CLASSROOM SUPPORTS FOR GENERALIZING

Amy Ellis
University of Georgia
amyellis@uga.edu

Yufeng Ying
University of Georgia
yufeng.ying@uga.edu

Anne Wawsa
University of Georgia
anne.wawsa@uga.edu

Kevin Moore
University of Georgia
kvcmoore@uga.edu

Michael Hamilton
University of Georgia
mwhamilton@uga.edu

Halil I. Tasova
University of Georgia
halil.tasova25@uga.edu

Aytuğ Özaltun Çelik
Pamukkale University
aozaltun@pau.edu.tr

Generalization is a critical component of mathematics learning, but it can be challenging to foster generalization in classroom settings. Teachers need access to better tools and resources to teach for generalization, including an understanding of what tasks and pedagogical moves are most effective. This study identifies the types of instruction, student engagement, and enacted tasks that support generalizing in the classroom. We identified three categories of Classroom Supports for Generalizing (CSGs): Interactional Moves, Structuring Actions, and Instructional Routines. The three categories operate at different levels to show how teachers, students, tasks, and artifacts work in interaction to mutually support classroom generalizing.

Keywords: algebra and Algebraic Thinking, Classroom Discourse, Instructional Activities and Practices

Understanding Classroom Generalization

Generalization is a central component of mathematical learning, with researchers arguing that it serves as the origin of mathematical ideas (Vygotsky, 1986; Peirce, 1902). The importance of generalization is reflected in national standards documents across North America (Council of Chief State School Officers, 2010; Ontario, 2005; Secretaría de Educación Pública, 2017), as well as in curricular materials (e.g., Hirsch et al., 2007). However, research shows pervasive student difficulties in creating and understanding correct general statements (e.g., adez & Kolar, 2015; English & Warren, 1995), creating further challenges in fostering success in many domains, including function, geometry, and combinatorics (e.g., Ellis & Grinstread, 2008; Pytlak, 2015; Lockwood & Reed, 2016).

Although students' challenges with generalizing is well documented, less is known about how to better support generalization, particularly in classroom settings. The majority of research on generalizing has occurred in laboratory settings, such as clinical interviews and small-scale, researcher-led teaching experiments. The field knows less about how productive generalization occurs in school settings with practicing teachers teaching everyday topics. Furthermore, the limited research on teachers' abilities to foster generalization shows that effectively supporting generalization is challenging for teachers (e.g., Callejo & Zapatera, 2017; Mouhayar & Jurdack, 2012). Teachers need support in learning how to help students generalize, including increased access to research-based tools and resources that build on the field's knowledge of students' productive generalizing. In response to these needs, this paper investigates the state of student generalizing in middle-school and secondary classrooms. In particular, we addressed the following questions: What are the opportunities for generalizing in classroom settings? Specifically, what types of instructional moves, student engagement, and enacted tasks support classroom generalizing?

Literature Review and Theoretical Framework

Researchers have identified both cognitive activities and pedagogical strategies that can foster generalization. The cognitive activities include visualizing properties beyond what is perceptually available (Becker & Rivera, 2007; Yeap & Kaur, 2008), attending to particular characteristics or relationships above others (Rivera & Becker, 2007), identifying relationships between tasks, representations, or properties (Cooper & Warren, 2008; Johanning, 2004), and describing general relationships or processes verbally or in written form (Ellis, 2007; Rivera & Becker, 2008). Research on pedagogical strategies has identified potentially productive moves to foster generalization, which includes having students consider big numbers (Zazkis et. al, 2008), showing variation across tasks (Mason, 1996), guiding students to reflect on their mathematical operations (Doerfler, 2008; Ellis, 2007), providing access to physical or visual representations (Amit & Neria, 2008), emphasizing similarity across contexts (Radford, 2008), and ordering tasks in a progressive sequence (Ellis, 2011; Steele & Johanning, 2004).

There are two caveats to consider in relation to the above findings concerning teachers supporting generalizing. The first is that the bulk of these studies were conducted in small-scale laboratory teaching settings, and the degree to which their findings might translate to whole-classroom activity is not well understood. A couple of studies, however, did detail the classroom factors influencing how middle-school students engaged with a generalization problem (Jurow, 2004; Koellner et al., 2008). For instance, Koellner and colleagues found that working with an open-ended problem with multiple entry points, having opportunities to visualize a concrete representation, and being able to work collaboratively fostered students' generalizing, along with the teacher's discursive moves of pushing for algebraic generalizations without supplying answers. The second caveat is that although many of the above studies have addressed specific instructional moves, fewer have explicitly addressed to the role that interaction can play in fostering generalizing. There are two notable exceptions. Ellis (2011) identified a number of generalizing-promoting actions representing how teachers and students can interact to foster generalizing, including publicly generalizing, encouraging justification, building on ideas, and focusing attention on mathematical relationships. This study, however, was situated in a teaching experiment setting rather than a classroom setting. In a classroom-based study, Jurow (2004) introduced the notion of participation frameworks to account for how students generalized in small groups. Both studies suggest that generalizing can occur as a consequence of processes distributed across tasks, students, and tools.

Defining and Situating Generalizing

While definitions of generalization vary, most characterize it as a claim that some property holds for a set of mathematical objects or conditions larger than the set of original cases (Carraher et al., 2008). For instance, Radford (2006) described generalizing as identifying a commonality based on particulars and then extending it to all terms, and Harel and Tall (1991) characterized generalization as the process of applying a given argument to a broader context. These definitions situate generalization as an individual, cognitive construct, but as seen with Jurow's (2004) work, one can also consider generalizing as a collective act distributed across multiple agents (Ellis, 2011; Tuomi-Gröhn & Engeström, 2003). This perspective attends to how social interactions, tools, and classroom environments can shape students' generalizing actions, positioning generalization as a fundamentally social practice. We follow this perspective to define generalizing as an activity in which learners in specific sociomathematical contexts engage in at least one of the following actions: (a) identifying commonality across cases, or (b) extending reasoning beyond the range in which it originated (Ellis, 2011).

We use the symbolic interactionist perspective, considering classroom learning to be a social process that occurs in interaction (Bauersfeld, 1995). From this perspective, learning is examined through the lens of multiple processes of interactions, in which students' interactions with tasks, artifacts, one another, and with their teacher all co-contribute to the activity of generalizing. This can occur through conversation, shared problem-solving activity, and negotiated meaning of problems and solutions. We view the learning environment as a system, made up of mutually interacting agents, and then consider how that system supports students' shared construction of meaning as they generalize. Reflecting the foci of our research questions, the symbolic interactionist perspective enables us to privilege both individual students' reasoning and the processes of interaction that supported that reasoning (Blumer, 1969; Voigt, 1995).

Methods

We conducted a series of classroom observations in one middle-school and two high-school classrooms. Prior to scheduling the observations, we asked each teacher to choose a unit that they thought would offer opportunities to observe generalizing. Mr. J was a third-year teacher who taught advanced algebra and precalculus, Ms. R was a sixth-year teacher who taught high-school algebra, and Ms. N was a third-year teacher who taught sixth-grade mathematics. In each classroom we conducted videoed observations with two cameras. One camera focused on the teacher and whole-class setting, and the other recorded a focus group of three to four students, capturing the entirety of their engagement including conversations, gestures, and written work.

In Mr. J's tenth-grade advanced algebra class we observed a three-day unit on exponents and roots, culminating in the development of the rule $\sqrt[b]{x^a} = \left(\sqrt[b]{x}\right)^a$. In Ms. R's ninth-grade algebra class we observed a four-day unit on using algebraic symbols and equation solving techniques to represent word problems. In Ms. N's sixth-grade class we observed a four-day unit on the coordinate plane, basic properties of quadrants, determining horizontal and vertical distances between points, and determining reflections over the *x*- and *y*-axes. We also interviewed each teacher twice after the observed units in order to explore their definitions of generalization, their beliefs about generalization, and their beliefs about how to foster generalization in the classroom. For the purposes of this paper, we draw specifically on the classroom observation data in order to determine student opportunities to generalize in classroom settings.

To analyze the data, we relied on both transcripts and video recordings, considering the participants' talk, gestures, intonations, and use of tools, drawings, and physical objects. We first coded all instances of generalization using Ellis et al.'s (2017) RFE Framework, and then turned to Ellis' (2011) categories of generalizing-promoting actions as an initial scheme to code instances of classroom interaction that supported the generalizations. In addition to using the generalizing-promoting actions categories, we revisited all classroom interactions to identify those that potentially contributed to the generalizations but were not captured by existing codes. We coded actions as fostering generalizing if generalizing occurred in direct response to an action, if a generalization mirrored or responded to a new idea introduced by an action, or if we could identify a conceptual chain linking the ideas or structure introduced by an action and a generalization that followed it. A number of interaction instances yielded novel codes, which contributed to the Classroom Supports for Generalizing (CSG) presented in this paper. Three members of the project team then independently re-coded every transcript, collaboratively resolving any discrepancies through consensus. Following the approaches others have used to investigate discourse (e.g., Pierson & Whitacre, 2010), the codes do not distinguish between teachers' and students' utterances. This is consistent with the interactionist framework, in which the students and teachers jointly contribute to a shared understanding (Cobb & Bauersfeld, 1995).

Results: Classroom Supports for Generalizing

We found three major CSG categories: (a) Interactional Moves, (b) Structuring Actions, and (c) Instructional Routines (Figure 1). Interactional Moves refer to the questions, initiations, responses, or ideas that people, task prompts, artifacts, or representations can introduce into the conversation. These moves are not limited to teacher moves; students can also initiate questions, share ideas or strategies, or encourage one another to generalize, justify, or share. In addition, specific task prompts or even one's use of a representation can constitute an Interactional Move, if they play an in-the-moment role of fostering generalizing during a classroom conversation.

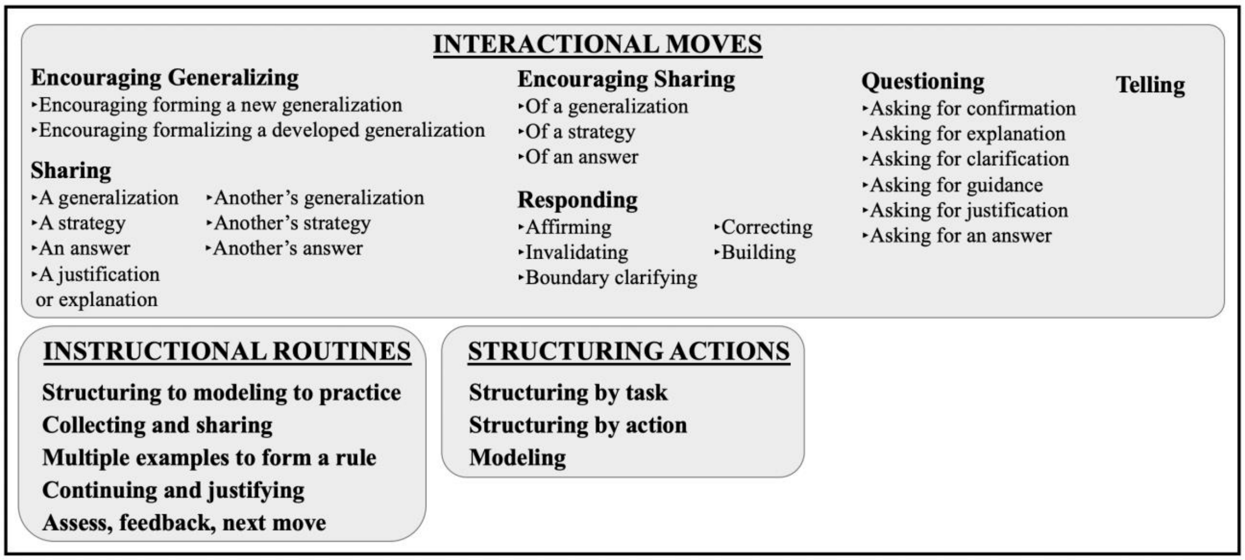

Figure 1: Interactional Moves, Instructional Routines, and Structuring Actions

In contrast to Interactional Moves, which are spontaneous and localized, Structuring Actions typically address the aspects of a teacher's instruction that are more systematic and intentional. They are the actions one employs to implicitly or explicitly structure students' activity in a manner designed to lead to a generalization. This can include developing and implementing task sequences with the aim of fostering a generalization, explicitly drawing students' attention to sameness across problem types or ideas, or choosing to organize a series of representations in a manner that highlights a generalizable feature. It can also include modeling the process of developing a generalization for other members of the community, an action that students may sometimes engage in as well as teachers.

The third category, Instructional Routines, depicts the patterned and recurrent ways that instruction unfolds in a classroom (Horn & Little, 2010). Following the work of those who have studied professional routines in teaching (e.g., Leinhardt et al., 1987; Rösken et al., 2008), we consider these routines to entail a stable schematic core with a more fluid shell, allowing for variable responses to demands of the moment. The Instructional Routines we identified were those stable, repeatable series of pedagogical moves that fostered student generalizing. These are processes such as collecting a range of student strategies to share for whole-class discussion and to serve as a source for forming a generalization (*collecting and sharing*), or visiting a small group, assessing their progress towards a generalization, providing feedback and guidance based on their progress, and then leaving them with a specific next step to achieve (*assess, feedback, next move*). Each of the routines we identified appeared repeatedly in one teacher's class but not in others', indicating that many routines may be somewhat teacher specific.

Developing a Generalization in Interaction: Horizontal Distance

Due to length constraints, rather than defining and discussing each CSG, we instead offer an extended data episode illustrating the manner in which multiple CSGs operate together in order to support the classroom development of a generalization. This episode draws from Ms. N's 6th-grade classroom and takes place during a lesson about the horizontal and vertical distance on a coordinate plane. The excerpt illustrates one of Ms. N's Instructional Routines, *multiple examples to form a rule*. In this routine, a teacher shares and discusses multiple examples of the same phenomenon, and then directs students to consider what remains invariant across the examples with the aim of developing a mathematical rule as an articulation of the invariance.

In launching the routine, Ms. N projected a coordinate plane on the board and placed a magnetic dart at the point (7, 5). She then asked a student to place a second dart a horizontal distance of 8 units from the first dart. The student placed the dart at the point (-1,5), and Ms. N encouraged the students to note the ordered pairs of the two points. She then repeated this process, placing a dart at (-1, 1) and asking a student to place a second dart at a horizontal distance of 3 units away. The student placed the dart at (-4, -1), and Ms. N again asked the students to attend to the ordered pairs of the two points. Ms. N then repeated this process a third time, placing the dart at (7, -4) and asking a student to place the second dart a horizontal distance of 10 units away. The student placed the dart at (-3, -4). At this point, Ms. N also engaged in the Structuring Action CSG of *structuring by action*: She wrote the three pairs of ordered pairs together on the board in a manner that made it visually salient that the *y*-values of each pair of ordered pairs was the same (Figure 2). The written representation itself played the role of *encouraging generalizing (forming)* by directing students' attention to the structure of each pair of points.

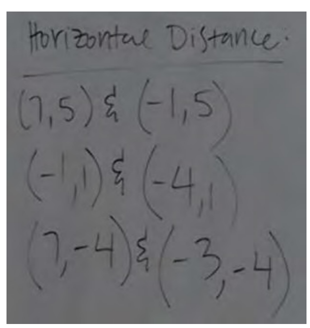

Figure 2: Ms. N's Representation of Three Pairs of Ordered Pairs

In the following table (Table 1), we provide each classroom member's utterance with the accompanying CSG it represents. The excerpt begins with Ms. N explicitly asking the students what the ordered pairs have in common:

Table 1: First Excerpt Utterances and CSGs

Utterance	CSG
Ms. N: Who can tell me what looked at these two ordered pairs to start [points to the first pair]. What do they have in common? What are these ordered pairs have in common?	Encouraging Generalizing (forming)
Ari: They both have the same *y*-axis coordinate?	Sharing (a generalization)
Ms. N: *y*-coordinate. Good. What is their *y*-coordinate?	Questioning (asking for answer)
Ari: Five.	Sharing (an answer)
Ms. N: Five. All right, what do these two points (points to the next pair) have in common? Rayna?	Encouraging Generalizing (forming)
Rayna: They have the same *y*-coordinate?	Sharing (a generalization)
Ms. N: What is the *y*-coordinate?	Questioning (asking for answer)
Rayna: One.	Sharing (an answer)
Ms. N: They both have a one in common in the *y*-coordinate place, and what do these two points have in common (points to the last pair)? Wesley.	Encouraging Generalizing (forming)
Wesley: They both have the same *y*-axis coordinate which is negative four.	Sharing (a generalization)
Ms. N: Perfect. So, what do they *not* have in common? What are they not sharing?	Encouraging Generalizing (forming)
Parker: *x*-coordinate.	Sharing (a generalization)
Ms. N: Their *x*-coordinates, right? So that is going to be a pattern that you will always notice whenever we are talking about horizontal distance between two points.	Sharing (a generalization)

Ms. N was *structuring by action* throughout the above exchange by explicitly drawing students' attention to sameness across the three pairs of ordered pairs. This occurred not only through the above exchange, by also by Ms. N's actions of finger pointing and underlining the *y*-coordinates of each ordered pair on the board. Those actions were to support the generalization that when determining a horizontal distance, each pair of points will have the same *y*-value. Ms. N then *encouraged generalizing* by asking the class, "Is it possible that I could look at these ordered pairs and without even plotting them, know the distance between them?" Jonah proposed the idea that you can simply take the sum of the absolute value of the *x*-values of each pair of points to find the difference:

Jonah: You just need to add them together. You can get how many things you go over. Because the top [pointing to (7, 5) and (-1, 5)] like if you, you add them together, but you get rid of the negative sign, it equals eight. Second [pointing to (-1, 1) and (-4, 1)] you move five.

Ms. N: Okay. So be careful with, with saying add them together. I think I know what you mean. But be careful with say add them.

With this proposal Jonah *shared* a generalization. He subsequently added that he meant the absolute value: "Absolute value. Just add them together." In response, Ms. N asked the students to consider the second case Jonah mentioned, with (-1, 1) and (-4, 1). In doing so, Ms. N engaged in a form of *responding* that was *boundary clarifying*: Her intent was to help the students determine when Jonah's generalization would work and when it would not. The students determined that it worked for the first and third pair, but not the middle pair of (-1, 1) and (-4, 1);

they concluded this by physically counting the number of units between the two points on the coordinate plane. In the next excerpt, the students and Ms. N together began with Jonah's incorrect generalization and transformed it into a correct one (Table 2):

Table 2: Second Excerpt Utterances and CSGs

Utterance	CSG
Ms. N: So what, what happened with your theory? I like the theory, it's almost there. But we need to tweak it a little bit going.	Encouraging Generalizing (forming)
Jonah: I think we are going negatives to positives. I think it only works with positive negative, positive positive.	Sharing (a generalization)
Ms. N: And try them if my two coordinates are not the same sign, you mean?	Questioning (asking for clarification)
Jonah: You change the negative, you just kind of do the opposite.	Sharing (a generalization)
Ms. N: Okay, cool, can be something to add to our rule.	Responding (affirming)
Riley: This one, like go, go ones that he's talking about adding. They start with the positive number. And when we, with this [(points to (-1, 1) and (-4, 1)], and it starts with negative number, you can subtract it from before, and equals three.	Responding (building)
Jonah: Yeah, that's what I mean by like negative, negative.	Responding (affirming)
Ms. N: Okay, so in general, what am I looking for? Absolute value is asking us for a, what do we say? What kind of measurement?	Questioning (asking for an answer)
Robin: Distance.	Sharing (an answer)
Ms. N: A distance. So in general, this is always going to be true. What am I looking for between the two points that aren't the same?	Questioning (asking for an answer)
Quinn: Positive number.	Sharing (an answer)
Ms. N: I'm looking for, the word you just said -	Questioning (asking for an answer)
Riley: (Interrupts) Distance.	Sharing (an answer)
Ms. N: I'm looking for the distance between them, right? So if I'm finding the distance, Jonah, between a positive number and a negative number, you're right, I am going to need to know their absolute value so that I can combine them. But if they're already on the same side of zero, I can literally just do what I can count one, two, I can just count the distance, right? Like I know from negative one to negative four. It's how far -	Telling
Jonah: (Interrupts) I think that only works when they are both at opposite sides.	Responding (building) Sharing (a generalization)
Ms. N: Yeah, I think that's true if they don't have the same sign, I like your strategy.	Responding (affirming)

The excerpt began with Jonah's initially incorrect generalization, that you add the absolute value of the *x*-coordinates for any two points. Through a series of transformations, Ms. N and the students built on one another's statements to develop a modified generalization, which was that if the two points are on the opposite side of the origin, the absolute values can be combined to determine the distance, but if they are on the same side of the origin, one can count the distance between them. Riley did propose a modification to Jonah's generalization, that one can subtract the absolute values for the pair of points that were both on the same side of the origin, the teacher did not take it up. In her interview, Ms. N shared that she did not do so because her students had not yet learned arithmetic with negative numbers. So, she instead highlighted that one could just count to determine the distance.

Note that all of the CSGs in each of the two excerpts were from the Interactional Moves category. The CSGs in this category are ones that lend themselves readily to teachers' and students' utterances in conversation, as well as particular task prompts or representation choices, such as Ms. N's organization of the three pairs of points in Figure 2. These Interactional Moves, however, occurred within the broader Instructional Routine of *multiple examples to form a rule*. Ms. N enacted a what was for her a common routine, that of sharing and discussing multiple cases of the same phenomenon, before then directing the students to consider what was the same across the examples in order to develop a general rule. Within this routine, she also engaged in a Structuring Action, structuring generalizing by drawing students' attention to sameness across the three ordered pairs. Within the Structuring Action and Instructional Routine, the Interactional Moves were the more immediate, localized moves made by both the teacher and the students that worked together to build up to the final generalization for determining the horizontal distance between two points.

Discussion

The three categories of CSGs enable attention to classroom interactions simultaneously at three different grain sizes. We found that the manner in which the Interactional Moves supported particular generalizations needed to be considered in light of the larger Structuring Actions and Instructional Routines in which they occurred. For instance, a specific move such as *sharing a generalization*, *boundary clarifying*, or *asking for an explanation* may or may not be effective in supporting generalizing depending on the immediate structure of interaction in which it takes place, as well as the larger structure of pedagogical actions and routines that form the sociomathematical milieu of the classroom. By considering the classroom environment to be a system of mutually interacting agents (Voigt, 1995), we have been able to identify simultaneous levels of support in order to better understand how generalization emerges in classroom contexts.

Similar to other studies attending to aspects of interaction in supporting generalizing (Ellis, 2011; Jurow, 2004), we found that the teacher, the students, the enacted tasks, the students' use of tools and artifacts, and the nature of representations worked in concert to support generalizing. Ms. N's representation of the pairs of points on the board worked together with her guiding remarks and the students' contributions to build up to the final generalization for determining horizontal distance. This illustrates the collective nature of generalizing, and the manner in which members of the classroom community can collaboratively build upon one another's ideas to introduce, reflect on, and refine generalizations.

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