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Two students' mathematical thinking and activity across representational registers in a combinatorial setting

Elise Lockwood[1] · Amy B. Ellis[2]

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Abstract
In recent decades, there has been considerable research that explores the teaching and learning of combinatorics. Such work has highlighted the fact that understanding and justifying combinatorial formulas can be challenging for students, and there is a need to identify ways to support students' combinatorial reasoning. In this paper, we contribute to research that explores effective ways to foster students' combinatorial reasoning by highlighting the role that shifts in representational registers can play in supporting students' combinatorial reasoning and activity. In particular, we present a case in which two students, aged 12 and 14, reasoned about, and developed a formula for, binomial coefficients. This occurred in a teaching experiment designed to examine their generalizing activity. In these interviews, the students first solved problems about determining the volume of a cube and then shifted to a combinatorial interpretation of this task, leveraging binomial coefficients as a way to consider growth in higher dimensions. The students demonstrated sophisticated reasoning about combinatorial tasks and were ultimately able to generate and understand a formula for binomial coefficients. In framing this case, we focus on the students' use of different representations, highlighting the power of being able to transition between representational registers as a way of supporting students' combinatorial reasoning. In this way, our case demonstrates the value of representational registers specifically as a mechanism by which to potentially improve the teaching and learning of combinatorics.

Keywords Combinatorics · Representation · Binomial coefficients · Discrete mathematics

1 Introduction and motivation

Counting problems are important yet are known to be difficult for students (e.g., Batanero et al., 1997), and there is a perennial need to investigate ways to foster rich combinatorial thinking for students. We are motivated generally by a desire to improve the teaching and learning of combinatorics by improving students' success in solving counting problems and understanding combinatorial topics. In this paper, we present a case study in which two secondary students transitioned from an algebraic to a discrete combinatorial setting as they reasoned about and developed a general formula for binomial coefficients. To do this, the students moved between different representational registers (Duval, 2006), which was an explicit aspect of our design that we used to facilitate and support the students' combinatorial reasoning. Our goal in this paper, then, is to demonstrate a case in which shifting across representational registers supported secondary students in successfully developing a formula for binomial coefficients in a relatively short amount of time.

In a recent commentary that made a case for research on combinatorics education, Lockwood et al. (2020) encouraged developments in the field of combinatorics education, articulating a need for more evidence-based research that highlights ways to support students' combinatorial reasoning. We address this need by examining secondary students' thinking about the particular topic of binomial coefficients. Although the scope of this research is fairly narrow, our detailed investigation offers specific insights about representational registers. In particular, we make explicit connections to representations in combinatorics, which are commonly used in solving counting problems but are rarely examined in and of themselves.

✉ Elise Lockwood
elise.lockwood@oregonstate.edu

[1] Oregon State University, Corvallis, USA

[2] University of Georgia, Athens, USA

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2 Literature review

Work exploring students' reasoning about different types of counting problems highlights differences among problems with certain constraints, such as whether or not repetition is allowed when constructing outcomes, and whether or not differently ordered elements count as distinct outcomes. Such investigations began with work by Piaget and Inhelder (1975) and was continued by Fischbein and colleagues (e.g., Fischbein et al., 1970), who focused on developmental stages and orders of difficulty of problem types for young students. Since then, more studies have explored students' reasoning about problem types (Batanero et al., 1997), including having undergraduate students develop formulas (Reed & Lockwood, 2021) and leveraging computer programming to reason about distinctions among problems (Lockwood & De Chenne, 2020). We focus on combinations in this paper, noting the tradition to examine students' reasoning about a variety of problem types.

2.1 Student reasoning about binomial coefficients

Binomial coefficients $\binom{n}{k} = \frac{n!}{(n-k)!k!}$ are so named because they are the coefficients of the terms in resulting expansions when binomials are multiplied together. The binomial theorem, which is stated as follows (for real numbers *x, y*, and nonnegative integers *n*), shows these terms as being the coefficients of the resulting polynomial:

$$(x+y)^n = \sum_{k=0}^{n} \binom{n}{k} x^k y^{n-k}$$

One way to interpret these coefficients is that of n binomials that are being multiplied together, any coefficient of $x^k y^{n-k}$ will consist of k x's and $n-k$ y's. To get such a term, we can simply select an x from k of the n terms, and the rest of the $n-k$ terms will contribute a y. There are $\binom{n}{k}$ such possibilities for how many ways this selection can happen, and there is a combinatorial interpretation for this solution. Binomial coefficients count the number of unordered selections, or *combinations*, of k elements from an n-element set. Thus, binomial coefficients can be thought of as another term for combinations, and we use both terms interchangeably in this paper.

2.1.1 K-12 students' reasoning about binomial coefficients

Understanding that there is much available research on K-12 students' combinatorial reasoning (e.g., Batanero et al., 1997; English, 1991; Tillema, 2013), we focus on students' understanding of binomial coefficients; most relevant are findings from the longitudinal study conducted by Maher and colleagues (see Maher et al., 2011). Their work demonstrated that students could make connections among a variety of contexts involving combinations, including colored towers, pizzas, and patterns in Pascal's triangle. There are similarities with our work presented here, in that, like Maher et al., we have investigated K-12 students in the context of solving combination problems. However, there are also key differences. Firstly, our data come not from students' long-term, repeated exposure to problems, but from two sessions of a 5-session teaching experiment in which the students had no prior combinatorial experience, demonstrating that it is possible for students to engage with this material in a short amount of time. Secondly, our students' work began with and emerged from a context that is not typically viewed as being combinatorial, namely dimensions and volume growth in cubes. Finally, we characterize students' understanding and development of a formula for combinations, which has not been the focus of previous work with middle and secondary students.

2.1.2 Undergraduate students' reasoning about binomial coefficients

Lockwood and colleagues highlighted some potential challenges that undergraduate students might face when reasoning about binomial coefficients. Even though we worked with younger students, these studies highlight findings that framed our work. For example, Lockwood and Caughman (2016) explored students' reasoning about set partition problems, which involve binomial coefficients; they found that because students associate order "not mattering" with combinations, they may find it difficult to address subtleties that arise in problems involving binomial coefficients. Lockwood et al. (2018) explored whether undergraduate students differentiate between different kinds of combination problems. The authors found that students were more likely to interpret what they called Category 1 combination problems (problems whose solutions are modeled as *an unordered selection of distinguishable objects*) as involving binomial coefficients than Category 2 combination problems (problems whose solutions are modeled as *an ordered sequence of two (or more) indistinguishable objects*). Their findings suggest that binomial coefficients are indeed a subtle and complex topic. Together, these studies highlight the importance of exploring students' reasoning about combination problems.

In light of such prior work at the K-12 and undergraduate levels, we were motivated to explore whether younger students could reason about and develop a formula for combinations. Thus, our overall research aim is to report on the students' success in formulating a correct formula for binomial coefficients by building out of an algebraic context involving dimensions of a cube, and to examine what role shifts in representational registers played in that successful formulation. To analyze and interpret the students' work, we sought a framework that would help us account for some key shifts in representations they used as they progressed through tasks. We elaborate these perspectives in the following section.

3 Theoretical perspective

3.1 Mathematical representations

We explicitly designed the teaching experiment to involve key shifts in representations to facilitate the students' progression in their combinatorial reasoning, and, in observing the students' work, we saw that representations indeed played a key role for their progress. We thus sought to understand what sense students made of different representational contexts. Because we saw shifts within and across representations, we found that Duval's (2006) framework for representational registers, specifically his constructs of treatments and conversions, provided particularly useful ideas for our analysis.

Duval (2006) described a representation as something that stands for something else. Mental representations can be individuals' beliefs, ideas, and conceptions, which can be accessed through their verbal or schematic productions. Examples of representations are "diagrams, number lines, graphs, arrangements of concrete objects or manipulatives, physical models, mathematical expressions, formulas and equations, or depictions on the screen of a computer or calculator—that encode, stand for, or embody mathematical ideas or relationships" (Goldin, 2014, para. 1). They are produced according to a set of rules, and allow the description of a system, a process, or a phenomenon (Duval, 2006).

3.2 Representational registers, treatments, and conversions

In considering how learners navigate different representational systems, Duval (2006) relied on the notion of a representational register, that is, a system that permits a transformation of representations. There are two types of transformations: treatments and conversions. Treatments are transformations of representations that occur within the same register. This can entail, for instance, manipulating geometric figures such as showing that the area of a parallelogram is equivalent to the area of a rectangle with the same length and height, or making transformations to an equation in order to solve it. Conversions, in contrast, are transformations of representations that consist of changing a register without changing the objects being denoted. This can include moving from an equation to a graph (moving from a symbolic register to a graphical register), or translating a word problem into a set of equations (moving from a natural language register to a symbolic register).

Duval (2006) noted that although treatment is generally emphasized in mathematics teaching, conversion is more complex because the change of register "first requires recognition of the same represented object between two representations whose contents have very often nothing in common" (p. 112). This recognition is nontrivial, as it can change the properties that are made explicit in each register. Researchers investigating students' representational transformations have reported that the ability to make transformations and, in particular, conversions is important for meaningful conceptualization of mathematical objects (Deelice & Kertil, 2014).

In this paper, we use the constructs of treatments, conversions, and representational registers to frame the students' work. In light of this theoretical perspective, and building on existing literature in combinatorics education, we now frame the research questions we seek to address in this paper. Our overall goal is to identify ways to improve the teaching and learning of combinatorics, and we want to investigate the ways in which shifts in representational registers may support students' combinatorial reasoning. We situate these two research specific questions within this broader goal.

(1) As two secondary students reasoned about and developed a formula for binomial coefficients, how did the students' work within and across different representational registers support their combinatorial reasoning about binomial coefficients?
(2) In particular, what treatments and conversions did the students make between and across registers, and how did those treatments and conversions support specific aspects of their combinatorial reasoning about binomial coefficients?

4 Methods

4.1 Data collection

This study was part of a three-year project investigating students' processes of generalizing in algebra, precalculus, and combinatorics. For that project, we conducted a series of teaching experiments (Steffe & Thompson, 2000) with middle-school, high-school, and undergraduate students (see Ellis et al. 2021, for more details). The data reported in this paper come from one paired teaching experiment with a 12-year-old middle-school student and a 14-year-old high school student (we call these secondary students, where secondary encompasses middle and high school (grades 6–12); this is in contrast to elementary and undergraduate students).

Table 1 Representational registers and instructor moves to promote them

Representationalregister	Instructor moves
Physical (Days 1, 2, 3, and 4)	• Provision of physical cubes to model volume • Allowance of open exploration with the cubes • Prompting to explain volume representations by showing pieces on the cubes
Algebraic (Days 3 and 4)	• Explicit direction to write and represent added volume for a generic N x N x N cube in terms of expressions involving 1s and Ns
Arrangements (Day 4)	• Explicit direction to write and represent dimensions of pieces as arrangements of 1s and Ns
Sets (Days 4 and 5)	• Explicit direction to write and represent arrangements of 1s and Ns as sets of numbers that represent positions of the Ns in a given arrangement

Previous work (e.g., Lockwood et al. 2016) had explored and established undergraduate students' engagement with such tasks. Given our research aim of examining the extent to which younger students could also reason about binomial coefficients via shifts in representational registers (and leverage algebraic contexts and representations to explore combinatorial ideas), we focused on this particular teaching experiment within our broader data set. This was the only teaching experiment in which we used these combinatorial tasks with students of this age; these data are thus unique among our broader set, which is why we focused on them for this paper.

4.1.1 Participants

The participants were Barney, a 12-year-old 7th-grade student, and Homer, a 14-years old 9th-grade student. (Barney and Homer are pseudonyms the participants chose.) Our aim was to conduct a paired teaching experiment, and Barney and Homer were the first two available students who were able to participate in a series of summer sessions which coincided with the researchers' availability. Our only requirement for participation was the completion of (at minimum) a pre-algebra course. At the time of the teaching experiment, Barney had just completed a pre-algebra course, which was part of the advanced mathematics track at his school. Barney had studied linear equations and functions, but had not yet encountered quadratic or higher-order polynomial functions or learned algebraic manipulations such as multiplying binomials. Homer had just completed Algebra I, and was familiar with quadratic and exponential functions, but had not yet encountered higher-order polynomial functions. The two students were comfortable with one another and communicated well together and with the teacher-researcher.

4.1.2 The teaching experiment and data sources

The teaching experiment consisted of five sessions that occurred over the course of 8 days, with each session ranging from 60 to 90 min. The sessions were audio and video recorded, and the second author was the teacher-researcher. This particular teaching experiment was the third such instantiation from the larger project, and thus the teacher-researcher had already developed and revised an initial set of tasks during the first two rounds with different participants. The tasks for the first three days were revisions of those previously developed (and focused on growth of dimensions of rectangles and cubes), and those for the final two days were new (and focused on connecting such growth to combinatorial contexts). We address these final two days in this paper. The teaching experiment setting allowed for the creation and testing of hypotheses in real time while engaging in teaching actions, as well as in reflection between each session. Between sessions, the second author met with the first author to review the data, debrief the session, and then revise or develop the tasks for the next session. The data sources consisted of the video and audio recordings of the teaching sessions, the students' written work, and transcripts from the video recordings.

4.2 Representational registers and tasks

4.2.1 Representational registers

Drawing on Duval (2006), we frame the students' work in terms of four different registers, each relying on its own method of representation. Table 1 shows each register and when it occurred, as well as the instructor moves we employed to elicit students' thinking and activity within and across those registers. We briefly describe each register and the associated types of tasks we elicited within the register.

The *physical register* involves reasoning about component pieces of volume while manipulating physical cubes. In this register, the students worked with pieces of a cube that were part of a physical manipulative. Figure 1a shows the original cube, which the students reasoned with as a generic cube with a side-length N. They added three $N \times N \times 1$ pieces (1b), then three $N \times 1 \times 1$ pieces (1c), and a $1 \times 1 \times 1$ piece (1d). Treatments within this register entailed comparing two different decompositions of a cube as ways to represent the same amount of added volume, or building up component pieces of the cube in multiple

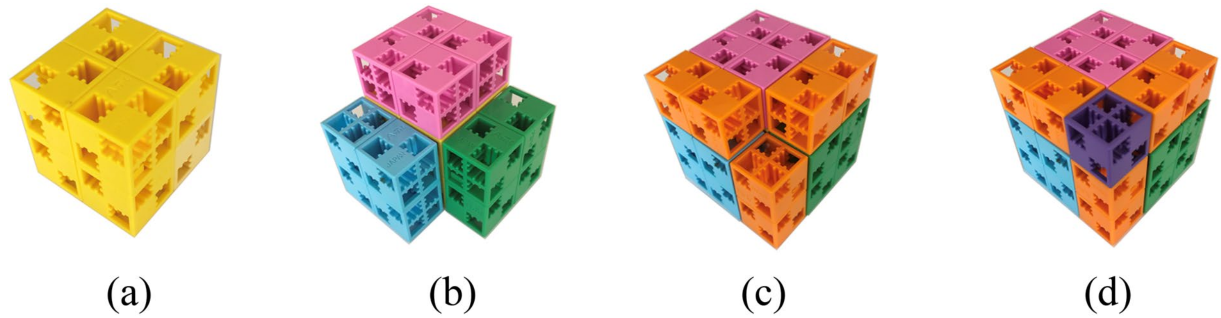


Fig. 1 Constituent parts of a cube that has grown by 1 cm in height, length, and width

ways. Students could also make conversions by algebraically representing the added volume that they had built on the cube.

The *algebraic register* involves writing and reasoning about symbolic, algebraic expressions such as $N^3+3N^2+3\ N+1$ (as opposed to manipulating physical pieces of a cube). Here, the students made sense of algebraic symbols, and they often made conversions between this and other registers. The *arrangements register* involves representing ideas in terms of arrangements of characters (in our case, arranging some number of 1s and some number of Ns), and not physical pieces or algebraic expressions. For example, work within the arrangements register might involve counting how many arrangements there are of four Ns and two 1s. Students could list arrangements and consider what counts as an outcome, and what makes outcomes distinct. Finally, the *sets register* involves counting an unordered set of numbers. Here, the objects of focus are not arrangements or symbolic expressions, but sets of numbers, such as {1, 2, 3, 4} or {3, 4, 5, 6}.

Importantly, there is a bijection between the arrangements of Ns and 1s and sets of numbers, where the sets of numbers correspond to the positions of the Ns in an arrangement of 1s and Ns. So, {1, 2, 3, 4} represents the arrangement NNNN11, and the arrangement 11NNNN represents {3, 4, 5, 6}. We leveraged this bijection in designing our tasks, as we hypothesized that it would be easier for the students to notice structure and patterns by considering lists of sets.

Although we are articulating different registers, they are all related. Let us consider as an example the term $3N^2$, which can be interpreted within all four registers. In the physical register, there are three 2-dimensional N^2 pieces of volume that are added to the original cube. In the algebraic register, the symbolic representation is the written expression, "$3N^2$". Students can represent these terms as algebraic symbols of numbers and variables. In the arrangements register, there are three ways of arranging one 1 and two Ns: NN1; N1N; 1NN. In the sets register, this represents that there are three 2-element sets of the number 1, 2, and 3: {1,2}, {1,3}, and {2, 3}. These four registers offer four different ways of representing the same idea.

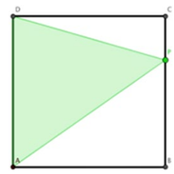
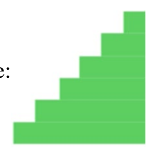
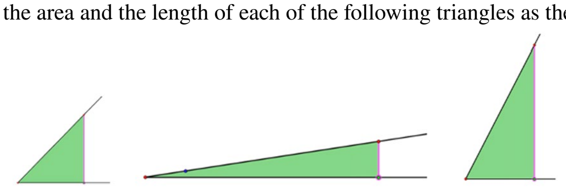
4.2.2 Tasks

The first two days of the teaching experiment emphasized linear growth as a representation of a constant rate of change, and quadratic growth as a representation of a constantly-changing rate of change. We created a task sequence exploring rates of change within the contexts of speed, area, and volume, from which students could develop a foundation of thinking about functional growth from a covariation perspective to leverage when exploring cubic and higher-order polynomial functions (Carlson et al., 2002; Ellis et al. 2022)—see Table 2 for representative tasks presented each day. On Day 3, we transitioned to volume tasks to explore the students' generalizations about volumes of growing cubes and rectangular prisms. The students manipulated a dynamic geometry software image of a smoothly growing cube, and also worked with physical cubes to build up models to think about the parts of the added volume.

On Day 4, we encouraged a shift in representational registers and asked the students to think about volume growth for higher dimensions that they could not visualize (4, 5, 6, and *N* dimensions). This activity involved shifting from working with the dimensions of a physical cube to working with arrangements of the characters 1 and N (where 1 and N each represented dimensions of a growing piece of a physical cube). Our intention was to have the students connect the pieces of the cube to arrangements of Ns and 1s. In this way, we prompted a shift from a physical register to an arrangements register that would foster combinatorial reasoning.

Day 5 saw an additional shift in registers, from reasoning about arrangements of 1s and Ns to sets of numbers. We fostered student reasoning about a particular bijection in which the students could connect a particular arrangement of 1s and Ns (such as 1N1NN) with a set of numbers that represented the positions of the Ns in the arrangement. Because binomial coefficients exactly count such sets, we were trying to prepare the students to develop a formula for counting such sets.

Table 2 Representative tasks presented during the teaching experiment

Day	Representative tasks
1	*Task 1*. Graph the area of the triangle (y-axis) versus the total distance traveled by point p (x-axis). Then, graph the area of the triangle (y-axis) versus the shortest distance along the square between point p and A (x-axis).  *Task 2*. Imagine you are sweeping out a rectangle with a fixed height that increases in length from left to right. Graph the amount of area versus the length swept. *Task 3*. Imagine that you sweep out a rectangle with a fixed height as before. It glides along for 2 seconds, then immediately extends to be twice as tall and sweeps out for another two seconds. a) Draw a picture of this scenario. b) Graph the amount of area versus the length swept. *Task 4*. Say you sweep out one rectangle with a height of 10 cm. Write an expression for the area of the rectangle after it has swept out x cm in length.
2	*Task 1*. Graph the area versus the length of the stairstep figure:  *Task 2*. Graph the relationship between the area and the length of each of the following triangles as they sweep out from left to right:  *Task 3*. a) Imagine a triangle that sweeps out with a height to length ratio of 1:1. What would be the area of the triangle for any swept length L? b) Now suppose the ratio of height to length was 2:3. What is the area of the triangle for any swept length L? c) Now suppose the ratio of height to length was 3:2. What is the area of the triangle for any swept length L? d) What would be the area of a triangle for any swept length L when the ratio of height to length is a:b?
3	*Task 1*. a) Let's think about a cube that starts growing from a point, and it grows the way we saw in the movie, from 0 cm x 0 cm x 0 cm to a cube that is 10 cm x 10 cm x 10 cm. How much volume did the cube gain as each side grew from 0 cm to 10 cm? b) Imagine that cube on the journey from 0 cm x 0 cm x 0 cm to N cm x N cm x N cm. It has grown to be some size and we don't know how big it is. How much volume will the cube gain when it grows 1 more cm on each side? *Task 2*. a) Now instead of a cube, let's think about a rectangular prism (we'll call it a box) that has the dimensions 1 cm by 2 cm by 3 cm. Imagine this box growing from a point (0 cm x 0 cm x 0 cm) to a 10 cm by 20 cm by 30 cm box. How much volume did the box gain on its journey? *Task 3*. a) How does the volume of an N x N x N cube grow when each side grows by x cm? b) How does the cube's volume grow if it grows by a cm in one dimension, b cm in the second dimension, and c cm in the third dimension?
4	*Task 1*. Now say you have an N x N x N x N hypercube. How much would its "volume" grow if the hypercube grew by 1 cm in each direction? *Task 2*. How many ways you can arrange two 1s and two Ns? *Task 3*. a) Extend this task to the 5^{th} dimension. Write an expression for the total volume of an N x N x N x N x N cube that grew by 1 cm in each direction. b) How many ways can you arrange three 1s and two Ns?
5	*Task 1*. How many ways can you arrange 3 1s and 3 Ns? *Task 2*. a) Say you have the numbers from 1 to 6. How many 2-number combinations are there, where order doesn't matter? b) How many 3-number combinations are there, where order doesn't matter? *Task 3*. a) Say you have six numbers, and two slots. How many ways can you put the numbers in the slots, if order doesn't matter? b) Say you have six numbers, and two slots. How many ways can you put the numbers in the slots, if order doesn't matter? *Task 4*. Say you have a list of seven numbers, and you are going to choose two numbers. How many different combinations are there, if order doesn't matter?

Finally, the students developed a formula for counting sets of numbers—that is, for binomial coefficients, or the number of k-element subsets of an n-element set.

4.3 Data analysis

A member of the research team created enhanced transcripts from the video recordings, which included relevant time stamps, images from the video data and students' written work, and descriptions of students' pauses and gestures. To analyze the data, the first author re-watched Days 4 and 5 and reviewed the enhanced transcripts of the teaching experiments. From the transcripts, she pulled relevant episodes into a document, writing a narrative and aiming to tell a comprehensive story of what had occurred. She recorded overall shifts in representations that the students had made (not yet using Duval's (2006) framework of representational registers), and she also identified particular combinatorial insights that were afforded for the students as they made progress in the teaching experiment.

Then, she shared this narrative with the second author, who reviewed the document and made sure it aligned with her viewing of the video and reading of the transcripts (and her experience as teacher-researcher in the experiment). They discussed the data and made sense of the big ideas presented in the narrative, ultimately agreeing on the accuracy of the overall narrative. The second author then used Duval's (2006) framework of representational registers to articulate the main shifts that occurred. Based on the narrative and on her reading of representational registers, the second author created Table 3 (shown in Sect. 5.4), which summarizes and presents the characterizations of conversions and treatments within the various representational registers. The first author then went through and compared the table with the narrative, adjusting and finalizing the table as an accurate representation of the data. Together, the authors resolved any discrepancies as they finalized the results section.

5 Results

In Sect. 5.1 we highlight initial student work about growth of a cube, which led to the set of tasks in this paper. Then, in Sect. 5.2 we share the bulk of our results, in which we account for the variety of representational registers in which students engaged as they came to understand a formula for binomial coefficients. Due to space, this is not a comprehensive account of every shift and all of the episodes in our data; instead, we focus on two major shifts that illuminate the ways in which different representational registers might be leveraged to foster combinatorial reasoning.

5.1 The physical and algebraic registers—homer and barney's initial work on dimensions of a cube

On Day 3, Barney and Homer investigated the change in volume of a cube that grew in 1-cm increments. They worked in the physical register to establish the component pieces of a larger cube (Fig. 1). The teacher-researcher asked the students to make a conversion between the physical register and the algebraic register. Pointing to a $4\times4\times4$ cube, she asked, "Can you express that algebraically if this was, instead of a $4\times4\times4$, it was an $N\times N\times N$?" Both students wrote the expression $3N^2+3N+1$, and then they explained each term of the expression in relation to the component pieces of the cube. The teacher-researcher asked Homer, "Where's the $3N^2$?" Homer pointed to the three $N\times N\times1$ pieces on the physical cube (Fig. 1b), and said, "The three little extra." Barney pointed to the three $N\times1\times1$ pieces (Fig. 1c), and said, "These are the Ns." Homer then pointed to the $1\times1\times1$ piece (Fig. 1d) and said, "And then the one's just this one." In this manner, the students began to establish relationships between component pieces of the cube in the physical register and analogous terms of an expression in the algebraic register.

5.2 The students' treatments among and conversions between the physical, algebraic, and arrangements registers

In the following sections we discuss the students' treatments among and conversions between the physical, algebraic, and arrangements register, and their treatments within the arrangement register. In doing so, we describe what treatments and conversions among registers the students made, and these episodes set the stage for us describing how these treatments and conversions ultimately supported certain aspects of their combinatorial reasoning.

5.2.1 Conversion from the physical to the algebraic register

Given an $N\times N\times N$ cube, the teacher-researcher asked the students to recall from their activity in the prior day what pieces were being added and what the dimensions were if the cube grew by 1 unit in each direction. The students wrote the expression for the added volume (Fig. 2), and the teacher-researcher started to transcribe what they had written onto a new page (Fig. 3a, b).

Notice that the teacher-researcher prompted the algebraic register, in which each term in the expression was re-written in terms of 1s and Ns, which stood for dimensions of the cube. Notably, the students spoke flexibly about the two

Table 3 A summary of the registers in the results, including the transformation type, the register the students used, and a description of the register

Transfor-mation	Register(s)			Description
Conversion	Physical	Algebraic		Students established relationships between terms of an algebraic expression and component pieces of the cube.
Conversion	Physical	Algebraic	Arrange-ments	Students explained each term of $3N^2 + 3N + 1$ on the cube and in terms of the dimensions 1 and N.
Conversion	Algebraic	Arrangements		Barney noted that the 4 of $4N^3$ represented "every possible permutation", i.e. (N*N*N*1), (N*1*N*N), (N*N*1*N), (1*N*N*N). Homer used similar reasoning for the 4N term.
Conversion	Algebraic	Arrangements		Barney and Homer reasoned that the second term for the expression of added hypervolume for a 4-dimensional cube should be $6N^3$ because there are 6 ways to arrange two Ns and two 1s.
Conversion	Algebraic	Arrangements		Barney and Homer determined coefficients from Pascal's triangle to write algebraic expressions for added hypervolume, and then justified the coefficients through listing Ns and 1s.
Treatment	Arrangements			Barney took the listing outcome of arranging three Ns and two 1s and used it to determine the listing outcome of arranging three 1s and two Ns by switching the 1s and Ns. Homer did the same with four Ns and one 1.
Treatment	Arrangements			Students determined the number of outcomes for three Ns and two 1s can be represented as 4 + 3 + 2 + 1.
Conversion	Sets	Arrangements		Barney transformed "145" in the sets register to "NIINNI" in the arrangements register.
Treatment	Sets			Homer took the listing outcome of 6 choose 2 and represented it as groups of 5, 4, 3, 2, and 1.
Treatment	Sets			Homer and Barney took the listing outcome of 6 choose 3 and represented it as groups of 10, 6, 3, and 1.
Conversion	Sets	Arrangements		Homer equated the first 10 sets of numbers that included a 1 for 6 choose 3 to combinations starting with 1 when listing 1s and Ns, and the second 10 sets of numbers as those that would begin with N.
Treatment	Sets			The students conduct calculations within the sets register to determine a way to find 6 choose 3 without listing.
Conversion	Sets	Arrangements		Students knew 7 choose 4 was the same as 7 choose 3 because it is equivalent to substituting 3 and 4 with I and N.

representations—the physical cube itself and the written algebraic expressions involving 1s and Ns. Even though 1s and Ns were written here, we view this expression as within the algebraic register because the students viewed them as algebraic representations of the dimensions, not yet as a new combinatorial representation of arrangements of 1s and Ns.

Teacher-Researcher: is there anything you notice about the way I wrote it?
Barney: Uh, yes. So it always goes height, length, width or height, width, length I think.

[…]

Teacher-Researcher: Okay. And what about the order in which I wrote the 1s and the Ns?
Barney: It goes, height, then width, then length.
Homer: And they're all different because.
Barney: 'Cause this is the 1 height, this is the 1 wide, this is the 1 long.
Teacher-Researcher: Okay. And you said they're all different Homer?
Homer Well, no they're not different pieces but they're represented differently, yeah.

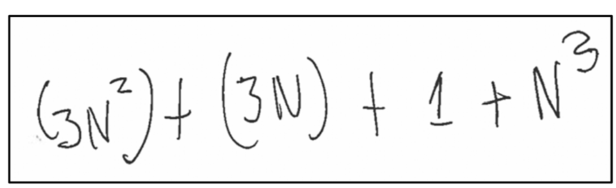


Fig. 2 Barney's expression for the volume of growth of a 3-dimensional cube

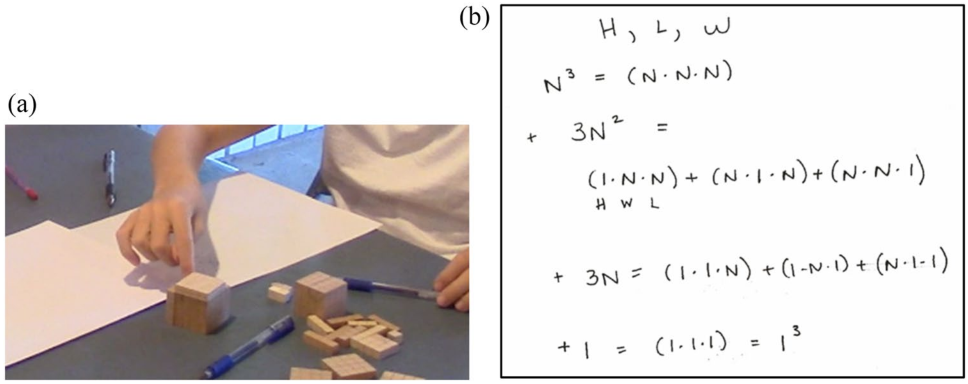


Fig. 3 **a**, **b** The physical cube and the volume expressed as 1s and Ns

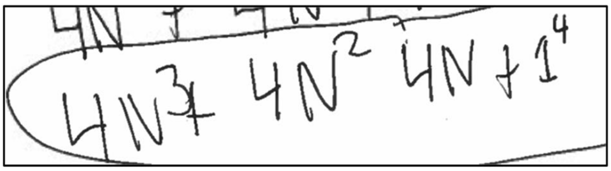


Fig. 4 The students' guess at the 4-dimensional case

Here, Homer noted that they were "represented" differently. We interpret that he meant that the three N pieces 1NN, N1N, NN1 were all being represented differently, even though they were all the same shape, and they had different positions on the cube. The students considered the different algebraic terms, which they connected to the physical cube, and had the following exchange:

Teacher-Researcher:	Okay, and what are the dimensions of those [the pieces that represent the 3N]?
Barney:	Uh, they are 1, by 1 by N.
Teacher-Researcher:	Okay.
Barney:	And then it's 1 by N by 1, and then no N by 1 by 1.

This exchange suggests that Homer and Barney connected the sequences 11N, 1N1, and N11 with the dimensions of pieces of the cube that were being added, a conversion between the physical and algebraic registers.

From this initial work with the 3-dimensional cube, we see that the students seemed to understand the relationship between the physical register and the algebraic register. Some listing occurred here when the teacher-researcher wrote arrangements of 1s and Ns, but we interpret that this activity was viewed as representing the pieces algebraically. This would lay the groundwork for the arrangements register, but these representations as arrangements was not the focus here. At this point, they had completed the 3-case, and written, as shown in Fig. 3b, the expression N^3+3N^2+3N+1 as a way to describe the total volume of an $N \times N \times N$ cube that was increased by one unit in each direction.

5.2.2 Conversion to arrangements register, and focusing on systematic listing in the 4-dimensional case

We now highlight episodes that occurred during the students' exploration on the 4-dimensional case, in which they were predicting an expression for extending the previous problem to an $N \times N \times N \times N$ situation. The teacher-researcher asked the students to consider what would happen if they extended the problem to 4 dimensions. The teacher-researcher and the students discussed the physical limitations of this scenario, and Barney noted, "It's literally impossible to imagine." This highlights their understanding of a need for a way to conceive of growth in dimensions that they could not visualize. Homer made an astute observation, saying, "So the real question is, how many, how many extra pieces would we have to add on like these (referring to pieces of the growing cube)?" This suggests that they understood that they were still connecting their work to dimensions in the physical register.

The students then made an incorrect but reasonable prediction for an algebraic expression of the added hypervolume when a hypercube grew 1 cm in each direction (see Fig. 4): $4N^3+4N^2+4N+1^4$. In their guess, only the coefficient of N^2 is incorrect; the correct answer is $4N^3+6N^2+4N+1^4$.[1]

To resolve this incorrect term, the teacher-researcher asked the students to inspect the accuracy of their expression, drawing their attention to listing combinations of 1s and Ns to reason about how many of each term there might be (that is, to check whether their prediction was correct). For instance, she asked the students to think about the first term, $4N^3$, in terms of the dimensions of the pieces added to the hypercube.

[1] We note the similarity between their guess and the expression for the 3-dimensional case, which was $3N^2+3\,N+1$.

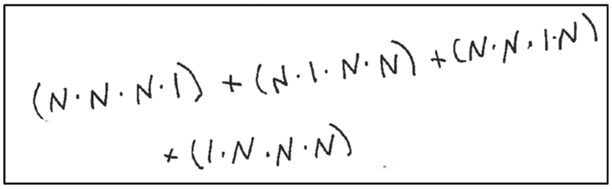


Fig. 5 Expressing the four pieces in terms of 1s and Ns

This is in line with their previous work in the 3-dimensional case (see Fig. 3), but here the teacher-researcher intentionally reoriented the students to writing out 1s and Ns. Homer and Barney noted that there would be four different pieces to add with four different dimensions, which, through discussion, the teacher-researcher wrote down (Fig. 5). Throughout this episode, the teacher-researcher's choice of representation was an intentional instructional move to foster conversion—this was manifest not by the teacher-researcher telling the students to use 1s and Ns, but rather by modeling that representation through her inscription.

Referring to the teacher-researcher's inscription in Fig. 5, Barney noted, "So the 4 represents every possible thing we could do, every possible permutation." Homer responded by saying, "Oh okay, oh okay that makes sense…Yeah I thought it was 4, but I did not know how to prove it." This suggests that seeing the four ways in which to write three Ns and one 1 (as indicated in Fig. 5) helped to convince Homer why four made sense as the coefficient of N^3. This is the first explicit evidence for the students enacting a conversion between the algebraic register and the arrangements register; the student's language suggested that they made sense of the coefficient, 4, of $4N^3$ (a term in the algebraic register) in terms of the representation in Fig. 5 (a representation in the arrangements register).

The students and teacher-researcher then moved on to the coefficient of N (which is also 4), with the teacher-researcher asking them to explain why they got 4. Homer explained it as follows, describing 4 arrangements of three 1s and one N:

Homer: Well yes, wait let me explain it. Because it's 4 by N because it's […] 1 by 1 by 1 by N then N by 1 by 1 by 1, and N by no then 1 by N by 1 by 1 and then 1 by 1 by N by 1. And there's 4 of them.

We note that here, Homer was still referring to dimensions, and it seems that the 1s and Ns were representing to him those dimensions, simply expressed as 1s and Ns. This reinforced Homer's relationship between the physical and algebraic registers, but it is not clear the extent to which he was thinking of arrangements in this statement. We note that to this point, the students spoke in terms of dimensions, but they also referred to inscriptions of 1s and Ns and, at times, to arrangements of Ns and 1s. Here there seemed to be a tacit understanding of the idea that the dimensions, the algebraic expressions, and the arrangements of Ns and 1s were connected, but to this point the students had not yet explicitly addressed these relationships in their discussion of the problem. Further, while the students were describing Ns and 1s, it was not always clear whether they viewed those as arrangements of characters or simply as ways of representing (through characters of 1s and Ns) the dimensions from the physical register.

The students then turned to the N^2 term, which they had guessed was $4N^2$. The teacher-researcher asked them to list out ways in which two of the dimensions were growing, which is equivalent to arranging two 1s and two Ns. Importantly, the teacher-researcher introduced language that drew attention to arrangements of 1s and Ns—referring to slots (or, positions) that would contain 1s or Ns:

Teacher-Researcher: So, the only one we haven't worked through is the $4N^2$. So, I want you both to list out all the ways you can arrange the N squared piece, so 2 dimensions are growing 1…So two of the slots are going to be 1s, and two of the slots are going to be Ns. So, list out all the ways.

Here the teacher-researcher, in one utterance, connected three different representational registers, the physical, the algebraic, and the arrangements registers. This was a deliberate attempt to engender a conversion to the arrangement register; as we will see, even though the students made this conversion, they did not lose track of the overall picture, which was related to questions of dimensionality. In doing what the teacher-researcher asked, the two students each listed such arrangements systematically, now working in the arrangements register, and arrived at an answer of 6 (Fig. 6).

The teacher-researcher then asked the students to revise their equation for the added volume. Barney adjusted the $4N^2$ to be a $6N^2$, writing $4N^3+6N^2+4N+1^4$, which we see as evidence that he could interpret his work in the arrangements register in terms of the algebraic register. Barney then suggested that they write an expression for the second dimension, and they wrote the expression N^2+2N+1 for the total area of the 2-dimensional figure. The teacher-researcher included this in her summary and reorganization of their work so far, which we discuss in the next section.

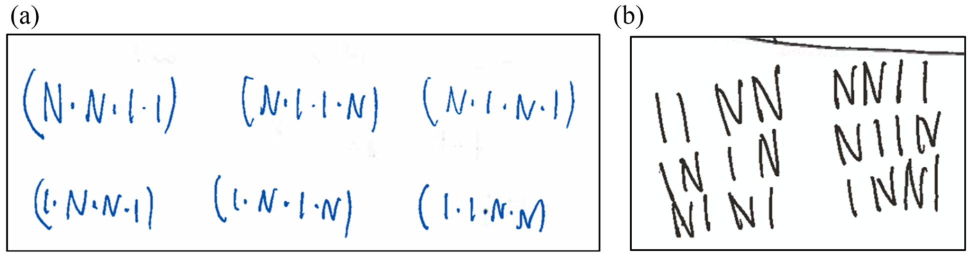


Fig. 6 **a**, **b** Homer's and Barney's lists of arrangements of two Ns and two 1s

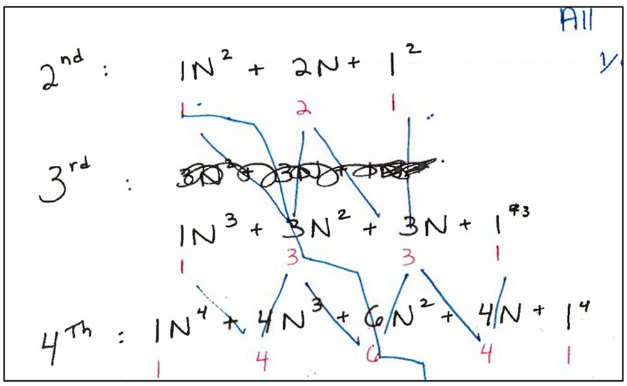


Fig. 7 A summary of the students' work so far

5.2.3 Focusing on patterns and relationships among cases in the algebraic register

The students had now established correct expressions for the 2, 3, and 4-dimensional cases. After this discussion, the teacher-researcher began to write down each of the expressions for dimensions 2, 3, and 4 in a particular way; as shown in Fig. 7, she listed them out as rows on top of each other (the red numbers and blue lines were added later). Here, as with systematic listing, we see an intentional instructional move to focus on certain properties and relationships *within* the algebraic representational register that directs their focus on observing numerical patterns. Once the students observed these numerical relationships, we will see that they could still convert to both the physical register and the arrangements register.

When the teacher-researcher drew the work in Fig. 7, Homer became excited and recognized Pascal's triangle, which both students recognized from their school experiences:

Homer:	Oh, I know what's happening, I know what's happening, I know what's happening, I know what's happening, I know what's happening
Teacher-Researcher:	Okay.
Barney:	Wow chill bro, settle down.

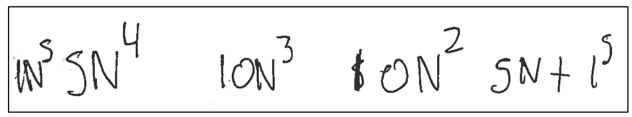


Fig. 8 Barney's expression for 5 dimensions

Homer:	I KNOW what is happening, Barney. It is simple, as 2 sorry I'm writing on it [begins to add the blue lines]. 2 plus 1 is 3 and 2 plus 1 is 3, 3 plus 3 is 6, 3 plus 1 is 4, 1 plus 3 is 4.

The fact that the students understood how to generate the next row in Pascal's triangle was useful, because it allowed them to hypothesize the values for the 5th dimensional case, which they could then check via listing. The teacher-researcher asked them to determine an expression for the 5th dimensional case. Homer wrote out the 1, 5, 10, 10, 5, 1 as the fifth row in Pascal's triangle, and, notably, both he and Barney started to list to check their work. Here the students shifted to engaging in activity that was entirely within the algebraic and arrangements registers, and they were no longer focusing on the physical register.

Barney used the coefficients from the fifth row of Pascal's triangle to write out the terms in Fig. 8. Barney seemed to make sense of the $1N^5$ and $5N^4$ terms and the 5N and 1^5 terms fairly quickly, and, importantly, when he was thinking about why there might be 5 for arranging one N and four 1s, he said, "Yeah, there's five places the N can go. Okay, that's easy enough." This suggests that in reasoning about the term 5N, Barney demonstrated a conversion, justifying a term in the algebraic register via the arrangements register (reasoning combinatorially about arranging 1s and Ns). That is, he articulated why that made sense in terms of producing combinations of two kinds of characters, and he justified the 5 in terms of arranging 1s and Ns.

The middle two cases of N^3 and N^2 were the more difficult coefficients. To justify the accuracy of the coefficient 10, Barney and Homer made a conversion to the arrangements register, both engaging in listing on these cases, producing lists for three Ns and two 1s (the N^3 case) in Fig. 9. As they

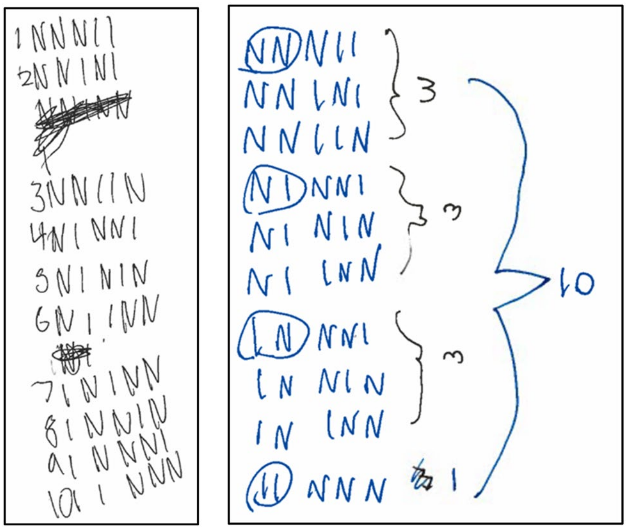


Fig. 9 Barney's and Homer's lists of arranging three Ns and two 1s

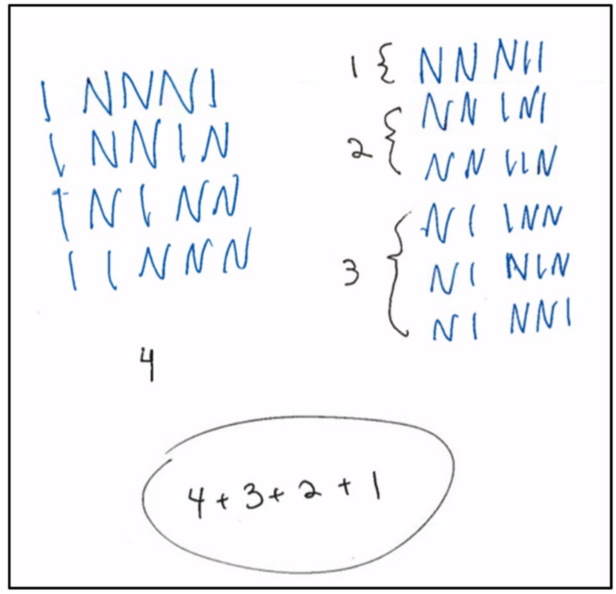


Fig. 10 A structured list of three Ns and two 1s

wrote and discussed their lists, they were operating fully within the arrangements register. Their exchanges suggest that they were reasoning about and justifying how many arrangements there are of three Ns and two 1s and engaging in systematic listing.[2] Seeing the list of 10 confirmed the result they had predicted based on the pattern in Pascal's triangle.

5.2.4 Finalizing day 4, and a brief discussion of a formula

Toward the end of this Day 4 work, the students reasoned about a formula for the number of 5-character arrangements of three Ns and two 1s (there are 10). We briefly summarize what occurred because it relates to their eventual articulation of the formula for binomial coefficients during Day 5. As the students worked, they started to hypothesize a formula for the number of arrangements of Ns and 1s. This began by noticing structure and regularity within their list of three Ns and two 1s, which was prompted by the teacher-researcher. In particular, she directed them to list these outcomes by attending to how many outcomes started with a 1. After some discussion, they created the following list (Fig. 10), noting that there were 10 total. They all discussed their systematic process (listing arrangements that started with 1, and with N, with NN, and with NNN) and articulated that the total would be $4+3+2+1$.

The teacher-researcher asked them to determine another way to express that sum, and they drew on previous experience to come up with a formula, $N(N+1)/2$ (they had developed this formula in Day 2 in an area context). They noted that there would be $\frac{5\bullet 4}{2}=10$ outcomes. Thus, through their recollection of a prior formula, the students conceived of the 10 in Pascal's triangle both as $10=4+3+2+1$ and as $10=\frac{(4+1)\bullet 4}{2}$, two instances of treatment transformations within the arrangements register (which were communicated via the algebraic register).

However, when Homer and Barney went to consider the third entry in the 6[th] row of Pascal's triangle, they experienced a perturbation. As they generalized their two options, on the one hand they thought $\binom{6}{3}$ should be $21=6+5+4+3+2+1$ (adding a 6 to the 15), but in Pascal's triangle the number of outcomes should be 20, since it is the sum of the previous two entries, 10 and 10. So, they reasoned that something was incorrect about their sum, and they wanted to explore it more. Their goal for Day 5 was to determine the correct coefficient for N^3 in the 6th-dimensional case.

5.3 The sets register

5.3.1 Second shift among representational registers: from the arrangements register to the sets register

At the beginning of Day 5, Barney and Homer used the Pattern from Pascal's triangle to write the following expression for the 6th dimension: $1N^6+6N^5+15N^4+20N^3+15N^2+6N+1^6$. Based on this, their guess was 20 for the N^3 term.

[2] We note that because of how the students wrote 1s, they sometimes referred to the 1s as Is.

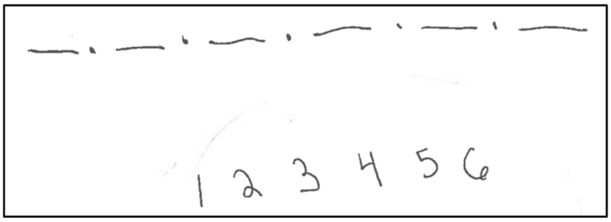


Fig. 11 Slots as a way to consider options for arrangements

Next, an important exchange occurred in which the teacher-researcher explicitly directed the students' attention toward another way of thinking about and representing outcomes. We elaborate this episode to highlight the students' thinking about this new register.

Teacher-Researcher: I'm going to introduce some new notation. So if you want to figure out that it's $20N^3$, remember we're thinking about dimensions right. So how many dimensions does this cube exist in?

Homer: Six.

Teacher-Researcher: Six dimensions. And so, basically we're saying okay, there's six different dimensions it can grow in so sort of six slots (Fig. 11), I'm going to kind of do it like this 3, 4, 5, 6 and it's growing in one centimeter each direction. So for this one, how many N's would there be and how many 1's would there be?

Barney: There'd be three N's and three 1's. Because it's N^3.

At this point, the students volunteered that they wanted to list all of the outcomes (Fig. 12).

The students' listing activity took place within the arrangements register. Wanting to help them move toward a more general formula, the teacher-researcher then made an explicit connection between the students' lists and a set of three numbers chosen from the numbers 1 through 6. She hypothesized that such a representation would highlight properties that would make generating the formula for binomial coefficients easier for the students. (Note that at this point, the students had begun to use the term "I" instead of "1", confusing the two symbols due to their visual similarity. In the transcript below, the teacher-researcher had taken up the students' use of the term "I", but she was referring to the 1s.)

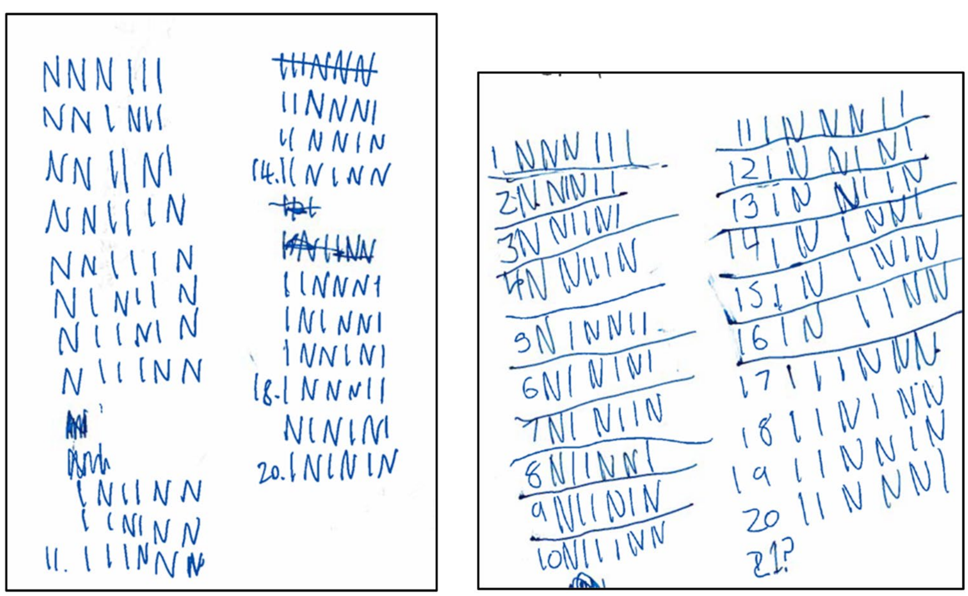
Teacher-Researcher: As you were listing what you were doing is, 1 2 3 4 5 6 there were six spots right? And you were choosing where to put the three Ns right? […] For instance, what's one 3-number string that you could choose?

Homer: 145.

Teacher-Researcher: 145. So, do you believe that choosing all of the different 3-number strings out of 6 is the same as what you were doing here, listing the Is and the Ns?

Homer: Mhm. Well, assuming that all the numbers are in, so like, if there was 541 [TR writes 541] that wouldn't be another one because--

Fig. 12 Homer and Barney's lists of three Ns and three 1s



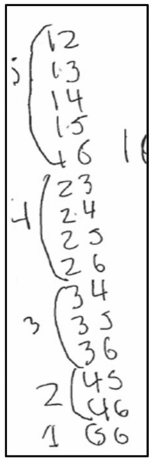


Fig. 13 2-elements subsets of the integers 1 through 6

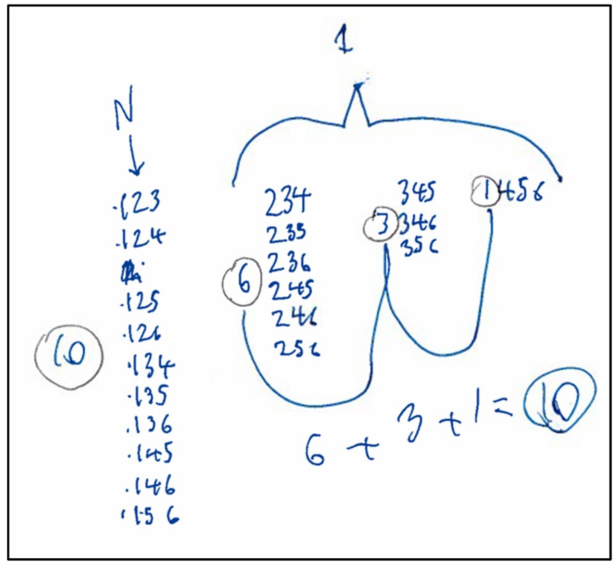


Fig. 14 An organized list of 3-element subsets of the integers 1 through 6

Teacher-Researcher: Correct, yes say why.

Homer: --because, we could say, alright, this is 1, 2, and 3 [points to arrangement NNN111], like for this one but we could also say oh it's 321 because so. Uh these [points to 145 and 541] are the same.

Teacher-Researcher: Yes.

Barney: These [the numbers] are the spots where you're going to put Ns.

Teacher-Researcher: Yes. The numbers are the spots where you're going to put Ns. That's right. So, if you chose 145, what would that look like in the string that you listed?

Barney: That would be, this one, number eight, NIINNI.

In asking what 145 would mean in the string the students had listed, the teacher-researcher explicitly asked the students to make a conversion between the new sets register and the arrangements register. Barney was able to transform "145" into "NIINNI" (here he used the symbol "I" instead of "1"). Furthermore, we consider this conversion to be a bijection between arrangements of 1s and Ns and sets of numbers. This relates to encoding outcomes, where students can think of counting one type of object (an arrangement of 3 Ns and 3 1s) as another type of object (a set of three numbers that represents positions of the Ns). Further, Barney articulated a specific mapping within that bijection between the set {1,4,5} and the arrangement NIINNI.

This is an impressive realization for any student, particularly one of Barney's age, as it is a difficult relationship to see. It is exactly related to the Category 2 combination problems that Lockwood et al. (2018) discussed, and there is evidence that students find it difficult to make connections between subsets and selecting positions. While the teacher-researcher did make that connection, it is noteworthy that Barney and Homer could articulate that connection and seemed to understand why making note of the order was relevant. This provides further evidence that they were reasoning combinatorially throughout the teaching experiment.

Continuing to draw attention to the sets of numbers, the teacher-researcher asked the students to list out 2-element subsets. (Note, she called them strings, but she also specified that order did not matter, so we consider this to be within this sets register.) Working within the sets register, the students produced the list in Fig. 13, counting 15 total, which Homer organized as groups of 5, 4, 3, 2, and 1. Thus, within the sets register as in within the arrangements register, the students could make a treatment transformation by representing the organized list of 2-element sets as $5+4+3+2+1$.

They then went about investigating the three case, and they wrote the list in Fig. 14. Note that the students performed another treatment by organizing the outcomes into groups of 10, 6, 3, and 1.

Importantly, right after they wrote these out, Homer articulated a profound insight that again highlights the ways in which they converted across representational registers. This episode shows a sophisticated level of reasoning about outcomes and understanding how one representation (an arrangement of 1s and Ns) is related to another in a different representational system (a set of numbers). Here, Homer looked at the sets of numbers he had just listed, and he realized that the 10 sets of numbers that included a 1 represented the 10 arrangements of 1s and Ns in which an N is in the first position:

Homer: Oh, what I like to think, how I like to think about this is, these are combinations starting with 1 [right part of Fig. 14], and these are combinations starting with N [left part of Fig. 14]. Because if there's an N in the first spot, it'd be the first 10 combinations…

Here, not only did Homer demonstrate that he could associate the numbers he was listing as being associated with positions (a conversion between the sets and arrangements registers), but he also actually volunteered that he could think of any set that includes a 1 as representing an arrangement that starts with an N. Given documented difficulties that undergraduate students face with encoding outcomes, this is extremely impressive. It also highlights how the particular conversion between the arrangements and sets registers supported an important and nuanced combinatorial insight.

5.3.2 Positional reasoning and equivalent outcomes in deriving a formula for binomial coefficients

We now describe the students' reasoning about a formula for binomial coefficients, and they again leveraged particular properties within the sets register to do so. Once they realized they could think about subsets as being related to their arrangements, we encouraged them to come up with a formula for binomial coefficients. They discussed the fact that they could use Pascal's triangle to determine any binomial coefficient recursively; however, they realized this had limitations, and they were thus motivated to derive an explicit formula.

In the following exchange, the teacher-researcher provided explicit guidance, but the students jumped on it right away, making connections for themselves. We view this as another instructional move, within this representation, to draw attention to important structure and properties of the representation that can help them gain insight about a more general formula for binomial coefficients. Barney and Homer determined a way to find the binomial coefficient $\binom{6}{2}$ without listing by arguing that they would multiply 6 options for the first number and 5 options for the second, and Homer noted, "But you divide by two. Because, because there's two and so you would have them backwards, like you'd have 12 and then you'd have 21 […] So it's 15." They then reasoned about what it would be for $\binom{6}{3}$, and they recognized that they had to decide between dividing the product of $6 \bullet 5 \bullet 4$ by 3 or dividing by 3!. The teacher-researcher had them consider an arbitrary 3-digit combination, {2, 4, 6}, and determine how many ways there were to rearrange those three numbers. The students were able to reason about why they needed to divide by 3!, noting that there were $3 \bullet 2 \bullet 1 = 6$ options for arranging three objects. Note that the students' ability to determine $\frac{6 \bullet 5 \bullet 4}{3!}$ was a consequence of multiple treatments within the sets register, as they were able to reason about why the 3! could be thought of as arranging three objects.

They then tested their formula on a few specific examples, using Pascal's triangle as verification for $\binom{7}{2}$, $\binom{7}{3}$, and $\binom{7}{4}$. To confirm they had a stable understanding, the teacher-researcher asked them to explain the components of the formula for $\binom{8}{3}$ and why it would be $\frac{8 \bullet 7 \bullet 6 \bullet 5 \bullet 4}{5!}$. We contend that the following quote indicates that the students understood the formula and could explain it combinatorially:

Barney: So there's 8 possible numbers that can go here. And if there's uh, a single number that goes here it can't be that number so there's only 7 numbers that could go here, and there's 2 numbers here so only 6 of the numbers can go here, and 5 here, and 4 here. And then you divide it by 5 factorial because, it's hard to explain without writing it out but we already explained it with the other one with the 6 and the stuff.

Barney then went on to explain that, like the case in which they argued for dividing by 3! (which is the 6 he referred to), they would divide by 5! because there are 5! ways to arrange 5 objects. Toward the end of the session, the teacher-researcher asked the students about a formula for $\binom{n}{p}$: "And so if it's n choose p, what we do is, what would go on top?" Homer and Barney correctly described the formula as being $\frac{n!}{p!(n-p)!}$.

6 Summary of results

We now briefly summarize the results, emphasizing takeaways from Homer and Barney's work and connecting to our research questions. We begin by providing Table 3, which gives an overview of the students' trajectory and highlights the transformations and registers in which the students engaged across the sessions reported in this paper.

In answering our first research question, we saw the power of students being able to transition between representational registers as a way of supporting their combinatorial reasoning. The intentional shift between registers enriched their combinatorial reasoning in important ways, namely by framing the work in terms of a combinatorial activity (listing 1s and Ns), and by recasting problems in a way that highlighted salient structure (listing subsets of numbers rather than sequences of 1s and Ns). Although we focus on the students' combinatorial reasoning in this paper, we also gain insight more broadly into supporting students' abilities to make sense of a mathematical pattern in one register via another register. We designed two intentional shifts in representational registers in our task sequence. The first was from conceiving of dimensions not just as sequences of 1s and Ns where order mattered but as 1s and Ns that were indistinguishable (a transformation from the physical to algebraic to arrangements register). The second shift was conceiving of arrangements of 1s and Ns as sets of numbers (a transition from the arrangements register to the sets register). These overall shifts can be seen in Table 3. As one progresses through the registers column, we see overall the conversions from physical to algebraic, from algebraic to arrangements, and then conversions among sets and arrangements registers over time.

We also saw that students were able to maintain contact with the big picture. We saw many instances of this, such as when Homer or Barney used the word dimensions even when speaking of arranging 1s and Ns. We hypothesize that an important factor in the students' success was a rich conceptual understanding of their goals within each register, and how that related to their main goal of determining volume growth. As the teaching experiment progressed, the students at times shifted goals, but even with a translation of goals from one representational register into another, the students consistently demonstrated an understanding of how the different registers were related to each other. These insights about the role that the conversions among registers played in their success also gives insight into our second research question, as we see how specific treatments and conversions supported particular insights about the formula for binomial coefficients.

Our second research question addressed what treatments and conversions students made between and across registers, as well as how those treatments and conversions supported student's combinatorial reasoning. We not only saw a variety of specific treatments and conversions (Table 3), we also saw how transformations among and within registers supported the students' combinatorial reasoning. In particular, there were times when conversions played a key role in their progression toward reasoning about binomial coefficients. For example, the shift from reasoning within the arrangements register to reasoning within the sets register helped the students organize their lists of outcomes and determine an expression to count the total number of outcomes. There were also times when treatments were particularly important for offering combinatorial insight. For instance, there were occasions in which it was useful to rearrange written lists in order to see some structure that offered combinatorial insight. This happened in both the arrangements and the sets registers, when Homer and Barney realized that they could organize 10 arrangements of Ns and 1s as groups of $4+3+2+1$, or when they organized lists of 20 sets as groups of 10, 6, 3, and 1. Such treatments facilitated recognition of regularity that they used to develop their formula for binomial coefficients. Overall, then, in our findings we have highlighted ways in which shifts within and among representational registers supported secondary students in their combinatorial reasoning and development of a formula for binomial coefficients.

7 Limitations, discussion, and avenues for future research

One limitation of our study was that we did not get to explore the students' understanding and application of the general formula for binomial coefficients further and on additional problems. Due to time constraints, we needed to end the teaching experiment when we did, and it would have been beneficial to see how the students would have gone on to use and apply the formula in new settings. In future studies, researchers could build in more time to explore how students subsequently use and apply the formula. Also, with more time, we might have explored additional formulas and combinatorial ideas with the students, including giving more time for investigating Pascal's triangle and justifying why certain identities hold.

Another limitation is that the study only involves two students, and thus there are some limitations to overall general claims we can make about students more broadly. However, our study offers one potential model of how students can leverage treatments and conversions (in the sense of Duval, 2006) within and across representational registers to build up powerful combinatorial reasoning. In this way, it serves as an existence proof of secondary students meaningfully engaging in combinatorial reasoning and developing a general formula for combinations via shifts in representational registers, following a rich tradition of existence proofs in mathematics education that addresses both students' understanding of particular mathematical ideas (e.g., Carpenter et al., 1998) and successful instructional interventions (e.g., Kramer & Keller, 2008). We have offered a non-traditional starting point for students to reason about discrete mathematics—namely, dimensions of a cube. The students made meaningful connections between combinatorics and an algebraic context of volume growth and dimensionality that is not traditionally the source of combinatorial thinking and activity (see Tillema, 2013, for other work that connects combinatorics and algebra). We demonstrated an instance in which students meaningfully moved between productive quantitative reasoning in an algebraic context and sophisticated combinatorial reasoning in a discrete context. This highlights potential implications for instructional interventions that highlight and support relationships (and not just differences) between algebraic contexts and combinatorial contexts.

Our findings could be viewed in light of Harel's (2008) *necessity principle*, which states that "For students to learn the mathematics we intend to teach them, they must have a need for it, where 'need' here refers to intellectual need" (p. 900). We suggest that our shifts to new registers were *necessitated* by the problems and tasks we gave to our students. They could not visualize hypervolume, and so there was an intellectual need to develop a new way to think about dimensionality via arranging Ns and 1s. Then, again, large lists of such arrangements can become too cumbersome, and the sets register offered an alternative way to keep track of such lists. We maintain that each register was necessitated for the students, helping them to resolve a problem in the prior register. Future work could explore more explicitly the role that necessity can and does play in these conversions and treatments of representational registers.

We also see a connection to our work in this paper and *empirical reconceptualization* (Ellis et al., 2021), which is "the process of re-interpreting an empirical generalization from a structural perspective" (p. 13). We saw that Pascal's triangle served as a tool that facilitated numerical verification, which helped to confirm some of the students' work. First, it allowed for them to have some numerical check against which to confirm the results of their listing. Further, Pascal's triangle highlighted numerical regularity and patterns, but in a way that could help the students to reason carefully and thoughtfully about the relationships they were observing. Given the difficulty involved in verifying combinatorial ideas (e.g., Eizenberg & Zaslavsky, 2004), the use of Pascal's triangle here was compatible with our goals of having students develop and use quantitative reasoning. Even though they used Pascal's triangle to create empirical generalizations, the connections to the combinatorial work they did helped them make sense of and reason about the generalizations they had identified.

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