsymbol{u}_1 - \boldsymbol{B}\nabla\cdot\boldsymbol{u}_1 - \boldsymbol{u}_1\cdot\boldsymbol{\nabla}B + \boldsymbol{b}^*\cdot\nabla\boldsymbol{U} - \boldsymbol{b}^*\nabla\cdot\boldsymbol{U} - \boldsymbol{b}^*\nabla\cdot\boldsymbol{u}^\infty, \quad (A2)$$

where ρ^∞ denotes the leading-order correction for the gas density and p^* is the high-order correction for the gas thermal pressure.

The fluctuating nearly incompressible Elsässer variables are defined as

$$\boldsymbol{z}^{*\pm} \equiv \boldsymbol{u}_1 \pm \frac{\boldsymbol{b}^*}{\sqrt{4\pi\rho}} = \boldsymbol{u}_1 \pm \boldsymbol{V}_A^*. \quad (A3)$$

On combining Equations (A1)–(A2), and using the identities $\boldsymbol{u}_1 = (\boldsymbol{z}^{*+} + (\boldsymbol{z}^{*-})/2$ and $\boldsymbol{V}_A^* = (\boldsymbol{z}^{*+} - \boldsymbol{z}^{*-})/2$, we obtain the transport equations for $z^*\pm$ as

$$\frac{\partial \boldsymbol{z}^{*\pm}}{\partial t} + (\boldsymbol{U}\mp\boldsymbol{V}_A)\cdot\nabla\boldsymbol{z}^{*\pm} + \boldsymbol{z}^{*\mp}\cdot\nabla\boldsymbol{U} + \frac{1}{2}\boldsymbol{z}^{\infty\mp}\cdot\nabla(\boldsymbol{z}^{*+}+\boldsymbol{z}^{*-}) \pm \frac{1}{4}(\boldsymbol{z}^{\infty+}+\boldsymbol{z}^{\infty-})\cdot\nabla(\boldsymbol{z}^{*+}-\boldsymbol{z}^{*-}) + \frac{1}{2}\boldsymbol{z}^{*\mp}\cdot\nabla(\boldsymbol{z}^{\infty+}+\boldsymbol{z}^{\infty-}) \pm \frac{1}{4}(\boldsymbol{z}^{*+}+\boldsymbol{z}^{*-}) \cdot\nabla(\boldsymbol{z}^{\infty+}-\boldsymbol{z}^{\infty-}) \pm \frac{(\boldsymbol{z}^{*+}+\boldsymbol{z}^{*-})\cdot\nabla\rho}{4\rho}(\boldsymbol{z}^{\infty+}-\boldsymbol{z}^{\infty-}) \pm \frac{\nabla\cdot\boldsymbol{U}}{4}(\boldsymbol{z}^{*+}-\boldsymbol{z}^{*-}) \mp \frac{\boldsymbol{n}_{u\infty}\cdot\nabla\rho}{4\sqrt{2}\,\rho} \times (E_T^\infty + E_D^\infty)^{1/2}(\boldsymbol{z}^{*+}-\boldsymbol{z}^{*-}) \pm \boldsymbol{z}^{*\mp}\cdot\nabla\boldsymbol{B}\big/\sqrt{4\pi\rho} + \frac{1}{2}\nabla\cdot\boldsymbol{V}_A(\boldsymbol{z}^{*+}-\boldsymbol{z}^{*-}) = -\frac{1}{\rho}\left[\nabla p^* + \frac{\nabla(\boldsymbol{B}\cdot\boldsymbol{b}^*)}{4\pi}\right]. \quad (A4)$$

We restrict attention to the incompressible modes $\nabla\cdot\boldsymbol{u}_1 = 0$, and neglect the source term for ρ^∞.

Following Zank et al. (2012, 2017), by taking the dot product of Equation (A5) with respect to $\boldsymbol{z}^{*\pm}$ ($\boldsymbol{z}^{*\mp}$) and constructing the ensemble average, we can derive the evolution equations for $\langle z^{*\pm 2}\rangle$ (E_D^*). The following assumptions are made. The nonlinear terms are modeled as

$$\boldsymbol{z}^{\infty\mp}\cdot\nabla\boldsymbol{z}^{*\pm} = \boldsymbol{z}^{*\pm}\frac{\langle z^{\infty\mp 2}\rangle^{1/2}}{\lambda_\perp^\pm}. \quad (A5)$$

We introduce an approximation for the variance of the components as some fraction of the variance of the trace. This structural similarity assumption shows that

$$\langle z_i^{*+} z_j^{*-}\rangle \approx bE_D^*(\delta_{ij} - \mathbf{ss}); \quad \langle z_i^{*\pm} z_j^{*\pm}\rangle \approx b\langle z^{*\pm 2}\rangle(\delta_{ij} - \mathbf{ss}), \quad (A6)$$

where b is the structural similarity parameter. The resulting evolution equations for $\langle z^{*\pm 2}\rangle$ are

$$\frac{\partial\langle z^{*+2}\rangle}{\partial t} + (\boldsymbol{U} - \boldsymbol{V}_A)\cdot\nabla\langle z^{*+2}\rangle + 2bE_D^*(\nabla\cdot\boldsymbol{U} - \mathbf{ss}:\nabla\boldsymbol{U}) + \frac{\nabla\cdot\boldsymbol{U}}{2}(\langle z^{*+2}\rangle - E_D^*) - \frac{\boldsymbol{n}_{u\infty}\cdot\nabla\rho}{2\sqrt{2}\,\rho}(E_T^\infty + E_D^\infty)^{1/2}(\langle z^{*+2}\rangle - E_D^*) - 2bE_D^*\mathbf{ss}:\nabla\boldsymbol{B}\big/\sqrt{4\pi\rho} + \nabla\cdot\boldsymbol{V}_A(\langle z^{*+2}\rangle - E_D^*) = -\frac{\langle z^{\infty+2}\rangle^{1/2}}{\lambda_\perp^-}\left(\frac{\langle z^{*+2}\rangle}{2} - \frac{E_D^*}{2}\right) - \frac{\langle z^{\infty-2}\rangle^{1/2}}{\lambda_\perp^+}\left(\frac{3\langle z^{*+2}\rangle}{2} + \frac{E_D^*}{2}\right); \quad (A7)$$

$$\frac{\partial\langle z^{*-2}\rangle}{\partial t} + (\boldsymbol{U} + \boldsymbol{V}_A)\cdot\nabla\langle z^{*-2}\rangle + 2bE_D^*(\nabla\cdot\boldsymbol{U} - \mathbf{ss}:\nabla\boldsymbol{U}) + \frac{\nabla\cdot\boldsymbol{U}}{2}(\langle z^{*-2}\rangle - E_D^*) - \frac{\boldsymbol{n}_{u\infty}\cdot\nabla\rho}{2\sqrt{2}\,\rho}(E_T^\infty + E_D^\infty)^{1/2}(\langle z^{*-2}\rangle - E_D^*) + 2bE_D^*\mathbf{ss}:\nabla\boldsymbol{B}\big/\sqrt{4\pi\rho} - \nabla\cdot\boldsymbol{V}_A(\langle z^{*-2}\rangle - E_D^*) = -\frac{\langle z^{\infty-2}\rangle^{1/2}}{\lambda_\perp^+}\left(\frac{\langle z^{*-2}\rangle}{2} - \frac{E_D^*}{2}\right) - \frac{\langle z^{\infty+2}\rangle^{1/2}}{\lambda_\perp^-}\left(\frac{3\langle z^{*-2}\rangle}{2} + \frac{E_D^*}{2}\right), \quad (A8)$$

where $E_D^* = \langle \boldsymbol{z}^{*+}\cdot\boldsymbol{z}^{*-}\rangle$. Note that $E_T^* = (\langle z^{*+2}\rangle + \langle z^{*-2}\rangle)/2$ and $E_C^* = (\langle z^{*+2}\rangle - \langle z^{*-2}\rangle)/2$, such that the evolution Equations (50) and (51) are obtained by combining Equations (A7) and (A8).

The evolution equation for E_D^* is given by Equation (52). The correlation functions and correlation lengths are related through

$$L_*^\pm = \int\langle \boldsymbol{z}^{*\pm}\cdot\boldsymbol{z}^{*\pm\prime}\rangle dr = \langle z^{*\pm 2}\rangle\lambda_*^\pm; \quad (A9)$$

$$L_D^* = \int\langle \boldsymbol{z}^{*\pm}\cdot\boldsymbol{z}^{*\mp\prime} + \boldsymbol{z}^{*\pm\prime}\cdot\boldsymbol{z}^{*\mp}\rangle dr = E_D^*\lambda_D^*, \quad (A10)$$

where r is the spatial lag between fluctuations, and $\boldsymbol{z}^{*\pm\prime} = \boldsymbol{z}^{*\pm}(\boldsymbol{x}+\boldsymbol{r})$ denotes the lagged Elsässer variable at a location $\boldsymbol{r}$ from $\boldsymbol{x}$. The evolution equation for the correlation functions can be obtained by applying the same procedure used in Zank et al. (2012, 2017). Taking appropriate moments of Equation (A4) and using the assumptions (A5) and (A6), we obtain Equations (53) and (54).

ORCID iDs

B.-B. Wang https://orcid.org/0000-0002-6000-1262
G. P. Zank https://orcid.org/0000-0002-4642-6192

L. Adhikari https://orcid.org/0000-0003-1549-5256
L.-L. Zhao https://orcid.org/0000-0002-4299-0490

References

Adhikari, L., Zank, G. P., Bruno, R., et al. 2015, ApJ, 805, 63
Adhikari, L., Zank, G. P., Hunana, P., et al. 2017, ApJ, 841, 85
Adhikari, L., Zank, G. P., & Zhao, L. L. 2020a, ApJ, 901, 102
Adhikari, L., Zank, G. P., Zhao, L. L., et al. 2020c, ApJS, 246, 38
Adhikari, L., Zank, G. P., & Zhao, L. L. 2021, Fluid, 6, 368
Adhikari, L., Zank, G. P., Zhao, L. L., & Webb, G. M. 2020b, ApJ, 891, 34
Bell, A. R., & Lucek, S. G. 2001, MNRAS, 321, 433
Breech, B., Matthaeus, W. H., Minnie, J., et al. 2008, JGRA, 113, A08105
Caprioli, D., Blasi, P., Amato, E., & Vietri, M. 2009, MNRAS, 395, 895
Chandran, B. D. G., & Hollweg, J. V. 2009, ApJ, 707, 1659
Cranmer, S. R., van Ballegooijen, A. A., & Edgar, R. J. 2007, ApJS, 171, 520
Dmitruk, P., Matthaeus, W. H., Milano, L. J., et al. 2002, ApJ, 575, 571
Dmitruk, P., Milano, L. J., & Matthaeus, W. H. 2001, ApJ, 548, 482
Heinemann, M., & Olbert, S. 1980, JGR, 85, 1311
Hollweg, J. V. 1973, JGR, 78, 3643
Hunana, P., & Zank, G. P. 2010, ApJ, 718, 148
Isenberg, P. A. 2005, ApJ, 623, 502
Isenberg, P. A., Smith, C. W., & Matthaeus, W. H. 2003, ApJ, 592, 564
Jones, T. W. 1993, ApJ, 413, 619
Ko, C. M. 1995, AdSpR, 15, 149
Landau, L. D., & Lifshitz, E. M. 1987, Fluid Mechanics (2nd ed.; Oxford: Pergamon), 10
Le Roux, J. A., Zank, G. P., & Khabarova, O. V. 2018, ApJ, 864, 158
Lugones, R., Dmitruk, P., Mininni, P. D., Pouquet, A., & Matthaeus, W. H. 2019, PhPl, 26, 122301
Marsch, E., & Tu, C. Y. 1989, JPlPh, 41, 479
Marsch, E., & Tu, C. Y. 1990, JGR, 95, 8211
Matthaeus, W. H., Oughton, S., Pontius, D. H. J., & Zhou, Y. 1994a, JGR, 99, 19267
Matthaeus, W. H., Zank, G. P., Oughton, S., Mullan, D. J., & Dmitruk, P. 1999a, ApJL, 523, L93
Matthaeus, W. H., Zank, G. P., Smith, C. W., & Oughton, S. 1999b, PhRvL, 82, 3444
Matthaeus, W. H., Zhou, Y., Zank, G. P., & Oughton, S. 1994b, JGR, 99, 23421
McKee, C. F., & Zweibel, E. G. 1995, ApJ, 440, 686
McKenzie, J. F., & Völk, H. J. 1982, A&A, 116, 191
Montagud-Camps, V., Grappin, R., & Verdini, A. 2018, ApJ, 853, 153
Ng, C. S., Bhattacharjee, A., Munsi, D., Isenberg, P. A., & Smith, C. W. 2010, JGRA, 115, A02101
Oughton, S., Matthaeus, W. H., Dmitruk, P., et al. 2001, ApJ, 551, 565
Oughton, S., Matthaeus, W. H., Smith, C. W., Breech, B., & Isenberg, P. A. 2011, JGRA, 116, A08105
Parker, E. N. 1965, SSRv, 4, 666
Shiota, D., Zank, G. P., Adhikari, L., et al. 2017, ApJ, 837, 75
Smith, C. W., Matthaeus, W. H., Zank, G. P., et al. 2001, JGR, 106, 8253
van Ballegooijen, A. A., & Asgari-Targhi, M. 2016, ApJ, 821, 106
Verdini, A., Velli, M., Matthaeus, W. H., Oughton, S., & Dmitruk, P. 2010, ApJL, 708, L116
Woolsey, L. N., & Cranmer, S. R. 2014, ApJ, 787, 160
Zank, G. P. 2014, Transport Processes in Space Physics and Astrophysics, Vol. 877 (Berlin: Springer)
Zank, G. P., Adhikari, L., Hunana, P., et al. 2017, ApJ, 835, 147
Zank, G. P., Adhikari, L., Hunana, P., et al. 2018a, ApJ, 854, 32
Zank, G. P., Adhikari, L., Zhao, L. L., et al. 2018b, ApJ, 869, 23
Zank, G. P., Dosch, A., Hunana, P., et al. 2012, ApJ, 745, 35
Zank, G. P., & Matthaeus, W. H. 1992, JPlPh, 48, 85
Zank, G. P., & Matthaeus, W. H. 1993, PhFlA, 5, 257
Zank, G. P., Matthaeus, W. H., & Smith, C. W. 1996, JGR, 101, 17093
Zank, G. P., Nakanotani, M., Zhao, L. L., Adhikari, L., & Telloni, D. 2020, ApJ, 900, 115
Zank, G. P., Zhao, L. L., Adhikari, L., et al. 2021, PhPl, 28, 080501
Zhao, L. L., Zank, G. P., Adhikari, L., et al. 2020, ApJ, 898, 113
Zhou, Y., & Matthaeus, W. H. 1989, GeoRL, 16, 755
Zhou, Y., & Matthaeus, W. H. 1990a, JGR, 95, 14863
Zhou, Y., & Matthaeus, W. H. 1990b, JGR, 95, 10291