



Using combinatorics problems to support secondary teachers understanding of algebraic structure

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Abstract

This paper presents data from the first of three iterations of teaching experiments conducted with secondary teachers. The purpose of the experiments was to investigate how teachers' combinatorial reasoning could support their development of algebraic structure, specifically structural relationships between the roots and coefficients of polynomials. The data in this paper examines the learning that occurred as one teacher transitioned from making a generalization from a sequence of contextualized combinatorics problems to applying her combinatorial reasoning to symbolic problems common in algebra curricula. The findings from the study include the identification of three planes of learning that can be used to differentiate among ways that combinatorial reasoning can be used to engage in binomial expansion. The highest plane involved constructing a *combinatorial scheme for binomial expansion*, a scheme that supported the teacher to produce the equivalence, $(x + a)(x + b)(x + c) = x^3 + (a + b + c)x^2 + (ab + ac + bc)x + abc$, and to see important algebraic structure in it. The contributions of the study include: (a) expanding earlier arguments about the ways that combinatorics can be integrated into goals of extant curricula (e.g., Maher et al. in *Combinatorics and reasoning: Representing, justifying and building isomorphisms*. Springer, 2011); and (b) proposing how reflecting abstraction can be used to study the transition between generalizations learners make from contextualized problem situations to operating with and on generalizations expressed with conventional algebraic symbols. This second contribution is an under-researched area in the algebra literature (Dörfler in *ZDM - Int J Math Educ* 40(1):143–160, 2008), and points to an important role that combinatorial reasoning can play in algebra learning.

1 Introduction

Mathematics educators have routinely called for an increase in discrete mathematics in K-12 curricula (e.g., Kapur, 1970). As part of making these calls, researchers interested in integrating combinatorics, one branch of discrete mathematics, into K-12 curricula have provided arguments for how combinatorics can support mathematical processes (e.g., Maher et al., 2011). To a lesser extent, researchers have made arguments for how combinatorics can be integrated with current content standards, with the strongest arguments being presented for connections between combinatorics and probability (e.g., Batanero et al., 2016).

Given the centrality of algebra in curricula internationally, we use this paper to present empirical data about how

combinatorial reasoning can support powerful forms of algebraic reasoning at the secondary level (grades 9–12). The data we present comes from the first of three teaching experiments (Steffe & Thompson, 2000) whose purpose was to understand how combinatorial reasoning could support secondary teachers to engage with advanced algebraic identities (e.g., $(x + a)(x + b)(x + c) = x^3 + (a + b + c)x^2 + (ab + ac + bc)x + abc$) in novel ways. This paper focuses on the learning of a single pre-service secondary teacher (PSST). We focus on this PSST to illustrate one facet of the learning that we aimed to engender with the teachers more broadly. We respond to the following research question in this paper: What learning occurred as a PSST transitioned from symbolizing regularities and constraints of contextualized combinatorics problems to applying her combinatorial reasoning to operate on conventional algebraic symbols?

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Core Aspects and Strands	
<i>Two Core Aspects</i>	
(A) Algebra as systematically symbolizing generalizations of regularities and constraints.	
(B) Algebra as syntactically guided reasoning and actions on generalizations expressed in conventional symbolic systems.	
<i>Core Aspects A & B are Embodied in Three Strands</i>	
1. Algebra as the study of structures and systems abstracted from computations and relations, including those arising in arithmetic (algebra as generalized arithmetic) and in quantitative reasoning.	
2. Algebra as the study of functions, relations, and joint variation.	
3. Algebra as the application of a cluster of modeling languages both inside and outside of mathematics.	

Fig. 1 Kaput's (2008/2017) framework for algebraic reasoning (p. 11)

2 Literature review

Research on students' combinatorial reasoning has identified how combinatorics problems can support mathematical processes like problem solving and posing (English, 1999), generalization (Speiser et al., 2007), and justification and proof (Maher et al., 2011). These processes are often identified as lacking in algebra courses at the secondary level. For example, many secondary algebra textbooks in the United States present symbolic problems first, delaying or omitting application problems designed for students to engage in problem solving (Sherman et al., 2016). Moreover, U.S. textbooks routinely present pre-given formulas as "useful" rather than provide opportunities for students to generate, justify, or prove their own generalizations (Thompson et al., 2012).¹ Thus, the domain of combinatorics, with its potential to support these processes, has significant promise for enhancing algebra curricula at the secondary level.

Kaput (2008/2017) has framed algebraic reasoning around three strands and two core aspects (Fig. 1). An important component of secondary school algebra is for students to engage in Core Aspect B of Kaput's framework (Kieran, 2007). Mathematics education researchers, however, have tended to focus on Core Aspect A, leaving open the question as to how work on Core Aspect A could support work on Core Aspect B (Dörfler, 2008). We use this paper to illustrate how symbolizing regularities and constraints in the solution of contextualized combinatorics problems (Core Aspect A) can support powerful forms of reasoning on

conventional algebraic symbols (Core Aspect B). Our work is situated within Strand 1 of Kaput's framework. Strand 1 has received a significant amount of attention at the elementary and middle grades level (e.g., Stephens et al., 2017), but less attention at the secondary level where more attention has been given to Strand 2, functions (e.g., Thompson & Carlson, 2017). To address Strand 1, we build from work where we have articulated connections between combinatorial and quantitative reasoning (e.g., Tillema, 2018; Tillema & Gatz, 2016).

3 Conceptual framework

We use this section to: (a) define the conceptual constructs we use to study learning; (b) outline the two combinatorial schemes we consider central to the learning that we document; and (c) discuss our use of reflecting abstraction to study the transition from Core Aspect A to B.

3.1 Constructs for studying learning

We use operations and schemes to investigate mathematical learning. Following Glasersfeld (1995), an *operation* is a mental action. Operations are the building blocks of schemes. A *scheme* contains three parts: an assimilatory mechanism, activity, and result. A scheme *opens* when a person assimilates a problem situation by making an interpretation of it. Assimilation triggers an activity, which involves carrying out operations on figurative material that may be perceptually present (e.g., a deck of playing cards or a physical array) or mentally generated (e.g., imagined cards or an imagined array). The operations a person carries out produces a result.

Schemes are goal-directed: a person establishes a goal in the act of assimilation, monitors the goal during the activity, and determines whether the result satisfies the goal. If a scheme's result satisfies the goal, the scheme *closes*. If the result does not satisfy the goal, or the activity produces an

¹ We note that our argument about curricula is rooted in the United States context and may not be the case for countries that have worked to more substantively integrate discrete mathematics into their curricula (e.g., Spain, Hungary, Germany, Israel, etc.). However, we consider this issue an important point for discussion among international researchers—understanding in what ways and how different countries have taken up calls to integrate discrete mathematics in curricular materials.

unexpected result, a person may experience a perturbation. In such cases, a person may make changes to their scheme. When a more or less permanent change occurs *while the schemes are in use* (Steffe, 1991), we consider these changes to be *functional* accommodations and functional accommodations to be acts of learning.

As part of characterizing a person's schemes, we attend to the extent to which a person carries out the activity of their schemes *and* the kind of figurative material on which they carry them out. We use the term *implemented* to discuss the extent to which a person carries out the activity of their scheme. At times, all operations of a scheme may be implemented, while at other times, operations may be evoked but not implemented. We use the term *enactive scheme* when a person is constrained to carrying out the scheme with the support of perceptually available figurative material. By constrained, we mean that, without the perceptually available material (e.g., a set of cards, a list, or a physical 3-D array), the person cannot carry out the activity of their scheme and accomplish their goal. We use the term *interiorized scheme* when a person can carry out a scheme on mentally generated figurative material produced in visualized imagination (cf. Steffe, 1992). Mentally generated figurative material may or may not manifest as perceptually present. For example, a person who operates on a 3-D array in visualized imagination, may operate exclusively in visualized imagination or they may create some perceptually present figurative material (e.g., draw part of a 3-D array they are imagining) as a record of having operated in visualized imagination.

3.2 Combinatorial schemes

We use the Password Problem to illustrate two combinatorial schemes whose coordination was important for characterizing the learning we aimed to engender with teachers.

Password Problem. Using the characters A and B, how many three-character passwords could you make (order matters)?

Upon reading the Password Problem, a person may form a goal of finding all possible three-character passwords, which is when the scheme opens. We provide one example of how a person may proceed with that goal in mind. A person may interpret the situation as involving three composite units, where a composite unit involves a person treating multiple units (e.g., letters A and B) as a single entity (e.g., a set of letter possibilities). The activity of the person's scheme can involve *ordering* the three composite units (a first set of letters, a second set, a third set); *ordering* the units of each composite unit (e.g., A is first, B is second); *pairing* the first unit of each composite unit to produce an ordered triple (i.e., AAA); *pairing* the first unit of the first

Fig. 2 Listing passwords using lexicographic ordering

A	A	A
A	A	B
A	B	A
A	B	B
B	A	A
B	A	B
B	B	A
B	B	B

Exactly three A's	A A A
Exactly two A's	A A B
	A B A
	B A A
Exactly one A	A B B
	B A B
	B B A
No A's	B B B

Fig. 3 Listing passwords by the number of A's

two composite units with the second unit of the third composite unit to produce another ordered triple (i.e., AAB). The person may then continue with this activity in a lexicographic order where the first two positions are fixed and all possibilities are put into the third position before changing the character in the second position (Fig. 2, English's [1991] odometer method).

With this example, the operations that make up the activity of the person's scheme are ordering and pairing operations (Tillema, 2014). The figurative material could be perceptually present (e.g., actual cards with letters) or mentally generated (e.g., letters a person imagines). The person's activity transforms the three composite units into the result of the scheme, eight ordered triples (i.e., eight three-character passwords). A person may then use the lexicographic order to confirm that they have satisfied their goal to produce all possible ordered triples at which point the scheme closes. We call this scheme a *multiplicative pairing scheme* (MPS) for 3-D combinatorics problems.

The second scheme we discuss is similar to the first in that: (a) a person assimilates the Password Problem using three composite units and (b) the activity of the scheme involves ordering and pairing operations. The primary difference is the organizing principle of this scheme. The person organizes the passwords according to the number of A's in them (i.e., exactly three, two, one, or no A's) (Fig. 3). With

this organizing principle in mind, the person may then use ordering and pairing operations to accomplish their goal of producing all possible ordered triples; with the goal satisfied, the scheme closes. We call this scheme a *binomial coefficient scheme* (BCS).

3.3 Reflecting abstraction: moving from Kaput's core aspect A to B

We use reflecting abstraction (Piaget, 1977/2001) to account for how a person transitions from Core Aspect A to B of Kaput's (2008/2017) framework. *Reflecting abstraction* involves a person abstracting the operations of a scheme or a coordination among schemes as opposed to abstractions related to specific figurative material on which the operations or coordinations are carried out. As such, the scheme or coordination among schemes is freed from specific figurative material (Glaserfeld, 1995). This criterion of reflecting abstraction is focused on *abstraction* in that a person can impose a scheme or coordination among schemes on new situations or contexts, which involve different figurative material. To claim that a person has moved from Core Aspect A to B means that their scheme is freed from both the specific figurative material and context in which it was developed and that the coordination is being applied to new figurative material, in this case algebraic symbols. It is in this way that we consider reflecting abstraction as a way to operationalize moving from Core Aspect A to B of Kaput's framework.

Piaget (1977/2001) uses two additional features to characterize reflecting abstraction: (a) there is a projection of a scheme or coordination among schemes from a lower to higher plane of learning and (b) a reorganization of the schemes at the higher plane of learning. Our data analysis focuses on a functional accommodation involving a coordination among the MPS and BCS. Therefore, we differentiate between two ways a person can coordinate schemes—sequentially or by inserting one scheme inside another. A *sequential* coordination of schemes means that the schemes are activated in sequence with one scheme closing before the next scheme opens. An *insertion* of schemes means a person embeds one scheme inside another scheme where the two schemes become a single scheme. Identifying an insertion, rather than a sequential use, of schemes is how we operationalize the notion of *projection* in the definition of reflecting abstraction. That is, one scheme may be inserted into another to become a single scheme where the single scheme is at a higher plane of learning precisely because the new single scheme was previously multiple schemes. The *reorganization* part of reflecting abstraction, then, entails how the new singular scheme gets integrated with a person's prior schemes.

4 Methods

4.1 Methodology, participants, and data collection

We report on Olive, a PSST, who participated in the first of three iterations of teaching experiments that we conducted with secondary teachers. One purpose of a teaching experiment is to identify and document authentic learning that occurs in the context of teaching (Steffe & Thompson, 2000). To this end, a teacher-researcher plans tasks for participants in order to learn about and test conjectures related to their ways of reasoning.

Olive participated in 13 teaching episodes that ranged from 60 to 90 min. The first author served as the teacher-researcher. All episodes occurred with a partner, Aaron, and were videotaped using three cameras: one to capture interaction, one camera to capture each PSSTs' written work. The video files were mixed into a single file for analysis. Five mathematics education graduate students were also part of the research team, including the second author. They attended all teaching episodes, assisted in data collection, and supported the planning of tasks to test conjectures about the PSSTs' reasoning.

4.2 Design and goals of the teaching episodes

We now outline relevant portions of the design and goals of the problems we presented to the PSSTs. Our teaching experiments have begun with the Card Problem.

Card Problem. You have the 2, 3, and K of Diamonds in a draw pile. A three-card hand is created by drawing a card, replacing it, drawing a card, replacing it, and drawing a third card. How many possible three-card hands could you make? How many three-card hands have no face cards, exactly one face card, exactly two face cards, and exactly three face cards?

The intent of this problem is to support PSSTs to develop the equivalence that $3^3 = (2 + 1)^3 = 1(2^3) + 3(2^2 \cdot 1^1) + 3(2^1 \cdot 1^2) + 1(1^3)$. We intend for this equivalence to develop as follows: 3^3 as an outgrowth of counting all possible three-card hands (hereafter, hands) (response to the first question in the Card Problem); $(2 + 1)^3$ as an equivalent way to express all possible hands that shows a binary choice between the number of non-face and face cards in the draw pile; and $1(2^3) + 3(2^2 \cdot 1^1) + 3(2^1 \cdot 1^2) + 1(1^3)$ as an equivalent way of counting all possible hands based on the number of face cards in a hand (response to the second question in the Card Problem). The final way of counting the hands involves reasoning about the number of ways to get a particular kind of hand (e.g., there are three ways to get a hand with one

Fig. 4 3-D array representation (a) as constructed by PSSTs and (b) highlighted according to the number of face cards

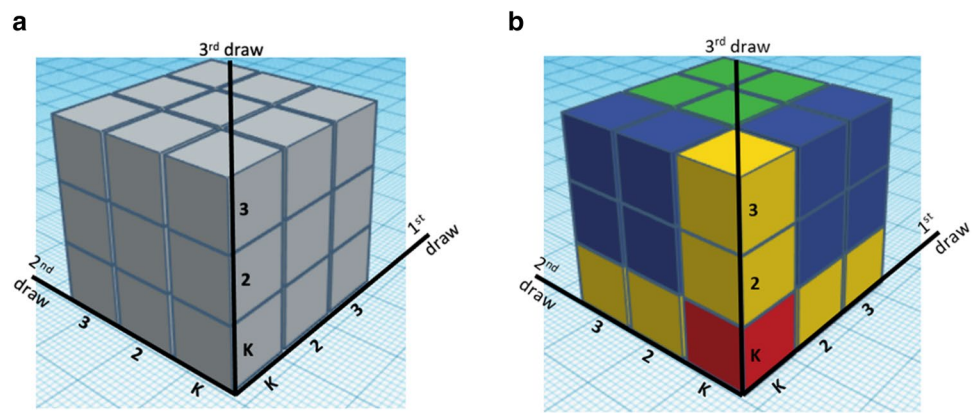


Fig. 5 Selection scenarios and partial products corresponding to $(x^2 + 3y)^3$

Exactly three x^2 factors	$(x^2 + 3y)(x^2 + 3y)(x^2 + 3y) \rightarrow x^2 \cdot x^2 \cdot x^2$
Exactly two x^2 factors	$(x^2 + 3y)(x^2 + 3y)(x^2 + 3y) \rightarrow x^2 \cdot x^2 \cdot 3y$
	$(x^2 + 3y)(x^2 + 3y)(x^2 + 3y) \rightarrow x^2 \cdot 3y \cdot x^2$
	$(x^2 + 3y)(x^2 + 3y)(x^2 + 3y) \rightarrow 3y \cdot x^2 \cdot x^2$
Exactly one x^2 factor	$(x^2 + 3y)(x^2 + 3y)(x^2 + 3y) \rightarrow x^2 \cdot 3y \cdot 3y$
	$(x^2 + 3y)(x^2 + 3y)(x^2 + 3y) \rightarrow 3y \cdot x^2 \cdot 3y$
	$(x^2 + 3y)(x^2 + 3y)(x^2 + 3y) \rightarrow 3y \cdot 3y \cdot x^2$
No x^2 factors	$(x^2 + 3y)(x^2 + 3y)(x^2 + 3y) \rightarrow 3y \cdot 3y \cdot 3y$

face card because the face card could be in the first, second, or third position) and reasoning about the total number of hands within a particular way (e.g., there are a total of $(2^2 \cdot 1^1)$ hands for each way because there are two non-face card options [drawn twice] and one face card option [drawn once]).

As part of solving the problem, PSSTs use snap cubes to represent the ordered triples as a 3-D array (Fig. 4a). They then locate regions in their 3-D arrays representing hands with no (green), exactly one (blue, only two shown), exactly two (yellow), and exactly three face cards (red) (Fig. 4b). We provide a multi-colored array (Fig. 4b) for the ease of the reader. However, in the teaching experiment, participants continue to use monochrome arrays (Fig. 4a) because we want them to establish regions in the array from the implementation and abstraction of mental operations rather than introducing a color pattern that might suggest a pre-given structure. PSSTs then work on cases of the Card Problem, where the number of face and non-face cards can vary, to develop the generalization that $(x + y)^3 = 1 \cdot (x^3) + 3 \cdot (x^2 \cdot y^1) + 3 \cdot (x^1 \cdot y^2) + 1 \cdot (y^3)$.

Once the teachers use the Card Problem to make a generalization of regularities and constraints (Core Aspect A), we have them work on applying their combinatorial reasoning to symbolic problems common in algebra curricula, like expanding $(x^2 + 3y)^3$ (Core Aspect B). By “applying their combinatorial reasoning”, we mean it is possible to conceive

of the three binomials as ordered (i.e., a first, second, and third binomial), where each binomial represents a binary choice between two variable expressions, x^2 and $3y$ (Fig. 5). Given that there are three binomials and for each there is a choice between two options, there are a total of eight partial products (i.e., 2^3) where each partial product consists of three factors (i.e., one factor from each binomial). When expanding, there is only one way to create an $(x^2)^3$ -term, selecting x^2 from each of the three binomials and multiplying these terms. There are three ways to create an $(x^2)^2 \cdot (3y)^1$ -term, selecting x^2 from two of the three binomials and $3y$ from the other binomial. Continuing with this reasoning produces $(x^2 + 3y)^3 = (x^2)^3 + 3((x^2)^2 \cdot (3y)^1) + 3((x^2)^1 \cdot (3y)^2) + (3y)^3$. Once teachers work on this kind of reasoning, we have them extend their combinatorial reasoning to the binomial theorem and problems like $(x + a)(x + b)(x + c)$.

One reason we have started the teaching experiments with cases of the Card Problem prior to the work with algebraic symbols is that we have considered it to be one way to support teachers to develop algebraic symbols as representing a variable number of possible options rather than treating algebraic symbols simply as letters. That is, we have considered working on cases of the Card Problem as a way to support teachers to establish algebraic symbols, x and y , in their generalization, $(x + y)^3 = 1 \cdot (x^3) + 3 \cdot (x^2 \cdot y^1) + 3 \cdot (x^1 \cdot y^2) + 1 \cdot (y^3)$,

as representing a possible number of options for non-face and face cards, respectively. Then, when they transition to problems where they work exclusively with algebraic symbols, like expanding $(x^2 + 3y)^3$, our aim is for them to maintain, x^2 and $3y$, as representing a variable number of possible options rather than treating, x^2 and $3y$, simply as letters. We note that it is possible (not preferable) to apply the combinatorial reasoning shown in Fig. 5 by treating the problem simply as one about making a choice between two kinds of letters, either one chooses x^2 or one chooses $3y$, where the letters do not represent a possible number of options. Thus, the organization of tasks is linked to our goal of working with teachers to develop algebraic symbols that represent a variable number of possible options.

4.3 Data analysis

After the teaching episodes concluded, the research team members independently watched the teaching episodes, took low-inference notes, applied and developed codes, and documented conjectures about the data (Saldana, 2013). At bi-weekly meetings, they discussed and triangulated individual interpretations of data. Once the data had been partitioned using codes, the research team established second order models of the PSSTs' reasoning. A second order model is the constellation of constructs used to account for another person's reasoning (e.g., schemes, reflecting abstraction, etc.) (Steffe et al., 1983). In our case, we examined the data for evidence that Olive had constructed the two schemes in the Conceptual Framework. We identified how she was coordinating these two schemes in different instances of the data, examined the data for changes she made to her schemes, and iteratively proposed and refined our account of her reasoning based on making a consistent and coherent account across the data set.

5 Results

We now present results from Olive's case to illustrate the learning that enabled her to transition from a generalization in the Card Problem to work on symbolic problems.

5.1 Episode 2: Olive's MPS and BCS

During the second teaching episode, Olive solved the "three case" of the Card Problem (two non-face and one face card) without using a 3-D array. During her solution, there was evidence that her MPS was enactive and BCS interiorized. Evidence that her MPS was enactive came from her response to determining the total number of hands. Olive stated that

KKK	222	333
KK2	22K	33K
KK3	223	332
K2K	2KK	3KK
K22	2K2	3K2
K23	2K3	3K3
K3K	23K	32K
K32	232	322
K33	233	323

Fig. 6 Olive's MPS list

there were three possible options for the first, second, and third positions, so the total number of hands was $3 \times 3 \times 3$. She then created a list (Fig. 6), explaining it as follows.

Excerpt 1: Olive's explanation

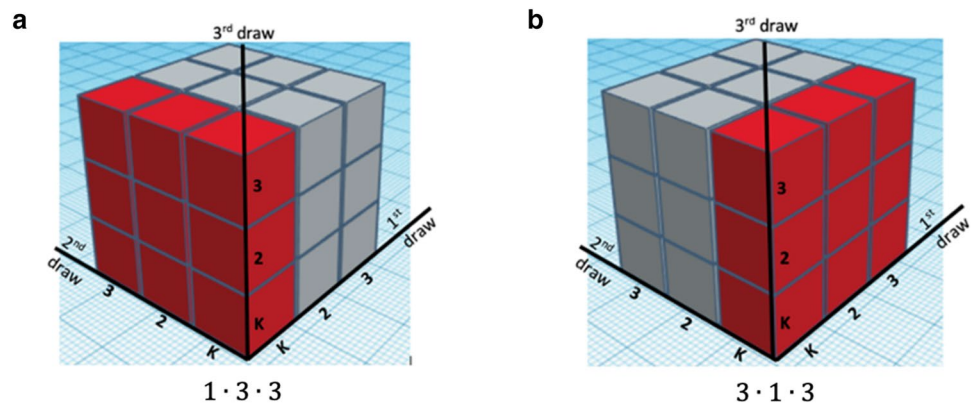
O: As you can see this whole column starts with K [the first column], this whole column starts with a 2 [the second column], and this whole column starts with a 3 [the third column]and then when you look at the second spot [pointing back to the first column], I rotated through....I exhausted all the ones that have king in the second spot, and then all the ones that have 2, and all the ones that have 3. And then I went in the same pattern (in the third spot) king, 2, 3; king, 2, 3; king, 2, 3....I kind of wish [describes how she would reorganize her second and third columns so the ordered triples are in a lexicographic order] So I could look at each column and they would have the exact same order.

Olive's explanation indicated that she connected her multiplication statement to her list; she saw three possible options for each position in her list (e.g., "I went in the same pattern (in the third spot) king, 2, 3"). We take this connection as one key criterion for inferring the construction of an MPS. The second criterion is organizing the list according to

Fig. 7 Olive's BCS list

333
332
323
233
322
232
223
222

Fig. 8 Ordered products corresponding to Kings in **a** first draw and **b** second draw



a lexicographic order. Olive described a lexicographic order in her first column, and then outlined how she would change the second and third columns to have the “exact same (lexicographic) order.” Therefore, establishing a lexicographic order occurred as she looked at her list (i.e., in the context of perceptually present figurative material) rather than as an organizing principle for producing it, which is why we conclude that initially her MPS was enactive.

There was strong evidence of Olive’s construction of an interiorized BCS within the “three case” of the Card Problem. As part of her solution of the “three case,” Olive created a list (Fig. 7) for the number of hands that contained no face cards. When Olive finished her list, she stated, “I got eight, and I’m confident with that answer,” explaining, “I started with all threes. And then all the ways there could be two threes. And then all the ways there could be one three. And then no threes.” We interpret her list as a record of hands that she was creating in visualized imagination and this record was organized according to the number of threes in a hand. Thus, her explanation was evidence she had constructed an interiorized BCS.

5.2 Episodes 3: Ordered products emerge from work on the 3-D array

We conjectured that Olive’s explanation of the lexicographic order in her list (Fig. 6) during episode two was a possible site for her to interiorize her MPS. We tested this conjecture with two problems at the beginning of the third teaching episode. Olive assimilated these using her MPS where she first wrote multiplication statements, and then produced the ordered triples in a lexicographic order. We took that as sufficient evidence to infer she had interiorized her MPS.

At that point, Olive transitioned to solving additional cases of the Card Problem where she represented *all* hands using a 3-D array. To work with her 3-D array, she ordered the units of each composite unit along the axes of the array (king was first, two was second, and three was third) and

ordered the composite units themselves (there was a first draw, a second draw, and a third draw). However, she did not actually pair the king with the king with the king; the king with the king with the two, etc. Thus, the operations of her MPS were evoked, but not fully implemented in working with the 3-D array.²

As part of the PSSTs’ work with the 3-D array, they located subsets of the ordered triples. For example, Olive located all hands with a face card in the first position (Fig. 8a) and Aaron located all hands with a face card in the second position (Fig. 8b). During this time, Aaron proposed that they use ordered products to show in their multiplication statements that subsets of ordered triples were located in different places in the 3-D array. For example, Aaron offered that they should write the multiplication statement $1 \cdot 3 \cdot 3$ for the number of hands with king in the first draw (one possible option for the first draw, three options for the second and third draws) (Fig. 8a) and $3 \cdot 1 \cdot 3$ for the number of hands with king in the second draw (Fig. 8b). Olive agreed, saying “if we are being precise” we would “put the *one* in the correct position.” The 3-D array, then, supported the PSSTs to use ordered products because they wanted the product to indicate the location of a subset of ordered triples. Using ordered products in this way opened a new opportunity for them.

5.3 Episode 4: A new opportunity

During the fourth teaching episode, Olive continued solving cases of the Card Problem. We illustrate how her reasoning developed with the “five case” (one face and four non-face cards). Olive began by selecting a $5 \times 5 \times 5$ 3-D array and organizing the first, second, and third axes in the same way

² Establishing the 3-D array as a representation for *all* ordered triples involves spatial operations, rotation and translation. We do not analyze those operations, here, because this paper is focused on the way Olive coordinated her MPS and BCS.

a $5^3 = 125$ $(1+4)^3 = 125$	b $4^3 = 64$ $16 + 16 + 16 = 48$ $(1 \cdot 1 \cdot 4) \cdot 3$ $1 \cdot 1 \cdot 1 = 1$
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Fig. 9 Olive's expressions to determine **a** the total number of three-card hands and **b** the number of each type of hand

as in Fig. 8a, b. She recorded the total number of hands in two different ways (Fig. 9a). She then located in her 3-D array the zero- and one-face-card regions and wrote notation for them (Fig. 9b, green, blue, respectively). She explained her notation for the one-face-card regions by saying that the multiplication problem would be “four times four.” Aaron had written the notation, $(1 \cdot 4 \cdot 4) \cdot 3$, for the one-face-card regions, explaining his multiplication statement as “you have one option of a face card, call that the first draw, and then the second draw you have four options of a non-face card, and then the third draw you have four options of a non-face card. And then you have to rotate where the face card is,” which gives you three possible ways to have one face card. After hearing Aaron, Olive proceeded to locate the ordered triples in her array with a face card in the first and second positions and a non-face card in the third position. She then recorded notation for *all* ordered triples with two face cards (Fig. 9b, yellow).

Excerpt 2: Olive explains her notation

TR [pointing to Fig. 9b, yellow]: Do you want to explain the multiplication problem?

O: If we want to make sure we for sure have two kings, there is only one option for two of the separate draws. Either the first and second draw both have to be a king, or the first and third both have to be a king, or the second and third both have to be a king. But either way we're multiplying one times one because two of them are fixed, and then times four because there are four more options for it to not be a king for the third draw. I could have expanded it out to show all of the different ways, but I know they are all going to be equivalent, so one times one times four. So I just did times three.

TR: [asks O and A to locate the three-face-card region. O does and records the notation in Fig. 9b, red. TR asks if she can make an equivalence between Fig. 9a and Fig. 9b. Instead, Olive writes the following equivalence

lence $5^3 = (1 + 4)^3 = 125$. She then looks at Fig. 9b and says she cannot remember what 4^3 represented.]

When Olive was done locating and writing notation for the zero-, one-, two-, and three-face-card regions in her array (Fig. 9b), she did not equate the sum of these regions with 5^3 or $(1 + 4)^3$. We take that as evidence that she began her solution of the “five case” by *sequentially* using her MPS, meaning that her scheme closed after each time she found a particular kind of hand. Her goal was simply to locate a specific kind of hand and record a multiplication problem for it. Doing so was not situated within a broader goal of finding hands with zero, one, two, and three face cards as another way to produce the total number of hands. Thus, she used her MPS, first to determine the total number of ordered triples (i.e., 5^3), then to determine the number of ordered triples with zero face cards, then the number of ordered triples with one face card.

Olive introduced a novelty into her way of operating when she located the ordered triples with two face cards. She located one region in her 3-D array where there were ordered triples with two face cards, identifying the ordered product as $1 \times 1 \times 4$. Having done so, she explained, “one option for two of the separate draws...four more options for it to not be a king for the third draw.” She then took the numerals 1, 1, and 4, using the position of the numerals to indicate the three different ways she could get two kings in a hand. We infer that moving the numerals' positions evoked the operations she used to locate ordered triples in her 3-D array, but that she only implemented those operations when she found the region with a face card in the first and second positions. Her ordered product rather than the array, then, became the figurative material on which she was operating. We consider her operating on the numerals as evidence that she *inserted* her MPS into a *portion* of her BCS (i.e., the portion of her BCS related to hands that included two face cards). We note that she operated on numerals that represented possible options rather than actual cards in making this insertion. Thus, when she inserted her MPS into her BCS the material she operated on was different than it had been during the second teaching episode when she was imagining cards (see Fig. 6).

When the teacher-researcher asked her to write an equivalence between the non-case breakdown of the problem and the case breakdown of the problem (Fig. 9a, b), Olive could not remember which region 4^3 represented. She continued to say, “I don't remember,” indicating she was experiencing a perturbation. The teacher-researcher encouraged her to figure it out. She spent about six minutes producing notation (Fig. 10) without locating the regions on her array.

From her notation, we infer that Olive inserted her MPS into her BCS. She implemented even fewer of the operations of her MPS than she had when locating ordered triples in

Handwritten work showing the breakdown of 4^3 into cases based on the number of face cards (0, 1, 2, 3). The work is organized into four rows, each representing a case:

- Row 1: $4^3 = 4 \cdot 4 \cdot 4 = 64$ (Total). To the right, a box contains "Non face" and another box contains "110 King".
- Row 2: $(1 \cdot 4 \cdot 4) + (4 \cdot 1 \cdot 4) + (4 \cdot 4 \cdot 1) = (1 \cdot 4 \cdot 4) \cdot 3 = 48$ (1 King). The term $(1 \cdot 4 \cdot 4)$ is underlined in blue.
- Row 3: $(1 \cdot 1 \cdot 4) + (1 \cdot 4 \cdot 1) + (4 \cdot 1 \cdot 1) = (1 \cdot 1 \cdot 4) \cdot 3 = 12$ (2 King). The term $(1 \cdot 1 \cdot 4)$ is underlined in yellow.
- Row 4: $(1 \cdot 1 \cdot 1) = 1$ (3 King). The term $(1 \cdot 1 \cdot 1)$ is underlined in red.

Fig. 10 Olive's revised notation for the "five case"

her array, while fully implementing the operations of her BCS (cf. Excerpt 2). We infer the only operation she implemented of her MPS was ordering the positions in a three-part product. Having done so, Olive took the number of possible options for non-face and face cards in a three-part product as the figurative material on which she fully implemented her BCS. We make the inference that she fully implemented her BCS based on her notation: selecting zero 1's (Fig. 10, green); selecting one 1 for either the first, second, or third position (Fig. 10, blue); selecting two 1's for either the first and second, first and third, or second and third positions (Fig. 10, yellow); selecting three 1's (Fig. 10, red). We infer that this coordination of schemes allowed her to keep track of which *ordered products* she had created until she produced all eight of them. We note that inserting her MPS into her BCS meant the result of her BCS was eight *ordered products* (e.g., $1 \cdot 4 \cdot 4$, $4 \cdot 1 \cdot 4$) not eight *ordered triples* (e.g., Fig. 7 where she listed three card hands like 333).

We leave open whether the insertion of her MPS into her BCS was enactive or interiorized. The figurative material that she carried this insertion out on was numerals, which she certainly could carry out in visualized imagination (i.e., leading to the conclusion that it was interiorized). However, she had relied extensively on the physical 3-D array to determine the ordered products. Therefore, it was unclear how tied her work was to the physical 3-D array; if it was tied to the physical 3-D array then we would conclude the insertion was enactive.

After she finished writing her notation, the following interaction took place.

Excerpt 3: Olive continues work on the "five case"

TR: We have ones with no kings, one king, two kings, and three kings [points to Fig. 10]. How much is that total?

O: All of this [points to Fig. 10.] Sixty-four plus forty-eight plus twelve plus one, I think it is probably one (hundred) twenty-five.

TR: Why do you think 'it is probably' that?

O: Because there is no other option. Either you have no king, one king, two king, or three king. There is

no other alternative. So that should encompass all the total options. You could add it up to check them.

TR: Do you feel compelled to add them up?

O: No, I'm confident.

Olive verbally equated the case and non-case breakdowns, saying that the sum of her case breakdown was "probably one (hundred) twenty-five". We take Olive's subsequent justification, coupled with her confidence, to indicate an expectation of equivalence, rather than an uncertainty. We infer from this data that her goal changed. Her goal was now to use the case break down to determine the total number of hands, which differs from her earlier goal to determine the total number of hands within a single case (e.g., how many hands have no face cards?). We attribute this change in goal to her new coordination of schemes, inserting her MPS into her BCS.

We further infer that she had two distinct ways to produce all ordered triples. The first was using her MPS, which she symbolized either as 5^3 or $(1 + 4)^3$. The second was by inserting her MPS into her BCS. We infer that these two ways of operating occurred *sequentially*—she could determine the total number of ordered triples was 5^3 or $(1 + 4)^3$ using her MPS, then that scheme closed. She then had a second way to determine the total number of ordered triples, which was to insert her MPS into her BCS to produce eight ordered products, which produced all hands because "there is no other alternative." We make the inference that these two ways of determining the total occurred sequentially for two reasons. First, there was an extended period of time between when she produced her notation for each, indicating that they were likely two distinct ways of producing the total number of ordered triples. Second, in stating the equivalence, she focused exclusively on the fact that she had produced a second way to count the total number of hands, the case break down. We see this explanation as indication that she was focused on two distinct ways of counting the hands.

We could attribute these ways of operating to Olive during the third and fourth teaching episodes. During this time, she continued to refine and condense her notation. For example, she solved a problem with a draw pile containing two face cards and three non-face cards, working with the 3-D array and producing notation (Fig. 11a). At the end of the fourth episode, she wrote the general statement (Fig. 11b) where we infer $(f + n)^3$ symbolized her first way of counting all possible ordered triples using her MPS and $f^3 + 3(n \cdot f^2) + 3(n^2 \cdot f) + n^3$ symbolized her second way of counting all possible ordered triples where she inserted her MPS into her BCS. We note that she defined f as the *number* of face cards and n as the *number* of non-face cards, indicating she was treating the letters as whole number variables *not* simply as letters. However, her choice of which letters to use as whole number variables (f and n) were still

Fig. 11 Olive's **a** quantitative equivalence and **b** general statement

a

$$\begin{aligned} \text{total options} \\ 5^3 &= (2+3)^3 = \boxed{125} = 3^3 + 3(2 \cdot 3 \cdot 3) + 3(2 \cdot 2 \cdot 3) + 2^3 \\ &= 3^3 + 3(2 \cdot 3^2) + 3(2^2 \cdot 3) + 2^3 \end{aligned}$$

b

$$x^3 = (f+n)^3 = f^3 + 3(n \cdot f^2) + 3(n^2 \cdot f) + n^3$$

closely connected to the Card Problem context, a hallmark of symbolizing regularities and constraints of contextualized problems.

5.4 Episode 5 and 6: A surprise

At the end of the fourth teaching episode, the teacher-researcher anticipated that Olive was ready to begin the process of transitioning from the generalization she made in the Card Problem to solving problems intended to expand the values that she considered for her variable (i.e., so that the variable could take on integer and rational number values). However, the fifth and sixth teaching episodes brought a surprise: Olive showed she was not ready for this work. We use a data excerpt from the sixth teaching episode to illustrate this surprise. The teacher-researcher asked Olive to recall the generalization she had symbolized or cases of it (e.g., Fig. 11a, b).

Excerpt 4: Olive begins the sixth teaching episode

O: Um, can you remind me what you mean by the identity because it's not, I don't know what you are referring to?

TR: Yeah, ok. So we did things like finding what five cubed was. And how did we find what five cubed was? What were we doing?

O: Five times five times five.

TR: That's one way. But then we also found another way.

O: Were we like expanding it?

TR: Mmhmm (yes).

....

O: Sorry I just really don't remember.

....

O [The TR begins to talk about a "five card" case of the Card Problem with two face and three non-face cards. O makes a first attempt at writing a symbolic statement, writing $5^3 = (2+3)^3 = 2^3 + (2^2 + 3^1) + (2^1 + 3^2) + 3^3$. She expresses uncertainty about it, so the TR

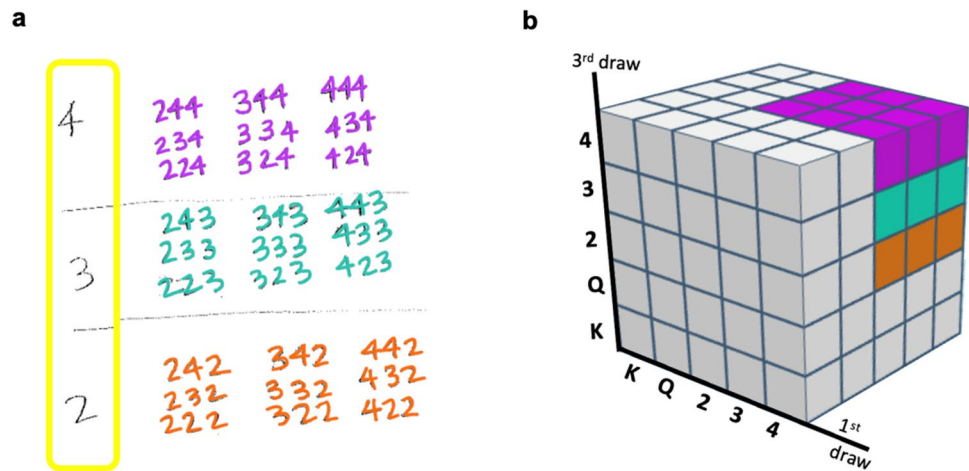
asks O to think further about it using a $5 \times 5 \times 5$ array. O locates the hands with three face cards, then the hands with face cards in the first and second positions, and finally the hands with face cards in the second and third positions. After locating these three regions, she stops and writes, $5^3 = (2+3)^3 = 2^3 + 3(2^2 \cdot 3^1) + 3(2^1 \cdot 3^2) + 3^3$.

We interpret this data excerpt as showing that Olive's insertion of her MPS into her BCS was enactive; she still needed the physical 3-D array to make the insertion. That is, we infer that locating the three regions on her array, and their associated ordered products, entailed Olive inserting her MPS into a portion of her BCS. Once she had located three regions and produced the first three ordered products, she was able to correct her symbolic statement without implementing the remainder of the insertion of her MPS into her BCS. Nonetheless, producing her equivalence, $5^3 = (2+3)^3 = 2^3 + 3(2^2 \cdot 3^1) + 3(2^1 \cdot 3^2) + 3^3$, was contingent upon having the physical 3-D array present. For this reason, we infer that she could not operate solely on the numeric symbols themselves to call forth this coordination.

To address this issue, the teacher-researcher made an on-the-spot conjecture that he would ask Olive to interpret each of her notational statements: 2^3 , $3(2^2 \cdot 3^1)$, $3(2^1 \cdot 3^2)$, and 3^3 . In particular, he pressed her to find a way to show all of the *hands* for each notational statement. He anticipated that this would necessitate Olive implementing more fully the operations of her MPS in the context of inserting it into her BCS. Olive spent 2.5 min creating Fig. 12a for her notation 3^3 , explaining, "I visualized the axes (of my 3-D array). Like the first choice, second choice, and third choice axes [motioning to show the direction of each axis]." She indicated the large numbers (Fig. 12a, yellow) represented the card in the third draw and, for example, all the ordered triples with 4 in the third draw (Fig. 12a, purple) represented a "slice" in her 3-D array (Fig. 12b, color-coordinated with Fig. 12a).

In responding to the teacher-researcher's request, Olive transitioned from working with a physical 3-D array to

Fig. 12 Olive **a** records and **b** imagines three-card hands with no face cards



working on a 3-D array in visualized imagination. Thus, we interpret Fig. 12a as a record of the hands with no face cards in a region of an array that she produced in visualized imagined (Fig. 12b). While the teacher-researcher anticipated that she would need to implement the operations of her MPS more fully while inserting it into her BCS, he did not anticipate that she would necessarily carry out these operations on an array in visualized imagination. Because she was operating on an array in visualized imagination, we take producing Fig. 12a as initial indication of interiorizing her insertion of her MPS into her BCS; that is, she took Fig. 12b (colored slices altogether), as the first of eight possible ordered products that she would produce. An alternative explanation would be that Olive simply used her MPS to produce these ordered triples (i.e., that she did not insert her MPS into her BCS). Because there is only one way to have no face cards, it is difficult to know from this data alone if her BCS was activated. To examine this issue, we present data from how she interpreted the request to show all the hands for her notational statement $3(2^2 \cdot 3^1)$ (i.e., hands with two face cards).

Excerpt 5: Olive interprets $3(2^2 \cdot 3^1)$

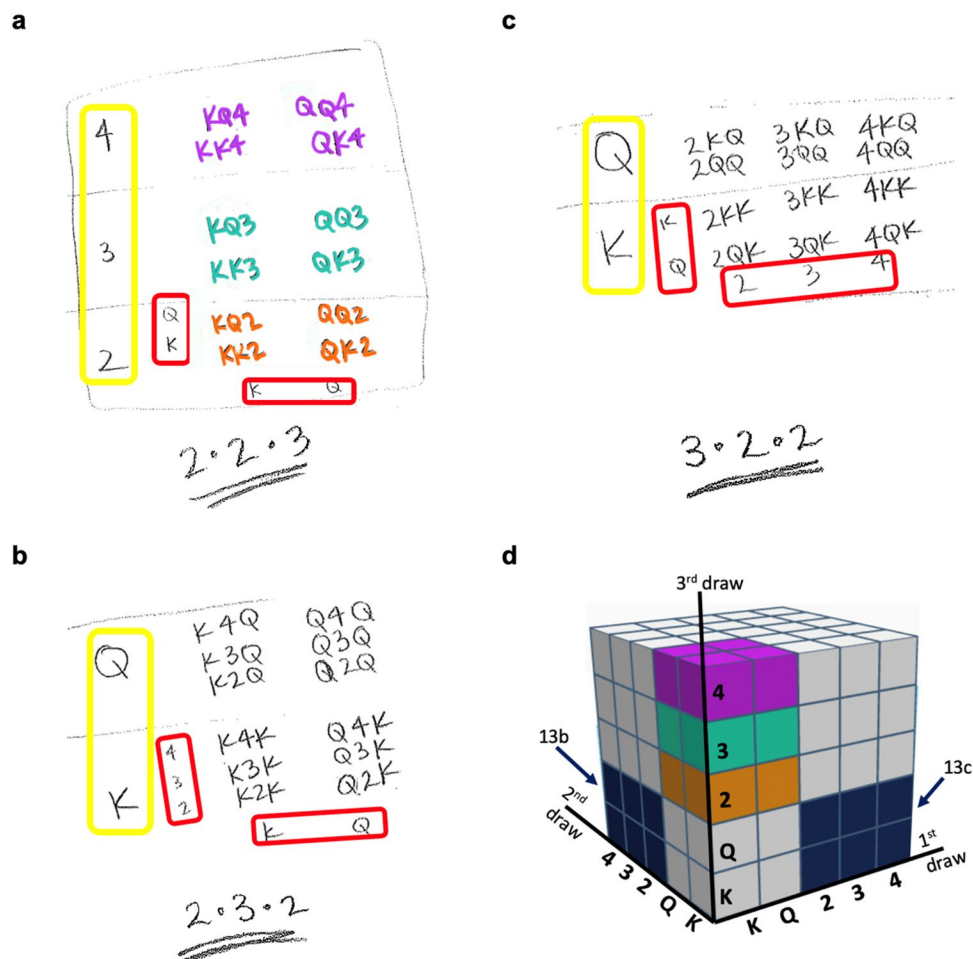
O [makes two unsuccessful attempts at recording the hands with two face cards. The attempts make apparent she is confusing the two different meanings of 'three' in the notational statement, $3(2^2 \cdot 3^1)$ where the first three means the three different ways to have two face cards and the second three means the three possible non-face card options. She then makes Fig. 13a–c without a physical array. Her explanation, however, indicates she is imagining one. She relates Fig. 13a–c to Fig. 12a and her imagined array, explaining]: For this one [points to Fig. 12a] I only had to make one because when I'm thinking of a 3-D axis, and when we look at like this [grabs the physical array] this is all in one section so that's why there is only one. But

this [points to Fig. 13a–c] is three different sections of this whole (array). So you have to, if I did it the same way, I had to do three different diagrams. So, this is the one [points to Fig. 13a, yellow] where my third axis or the last spot in my sequence is the non-face card, and there are three options. [O continues to explain that she is imagining the face card options on the first and second axes of her 3-D array (Fig. 13a, red). She then says the ordered triples in Fig. 13a are slices in her array that she is imagining (Fig. 13d, color coded to match Fig. 13a). She makes similar explanations for Fig. 13b, c.]

We consider this data excerpt to be confirmation that Olive was inserting her MPS into her BCS, while fully implementing the operations of her MPS to produce ordered triples in a lexicographic order. Given Olive's two unsuccessful attempts to make Fig. 13a where it was evident she was conflating the two meanings of three in her notation, $3(2^2 \cdot 3^1)$, we infer that fully implementing her MPS in the context of inserting it into her BCS was occasion for her to reflect on how she was coordinating these two schemes. Like Fig. 12a, Fig. 13a–c were a record of Olive operating on an array that she imagined rather than her operating on a physical array. Therefore, we take this data excerpt as indication that Olive interiorized the insertion of her MPS into her BCS; she was now working only on an array in visualized imagination.

Beyond Olive's interiorization of this coordination, we make an additional inference about Olive's coordination of schemes in Excerpt 5 that is different from how we interpreted Olive's coordination of schemes in Excerpt 3. To support making this inference, we first point out that, as Olive created Fig. 13a–c, she seemed to maintain that each axis in her array had five options and appropriately selected either two or three options from each axis. We draw this conclusion based on comparing across Fig. 13a–c where five choices are

Fig. 13 Olive a–c records ordered triples from an imagined 3-D array



represented along each axis. For example, Fig. 13a (yellow) has the 2, 3, 4 as choices for the third axis while Fig. 13b (yellow) has K, Q as the choices on her third axis. A similar comparison across Fig. 13a–c shows that all five choices are represented for the first and second axis as well (Fig. 13a–c, red). We take the fact that Olive maintained all five possibilities on each axis as *initial* evidence for the inference that she inserted the case breakdown into her way of counting the total number of hands without the case breakdown. Thus, her way of counting the case breakdown, symbolized as $2^3 + 3(2^2 \cdot 3^1) + 3(2^1 \cdot 3^2) + 3^3$, was contained inside her way of counting the non-case break down, symbolized as 5^3 or $(2 + 3)^3$. This insertion meant that she could use the number of possible face options, 2, and number of possible non-face options, 3, *either* to generate the total number of three card hands without the case breakdown (i.e., 5 possible options that could be used to produce, 5^3 hands) *or* to produce the case breakdown through an appropriate selection of the number of face card options or number of non-face card options.

This inference differs from the explanation we gave in Excerpt 3 when Olive solved the “five case” with one face

card and four non-face cards. In that case, we interpreted Olive as having two *sequential* ways of counting the total number of hands. The first was a result of her MPS which she expressed as 5^3 or $(1 + 4)^3$ and the second was the case breakdown that involved inserting her MPS into her BCS (see Fig. 10 for her notation). They were equivalent because both counted *all* of the hands. However, they were still two distinct ways to count all of the hands. We explain Excerpt 5 in terms of schemes in the following way: Olive inserted into her MPS (i.e., her way of counting the total number of hands not using the case breakdown) the insertion of her MPS into her BCS (i.e., her way of counting the case breakdown). Doing so was a recursive insertion of her schemes where the case and non-case ways of counting hands were contained in a singular scheme. We consider this recursive insertion to be a functional accommodation where we call the single scheme it produced a *combinatorial scheme for binomial expansion* (CSBE). We infer this scheme was interiorized given that Olive operated on an array in visualized imagination.

We note that Olive’s recursive insertion of schemes, and construction of a CSBE, meant that the equivalence (i.e.,

Fig. 14 Olive **a** expands $(x^2 + 3y)^3$ and **b** records corresponding ordered products

a

$$(x^2 + 3y)^3 = (x^2)^3 + 3(x^2)^2(3y) + 3(x^2)(3y)^2 + (3y)^3$$

b

$$x^2 \cdot x^2 \cdot x^2 \quad x^2 \cdot 3y \cdot x^2 \quad 3y \cdot x^2 \cdot x^2 \quad x^2 \cdot 3y \cdot 3y \quad 3y \cdot x^2 \cdot 3y \quad 3y \cdot 3y \cdot x^2 \quad 3y \cdot 3y \cdot 3y$$

$$(x^2 + 3y)(x^2 + 3y)(x^2 + 3y)$$

Fig. 15 Olive writes $(x^2 + 3y)^3$ as the product of three binomials

$(2 + 3)^3 = 2^3 + 3(2^2 \cdot 3^1) + 3(2^1 \cdot 3^2) + 3^3$ would no longer grow out of having two different ways of counting the total number of hands, as it had when she sequentially used her MPS to find the total number of hands, and then inserted her MPS into her BCS to find the total number of hands through the case breakdown. Instead, she would be able to use an appropriate process of selecting options from $(2 + 3)^3$ to produce $2^3 + 3(2^2 \cdot 3^1) + 3(2^1 \cdot 3^2) + 3^3$. We look for supporting evidence of Olive's construction of a CSBE from later in the sixth teaching episode.

5.5 Episodes 6: Olive's CSBE

The teacher-researcher asked the PSSTs to consider what the expansion of $(x^2 + 3y)^3$ would be. Olive wrote Fig. 14a. The teacher-researcher asked her how she would explain her equivalence. She wrote the ordered products in Fig. 14b, explaining the different ways to arrange the terms. Aaron gave a similar justification, but then asked how the binomials to be expanded, i.e., $(x^2 + 3y)^3$, were connected to the ordered products (Fig. 14b). Olive responded to Aaron's question.

Excerpt 6: Olive explains how $(x^2 + 3y)^3$ can be used to generate Fig. 14b

O: Um, I'm thinking of when you write out, like x squared, like this [writes Fig. 15].

Cause, I don't know, when you think of FOIL, you know—like, when you think of FOIL, first [covers up the third binomial and only focuses on the first two binomials], you like multiply this [points to the x^2 in the first binomial] by the first one [points to x^2 in the

second binomial] and then you multiply it [x^2 from the first binomial] by the second one [$3y$ in the second binomial]. And so like, the order in which you multiply them changes like—you have to make sure that you account for each one. [Olives attempts twice to make a description of how she would apply this thinking to all three binomials. She loses track of which partial products she has produced, saying] It (FOIL) doesn't make sense when I'm looking at all (three) of them (binomials) at once.

....

O [TR encourages her to continue. O seems to have an insight and says excitedly] Yeah, yeah, ok! And then, yeah, so you have to multiply this [x^2 in the first binomial] by each thing [gestures across the second and third binomials]. And so you have to—it's like drawing cards! Like you're—it's like drawing cards, you—you start with this one [x^2 in first binomial], this one [x^2 in second binomial, and this one [x^2 in third binomial]. Then you have to do this one [x^2 in first binomial], this one [x^2 in second binomial], and this one [$3y$ in third binomial. Continues her explanation about how appropriate selection from the binomials produces each term of the polynomial.] So you have to make sure you multiply—you're taking care of all the terms multiplied by every single term. So that is why there's three ways to do x squared x squared three y . [O exclaims] When I was in high school, I wouldn't have realized that. That was fun! That was worth really struggling.

We consider this excerpt to be confirmation of Olive's construction of a CSBE because she *connected* the initial binomials (Fig. 15) to the ordered products they produced (Fig. 14b), indicating that she was no longer considering her equivalence as standing for two distinct ways of counting the same set of ordered triples. Had that been the case, she would have seen the expression, $(x^2 + 3y)^3$, as similar to her

Fig. 16 **a** Aaron's and **b** Olive's equivalences for $(x+1)(x+2)(x+3)$

a

$$(x+1)(x+2)(x+3) = x^3 + [(x \cdot x \cdot 3) + (x \cdot 2 \cdot x) + (1 \cdot x \cdot x)] + [(x \cdot 2 \cdot 3) + (1 \cdot x \cdot 3) + (1 \cdot 2 \cdot x)] + 1 \cdot 2 \cdot 3$$

b

$$\begin{aligned} (x+1)(x+2)(x+3) &= x^3 + x^2 \cdot 3 + x^2 \cdot 2 + x^2 \cdot 1 + x \cdot 2 \cdot 3 + x \cdot 2 \cdot 1 + x \cdot 3 \cdot 1 + 1 \cdot 2 \cdot 3 \\ &= x^3 + 3x^2 + 2x^2 + x^2 + 6x + 2x + 3x + 6 \\ &= x^3 + 6x^2 + 11x + 6 \end{aligned}$$

first way of finding the total number of hands and the expression $(x^2)^3 + 3((x^2)^2 \cdot (3y)^1) + 3((x^2)^1 \cdot (3y)^2) + (3y)^3$ as similar to her second way of counting the total number of hands, but she would have been unable to produce one way of counting from the other. Instead, she was able to use the binomials to produce the ordered products. Thus, we take this excerpt as confirmation of her recursive insertion of her schemes (i.e., her construction of a CSBE), which characterizes the *projective* aspect of reflecting abstraction.

This excerpt also illustrates Olive's *abstraction* of the operations of her CSBE, and application of them to novel figurative material in the form of algebraic symbols. We infer from Fig. 15 that she ordered the binomials as a first, a second, and a third binomial, which was similar to how she operated in the Card Problem ordering the draws. She, then, tried using two binomial to explain why there would be different arrangements of the ordered products, stating that she was thinking about FOIL. Thinking about "FOIL", however, did not help her explain the product of three binomials. She got lost twice, and then stated, "It doesn't make sense when I'm looking at all (three) of them (binomials) at once." We take this as indication that she had entered a state of perturbation. She resolved her perturbation on the spot by envisioning x^2 and $3y$ as the possible number of face or non-face card options in the Card Problem. She then produced all of the possible ways to select from these two options, differentiating between when she selected a term from the first, second, or third binomial in order to produce all of the possible partial products that made up the final polynomial. We consider this excerpt as evidence that she abstracted the operations from the Card Problem context and used them to re-interpret how she thought about binomial expansion.

We look to teaching episode seven for evidence of how her CSBE supported her to *reorganize* her extant schemes for binomial expansion, the third component of reflecting abstraction.

5.6 Episodes 7: Olive's reorganization

The seventh teaching episode began with Olive explaining two symbolic problems involving cubing binomials similar to the one in Excerpt 6. She suggested that she could generalize her reasoning to $(a+b)^z$ where z represented a variable number of draws. She and Aaron spent much of the seventh teaching episode justifying the $z = 4$ and 5 cases. At the end of the seventh teaching episode, Olive solved two problems involving binomial expansion of non-identical binomial factors, for example, of the form, $(x+a)(x+b)(x+c)$, where x is a variable; a, b, c are parameters; and $a \neq b \neq c$. Her solution of these problems showed the power of her CSBE in supporting her to operate on problems involving conventional algebraic symbols and to develop novel algebraic structure.

Excerpt 7: Olive's expansion of non-symmetric binomials

TR [Asks the PSSTs to consider how they could expand $(x+1)(x+2)(x+3)$. A writes Fig. 16a; O writes Fig. 16b]: What's hiding in this last expression [circles where O has written $x^3 + 6x^2 + 11x + 6$]?

O: It doesn't show you what order you are picking the three different ways you can get two x 's. And then the three different ways you can get one x . It's just hiding that. Like that fixing process.

Fig. 17 Olive's equivalence for $(x+a)(x+b)(x+c)$

$$(x+a)(x+b)(x+c) = x^3 + (a+b+c)x^2 + (ab+ac+bc)x + (abc)$$

....

O: And also the parentheses [referring to A's expression (Fig. 16a)] shows you, kind of like, each choice. It is separating like 'this is how you can get two x 's one way [points to $x \cdot x \cdot 3$]. This is how you can get two x 's another way [points to $x \cdot 2 \cdot x$]. This is how you can get two x 's another way [points to $1 \cdot x \cdot x$].

....

TR: So we lost something [referring to the expression $x^3 + 6x^2 + 11x + 6$]. Is there some way you could write that coefficient [pointing to the 6 in $6x^2$] and still show some relationship to this [points to the initial binomials]?

O: Yeah [writes $x^3 + (3 + 2 + 1)x^2 + (2 \cdot 3 + 2 \cdot 1 + 3 \cdot 1)x + 1 \cdot 2 \cdot 3$].

....

TR [Asks the PSSTs to determine $(x+a)(x+b)(x+c)$.

O writes Fig. 17]: Did you guys know that that was true?

O & A: No.

O: Nope! That was fun!

TR: Does what we are doing feel different from "FOILing" to you?

....

O: Yeah. So when I'm multiplying, I don't normally think about taking out the three and the two and the one and adding them together and multiplying that by the x squared. I kind of think of just multiplying three things together over and over again. So this kind of makes it appear simpler for sure because there is this many times [pointing to the coefficient $(a+b+c)$ in Fig. 17], yeah, this many times I'm going to have an x squared. Cause you're doing x times x times three, and you're also doing x times two times x , and you're also doing one times x times x . You're kind of like simplifying. Instead of doing the x times x three different times you're seeing, you're looking 'I'm going to have it three times' and this (referring to $(a+b+c)$) is what I'm going to be multiplying it by.

We consider this data excerpt to illustrate how Olive was reorganizing her extant schemes in relation to her CSBE. Her CSBE allowed her to interpret binomial expansion as

more than "just multiplying three things together over and over again," which is how she had thought about it before. Instead, she saw a common structure that would be involved in expanding *any* three binomials—namely, that there would be one way to get an x^3 -term, three ways to get an x^2 -term, three ways to get an x -term, and one way to get a constant term. We infer that this common structure is what allowed her to move immediately from a specific case of a cubic expansion, $(x+1)(x+2)(x+3)$, to the general case, $(x+a)(x+b)(x+c)$.

Her work on these two cubic expansions also involved her in producing a *new* algebraic structure. The new structure was that the coefficients for each term of the polynomial had a specific relationship to the constant terms of the binomials ("and this [referring to $(a+b+c)$] is what I'm going to be multiplying it [the x^2 -term] by"). This new structure made binomial expansion "appear simpler"; we infer that it appeared simpler because there was the same structure for any given set of three binomials that allowed her to move more directly to a final polynomial. Her work here involved producing this new algebraic structure in the context of working within a conventional algebraic symbol system.

6 Discussion

6.1 Three planes of learning

Olive's case helps us to identify three "planes of learning" that are important in the construction of a CSBE and, in turn, helps us to operationalize the projective aspect of reflecting abstraction. The first plane of learning entails the plane at which a person can use their MPS and their BCS to solve combinatorics problems. We identified that Olive had entered the teaching experiment having an enactive MPS (that quickly became interiorized) and an interiorized BCS. It is possible to solve the Card Problem (or similar problems) in two different ways at this first plane of learning. The first way is that a person may *sequentially* use their MPS to find each specific kind of hand where their MPS closes after they find each kind. From the problem solver's perspective, their goal of finding how many hands, for example, have no face cards is not situated in a broader

	a	b
Exactly three 1's	1 1 1	$1 \times 1 \times 1$
Exactly two 1's	1 1 4 1 4 1 4 1 1	$1 \times 1 \times 4$ $1 \times 4 \times 1$ $4 \times 1 \times 1$
Exactly one 1	1 4 4 4 1 4 4 4 1	$1 \times 4 \times 4$ $4 \times 1 \times 4$ $4 \times 4 \times 1$
No 1's	4 4 4	$4 \times 4 \times 4$

Fig. 18 Three choices between 1 and 4: **a** as ordered triples, **b** as ordered products

goal that the case breakdown will produce *all* of the possible hands. Olive operated in this way in her first solution of the “five case” of the Card Problem with one face and four non-face cards. The second way is that a person may only use their BCS. This kind of reasoning in the “five case” of the Card Problem would amount to treating one possible option and four possible options simply as digits 1 and 4, where the possible outcomes are ordered triples (Fig. 18a) rather than products of possible options (Fig. 18b). We consider it unlikely for a person to operate in this way in the context of the Card Problem where a researcher can easily problematize this meaning by, for example, asking a person to list hands symbolized by the digits “1 4 4”. However, our studies have shown that this way of operating is common when teachers start with or move to symbolic tasks in algebra. In these tasks, some teachers interpret the variables simply as letters (e.g., the letters x and y). In the context of symbolic algebra tasks, it is difficult to problematize this meaning because it is not possible to represent the set of ordered triples that is symbolized by, for example, xyy .

The second plane of learning is characterized by having two sequential ways of counting all possible ordered triples. First, a person uses their MPS to determine the total number of ordered triples; for example, 5^3 ordered triples in the “five case” of the Card Problem. With the goal of their MPS satisfied, this scheme closes before they initiate the second way of counting all possible ordered triples. This second way involves a person inserting their MPS into their BCS to count the zero-, one-, two-, and three-face-card hands where the goal of this insertion arises from the person’s establishment of the case breakdown as a second way to count all possible ordered triples. We emphasize that the two ways of counting (the case and non-case break down) are done *sequentially*. Thus, the equivalence that arises is derived from having two distinct ways of counting the same ordered triples. Olive engaged in this kind of reasoning when she

produced 8 *ordered products* in the “five case” (one face and four non-face) and equated them with her way of counting all of the ordered triples for the non-case break down.

The third plane of learning is characterized by a recursive insertion of schemes; a person inserts into their MPS (the non-case breakdown) the insertion of their BCS into their MPS (the case breakdown), which yields the construction of a CSBE. At this plane of learning, the case breakdown is no longer a distinct way of counting ordered triples. Instead, it is contained within the non-case breakdown of the situation. A marker of this kind of reasoning is maintaining the binary choice as a sum of the total number of possible options. For example, Olive maintained five as composed of two options and three options, and used these options to produce all of the possible ordered products. For Olive, one consequence of constructing a CSBE was that she could see a connection between the binomials she started with (e.g., $(x^2 + 3y)^3$) and the final polynomial she produced (e.g., $(x^2)^3 + 3((x^2)^2 \cdot (3y)^1) + 3((x^2)^1 \cdot (3y)^2) + (3y)^3$). This consequence was profound: she reorganized how she thought about binomial expansion in the cubic, quartic, quintic, etc. cases and in non-symmetric binomial multiplication problems where she could see what she considered a novel algebraic structure.

6.2 Contributions and future directions

We propose two inter-related contributions of this study. First, we have illustrated how developing combinatorial reasoning was instrumental in supporting Olive’s algebraic reasoning. Once she had constructed a CSBE, she established the equivalence $(x + a)(x + b)(x + c) = x^3 + (a + b + c)x^2 + (ab + ac + bc)x + abc$ with ease. Olive interpreted this equivalence as a statement about a relationship between the constant terms of binomials and the coefficients of a polynomial. We consider this relationship to be a significant step towards seeing how the roots and coefficients of a polynomial are connected (i.e., Viète’s formulas). Establishing this connection *could be* a powerful basis for the study of polynomials and their roots at the secondary level. However, in the U.S. context, textbooks tend to emphasize sequential distribution for binomial expansion, which in Olive’s words involves, “multiplying three things together over and over again,” rather than emphasizing combinatorial reasoning that can support seeing how the roots and coefficients of a polynomial are connected (Burch et al., 2021). We see this contribution as opening a conversation with international audiences to compare the ways in which discrete mathematics has or has not been integrated into extant curricular materials to support students study of polynomials and their roots. We consider this comparison to be an important area for future research.

The second contribution of this study is that we have provided a way to operationalize how teachers and students make the transition from Core Aspect A to B of Kaput's (2008/2017) framework vis-à-vis Piaget's reflecting abstraction. Kieran (2007) noted the centrality of Core Aspect B to secondary school algebra (what she calls the transformational aspect of algebra), while also identifying a limited number of studies that have investigated how students or teachers engage in this work. Dörfler (2008) echoed this observation suggesting that studies on, what he called descriptive algebra (similar to Core Aspect A), were only half of the story, and that further work was needed on operative aspects of algebra (similar to Core Aspect B). Reflecting abstraction provides a tool to study how these two Core Aspects of algebra might be connected so that students and teachers can use their work in Core Aspect A to transition to powerful work related to Core Aspect B. One important question for further research in this arena is when and how to move back and forth between these two Core Aspects so that the two are mutually supportive of each other.

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