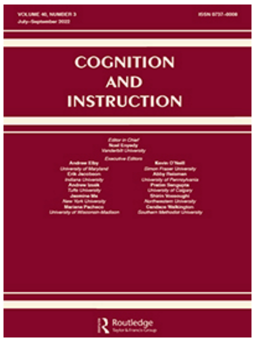

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# Generalization Across Multiple Mathematical Domains: Relating, Forming, and Extending

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**ABSTRACT**

Generalization is a critical component of mathematical reasoning, with researchers recommending that it be central to education at all grade levels. However, research on students' generalizing reveals pervasive difficulties in creating and expressing general statements, which underscores the need to better understand the processes that can support more productive generalizations. In response, we report on results from 146 interviews with 93 participants in middle school through college in the domains of algebra, advanced algebra, trigonometry/pre-calculus, and combinatorics while solving complex problems. Our findings yielded the Relating-Forming-Extending (RFE) Framework, which distinguishes multiple related forms and types of generalizing. We also present two aspects of mental activity that promote generative generalizations: operative activity, and building and refining activity.

## Introduction

Generalizing is widely acknowledged to be a critical component of mathematical activity. Researchers argue that not only does generalization serve as the means for constructing new knowledge, but that mathematical thought cannot occur in its absence (e.g., Kaput, 1999; Mason, Burton, & Stacey, 2010; Mata-Pereira & da Ponte, 2017; Peirce, 1902; Sfard, 1995; Vygotsky, 1986). For these reasons, Davydov (1972/1990) noted that "[d]eveloping children's generalizations is regarded as one of the principal purposes of school instruction" (p. 10). Accordingly, researchers have studied the importance of generalization for promoting algebraic reasoning (Amit & Klass-Tsirulnikov, 2005; Cooper & Warren, 2008; Dougherty, Bryant, Bryant, Darrough, & Pfannenstiel, 2015; Radford, 2006, 2008), geometric thinking (Lehrer, Kobiela, & Weinberg, 2013; Pytlak, 2015), and functional thinking (Ellis, 2011; Rivera & Becker, 2007), among other areas. Current curricula and standards documents also emphasize the importance of generalization, with curricular materials including tasks designed to promote generalization (e.g., Hirsch, Fey, Hart, Harold, & Watkins, 2015; Lappan, Phillips, Fey, & Friel, 2014), and the Common Core State Standards in Mathematics highlighting generalization in both the content and practice standards (National Governors Association Center/Council of Chief State School Officers, 2010).

The ability to productively generalize has been identified as a characteristic of mathematically-capable students (Amit & Neria, 2008), but research on students' generalizing reveals pervasive difficulties. Students experience challenges in creating general statements that are mathematically correct (English & Warren, 1995; Kieran, 1992; Lannin, 2005), attending to patterns that are generalizable (Blanton & Kaput, 2002; Čadež & Kolar, 2015; Lee, 1996; Radford, 2003, 2006, 2008; Rivera & Becker, 2008), and using generalized language (Mason, 1996). When they are able to

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generalize patterns, students struggle to shift from recursive patterns to generating a rule to determine the *n*th case (English & Warren, 1995; Moss, Beatty, McNab, & Eisenband, 2006; Schliemann, Carraher, & Brizuela, 2001). Reporting the results of 5 years of performance assessments on generalization given to more than 60,000 students, Rivera (2008) found a stable ceiling value of only a 20% success rate in students' abilities to correctly construct general formulas. These challenges underscore the fact that much remains to be understood about the nature of student generalizing and the processes that support more productive generalizations.

Definitions of generalization vary, but most characterize it as a claim that some property holds for a set of mathematical objects or conditions that is larger than the set of individually verified cases (Carraher, Martinez, & Schliemann, 2008). For instance, Harel and Tall (1991) described generalization as the process of "applying a given argument in a broader context" (p. 38), and Radford (2006) argued that the generalization of a pattern involves identifying a commonality based on some particulars and then extending that commonality to all terms. More recent sociocultural examinations situate generalization within activity and context, describing it as an act that can be distributed across multiple agents or that can occur collectively (Ellis, 2011; Reid, 2002; Tuomi-Gröhn & Engeström, 2003). Due to the nature of our data set, for the purposes of this study we follow the cognitive perspective to define generalizing as an activity in which learners engage in at least one of the following actions: (a) identifying commonality across cases (Dreyfus, 1991), (b) deriving broader results from particular cases to form general relationships, rules, concepts, or connections (Ellis, 2007a; Kaput, 1999), and/or (c) extending one's reasoning beyond the range in which it originated (Carraher et al., 2008; Dubinsky, 1991; Harel & Tall, 1991; Radford, 2006). Following Ellis (2007a), we call these three processes *generalizing*, and we reserve the term *generalization* for the outcomes of these processes.

### *The need to better understand students' generalizing activity*

The majority of the research on students' generalizing in mathematics has occurred in the domains of early algebra (e.g., Carraher et al., 2008; Cooper & Warren, 2008; Dougherty et al., 2015; El Mouhayar, 2018a; Mulligan & Mitchelmore, 2009), patterns (Amit & Neria, 2008; Becker & Rivera, 2006; Cooper & Warren, 2008; Jureczko, 2017; Moss et al., 2006; Mulligan & Mitchelmore, 2009; Radford, 2002, 2006; Rivera, 2010; Rivera & Becker, 2008; Steele, 2008; Warren, Miller, & Cooper, 2013), and linearity (e.g., Carraher et al., 2008; Ellis, 2007a, 2007b; Radford, 2008; Rivera & Becker, 2008). There have been some studies examining students' generalizing in topics beyond linear function, such as quadratics (Amit & Neria, 2008; Ellis, 2011; Lee & Wheeler, 1987; Rico, 1996; Rivera & Becker, 2007), advanced algebra (Mason, Drury, & Bills, 2007), calculus (e.g., Font & Contreras, 2008; Harel & Tall, 1991; Park & Lee, 2016) and combinatorics (Lockwood, 2011; Lockwood & Reed, 2016; Tillema & Gatza, 2017), but in general, less is known about how students generalize beyond the domains of patterning and early algebra. In particular, fewer studies have addressed students' generalizing activity in domains such as advanced algebra, pre-calculus, trigonometry, and discrete mathematics, particularly in upper secondary school and at the undergraduate level.

In addition, the manner in which typical pattern generalization tasks are presented tend to have a strong regulating effect on students' generalizing (Doerfler, 2008). These tasks often provide figural cues and expected progressions that encourage a narrow range of potential generalizations. In contrast, more open-ended generalizing tasks that are not restricted by pre-determined goals could provide valuable information about how students generalize when engaging in the types of problem-solving activities they are likely to encounter in school mathematics (see Rivera & Becker, 2016, for an example). Thus, there remains a need to better understand how students construct generality not only in more varied and advanced mathematical domains, but also in more open-ended problems.

In response to the need to study generalization in open-ended tasks, Ellis (2007a) explored the generalizations middle-school students made in a teaching experiment setting investigating ratios and linear functions. This work yielded a generalization taxonomy, which categorized three major forms of generalizing activity: Relating, Searching, and Extending. Ellis' taxonomy provides the foundation for the current study, but it was restricted to a small group of students studying one early algebra topic. We seek to build on this work while considering the need to (a) better understand generalizing in more varied and advanced domains, and (b) identify what aspects of mathematical activity are productive for generalization across domains. Consequently, we developed the following research questions:

1. How can Ellis' generalization taxonomy be refined and extended to apply to the domains of algebra, advanced algebra, trigonometry/pre-calculus, and combinatorics?
2. What aspects of mathematical activity foster productive generalizing across the domains of algebra, advanced algebra, trigonometry/pre-calculus, and combinatorics?

In addressing these questions, our aim is to better understand the character of student generalizing across multiple content domains and grade levels (middle-school, high-school, and college), and across a varied range of tasks and activities beyond those presenting patterns.

Our findings yield a broader, more comprehensive framework of generalizing activity called the *Relating-Forming-Extending (RFE) Framework*. The RFE Framework distinguishes related forms and types of students' generalizing across multiple content domains, spanning student thinking in middle-school, high-school, and university topics. In the results section of this paper, we describe the RFE Framework, then we present findings on two aspects of mathematical activity that foster productive generalizing activity, which we call generative generalizing. The first aspect addresses students' initial focus when investigating a task, and the second addresses students' abilities to leverage initial generalizations into more powerful generalizations as they grapple with novel tasks. We close with instructional recommendations for better supporting students' productive generalizing.

## Review of the literature: characterizing generalization

### *Prior characterizations of generalization in the literature*

In framing our contribution to the literature, we first briefly review other ways in which generalization has been characterized in the literature. We do this to highlight the fact that distinctions among forms and types of generalizing (e.g., Harel, 2001), as well as generalizing strategies and processes (e.g., Radford, 2008, 2010), have long been of interest to the mathematics education community. In elaborating existing distinctions in the literature, we note three main categories of studies: studies of students in one particular setting or focused on one given domain; studies of students' strategies across grade levels that are focused on one specific kind of generalizing (such as pattern generalization); and studies that rely on theoretical analyses.

Researchers have highlighted different generalization types, focusing particularly on whether the source of the generalization is based on empirical results or reasoning processes. For instance, in studying undergraduates and proof, Harel (2001) identified two forms of generalization: result-pattern generalization and process-pattern generalization. The former is the development of generality from regularity in the result of a process, and the latter is the development of generality from regularity in the process of moving from one term or outcome to the next. Rivera (2008) noted that Harel's distinction shadows what Doerfler (1991) referred to as empirical generalization and theoretical generalization, respectively. Empirical generalization is the recognition of common features, whereas theoretical generalization relies on a system of action in which generalizations are constructed through the abstraction of invariants. Theoretical generalization, like process-pattern generalization, foregrounds commonalities that emerge in reasoning about actions, rather than about their outcomes. In working with third grade students in the context of patterning in functions, Carraher et al. (2008) also used the terms empirical and theoretical generalization, the former of which arises from examining data for underlying trends and structure, and the latter emerges from ascribing models to data (see also, Bills & Rowland, 1999). There is a loose implied hierarchy in many of these distinctions, in which generalizations that are theoretical (or structural, or process-pattern) are viewed as more mathematically powerful than their empirical counterparts.

A number of researchers have also distinguished more explicit stages or levels of sophistication in generalizing. For example, in a theoretical paper, Harel and Tall (1991) characterized a hierarchy of conceptual effort in which they distinguished between expansive, reconstructive, and disjunctive generalization. Expansive generalization occurs when one expands the applicability range of an existing scheme without reconstructing it. Reconstructive generalization involves reconstructing a scheme to widen its applicability range, and disjunctive generalization is the construction of a new scheme. Other researchers have also explored hierarchies in a variety of contexts; one example is the distinction between reflective (or constructive), eliminative, and additive abstraction (Font & Contreras, 2008), which was introduced in a theoretical analysis of the derivative function.

Much of the work describing progressions in generalization has been situated within the context of pattern generalization (e.g., Amit & Neria, 2008; El Mouhayar & Jurdak, 2015; Radford, 2003; Rivera & Becker, 2008, 2016). For instance, Rivera and Becker (2008) distinguished constructive versus deconstructive generalizations among middle school students' linear figural patterns, and in a series of studies, Radford (2003, 2006, 2008, 2010) offered elaborations of constructs such as factual, contextual, and symbolic generalizations that contribute to young students' patterning activity. Researchers have also explored differences between elementary, middle-school, and secondary students' generalizing in patterning contexts, addressing distinctions such as inductive versus deductive strategies (Rivera, 2013), near versus far generalizations (El Mouhayar & Jurdak, 2015), function type (Jurdak & El Mouhayar, 2014), and figural versus numerical generalizations (El Mouhayar, 2018a, 2018b). Each of these distinctions captures important insights into students' cognitive processes within the context of pattern generalization.

Finally, a few researchers have described students' generalization strategies, rather than stages or levels (e.g., Becker & Rivera, 2007; Park & Lee, 2016; Rivera & Becker, 2005). For instance, Lannin, Barker, and Townsend (2006) described four generalizing strategies that emerged in their work with two fifth grade students in algebra: recursive, chunking, unitizing, and explicit. Their work has since been replicated by Yeap and Kaur (2008), also with fifth grade students. Warren et al. (2013) found that the use of three strategies—gesturing, self-talk, and concrete acting out—supported primary students' generalizations about functional relationships. Becker and Rivera (2006, 2007) distinguished between figural and numerical strategies as two ways to express generality, characterizing students' shifts from figurative to numerical strategies over time. These reported strategies highlight that researchers see merit in articulating differences between how students might engage in generalizing and that there may be strategies that are more or less effective for students in this process.

These previous studies have articulated meaningful distinctions between ways in which students generalize and progress in generalizing activity. As we have noted, many of these distinctions emerged from a single mathematical level and domain, from studies of students' strategies across grade levels that are focused on one specific kind of generalizing (such as pattern generalization), or through theoretical arguments that were based in mathematical examples. We now turn to Ellis' (2007a) taxonomy, upon which our work builds directly, and discuss the manner in which it identifies multiple types of generalizing grouped into thematically-related categories.

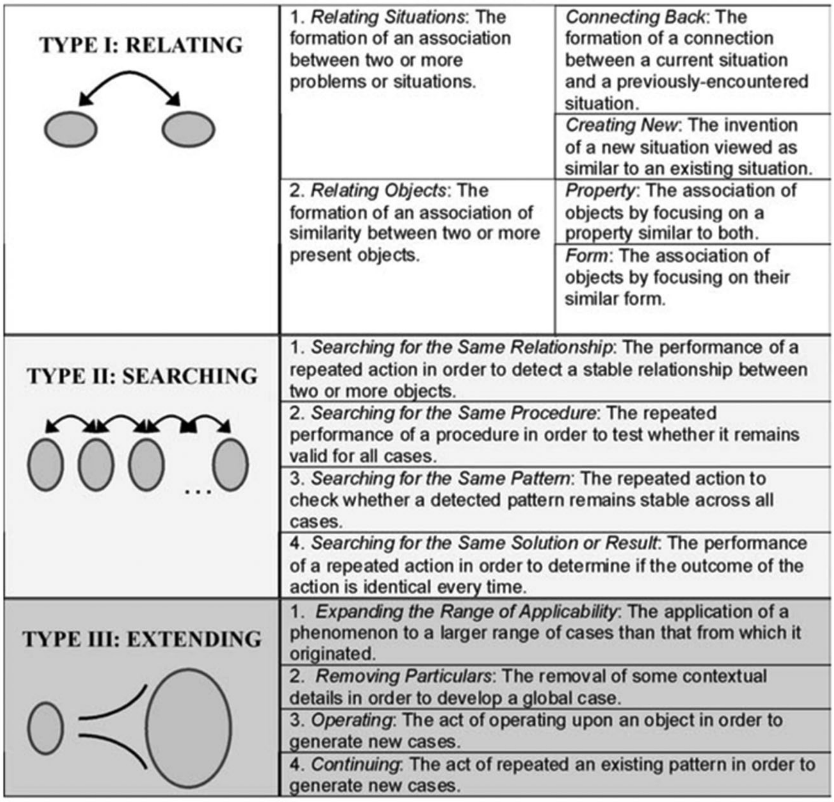



**Figure 1.** Ellis' (2007a) Taxonomy of Generalizing Actions.

### *Ellis' generalization taxonomy*

Like many of the other studies we have reviewed, Ellis' (2007a) generalization taxonomy was developed from a data set that was situated in a single mathematical domain and level, namely, an examination of seven 7th-grade students studying ratios and linear functions. In it, Ellis differentiated between generalizing actions and reflection generalizations, describing a generalizing action as an observer's inference of a mental action, and a reflection generalization as a final statement of generalization that is shared publicly. The generalization taxonomy includes three broad forms of generalizing action: Relating, Searching, and Extending (Figure 1). *Relating* is the formation of an association of similarity between two or more contexts, tasks, or objects. Relating can entail what a student perceives as two distinct problem situations (relating two contexts or tasks) or can occur within a single problem situation (relating two objects). *Searching* is the attempt to find an element of similarity, and can encompass a focus on a pattern, procedure, relationship, or solution. *Extending* is the expansion of the rule (or pattern or relationship) to account for new cases.

One purpose of the taxonomy was to identify ways in which students' initial generalizing activity, (i.e., their first attempts at generalizing), could be leveraged to create more mathematically powerful generalizations over time. It has since informed work in a number of other domains, including number theory and arithmetic (Chinnappan & Pandian, 2009; Vale, Widjaja, Herbert, Bragg, & Loong, 2017); patterning activities (Maj-Tatsis & Tatsis, 2018; Oflaz, 2019), algebraic thinking and functional relationships (Ellis & Grinstead, 2008; Napaphun, 2012; Strachota, Knuth, & Blanton, 2018; Tillema & Gatza, 2017), geometry (Yao, 2018); trigonometry (Tasova & Moore, 2018); calculus (Dorko, 2015; Dorko & Lockwood, 2016; Dorko & Weber, 2014), linear algebra

(Aranda & Callejo, 2010), and real analysis (Reed, 2018). These studies have increased the broader applicability of the taxonomy by applying it to new domains. However, they did not challenge, refine, or extend the taxonomy itself. It was precisely for this purpose that we developed the RFE Framework, particularly in light of the anticipation that we might see changes to the taxonomy as a consequence of investigating students' generalizing across a broader range of mathematical domains and levels. In the following sections, we outline how we have challenged, extended, and refined Ellis' (2007a) taxonomy, including what new forms and types of generalizing emerged and how particular constructs enabled us to characterize the mathematical activity that led to productive generalizing.

## Theoretical framework

We situate the RFE framework within Lobato's (2003) actor-oriented approach by focusing on the perspective of the learner (see also, Ellis, 2007a). Consistent with that approach, we sought evidence of generalizing by attempting to identify the similarities and extensions that our students perceived as general, whatever those may be, rather than relying on a predetermined set of mathematically correct general statements. Thus, we did not require a formal description of a correct rule as evidence of generalizing, although we did attend to what mathematical activities promoted productive generalizing. Instead, we probed students' sense-making of problem situations and investigated their conceptions of general processes and relationships, enabling a focus on what appeared to be salient features of tasks from the student's point of view. This does not mean that we do not value mathematical correctness, but rather that we value what students perceive as general and seek to understand the processes by which they develop generalizations.

### *Key cognitive constructs*

As noted, Ellis' (2007a) generalization taxonomy, which was also grounded in the actor-oriented perspective (Lobato, 2003), distinguishes between generalizing actions and reflection generalizations. We drew from both parts of the taxonomy, integrating them into a single framework in which our focus was the mental actions that support them. The key cognitive constructs we rely on come from this attention to students' mental actions and the need to describe, characterize, and differentiate them. By mental action we follow Piaget's (1970) notion of action as consisting of goal-directed mental operations that transform a problem situation into a result. We use "way of operating" to capture when, from our perspective, a student enacts a system of mental operations when engaged in problem solving activity. Such a system need not be generalizing, as the enacted way of operating could be the result of a previous generalization or merely an in-the-moment enactment that solves the problem at hand.

Piaget and others (e.g., Piaget & Inhelder, 1971; Steffe, 1991; von Glasersfeld, 1997) have been careful to distinguish the perspective of the subject from the perspective of the observer, noting that constructs must originate from the observer's perspective. For instance, von Glasersfeld (1997) remarked that:

> The constructivists who follow Piaget, attempt to think in a way that includes the observer. They are trying to devise a model that may show one way in which intelligent organisms, who start their thinking career in the middle of their own experience, could possibly come to have concepts of others, of themselves, and of an environment, and could ultimately arrive at a comprehensive non-contradictory complex of livable ideas (p. 305).

We see the attention to distinguishing the observer from the subject (or actor) as useful in acknowledging that our descriptions and constructs of students' activity can only ever be made by us as observers of that activity. We see this as compatible with the actor-oriented perspective in that as observers of student activity, we are trying to construct models of student thinking that reflect our hypotheses about students' perceptions and experiences of generality. These hypotheses are informed by our observations of student activity, by students' descriptions of their thinking, and by our interactions with students. If we can construct a viable hypothesis that a student perceives two situations as similar, for instance, then we consider this a case of generalizing, even if we do not perceive the situations as similar from our own mathematical perspectives. Thus, the constructs that we describe below are observers' second-order models (Steffe, von Glasersfeld, Richards, & Cobb, 1983) of students, but those models are ones that are attempts to depict students' perspectives—i.e., they are explanatory models of students' mathematics.

An important distinction addressing students' ways of operating is that of *figurative* and *operative* forms of thought (Piaget, 2001; Piaget & Inhelder, 1971; Steffe, 1991). Mental operations are always performed on either perceptually present or mentally generated (i.e., in visualized imagination) figurative material. For example, a student might envision a rectangle continuously growing along its length dimension; in this case, the figurative material is the rectangle, and the mental operations are those that enable the student to envision the growth of the rectangle and its properties. A student who bases judgments of growth on perceptual or sensorimotor characteristics or the results of actions is said to be foregrounding figurative aspects of thought (Thompson, 1985). A student who bases judgments of growth on the coordination of the mental operations that produce growth, including its measurable attributes, is foregrounding operative aspects of thought. In the context of a rectangle growing in length, an example of figurative thought would be a student concluding that a graph relating area and length should be "smooth" because the figure appears to be changing smoothly. An example of operative thought would be a student considering the length of the rectangle varying by particular amounts, making multiplicative comparisons with the amount the area varies, and then constructing a graph that representing the identified covariational relationship (e.g., a linear relationship). For the purposes of characterizing students' generalizing across domains, we distinguish between thought that foregrounds (a) perceptual or sensorimotor characteristics, (b) the coordination of mental operations, and (c) comparisons of mental activity with the terms figurative, operative, and activity-based, respectively. These are researchers' second-order models of students' mathematics, but they are attempts to characterize what students are attending to and experiencing as salient.

Another set of constructs characterizing students' activity is that of assimilation and accommodation (Piaget, 2001; Steffe, 1990; von Glasersfeld, 1995). Assimilation is the mental action by which the actor conceives a situation, and accommodation entails a change in their way of operating. Accommodation necessarily entails an act of assimilation, as it affords an act of assimilation that reconciles some experienced perturbation (e.g., a problem to be solved). To illustrate the relationship between assimilation and accommodation relative to generalizing actions, consider an individual identifying a commonality across situations. Again, our characterization of this action reflects the actor-oriented perspective in that we are concerned with the individual's perception of commonality, rather than our own. Initially, a person conceiving two situations may not perceive a commonality among them; they might not see them as cases of the same thing. In the context of particular goal-oriented activity, that person might experience a perturbation stemming from those differences and sensing or anticipating that there should be commonalities not yet conceived. In working to reconcile that perturbation, one might construct new ways of operating, enact extant ways of operating not yet used, or combine extant ways of operating in some novel way. Such a construction is an accommodation, and if that accommodation involves the person now assimilating two situations as entailing some commonality not previously conceived, then their accommodation has entailed a generalizing action.

We distinguish between *minor* and *major* accommodations when operating on a generalization to extend it. For the purposes of this paper, a minor accommodation does not result in a change to the ways of operating involved in the original generalizing action, whereas a major accommodation involves such a change. This can result in two distinct outcomes—the first is that some key aspects of the original generalization are maintained, or it can result in a student forming a new generalization. We provide an example of this distinction below in the Results section.

### *Ellis' generalization taxonomy and the RFE framework*

We built on Ellis' (2007a) taxonomy in two ways in creating the RFE framework. First, we viewed the forms of generalizing actions as a viable initial set of categories from which to develop the RFE framework. Second, we aimed to integrate reflection generalizations into the forms of generalizing, because we viewed general statements as based on mental operations, an orientation for reflection generalizations that was not explicit in the original taxonomy. With these two uses of the taxonomy as a starting place, our aim was to inspect whether a broader range of students, mathematical domains, and problem types would yield different forms and types of generalizing. In the Results section, we discuss the ways in which the RFE Framework re-conceived the major forms of generalizing identified in Ellis' (2007a) taxonomy, as well as the new generalization types that emerged as a consequence of our broader data set.

## Methods

This study is part of a larger project investigating students' generalizing in interview settings, teaching experiments, and design experiments. For the purposes of this paper, we draw on a series of individual semi-structured interviews (Bernard, 1988) with middle-school, high-school, and undergraduate students in the domains of algebra, advanced algebra, trigonometry/pre-calculus, and combinatorics in order to elicit and characterize students' generalizations in domains and tasks that go beyond the typical patterning activities often seen in the generalization literature. Below we discuss the participants, instruments, and analysis methods for the study.

### *Participants and data collection*

We conducted a total of 146 individual semi-structured interviews with 93 participants across 4 research sites (see Table 1). At sites 1 and 2, we recruited from local middle and high school classrooms by sending fliers home with students. At sites 3 and 4, we recruited undergraduate students in relevant courses through in-class and email announcements. At each site, we interviewed the first students who indicated that they wanted to participate (that is, we did not restrict eligibility for participation beyond enrollment in the class in which we recruited). Thus, we did not narrow the interview pool based on mathematical backgrounds or achievement levels, other than at Site 3, in which we sought out a pool that consisted both of students who had and had not taken discrete mathematics. This recruitment method aligned with our goal of intentionally interviewing a broad swath of students across and within grade levels. It was not our intent to

**Table 1.** Distribution of topics, interviews, and participants across the 4 research sites.

| Site | Topic | Interviews | Participants |
|---|---|---|---|
| 1 | Algebra | 1 individual 60–90 min interview | 10 middle-school students (5 M, 5 F) |
| | Algebra and Advanced Algebra | | 11 high-school students (7 M, 4 F) |
| 2 | Combinatorics (MS) | 2 individual 60-min interviews | 19 middle-school students (11 M, 8 F) |
| | Combinatorics (HS) | | 13 high-school students (9 M, 4 F) |
| 3 | Combinatorics (pre-discrete) | 1 individual 60–90 min interview | 21 pre-discrete undergraduate students (17 M, 4 F) |
| | Combinatorics (post-discrete) | | 6 post-discrete undergraduate students (6 M) |
| 4 | Trigonometry/Pre-calculus | 3 individual interviews, each 60–120 min | 10 undergraduate students (8 F, 2 M) |
| | Trigonometry/Pre-calculus | 2 individual interviews, each 60–120 min | 1 undergraduate student (1 F) |
| | Trigonometry/Pre-calculus | 1 individual interview, 60–120 min | 2 undergraduate students (1 F, 1 M) |

make associations based on students' background knowledge. All interviews were video recorded and transcribed, and we used gender-preserving pseudonyms for all participants.

### Instruments

Given our definition of generalizing as identifying commonality, deriving broader results from particular cases to form general relationships, rules, concepts, or connections, or extending one's reasoning beyond the range in which it originated, we chose task-based, semi-structured interviews in order to engineer opportunities to observe students engaging in these activities. During the interviews, we presented the participants with tasks to elicit generalizations, and we asked them to identify patterns and themes, discuss any elements of similarity they noticed, and explain and discuss the generalizations they formed. Table 2 presents a representative sample of the interview tasks across the mathematical domains. We include these specific tasks in the table because we refer to them in the Results section to discuss specific data episodes.

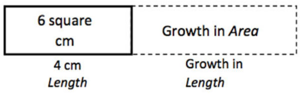
Consistent with the actor-oriented approach, our aim was to probe the students' identified similarities and relationships, regardless of their mathematical correctness from our perspective. The algebra, advanced-algebra, and trigonometry/pre-calculus topics included linear, quadratic, higher-order polynomial, and trigonometric functions. The combinatorics topics included counting tasks involving quadratic and cubic relationships and questions about the binomial theorem. We chose tasks to reflect a variety of topics as well as different types of generalizing based on Ellis' (2007a) taxonomy. A key feature of the interview tasks is that they were not restricted to the typical patterning activities commonly reported in the generalization literature, in which students are presented with a series of mathematical terms or figures and are then asked to develop expressions for the nth term. We see those tasks as valuable, but we were also interested in exploring students' generalizing in other types of tasks as well. The interviews included some tasks of this nature—for instance, see the advanced algebra and combinatorics examples in Table 2—but they also included open-ended tasks that challenged students' mathematical thinking. Examples of those can be seen in the middle-school algebra and precalculus tasks in Table 2.

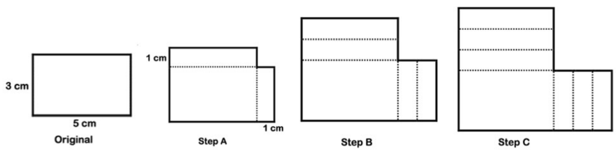
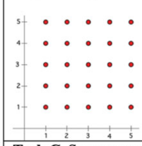
### Analysis

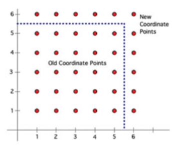
We relied on the constant comparative method (Strauss & Corbin, 1990) to analyze the interview data in order to identify forms and types of generalization. For the first round of analysis, we drew on Ellis' (2007a) taxonomy to infer categories of generalizing based on students' talk, gestures, diagrams and task responses. Three questions guided our initial analysis. The first was, "What is the content of the generalization?" For instance, an example of a generalization we could describe is, "Each time the rectangle's side length grows by 1 in, the area grows by 3 $in^2$." In operationalizing the actor-oriented perspective, we identified the content of the students' generalizations regardless of their mathematical correctness. Our aim was to find all of the similarities, connections, relationships, rules, etc. that the students identified, even if they may not have corresponded to our predetermined expectations of the particular generalizations we anticipated based on the task structure. Many of the participants' generalizations were not ones we anticipated based on the task structure.

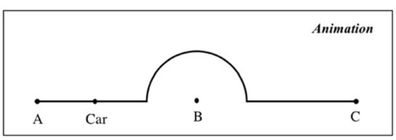
The second question was, "What is the generalization based on, and can we articulate the mental operations that formed it?" For instance, is the student generalizing based on reflecting on their mathematical activity, based on noticing a pattern in the numbers or figures, based on identifying a relationship between quantities, or based on some other activity? Consistent with the actor-oriented perspective, we sought to identify these operations based on the students' observed activity, language, gestures, etc., rather than identifying operations based on an a priori determination of the type of mental operation that should be leveraged based on the task. As an example,

**Table 2.** Concise versions of sample interview tasks.

<table>
<tr><th>Interview Task</th><th>Domain and grade level</th></tr>
<tr><td>The rectangle below grows along the dotted path as shown:


Complete the following statement: When the length of the rectangle's side grows by ______, the area grows by ______.</td><td>Algebra, middle school</td></tr>
<tr><td>Consider the following sequence of figures. How much does the area grow from one step to the next, and what is the new total area each time? How big would the total area be for (a) Step 10, (b) Step 100, (c) Step $x$?

</td><td>Advanced Algebra, high school</td></tr>
<tr><td>**Task A.** You have a deck of number cards numbered 1-5. You create a coordinate point by drawing a card from the deck, replacing it, and drawing a second card. Create a list and then an array to show how many possible coordinate points you could make. (Array was not provided, it is what students should produce.)


**Task B.** Suppose you added one card, the card numbered 6, to your deck. List the new coordinate points you could make, and add them to your array. How many old coordinate points would there be? How many total coordinate points are there?


**Task C.** Suppose again you added one card, the card numbered 7, to your deck. List the new coordinate points you could make, and add them to your array. How many old coordinate points would there be? How many total coordinate points are there?
**Task D.** Suppose you added one card, the card numbered 8, to your deck. Can you predict the number of new coordinate points you could make without listing them?</td><td>Combinatorics, middle school</td></tr>
<tr><td>I have a password consisting of the letters A and B. I want to make all possible passwords. How many such passwords are there of length 3? Length 4? Length $n$?<br>I have a password consisting of the characters A, B, and 1. I want to make all possible passwords. How many such passwords are there of length 3, 4, and $n$?</td><td>Combinatorics, undergraduate</td></tr>
<tr><td>

You've decided to road trip to C for spring break. Of course, this means traveling around B on your way down and back, because who would want to go through B? The animation represents a simplification of your trip there and back. (a) Create a graph relating your total distance travelled ($x$-axis) and your distance from B ($y$-axis) during your trip. (b) Create a graph relating your distance from B ($x$-axis) and your distance from A ($y$-axis) during your trip.</td><td>Precalculus, Undergraduate</td></tr>
</table>

for the generalization that the area of a rectangle grows by 3 $in^2$ for each 1-in increase in side length, the student may have generalized based on noticing a pattern in the associated side length and area pairs, after developing multiple examples.

The final question was, "What is the form and type of generalizing?" This question was related to discerning forms of generalization as identified in Ellis' (2007a) taxonomy, such as relating, searching, and extending, as well as identifying new potential forms and types based on our data. For the above example, the student may have developed a generalization based on searching for the same relationship between the side length and the area across multiple cases. As with the other two questions, our analysis of this question had to be emergent in order to honor the actor-oriented perspective. We sought to abandon preconceived assumptions about the forms and types of generalizations a task should elicit in favor of simply exploring the forms and types of generalizations each task did elicit.

Beginning with 1–3 sample interviews from each of the four sites, the project team met and analyzed each of the interviews together as a group for an initial round of coding. This first round, based on the above three guiding questions, led to an initial set of emergent forms and types. Then, for the mental operation constituting the basis of the generalization, we developed emergent categories such as operating on the structure of a prior generalization, on an extant scheme, on reflecting on activity (such as counting, forming relationship between quantities, or other activities), or on operative or figurative material, among others. The results of this round of analysis are presented in the second part of the Results section, in which we discuss the utility of the framework. In particular, by identifying the mental operations constituting the basis of each generalization, we were then able to examine which operations undergirded those generalizations that we deemed generative.

The project team met weekly to refine and adjust the codes in relation to one another. This iterative process continued until no new codes emerged, and none of the existing codes required refinement. At each of the four sites, the local project team collaboratively coded all of the site interviews via the qualitative software program MaxQDA, meeting weekly to discuss and resolve any discrepancies in the coding. At the completion of this process, the entire project team across all four sites then coded two interviews from each of the sites, meeting together in order to discern coding agreement across the entire project team as well as collaboratively discuss and resolve any remaining discrepancies. Our process of collaboratively resolving discrepancies and establishing common understandings of the coding categories followed Syed and Nelson's (2015) recommendations for establishing reliability in qualitative research. Namely, they argue for adherence to rigor rather than reliability, in which research teams develop and implement a coding system in a manner that emphasizes multiple readings of the data and relies on group consensus.

## Results

Our analysis of 146 interviews with 93 participants yielded two major results. The first is an empirically-grounded updated framework that captures the broad range of generalizing activity for multiple content domains and levels. We introduce the RFE Framework in Table 3 and then discuss in detail the elements that differ from Ellis' (2007a) generalization taxonomy. Our second result identifies two key elements of what we call generative generalizing activity, namely, (a) operative activity, and (b) building and refining activity. We elaborate on each of these results below.

### *The relating-forming-extending (RFE) framework*

The RFE framework identifies three major categories of generalizing: *relating*, *forming*, and *extending*. We call these major categories forms of generalizing, and then each of the

Table 3. The relating-forming-extending framework.

| | | |
|---|---|---|
| RELATING | ***Relating Situations***: Forming a relation of similarity across contexts, problems, or situations. | *Connecting back*: Forming a connection between a current and previous problem or situation.<br>*Analogy invention*: Creating a new situation or problem to be similar to the current one.<br>*Recursive embedding:* Embedding a previous situation into a new one as a key component of the new task. |
| | ***Relating Ideas or Strategies (Transfer)***: Influence of a prior context or task is evident in a student's current operating. | |
| FORMING | ***Associating Objects***: Forming a relation of similarity between two or more present mathematical objects. | *Operative*: Associating objects by isolating a similar property or structure.<br>*Figurative*: Associating objects by isolating similarity in form.<br>*Activity-based:* Relating objects or ideas based on identifying the product of one's activity as similar. |
| | ***Searching for Similarity or Regularity***: Searching to find a stable pattern, regularity, or element of similarity across cases, numbers, or figures. | |
| | ***Isolating Constancy***: Focusing on and isolating a constant feature across varying features without reaching the final stage of fully identifying a regularity, pattern, or relationship across those cases. | |
| | ***Establishing a Way of Operating***: Establishing a new way of operating that has the potential to be repeated. | |
| | ***Identify a Regularity***: Identification of a regularity or pattern across cases, numbers, or figures. | *Extracted*: Extracting regularity across multiple cases.<br>*Anticipated*: Describing a predicted stable feature that the student anticipates will hold for future cases. |
| EXTENDING | ***Continuing***: Continuing an existing pattern or regularity to a new case, instance, situation, or scenario beyond the one in which the generalization was developed. | |
| | ***Operating***: Operating on an identified pattern, regularity, or relationship in order to extend it to a new case, instance, situation, or scenario beyond the one in which the generalization was developed. | *Minor accommodation*: Making a minor change to a regularity in order to extend it to a new case.<br>*Major accommodation*: Making a significant change to the structure of a regularity in order to project it to a far case or to make sense of a new relationship. |
| | ***Transforming***: Extending a generalization by changing the generalization to be extended; in contrast to operating, the generalization itself changes in the act of transforming. | |
| | ***Removing Particulars***: Extending a specific relationship, pattern, or regularity by removing particular details to express the relationship more generally. | |

sub-categories within them types. This is to distinguish between the broader structure of the generalizing actions (the form), and the traits of each of the generalizing actions (the type) grouped together within one form. *Relating* is the identification of similarity across situations, problems, or strategies that the learner perceives as distinct contexts. *Forming* is the development of an initial, sometimes tentative generalization, and *extending* is the use of that generalization, sometimes to a broader domain. These three forms are not necessarily hierarchical in sophistication or in temporal occurrence; relating does not necessarily have to occur before forming and extending. Forming and extending actions, however, do typically occur as part of a two-part process of generalizing. In the first part, one might develop an initial generalization, which we characterize as forming, and then when extending, students may apply the generalization they formed. Thus, extending does presuppose an existing generalization that has been formed, but forming does not necessarily always lead to extending. It is often through the extending process that a generalization becomes solidified. This does not mean, however, that extending is necessarily more complex than the action of relating or of forming. In the *Utility of the Framework* section below, for instance, we detail one extended example of a student, Willow, who seamlessly shifts back and forth between forming and extending, while also shifting between types within each of those forms.

Table 3 provides an overview of the generalizing forms and types along with their definitions. The shaded cells are the types that are novel, in that they capture generalizing that was not present in Ellis' (2007a) generalization taxonomy. Below we explore those novel types in more detail for each of the three forms. We also address, within each form, the ways in which the types of generalizing may or may not be hierarchically related.

### Relating

One way in which we refined the relating category as part of developing the RFE Framework is that we use relating only to identify generalizing actions that occur between what the student perceives as two distinct tasks or contexts. In that way, relating is intended to capture a larger grain size of cognitive acts that occur over a sequence of tasks and period of time. There are two major types of relating, *relating situations* and *relating ideas or strategies*. When relating situations, students form a connection between what they perceive as the current situation and a different situation. The different situation may be one they have previously encountered (i.e., *connecting back*), or they may generate a new situation that entails similar features to the current situation (i.e., *analogy invention*). This connection is explicit, and we coded activity as relating situations when students could verbally describe a connection. In contrast to relating situations, *relating ideas or strategies* may not be explicit or verbalized, but occurred when we found evidence that a student's prior ways of operating influenced their current ways of operating with the task at hand. This influence is typically described as transfer, and Ellis (2007a) characterized it as such in the reflection generalizations part of the generalization taxonomy. As an example, an advanced-algebra high-school student considered the L-shaped rectangle task from Table 2. This was a pattern task in which the students were asked to determine the nature of growth from one figure to the next. The student distinguished the area of the rectangle in the original step as "P" from the combined area of the two added pieces in Step A as "V": "Let's do V for valence because that's one word I know for outer ring." When the interviewer asked the student if that term was from his chemistry class, the student responded, "Yep. Like the valence electrons … how much that equals plus the previous one, would equal your new answer." The student understood valence as a recursive relationship and interpreted the new task as a relationship with the same recursive structure.

We have also identified a new type of relating situations that was not present in the generalization taxonomy or in other generalization constructs, called *recursive embedding* (also described in Lockwood & Reed, 2016). When recursively embedding, a student reflects on prior activity (often as a consequence of *connecting back*) and inserts that activity into one part of a new situation. Because recursive embedding is a novel type of generalizing, we present an example with an undergraduate vector calculus student, Tyler, who worked on the Passwords Task (see Table 2).

In the Passwords Task, students had to count 3, 4, and 5-character passwords consisting of the letters A and/or B, and then create tables that organized the passwords according to the number of As in them. Then, students would count 3, 4, and 5-character passwords consisting of the characters A, B, and/or 1, again creating tables that organized passwords but according to the number of 1 s in the passwords. (Note that the presentation of this task was accompanied by prompts for students to engage in listing, and thus the students did not interpret them as textbook tasks that could be answered with the use of a formula.) We highlight this example in order to characterize the nature of recursive embedding as a student works with a complex, hierarchical series of tasks.

In order to determine the total number of 3-character AB passwords, Tyler created a systematic, organized list of outcomes (Figure 2). Tyler used his list to correctly create a table of the number of passwords that contained a certain number of As in a 3-character password, and then completed similar tasks for 4- and 5-character passwords.

In the episode relevant to recursive embedding, Tyler was working on a problem that asked for the number of 4-character passwords using As, Bs, and 1s. As part of this, he created a table according to the number of passwords that contained a certain number of 1s in it, and established that there was one password with four 1s and 16 passwords with no 1s. In working on the row for one 1, Tyler introduced a way of describing a general outcome involving 1s and *x*s. Specifically, he wrote four general outcomes 1*xxx*, *x*1*xx*, *xx*1*x*, and *xxx*1 (Figure 3(a)), and he used those generalized outcomes to fill out his table. Tyler discussed his reasoning while referring back to his three-character AB table (Figure 3(b)), explaining that "There is only a certain amount of spots for it. Like it has to be … like, I'm just going to use *x* because, has to be in one of these spots":

Tyler: So, there's – now there's just three *x*s, and I know that for … 3 spots with 2 different letters there's going to be eight different ways to do it [points back to the previous 3-character AB table, see Figure 2]. Um, so I guess eight, there's eight different of each of those, just using this same table [points to the 3-character AB table]. Um, there's just 32. So, I want to say there's gonna be, um, 32 for just the one.

Int.: Okay. And you got – you're thinking of that as kind of the 4 times 8?

Tyler: Yeah. I just – adding them all up [referring to adding the four groups of 8].

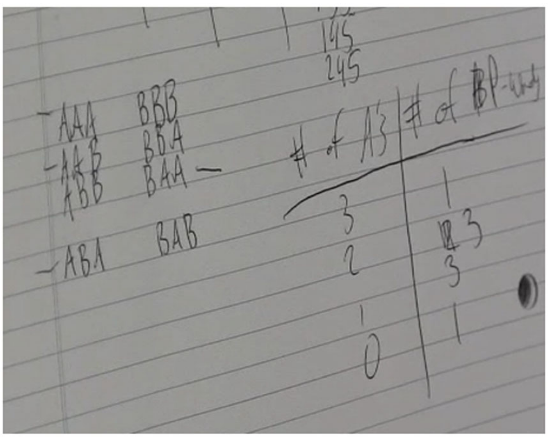



**Figure 2.** Tyler's List of the Eight 3-Character AB Passwords and the 3-Character AB Table.

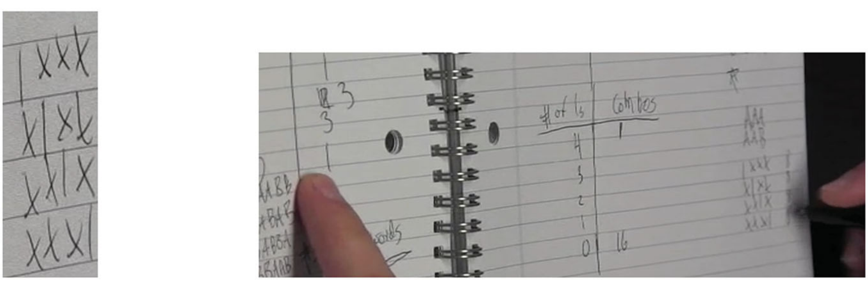



**Figure 3.** Tyler's List of One 1 and Three xs (a), and Tyler Explicitly Refers Back to the 3-Character AB Table (b).

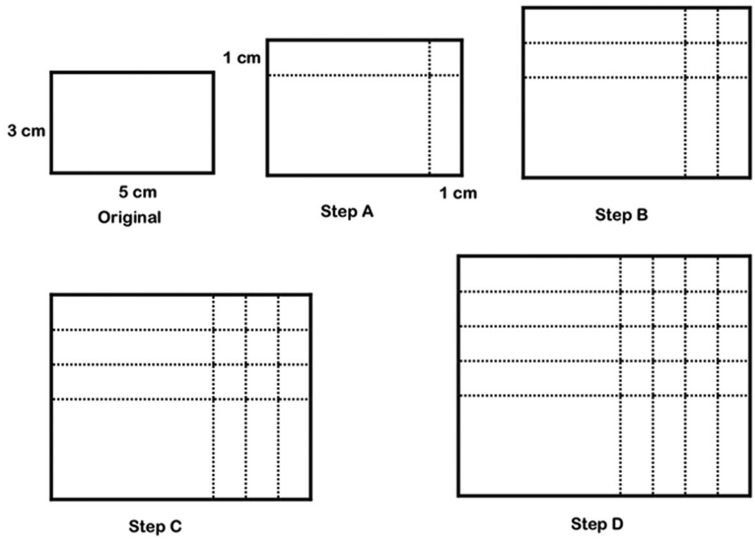



**Figure 4.** Quadratic Growing-Rectangle Task.

Tyler recognized the *x*s in the 1*xxx* structure in the AB1 situation as representing placeholders into which he could embed activity from a previous situation (listing AB passwords). Tyler perceived the AB and AB1 passwords contexts as distinct but related via this embedding activity. He was also explicit in his use of his prior work in making sense of a new, more complex task. In doing so, Tyler was able to create a more general representation relying on 1s and *x*s.

Recursive embedding initially emerged from analysis of the undergraduate and high-school students' work on combinatorics tasks. We posit that recursive embedding emerged in this context because many of those tasks were hierarchical in nature, with repeated elements across many subtasks as in the Passwords task. Recursive embedding is not unique to the domain of combinatorics, however, and once this category was identified, we then found instances of it in other domains, such as advanced algebra. These instances often occurred when students worked with tasks with repeated elements, such as in the L-shaped rectangle task and the quadratic growing rectangle task (Figure 4). In that case, we saw instances of students embedding the growth pattern from the prior L-shaped task into the new task and then adjusting the pattern to relate it to the corner squares.

### *Forming*

As we described above, if the participants indicated that they saw their actions as spanning two or more different situations, we coded their generalizing as relating, even if, from our perspective, those situations were the same. If the participants identified relations of similarity but they perceived those relations to be occurring within one situation, we coded their generalizing as forming. Consequently, the forming activity of *associating objects*, which is the formation of a relation of similarity between two or more mathematical tasks or other representations, is similar to the category "relating objects" from Ellis (2007a) generalization taxonomy, but in that taxonomy, it was situated within relating. There are three types of associating objects: *operative*, *figurative*, and *activity-based*. These three types are based on the locus of mental activity that forms the association: Is the student attending to similarity in structure or function through the coordination and transformation of mental operations (operative), similarity in perceptual or sensorimotor characteristics (figurative), or similarity in the products of their own mental operations (activity-based)? The operative and figurative types are similar to constructs that appeared in the generalization taxonomy, but the activity-based type is new.

As a brief example of these three types of associating objects, a student could associate the sine curve with circular motion through conceiving each as representing an invariant relationship of co-varying quantities (operative), through conceiving the sine curve as smooth because the motion is perceived as continuous (figurative), or through conceiving two objects being composed of points with the same numerical pairs (activity-based). We discuss the activity-based type in more detail below in the *Utility of the Framework* section.

Associating objects is not a searching action, but there are other searching actions that do occur within forming, similar to those depicted in the generalization taxonomy. Drawing on the constructs of assimilation, accommodation, and perturbation, we understand searching to occur when a student has experienced a perturbation that leads to a search for an element of similarity. Thus, *searching for similarity or regularity* involves the attempt to find a stable pattern or regularity across cases. This search can sometimes result in the isolation of a property that remains unchanged across variation of other elements; we term this *isolating constancy*. Isolating constancy can also occur when students simply notice a feature or characteristic that remains unchanged across multiple iterations of the same situation. This does not necessarily have to be the result of a searching action, but it can instead reflect the activity of attending to a constant feature across variation. This was a typical occurrence, for instance, in students' exploration of the L-shaped rectangle task (Table 2), in which they isolated the area of the base rectangle in each of the steps to be $15\,cm^2$.

Isolating constancy often occurs as a precursor to *identifying a regularity*. This type of generalizing was present in the original generalization taxonomy, but as a reflection generalization. In the RFE Framework, we identified two new subtypes of identifying a regularity: *extracted*, in which one explicitly identifies or describes a common feature across multiple observed cases, and *anticipated*, in which one predicts that a feature will continue to future cases, even if this has not been observed in multiple cases. Whereas an extracted generalization can occur as a result of searching, an anticipated generalization stems from an already known feature. As an example of this distinction, consider the work of Joyce and Cordelia. Joyce was a 9th grade student in Honors Algebra, and Cordelia was a 10th grade student in Geometry. When working with the L-shaped rectangle task, Joyce and Cordelia both identified a regularity, but we interpret that Joyce's regularity was extracted and Cordelia's was anticipated.

Joyce calculated the area of the original figure to be $15\,cm^2$, calculated the area of the Step A figure to be $23\,cm^2$, and then calculated the area of the Step B figure to be $31\,cm^2$. She noticed that each new area value was $8\,cm^2$ greater than the previous, which she used when the interviewer asked her to determine the growth in area from Step B to Step C. Joyce then *identified a regularity* when she explained how she noticed the growth of $8\,cm^2$: "Because it's a pattern. You're adding the same amount each time, so it's growing by the same amount each time, which is 8." She extracted the pattern of adding 8 from the original figure to Step A, and from Step A to Step B, and then was able to use this pattern to determine the area for other figures.

Cordelia, in contrast, did not determine that the area grew by $8\,cm^2$ from one figure to the next through calculation. In fact, she did not calculate any area values before anticipating that the growth would always be $8\,cm^2$:

| | |
|---|---|
| Int: | How much does the area grow by from the original to Step A? |
| Cordelia: | 8 centimeters. I think. Yes. That is my final answer. |
| Int: | Final answer, and will you explain how you thought about that one? |
| Cordelia: | Well, it says that each of these are 1 centimeter (points to the added rectangular sections in Step A), and so if it grows by 1 centimeter like that, it's still multiplied by the bottom, like, base. And so, 1 by 5 is 5 and then the same thing with the other side. So, 1 by 3 is 3 and then you just add 8 onto the original square. |

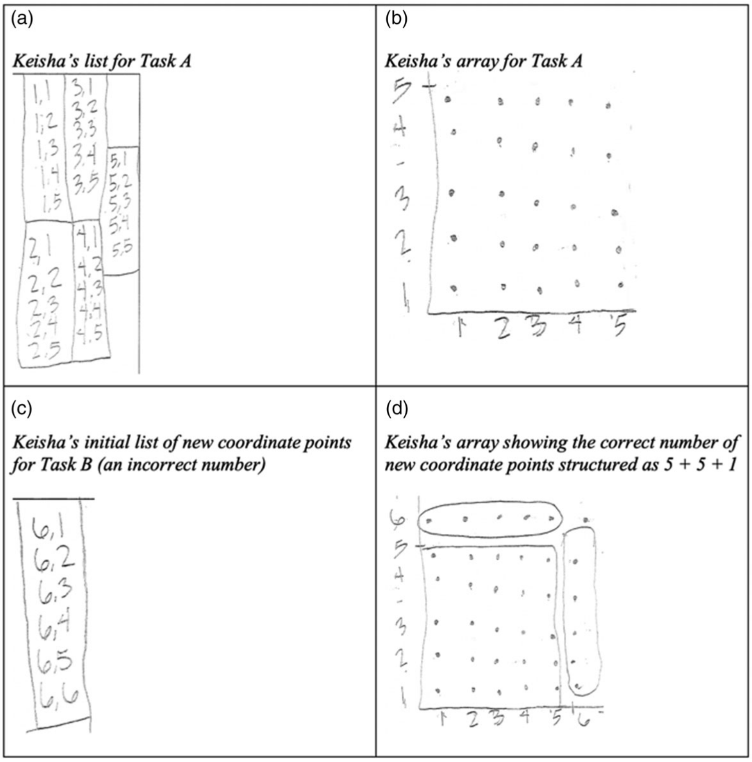



**Figure 5.** Keisha's Lists and Arrays for the Number Cards Task.

When next asked to determine the growth from Step A to Step B, Cordelia did not calculate, but she anticipated that the growth would be 8 $cm^2$ again: "I don't have measurements but if it's the same then yeah, 8 centimeters." Cordelia then began operating with the $+8$ pattern as a regularity, determining that she could multiply 8 by the number of steps to find the amount of added area from any one given step to a later step. When asked to find the amount of area for the figure in the 10th step, Cordelia jumped straight to writing the rule $15 + 8x$. She did not extract a generalization that the area would always grow by 8 from multiple instances of calculation across the figures, but instead she *anticipated* the regularity across all future cases, based on her understanding of the nature of each figure's growth. Note that this type of generalizing occurred in other domains as well, particularly when students could reflect upon multiple events or outcomes (whether those were figural patterns, numerical relationships, or another kind of mathematical object).

The final generalizing type within forming is *establishing a way of operating*, which is also a novel construct that was not present in Ellis' (2007a) taxonomy. When establishing a way of operating, students form novel mental operations that become the foundation for generalizing that can be applied to a broader domain. As an example, we present Keisha's work on the number cards task, the middle school combinatorics Task A and B from Table 2. Keisha was taking a 7th grade mathematics class that prepared her to take pre-algebra in 8th grade.

To solve Task A, Keisha first created six coordinate points by physically drawing cards from a deck of numbered cards. The interviewer then asked her to create a list of all of the possible coordinate points. Initially she listed pairs of coordinate points (e.g., (1, 2) and (2, 1), then (3, 5) and (5, 3), etc.) until she realized that she was unable to tell whether she had created all of the possible coordinate pairs. She abandoned this method and introduced a systematic method for listing (Figure 5(a)), eventually making an array (Figure 5(b)).

Keisha was then asked to respond to Task B, specifically how many new coordinate points she could make if she were given one additional card, numbered 6. She began by listing the coordinate points shown in Figure 5(c), indicating that those were all of the coordinate points she could make with six. The interviewer asked her to check by making further coordinate points with the actual cards. Keisha drew one from the deck, and then said, "Oh you could do one, six." She continued, "I was just thinking of like you can add one, six [pointing in Figure 5(a) to where it would be added to the column of coordinate points that have one in the first position, then continuing to point to the appropriate location in Figure 5(a), while saying]; two, six; three, six; four, six; and five, six." Keisha then noted that she had added one new coordinate point to "each of five boxes" in Figure 5(a) and one "new box of six" coordinate points (Figure 5(c)). She then identified the coordinate point (6, 6) as special because "it had six in the first and second spot." Keisha illustrated this observation on her array (Figure 5(d)), and wrote that the total number of coordinate points could be expressed as "$36 = 25 + 5 + 5 + 1$."

We coded Keisha's work as establishing a way of operating. We used two criteria for this code: (a) when we could infer that a student was engaged in a significant coordination of novel mental operations; and (b) when we could infer that it was a baseline way of operating that supported students to engage in subsequent generalizing actions. In Keisha's case, she initially considered an increase in the coordinate points along only a single dimension of the array, but subsequently considered an increase in the coordinate points along both dimensions, structuring this increase as $5 + 5 + 1$, according to whether the new digit was in the first position, the second position, or both positions. Further, once she had established this way of operating, it was the basis for a number of additional opportunities to generalize. For instance, it enabled her to identify a regularity (extracted) when she worked on Tasks C and D, during which she stated that she produced new coordinate points "each time by adding one to each of my original boxes, adding a whole new box" that had the same number of coordinate points as the number she had added when adding one to each box "plus one special one (the coordinate point that had the same digit in the first and second position)."

Establishing a way of operating can be a challenging type of generalizing to identify, because finding evidence of it requires observation of a student's extended engagement in a task or a series of tasks over time. Many of the initially-determined cases of this category occurred at the undergraduate level, in advanced algebra and combinatorics, perhaps due to the students' increased capacity to explain their thinking compared to the high-school and middle-school students. Once we established this type of generalizing, we then revisited the remainder of the data corpus and found additional instances of establishing a way of operating across domains and at all levels.

### *Extending*

Once formed, a generalization can be extended to a new case. One of the central differences between the extending category in Ellis' original taxonomy and extending in the RFE Framework is that in the RFE Framework we have accounted for when a generalizing action includes learning (accommodation), as well as the nature of the learning involved in the extending action. By doing so, all extending actions are differentiated based on specific qualities of learning that occur.

We found four major types of extending: *continuing*, *operating*, *transforming*, and *removing particulars*. Continuing, which existed in Ellis (2007a) taxonomy, is the most straightforward type of extending: when continuing, a student does not alter the generalization, but simply applies an already-formed regularity to a new case. When operating, a student will alter the expression of the generalization in order to extend it to a new case. Operating was present in the original taxonomy, but we identified a novel distinction between two subtypes of operating, specifically *minor accommodation* from *major accommodation*. Recall that the difference between a minor and major accommodation depends on a researcher's model of the student; taking the actor-

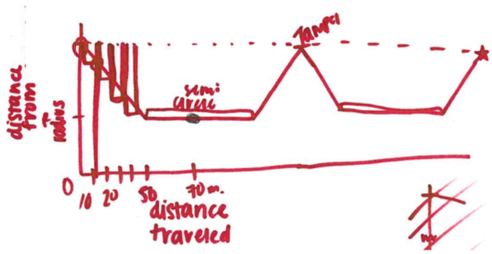



**Figure 6.** Patty's Graph for Part (a) (*y*-axis is "Distance From B," *x*-axis is Total Distance Traveled).

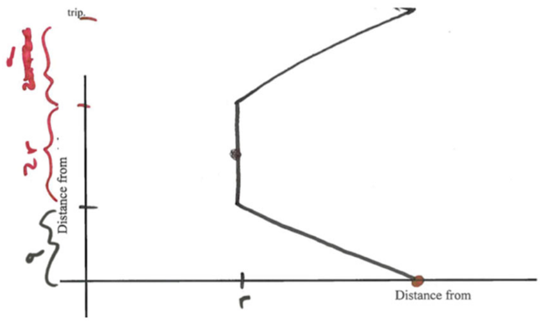



**Figure 7.** A Sample Graph of a Normative Solution for Part (b) (Distance from B is on the *x*-axis and Distance from A is on the *y*-axis).

oriented perspective, we judge an accommodation relative to the significance it has for the student's understanding of a task and the extent to which it requires a structural change to their current ways of operating. To illustrate this difference, we draw on the work of Patty, an undergraduate student in a trigonometry/pre-calculus course, in the "Going Around Gainesville" task (Table 2) (Moore 2019). Although Patty only experienced a temporary accommodation, her activity illustrates the potential of a major accommodation. Against that backdrop we identify examples of minor accommodations.

In Part (a), Patty drew a graph (Figure 6) indicating that, for the first portion of the trip, the distance from B was decreasing at a constant rate as total distance increased from 0. Patty also provided evidence that she was able to reason about the relationships between these two distances as co-varying quantities.

Patty experienced difficulty in solving Part (b), and we inferred she had generalized that graphing a relationship such that a quantity decreases at a constant rate as another quantity increases from 0 entails "starting" on the *y*-axis and then progressing down and to the right. To graph such a relationship for Part (b), one must begin graphing on the *x*-axis as seen in another student's work (Figure 7). Patty struggled to extend her generalization to this new case, and she spent a sustained period of time attempting to "start" her graph on the *y*-axis in order to *extend by continuing* her generalization to Part (b). Patty eventually drew a graph with the axes' referents switched, which allowed her to use her previous generalization. Suggesting a potential accommodation, Patty then operated on that generalization to consider beginning her graph on the *x*-axis and drawing it right to left. However, she questioned the viability of her solution, saying, "It's backwards so I don't like it," "My graph is from right-to-left, which is probably not right," and "I think I'm missing something." She then abandoned a graph that starts on the *x*-axis and is drawn upward from right to left.

Patty's goal and understanding of drawing a graph as a representation of two co-varying quantities did not change from Part (a) to Part (b), nor did she have difficulty sustaining an image of the relationship she intended to graph. Rather, her difficulty was in attempting to operate on her extant generalization for how to draw a graph in order to extend it to an alternate orientation of axes. Suggesting a potential major accommodation, Patty's operating on her generalization was effortful and indicates she was in the process of re-conceiving graphing a relationship as a process not constrained to a specific axis orientation. As further evidence of Patty being on the verge of a major accommodation, she questioned the viability of her temporary accommodation, which highlights that such an accommodation carried with it a significant change to her extant generalization regarding graphing.

Although Patty's accommodation was only temporary, her actions illustrate that a major accommodation results in a structural change to a generalization. In her case, and had her accommodation been permanent, that structural change would have involved a shift from foregrounding the direction a graph is drawn to a generalization that privileges the covariational relationship being represented. As stated above, we judge minor versus major accommodations based on the relative importance that it has for a particular student. As an example of a minor accommodation, a different student might solve Part (a) of the task, making a similar generalization as Patty. In solving Part (b) of the task, this student might treat the $x$-axis like the $y$-axis through a mental rotation of the axes, and then draw the graph shown in Figure 7 with relative ease. In this case, the student would have operated on their initial generalization by combining it with envisioning a rotation of the axes. Such an accommodation would be minor because the structure of the regularity would remain essentially equivalent except for conceiving it in combination with mentally rotating axes.

The above example highlights that content involving covarying relationships and graphing conventions could be particularly well-suited to elicit minor and major accommodations, but, again, we saw instances of this phenomenon among other domains. The specific features of an accommodation depend on the particular mathematical concepts at play, but the occurrence of an accommodation can occur in any content area.

The remaining two types of extending are *removing particulars* and *transforming*. As in Ellis' (2007a) generalization taxonomy, removing particulars involves expressing a regularity more generally, which could occur as a verbal or an algebraic description. Transforming involves changing the generalization itself. This is a novel type of activity that has some similarities to Ellis' "extending by operating," in which students would operate on a generalization in order to generate new cases, but it also has some important differences. In order to clarify the distinction, we ask the reader to recall the quadratic growing rectangle task from Figure 4, in which students are presented with a sequence of rectangular figures and must determine the total area for any given step. One participant, Xander, wrote "Area $= 5x + 3x + 15 + x^2$" to express the area of a figure at Step $x$. If Xander were to extend by operating, he could consolidate his actions of separately adding 5 square units times the step number and then adding 3 square units times the step number into adding 8 square units times the step number, which could be expressed as $8x + 15 + x^2$. The consolidation of $5x + 3x$ into $8x$ is a minor accommodation to his existing generalization, and the generalization itself did not change. In other cases, however, we found instances in which our participants actually changed the content of the generalization itself.

For instance, when Caleb worked on the linearly-growing rectangle task shown in Table 2 (second row), he wrote "A $= (5 + n)(3 + n) - n^2$" to express the area of a Step $n$ figure by imagining the entire rectangle and then subtracting the missing portion. When Caleb then shifted to the quadratic growing rectangle task, he *transformed* his prior expression, "A $= (5 + n)(3 + n) - n^2$", to "A $= (5 + n)(3 + n)$", reasoning that he no longer had to subtract the missing corner. In this transformation act, the content of the generalization itself changed.

As we described when introducing the RFE Framework, extending a generalization means that there had to have been, prior to the extending activity, the action of forming a generalization. Within extending, the different generalizing types can at times follow a loose progression, but this is not always the case. Operating via a major accommodation, for instance, may be a more conceptually difficult act than transforming, particularly if the transformation is not a challenging one. Students may continue without ever operating or transforming. Removing particulars is an extending action that can follow multiple instances of continuing, operating, or transforming, but this did not always occur in our data set. We found multiple instances of students forming a generalization and then shifting to removing particulars in order to express a formal algebraic relationship, all before ever engaging in any of the other extending actions.

### *Utility of the framework: mental activity undergirding generative generalizations*

In order to further examine the framework's utility, we now shift to a consideration of our third research question: What aspects of mathematical activity foster productive generalizing? Across all domains, we observed episodes of generalizing that we perceived to be generative in nature. The generative learning model was first introduced by Osborne and Wittrock (1983), in which they characterized generative learning as combining previous knowledge with new knowledge; new ideas are integrated with a student's existing ways of thinking. For our purposes, we characterize *generative generalizations* as having two main attributes. They facilitate extending, in that they allow for the generalization to be expanded to accommodate new cases, and they can be effectively justified, which we consider an indication that students have a conceptual understanding of what is being generalized. Note that a characterization of which generalizations are generative are ones that we established as observers. Nevertheless, we still attempted to adopt the actor's perspective when considering how students viewed their generalizations as amenable to extending, as well as what students viewed to be convincing justifications. Thus, a given generalization may be more or less generative depending on how a student makes sense of it: what might appear to be the same act of generalizing to the observer may be more meaningful for a student who can justify that generalization.

The above attributes are how we characterized the generalizations that the students produced as generative. Once these generalizations were identified, we then turned to investigating students' mathematical activity when producing them. By mathematical activity, we mean an individual's ways of operating when engaging in mathematical thinking. Our aim was to determine whether there were any common aspects of mental activity across the instances of generative generalizations that were not domain dependent. Specifically, were there aspects of students' mental activity that were typically present in the production of generative generalizations, but not typically present in the production of the non-generative generalizations? We found two aspects: (1) operative activity, and (2) building and refining activity. We discuss each aspect in turn with exemplifying data episodes drawn from different domains.

#### *Operative activity*

A key distinguishing factor in instances of generative generalizations was whether students' initial activity was *figurative* or *operative*. Recall that operative associations involve the coordination and transformation of mental operations (i.e., function or structure), while figurative associations involve perceiving similarity in perceptual or sensorimotor characteristics, often in ways tied to the results of activity. We offer an example from two undergraduate combinatorics students, Jonas and James, to clarify this distinction as it relates to the generativity of students' generalizations.

Jonas, a vector calculus student, and James, a student who had completed discrete mathematics, both worked on the Passwords Task. Both students *associated objects*, but there were

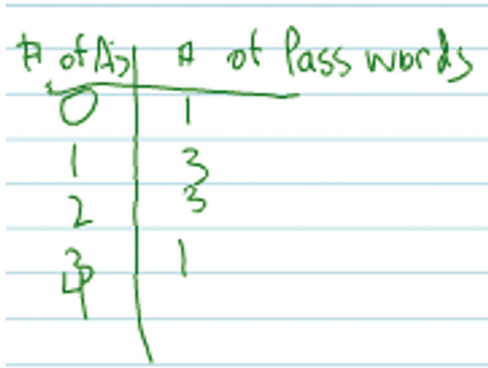



**Figure 8.** Jonas' 3-Character AB Table.

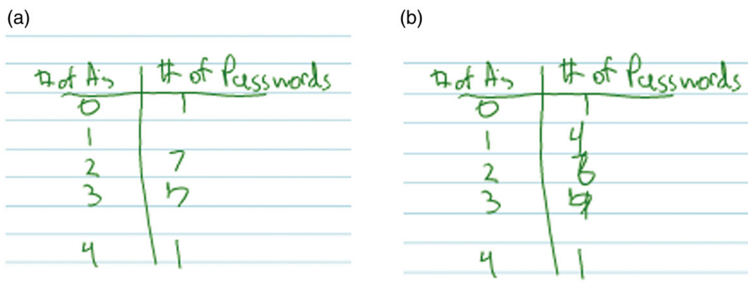



**Figure 9.** Jonas' 4-Character First Attempt (a) and Second Attempt (b) 4-Character AB Tables.

important differences in their mental activity when filling out the AB tables. In completing the 3-character AB password task, Jonas correctly created the table in Figure 8 by listing outcomes. He then conjectured that for a length-$n$ password there were $2^n$ possible outcomes (an *identification of a regularity* (*anticipated*)), and he observed the symmetric nature of the table.

The interviewer then asked Jonas to create a 4-character AB table, organized according to the number of As in the password. In his first attempt (Figure 9(a)), Jonas copied the form of the 3-character AB table in an attempt to ensure that the total would be 16 ($2^4$), even though the numbers did not properly align with the number of rows. After some examination he corrected the table by crossing out numbers and arriving at 1, 4, 6, 4, 1 (Figure 9(b)), but we infer that this was based on a pattern of totaling 16 and maintaining symmetry. That is, he engaged in *forming* by *associating objects*, in this case between the 3-character and the 4-character tables, but that association was based on figurative activity, namely, the result of the table entries in the right-hand column – they maintained the symmetric structure of numbers obtained previously.

As further evidence of this claim, when Jonas filled out the 5-character table, he guessed and checked to create a pattern that resembled the symmetric form of the previous tables, explaining, "I'm just trying to follow the same pattern." Jonas' attention was to the result of his activity in producing outcomes with symmetry in the table entries. In addition, Jonas could not justify why that symmetry made sense beyond maintaining a result achieved in the first task. Ultimately, he was limited in the kinds of generalizing he could do beyond the 3-character AB case. Because he understood the tables in terms of numerical patterns that must maintain a symmetric form, Jonas struggled to make sense of what was happening combinatorially in future activity. Jonas' generalizations were correct, but they were not generative because they did not facilitate extending and he could not justify them.

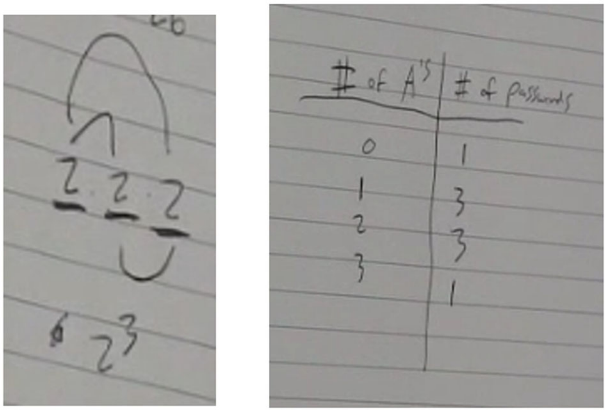



**Figure 10.** James' Counting Strategy and Table for Number of Passwords Containing As.

In contrast, we characterize James' initial activity when filling out his tables as operative because it foregrounded the underlying structure of his counting activity. The following exchange occurred when James filled out a table for the 3-character AB passwords, and it demonstrates that he thought about the combinatorial context as he completed the table (Figure 10). He determined there would be one password containing no As, and then reasoned about how to find the number of passwords with one and two As:

James: Well, how many of them there are, which is one A, is you can put A in any of the 3 places, and then the other 2 places would have to be B [writes 3 in the table].

Int.: Okay.

James: For 2 As, the 2 As could be here, here or here [draws lines as in Figure 10(a)]. So, I guess that's 3? And 3 As there's only 1, because all 3 were A.

James engaged in similar reasoning for the 4-character AB case. He did not need to guess and check for the purposes of maintaining particular numerical patterns, but rather he provided combinatorial explanations for the respective entries and used systematic listing to complete the tables. When asked about the symmetry of his tables, James replied:

James: When you have 0 As there's going to be all – it's all going to be Bs. And there's only 1 way to do that, because the Bs are indistinguishable.

Int.: Uh-huh.

James: And then when you have 5 As it's the same thing, the 5 As are indistinguishable. So, there's only 1 way to arrange them. And then the same thing, concept with like one A, 4 Bs, and 4 As and one B.

Here James isolated not just the symmetric results of prior tables, but he recognized the underlying combinatorial property of why the tables must be symmetric – namely the fact that As and Bs will behave similarly with respect to counting strategies. Thus James, like Jonas, engaged in *forming* by *associating objects*, but his association was operative. He associated the tables based on identical counting structures between 0 As and 5 As. James was able to justify his forming activity combinatorially, and he went on to successfully use his combinatorial reasoning to form and extend new generalizations.

We found that a student's initial mental activity provided an influential foundation for the generativity of their generalizations. If a student's initial activity is strictly figurative, then the generalizing is likely to be limited in its generativity, particularly in one's ability to extend and justify subsequent generalizations. If one's initial activity is operative, a student can potentially make meaningful extensions that they can justify based on the coordination of mental operations. Returning to Patty's work above on the Going Around B graphing task, her work provides additional insights into both the limitations of activity that is figurative and the potential affordances of activity that is operative, but in the context of precalculus concepts. In the event that a student's initial activity foregrounds perceptual or sensorimotor features when graphing, their extending of those generalizations are limited to those instances that enable maintaining those features. On the other hand, in the event that a student' activity foregrounds the coordination of mental actions, their extending of those generalizations can involve transformations that accommodate to any context in which one has conceived of quantities that vary. Particularly, generalizing rooted in the initial operative activity based on quantitative and covariational reasoning supports students in conceiving invariance among their actions despite perceptual or sensorimotor differences (Liang & Moore, 2021; Moore, Silverman, Paoletti, Liss, & Musgrave, 2019; Moore, Stevens, Paoletti, Hobson, & Liang, 2019).

### Building and refining activity

Students who were able to produce generative generalizations did not always begin with well-formed, correct generalizations that they could extend and justify. This was particularly the case when students encountered novel tasks that challenged their mathematical thinking. Instead, the development of generative generalizations was an iterative process of building on and refining prior generalizing, often in a series of repeated cycles of forming and extending. We found that one important part of the process of refining prior generalizing involved extending by transforming. As an example of this process, we present Willow's work on a proportional area task.

Willow: Unless it will start at zero! Because if you start it at 0, if you start from 0 to find the actual growth, then, say this is like the first theygrew [gestures along the length], and this kind of, so this [the length] grew by 4 first. And then this [gestures along the area] grew by 6. So this [the length] could grow by 4 again, and this [the area] could grow by 6 again.

Willow, a 6th grade student, began investigating the growing rectangle task (Table 2, first row), in which a rectangle with a side length of 4 cm and an area of 6 $cm^2$ grew in length but maintained its (unlabeled) width. The interviewer asked Willow to determine a relationship between the growth in side length and the growth in area by filling in the blanks for the statement, "When the length grows by ____, the area grows by ____." Willow initially made an additive comparison, explaining that "the area is 2 more than the length", and generalizing that this would always be the case: "It would always be 2 more if they grew in the same, like, the same amount." Willow's initial generalization was in the *forming* category, an *identification of a regularity (anticipated)*.

Willow remained convinced that the relationship between side length and area was additive for a significant amount of time, and she *extended by continuing* her regularity to a number of different new cases, as well as ultimately *extending by removing particulars* in order to state, "If the length grew by *x*, then the area would be 2 more than the total length." At this point, however, Willow suddenly evidenced a shift in her thinking about the nature of the task:

Willow re-conceived the situation dynamically, imagining the rectangle growing from a starting point of zero. This re-imagining then enabled her to project an image of continued growth beyond the initial 4 cm:6 $cm^2$ growth. She did so by iterating, repeating the imagined growth from 0 cm to 4 cm in length. Willow then constructed a new generalization, another *identification of a regularity* (*anticipated*), that a growth in 4 (cm) for side length would always be associated with a growth in 6 ($cm^2$) for area.

Willow then *extended* her regularity by *continuing*, saying, "When the length grows by 8, then the area grows by 12," and then, when asked to determine the added area for an added length of 16 cm, she responded immediately:

Willow: 24. Oh so it's, so we're trying to find the growth in area, so this will be the growth in area, equals, well I was kind of thinking, like, if you kind of made it like the growth in length, if it grew like, like 16 is 4 times 4 so it grew 4 times the original length kind of, then the area grows 4 times the original length.

Willow was able to take her generalization based on the 4:6 ratio and *extend* it to a new case by *operating*. Rather than iterating 4 times, Willow could consolidate this process by multiplying the side length by 4, preserving the ratio by also multiplying the area by 4. This appeared to be a *minor accommodation* for Willow, evidenced by her immediate response. She now had a generalization that she could take the amount of additional length, divide it by 4, and multiply that result by 6 to find the new area. Willow was then able to extend this generalization by *continuing* to a new case of a growth of 40 cm in side length, determining that the area would grow by 60 $cm^2$, and again by *operating* when asked to determine the additional area for a side length growth of 2 cm.

The interviewer then asked Willow to consider the area for a growth of 1 cm in length:

Willow: Well, 1 goes into 4 four times. [Pause]. One point 5? Because 1.5 goes into 6 four times. Hmm. [Calculates 1.5 + 1.5 + 1.5 + 1.5] Yeah. But I don't know if that's … I don't know if that's right though.

Willow's statement represented another extending generalization, *operating*, and appeared to be a *major accommodation* given her effort in making a structural change to the regularity. Namely, given a new value of added length $x$ that was less than 4 cm, she determined the quotient $4/x$, and then found the proportional amount that would "go into" 6 four times. The result, 1.5 $cm^2$, is equivalent to dividing 6 by 4, but Willow's strategy for this was to guess an amount and then iterate it 4 times to confirm that the result was 6.

Willow was able to again *extend* to new cases by *continuing* her adjusted generalization, creating length:area pairs of 3:4.5 and 5:7.5. In each case, she relied on her knowledge that $\frac{1}{4}$ of 6 is 1.5 in order to take $\frac{3}{4}$ of 6 and 5/4 of 6, each time iterating 1.5 three times and five times, respectively. Willow then realized an element of similarity across her activity with these tasks, exclaiming, "Oh! I think I see something!":

Willow: So, this would be, this was $\frac{1}{4}$, right? One point five is $\frac{1}{4}$ of 6. So, 1 would be $\frac{1}{4}$ [of 4], so it [the associated area for 1 cm in length] would be 1.5. And then this [the length] grew by 1. Hmm. Well, if it [side length] grew by 3, this [the area] is 4.5, so I think, it's, like, 1.5 bigger *each time*.

Even though Willow had previously solved a task in which the length grew by 1 cm and had found an associated growth of 1.5 $cm^2$ for area, she had not conceived of 1:1.5 as a unit ratio. Here, we see evidence of the third type of forming by associating, *activity-based associating*. Namely, in reflecting back over her mental activity of repeatedly iterating 1.5 in order to create multiple equivalent ratios, Willow realized that 1 is $\frac{1}{4}$ of 4, and 1.5 is therefore $\frac{1}{4}$ of 6. She shifted back to forming, *associating objects* based on her *activity*. Willow could now express a new regularity for this 1:1.5 ratio, saying, "it's 1.5 bigger *each time*", which suggests the image of adding 1.5 $cm^2$ of area for each additional 1 cm in side length. Willow also expressed this relationship more formally, writing "$A = 1.5 \times L$". In writing this expression, Willow extended by *transforming* her prior generalization into a unit ratio statement. Prior to this moment, Willow had generalized a strategy that relied on repeatedly iterating 1.5, but it is only now that she was able to reconceive of the area as 1.5 times any length.

Willow began her generalizing activity by making an additive comparison. When she shifted to viewing the rectangle as dynamically growing "from zero", Willow reconceived the situation as a multiplicative relationship. We characterize Willow's generalizing from this point forward as generative because she could extend her generalizations to new cases, and she bootstrapped her repeated iterating and partitioning activity into the development of a unit ratio. Further, Willow understood and could justify her generalizations; indeed, we saw an iterative relationship between generalizing and justifying in Willow's extended activity (Ellis, 2007b). Willow's final generalization, $A = 1.5 \times L$, required significant intellectual work to achieve and was the result of a series of shifts between forming and extending, and between types of forming and extending. It is when she reflected on her activity and made an association of similarity that she was ultimately able to transform 1.5 into a unit ratio, an amount of area associated with a 1-cm increase in side length.

We observed many cases of more mathematically advanced students easily forming the same generalization, $A = 1.5 \times L$. It is not the case that a student must engage in repeated cycles of forming and extending in order to produce a generative generalization. Rather, this is a type of activity we observed when students grappled with novel mathematical ideas that posed a challenge for them. We consider this process noteworthy because it characterizes students' generalizing activity when engaged in "productive struggle" (Warshauer, 2014), providing insight into how meaningful generalizations can occur when working on challenging, complex tasks. The RFE Framework provides a way to understand and characterize these instances in which students are observed bootstrapping their initial generalizations (which may be of limited utility or correctness) into more generative generalizations. In particular, the framework's distinction between forming and extending actions enabled us to identify a pattern of cyclical generalizing in our participants, in which they shifted back and forth between the two forms in order to deepen and expand the content of their generalizations over an extended sequence of problem solving.

## Discussion

Much of the research on students' generalizing has occurred in the domains of algebra, (e.g., Becker & Rivera, 2006; Carraher et al., 2008; Cooper & Warren, 2008; Mulligan & Mitchelmore, 2009), and much of the rich work examining students' progressions with generalizing activity has occurred specifically within a focus on patterning tasks (e.g., 2008; 2010; El Mouhayar, 2018a, 2018b; El Mouhayar & Jurdak, 2015; Jurdak & El Mouhayar, 2014; Radford, 2003; 2006; Rivera, 2013; Rivera & Becker, 2016). Following the lead of other researchers (e.g., Font & Contreras, 2008; Harel & Tall, 1991; Mason et al., 2007; Rivera & Becker, 2016), we have extended the research base to more advanced mathematical domains, and have also incorporated more varied, open-ended, and complex tasks to afford an exploration of how students generalize when engaged in problem-solving contexts that are more similar to those students might encounter in a classroom setting. In doing so, we responded to Doerfler's (2008) call to leverage "free" generalization tasks unrestricted by pre-determined goals.

The RFE framework confirmed a number of existing constructs delineating types of generalizations, including the categories Ellis (2007a) initially discovered about generalizing when considering a narrow domain. This can be seen with our categories of extending by removing particulars (similar to symbolic generalization; Radford, 2008), establishing a regularity (similar to result-pattern generalization; Harel, 2001) and empirical generalization (Bills & Rowland, 1999; Doerfler, 1991), forming by establishing a way of operating (similar to reconstructive generalization; Harel & Tall, 1991), extending by continuing (similar to expansive generalization; Harel & Tall, 1991), and other processes of forming and extending, such as establishing and continuing a regularity (similar to constructive generalization; Rivera & Becker, 2008) and reflective or eliminative abstraction (Font & Contreras, 2008).

We also found a number of instances of student activity that could not be explained by previous constructs. One example is the activity of extending by transforming, in which the content of the generalization itself is changed. Instances in which students change the content of generalizations are not well specified in the literature, likely because many existing constructs address students' statements of generalization more than the ongoing process of creating and modifying generalizations over time. Other new generalization types include relating by recursive embedding, and the more fine-grained distinctions between types of forming and extending, such as the distinction between extracted and anticipated regularities, the distinction between operative, figurative, and activity-based forming by associating objects, and the distinction between minor and major accommodations. We found that taking the actor's perspective supported our ability to identify aspects of generalizing that may not have been possible to discern from the expert's perspective. Further, many of these distinctions exemplify the RFE Framework's ability to help the researcher identify operations that are new and/or conceptually challenging to students. As such, the framework emphasizes the conceptual mechanisms that undergird generalizing activity, rather than generalization strategies or statements. Further, the RFE Framework organizes these constructs into thematically-related forms and types of generalizing, which enables an examination of students' generalizing activity throughout the process of problem solving, beginning with students' initial forays into new tasks and continuing as students grapple with revisiting and refining their ideas over time. Thus, the RFE Framework offers a way to characterize and understand students' generalizing activity when engaged in complex, open-ended problems with multiple solution strategies.

### *The importance of fostering repeated cycles of generalizing*

One of our aims in designing this study was to better understand how students initially form and develop new generalizations, how they adjust and expand those generalizations, and how they ultimately settle on final versions of their generalizations. Of particular interest was the manner in which students bootstrap their initial generalizing activity, which might be of limited use, into more generative generalizations. Students often develop initial generalizations or generalizing attempts, abandon them for a new direction or adjust them over time, circle back to prior ideas and revise or expand them, and make connections between different aspects of their conceptual activity over the course of solving a problem. In trying to understand these processes, we found that when engaged in problem solving, students shifted between forms of generalizing and also engaged in multiple types of generalizing activity within each form. This led to an important finding characterizing generative generalizations, which is that students can cycle back and forth between forming and extending, leveraging their initial activity into more powerful generalizations over time.

Many students require the time and space to engage in repeated cycles of forming and extending. This is particularly true for students' first introduction to mathematical concepts—their initial generalizations may be of limited utility or may even be incorrect, but with proper instructional support, they can serve as productive launch points for subsequent generalizing. Designing tasks that enable and encourage repeated cycles of forming and extending can be productive for fostering more sophisticated generalizations over time. Some potential features of such tasks could include prompting for prediction before allowing calculation (as seen in the Passwords Task), leveraging situations with surface similarities that contain structural differences (such as in the Going Around B Task), and designing questions that perturb prior problematic generalizations (as seen in the Growing Rectangle Task). When implementing tasks, teachers can also foster more extended engagement in generalizing by encouraging students to reflect back on their work and to articulate the similarities and differences they notice in their activity, both within and across contexts. When doing so, teachers should emphasize operative rather than figurative aspects of students' activity.

### *Generalizing across mathematical domains*

Studying a common mathematical practice, such as generalizing, across multiple domains and levels enables us to distill characteristic activity about that practice in a manner that is not bound to reasoning about a particular mathematical area. One interest guiding our study design was the question of whether participants would engage in similar generalizing activity across different domains, or whether generalizing is more domain-specific. Would we see particular generalizing types occur only in specific domains and not others? What we found is that generalizing is both domain-specific *and* domain-transcending. Generalizing is domain-transcending in that we did not find any generalizing types or forms that only occurred within one domain, or for one age level. In fact, all of the types and forms occurred in multiple domains. In the rare cases in which we saw few instances of a particular type within a specific domain, we saw this as a consequence of the limitation of our data set and our task bank. For example, we did not see many instances of isolating constancy within the undergraduate trigonometry/pre-calculus domain. This is, however, likely due to the nature of the tasks we used within that domain, in which students explored dynamic covarying quantities and their graphs. Those tasks encouraged relating situations, associating objects, establishing a way of operating, and extending more than other forms and types of generalizing. This was more a constraint of the tasks than of the domain itself, as there are other types of trigonometric tasks that could better encourage isolating constancy. However, despite these differences, we did see each of the generalizing types occur across domains, suggesting that the generalizing identified in the RFE Framework depicts common conceptual mechanisms that can occur in many areas of mathematics and for students in different grade levels.

How then, is generalizing domain-specific? We did find generalizing types that occurred more often within particular domains than others. For instance, as we have discussed, recursive embedding occurred most commonly in the combinatorics domain, which is unsurprising given that recursion of both an additive and multiplicative nature that is typical in combinatorics problems. Similarly, searching for a similarity or regularity and isolating constancy occurred often in the algebra tasks we presented to middle-school and high-school students, likely due to the presentation of these tasks as consecutive sequences of numbers and figures. In contrast, when students explored graphs and variation, they were less likely to engage in searching, recursive embedding, or isolating constancy. These examples underscore that it is difficult to truly disentangle domain from task, because particular domains are often coupled with specific types of tasks.

Additionally, domain-specific activity did play a critical role in supporting generalizing. We argue that generalization, like proof, should not be treated as a domain-independent heuristic (Dawkins & Karunakaran, 2016). Different domains involve different kinds of initial conceptual activity, and research in those domains can inform potential opportunities to leverage domain-specific thinking that could facilitate productive generalizing activity. In our study, we observed that students' generalizing could be deeply connected with the concepts and practices that are specific to a given domain. As an example, in combinatorics, our data support previous findings that it is critical to focus on sets of outcomes when counting (e.g., Lockwood, 2013, 2014; Lockwood & Gibson, 2016; Wasserman & Galarza, 2019). When students can connect their counting processes to sets of outcomes, they can gain insight into which formula might be appropriate, whether or not they have all of the outcomes, and how to avoid common traps such as overcounting. In advanced algebra, trigonometry, and pre-calculus, in contrast, identifying relevant quantities (Thompson & Carlson, 2017) and determining their dependency relationships were important ways of operating that supported the development of generative generalizations.

Students' generalizing does not occur in decontextualized spaces, and thus students may leverage ways of thinking that can allow for intentional generalizing activity that aligns with the content and practices relevant to a given domain. Furthermore, although we observed the generalizing forms and types occurring across domains, how those forms and types might manifest in student activity can differ depending on domain. By exploring generalization across domains and levels, the RFE Framework provides a way to characterize how particular types of generalizing can manifest in different ways across different tasks, contexts, and domains.

### *Future directions*

This study suggests some potential avenues for future directions of research. We expanded the research base to explore generalizing in more advanced domains, but it would be fruitful to also characterize generalizing among young children. In addition, although we have considered the degree to which some types of generalizing may or may not need to precede others, our data sets were not specifically organized to afford an in-depth consideration of this question. Future studies exploring students' generalizing in extended teaching settings, be that teaching experiments, design experiments, or whole-classroom studies, could more thoroughly investigate the hierarchy of students' generalizing types within the RFE framework. Situating future research within extended teaching settings could also support greater examination of the types of activities that are conducive to generative generalizing, as well as characterize the instructional moves that teachers can leverage in their own classrooms to better support powerful generalizing.

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## References

Amit, M., & Klass-Tsirulnikov, B. (2005). Paving a way to algebraic word problems using a nonalgebraic route. *Mathematics Teaching in the Middle School*, *10*(6), 271–276. doi:10.5951/MTMS.10.6.0271

Amit, M., & Neria, D. (2008). "Rising to the challenge": Using generalization in pattern problems to unearth the algebraic skills of talented pre-algebra students. *ZDM*, *40*(1), 111–129. doi:10.1007/s11858-007-0069-5

Aranda, C., & Callejo, M.L. (2010). Construction of the concept of linear dependence in a context of dynamic geometry: A case study. *Relime*, *13*(2), 129–158.

Becker, J. R., & Rivera, F. (2007). Factors affecting seventh graders' cognitive perceptions of patterns involving constructive and deconstructive generalizations. In J.-H. Woo, H.-C. Lew, K.-S. Park, & D.-Y. Seo (Eds.),

*Proceedings of the 31th conference of the international group for the psychology of mathematics education* (Vol. 4, pp. 129–136). Seoul, Korea: The Korea Society of Educational Studies in Mathematics.
Becker, J. R., & Rivera, F. D. (2006). Sixth graders' figural and numerical strategies for generalizing patterns in algebra (1). In S. Alatorre, J. L. Cortina, M. Sáiz, & A. Méndez (Eds.), *Proceedings of the 28th annual meeting of the North American chapter of the international group for the psychology of mathematics education* (pp. 95–101). Mérida, México: Universidad Pedagógica Nacional.
Bernard, H. E. (1988). *Research methods in cultural anthropology*. Newbury Park, CA: Sage.
Bills, L., & Rowland, T. (1999). Examples, generalization and proof. In L. Brown (Ed.), *Making meaning in mathematics. Advances in mathematics education* (pp. 103–116). York, UK: QED. doi:10.1080/14794809909461549
Blanton, M., & Kaput, J. (2002). Developing elementary teachers' algebra "eyes and ears": Understanding characteristics of professional development that promote generative and self-sustaining change in teacher practice. Paper presented at the annual meeting of the American Educational Research Association, New Orleans, LA.
Čadež, T. H., & Kolar, V. M. (2015). Comparison of types of generalizations and problem-solving schemas used to solve a mathematical problem. *Educational Studies in Mathematics*, *89*(2), 283–306.
Carraher, D. W., Martinez, M. V., & Schliemann, A. D. (2008). Early algebra and mathematical generalization. *ZDM*, *40*(1), 3–22. doi:10.1007/s11858-007-0067-7
Chinnappan, M., & Pandian, A. (2009). Malaysian and Australian children's representations and explanations of numeracy problems. *Educational Research for Policy and Practice*, *8*(3), 197–209. doi:10.1007/s10671-009-9071-8
Cooper, T. J., & Warren, E. (2008). The effect of different representations on Years 3 to 5 students' ability to generalise. *ZDM*, *40*(1), 23–37. doi:10.1007/s11858-007-0066-8
Davydov, V. V. (1990). *Soviet studies in mathematics education: Vol. 2. Types of generalization in instruction: Logical and psychological problems in the structuring of school curricula* (J. Kilpatrick, Ed., & J. Teller, Trans.). Reston, VA: National Council of Teachers of Mathematics. (Original work published 1972).
Dawkins, P., & Karunakaran, S. (2016). Why research on proof-oriented mathematical behavior should attend to the role of particular mathematical content. *The Journal of Mathematical Behavior*, *44*, 65–75. doi:10.1016/j.jmathb.2016.10.003
Doerfler, W. (2008). En route from patterns to algebra: Comments and reflections. *ZDM Mathematics Education*, *40*, 143–160.
Doerfler, W. (1991). Forms and means of generalization in mathematics. In A. Bishop (Ed.), *Mathematical knowledge: Its growth through teaching* (pp. 63–85). Mahwah, NJ: Erlbaum.
Dorko, A. (2015, November 5–8). *Generalizing average rate of change from single- to multivariable functions*. Paper presented at the Annual Meeting of the North American Chapter of the International Group for the Psychology of Mathematics Education, 37th, East Lansing, MI.
Dorko, A., & Lockwood, E. (2016, November 3–6). *What do students attend to when first graphing planes in R3? North American Chapter of the International Group for the Psychology of Mathematics Education.* Paper presented at the Annual Meeting of the North American Chapter of the International Group for the Psychology of Mathematics Education, 38th, Tucson, AZ.
Dorko, A., & Weber, E. (2014). Generalizing calculus ideas from two dimensions to three: How multivariable calculus students think about domain and range. *Research in Mathematics Education*, *16*(3), 269–287. doi:10.1080/14794802.2014.919873
Dougherty, B., Bryant, D. P., Bryant, B. R., Darrough, R. L., & Pfannenstiel, K. H. (2015). Developing conceptions and generalizations to build algebraic thinking: The reversibility, flexibility, and generalization approach. *Intervention in School and Clinic*, *50*(5), 273–281. doi:10.1177/1053451214560892
Dreyfus, T. (1991). Advanced mathematical thinking processes. In D. Tall (Ed.), *Advanced mathematical thinking* (pp. 25–41). Dordrecht, The Netherlands: Kluwer.
Dubinsky, E. (1991). Reflective abstraction in advanced mathematical thinking. In D. Tall (Ed.), *Advanced mathematical thinking* (pp. 95–123). Dordrecht, The Netherlands: Kluwer.
Ellis, A. B. (2011). Generalizing-promoting actions: How classroom collaborations can support students' mathematical generalizations. *Journal for Research in Mathematics Education*, *42*(4), 308–345. doi:10.5951/jresematheduc.42.4.0308
Ellis, A.B. (2007a). A taxonomy for categorizing generalizations: Generalizing actions and reflection generalizations. *Journal of the Learning Sciences*, *16*(2), 221–262. doi:10.1080/10508400701193705
Ellis, A.B. (2007b). The influence of reasoning with emergent quantities on students' generalizations. *Cognition and Instruction*, *25*(4), 439–478. doi:10.1080/07370000701632397
Ellis, A.B., & Grinstead, P. (2008). Hidden lessons: How a focus on slope-like properties of quadratic functions encouraged unexpected generalizations. *The Journal of Mathematical Behavior*, *27*(4), 277–296. doi:10.1016/j.jmathb.2008.11.002
El Mouhayar, R. (2018a). Trends of progression of student level of reasoning and generalization in numerical and figural reasoning approaches in pattern generalization. *Educational Studies in Mathematics*, *99*(1), 89–107. doi:10.1007/s10649-018-9821-8

El Mouhayar, R. (2018b). Levels of generalization and the SOLO taxonomy. In *Proceedings of the 42nd conference of the international group for the psychology of mathematics education* (Vol. 5). Umeå, Sweden: PME.

El Mouhayar, R., & Jurdak, M. (2015). Variation in strategy use across grade level by pattern generalization types. *International Journal of Mathematical Education in Science and Technology*, *46*(4), 553–569. doi:10.1080/0020739X.2014.985272

English, L., & Warren, E. (1995). General reasoning processes and elementary algebraic understanding: Implications for instruction. *Focus on Learning Problems in Mathematics*, *17*(4), 1–19.

Font, V., & Contreras, A. (2008). The problem of the particular and its relation to the general in mathematics education. *Educational Studies in Mathematics*, *69*(1), 33–52. doi:10.1007/s10649-008-9123-7

Harel, G. (2001). The development of mathematical induction as a proof scheme: A model for DNR-based instruction. In S. Campbell & R. Zaskis (Eds.), *Learning and teaching number theory* (pp. 185–212). Westport, CT: Ablex.

Harel, G., & Tall, D. (1991). The general, the abstract, and the generic. *For the Learning of Mathematics*, *11*, 38–42.

Hirsch, C. R., Fey, J. T., Hart, E. W., Harold, L. S., & Watkins, A.E. (2015). *Core-plus mathematics: Contemporary mathematics in context, course 1*. New York, NY: McGraw-Hill Education.

Jurdak, M. E., & El Mouhayar, R. R. (2014). Trends in the development of student level of reasoning in pattern generalization tasks across grade level. *Educational Studies in Mathematics*, *85*(1), 75–92. doi:10.1007/s10649-013-9494-2

Jureczko, J. (2017). The strategies of using a special kind of number patterns in different stages of education. *Educational Research and Reviews*, *12*(12), 643–652.

Kaput, J. (1999). Teaching and learning a new algebra with understanding. In E. Fennema & T. Romberg (Eds.), *Mathematics classrooms that promote understanding* (pp. 133–155). Mahwah, NJ: Erlbaum.

Kieran, C. (1992). The learning and teaching of school algebra. In D. Grouws (Ed.), *Handbook of research on mathematics teaching and learning* (pp. 390–419). New York: Macmillan Publishing Company.

Lannin, J. K. (2005). Generalization and justification: The challenge of introducing algebraic reasoning through patterning. *Mathematical Thinking and Learning*, *7*(3), 231–258. doi:10.1207/s15327833mtl0703_3

Lannin, J., Barker, D., & Townsend, B. (2006). Algebraic generalization strategies: Factors influencing student strategy selection. *Mathematics Education Research Journal*, *18*(3), 3–28. doi:10.1007/BF03217440

Lappan, G., Phillips, E. E., Fey, J. T., & Friel, S. (2014). *A guide to Connected Mathematics 3: Understanding, implementing and teaching*. Boston, MA: Pearson.

Lee, L. (1996). An initiation into algebraic culture through generalization activities. In N. Bednarz, C. Kieran, & L. Lee (Eds.), *Approaches to algebra* (pp. 87–106). Boston, MA: Kluwer.

Lee, L., & Wheeler, D. (1987). *Algebraic thinking in high school students: Their conceptions of generalization and justification (Research Report)*. Montreal, Canada: Concordia University, Department of Mathematics.

Lehrer, R., Kobiela, M., & Weinberg, P. (2013). Cultivating inquiry about space in a middle school mathematics classroom. *ZDM*, *45*(3), 365–376. doi:10.1007/s11858-012-0479-x

Liang, B., & Moore, K. C. (2021). Figurative and operative partitioning activity: Students' meanings for amounts of change in covarying quantities. *Mathematical Thinking and Learning*, *23*(4), 291–317. doi:10.1080/10986065.2020.1789930

Lobato, J. (2003). How design experiments can inform a rethinking of transfer and vice versa. *Educational Researcher*, *32*(1), 17–20. doi:10.3102/0013189X032001017

Lockwood, E. (2011). Student connections among counting problems: An exploration using actor-oriented transfer. *Educational Studies in Mathematics*, *78*(3), 307–322. doi:10.1007/s10649-011-9320-7

Lockwood, E. (2013). A model of students' combinatorial thinking. *The Journal of Mathematical Behavior*, *32*(2), 251–265. doi:10.1016/j.jmathb.2013.02.008

Lockwood, E. (2014). A set-oriented perspective on solving counting problems. *For the Learning of Mathematics*, *34*(2), 31–37.

Lockwood, E., & Gibson, B. R. (2016). Combinatorial tasks and outcome listing: Examining productive listing among undergraduate students. *Educational Studies in Mathematics*, *91*(2), 247–270. doi:10.1007/s10649-015-9664-5

Lockwood, E., & Reed, Z. (2016). Students' meanings of a (potentially) powerful tool for generalizing in combinatorics. In T. Fukawa-Connelly, K. Keene, & M. Zandieh (Eds.), *Proceedings for the nineteenth special interest group of the MAA on research on undergraduate mathematics education*. Pittsburgh, PA: West Virginia University.

Maj-Tatsis, B., & Tatsis, K. (2018). The use of variables in a patterning activity: Counting dots. *Center for Educational Policy Studies Journal*, *8*(2), 55–70. doi:10.26529/cepsj.309

Mason, J. (1996). Expressing generality and roots of algebra. In N. Bednarz, C. Kieran, & L. Lee (Eds.), *Approaches to Algebra* (pp. 65–86). Dordrecht, The Netherlands: Kluwer.

Mason, J., Burton, L., & Stacey, K. (2010). *Thinking mathematically*. London: Prentice Hall.

Mason, J., Drury, H., & Bills, E. (2007). Explorations in the zone of proximal awareness. In J. Watson & K. Beswick (Eds.), *Mathematics: Essential research, essential practice. Proceedings of the 30th annual conference of the Mathematics Education Research Group of Australasia* (Vol. 1, pp. 42–58). Adelaide, Australia: MERGA.

Mata-Pereira, J., & da Ponte, J.P. (2017). Enhancing students' mathematical reasoning in the classroom: Teacher actions facilitating generalization and justification. *Educational Studies in Mathematics*, *96*(2), 169–186. doi:10.1007/s10649-017-9773-4

Moore, K. C., Silverman, J., Paoletti, T., Liss, D., & Musgrave, S. (2019). Conventions, habits, and U.S. teachers' meanings for graphs. *The Journal of Mathematical Behavior*, *53*, 179–195. doi:10.1016/j.jmathb.2018.08.002

Moore, K. C., Stevens, I. E., Paoletti, T., Hobson, N. L. F., & Liang, B. (2019). Pre-service teachers' figurative and operative graphing actions. *The Journal of Mathematical Behavior*, *56*, 100692. doi:10.1016/j.jmathb.2019.01.008

Moss, J., Beatty, R., McNab, S. L., & Eisenband, J. (2006, April). *The potential of geometric sequences to foster young students' ability to generalize in mathematics*. Paper presented at the annual meeting of the American Educational Research Association, San Francisco, CA.

Mulligan, J., & Mitchelmore, M. (2009). Awareness of pattern and structure in early mathematical development. *Mathematics Education Research Journal*, *21*(2), 33–49. doi:10.1007/BF03217544

National Governors Association Center/Council of Chief State School Officers (2010). *Common core state standards for mathematics*. Washington, DC: Council of Chief State School Officers.

Napaphun, V. (2012). Relational thinking: Learning arithmetic in order to promote algebraic thinking. *Journal of Science and Mathematics Education in Southeast Asia*, *35*(2), 84–101.

Oflaz, G. (2019). An aspect of generalization act from an actor-oriented transfer perspective. *Başkent University Journal of Education*, *6*(1), 1–12.

Osborne, R. J., & Wittrock, M. C. (1983). Learning science: A generative process. *Science Education*, *67*(4), 489–508. doi:10.1002/sce.3730670406

Park, J. H., & Lee, K. (2016). How can students generalize the chain rule? The roles of abduction in mathematical modeling. *EURASIA Journal of Mathematics, Science & Technology Education*, *12*(9), 2331–2352.

Peirce, C. S. (1902). The essence of mathematics. In J. R. Newman (Ed.), *The world of mathematics*. New York, NY: Simon and Schuster.

Piaget, J. (2001). *Studies in reflecting abstraction*. Hove, UK: Psychology Press Ltd.

Piaget, J. (1970). *Genetic epistemology* (E. Duckworth, Trans). New York: Columbia University Press.

Piaget, J., & Inhelder, B. (1971). *Mental imagery in the child: A study of the development of imaginal representation*. London: Routledge & Kegan Paul.

Pytlak, M. (2015). Learning geometry through paper-based experiences. In K. Krainer & N. Vondrová (Eds.), *Proceedings of the ninth congress of the European Society for research in mathematics education* (pp. 571–577). Prague, Czech Republic: Charles University in Prague.

Radford, L. (2002). The seen, the spoken and the written: A semiotic approach to the problem of objectification of mathematical knowledge. *For the Learning of Mathematics*, *22*(2), 14–23.

Radford, L. (2003). Gestures, speech and the sprouting of signs. *Mathematical Thinking and Learning*, *5*(1), 37–70. doi:10.1207/S15327833MTL0501_02

Radford, L. (2006). Algebraic thinking and the generalizations of patterns: A semiotic perspective. In S. Alatorre, J. L. Cortina, M. Sáiz, & A. Méndez (Eds.), *Proceedings of the 28th annual meeting of international group for the psychology of mathematics education, North American Chapter* (Vol. 1, pp. 2–21). Mérida: Universidad Pedagógica Nacional.

Radford, L. (2008). Iconicity and contraction: A semiotic investigation of forms of algebraic generalization of patterns in different contexts. *ZDM*, *40*(1), 83–96. doi:10.1007/s11858-007-0061-0

Radford, L. (2010). Algebraic thinking from a cultural semiotic perspective. *Research in Mathematics Education*, *12*(1), 1–19. doi:10.1080/14794800903569741

Reed, Z. (2018). *Undergraduate students' generalizing activity in real analysis: Constructing a general metric* (Unpublished doctoral dissertation). Retrieved from https://ir.library.oregonstate.edu/concern/graduate_thesis_or_dissertations/nz8064938.

Reid, D. (2002). Conjectures and refutations in grade 5 mathematics. *Journal for Research in Mathematics Education*, *33*(1), 5–29. doi:10.2307/749867

Rico, L. (1996). The role of representation systems in the learning of numerical structures. In L. Puig & A. Gutierrez (Eds.), *Proceedings of the 20th conference of the international group for the psychology of mathematics education* (Vol. 1, pp. 87–102). Valencia, Spain: University of Valencia.

Rivera, F. D. (2013). Pattern generalization processing of younger and older students: Similarities and differences. In *Proceedings of the 37th Conference of the International* (Vol. 4, pp. 97–104). Kiel, Germany: PME.

Rivera, F. D. (2010). Visual templates in pattern generalization activity. *Educational Studies in Mathematics*, *73*(3), 297–328. doi:10.1007/s10649-009-9222-0

Rivera, F. D. (2008). On the pitfalls of abduction: Compolicities and complexities in patterning activity. *For the Learning of Mathematics*, *28*(1), 17–25.

Rivera, F. (2007). Visualizing as a mathematical way of knowing: understanding figural generalization. *The Mathematics Teacher*, *101*(1), 69–75. doi:10.5951/MT.101.1.0069

Rivera, F. D., & Becker, J. R. (2016). Middle school students' patterning performance on semi-free generalization tasks. *The Journal of Mathematical Behavior*, *43*, 53–69. doi:10.1016/j.jmathb.2016.05.002

Rivera, F. D., & Becker, J. R. (2008). Middle school children's cognitive perceptions of constructive and deconstructive generalizations involving linear figural patterns. *ZDM*, *40*(1), 65–82. doi:10.1007/s11858-007-0062-z

Rivera, F. D., & Becker, J. R. (2007). Abduction-induction (generalization) processes of elementary majors on figural patterns in algebra. *The Journal of Mathematical Behavior*, *26*(2), 140–155. doi:10.1016/j.jmathb.2007.05.001

Rivera, F., & Becker, J. R. (2005). Figural and numerical modes of generalizing in algebra. *Mathematics Teaching in the Middle School*, *11*(4), 198–203. doi:10.5951/MTMS.11.4.0198

Schliemann, A.D., Carraher, D.W., & Brizuela, B.M. (2001). When tables become function tables. *Proceedings of the XXV conference of the international group for the psychology of mathematics education* (pp. 145–152). Utrecht, The Netherlands: Kluwer.

Sfard, A. (1995). The development of algebra: Confronting historical and psychological perspectives. *The Journal of Mathematical Behavior*, *14*(1), 15–39. doi:10.1016/0732-3123(95)90022-5

Steele, D. (2008). Seventh-grade students' representations for pictoral growth and change problems. *ZDM*, *40*(1), 97–110. doi:10.1007/s11858-007-0063-y

Steffe, L. P. (1991). The learning paradox: A plausible counterexample. In L. P. Steffe (Ed.), *Epistemological foundations of mathematical experience* (pp. 26–44). New York, NY: Springer-Verlag.

Steffe, L. P. (1990). Mathematics curriculum design: A constructivist's perspective. In L. P. Steffe & T. Wood (Eds.), *Transforming children's mathematics education: International perspectives* (pp. 389–398). Hillsdale, NJ: Lawrence Erlbaum.

Steffe, L.P., von Glasersfeld, E., Richards, J., & Cobb, P. (1983). *Children's counting types: Philosophy, theory, and application*. New York, NY: Praeger Scientific.

Strachota, S., Knuth, E., & Blanton, M. (2018). Cycles of generalizing activities in the classroom. In C. Kieran (Ed.), *Teaching and learning algebraic thinking with 5- to 12-year-olds. ICME-13 monographs*. Cham: Springer.

Strauss, A., & Corbin, C. (1990). *Basics of qualitative research: Grounded theory procedures and techniques*. Newbury Park, CA: Sage Publications.

Syed, M., & Nelson, S.C. (2015). Guidelines for establishing reliability when coding narrative data. *Emerging Adulthood*, *3*(6), 375–387. doi:10.1177/2167696815587648

Tasova, H., & Moore, K. (2018). Generalization of an invariant relationship between two "quantities". In T. E. Hodges, G. J. Roy, & A. M. Tyminski (Eds.), *Proceedings of the 40th annual meeting of the North American chapter of the international group for the psychology of mathematics education* (pp. 588–595). Greenville, SC: University of South Carolina & Clemson University.

Tillema, E., & Gatza, A. (2017). The processes and products of students' generalizing activity. In Galindo, E., & Newton, J., (Eds.), *Proceedings of the 39th annual meeting of the North American chapter of the international group for the psychology of mathematics education* (pp. 259–266). Indianapolis, IN: Hoosier Association of Mathematics Teacher Educators.

Thompson, P. W. (1985). Experience, problem solving, and learning mathematics: Considerations in developing mathematics curricula. In E. A. Silver (Ed.), *Teaching and learning mathematical problem solving: Multiple research perspectives* (pp. 189–243). Hillsdale, NJ: Erlbaum.

Thompson, P. W., & Carlson, M. P. (2017). Variation, covariation, and functions: Foundational ways of thinking mathematically. In J. Cai (Ed.), *Compendium for research in mathematics education* (pp. 421–456). Reston, VA: National Council of Teachers of Mathematics.

Tuomi-Gröhn, T., & Engeström, Y. (2003). *Between school and work: New perspectives on transfer and boundary crossing*. Amsterdam: Pergamon.

Vale, C., Widjaja, W., Herbert, S., Bragg, L., & Loong, E.Y. (2017). Mapping variation in children's mathematical reasoning: The case of 'what else belongs?' *International Journal of Science and Mathematics Education*, *15*(5), 873–894. doi:10.1007/s10763-016-9725-y

von Glasersfeld, E. (1997). Homage to Jean Piaget (1896–1980). *The Irish Journal of Psychology*, *18*(3), 293–306. doi:10.1080/03033910.1997.10558148

von Glasersfeld, E. (1995). *Radical constructivism: A way of knowing and learning*. Washington, DC: Falmer Press.

Vygotsky, L. S. (1986). *Thought and language* (A. Kozulin, Trans and ed.). Cambridge, MA: MIT Press.

Warren, E., Miller, J., & Cooper, T.J. (2013). Exploring young students' functional thinking. *PNA*, *7*(2), 75–84.

Warshauer, H. K. (2014). Productive struggle in teaching and learning middle school mathematics. *Journal of Mathematics Education*, *17*(4), 3–28.

Wasserman, N. H., & Galarza, P. (2019). Conceptualizing and justifying sets of outcomes with combination problems. *Investigations in Mathematics Learning*, *11*(2), 83–102. doi:10.1080/19477503.2017.1392208

Yao, X. (2018). *An exploration of the interplay among teacher intervention, dynamic geometry software and students' mathematical generalizing practices about properties of geometric transformations* (Electronic thesis or dissertation). Retrieved from https://etd.ohiolink.edu/.

Yeap, B., & Kaur, K. (2008). Elementary school students engaging in making generalisation: A glimpse from a Singapore classroom. *ZDM*, *40*(1), 55–64. doi:10.1007/s11858-007-0072-x