FROM NUMBER LINES TO GRAPHS: A MIDDLE SCHOOL STUDENT'S RE-ORGANIZATION OF THE SPACE

Halil I. Tasova
CSU, San Bernardino
halil.tasova@csusb.edu

Kevin C. Moore
University of Georgia
kvcmoore@uga.edu

We report on developmental shifts of a middle school student's (Ella) graphing activity as we implement an instructional sequence that emphasizes quantitative and covariational reasoning. Our results suggest that representing quantities' magnitudes as varying length of directed bars on empty number lines supported Ella re-organizing the space consistent with a Cartesian plane.

Keywords: Algebra and Algebraic Thinking, Cognition, Representations and Visualization

Constructing and interpreting graphs represents a "critical moment" in middle school mathematics for its opportunity to foster powerful learning (Leinhardt et al., 1990). Students, however, experience a number of challenges (e.g., conceiving graphs as picture of situation, event phenomena, literal motion of an object) in interpreting graphs (see Johnson et al., 2020 for a summary of these challenges). One potentially promising way to support students to develop productive meanings for graphs is to emphasize the role of seeing a graph as an emergent trace of how two quantities' measures vary simultaneously (Moore & Thompson, 2015). Although numerous researchers have investigated students' ability to interpret and construct graphs by plotting points and scaling axes, using slope and y-intercept, incorporating embodiment-based learning opportunities, and connecting with the other forms of multiple representations of functions, far fewer researchers have focused investigating how students construct graphs as emergent traces of quantities' covariation. Thus, we investigate the following questions: What ways of thinking do middle school students engage in graphing activities intended to emphasize quantitative reasoning? How can modeling with a quantitative reasoning approach support students' ability to develop productive and powerful ways of graphing?

Theoretical Framework: Quantitative Reasoning

This study focuses on middle school students' graphing activities involved in reasoning with relationships between quantities in real-world situations. We use *quantity* to refer to a conceptual entity an individual construct as a measurable attribute of an object (Thompson, 2011). In this study, we demonstrate ways in which students make sense of quantitative relationships in dynamic events and in graphs by reasoning with quantities' *magnitudes* (i.e., the quantitative size of an object's measurable attribute) independent of numerical values.

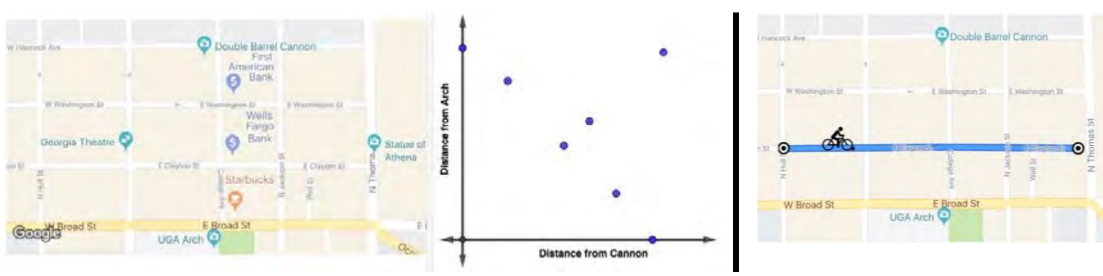
In the context of graphing, a relationship between two quantities is often represented in a coordinate system. Lee (2017) pointed out that researchers and educators have often taken coordinate systems for granted in students' graphing activity. Until recently, researchers did not question the importance of constructing a coordinate system because most researchers did not view it as a mental structure that students needed to construct (Lee, Moore, & Tasova, 2019). Furthermore, the idea of representing quantities' values or magnitudes on number lines is often taken for granted in students' construction of coordinate systems (and, in turn, in students' graphing activities), which is problematic because the construction of a plane requires conceiving of two number lines and using them to create a two-dimensional space (Lee, Hardison, & Paoletti, 2018). Thus, in this study, we investigated the nature and extend of students' abilities to represent varying quantities' magnitudes on number lines, and whether/how those abilities influence their construction of coordinate systems and graphs.

Methods

This study is situated within a larger study that examined four seventh-grade students' graphing activities in a teaching experiment (Steffe & Thompson, 2000) that occurred at a public middle school in the southeast United States. This study focuses on Ella's meanings for graphs and her developmental shift of those meanings over the teaching experiment.

Ella participated in 6 teaching sessions each of which last for approximately one-hour. Data sources included video and transcripts of each session that captured her exact words, gestures, and drawings. We conducted a conceptual analysis in order to understand her verbal explanations and actions and develop viable models of her mathematics (Steffe & Thompson, 2000). Our analysis relied on generative and axial methods (Corbin & Strauss, 2008), and it was guided by an attempt to develop working models of Ella's thinking.

Before conducting the teaching experiment, we developed an initial sequence of tasks each of which was designed with a dynamic geometry software and displayed on a tablet device (see https://www.geogebra.org/m/w9n4hn7r for digital versions of the tasks). *Downtown Athens Task* (DAT) includes a map with seven locations highlighted and labeled (see Figure 1a). We also present a Cartesian plane whose horizontal axis is labeled as Distance from Cannon (DfC) and vertical axis is labeled as Distance from Arch (DfA). Seven points are plotted without labelling in the coordinate plane to represent the seven locations' DfA and DfC (see Figure 1a, right). We asked students what each of these points on the plane might represent with an intention to observe their spontaneous responses and to explore students' meanings of points.



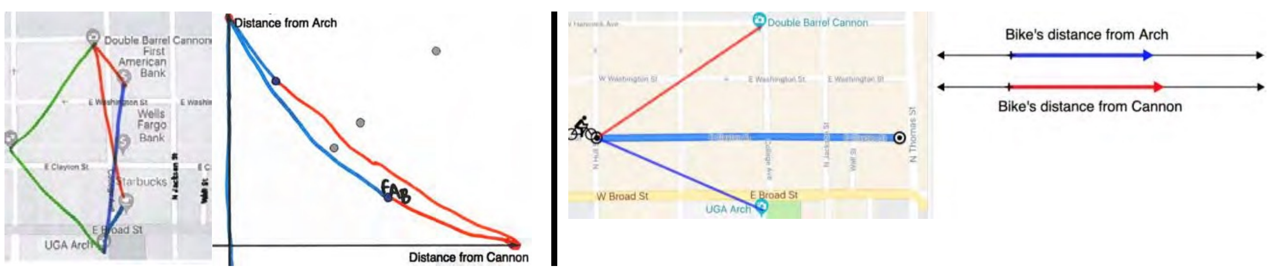
(a) (b)

Figure 1. (a) Downtown Athens Task (b) Downtown Athens Bike Task

In *Downtown Athens Bike Task* (DABT), we present the students with the same map of Downtown Athens highlighting a straight road (i.e., Clayton St.). We asked students to graph the relationship between the bike's DfA and DfC as the bike moves at a constant speed back and forth along the road. We also designed numerous tasks where students engage with quantities' magnitudes represented by varying length of directed bars placed on empty number lines (also called *magnitude lines*, see Figure 2b). The length of directed bars on the magnitude lines vary according to the bike's movement in the map. We conjectured that this representation might help students when they move to the two-dimensional space to represent two quantities by a single point in a coordinate plane. Note that we call the line "empty number line" in order to emphasize magnitude reasoning as opposed to numerical or value reasoning.

Results

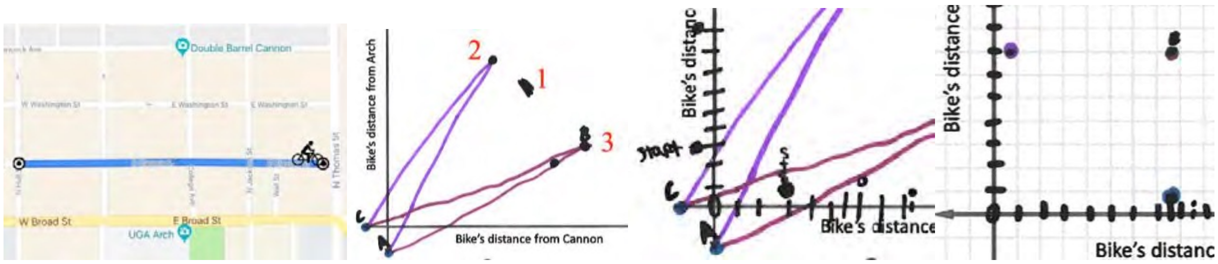
Initial meanings of the points and the organization of the space. We illustrate Ella's initial meanings by using her activity during DAT. Ella assimilated points in the plane as a location/object, however, her meanings were based in focusing on object's quantitative properties. After conceiving Arch and Cannon physically located on each axis as implied by the labels (see orange dots on each axis in Figure 2a), Ella made sense of the rest of the space by coordinating the radial distances between "places" on the plane and Arch and Cannon on each axis. For example, Ella labeled a point as "FAB" on the plane (see Figure 2a) to indicate First American Bank, and she conceived the point as FAB based on the orange and blue line segments that she drew on the plane. She stated, "the orange is shorter, and the blue is longer… [*referring to the orange and blue line segments on the map*] over here, like the same thing." Ella perceived FAB is closer to Cannon and farther from Arch in the map as well as in the plane. Therefore, we infer that Ella's meanings of the points included determining quantitative features of an object in the situation (i.e., its DfA and DfC as indicated by segments) and subsequently preserving these quantitative properties via the location of a point in the plane.



(a) (b)

Figure 2. (a) Ella's bipolar coordinate system, (b) DABT with magnitude lines

Representing a quantity's magnitude on an empty number line. In order to aid Emma in developing particular meanings for representing quantities in Cartesian plane, we engaged her in a dynamic tool that represented quantities' magnitudes as directed bars of varying length (see Figure 2b, right). We first wanted to get insights to how Ella could conceive this representation. While moving the bike to the right from its position seen in Figure 2b (red segment in the map and the corresponding red bar on the line were hidden at the moment), we drew Ella's attention to the fact that the right end side of blue bar on the magnitude line was moving to the left (indicating the bike's DfA was decreasing from our perspective). Ella determined that the bike's DfA is decreasing while moving the bike to the right in the map. She explained "it [*pointing to the blue bar*] is gonna get smaller because distance is smaller on the number line too." Moreover, Ella labeled the starting point as "zero." From this activity, we infer that Ella conceived the length of the blue bar on the magnitude line as a representation of the bike's DfA.



(a) (b) (c) (d)

Figure 3. (a) Map showing the bike's position when questioned, (b) Ella's graph, (c) marks and dots on axes, and (d) points in the plane.

Graphing bike's DfA and DfC. After Ella engaged with the magnitude line activity, we asked her to sketch a graph to represent the bike's DfC and DfA using a given piece of paper with two orthogonal axes. Ella re-organized the space different than her earlier actions in the teaching experiment (see Figure 3b vs. Figure 2a). For example, Ella conceived Cannon at very left side of the horizontal axis (labeled C) because "farther it is here [*sweeping her finger to the right from left over the horizontal axis*] means that farther it is from Cannon." This may show that Ella's re-organization of the space was an implication of her engagement with the magnitude line activity. Ella still assimilated the dot she drew in the plane as the bike (labeled B, #3 in Figure 3b) whose location was determined by coordinating the radial distances between the bike's DfA and DfC. Note that Ella wanted to change the location of the dots (see her earlier attempts in Figure 3b with the numbers showing the order in which she drew) "because it [the dot labeled as B] is like farther away from Cannon than it is Arch."

Ella's shift. Note that Ella plotted only one point on the plane (see Figure 3b), although the prompt was to graph the relationship as the bike traveled. We asked her whether her graph (i.e., the dot she plotted) illustrated the relationship between the bike's DfA and DfC as we animated the bike—the length of the bars on the magnitude lines also varied accordingly—in the tablet screen. She said no. Ella claimed, "I probably could have put a number line right here [*referring to the axes of the plane*]" to show how the bike's DfC and DfA changed as it moved. To illustrate this, she plotted tick marks on each axis in conjunction with tick marks plotted on the magnitude lines. She added dots near certain (and somewhat arbitrary) tick marks on each axis (see black dots in Figure 3c) to represent certain states of bike's DfA and DfC as the bike changed its location. During this activity, Ella did not focus on her purple line segments or the points that she drew earlier in the plane (see Figure 3b). She only worked on the axis to represent each quantity, and she did not plot points in the plane to represent them simultaneously. So, we repeated the same task with grid paper to see if she could join those quantities in the plane. By describing "this is what I did earlier" referring to her latest activity, Ella began plotting a dot on each axis to show the bike's DfA and DfC (Figure 3d). Then, she plotted a point in the plane "where those two [*tracing the pen in the air from the dots on each axis to the dot in the plane horizontally and vertically, respectively*] would meet up if they have like a little line." When asked to explain what that point represented to her, Ella said, "that is where the bike is." We infer that Ella seemed to establish a way to represent two quantities in her newly organized space as a single point; although she seemed to conceive the point that she plotted in the plane as the physical location of the bike.

Discussion

In this study, we illustrated different ways a student's graphing activity involved representing quantitative relationships. These examples illustrate alternative meanings of a coordinate system and coordinate points. Ella initially assimilated the points on the plane in relation to the physical objects that appear in the situation, and her meanings for points were based in quantitative properties (i.e., magnitudes from a fixed point). Ella conceived the length of the bar on the magnitude line as a proxy for the quantity that she conceived in the situation (i.e., the bike's DfA). In doing so, she conceived a constrain regarding how to represent the variation of a quantity on a magnitude line (e.g., only left and right on a horizontal line). Thus, she organized the space accordingly in later activities when considering two-dimensional space (see her shifts along Figure 2a, Figure 3b, and Figure 3d). Our results illustrate that explicit attention to quantities in the situation and mapping those quantities' magnitudes onto the empty number lines supported Ella's re-organization of the space consistent with a Cartesian plane.

References

Corbin, J. M., & Strauss, A. (2008). *Basics of Qualitative Research: Techniques and Procedures for Developing Grounded Theory* (3rd ed.). Thousand Oaks, CA: Sage.

Earnest, D. (2015). From number lines to graphs in the coordinate plane: Investigating problem solving across mathematical representations. *Cognition and Instruction, 33*(1).

Johnson, H. L., McClintock, E., & Gardner, A. (2020). Opportunities for reasoning: Digital task design to promote students' conceptions of graphs as representing relationships between quantities. *Digital Experiences in Mathematics Education*.

Lee, H. (2017). *Students' construction of spatial coordinate systems*. Unpublished doctoral dissertation. Athens, GA: University of Georgia.

Lee, H. Y., Hardison, H., & Paoletti, T. (2018). Uses of coordinate systems: A conceptual analysis with pedagogical implications. In T. E. Hodges, G. J. Roy, & A. M. Tyminski (Eds.), P*roceedings of the 40th annual meeting of the North American Chapter of the International Group for the Psychology of Mathematics Education* (pp. 1307–1314). Greenville, SC: University of South Carolina & Clemson University.

Lee, H. Y., Moore, K. C., & Tasova, H. I. (2019). Reasoning within quantitative frames of reference: The case of Lydia. *Journal of Mathematical Behavior, 53*, 81–95

Leinhardt, G., Zaslavsky, O., & Stein, M. K. (1990). Functions, graphs, and graphing: Tasks, learning, and teaching. *Review of Educational Research, 60*(1), 1–64.

Moore, K. C., & Thompson, P. W. (2015). Shape thinking and students' graphing activity. In T. Fukawa-Connelly, N. I. Engelke, K. Keene, & M. Zandieh (Eds.), *Proceedings of the 18th Annual Conference on Research in Undergraduate Mathematics Education*. Pittsburgh, PA: West Virginia University.

Schliemann, A.D., Carraher, D.W., & Caddle, M. (2013). From Seeing Points to Seeing Intervals in Number Lines and Graphs. in B. Brizuela & B. Gravel (Eds.) *Show me what you know: Exploring Representations across STEM disciplines*. Teachers College Press.

Steffe, L. P., & Thompson, P. W. (2000). Teaching experiment methodology: Underlying principles and essential elements. In R. A. Lesh & A. E. Kelly (Eds.), *Handbook of research design in mathematics and science education* (pp. 267–307). Hillside, NJ.

Thompson, P. W. (2011). Quantitative reasoning and mathematical modeling. In S. Chamberlin, L. L. Hatfield, & S. Belbase (Eds.), *New perspectives and directions for collaborative research in mathematics education: Papers from a planning conference for WISDOM^e* (pp. 33-57). Laramie, WY: University of Wyoming.

Thompson, P. W., Carlson, M. P., Byerley, C., & Hatfield, N. (2014). Schemes for thinking with magnitudes: A hypothesis about foundational reasoning abilities in algebra. In L. P. Steffe, K. C. Moore, L. L. Hatfield, & S. Belbase (Eds.), *Epistemic algebraic students: Emerging models of students' algebraic knowing* (pp. 1-24). University of Wyoming.