Algebras and Representation Theory (2021) 24:81–133
https://doi.org/10.1007/s10468-019-09934-z

A Kazhdan-Lusztig Algorithm for Whittaker Modules

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Received: 16 March 2019 / Accepted: 23 October 2019 / Published online: 21 January 2020
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Abstract
We study a category of Whittaker modules over a complex semisimple Lie algebra by realizing it as a category of twisted $\mathcal{D}$-modules on the associated flag variety using Beilinson–Bernstein localization. The main result of this paper is the development of a geometric algorithm for computing the composition multiplicities of standard Whittaker modules. This algorithm establishes that these multiplicities are determined by a collection of polynomials we refer to as Whittaker Kazhdan–Lusztig polynomials. In the case of trivial nilpotent character, this algorithm specializes to the usual algorithm for computing multiplicities of composition factors of Verma modules using Kazhdan–Lusztig polynomials.

Keywords Whittaker modules · D-modules · Localization of representations · Kazhdan–Lusztig polynomials

Mathematics Subject Classification (2010) 22E47 · 14F10

1 Introduction

A fundamental goal in representation theory is to understand all representations of complex semisimple Lie algebras. However, the category of all modules for a given Lie algebra is so large that a full classification has only been obtained for the simplest example, the Lie algebra $\mathfrak{sl}(2, \mathbb{C})$ [3]. In light of this, one way to approach this goal is to study well-behaved categories of representations subject to certain restrictions, then relax the restrictions to expand the categories and observe what aspects of the structure carry over into the larger category. A classic example of such a well-behaved category is Bernstein-Gelfand–Gelfand's category $\mathcal{O}$, which has been studied extensively in the past 40 years and found to display deep connections across representation theory. The category $\mathcal{N}$ of Whittaker modules introduced by Miličić–Soergel in [16] is a generalization of category $\mathcal{O}$ which also contains

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a collection of nondegenerate Whittaker modules introduced by Kostant [10]. In category $\mathcal{O}$, characters of simple modules are determined by Kazhdan–Lusztig polynomials. In this paper, we show that the same is true in the category of Whittaker modules, and we develop an algorithm for computing these characters. The main result of this paper is the following theorem.

Theorem 1 (Theorem 11, Corollary 4, Eq. 33) *For any irreducible Whittaker module L and standard Whittaker module M with the same regular integral infinitesimal character, there exists a polynomial $Q_{ML} \in q\mathbb{Z}[q] \cup \{1\}$ such that the multiplicity of L in the composition series of M is given by $Q_{ML}(-1)$. Moreover, the polynomials Q_{ML} can be computed through a combinatorial recursive algorithm.*

Our approach to studying Whittaker modules is to use the localization of Beilinson–Bernstein [2] to relate $\mathcal{N}$ to a certain category of holonomic $\mathcal{D}$-modules (so-called twisted Harish-Chandra sheaves) on the associated flag variety. This geometric approach gives us access to powerful tools such as the decomposition theorem for arbitrary holonomic $\mathcal{D}$-modules [18] which are essential in the development of the algorithm for computing the polynomials of Theorem 1.

The four main contributions of this paper to the existing literature on Whittaker modules are the following. First, we develop a theory of formal characters for Whittaker modules which generalizes the theory of formal characters of highest weight modules and distinguishes isomorphism classes of objects in the Grothendieck group of the category (Section 2.2). Second, we give a detailed description of the structure of the category of twisted Harish-Chandra sheaves (Section 3). Irreducible objects in this category were classified in [17], but this paper includes a collection of new results describing the action of intertwining functors on certain costandard sheaves, which were originally introduced by Miličić–Soergel in [17]. The third and most significant contribution of the current paper is the development of an algorithm for computing the composition multiplicities of standard Whittaker modules, which establishes that the formal characters of simple Whittaker modules are given by a collection of polynomials that we refer to as Whittaker Kazhdan–Lusztig polynomials (Section 5). Finally, we give a comparison of the Whittaker Kazhdan–Lusztig polynomials which arise in our algorithm to other types of Kazhdan–Lusztig polynomials in the existing literature (Section 6). This places Theorem 1 in the context of the Kazhdan–Lusztig combinatorics of the Hecke algebra and establishes a connection between Whittaker modules and other representation theoretic objects such as generalized Verma modules.

We will spend the rest of the introduction describing the main results of this paper in more detail. Let $\mathcal{U}(\mathfrak{g})$ be the universal enveloping algebra of a semisimple Lie algebra $\mathfrak{g}$ over $\mathbb{C}$ and $\mathcal{Z}(\mathfrak{g})$ the center of $\mathcal{U}(\mathfrak{g})$. Let $\mathfrak{b}$ be a fixed Borel subalgebra of $\mathfrak{g}$ with nilpotent radical $\mathfrak{n} = [\mathfrak{b}, \mathfrak{b}]$ and $\mathfrak{h} \subset \mathfrak{b}$ a Cartan subalgebra. The category $\mathcal{N}$ of Whittaker modules consists of all $\mathcal{U}(\mathfrak{g})$-modules which are finitely generated, $\mathcal{Z}(\mathfrak{g})$-finite, and $\mathcal{U}(\mathfrak{n})$-finite. For a choice of $\lambda \in \mathfrak{h}^*$ and a Lie algebra morphism $\eta : \mathfrak{n} \to \mathbb{C}$, McDowell [12] constructed a standard Whittaker module $M(\lambda, \eta)$ (Definition 2), which has a unique irreducible quotient $L(\lambda, \eta)$, and showed that all irreducible Whittaker modules appear as such quotients. When $\eta = 0$, the $M(\lambda, 0)$ are Verma modules and the $L(\lambda, 0)$ are simple highest weight modules. When η acts non-trivially on all root subspaces of $\mathfrak{g}$ corresponding to simple roots (we say such η are *nondegenerate*), the $M(\lambda, \eta)$ are the irreducible modules studied by Kostant in [10].

Unlike highest weight modules, Whittaker modules don't decompose into generalized $\mathfrak{h}$-weight spaces. However, in blocks of $\mathcal{N}$ where the nilpotent radical $\mathfrak{n}$ acts by a specific character η, Whittaker modules do decompose into generalized weight spaces for a certain subalgebra $\mathfrak{h}^{\Theta} \subset \mathfrak{h}$, which is the center of a Levi subalgebra of $\mathfrak{g}$ determined by the character η (Section 2.1). In contrast to the generalized $\mathfrak{h}$-weight spaces of category $\mathcal{O}$, the generalized $\mathfrak{h}^{\Theta}$-weight spaces in this decomposition are not finite-dimensional, but they are of finite length in the category of modules over the specified Levi subalgebra. We can capture the structure of this $\mathfrak{h}^{\Theta}$-weight space decomposition by defining the *formal character* (Definition 3) of a Whittaker module in a way that generalizes the formal character of highest weight modules. Then a natural problem in understanding the structure of the category of Whittaker modules is to compute the formal characters of irreducible modules in $\mathcal{N}$, which reduces to computing the multiplicities of the irreducible constituents of a standard Whittaker module.

These multiplicities were first determined for integral λ by Miličić and Soergel in [16] and for arbitrary λ by Backelin in [1] by relating subcategories of Whittaker modules to certain blocks of category $\mathcal{O}$ and using the classical Kazhdan–Lusztig algorithm for Verma modules. The current paper provides a more efficient procedure for calculating these multiplicities by using a geometric realization of Whittaker modules as twisted sheaves of $\mathcal{D}$-modules on the flag variety. This geometric perspective allows us to relate the multiplicities to combinatorial data extracted from the associated Hecke algebra, providing a direct link between Whittaker modules and Kazhdan–Lusztig polynomials.

The first step in studying Whittaker modules geometrically is to realize $\mathcal{N}$ as a category of twisted Harish-Chandra modules. Let N be the unipotent subgroup of $\operatorname{Int}\mathfrak{g}$ such that $\operatorname{Lie}N = \mathfrak{n}$. For a Lie algebra morphism $\eta : \mathfrak{n} \to \mathbb{C}$, the category of η-twisted Harish-Chandra modules consists of $\mathfrak{g}$-modules which admit an algebraic action of N whose differential differs from the restricted $\mathfrak{g}$-action by η. We denote the category of such modules with infinitesimal character corresponding to a Weyl-group orbit $\theta \subset \mathfrak{h}^*$ (via the Harish-Chandra homomorphism) by $\mathcal{M}_{fg}(\mathcal{U}_\theta, N, \eta)$. In [17], Miličić and Soergel established a categorical equivalence between certain blocks of $\mathcal{N}$ and the categories $\mathcal{M}_{fg}(\mathcal{U}_\theta, N, \eta)$.

This description allows us to use the localization theory of Beilinson–Bernstein to study Whittaker modules. For each $\lambda \in \mathfrak{h}^*$, Beilinson and Bernstein [2] constructed a sheaf of twisted differential operators $\mathcal{D}_\lambda$ on the flag variety X of $\mathfrak{g}$ whose global sections $\Gamma(X, \mathcal{D}_\lambda)$ are equal to $\mathcal{U}_\theta$, where θ is the Weyl group orbit of λ in $\mathfrak{h}^*$ and $\mathcal{U}_\theta$ is the quotient of $\mathcal{U}(\mathfrak{g})$ by the corresponding ideal in $\mathcal{Z}(\mathfrak{g})$. Applying the localization functor $\Delta_\lambda = \mathcal{D}_\lambda \otimes_{\mathcal{U}_\theta} -$ to the category $\mathcal{M}_{fg}(\mathcal{U}_\theta, N, \eta)$, we obtain a geometric category $\mathcal{M}_{coh}(\mathcal{D}_\lambda, N, \eta)$ of *η-twisted Harish-Chandra sheaves* (Section 3), which are N-equivariant $\mathcal{D}_\lambda$-modules satisfying a compatibility condition determined by η. This category consists of holonomic $\mathcal{D}_\lambda$-modules, so its objects have finite length and there is a well-defined duality in the category. The morphism η determines a parabolic subgroup W_Θ of the Weyl group W of $\mathfrak{g}$, and from the parameters $\eta : \mathfrak{n} \to \mathbb{C}$, $C \in W_\Theta \backslash W$, and $\lambda \in \mathfrak{h}^*$, we construct a standard sheaf $\mathcal{I}(w^C, \lambda, \eta)$, costandard sheaf $\mathcal{M}(w^C, \lambda, \eta)$, and irreducible sheaf $\mathcal{L}(w^C, \lambda, \eta)$ (Section 3). Here w^C is the longest element in the coset C. The precise relationship between the algebraic category $\mathcal{N}$ and the geometric category $\mathcal{M}_{coh}(\mathcal{D}_\lambda, N, \eta)$ is given by the following theorem, which we prove in Section 4.

Theorem 2 (Theorem 9, Theorem 10) *Let $\lambda \in \mathfrak{h}^*$, $\eta : \mathfrak{n} \to \mathbb{C}$ a Lie algebra morphism, and $C \in W_\Theta \backslash W$. Let $\mathcal{M}(w^C, \lambda, \eta)$ be the corresponding costandard η-twisted Harish-Chandra sheaf and $M(w^C\lambda, \eta)$ the corresponding standard Whittaker module. Then*

(i) *if λ is antidominant,*

$$\Gamma(X, \mathcal{M}(w^C, \lambda, \eta)) = M(w^C\lambda, \eta), \text{ and}$$

(ii) *if* λ *is also regular, then*

$$\Gamma(X, \mathcal{L}(w^C, \lambda, \eta)) = L(w^C\lambda, \eta).$$

Hence to compute the composition multiplicities of standard Whittaker modules $M(\lambda, \eta)$, it suffices to compute the composition multiplicities of the costandard η-twisted Harish-Chandra sheaves $\mathcal{M}(w^C, \lambda, \eta)$ in the category $\mathcal{M}_{coh}(\mathcal{D}_\lambda, N, \eta)$. In the case of regular integral $\lambda \in \mathfrak{h}^*$, the structure of this category is completely determined by the parameter η, so we may further restrict our attention to the case $\lambda = -\rho$, where ρ is the half-sum of positive roots. In this setting, $\mathcal{D}_\lambda = \mathcal{D}_X$ is the sheaf of differential operators on X (without a twist). One way to better understand the structure of the irreducible $\mathcal{D}_X$-modules $\mathcal{L}(w^C, -\rho, \eta)$ (or indeed any $\mathcal{D}_X$-module in this category) is to utilize the stratification of the flag variety and to restrict them to Bruhat cells contained in their support. The resulting restricted $\mathcal{D}$-modules are easy to understand: the N-equivariance guarantees that they decompose into a direct sum of copies of the structure sheaf on the corresponding Bruhat cell. By keeping track of how many copies appear in the direct sum corresponding to each Bruhat cell (we refer to this integer as the "$\mathcal{O}$-dimension," denoted $\dim_{\mathcal{O}}$), we can construct a combinatorial object which captures all important structural information of each irreducible $\mathcal{D}_X$-module in the category $\mathcal{M}_{coh}(\mathcal{D}_X, N, \eta)$. For each coset $D \in W_\Theta\backslash W$, let δ_D be a formal variable parameterized by D, and let $\mathcal{H}_\Theta$ be the free $\mathbb{Z}[q, q^{-1}]$-module with basis $\{\delta_D, D \in W_\Theta\backslash W\}$. Let $i_{w^D} : C(w^D) \to X$ be the inclusion of the corresponding Bruhat cell into the flag variety. We define a map $\nu : \mathcal{M}_{coh}(\mathcal{D}_X, N, \eta) \to \mathcal{H}_\Theta$ by

$$\nu(\mathcal{F}) = \sum_{D\in W_\Theta\backslash W} \sum_{m\in\mathbb{Z}} \dim_{\mathcal{O}}(R^m i^!_{w^D}(\mathcal{F}))q^m\delta_D.$$

Here, $R^m i^!_{w^D}$ are the right derived functors of the $\mathcal{D}_X$-module extraordinary inverse image functor (Section A.2).

We use ν to develop our desired Kazhdan–Lusztig algorithm for Whittaker modules. Let Σ be the root system of $\mathfrak{g}$ and $\Pi \subset \Sigma$ the set of simple roots determined by our fixed $\mathfrak{b}$. Let $\Theta \subset \Pi$ be the subset of simple roots picked out by $\eta \in \operatorname{ch}\mathfrak{n}$, and let $W_\Theta \subset W$ be the corresponding parabolic subgroup of the Weyl group. For any $\alpha \in \Pi$, we define a certain $\mathbb{Z}[q, q^{-1}]$-module endomorphism $T_\alpha : \mathcal{H}_\Theta \to \mathcal{H}_\Theta$ (Section 5). The main result of this paper is the following theorem.

Theorem 3 (Theorem 11, Proposition 9) *The function* $\varphi : W_\Theta\backslash W \to \mathcal{H}_\Theta$ *given by* $\varphi(C) = \nu(\mathcal{L}(w^C, -\rho, \eta))$ *is the unique function satisfying the following properties.*

(i) *For* $C \in W_\Theta\backslash W$,

$$\varphi(C) = \delta_C + \sum_{D<C} P_{CD}\delta_D,$$

where $P_{CD} \in q\mathbb{Z}[q]$.

(ii) *For* $\alpha \in \Pi$ *and* $C \in W_\Theta\backslash W$ *such that* $Cs_\alpha < C$, *there exist* $c_D \in \mathbb{Z}$ *such that*

$$T_\alpha(\varphi(Cs_\alpha)) = \sum_{D\leq C} c_D\varphi(D).$$

The existence and uniqueness of a function satisfying equivalent conditions to (i) and (ii) was shown combinatorially by Soergel in [19].[1] By realizing the function φ explicitly in

[1] The formulation in [19] is in terms of the antispherical module of the Hecke algebra. We prove in Section 6.3 that this formulation is equivalent to conditions (i) and (ii) in Theorem 3.

terms of the category $\mathcal{M}_{coh}(\mathcal{D}_\lambda, N, \eta)$, Theorem 3 relates the Hecke algebra combinatorics established in [19] to the category of Whittaker modules, which is the main accomplishment of this paper. Theorem 3 determines a family $\{P_{CD}\}$ of polynomials in $q\mathbb{Z}[q]$ parameterized by pairs of cosets $C, D \in W_\Theta\backslash W$. We refer to these as *Whittaker Kazhdan–Lusztig polynomials*. In Section 6 we describe their relationship to other types of Kazhdan–Lusztig polynomials appearing in the literature. These polynomials determine the composition multiplicities of standard Whittaker modules. More precisely, if $(\mu_{CD})_{C,D\in W_\Theta\backslash W}$ is the inverse of the lower-triangular matrix $(P_{CD}(-1))_{C,D\in W_\Theta\backslash W}$, then we have the following corollary to Theorem 3.

Corollary 1 (Corollary 3, Corollary 4) *Let $\lambda \in \mathfrak{h}^*$ be regular, integral, and antidominant. Then the multiplicity of the irreducible Whittaker module $L(w^D(\lambda - \rho), \eta)$ in the standard Whittaker module $M(w^C(\lambda - \rho), \eta)$ is μ_{CD}.*

This paper is organized in the following way. We start by describing the structure of the algebraic category of Whittaker modules in Section 2, following [12]. In this section we recall McDowell's construction of standard and simple Whittaker modules and develop a new theory of formal characters for Whittaker modules. In Section 3, we describe the category of twisted Harish-Chandra sheaves, following [17]. We recall Miličić–Soergel's construction of standard and simple objects in this category, then introduce a class of costandard objects. These costandard objects were mentioned in [17] but not explicitly defined or studied. We prove some results about the action of intertwining functors on these costandard objects which are necessary for our arguments in Section 4. We dedicate Section 4 to explicitly relating the category $\mathcal{N}$ of Whittaker modules and the category $\mathcal{M}_{coh}(\mathcal{D}_\lambda, N, \eta)$ of twisted Harish-Chandra sheaves by proving that the global sections of costandard twisted Harish-Chandra sheaves are standard Whittaker modules. This result sets us up to work completely in the geometric category. Section 5 contains the proof of Theorem 3, which is the main result of this paper. In Section 6 we determine the relationship between Whittaker Kazhdan–Lusztig polynomials and Kazhdan–Lusztig polynomials, and we describe a combinatorial duality between the Kazhdan–Lusztig algorithm for generalized Verma modules found in [14] and the Kazhdan–Lusztig algorithm for Whittaker modules established in this paper. In Appendix A, we record our geometric conventions and include some fundamental facts about modules over twisted sheaves of differential operators.

2 A Category of Whittaker Modules

In this section, we introduce the category of representations which is the main focus of this paper and describe some key aspects of its structure. Let $\mathfrak{g}$ be a complex semisimple Lie algebra, $\mathcal{U}(\mathfrak{g})$ its universal enveloping algebra, and $\mathcal{Z}(\mathfrak{g})$ the center of $\mathcal{U}(\mathfrak{g})$. Let $\mathfrak{b}$ be a Borel subalgebra with nilpotent radical $\mathfrak{n} = [\mathfrak{b}, \mathfrak{b}]$ and $\mathfrak{h}$ the (abstract) Cartan subalgebra of $\mathfrak{g}$ [15, §2]. Let $\Pi \subset \Sigma^+ \subset \Sigma \subset \mathfrak{h}^*$ be the corresponding set of simple roots and positive roots, respectively, inside the root system of $\mathfrak{g}$. Let W be the Weyl group of $\mathfrak{g}$, and denote by $\rho \in \mathfrak{h}^*$ the half-sum of positive roots.

We begin by recalling some standard terminology. For a W-orbit $\theta \subset \mathfrak{h}^*$, there is a unique maximal ideal $J_\theta \subset \mathcal{Z}(\mathfrak{g})$, which can be obtained as the kernel of the Lie algebra morphism $\chi_\lambda : \mathcal{Z}(\mathfrak{g}) \to \mathbb{C}$ defined by $z \mapsto (\lambda - \rho)(\gamma(z))$, where $\gamma : \mathcal{Z}(\mathfrak{g}) \to \mathcal{U}(\mathfrak{h})$ is the untwisted Harish-Chandra homomorphism and λ is an element of the W-orbit θ [8, Ch. 1 §9]. All $\lambda \in \theta$ result in the same homomorphism χ_λ. We call such a Lie algebra morphism

χ_λ an *infinitesimal character*. We say a $\mathfrak{g}$-module V *has infinitesimal character* if it has the property that there exists an infinitesimal character χ_λ such that for any $z \in \mathcal{Z}(\mathfrak{g})$ and $v \in V$, $zv = \chi_\lambda(z)v$, or, equivalently, if it is annihilated by the ideal J_θ. We say a $\mathfrak{g}$-module *has generalized infinitesimal character* if there exists an infinitesimal character χ_λ and $k \in \mathbb{N}$ such that for all $v \in V$ and $z \in \mathcal{Z}(\mathfrak{g})$, $(z - \chi_\lambda(z))^k v = 0$, or, equivalently, if it is annihilated by a power of the ideal J_θ.

We are interested in the following category of $\mathfrak{g}$-modules, which was originally introduced by Miličić and Soergel in [16].

Definition 1 Let $\mathcal{N}$ be the category of $\mathfrak{g}$-modules which are

(i) finitely generated as $\mathcal{U}(\mathfrak{g})$-modules,
(ii) $\mathcal{Z}(\mathfrak{g})$-finite, and
(iii) $\mathcal{U}(\mathfrak{n})$-finite.

We refer to objects in this category as *Whittaker modules*.

Remark 1 In Kostant's original paper [10] the term *Whittaker module* is used to describe any $\mathfrak{g}$-module that is cyclically generated by a Whittaker vector. (These are vectors where $\mathfrak{n}$ acts by a nondegenerate Lie algebra morphism $\eta : \mathfrak{n} \to \mathbb{C}$.) We note that Definition 1 differs from Kostant's original terminology, though all irreducible Whittaker modules (in the sense of Kostant) are contained in $\mathcal{N}$.

McDowell showed that all objects in $\mathcal{N}$ have finite length [12] (a fact which follows immediately from their description as holonomic $\mathcal{D}$-modules in [17]). This category is a natural generalization of Bernstein–Gelfand–Gelfand's category $\mathcal{O}$. Indeed, if condition (ii) is replaced by the stronger condition that $\mathfrak{h}$ acts semisimply on the module, the resulting category is exactly category $\mathcal{O}$ [8], so $\mathcal{O}$ is a full subcategory of $\mathcal{N}$. A key difference between $\mathcal{N}$ and $\mathcal{O}$ is that when the $\mathfrak{h}$-semisimplicity condition is relaxed to $\mathcal{Z}(\mathfrak{g})$-finiteness, the existence of weight space decompositions is lost. However, the finiteness conditions (ii) and (iii) provide us with other useful decompositions of $\mathcal{N}$ which lead to structural results reminiscent of those in category $\mathcal{O}$. In particular, we have two categorical decompositions [17, §2 Lem. 2.1, Lem. 2.2]:

$$\mathcal{N} = \bigoplus_{\theta \in W\backslash\mathfrak{h}^*} \mathcal{N}_{\hat{\theta}} \text{ and } \mathcal{N} = \bigoplus_{\eta \in \mathfrak{n}^*} \mathcal{N}_\eta.$$

Here $\mathcal{N}_{\hat{\theta}}$ is the full subcategory of $\mathcal{N}$ consisting of modules with generalized infinitesimal character χ_λ for $\lambda \in \theta$, and $\mathcal{N}_\eta$ is the full subcategory of $\mathcal{N}$ consisting of modules where for any $X \in \mathfrak{n}$, $X - \eta(X)$ acts locally nilpotently on V. The only elements $\eta \in \mathfrak{n}^*$ for which $\mathcal{N}_\eta \neq 0$ are Lie algebra morphisms [4, Ch. VII §1.3 Prop. 9(iii)]. We call such a Lie algebra morphism $\eta : \mathfrak{n} \to \mathbb{C}$ an $\mathfrak{n}$-*character* and say that modules in $\mathcal{N}_\eta$ have *generalized* $\mathfrak{n}$-*character* η. We denote by $\operatorname{ch}\mathfrak{n} \subset \mathfrak{n}^*$ the set of $\mathfrak{n}$-characters.

Let $\mathcal{N}_\theta$ be the full subcategory of $\mathcal{N}$ consisting of modules with infinitesimal character χ_λ for $\lambda \in \theta$, and let $\mathcal{N}_{\theta,\eta}$ be the intersection $\mathcal{N}_\theta \cap \mathcal{N}_\eta$. Any irreducible Whittaker module lies in $\mathcal{N}_{\theta,\eta}$ for some Weyl group orbit θ and some $\eta \in \operatorname{ch}\mathfrak{n}$, so we will often restrict our attention to this full subcategory $\mathcal{N}_{\theta,\eta}$.

The category $\mathcal{N}_{\theta,\eta}$ is equivalent to a certain category of η-twisted Harish-Chandra modules, which is easier to relate to the geometric categories which appear later in this paper. We describe this equivalence now. Let N be the unipotent subgroup of $\operatorname{Int}\mathfrak{g}$ such that $\operatorname{Lie} N = \mathfrak{n}$.

Because N acts on the flag variety X of $\mathfrak{g}$ with finitely many orbits, the pair $(\mathfrak{g}, N)$ is a Harish-Chandra pair in the sense of [17, §1]. For a fixed $\mathfrak{n}$-character $\eta \in \operatorname{ch}\mathfrak{n}$, denote by $\mathcal{M}_{fg}(\mathfrak{g}, N, \eta)$ the category of triples (π, ν, V) such that:

(i) (π, V) is a finitely generated $\mathcal{U}(\mathfrak{g})$-module,
(ii) (ν, V) is an algebraic representation of N, and
(iii) the differential of the N-action on V induces a $\mathcal{U}(\mathfrak{n})$-module structure on V such that for any $\xi \in \mathfrak{n}$,

$$\pi(\xi) = d\nu(\xi) + \eta(\xi).$$

This is the category of *η-twisted Harish-Chandra modules* for the Harish-Chandra pair $(\mathfrak{g}, N)$. Let $\mathcal{M}_{fg}(\mathcal{U}_\theta, N, \eta)$ be the full subcategory of $\mathcal{M}_{fg}(\mathfrak{g}, N, \eta)$ consisting of modules which are also $\mathcal{U}_\theta = \mathcal{U}(\mathfrak{g})/(\mathcal{U}(\mathfrak{g})J_\theta)$-modules; that is, modules $V \in \mathcal{M}_{fg}(\mathfrak{g}, N, \eta)$ which are annihilated by J_θ. In [17, §2 Lem. 2.3], Miličić and Soergel show that the categories $\mathcal{N}_{\theta,\eta}$ and $\mathcal{M}_{fg}(\mathcal{U}_\theta, N, \eta)$ are equivalent. This association lets us use the localization functor of Beilinson and Bernstein (Section A.3) to study the category of Whittaker modules geometrically. In particular, by localizing objects in $\mathcal{M}_{fg}(\mathcal{U}_\Theta, N, \eta)$ one obtains a category of η-twisted holonomic $\mathcal{D}$-modules which are equivariant for the action of N. We will discuss the details of this construction in Section 3.

2.1 Standard and Simple Modules

In this section we briefly review McDowell's construction of standard Whittaker modules, which are a class of induced modules in $\mathcal{N}$ that generalize the Verma modules in category $\mathcal{O}$. For a choice of $\lambda \in \mathfrak{h}^*$ and $\eta \in \operatorname{ch}\mathfrak{n}$, we construct a standard Whittaker module $M(\lambda, \eta)$. When $\eta = 0$, these modules are Verma modules, and when η is nondegenerate, these modules are the irreducible modules studied by Kostant in [10]. For partially degenerate η, these modules share some structural properties with Verma modules and some structural properties with Kostant's nondegenerate modules. In particular, McDowell showed that the $M(\lambda, \eta)$ decompose into $\mathfrak{h}^\Theta$-weight spaces for the action of a certain subalgebra $\mathfrak{h}^\Theta \subset \mathfrak{h}$ depending on η. When $\eta = 0$, this subalgebra is equal to $\mathfrak{h}$ and McDowell's decomposition is the decomposition of a Verma module into finite-dimensional weight spaces. When η is nondegenerate, this subalgebra is trivial, so the entire module is a single infinite-dimensional weight space. After reviewing the construction of $M(\lambda, \eta)$, we generalize McDowell's result and show that all modules in $\mathcal{N}_\eta$ admit generalized $\mathfrak{h}^\Theta$-weight space decompositions. We also show that these $\mathfrak{h}^\Theta$-weight spaces are themselves Whittaker modules for a Levi subalgebra determined by η. This extra structure enables us to develop a new theory of formal characters for $\mathcal{N}$ in Section 2.2 which generalizes the theory of formal characters of highest weight modules (as described in [8, §1.15]).

For the remainder of this subsection, fix an $\mathfrak{n}$-character $\eta \in \operatorname{ch}\mathfrak{n}$. For $\alpha \in \Sigma$, let $\mathfrak{g}_\alpha$ be the root space corresponding to α. Then η determines a subset $\Theta \subset \Pi$ of the simple roots in the following way:

$$\Theta = \{\alpha \in \Pi : \eta|_{\mathfrak{g}_\alpha} \neq 0\}.$$

If $\Theta = \Pi$, we say that η is *nondegenerate*. We call a Whittaker module $V \in \mathcal{N}_\eta$ for η nondegenerate a *nondegenerate Whittaker module*. The cyclically generated Whittaker modules studied by Kostant in [10] are examples of nondegenerate Whittaker modules in our terminology.

Let $\Sigma_\Theta \subset \Sigma$ be the root subsystem generated by Θ, and $\Sigma_\Theta^+ = \Sigma^+ \cap \Sigma_\Theta$ the corresponding set of positive roots. Let W_Θ be the Weyl group of Σ_Θ, and $\rho_\Theta = \frac{1}{2}\sum_{\alpha\in\Sigma_\Theta^+}\alpha$.

Let

$$\mathfrak{n}_\Theta = \bigoplus_{\alpha\in\Sigma_\Theta^+} \mathfrak{g}_\alpha,\ \mathfrak{u}_\Theta = \bigoplus_{\alpha\in\Sigma^+-\Sigma_\Theta^+} \mathfrak{g}_\alpha,\ \bar{\mathfrak{n}}_\Theta = \bigoplus_{\alpha\in-\Sigma_\Theta^+} \mathfrak{g}_\alpha,\ \text{and } \bar{\mathfrak{u}}_\Theta = \bigoplus_{\alpha\in-\Sigma^+-(-\Sigma_\Theta^+)} \mathfrak{g}_\alpha.$$

In this way, the character η determines a reductive subalgebra $\mathfrak{l}_\Theta = \bar{\mathfrak{n}}_\Theta \oplus \mathfrak{h} \oplus \mathfrak{n}_\Theta$ of $\mathfrak{g}$ and a parabolic subalgebra $\mathfrak{p}_\Theta = \mathfrak{l}_\Theta \oplus \mathfrak{u}_\Theta$. The reductive Lie subalgebra $\mathfrak{l}_\Theta$ decomposes into the direct sum of a semisimple subalgebra $\mathfrak{s}_\Theta$ and its center $\mathfrak{z}_\Theta$. The semisimple subalgebra $\mathfrak{s}_\Theta$ in this decomposition is the derived subalgebra $[\mathfrak{l}_\Theta, \mathfrak{l}_\Theta]$, and it is easy to check that the center $\mathfrak{z}_\Theta$ is the subalgebra $\mathfrak{h}^\Theta = \{H \in \mathfrak{h} \mid \alpha(H) = 0, \alpha \in \Theta\} \subset \mathfrak{h}$.

Let $\gamma_\Theta : \mathcal{Z}(\mathfrak{l}_\Theta) \to \mathcal{U}(\mathfrak{h})$ be the untwisted Harish-Chandra homomorphism of $\mathcal{Z}(\mathfrak{l}_\Theta)$ [8, Ch. 1 §7]. Fix $\lambda \in \mathfrak{h}^*$, and define $\varphi_{\Theta,\lambda} : \mathcal{U}(\mathfrak{h}) \longrightarrow \mathbb{C}$ to be the homomorphism sending $H \in \mathfrak{h}$ to $(\lambda - \rho_\Theta)(H) \in \mathbb{C}$. The homomorphism

$$\Omega_{\Theta,\lambda} = \varphi_{\Theta,\lambda} \circ \gamma_\Theta : \mathcal{Z}(\mathfrak{l}_\Theta) \longrightarrow \mathbb{C} \tag{1}$$

is an infinitesimal character of $\mathcal{Z}(\mathfrak{l}_\Theta)$. This gives us a map associating elements of $\mathfrak{h}^*$ to maximal ideals in $\mathcal{Z}(\mathfrak{l}_\Theta)$:

$$\begin{aligned} \xi_\Theta : \mathfrak{h}^* &\longrightarrow \operatorname{Max}\mathcal{Z}(\mathfrak{l}_\Theta) \\ \lambda &\mapsto \ker(\Omega_{\Theta,\lambda}). \end{aligned}$$

From the data $(\lambda, \eta) \in \mathfrak{h}^* \times \operatorname{ch}\mathfrak{n}$, we construct an $\mathfrak{l}_\Theta$-module

$$Y(\lambda, \eta) = (\mathcal{U}(\mathfrak{l}_\Theta)/\xi_\Theta(\lambda)\mathcal{U}(\mathfrak{l}_\Theta)) \otimes_{\mathcal{U}(\mathfrak{n}_\Theta)} \mathbb{C}_\eta.$$

Here $\mathbb{C}_\eta$ is the one-dimensional $\mathcal{U}(\mathfrak{n}_\Theta)$-module where $\mathfrak{n}_\Theta$ acts by η. This induced module $Y(\lambda, \eta)$ is an irreducible $\mathfrak{l}_\Theta$-module [12, §2 Prop. 2.3].

Definition 2 The *standard Whittaker module* in $\mathcal{N}$ associated to $\lambda \in \mathfrak{h}^*$ and the character $\eta \in \operatorname{ch}\mathfrak{n}$ is the $\mathfrak{g}$-module

$$M(\lambda, \eta) = \mathcal{U}(\mathfrak{g}) \otimes_{\mathcal{U}(\mathfrak{p}_\Theta)} Y(\lambda - \rho + \rho_\Theta, \eta).$$

Here $Y(\lambda - \rho + \rho_\Theta, \eta)$ is viewed as a $\mathcal{U}(\mathfrak{p}_\Theta)$-module by letting $\mathfrak{u}_\Theta$ act trivially and $M(\lambda, \eta)$ is a $\mathfrak{g}$-module by left multiplication on the first factor.

To get a sense for this construction, it is useful to examine particular values of η. If $\eta = 0$, then Θ is empty, and $M(\lambda, 0) = \mathcal{U}(\mathfrak{g}) \otimes_{\mathcal{U}(\mathfrak{b})} Y(\lambda - \rho, 0)$ is a Verma module of highest weight $\lambda - \rho$. If η is nondegenerate, then $M(\lambda, \eta) = Y(\lambda, \eta)$ is an irreducible Whittaker module, as in [10].

Two such modules $M(\lambda, \eta)$ and $M(\mu, \eta)$ are isomorphic if and only if λ and μ are in the same W_Θ-orbit in $\mathfrak{h}^*$. McDowell showed that each standard Whittaker module $M(\lambda, \eta)$ has a unique irreducible quotient $L(\lambda, \eta)$, and all irreducible Whittaker modules appear as such quotients [12, §2 Thm. 2.9]. Clearly both $M(\lambda, \eta)$ and $L(\lambda, \eta)$ have infinitesimal character χ_λ and generalized $\mathfrak{n}$-character η, so they both lie in $\mathcal{N}_{\theta,\eta}$.

McDowell showed that the center $\mathfrak{h}^\Theta$ of $\mathfrak{l}_\Theta$ acts semisimply on $M(\lambda, \eta)$ [12, §2 Prop. 2.4(e)]. This decomposition will be necessary in the theory of formal characters established in the following section, so we briefly review it here. For any $\nu \in \mathfrak{h}^*$, we use bold to denote the restriction of ν to $\mathfrak{h}^{\Theta*}$; that is, $\boldsymbol{\nu} = \nu|_{\mathfrak{h}^\Theta} \in \mathfrak{h}^{\Theta*}$. There is a natural partial order on $\mathfrak{h}^{\Theta*}$ [12, §1 Prop. 1.8(a)]. Let $\Pi - \Theta = \{\alpha_1, \alpha_2, \cdots, \alpha_p\}$. Then $\{\boldsymbol{\alpha_1}, \cdots, \boldsymbol{\alpha_p}\}$ is a basis for $\mathfrak{h}^{\Theta*}$. For $\boldsymbol{\alpha}, \boldsymbol{\beta} \in \mathfrak{h}^{\Theta*}$, say that $\boldsymbol{\alpha} \le \boldsymbol{\beta}$ if

$$\boldsymbol{\beta} - \boldsymbol{\alpha} = c_1\boldsymbol{\alpha_1} + c_2\boldsymbol{\alpha_2} + \cdots + c_p\boldsymbol{\alpha_p}$$

for $c_i \in \mathbb{Z}_{\geq 0}$. For a module V in $\mathcal{N}_\eta$ and linear functional $\boldsymbol{\mu} \in \mathfrak{h}^{\Theta *}$, let $V_{\boldsymbol{\mu}} = \{v \in V | Xv = \boldsymbol{\mu}(X)v$ for all $X \in \mathfrak{h}^\Theta\}$ be the corresponding $\mathfrak{h}^\Theta$-weight space, and $V^{\boldsymbol{\mu}} = \{v \in V|$ for all $X \in \mathfrak{h}^\Theta, (X - \boldsymbol{\mu}(X))^k v = 0$ for some $k \in \mathbb{N}\}$ the corresponding generalized $\mathfrak{h}^\Theta$-weight space. If $V^{\boldsymbol{\mu}} \neq 0$, we say $\boldsymbol{\mu}$ is a $\mathfrak{h}^\Theta$-*weight* of V. Then we have the following decomposition:

$$M(\lambda, \eta) = \bigoplus_{\boldsymbol{\nu} \leq \boldsymbol{\lambda} - \boldsymbol{\rho}} M(\lambda, \eta)_{\boldsymbol{\nu}}.$$

Furthermore, $M(\lambda, \eta)_{\boldsymbol{\lambda}-\boldsymbol{\rho}} = Y(\lambda - \rho + \rho_\Theta)$, and $M(\lambda, \eta)_{\boldsymbol{\nu}} = \mathcal{U}(\bar{\mathfrak{u}}_\Theta)_{\boldsymbol{\mu}} \otimes_{\mathbb{C}} Y(\lambda - \rho + \rho_\Theta, \eta)$ for $\boldsymbol{\mu} \leq 0$ in $\mathfrak{h}^{\Theta *}$. (Here, we are using the fact that $\mathfrak{h}^\Theta$ acts semisimply on $\mathcal{U}(\bar{\mathfrak{u}}_\Theta)$ [12, §2 Lem. 2.2(a)].)

The $\mathfrak{h}^\Theta$-weight spaces of $M(\lambda, \eta)$ have a richer structure than just that of $\mathfrak{h}^\Theta$-modules, as the following proposition shows. Given an $\mathfrak{l}_\Theta$-module V, we denote by $\overline{V}$ the $\mathfrak{s}_\Theta$-module induced by the inclusion of $\mathfrak{s}_\Theta \subset \mathfrak{l}_\Theta$. Since $\mathfrak{s}_\Theta$ is semisimple, standard semisimplicity results apply to $\overline{V}$. Let $\mathcal{N}(\mathfrak{s}_\Theta)$ be the category of $\mathfrak{s}_\Theta$-Whittaker modules.

Proposition 1 *Let $M(\lambda, \eta) = \bigoplus_{\boldsymbol{\nu} \leq \boldsymbol{\lambda} - \boldsymbol{\rho}} M(\lambda, \eta)_{\boldsymbol{\nu}}$ be the decomposition of a standard Whittaker module in $\mathcal{N}_\eta$ into $\mathfrak{h}^\Theta$-weight spaces. For each $\boldsymbol{\nu} \in \mathfrak{h}^{\Theta *}$,*

(i) *$M(\lambda, \eta)_{\boldsymbol{\nu}}$ is a finite length $\mathfrak{l}_\Theta$-module, and*
(ii) *$\overline{M(\lambda, \eta)_{\boldsymbol{\nu}}}$ is an object in $\mathcal{N}(\mathfrak{s}_\Theta)$.*

Proof If $\eta = 0$, then $\mathfrak{h}^\Theta = \mathfrak{h}$ and $\mathfrak{s}_\Theta = 0$. In this setting, the assertion is trivially true, so we assume $\eta \neq 0$. The action of $\mathfrak{l}_\Theta$ commutes with the action of $\mathfrak{h}^\Theta$, so the $\mathfrak{h}^\Theta$-weight spaces of $M(\lambda, \eta)$ are $\mathfrak{l}_\Theta$-stable. This proves that $M(\lambda, \eta)_{\boldsymbol{\nu}}$ are $\mathfrak{l}_\Theta$-modules. The vector space $\mathcal{U}(\bar{\mathfrak{u}}_\Theta)_{\boldsymbol{\mu}}$ is finite dimensional because there are only finitely many ways that we can express a given $\boldsymbol{\mu} \leq 0$ in $\mathfrak{h}^{\Theta *}$ as a negative sum of roots in $\Pi - \Theta$. This implies that $M(\lambda, \eta)_{\boldsymbol{\nu}}$ is the tensor product of a finite dimensional $\mathfrak{l}_\Theta$-module with an irreducible Whittaker module. Such modules are of finite length and have composition factors which are irreducible Whittaker modules (for $\eta|_{\mathfrak{n}_\Theta}$) by [10, §4 Thm. 4.6]. Because categories of Whittaker modules are closed under extensions [16, §1], this in turn implies that $\overline{M(\lambda, \eta)_{\boldsymbol{\nu}}}$ is an object in $\mathcal{N}(\mathfrak{s}_\Theta)$. □

The $\mathfrak{h}^\Theta$-weight space structure of $M(\lambda, \eta)$ described in proposition 1 is also inherited by its unique irreducible quotient $L(\lambda, \eta)$. Moreover, because the unique maximal submodule $N \subset M(\lambda, \eta)$ has $\mathfrak{h}^\Theta$-weights which are strictly less than $\boldsymbol{\lambda} - \boldsymbol{\rho}$, $L(\lambda, \eta)$ has a unique maximal $\mathfrak{h}^\Theta$-weight, $\boldsymbol{\lambda} - \boldsymbol{\rho}$, with respect to the partial order on $\mathfrak{h}^{\Theta *}$, and all other weights of $L(\lambda, \eta)$ lie in a cone below this "highest" weight. The highest $\mathfrak{h}^\Theta$-weight space of a standard module in $\mathcal{N}$ and the highest $\mathfrak{h}^\Theta$-weight space of its unique irreducible quotient are both isomorphic to an irreducible $\mathfrak{l}_\Theta$-Whittaker module: $M(\lambda, \eta)_{\boldsymbol{\lambda}-\boldsymbol{\rho}} = L(\lambda, \eta)_{\boldsymbol{\lambda}-\boldsymbol{\rho}} = Y(\lambda - \rho + \rho_\Theta, \eta)$.

We finish this section by showing that all modules in $\mathcal{N}_\eta$ decompose into generalized $\mathfrak{h}^\Theta$-weight spaces, and these weight spaces are modules in $\mathcal{N}(\mathfrak{s}_\Theta)$.

Theorem 4 *Any object V in $\mathcal{N}_\eta$ admits a decomposition*

$$V = \bigoplus_{\boldsymbol{\mu} \in \mathfrak{h}^{\Theta *}} V^{\boldsymbol{\mu}}$$

where the generalized $\mathfrak{h}^{\Theta}$-weight spaces $V^{\boldsymbol{\mu}}$ are finite length $\mathfrak{l}_{\Theta}$-modules. Moreover, if we restrict the $\mathfrak{l}_{\Theta}$-action to the semisimple subalgebra $\mathfrak{s}_{\Theta} \subset \mathfrak{l}_{\Theta}$ and denote the resulting $\mathfrak{s}_{\Theta}$-module by $\overline{V^{\boldsymbol{\mu}}}$, the generalized $\mathfrak{h}^{\Theta}$-weight spaces $\overline{V^{\boldsymbol{\mu}}}$ of V are objects in $\mathcal{N}(\mathfrak{s}_{\Theta})$.

Proof It is enough to consider $V \in \mathcal{N}_{\theta,\eta}$. By [16, §1], these categories are stable under subquotients and extensions. The $\mathfrak{h}^{\Theta}$-semisimplicity of irreducible modules in $\mathcal{N}_{\theta,\eta}$ implies that all modules in $\mathcal{N}_{\theta,\eta}$ are $\mathcal{U}(\mathfrak{h}^{\Theta})$-finite. Because objects in $\mathcal{N}$ are finite length and exact sequences of $\mathfrak{g}$-modules in $\mathcal{N}_{\theta,\eta}$ descend to exact sequences of $\mathfrak{h}^{\Theta}$-weight spaces, the assertion follows from induction in the length of V. □

2.2 Character Theory

In this section, we use the decomposition of a module in $\mathcal{N}_{\eta}$ into generalized $\mathfrak{h}^{\Theta}$-weight spaces to develop a theory of formal characters in the category of Whittaker modules which generalizes the theory of formal characters of highest weight modules [8, Ch. 1 §13]. This character theory is new to the literature, though an alternate version of a character theory for Whittaker modules appeared in unpublished work [11]. The main result of this section is that the formal character of a module V in $\mathcal{N}_{\eta}$ completely determines its class in the Grothendieck group $K\mathcal{N}_{\eta}$.

Fix an $\mathfrak{n}$-character $\eta \in \operatorname{ch}\mathfrak{n}$, and let $K\mathcal{N}(\mathfrak{s}_{\Theta})$ be the Grothendieck group of the category $\mathcal{N}(\mathfrak{s}_{\Theta})$. For an object $V \in \mathcal{N}(\mathfrak{s}_{\Theta})$, we refer to the corresponding isomorphism class in $K\mathcal{N}(\mathfrak{s}_{\Theta})$ by $[V]$.

Definition 3 Let V be an object in $\mathcal{N}_{\eta}$. For $\eta \neq 0$, the *formal character* of V is

$$\operatorname{ch} V = \sum_{\boldsymbol{\mu} \in \mathfrak{h}^{\Theta *}} [\overline{V^{\boldsymbol{\mu}}}] e^{\boldsymbol{\mu}}$$

where $\overline{V^{\boldsymbol{\mu}}}$ is the restriction of the $\mathfrak{l}_{\Theta}$-module $V^{\boldsymbol{\mu}}$ to the semisimple subalgebra $\mathfrak{s}_{\Theta} \subset \mathfrak{l}_{\Theta}$, $[\overline{V^{\boldsymbol{\mu}}}]$ is the class of $\overline{V^{\boldsymbol{\mu}}}$ in the Grothendieck group $K\mathcal{N}(\mathfrak{s}_{\Theta})$, and $e^{\boldsymbol{\mu}}$ is a formal variable parameterized by $\boldsymbol{\mu} \in \mathfrak{h}^{\Theta *}$. For $\eta = 0$ and $V \in \mathcal{N}_0$ we define $\operatorname{ch} V = [V] \in K\mathcal{N}$.

A standard Whittaker module is completely determined by its formal character.

Proposition 2 *The following are equivalent.*

(i) $\operatorname{ch} M(\lambda, \eta) = \operatorname{ch} M(\nu, \eta)$.
(ii) $M(\lambda, \eta) = M(\nu, \eta)$.

Proof It is clear that (ii) implies (i). Assume that $\operatorname{ch} M(\lambda, \eta) = \operatorname{ch} M(\nu, \eta)$. Then $M(\lambda, \eta)$ and $M(\nu, \eta)$ have the same $\mathfrak{h}^{\Theta}$-weights, and $[\overline{M(\lambda, \eta)_{\boldsymbol{\mu}}}] = [\overline{M(\nu, \eta)_{\boldsymbol{\mu}}}]$ for any such $\mathfrak{h}^{\Theta}$-weight $\boldsymbol{\mu}$. This implies that $\boldsymbol{\lambda} - \boldsymbol{\rho}$ is an $\mathfrak{h}^{\Theta}$-weight of $M(\nu, \eta)$, so $\boldsymbol{\lambda} - \boldsymbol{\rho} \leq \boldsymbol{\nu} - \boldsymbol{\rho}$. But also, $\boldsymbol{\nu} - \boldsymbol{\rho}$ is an $\mathfrak{h}^{\Theta}$-weight of $M(\lambda, \eta)$, so $\boldsymbol{\nu} - \boldsymbol{\rho} \leq \boldsymbol{\lambda} - \boldsymbol{\rho}$ and thus $\boldsymbol{\lambda} - \boldsymbol{\rho} = \boldsymbol{\nu} - \boldsymbol{\rho}$. Because $M(\lambda, \eta)_{\boldsymbol{\lambda} - \boldsymbol{\rho}} = Y(\lambda - \rho + \rho_{\Theta}, \eta)$ and $M(\nu, \eta)_{\boldsymbol{\nu} - \boldsymbol{\rho}} = Y(\nu - \rho + \rho_{\Theta}, \eta)$, we have

$$[\overline{Y(\lambda - \rho + \rho_{\Theta}, \eta)}] = [\overline{Y(\nu - \rho + \rho_{\Theta}, \eta)}] \in K\mathcal{N}(\mathfrak{s}_{\Theta}).$$

Because the $\mathfrak{s}_{\Theta}$-modules $\overline{Y(\lambda - \rho + \rho_{\Theta})}$ and $\overline{Y(\nu - \rho + \rho_{\Theta})}$ are irreducible objects in $\mathcal{N}(\mathfrak{s}_{\Theta})$, the equality $[\overline{Y(\lambda - \rho + \rho_{\Theta}, \eta)}] = [\overline{Y(\nu - \rho + \rho_{\Theta}, \eta)}]$ of isomorphism classes in the Grothendieck group implies that $\overline{Y(\lambda-\rho+\rho_\Theta,\eta)} = \overline{Y(\nu-\rho+\rho_\Theta,\eta)}$ as $\mathfrak{s}_\Theta$-modules. Irreducible nondegenerate Whittaker modules are completely determined by their infinitesimal character [10, §3 Thm. 3.6.1], so both modules have infinitesimal character $\Omega_{\Theta,\lambda-\rho+\rho_\Theta}$. This is only possible if $W_\Theta \cdot \lambda = W_\Theta \cdot \nu$, which implies that $M(\lambda,\eta) = M(\nu,\eta)$. $\square$

Because any module V in $\mathcal{N}_{\theta,\eta}$ has infinitesimal character χ_λ for $\lambda \in \theta$, there are only finitely many irreducible modules in the category $\mathcal{N}_{\theta,\eta}$. Let $\{L(\lambda_1,\eta),\ldots,L(\lambda_m,\eta)\}$ be the distinct irreducible modules in $\mathcal{N}_{\theta,\eta}$, and let $S_0 = \{\boldsymbol{\lambda_1}-\boldsymbol{\rho},\ldots,\boldsymbol{\lambda_m}-\boldsymbol{\rho}\} \subset \mathfrak{h}^{\Theta*}$ be the collection of their highest $\mathfrak{h}^\Theta$-weights. Any module V in $\mathcal{N}_{\theta,\eta}$ must have composition factors on this list, so by Theorem 4, the $\mathfrak{h}^\Theta$-weights $\boldsymbol{\mu}$ of V that show up in the character must be of the form $\boldsymbol{\mu} = \boldsymbol{\lambda_i} - \boldsymbol{\rho} - \sum_{j=1}^{p} m_j \boldsymbol{\alpha_j}$ for $1 \le i \le m$ and $m_j \in \mathbb{Z}_{\ge 0}$.

Let $K\mathcal{N}_{\theta,\eta}$ be the Grothendieck group of the category $\mathcal{N}_{\theta,\eta}$. If V and W are isomorphic objects in $\mathcal{N}_{\theta,\eta}$, then $\operatorname{ch} V = \operatorname{ch} W$, and since character is additive on short exact sequences, we have a well-defined homomorphism

$$\operatorname{ch} : K\mathcal{N}_{\theta,\eta} \longrightarrow \prod_{\boldsymbol{\mu}\le S_0} K\mathcal{N}(\mathfrak{s}_\Theta)e^{\boldsymbol{\mu}}$$

given by $\operatorname{ch}[V] = \operatorname{ch} V$. Here $\boldsymbol{\mu} \le S_0$ means that $\boldsymbol{\mu} \le \boldsymbol{\lambda_i} - \boldsymbol{\rho}$ for some $\boldsymbol{\lambda_i} - \boldsymbol{\rho} \in S_0$. Our main result of this section is the following.

Theorem 5 $\operatorname{ch} : K\mathcal{N}_{\theta,\eta} \longrightarrow \prod_{\boldsymbol{\mu}\le S_0} K\mathcal{N}(\mathfrak{s}_\Theta)e^{\boldsymbol{\mu}}$ *is an injective homomorphism.*

Proof To show that ch is injective, it is enough to show that the set of characters $\{\operatorname{ch}[L(\lambda_1,\eta)],\ldots,\operatorname{ch}[L(\lambda_m,\eta)]\}$ is linearly independent. Consider a non-trivial linear combination

$$b_1 \operatorname{ch}[L(\lambda_1,\eta)] + \cdots + b_m \operatorname{ch}[L(\lambda_m,\eta)] = 0.$$

As before let $S_0 = \{\boldsymbol{\lambda_1}-\boldsymbol{\rho},\ldots,\boldsymbol{\lambda_m}-\boldsymbol{\rho}\} \subset \mathfrak{h}^{\Theta*}$ be the collection of the highest $\mathfrak{h}^\Theta$-weights of the irreducible objects in $\mathcal{N}_{\theta,\eta}$. Note that the elements $\{\lambda_i\}_{i=1}^m \subset \mathfrak{h}^*$ are distinct, but it is possible that when restricted to $\mathfrak{h}^\Theta$, $\boldsymbol{\lambda_i} = \boldsymbol{\lambda_j}$ for some $i \neq j$, so S_0 might have repeated elements. Choose a maximal element of this set, $\boldsymbol{\lambda_j} - \boldsymbol{\rho}$. Then $\boldsymbol{\lambda_j} - \boldsymbol{\rho}$ can only appear as a highest weight of modules in $\{L(\lambda_1,\eta),\ldots,L(\lambda_m,\eta)\}$.

Because the linear combination of irreducible characters vanishes, the coefficient of $e^{\boldsymbol{\lambda_j}-\boldsymbol{\rho}}$ must vanish as well. That coefficient is

$$b_{i_1}[\overline{L(\lambda_{i_1},\eta)}_{\boldsymbol{\lambda_j}-\boldsymbol{\rho}}] + \cdots + b_{i_n}[\overline{L(\lambda_{i_n},\eta)}_{\boldsymbol{\lambda_j}-\boldsymbol{\rho}}],$$

where $\{\lambda_{i_1},\ldots,\lambda_{i_n}\} \subset \{\lambda_1,\ldots,\lambda_m\}$ are the elements of $\mathfrak{h}^*$ so that $\boldsymbol{\lambda_{i_1}} - \boldsymbol{\rho} = \cdots = \boldsymbol{\lambda_{i_n}} - \boldsymbol{\rho} = \boldsymbol{\lambda_j} - \boldsymbol{\rho}$. Because the highest $\mathfrak{h}^\Theta$-weight space of an irreducible module in $\mathcal{N}$ is an irreducible Whittaker module for $\mathfrak{s}_\Theta$, we have a vanishing linear combination of isomorphism classes of irreducible objects in $K\mathcal{N}(\mathfrak{s}_\Theta)$:

$$b_{i_1}[\overline{Y(\lambda_{i_1}-\rho+\rho_\Theta,\eta)}] + \cdots + b_{i_n}[\overline{Y(\lambda_{i_n}-\rho+\rho_\Theta,\eta)}] = 0$$

Each of the classes in the above sum must be distinct because the corresponding irreducible modules are non-isomorphic, so we conclude that $b_{i_1} = \cdots = b_{i_n} = 0$, and ch must be injective. $\square$

This immediately implies the following corollary.

Corollary 2 *Let V and W be objects in $\mathcal{N}_{\theta,\eta}$. Then the following are equivalent:*

(i) $\operatorname{ch} V = \operatorname{ch} W$.
(ii) *V and W have the same composition factors.*

We complete this section with an explicit calculation of the formal character of a standard Whittaker module, which we will use in Section 4. Let $M(\lambda, \eta)$ be the standard Whittaker module determined by $\lambda \in \mathfrak{h}^*$ and $\eta \in \operatorname{ch} \mathfrak{n}$. Note that as an $\mathfrak{l}_\Theta$-module, $M(\lambda, \eta) = \mathcal{U}(\bar{\mathfrak{u}}_\Theta) \otimes_{\mathbb{C}} Y(\lambda - \rho + \rho_\Theta, \eta)$. The Cartan subalgebra $\mathfrak{h}$ acts semisimply on $\mathcal{U}(\bar{\mathfrak{u}}_\Theta)$, and the collection of $\mathfrak{h}$-weights of $\mathcal{U}(\bar{\mathfrak{u}}_\Theta)$ are

$$Q = \left\{ - \sum_{\alpha \in \Sigma^+ \setminus \Sigma_\Theta^+} m_\alpha \alpha : m_\alpha \in \mathbb{Z}_{\geq 0} \right\}.$$

As described in Section 2.1, $M(\lambda, \eta)$ decomposes into $\mathfrak{h}^\Theta$-weight spaces of the form

$$M(\lambda, \eta)_{\boldsymbol{\nu}} = \mathcal{U}(\bar{\mathfrak{u}}_\Theta)_{\boldsymbol{\mu}} \otimes_{\mathbb{C}} Y(\lambda - \rho + \rho_\Theta, \eta)$$

for $\boldsymbol{\mu} \leq 0$ in $\mathfrak{h}^{\Theta *}$. The $\mathfrak{h}^\Theta$-weight space of $\mathcal{U}(\bar{\mathfrak{u}}_\Theta)$ corresponding to a $\mathfrak{h}^\Theta$-weight $\boldsymbol{\mu} \leq 0$ is the sum of the $\mathfrak{h}$-weight spaces of $\mathcal{U}(\bar{\mathfrak{u}}_\Theta)$ corresponding to $\mathfrak{h}$-weights that restrict to $\boldsymbol{\mu}$ on $\mathfrak{h}^\Theta$; that is, for $\mu \in \mathfrak{h}^\Theta$,

$$\mathcal{U}(\bar{\mathfrak{u}}_\Theta)_{\boldsymbol{\mu}} = \sum_{\kappa \in Q, \kappa|_{\mathfrak{h}^\Theta} = \boldsymbol{\mu}} \mathcal{U}(\bar{\mathfrak{u}}_\Theta)_\kappa.$$

We define a function $p : Q \to \mathbb{N}$ by $p(\kappa) = \dim \mathcal{U}(\bar{\mathfrak{u}}_\Theta)_\kappa$. This function can be interpreted combinatorially as counting the number of distinct ways that $\nu \in \mathfrak{h}^*$ can be expressed as a sum of roots in $\Sigma^+ \setminus \Sigma_\Theta^+$. When $\Theta = \emptyset$, this is Kostant's partition function.

By [12, §2 Lem. 2.2(b)], each $\mathcal{U}(\bar{\mathfrak{u}}_\Theta)_{\boldsymbol{\mu}}$ is a finite-dimensional $\mathfrak{l}_\Theta$-module, so the $\mathfrak{s}_\Theta$-module $\overline{M(\lambda, \eta)_{\boldsymbol{\nu}}}$ is the direct sum of a finite-dimensional $\mathfrak{s}_\Theta$-module and an irreducible $\mathfrak{s}_\Theta$-module. This allow us to apply [10, §4 Thm. 4.6] and conclude that $\mathfrak{n}_\Theta$ acts on $\overline{M(\lambda, \eta)_{\boldsymbol{\nu}}}$ by the nondegenerate character $\eta|_{\mathfrak{n}_\Theta}$, and that $\overline{M(\lambda, \eta)_{\boldsymbol{\nu}}}$ has composition series length equal to $\dim \mathcal{U}(\bar{\mathfrak{u}}_\Theta)_{\boldsymbol{\mu}} = \sum_{\kappa \in Q, \kappa|_{\mathfrak{h}^\Theta} = \boldsymbol{\mu}} p(\kappa)$. Furthermore, [10, §4 Thm. 4.6] implies that the composition factors of $\overline{M(\lambda, \eta)_{\boldsymbol{\nu}}}$ are

$$\{Y(\lambda - \rho + \rho_\Theta + \kappa, \eta) \mid \kappa \in Q \text{ and } \boldsymbol{\kappa} = \boldsymbol{\mu}\}.$$

This implies that in the Grothendieck group $K\mathcal{N}(\mathfrak{s}_\Theta)$,

$$[\overline{M(\lambda, \eta)_{\boldsymbol{\nu}}}] = \sum_{\kappa \in Q, \kappa|_{\mathfrak{h}^\Theta} = \boldsymbol{\mu}} p(\kappa) [\overline{Y(\lambda - \rho + \rho_\Theta + \kappa, \eta)}].$$

Therefore,

$$\operatorname{ch} M(\lambda, \eta) = \sum_{\boldsymbol{\nu} \in \mathfrak{h}^{\Theta *}} [\overline{M(\lambda, \eta)_{\boldsymbol{\nu}}}] e^{\boldsymbol{\nu}} = \sum_{\kappa \in Q} p(\kappa) [\overline{Y(\lambda - \rho + \rho_\Theta + \nu, \eta)}] e^{\boldsymbol{\lambda} - \boldsymbol{\rho} + \boldsymbol{\kappa}}. \tag{2}$$

3 A Category of Twisted Sheaves

In this section, we introduce the geometric objects that correspond to Whittaker modules under Beilinson–Bernstein localization. Let X be the flag variety of $\mathfrak{g}$, and for $\lambda \in \mathfrak{h}^*$, let $\mathcal{D}_\lambda$ be the corresponding twisted sheaf of differential operators on X. (See Appendix A.3 for more details on this construction.) The geometric category that emerges as an analogue to the category $\mathcal{N}_{\theta,\eta}$ is a certain subcategory of the category $\mathcal{M}_{qc}(\mathcal{D}_\lambda)$ of quasi-coherent $\mathcal{D}_\lambda$-modules which is equivariant under the action of the Lie group $N = \operatorname{Int}\mathfrak{n}$. We start by describing this category of twisted Harish-Chandra sheaves[2] for a general Harish-Chandra pair $(\mathfrak{g}, K)$ to establish a parameterization of simple objects and to define standard and costandard objects. Then we specialize to the Harish-Chandra pair $(\mathfrak{g}, N)$ which describes our setting of Whittaker modules. The classification of simple η-twisted Harish-Chandra sheaves for an arbitrary Harish-Chandra pair $(\mathfrak{g}, K)$ appeared in [17], as did the idea of using holonomic duality to define costandard η-twisted Harish-Chandra sheaves. The results on costandard η-twisted Harish-Chandra sheaves in this section are new to the literature.

3.1 Twisted Harish-Chandra Sheaves

In this section we describe the category of twisted Harish-Chandra sheaves, following [17]. For details on our choice of notation and geometric conventions, see Appendix A. Fix a Harish-Chandra pair $(\mathfrak{g}, K)$ and linear form $\lambda \in \mathfrak{h}^*$. Let $\mathfrak{k}$ be the Lie algebra of K, and let $\eta : \mathfrak{k} \to \mathbb{C}$ be a Lie algebra morphism. We say that $\mathcal{V}$ is a $(\mathcal{D}_\lambda, K, \eta)$-module if

(i) $\mathcal{V}$ is a coherent $\mathcal{D}_\lambda$-module,
(ii) $\mathcal{V}$ is a K-equivariant $\mathcal{O}_X$-module, and
(iii) in $\operatorname{End}\mathcal{V}$, $\pi(\xi) = \mu(\xi) + \eta(\xi)$ for all $\xi \in \mathfrak{k}$, and the morphism

$$\mathcal{D}_\lambda \otimes \mathcal{V} \to \mathcal{V}$$

is K-equivariant. Here π is induced by the $\mathcal{D}_\lambda$-action and μ is the differential of the K-action.

We denote by $\mathcal{M}_{coh}(\mathcal{D}_\lambda, K, \eta)$ the category of $(\mathcal{D}_\lambda, K, \eta)$-modules, and we refer to the objects in this category as *η-twisted Harish-Chandra sheaves*. This category of twisted Harish-Chandra sheaves carries much of the same structure as the non-twisted category described in [14, Ch. 4]. In particular, any η-twisted Harish-Chandra sheaf is holonomic [17, Lem. 1.1] so all η-twisted Harish-Chandra sheaves have finite length [17, Cor. 1.2].

Irreducible η-twisted Harish-Chandra sheaves were classified in [17, §3]. An irreducible sheaf in $\mathcal{M}_{coh}(\mathcal{D}_\lambda, K, \eta)$ is uniquely determined by a pair (Q, τ) of a K-orbit $Q \subset X$ and an irreducible η-twisted connection τ on Q. All irreducible η-twisted Harish-Chandra sheaves $\mathcal{L}(Q, \tau)$ occur as unique irreducible subsheaves of standard η-twisted Harish-Chandra sheaves, which are defined as follows. Fix $x \in Q$, and let $\mathfrak{b}_x$ be the corresponding Borel subalgebra of $\mathfrak{g}$. Let S_x denote the stabilizer in K of x. Then the Lie algebra of S_x is $\mathfrak{k} \cap \mathfrak{b}_x$. Let $\mathfrak{c}$ be a Cartan subalgebra in $\mathfrak{g}$ contained in $\mathfrak{b}_x$, and $s : \mathfrak{h}^* \to \mathfrak{c}^*$ the specialization at x [15, §2]. Let μ denote the restriction of the specialization of $\lambda + \rho$ to $\mathfrak{k} \cap \mathfrak{b}_x$ and $i : Q \to X$ the inclusion of Q into X. Then in the notation of Appendix A, $(\mathcal{D}_\lambda)^i = \mathcal{D}_{Q,\mu}$ [7, App. A].

Definition 4 Let Q be a K-orbit in X, $i : Q \to X$ be the natural inclusion, and τ an irreducible $\mathcal{M}(\mathcal{D}_{Q,\mu}, K, \eta)$-module. Then $\mathcal{I}(Q, \tau) = i_+(\tau)$ is a holonomic $(\mathcal{D}_\lambda, K, \eta)$-module. We call $\mathcal{I}(Q, \tau)$ the *standard η-twisted Harish-Chandra sheaf* attached to (Q, τ).

[2] When $\eta = 0$, the twist disappears and this category is exactly the category of Harish-Chandra sheaves in [14, Ch. 4, §3].

Let us now see how holonomic duality can be used to define costandard objects in the category $\mathcal{M}_{coh}(\mathcal{D}_\lambda, K, \eta)$. For our fixed $\lambda \in \mathfrak{h}^*$ let $\theta \subset \mathfrak{h}^*$ be the Weyl group orbit of λ. Let $D^b_{coh}(\mathcal{M}(\mathcal{D}_\lambda))$ be the derived category of bounded complexes of coherent $\mathcal{D}_\lambda$-modules. We have a duality functor

$$\mathbb{D} : D^b_{coh}(\mathcal{M}(\mathcal{D}_\lambda)) \to D^b_{coh}(\mathcal{M}(\mathcal{D}_{-\lambda}))^{op}$$

given by the formula

$$\mathbb{D}(\mathcal{V}^\cdot) = RHom_{\mathcal{D}_\lambda}(\mathcal{V}^\cdot, \mathcal{D}_\lambda)[\dim X],$$

for $\mathcal{V}^\cdot \in D^b_{coh}(\mathcal{M}(\mathcal{D}_\lambda))$.

In the case of holonomic $\mathcal{D}_\lambda$-modules, we can use this duality on derived categories to define a notion of duality on modules. Let $\mathcal{M}_{hol}(\mathcal{D}_\lambda)$ be the thick subcategory of $\mathcal{M}_{coh}(\mathcal{D}_\lambda)$ consisting of holonomic $\mathcal{D}_\lambda$-modules. If $\mathcal{V}$ is an object in $\mathcal{M}_{hol}(\mathcal{D}_\lambda)$, then $\mathbb{D}(\mathcal{V})$ is a complex in $D^b_{coh}(\mathcal{M}(\mathcal{D}_{-\lambda}))$ with holonomic cohomology and $H^p(\mathbb{D}(\mathcal{V})) = 0$ for $p \neq 0$. Therefore, we can define a functor

$$^* : \mathcal{M}_{hol}(\mathcal{D}_\lambda) \to \mathcal{M}_{hol}(\mathcal{D}_{-\lambda})^{op}$$

by

$$\mathcal{V}^* = H^0(\mathbb{D}(\mathcal{V})).$$

This is the *holonomic duality functor*. We have the following result.

Theorem 6 (i) *The functor $\mathcal{V} \mapsto \mathcal{V}^*$ from $\mathcal{M}_{hol}(\mathcal{D}_\lambda)$ to $\mathcal{M}_{hol}(\mathcal{D}_{-\lambda})^{op}$ is an antiequivalence of categories.*
(ii) *The functor $\mathcal{V} \mapsto (\mathcal{V}^*)^*$ is isomorphic to the identity functor on $\mathcal{M}_{hol}(\mathcal{D}_\lambda)$.*

We use the holonomic duality functor to construct costandard objects in the category $\mathcal{M}_{coh}(\mathcal{D}_\lambda, K, \eta)$ as follows. Let Q be a K-orbit in X and τ an irreducible $\mathcal{M}(\mathcal{D}_{Q,\mu}, K, \eta)$-module. Let $\mathcal{L}(Q, \tau)$ be the corresponding irreducible η-twisted Harish-Chandra sheaf, and $\mathcal{I}(Q, \tau)$ the corresponding standard η-twisted Harish-Chandra sheaf. Then $\mathcal{L}(Q, \tau)$ is an irreducible holonomic $\mathcal{D}_\lambda$-module supported on the closure of the orbit Q. Therefore, by Theorem 6, $\mathcal{L}(Q, \tau)^*$ is an irreducible holonomic $\mathcal{D}_{-\lambda}$-module whose support is contained in the closure of Q.

Lemma 1

$$\mathcal{L}(Q, \tau^*)^* = \mathcal{L}(Q, \tau).$$

Proof Let $\partial Q = \overline{Q} - Q$ and $X' = X - \partial Q$. Then $j : Q \to X'$ is a closed immersion, and $k : X' \to X$ is an open immersion. We have an exact sequence of η-twisted Harish-Chandra sheaves

$$0 \to \mathcal{L}(Q, \tau) \to \mathcal{I}(Q, \tau) \to \mathcal{Q} \to 0,$$

where $\mathcal{Q} = \mathcal{I}(Q, \tau)/\mathcal{L}(Q, \tau)$. One can show that $\mathcal{Q}$ is supported on ∂Q [17, §3]. Because k is an open immersion, k^+ is exact, and for any $\mathcal{D}_\lambda$-module $\mathcal{V}$, $k^+(\mathcal{V}) = \mathcal{V}|_{X'}$. Therefore, by restricting to X' we see that $\mathcal{L}(Q, \tau)|_{X'} = \mathcal{I}(Q, \tau)|_{X'}$. Because duality is local, we have

$$\mathcal{L}(Q, \tau)^*|_{X'} = (\mathcal{L}(Q, \tau)|_{X'})^* = (\mathcal{I}(Q, \tau)|_{X'})^* = j_+(\tau)^*.$$

Moreover, by Kashiwara's equivalence of categories (Theorem 15), j_+ commutes with duality, so we have

$$\mathcal{L}(Q, \tau)^*|_{X'} = j_+(\tau^*).$$

On the other hand, τ^* is an irreducible η-twisted K-equivariant connection on Q compatible with $(-\lambda+\rho,\eta)$. Hence,

$$\mathcal{L}(Q,\tau)^*|_{X'} = j_+(\tau^*) = \mathcal{L}(Q,\tau^*)|_{X'},$$

and we see that

$$\mathcal{L}(Q,\tau)^* = \mathcal{L}(Q,\tau^*).$$

Dualizing, we obtain the desired result. □

This leads us to our definition of costandard objects in the category $\mathcal{M}_{coh}(\mathcal{D}_\lambda, K, \eta)$.

Definition 5 Let Q be a K-orbit in X, $i : Q \to X$ be the natural inclusion, and τ an irreducible $\mathcal{M}(\mathcal{D}_{Q,\mu}, K, \eta)$-module. The η-twisted Harish-Chandra sheaf $\mathcal{M}(Q,\tau) = \mathcal{I}(Q,\tau^*)^*$ is the *costandard η-twisted Harish-Chandra sheaf* attached to the geometric data (Q,τ).

There is a natural inclusion $\mathcal{L}(Q,\tau^*) \to \mathcal{I}(Q,\tau^*)$. By dualizing, we get a natural epimorphism $\mathcal{M}(Q,\tau) \to \mathcal{L}(Q,\tau)$, so $\mathcal{L}(Q,\tau)$ is a quotient of $\mathcal{M}(Q,\tau)$. The main properties of costandard η-twisted Harish-Chandra sheaves are the following.

Proposition 3 (i) *The length of $\mathcal{M}(Q,\tau)$ is equal to the length of $\mathcal{I}(Q,\tau)$.*
(ii) *The irreducible η-twisted Harish-Chandra sheaf $\mathcal{L}(Q,\tau)$ is the unique irreducible quotient of $\mathcal{M}(Q,\tau)$. The kernel of this projection is supported on ∂Q.*

Proof Duality preserves irreducibility and $\mathcal{L}(Q',\tau'^*)^* = \mathcal{L}(Q',\tau')$ for any irreducible η-twisted Harish-Chandra sheaf $\mathcal{L}(Q',\tau')$, so by Lemma 1, the composition factors of $\mathcal{M}(Q,\tau)$ must be equal to those of $\mathcal{I}(Q,\tau)$. This proves (i). Furthermore, we have a short exact sequence of $\mathcal{D}_{-\lambda}$-modules

$$0 \to \mathcal{L}(Q,\tau^*) \to \mathcal{I}(Q,\tau^*) \to \mathcal{Q} \to 0,$$

where $\mathcal{Q}$ is a holonomic $\mathcal{D}_{-\lambda}$-module supported in ∂Q. Applying holonomic duality to this, we get a short exact sequence of $\mathcal{D}_\lambda$-modules

$$0 \to \mathcal{Q}^* \to \mathcal{M}(Q,\tau) \to \mathcal{L}(Q,\tau) \to 0.$$

Because $\mathcal{L}(Q,\tau^*)$ is the unique irreducible submodule of $\mathcal{I}(Q,\tau^*)$ and duality preserves support, this implies that the kernel $\mathcal{Q}^*$ of the projection map $\mathcal{M}(Q,\tau) \to \mathcal{L}(Q,\tau)$ is the unique maximal submodule of $\mathcal{M}(Q,\tau)$ and is supported in ∂Q. This proves the proposition. □

We complete this section with a proposition (Proposition 4) which will be of use in computing global sections of η-twisted Harish-Chandra sheaves in Section 4. The proof of the proposition uses the following three lemmas.

Lemma 2 *If $\mathcal{V}$ is a object in $\mathcal{M}_{coh}(\mathcal{D}_\lambda, K, \eta)$ such that $[\mathcal{V}] = [\mathcal{I}(Q,\tau)]$ in the Grothendieck group $K\mathcal{M}_{coh}(\mathcal{D}_\lambda, K, \eta)$, then there exists a nontrivial morphism from $\mathcal{V}$ into $\mathcal{I}(Q,\tau)$.*

Proof Let $i : Q \to X$ be the natural inclusion. As in the proof of Lemma 1, we can write i as the composition of a closed immersion $j : Q \to X' := X - \partial Q$ and an open immersion

$k : X' \to X$. Because the quotient $\mathcal{Q} := \mathcal{I}(Q, \tau)/\mathcal{L}(Q, \tau)$ is supported on ∂Q and the restriction functor $k^+ = |_{X'}$ is exact, we have

$$\mathcal{I}(Q, \tau)|_{X'} = \mathcal{L}(Q, \tau)|_{X'}.$$

In $K\mathcal{M}_{coh}(\mathcal{D}_\lambda, K, \eta)$, $[\mathcal{V}] - [\mathcal{L}(Q, \tau)] = [\mathcal{Q}]$, so all other composition factors of $\mathcal{V}$ must be supported in ∂Q. Hence

$$\mathcal{V}|_{X'} = \mathcal{L}(Q, \tau)|_{X'}$$

as well. Since k_+ is right adjoint to $|_{X'}$, we have

$$\mathrm{Hom}(\mathcal{V}, \mathcal{I}(Q, \tau)) = \mathrm{Hom}(\mathcal{V}|_{X'}, j_+(\tau)) = \mathrm{Hom}(\mathcal{L}(Q, \tau)|_{X'}, \mathcal{L}(Q, \tau)|_{X'}) \neq 0.$$

This proves the lemma. □

Lemma 3 *If $\mathcal{V}$ is an object in $\mathcal{M}_{coh}(\mathcal{D}_\lambda, K, \eta)$ such that $[\mathcal{V}] = [\mathcal{M}(Q, \tau)]$ in the Grothendieck group $K\mathcal{M}_{coh}(\mathcal{D}_\lambda, K, \eta)$, then there exists a nontrivial morphism from $\mathcal{M}(Q, \tau)$ into $\mathcal{V}$.*

Proof By dualizing the morphism in Lemma 2, we know that if $[\mathcal{V}^*] = [\mathcal{M}(Q, \tau^*)]$ in $K\mathcal{M}_{coh}(\mathcal{D}_{-\lambda}, K, \eta)$, then there exists a nontrivial morphism from $M(Q, \tau^*)$ into $\mathcal{V}^*$. Applying this fact to $\mathcal{V}^*$ proves the lemma. □

Lemma 4 *If $\mathcal{V}$ is an object in $\mathcal{M}_{coh}(\mathcal{D}_\lambda, K, \eta)$ such that $[\mathcal{V}] = [\mathcal{M}(Q, \tau)]$ and $\mathcal{V}$ has $\mathcal{L}(Q, \tau)$ as a unique irreducible quotient, then $\mathcal{V} \simeq \mathcal{M}(Q, \tau)$.*

Proof By Lemma 3, there is a nontrivial morphism $f : \mathcal{M}(Q, \tau) \to \mathcal{V}$. Because $\mathcal{L}(Q, \tau)$ is the unique irreducible quotient of $\mathcal{M}(Q, \tau)$ (Proposition 3), the image of f has $\mathcal{L}(Q, \tau)$ as a composition factor. If the image of f is not all of $\mathcal{V}$, then it is contained in the unique maximal submodule of $\mathcal{V}$. But then the image of f cannot have $\mathcal{L}(Q, \tau)$ as a composition factor. Hence f must be surjective. The objects $\mathcal{V}$ and $\mathcal{M}(Q, \tau)$ have the same length, so the kernel of f is zero. We conclude that f is an isomorphism. □

We can use the preceding lemmas to relate global sections of η-twisted Harish-Chandra sheaves to η-twisted Harish-Chandra modules. For a regular W-orbit $\theta \subset \mathfrak{h}^*$ and Lie algebra morphism $\eta : \mathfrak{k} \to \mathbb{C}$, let $\mathcal{M}_{fg}(\mathcal{U}_\theta, K, \eta)$ be the category of η-twisted Harish-Chandra modules, as in [17][§1].[3]

Proposition 4 *Let $\lambda \in \theta \subset \mathfrak{h}^*$ be antidominant and regular, and $\{\mathcal{M}(Q, \tau)\} \subset \mathcal{M}_{coh}(\mathcal{D}_\lambda, K, \eta)$ the set of costandard η-twisted Harish-Chandra sheaves. Let $\{M(Q, \tau)\}$ be a family of modules in $\mathcal{M}_{fg}(\mathcal{U}_\theta, K, \eta)$ parameterized by the pairs (Q, τ) such that*

(i) *each $M(Q, \tau)$ has a unique irreducible quotient $L(Q, \tau)$, and*
(ii) *in $K\mathcal{M}_{fg}(\mathcal{U}_\theta, K, \eta)$, $[\Gamma(X, \mathcal{M}(Q, \tau))] = [M(Q, \tau)]$.*

Then $\Gamma(X, \mathcal{L}(Q, \tau)) = L(Q, \tau)$ and $\Gamma(X, \mathcal{M}(Q, \tau)) = M(Q, \tau)$.

[3] The definition in Section 2 is a special case of this category for $K = N$.

Proof We prove the proposition by induction on the dimension of Q. Assume that Q is of minimal dimension. Then $\mathcal{M}(Q,\tau)$ is irreducible. Because λ is antidominant and regular, $\Gamma(X,\mathcal{M}(Q,\tau))$ must be irreducible. The modules $\Gamma(X,\mathcal{M}(Q,\tau))$ and $M(Q,\tau)$ have the same composition factors because they have the same class in the Grothendieck group, so $\Gamma(X,\mathcal{M}(Q,\tau)) = M(Q,\tau)$. Because $\mathcal{M}(Q,\tau) = \mathcal{L}(Q,\tau)$, this proves the proposition in the base case.

Let Q be of dimension n, and assume that (i) and (ii) hold for all Q' of dimension less than or equal to n. Because $\mathcal{M}(Q,\tau)$ has $\mathcal{L}(Q,\tau)$ as its unique irreducible quotient, all other composition factors of $\mathcal{M}(Q,\tau)$ are of the form $\mathcal{L}(Q',\tau')$ for orbits Q' which are contained in ∂Q. By the induction assumption, the composition factors of $\Gamma(X,\mathcal{M}(Q,\tau))$ are $\Gamma(X,\mathcal{L}(Q',\tau')) = L(Q',\tau')$ and $\Gamma(X,\mathcal{L}(Q,\tau))$. But $\mathcal{L}(Q,\tau) \neq \mathcal{L}(Q',\tau')$ for $Q \neq Q'$, so $\Gamma(X,\mathcal{L}(Q,\tau)) \neq L(Q',\tau')$. Since $M(Q,\tau)$ has $L(Q,\tau)$ as a unique irreducible quotient and $[M(Q,\tau)] = [\Gamma(X,\mathcal{M}(Q,\tau))]$ in the Grothendieck group, we must have that $\Gamma(X,\mathcal{L}(Q,\tau)) = L(Q,\tau)$. This proves the first statement.

It follows that $\Delta_\lambda(M(Q,\tau))$ has unique irreducible quotient $\Delta_\lambda(L(Q,\tau)) = \mathcal{L}(Q,\tau)$. Therefore, by Lemma 4, $\Delta_\lambda(M(Q,\tau)) \simeq \mathcal{M}(Q,\tau)$. This completes the proof. □

3.2 The Harish-Chandra Pair $(\mathfrak{g}, N)$

Now we specialize to the setting of Whittaker modules. Let $K = N = \operatorname{Int}\mathfrak{n}$. Let $\mathfrak{b}$ be the unique Borel subalgebra of $\mathfrak{g}$ containing $\mathfrak{n} = \operatorname{Lie}N$. The pair $(\mathfrak{g}, N)$ is a Harish-Chandra pair. By the discussion in Section 3.1, standard objects in $\mathcal{M}_{coh}(\mathcal{D}_\lambda, N, \eta)$ are parameterized by pairs (Q,τ), where Q is an N-orbit and τ is an irreducible N-equivariant connection in $\mathcal{M}_{coh}(\mathcal{D}_{Q,\mu}, N, \eta)$. In the setting of the Harish-Chandra pair $(\mathfrak{g}, N)$, we can describe these pairs more explicitly.

The N-orbits on X are Bruhat cells $C(w)$, $w \in W$. Our fixed character $\eta \in \operatorname{ch}\mathfrak{n}$ determines a parabolic subgroup $P_\Theta \subset G$ such that $\operatorname{Lie}P_\Theta = \mathfrak{p}_\Theta$ as in Section 2.1. The P_Θ-orbits on X are unions of Bruhat cells [14, Ch. 6 §1 Lem. 1.9], and for each P_Θ-orbit, there is a unique Bruhat cell which is open in that orbit. There is a bijection between the P_Θ-orbits in X and the cosets $W_\Theta\backslash W$, and the partial order on orbits determined by closure corresponds to the partial order on $W_\Theta\backslash W$ inherited from the Bruhat order on longest coset representatives [14, Ch. 6 §1 Prop. 1.10, Prop 1.11]. Furthermore, the Weyl group element w parameterizing the unique open Bruhat cell in a P_Θ-orbit is the unique longest coset representative w^C in the corresponding coset C. In [17, §4], Miličić and Soergel established that the only N-orbits admitting compatible connections[4] are Bruhat cells $C(w)$ that are open in some P_Θ-orbit. They also established that the only irreducible η-twisted N-equivariant $\mathcal{O}_{C(w)}$-modules on such Bruhat cells are $\mathcal{O}_{C(w)}$. Therefore, our standard, simple, and costandard objects in the category $\mathcal{M}_{coh}(\mathcal{D}_\lambda, N, \eta)$ are the following.

Definition 6 For the parameters $C \in W_\Theta\backslash W$, $\lambda \in \mathfrak{h}^*$ and $\eta \in \operatorname{ch}\mathfrak{n}$, we define $\mathcal{I}(w^C, \lambda, \eta)$ to be the standard η-twisted Harish-Chandra sheaf corresponding to the N-orbit $C(w^C)$ and the compatible connection $\mathcal{O}_{C(w^C)}$ on $C(w^C)$. (Here w^C is the unique longest coset representative of C.) We refer to the corresponding irreducible η-twisted Harish-Chandra sheaf by $\mathcal{L}(w^C, \lambda, \eta)$ and the corresponding costandard η-twisted Harish-Chandra sheaf by $\mathcal{M}(w^C, \lambda, \eta)$.

[4]That is, the only orbits on which there exist nontrivial irreducible $(\mathcal{D}_{Q,\mu}, N, \eta)$-modules

Remark 2 The parameter $\lambda \in \mathfrak{h}^*$ in this definition emerges in the direct image functor, $i_+ : \mathcal{M}(\mathcal{D}_{Q,\mu}) \to \mathcal{M}(\mathcal{D}_\lambda)$, whose construction depends on λ. (See Appendix A.2 for more details.)

It is clear that the global sections of irreducible η-twisted Harish-Chandra sheaves for the Harish-Chandra pair $(\mathfrak{g}, N)$ are η-twisted Harish-Chandra modules for the same Harish-Chandra pair. Under the equivalence of the categories $\mathcal{M}_{fg}(\mathcal{U}_\theta, N, \eta)$ and $\mathcal{N}_{\theta,\eta}$ [17, §2 Lem. 2.3], these irreducible η-twisted Harish-Chandra modules correspond to irreducible Whittaker modules. Recall that the goal of this paper is to develop an algorithm for computing composition multiplicities of standard Whittaker modules. From the arguments above, we see that converting this multiplicity question to the geometric setting of twisted Harish-Chandra sheaves amounts to showing that the global sections of either costandard or standard η-twisted Harish-Chandra sheaves are standard Whittaker modules. We will do this in Section 4, but first we establish some useful results on the action of intertwining functors on costandard η-twisted Harish-Chandra sheaves.

3.3 Intertwining Functors and U-Functors

For $\lambda \in \mathfrak{h}^*$ and $w \in W$, one can construct an "intertwining functor" which sends $\mathcal{D}_\lambda$-modules to $\mathcal{D}_{w\lambda}$-modules. These functors play a crucial role in our geometric arguments in Section 5, so we use this section to record some of their key properties. Detailed development of these properties can be found in [14, Ch. 3 §3].

The orbits of the diagonal action of $G = \mathrm{Int}(\mathfrak{g})$ on $X \times X$ are smooth subvarieties, and can be parameterized in the following way. Given x, y in X and corresponding Borel subalgebras $\mathfrak{b}_x, \mathfrak{b}_y$, we can choose a Cartan subalgebra $\mathfrak{c}$ contained in $\mathfrak{b}_x \cap \mathfrak{b}_y$. Let $\mathfrak{n}_x = [\mathfrak{b}_x, \mathfrak{b}_x]$ and $\mathfrak{n}_y = [\mathfrak{b}_y, \mathfrak{b}_y]$. Then $\mathfrak{b}_x$ and $\mathfrak{b}_y$ determine specializations [15, §2] of $(\mathfrak{h}^*, \Sigma, \Sigma^+)$ into $(\mathfrak{c}^*, R, R_x^+)$, and $(\mathfrak{c}^*, R, R_y^+)$, respectively, where R is the root system of $(\mathfrak{g}, \mathfrak{c})$, $R_x^+ \subset R$ is the collection of positive roots determined by $\mathfrak{n}_x$, and $R_y^+ \subset R$ is the collection of positive roots determined by $\mathfrak{n}_y$. The positive root systems R_x^+ and R_y^+ are related by $w(R_x^+) = R_y^+$ for some Weyl group element $w \in W$, and this w does not depend on choice of Cartan subalgebra in $\mathfrak{b}_x \cap \mathfrak{b}_y$. We say that $\mathfrak{b}_y$ is in *relative position* w with respect to $\mathfrak{b}_x$. It is clear that $\mathfrak{b}_x$ is in relative position w^{-1} with respect to $\mathfrak{b}_y$. For $w \in W$, let

$$Z_w = \{(x, y) \in X \times X | \mathfrak{b}_y \text{ is in relative position } w \text{ with respect to } \mathfrak{b}_x\}. \tag{3}$$

This gives us a parameterization of G-orbits in $X \times X$.

Lemma 5 [14, Ch. 3 §3 Lem. 3.1]

(i) *Sets Z_w for $w \in W$ are smooth subvarieties of $X \times X$.*
(ii) *The map $w \mapsto Z_w$ is a bijection of W onto the set of G-orbits in $X \times X$.*

Denote by p_1 and p_2 the projections of Z_w onto the first and second factors of $X \times X$, respectively. Then p_i for $i = 1, 2$ are locally trivial fibrations with fibers isomorphic to affine spaces of dimension $\ell(w)$. Moreover, they are affine morphisms [14, Ch. 3 §3 Lem. 3.2]. Let $\omega_{Z_w|X}$ be the invertible $\mathcal{O}_{Z_w}$-module of top degree relative differential forms for the projection $p_1 : Z_w \to X$ and let $\mathcal{T}_w$ be its inverse sheaf. Then $\mathcal{T}_w = p_1^*(\mathcal{O}(\rho - w\rho))$, and there is a natural isomorphism [14, Ch. 3 §3 Lem. 3.3]

$$(\mathcal{D}_{w\lambda})^{p_1} = (\mathcal{D}_\lambda^{p_2})^{\mathcal{T}_w}.$$

The morphism $p_2 : Z_w \to X$ is a surjective submersion, so the inverse image functor

$$p_2^+ : \mathcal{M}(\mathcal{D}_\lambda) \to \mathcal{M}(\mathcal{D}_\lambda^{p_2})$$

is exact. Because twisting by an invertible sheaf is also an exact functor, we can define a functor

$$LI_w : D^b(\mathcal{M}(\mathcal{D}_\lambda)) \to D^b(\mathcal{M}(\mathcal{D}_{w\lambda}))$$

by the formula

$$LI_w(\mathcal{V}^\cdot) = p_{1+}(\mathcal{T}_w \otimes_{\mathcal{O}_{Z_w}} p_2^+(\mathcal{V}^\cdot))$$

for $\mathcal{V}^\cdot \in D^b(\mathcal{M}(\mathcal{D}_\lambda))$. This is the left derived functor of the functor

$$I_w : \mathcal{M}(\mathcal{D}_\lambda) \to \mathcal{M}(\mathcal{D}_{w\lambda}),$$

where for $\mathcal{V} \in \mathcal{M}(\mathcal{D}_\lambda)$,

$$I_w(\mathcal{V}) = H^0 p_{1+}(\mathcal{T}_w \otimes_{\mathcal{O}_{Z_w}} p_2^+(\mathcal{V})).$$

We call the right exact functor I_w the *intertwining functor* attached to $w \in W$.

In the case where w is a simple root, we can define a related collection of *U-functors*, which have desirable semisimplicity properties. Let $\alpha \in \Pi$ be a simple root, and denote by X_α the variety of parabolic subalgebras of type α. Let p_α be the natural projection of X onto X_α, and let $Y_\alpha = X \times_{X_\alpha} X$ be the fiber product of X with X relative to the morphism p_α. Denote by q_1 and q_2 the projections of Y_α onto the first and second factors, respectively. Then we have the following commutative diagram:

$$\begin{tikzcd} Y_\alpha \arrow[r, "q_2"] \arrow[d, "q_1"'] & X \arrow[d, "p_\alpha"] \\ X \arrow[r, "p_\alpha"] & X_\alpha. \end{tikzcd}$$

There is a natural embedding of Y_α into $X \times X$ that identifies Y_α with the closed subvariety $Z_1 \cup Z_{s_\alpha}$ of $X \times X$. Under this identification, Z_1 is a closed subvariety of Y_α, and Z_{s_α} is an open, dense, affinely embedded subvariety of Y_α [14, Ch. 3 §8 Lem. 8.1].

Let $\lambda \in \mathfrak{h}^*$ be such that $p = -\alpha^\vee(\lambda)$ is an integer. Let $\mathcal{L}$ be the invertible O_{Y_α}-module on Y_α given by

$$\mathcal{L} = q_1^*(\mathcal{O}((-p+1)s_\alpha\rho + \alpha)) \otimes_{\mathcal{O}_{Y_\alpha}} q_2^*(\mathcal{O}((-p+1)\rho))^{-1}.$$

This allows us to define functors

$$U^j : \mathcal{M}_{qc}(\mathcal{D}_\lambda) \to \mathcal{M}_{qc}(\mathcal{D}_{s_\alpha\lambda})$$

by the formula

$$U^j(\mathcal{V}) = H^j q_{1+}(q_2^+(\mathcal{V}) \otimes_{\mathcal{O}_{Y_\alpha}} \mathcal{L})$$

for $\mathcal{V} \in \mathcal{M}_{qc}(\mathcal{D}_\lambda)$ [14, Ch. 3 §8, Lem. 8.2]. These functors first appeared in [14] as geometric analogues to the U_α functors in [20], and they play a critical role in the algorithm of Section 5 for their semisimplicity properties. Because the fibers of q_1 are one-dimensional, $U^j = 0$ for $j \neq -1, 0, 1$. If $\mathcal{V}$ is irreducible, the relationship between $U^j(\mathcal{V})$ and $I_{s_\alpha}(\mathcal{V})$ is captured in the following theorem.

Theorem 7 [14, Ch. 3 §8 Thm. 8.4] *Let $\lambda \in \mathfrak{h}^*$ be such that $p = -\alpha^\vee(\lambda)$ is an integer, and $\mathcal{V} \in \mathcal{M}_{qc}(\mathcal{D}_\lambda)$ an irreducible $\mathcal{D}_\lambda$-module. Then either*

(i) *$U^{-1}(\mathcal{V}) = U^1(\mathcal{V}) = \mathcal{V}(p\alpha)$ and $U^0(\mathcal{V}) = 0$, and in this case $I_{s_\alpha}(\mathcal{V}) = 0$ and $L^{-1}I_{s_\alpha}(\mathcal{V}) = \mathcal{V}(p\alpha)$; or*

(ii) $U^{-1}(\mathcal{V}) = U^1(\mathcal{V}) = 0$, *and in this case* $L^{-1}I_{s_\alpha}(\mathcal{V}) = 0$ *and there exists a natural exact sequence*

$$0 \to U^0(\mathcal{V}) \to I_{s_\alpha}(\mathcal{V}) \to \mathcal{V}(p\alpha) \to 0.$$

The module $U^0(\mathcal{V})$ *is the largest proper quasicoherent* $\mathcal{D}_{s_\alpha\lambda}$*-submodule of* $I_{s_\alpha}(\mathcal{V})$.

3.4 Intertwining Functors on Standard and Costandard Sheaves

In this section we examine the action of intertwining functors on standard and costandard η-twisted Harish-Chandra sheaves in the category $\mathcal{M}_{coh}(\mathcal{D}_\lambda, N, \eta)$. These results will be critical in establishing the relationship between $\mathcal{N}_{\theta,\eta}$ and $\mathcal{M}_{coh}(\mathcal{D}_\lambda, N, \eta)$, and are new to the literature. Let $\alpha \in \Pi$, $w \in W$, and p_i for $i = 1, 2$ the projections of Z_{s_α} (3) onto the first and second coordinates, respectively. As in Section 3.2, let $\mathfrak{b}$ be the unique Borel subalgebra of $\mathfrak{g}$ containing $\mathfrak{n} = \mathrm{Lie}N$. We start with a useful lemma.

Lemma 6 *The projection* $p_1 : Z_{s_\alpha} \to X$ *induces an immersion of* $p_2^{-1}(C(w))$ *into* X*, and its image is equal to* $C(ws_\alpha)$.

Proof If $y \in C(w)$, then $\mathfrak{b}_x$ is in relative position s_α with respect to $\mathfrak{b}_y$ if and only if $x \in C(ws_\alpha)$. Therefore, $p_2^{-1}(C(w)) = C(ws_\alpha) \times C(w)$, which implies the result. □

Our first result is the following proposition.

Proposition 5 *Let* $C \in W_\Theta\backslash W$ *and* $\alpha \in \Pi$ *be such that* $Cs_\alpha > C$*, and let* $\lambda \in \mathfrak{h}^*$ *be arbitrary. Then*

$$LI_{s_\alpha}(\mathcal{I}(w^C, \lambda, \eta)) = \mathcal{I}(w^C s_\alpha, s_\alpha\lambda, \eta).$$

Proof The diagram

$$\begin{array}{ccc} p_2^{-1}(C(w^C)) & \xrightarrow{j} & Z_{s_\alpha} \\ {\scriptstyle pr_2}\downarrow & & \downarrow{\scriptstyle p_2} \\ C(w^C) & \xrightarrow{i_{w^C}} & X \end{array}$$

commutes. Furthermore, p_2 and $pr_2 = p_2|_{p_2^{-1}(C(w^C))}$ are surjective submersions and j and i_{w^C} are affine immersions, so p_2^+, pr_2^+, i_{w^C+}, and j_+ are all exact. Thus,

$$p_2^+(\mathcal{I}(w^C, \lambda, \eta)) = p_2^+(i_{w^C+}(\mathcal{O}_{C(w^C)})) \tag{4}$$
$$= j_+(pr_2^+(\mathcal{O}_{C(w^C)})) \tag{5}$$
$$= j_+(\mathcal{O}_{p_2^{-1}(C(w^C))}). \tag{6}$$

Here Eq. 4 is the definition of $\mathcal{I}(w^C, \lambda, \eta)$, Eq. 5 is base change, and Eq. 6 follows from the fact that $\dim Z_{s_\alpha} - \dim X = \dim p_2^{-1}(C(w^C)) - \dim C(w^C)$.

Applying the projection formula of Proposition 11 to the morphism p_1, the line bundle $\mathcal{L} = \mathcal{O}(\rho - s_\alpha\rho)$, and the twisted sheaf of differential operators $\mathcal{D}_\lambda$ on X, we obtain the following commutative diagram:

$$\begin{array}{ccc} \mathcal{M}(\mathcal{D}_\lambda^{p_1}) & \xrightarrow{p_{1+}} & \mathcal{M}(\mathcal{D}_\lambda) \\ {\scriptstyle p_1^*(\mathcal{L})\otimes_{\mathcal{O}_{Z_\alpha}}-}\downarrow & & \downarrow{\scriptstyle \mathcal{L}\otimes_{\mathcal{O}_X}-} \\ \mathcal{M}((\mathcal{D}_\lambda^{\mathcal{L}})^{p_1}) & \xrightarrow{p_{1+}} & \mathcal{M}(\mathcal{D}_\lambda^{\mathcal{L}}). \end{array}$$

We compute

$$\begin{aligned} LI_{s_\alpha}(\mathcal{I}(w^C,\lambda,\eta)) &= p_{1+}(\mathcal{T}_{s_\alpha} \otimes_{\mathcal{O}_{Z_{s_\alpha}}} p_2^+(\mathcal{I}(w^C,\lambda,\eta))) && (7)\\ &= p_{1+}(\mathcal{T}_{s_\alpha} \otimes_{\mathcal{O}_{Z_{s_\alpha}}} j_+(\mathcal{O}_{p_2^{-1}(C(w^C))})) && (8)\\ &= p_{1+}(p_1^*(\mathcal{O}(\rho - s_\alpha\rho)) \otimes_{\mathcal{O}_{Z_{s_\alpha}}} j_+(\mathcal{O}_{p_2^{-1}(C(w^C))})) && (9)\\ &= \mathcal{O}(\rho - s_\alpha\rho) \otimes_{\mathcal{O}_X} p_{1+}(j_+(\mathcal{O}_{p_2^{-1}(C(w^C))})). && (10) \end{aligned}$$

Here Eq. 7 follows from the definition of intertwining functors, Eq. 8 from the Eqs. 4–6 above, Eq. 9 from the fact that $\mathcal{T}_{s_\alpha} = p_1^*(\mathcal{O}(\rho - s_\alpha\rho))$, and Eq. 10 from the projection formula diagram.

By Lemma 6, we have a commutative diagram

$$\begin{array}{ccc} p_2^{-1}(C(w^C)) & \xrightarrow{j} & Z_{s_\alpha} \\ {\scriptstyle pr_1}\downarrow & & \downarrow{\scriptstyle p_1} \\ C(w^C s_\alpha) & \xrightarrow{i_{w^C s_\alpha}} & X. \end{array}$$

where $pr_1 = p_1|_{C(w)}$.

Picking up our previous computation, this lets us further conclude that

$$\begin{aligned} (10) &= \mathcal{O}(\rho - s_\alpha\rho) \otimes_{\mathcal{O}_X} i_{w^C s_\alpha +}(pr_{1+}(\mathcal{O}_{p_2^{-1}(C(w^C))})) && (11)\\ &= \mathcal{O}(\rho - s_\alpha\rho) \otimes_{\mathcal{O}_X} i_{w^C s_\alpha +}(\mathcal{O}_{C(w^C s_\alpha)}) && (12)\\ &= \mathcal{I}(w^C s_\alpha, s_\alpha\lambda, \eta). && (13) \end{aligned}$$

In this final computation, Eq. 11 follows from the commutative diagram immediately preceding it, Eq. 12 from Lemma 6, and Eq. 13 from the definition of $\mathcal{I}(w^C s_\alpha, s_\alpha\lambda, \eta)$ and [14, Ch.2 §2]. □

For $C \in W_\Theta\backslash W$, let $\mathcal{M}(w^C, \lambda, \eta)$ be the corresponding costandard η-twisted Harish-Chandra sheaf in the category $\mathcal{M}_{coh}(\mathcal{D}_\lambda, N, \eta)$. Our second result is the following.

Proposition 6 *Let $C \in W_\Theta\backslash W$ and $\alpha \in \Pi$ be such that $Cs_\alpha < C$, and let $\lambda \in \mathfrak{h}^*$ be arbitrary. Then*

$$I_{s_\alpha}(\mathcal{M}(w^C,\lambda,\eta)) = \mathcal{M}(w^C s_\alpha, s_\alpha\lambda, \eta),$$

and

$$L^p I_{s_\alpha}(\mathcal{M}(w^C,\lambda,\eta)) = 0 \textit{ for } p \neq 0.$$

Proof By Proposition 5 applied to the coset Cs_α and linear form $-\lambda \in \mathfrak{h}^*$, we have

$$\mathcal{I}(w^C, -\lambda, \eta) = LI_{s_\alpha}(\mathcal{I}(w^C s_\alpha, -s_\alpha\lambda, \eta)).$$

Applying holonomic duality, we get

$$\begin{aligned}\mathcal{M}(w^C, \lambda, \eta) &= \mathbb{D}(LI_{s_\alpha}(\mathcal{I}(w^C s_\alpha, -s_\alpha\lambda, \eta))) \\ &= (\mathbb{D} \circ LI_{s_\alpha} \circ \mathbb{D})(\mathcal{M}(w^C s_\alpha, s_\alpha\lambda, \eta))\end{aligned}$$

By [14, Ch. 3 §4 Thm. 4.4], $\mathbb{D} \circ LI_{s_\alpha} \circ \mathbb{D}$ is the quasi-inverse of the intertwining functor LI_{s_α}, so applying LI_{s_α} to both sides of the above equation proves the proposition. □

Combined with [14, Ch. 3 §3 Cor. 3.22], this implies the following result.

Theorem 8 *If $\lambda \in \mathfrak{h}^*$ is α-antidominant, and $C \in W_\Theta \backslash W$ is such that $Cs_\alpha < C$, we have*

$$H^p(X, \mathcal{M}(w^C, \lambda, \eta)) = H^p(X, \mathcal{M}(w^C s_\alpha, s_\alpha\lambda, \eta))$$

for any $p \in \mathbb{Z}_+$.

The final result of this section is a technical lemma which uses Proposition 6 to relate costandard η-twisted Harish-Chandra sheaves supported on arbitrary P_Θ-orbits to costandard η-twisted Harish-Chandra sheaves supported on the unique closed P_Θ-orbit. This lemma will be critical in the arguments of Section 4. Recall that every coset $C \in W_\Theta \backslash W$ has a unique longest coset representative w^C and unique shortest coset representative w_C [14, Ch. 6 §1 Thm. 1.4]. If $w_\Theta \in W_\Theta$ is the longest element, then by [14, Ch. 6 §1 Thm. 1.2 Thm. 1.4], we have $w_\Theta w_C = w^C$, and $\ell(w_\Theta w_C) = \ell(w_\Theta) + \ell(w_C) = \ell(w^C)$.

Lemma 7 *Let $\lambda \in \mathfrak{h}^*$ be arbitrary. For any $C \in W_\Theta \backslash W$,*

$$I_{w_C}(\mathcal{M}(w^C, \lambda, \eta)) = \mathcal{M}(w_\Theta, w_C\lambda, \eta),$$

and

$$L^p I_{w_C}(\mathcal{M}(w^C, \lambda, \eta)) = 0 \textit{ for } p \neq 0.$$

Proof We proceed by induction in $\ell(w_C)$. If $\ell(w_C) = 0$, then $C = W_\Theta$, and the assertion is trivially true. If $\ell(w_C) = 1$, then w_C is a simple reflection s_α for $\alpha \in \Pi - \Theta$. Then $\ell(w_\Theta s_\alpha) = \ell(w_\Theta) + 1$ and $W_\Theta s_\alpha > W_\Theta$. By Proposition 6,

$$I_{s_\alpha}(\mathcal{M}(w_\Theta s_\alpha, \lambda, \eta)) = \mathcal{M}(w_\Theta, s_\alpha\lambda, \eta),$$

and

$$L^p I_{s_\alpha}(\mathcal{M}(w_\Theta s_\alpha, \lambda, \eta)) = 0 \text{ for } p \neq 0.$$

Now let $C \in W_\Theta \backslash W$ be arbitrary and assume that

$$I_{w_C}(\mathcal{M}(w^C, \lambda, \eta)) = \mathcal{M}(w_\Theta, w_C\lambda, \eta) \text{ and } L^p I_{w_C}(\mathcal{M}(w^C, \lambda, \eta)) = 0 \text{ for } p \neq 0.$$

Let $\alpha \in \Pi$ be such that $Cs_\alpha > C$. By [14, Ch. 6 §1 Prop. 1.6], the shortest element w_{Cs_α} in Cs_α is $w_C s_\alpha$. Thus,

$$\begin{aligned}I_{w_C s_\alpha}(\mathcal{M}(w^C s_\alpha, \lambda, \eta)) &= I_{w_C}(I_{s_\alpha}(\mathcal{M}(w^C s_\alpha, \lambda, \eta))) \\ &= I_{w_C}(\mathcal{M}(w^C, s_\alpha\lambda, \eta)) \\ &= \mathcal{M}(w_\Theta, w_C s_\alpha\lambda, \eta).\end{aligned}$$

Here the first equality follows from the "product formula" for intertwining functors [14, Ch. 3 §3 Cor. 3.8] and the second equality from Proposition 6. This completes the proof of the lemma by induction. □

4 Geometric Description of Whittaker Modules

In this section we establish the connection between the category of Whittaker modules and the category of twisted Harish-Chandra sheaves by proving that global sections of costandard twisted Harish-Chandra sheaves are standard Whittaker modules. The theorem is proven in three steps: first, we establish the result for costandard sheaves where the parameter $\eta \in \operatorname{ch}\mathfrak{n}$ is nondegenerate; then, we prove that the formal characters align properly for costandard sheaves corresponding to the smallest P_Θ-orbit (where the parameter η is allowed to be arbitrary); finally, we extend the result to all costandard sheaves. This proof is new to the literature, though an alternate proof of this relationship was given in the unpublished work [11]. This allows us to use geometric arguments to draw conclusions about our algebraic category of Whittaker modules, which will be essential in the interpretation of the algorithm developed in Section 5. Our main tool in this section is the theory of formal characters developed in Section 2.2.

We begin by examining the nondegenerate case. Let w_0 be the longest element of the Weyl group W of $\mathfrak{g}$.

Proposition 7 *Let $\eta \in \operatorname{ch}\mathfrak{n}$ be nondegenerate and $\lambda \in \mathfrak{h}^*$. Then*

$$\Gamma(X, \mathcal{M}(w_0, \lambda, \eta)) = M(w_0\lambda, \eta).$$

Proof If η is nondegenerate, then $W = W_\Theta$, so by [17, Thm. 5.1], there exists a unique irreducible object $\mathcal{L}(w_0, \lambda, \eta) = \mathcal{I}(w_0, \lambda, \eta) = \mathcal{M}(w_0, \lambda, \eta) = \mathcal{D}_\lambda \otimes_{\mathcal{U}(\mathfrak{n})} \mathbb{C}_\eta$ in $\mathcal{M}_{coh}(\mathcal{D}_\lambda, N, \eta)$. Assume λ is antidominant, and let $\theta \subset \mathfrak{h}^*$ be the W-orbit of λ. Then by [17, Thm. 5.2],

$$\Gamma(X, \mathcal{M}(w_0, \lambda, \eta)) = \mathcal{U}_\theta \otimes_{\mathcal{U}(\mathfrak{n})} \mathbb{C}_\eta = M(w_0\lambda, \eta).$$

Now, in order to deal with general $\lambda \in \mathfrak{h}^*$, let $w \in W$ be arbitrary. By the preceding argument (first equality) and [14, Ch. 3 §3 Thm 3.23] (second equality), we have

$$M(w_0\lambda, \eta) = R\Gamma(\mathcal{M}(w_0, \lambda, \eta)) = R\Gamma(LI_w(\mathcal{M}(w_0, \lambda, \eta))) = R\Gamma(\mathcal{C}^\cdot),$$

where $\mathcal{C}^\cdot$ is a complex in $D^b(\mathcal{D}_{w\lambda})$ such that for any $i \in \mathbb{Z}$, $\mathcal{C}^i$ is a finite sum of copies of the unique irreducible object $\mathcal{M}(w_0, w\lambda, \eta)$. (The last equality follows from [17, §5 Thm. 5.6].) Because the image of $M(w_0\lambda, \eta)$ in the derived category is a complex with a single irreducible object in degree zero and zeros elsewhere and $R\Gamma$ is an equivalence of derived categories, the equality above implies that

$$LI_w(\mathcal{M}(w_0, \lambda, \eta)) = \mathcal{M}(w_0, w\lambda, \eta).$$

Therefore,

$$\Gamma(X, \mathcal{M}(w_0, w\lambda, \eta)) = M(w_0\lambda, \eta) = M(w_0 w\lambda, \eta).$$

This completes the proof of the proposition. □

Proposition 8 *Let $\eta \in \operatorname{ch}\mathfrak{n}$ be arbitrary, $\lambda \in \mathfrak{h}^*$, and $\theta \subset \mathfrak{h}^*$ the Weyl group orbit of λ. In the Grothendieck group $K\mathcal{M}_{fg}(\mathcal{U}_\theta, N, \eta)$,*

$$[\Gamma(X, \mathcal{M}(w_\Theta, \lambda, \eta))] = [M(w_\Theta\lambda, \eta)].$$

Here w_Θ is the longest element in the Weyl group W_Θ determined by Θ. We will prove the proposition in a series of steps. Our first step is to realize the standard sheaf corresponding to the smallest P_Θ-orbit as the direct image of a twisted Harish-Chandra sheaf for the flag variety of $\mathfrak{l}_\Theta$. Let $P(w_\Theta)$ be the P_Θ-orbit with open Bruhat cell $C(w_\Theta) \subset P(w_\Theta)$. Because w_Θ is minimal in the set of longest coset representatives [14, Ch. 6 §1 Lem. 1.7], $P(w_\Theta)$ is a closed subvariety of X. Because $P(w_\Theta)$ is an orbit of an algebraic group action it is also a smooth subvariety of X. In fact, $P(w_\Theta)$ is isomorphic to the flag variety of $\mathfrak{l}_\Theta$. In particular, by [14, Ch. 6, §1, Lem. 1.9], we have the following orbit decomposition $P(w_\Theta) = \bigcup_{t \in W_\Theta} C(tw_\Theta) = \bigcup_{w \in W_\Theta} C(w)$. Let

$$i_{w_\Theta} : C(w_\Theta) \to P(w_\Theta),\ j : P(w_\Theta) \to X, \text{ and } i : C(w_\Theta) \to X$$

be the natural inclusions, so $i = j \circ i_{w_\Theta}$ is the composition of an open immersion and a closed immersion. By definition, $\mathcal{I}(w_\Theta, \lambda, \eta) = j_+(\mathcal{F})$, where $\mathcal{F} = i_{w_\Theta +}(\mathcal{O}_{C(w_\Theta)})$, and $\mathcal{O}_{C(W_\Theta)}$ is the N-equivariant connection in $\mathcal{M}_{coh}(\mathcal{D}_\lambda^i, N, \eta)$ described in Section 3.2.

Lemma 8 *The sheaf $\mathcal{F}$ is the standard object $\mathcal{I}(w_\Theta, \lambda + \rho - \rho_\Theta, \eta|_{\mathfrak{n}_\Theta})$ in the category $\mathcal{M}_{coh}(\mathcal{D}_{P(w_\Theta),\lambda+\rho}, N_\Theta, \eta|_{\mathfrak{n}_\Theta})$ corresponding to the open Bruhat cell $C(w_\Theta) \subset P(w_\Theta)$.*

Proof As described above, we can view $P(w_\Theta)$ as the flag variety for $\mathfrak{l}_\Theta$, and the character $\eta|_{\mathfrak{n}_\Theta}$ is nondegenerate on $\mathfrak{l}_\Theta$. The irreducible N-equivariant connection $\mathcal{O}_{C(w_\Theta)}$ is compatible with $(\lambda, \eta) \in \mathfrak{h}^* \times \mathrm{ch}\,\mathfrak{n}$ by construction. We can restrict the N-action to $N_\Theta \subset N$, and consider $\mathcal{O}_{C(w_\Theta)}$ as an irreducible N_Θ-equivariant connection compatible with $(\lambda, \eta|_{\mathfrak{n}_\Theta}) \in \mathfrak{h}^* \times \mathfrak{n}_\Theta^*$. This allows us to interpret $\mathcal{F} = i_{w_\Theta +}(\mathcal{O}_{C(w_\Theta)})$ as the standard sheaf on the flag variety of $\mathfrak{l}_\Theta$ induced from the irreducible N_Θ-equivariant connection $\mathcal{O}_{C(w_\Theta)}$ on $C(w_\Theta)$ in $\mathcal{M}_{coh}((\mathcal{D}_\lambda^j)^i, N_\Theta, \eta|_{\mathfrak{n}_\Theta})$. (Note that because $\eta|_{\mathfrak{n}_\Theta}$ is nondegenerate, this is the only standard $\eta|_{\mathfrak{n}_\Theta}$-twisted Harish-Chandra sheaf in the category $\mathcal{M}_{coh}(\mathcal{D}_\lambda^j, N_\Theta, \eta|_{\mathfrak{n}_\Theta})$ by [17, Thm. 5.1].) Because

$$\mathcal{D}_\lambda^j = (\mathcal{D}_{X,\lambda+\rho})^j = \mathcal{D}_{P(w_\Theta),\lambda+\rho} = \mathcal{D}_{\lambda+\rho-\rho_\Theta},$$

we have that

$$\mathcal{F} = \mathcal{I}(w_\Theta, \lambda + \rho - \rho_\Theta, \eta|_{\mathfrak{n}_\Theta}).$$

This completes the proof. □

Our next step is to use the normal degree filtration (Appendix A.2) to analyze the global sections of the standard sheaf $\mathcal{I}(w_\Theta, \lambda, \eta)$. We will do so using the theory of formal characters established in Section 2.2. By Lemma 8, we can express our standard sheaf $\mathcal{I}(w_\Theta, \lambda, \eta) = j_+(\mathcal{F})$, where $\mathcal{F} = \mathcal{I}(w_\Theta, \lambda + \rho - \rho_\Theta, \eta|_{\mathfrak{n}_\Theta})$. Because $j : P(w_\Theta) \to X$ is a closed immersion, this implies that $\mathcal{I}(w_\Theta, \lambda, \eta)$ has a filtration by normal degree, $F_n\mathcal{I}(w_\Theta, \lambda, \eta)$. Let $Gr\mathcal{I}(w_\Theta, \lambda, \eta)$ be the associated graded sheaf. Let $ch : \mathcal{N}_{\theta,\eta} \longrightarrow \prod_{\mu \leq S_0} K\mathcal{N}([\mathfrak{l}_\Theta, \mathfrak{l}_\Theta])e^\mu$ be the formal character function described in Section 2.2.

Lemma 9 $ch\Gamma(X, Gr\mathcal{I}(w_\Theta, \lambda, \eta)) = ch\Gamma(X, \mathcal{I}(w_\Theta, \lambda, \eta))$.

Proof By construction, we have

$$\Gamma(X, \mathcal{I}(w_\Theta, \lambda, \eta)) = \varinjlim \Gamma(X, F_n\mathcal{I}(w_\Theta, \lambda, \eta)).$$

For each $n \in \mathbb{Z}_+$, we have an exact sequence

$$0 \to F_{n-1}\mathcal{I}(w_\Theta, \lambda, \eta) \to F_n\mathcal{I}(w_\Theta, \lambda, \eta) \to Gr_n\mathcal{I}(w_\Theta, \lambda, \eta) \to 0.$$

We claim that $H^p(X, Gr_n\mathcal{I}(w_\Theta, \lambda, \eta)) = 0$ for $p > 0$. To see this, note that by construction, $Gr_n\mathcal{I}(w_\Theta, \lambda, \eta)$ is the sheaf-theoretic direct image of a sheaf on $P(w_\Theta)$ which has a finite filtration such that the graded pieces are standard $\eta|_{\mathfrak{n}_\Theta}$-twisted Harish-Chandra sheaves on the flag variety $P(w_\Theta)$ of $\mathfrak{l}_\Theta$. These have vanishing cohomologies by the proof of Proposition 7, which implies the claim. The short exact sequence above gives rise to a long exact sequence

$$\begin{aligned}0 &\to \Gamma(X, F_{n-1}\mathcal{I}(w_\Theta, \lambda, \eta)) \to \Gamma(X, F_n\mathcal{I}(w_\Theta, \lambda, \eta)) \to \Gamma(X, Gr_n\mathcal{I}(w_\Theta, \lambda, \eta)) \to \\ &\to H^1(X, F_{n-1}\mathcal{I}(w_\Theta, \lambda, \eta)) \to H^1(X, F_n\mathcal{I}(w_\Theta, \lambda, \eta)) \to 0 \to \cdots\end{aligned}$$

Using induction on n and the preceding paragraph, we see that $H^p(X, F_n\mathcal{I}(w_\Theta, \lambda, \eta)) = 0$ for $p > 0$, and therefore $H^p(X, \mathcal{I}(w_\Theta, \lambda, \eta)) = 0$ for $p > 0$. This implies that for each $n \in \mathbb{Z}_+$, we have a short exact sequence

$$0 \to \Gamma(X, F_{n-1}\mathcal{I}(w_\Theta, \lambda, \eta)) \to \Gamma(X, F_n\mathcal{I}(w_\Theta, \lambda, \eta)) \to \Gamma(X, Gr_n\mathcal{I}(w_\Theta, \lambda, \eta)) \to 0.$$

Note that if $\lambda \in \mathfrak{h}^*$ is antidominant, the existence of this short exact sequence follows from the exactness of Γ, but this argument above holds for arbitrary $\lambda \in \mathfrak{h}^*$. This gives us a filtration of $\Gamma(X, \mathcal{I}(w_\Theta, \lambda, \eta))$, with associated graded module

$$\begin{aligned}\Gamma(X, Gr\mathcal{I}(w_\Theta, \lambda, \eta)) &= \bigoplus \Gamma(X, Gr_n\mathcal{I}(w_\Theta, \lambda, \eta)) \\ &= \bigoplus \Gamma(X, F_n\mathcal{I}(w_\Theta, \lambda, \eta))/\Gamma(X, F_{n-1}\mathcal{I}(w_\Theta, \lambda, \eta)).\end{aligned}$$

Because the formal character sums over short exact sequences, we have

$$ch\Gamma(X, Gr_n\mathcal{I}(w_\Theta, \lambda, \eta)) = ch\Gamma(X, F_n\mathcal{I}(w_\Theta, \lambda, \eta)) - ch\Gamma(X, F_{n-1}\mathcal{I}(w_\Theta, \lambda, \eta)).$$

Now we compute the formal character, using the fact that it distributes through direct sums.

$$\begin{aligned}ch\Gamma(X, Gr\mathcal{I}(w_\Theta, \lambda, \eta)) &= ch \bigoplus_{n\in\mathbb{Z}_+} \Gamma(X, Gr_n\mathcal{I}(w_\Theta, \lambda, \eta)) \\ &= \sum_{n\in\mathbb{Z}_+} (ch\Gamma(X, F_n\mathcal{I}(w_\Theta, \lambda, \eta)) - ch\Gamma(X, F_{n-1}\mathcal{I}(w_\Theta, \lambda, \eta))) \\ &= ch\Gamma(X, \mathcal{I}(w_\Theta, \lambda, \eta)).\end{aligned}$$

This completes the proof. □

This reduces our calculation of the formal character of $\Gamma(X, \mathcal{I}(w_\Theta, \lambda, \eta))$ to the calculation of the formal character of $\Gamma(X, Gr\mathcal{I}(w_\Theta, \lambda, \eta))$. Before completing this calculation, we need a few more supporting lemmas.

The adjoint action of the Borel $\mathfrak{b}$ on $\bar{\mathfrak{u}}_\Theta$ extends to an action of $\mathfrak{b}$ on the universal enveloping algebra $\mathcal{U}(\bar{\mathfrak{u}}_\Theta)$. The $\mathfrak{h}$-weights of this action are

$$Q = \left\{ -\sum_{\alpha\in\Sigma^+\setminus\Sigma_\Theta^+} m_\alpha\alpha \;\middle|\; m_\alpha \in \mathbb{Z}_{\geq 0} \right\}.$$

Let $\mathcal{N}_{X|P(w_\Theta)} = j^*(\mathcal{T}_X)/\mathcal{T}_{P(w_\Theta)}$ be the normal sheaf of $P(w_\Theta)$ in X and $S(\mathcal{N}_{X|P(w_\Theta)})$ the corresponding sheaf of symmetric algebras.

Lemma 10 *As $\mathcal{O}_{P(w_\Theta)}$-modules,*

$$S(\mathcal{N}_{X|P(w_\Theta)}) = \bigoplus_{\mu\in Q} \mathcal{O}(\mu).$$

Proof For any $x \in P(w_\Theta)$, there is an equivalence of categories between the category $\mathcal{M}_{qc}(\mathcal{O}_{P(w_\Theta)}, P_\Theta)$ of quasicoherent P_Θ-equivariant $\mathcal{O}_{P(w_\Theta)}$-modules and the category of algebraic representations of $B_x = stab_{P_\Theta}\{x\}$ given by taking the geometric fiber of a sheaf $\mathcal{F}$ in $\mathcal{M}_{qc}(\mathcal{O}_{P(w_\Theta)}, P_\Theta)$. Under this correspondence, the one-dimensional representation $\mathbb{C}_\mu$ of weight μ corresponds to the sheaf $\mathcal{O}_{P(w_\Theta)}(\mu)$.

Let $x_0 \in X$ be the point corresponding to B. The P_Θ-orbit of x_0 in X is the unique closed P_Θ-orbit, so it must be equal to $P(w_\Theta)$. In particular, $x_0 \in P(w_\Theta)$, so we have an equivalence of the category $\mathcal{M}_{qc}(\mathcal{O}_{P(w_\Theta)}, P_\Theta)$ with the category of algebraic representations of B. Under this equivalence, the normal sheaf $\mathcal{N}_{X|P(w_\Theta)}$ corresponds to the Adjoint representation of B on $\bar{\mathfrak{u}}_\Theta$, or, equivalently, the adjoint representation of $\mathfrak{b}$ on $\bar{\mathfrak{u}}_\Theta$.

Therefore to analyze the $\mathcal{O}_{P(w_\Theta)}$-module $S(\mathcal{N}_{X|P(w_\Theta)})$, we can examine the symmetric algebra $S(\bar{\mathfrak{u}}_\Theta)$, viewed as a $\mathfrak{b}$-module under the inherited action of the adjoint representation of $\mathfrak{b}$ on $\bar{\mathfrak{u}}_\Theta$. The universal enveloping algebra $\mathcal{U}(\bar{\mathfrak{u}}_\Theta)$ has a PBW filtration such that the associated graded module $Gr\mathcal{U}(\bar{\mathfrak{u}}_\Theta)$ is isomorphic to $S(\bar{\mathfrak{u}}_\Theta)$. Under the adjoint action, $\mathcal{U}(\bar{\mathfrak{u}}_\Theta)$ decomposes into $\mathfrak{h}$-weight spaces corresponding to weights in Q. Therefore, the $\mathfrak{b}$-module $S(\bar{\mathfrak{u}}_\Theta)$ decomposes into $\mathfrak{h}$-weight spaces corresponding to the same weights in Q.

For $k \in \mathbb{Z}_{\geq 0}$, consider $V = S^k(\bar{\mathfrak{u}}_\Theta)$. There is a $\mathfrak{b}$-invariant filtration

$$0 = F_0 V \subset F_1 V \subset \cdots \subset F_n V = V$$

such that $F_i V / F_{i-1} V = \mathbb{C}_\mu$, where $\mu \in Q$ is an $\mathfrak{h}$-weight of $S^k(\bar{\mathfrak{u}}_\Theta)$. This induces a filtration of $\mathcal{V} = S^k(\mathcal{N}_{X|P(w_\Theta)})$

$$0 = F_0 \mathcal{V} \subset F_1 \mathcal{V} \subset \cdots \subset F_n \mathcal{V} = \mathcal{V}$$

where each $F_i \mathcal{V}$ is a P_Θ-equivariant subsheaf and $F_i \mathcal{V} / F_{i+1} \mathcal{V} = \mathcal{O}_{P(w_\Theta)}(\mu)$. This proves the result. □

Lemma 11 *For* $\lambda, \mu \in \mathfrak{h}^*$,

$$\mathcal{I}(w_\Theta, \lambda, \eta|_{\mathfrak{n}_\Theta}) \otimes_{\mathcal{O}_{P(w_\Theta)}} \mathcal{O}(\mu) = \mathcal{I}(w_\Theta, \lambda + \mu, \eta|_{\mathfrak{n}_\Theta}).$$

Proof This follows immediately from the definition of $\mathcal{I}(w_\Theta, \lambda + \mu, \eta|_{\mathfrak{n}_\Theta})$ (Definition 4) and the projection formula (Proposition 11). □

Lemma 12 *As a left* $\mathcal{D}_\lambda$*-module, the graded sheaf*

$$Gr\mathcal{I}(w_\Theta, \lambda, \eta) = j_\bullet(\mathcal{F} \otimes_{\mathcal{O}_{P(w_\Theta)}} S(\mathcal{N}_{X|P(w_\Theta)}) \otimes_{\mathcal{O}_{P(w_\Theta)}} \mathcal{O}(2\rho_\Theta - 2\rho)).$$

Proof Recall the left $\mathcal{D}_\lambda^j$-module $\mathcal{F}$ of Lemma 8. By an application of Eq. 38 to the right $\mathcal{D}_\lambda^j$-module $\mathcal{F} \otimes_{\mathcal{O}_{P(w_\Theta)}} \omega_{P(w_\Theta)}$, we see that as a right $\mathcal{D}_\lambda$-module,

$$Gr\mathcal{I}(w_\Theta, \lambda, \eta) = j_\bullet(\mathcal{F} \otimes_{\mathcal{O}_{P(w_\Theta)}} S(\mathcal{N}_{X|P(w_\Theta)}) \otimes_{\mathcal{O}_{P(w_\Theta)}} \omega_{P(w_\Theta)}).$$

Twisting by ω_X gives us the left $\mathcal{D}_\lambda$-module structure

$$Gr\mathcal{I}(w_\Theta, \lambda, \eta) = j_\bullet(\mathcal{F} \otimes_{\mathcal{O}_{P(w_\Theta)}} S(\mathcal{N}_{X|P(w_\Theta)}) \otimes_{\mathcal{O}_{P(w_\Theta)}} \omega_{P(w_\Theta)|X}),$$

where $\omega_{P(w_\Theta)|X} = \omega_{P(w_\Theta)} \otimes_{\mathcal{O}_{P(w_\Theta)}} j^*(\omega_X^{-1})$ is the invertible $\mathcal{O}_{P(w_\Theta)}$-module of top degree relative differential forms for the morphism j. The result then follows from the fact that $\omega_{P(w_\Theta)|X} = \mathcal{O}(2\rho_\Theta - 2\rho)$. □

Now we are ready to prove Proposition 8.

Proof of Proposition 8 Using the preceding lemmas and the computation of the character of standard Whittaker modules from Section 2.2, we can show that the formal character of $\Gamma(X, \mathcal{I}(w_\Theta, \lambda, \eta))$ is equal to the formal character of $M(w_\Theta\lambda, \eta)$. By Corollary 2, this implies our result. Here $\lambda \in \mathfrak{h}^*$ and $\eta \in \operatorname{ch}\mathfrak{n}$ are arbitrary. We compute:

$$
\begin{aligned}
ch\Gamma(X,\mathcal{I}(w_\Theta, \lambda, \eta)) &= ch\Gamma(X, Gr\mathcal{I}(w_\Theta, \lambda, \eta)) && (14)\\
&= ch\Gamma(X, j_\bullet(\mathcal{F} \otimes_{\mathcal{O}_{P(w_\Theta)}} S(\mathcal{N}_{X|P(w_\Theta)}) \otimes_{\mathcal{O}_{P(w_\Theta)}} \mathcal{O}(2\rho_\Theta - 2\rho))) && (15)\\
&= ch\Gamma(P(w_\Theta), \mathcal{F} \otimes_{\mathcal{O}_{P(w_\Theta)}} S(\mathcal{N}_{X|P(w_\Theta)}) \otimes_{\mathcal{O}_{P(w_\Theta)}} \mathcal{O}(2\rho_\Theta - 2\rho)) && (16)\\
&= ch\Gamma(P(w_\Theta), \mathcal{F} \otimes_{\mathcal{O}_{P(w_\Theta)}} \bigoplus_{\mu \in Q} \mathcal{O}(\mu) \otimes_{\mathcal{O}_{P(w_\Theta)}} \mathcal{O}(2\rho_\Theta - 2\rho)) && (17)\\
&= ch\Gamma(P(w_\Theta), \bigoplus_{\mu \in Q} \mathcal{I}(w_\Theta, \lambda + \rho - \rho_\Theta + \mu + 2\rho_\Theta - 2\rho, \eta|_{\mathfrak{n}_\Theta})) && (18)\\
&= ch \bigoplus_{\mu \in Q} Y(\lambda - \rho + \rho_\Theta + \mu, \eta|_{\mathfrak{n}_\Theta}) && (19)\\
&= \sum_{\mu \in Q} [\overline{Y(\lambda - \rho + \rho_\Theta + \mu, \eta)}] e^{\boldsymbol{\lambda - \rho + \mu}} && (20)\\
&= chM(\lambda, \eta) = chM(w_\Theta\lambda, \eta). && (21)
\end{aligned}
$$

Here, Eq. 14 follows from Lemma 9, Eq. 15 from Lemma 12, Eq. 16 from Kashiwara's theorem, Eq. 17 from Lemma 10, Eq. 18 from Lemma 8, Eq. 19 from Proposition 7, Eq. 20 from Definition 3, and Eq. 21 from Eq. 2 and the fact that two standard Whittaker modules are isomorphic if their $\mathfrak{h}^*$ parameters are in the same W_Θ-orbit.

Because $\mathcal{I}(w_\Theta, \lambda, \eta) = \mathcal{M}(w_\Theta, \lambda, \eta)$, we conclude using Corollary 2 that in $K\mathcal{M}_{fg}(\mathcal{U}_\theta, N, \eta)$,

$$[\Gamma(X, \mathcal{M}(w_\Theta, \lambda, \eta))] = [M(w_\Theta\lambda, \eta)].$$

This completes the proof of Proposition 8. □

Before stating and proving the main result of this section, we record one final fact about tensor products of standard Whittaker modules with finite-dimensional $\mathfrak{g}$-modules. This lemma will be used in the proof of Theorem 9 to deal with the case of singular $\lambda \in \mathfrak{h}^*$.

Let $\lambda \in \mathfrak{h}^*$ be antidominant and $\mu \in P(\Sigma)$ be antidominant and regular. Then $\lambda + \mu$ is antidominant and regular. Let $Q(\Sigma)$ be the root lattice. Let

$$W_\lambda = \{w \in W \mid w\lambda - \lambda \in Q(\Sigma)\} \subset W$$

be the integral Weyl group of λ, which is the Weyl group of the root subsystem

$$\Sigma_\lambda = \{\alpha \in \Sigma \mid \alpha^\vee(\lambda) \in \mathbb{Z}\} \subset \Sigma.$$

For any $\mathfrak{g}$-module V, denote by $V_{[\lambda]}$ the generalized $\mathcal{Z}(\mathfrak{g})$-eigenspace of V corresponding to the infinitesimal character χ_λ.

Lemma 13 *Let F be the finite-dimensional $\mathfrak{g}$-module of highest weight $-\mu$. For $w \in W$,*

$$(M(w(\lambda + \mu), \eta) \otimes_{\mathbb{C}} F)_{[\lambda]} = M(w\lambda, \eta).$$

Proof By [16, Lem. 5.12], $T := M(w(\lambda + \mu), \eta) \otimes_{\mathbb{C}} F$ has a filtration by $\mathfrak{g}$-submodules

$$0 = T_0 \subset T_1 \subset \cdots \subset T_n = T$$

such that the associated graded module GrT is isomorphic to the direct sum

$$\bigoplus_{\nu \in P(F)} M(w(\lambda+\mu)+\nu, \eta),$$

where $P(F)$ is the set of weights of F, counted with multiplicity. We claim that there is exactly one standard Whittaker module appearing in this sum with infinitesimal character χ_λ, and it is equal to $M(w\lambda, \eta)$. Indeed, assume that for some $v \in W$ and $\nu \in P(F)$,

$$w(\lambda+\mu)+\nu = v\lambda.$$

Then $\lambda+\mu+w^{-1}\nu = w^{-1}v\lambda$, so $w^{-1}v\lambda-\lambda = w^{-1}\nu-(-\mu) \in Q(\Sigma)$. On one hand, since λ is antidominant, $w^{-1}v\lambda - \lambda$ must be a positive sum of positive roots in Σ_λ. On the other hand, since $-\mu$ is the highest weight of F and $w^{-1}\nu \in P(F)$, $w^{-1}\nu - (-\mu)$ is a negative sum of positive roots in Σ_λ. Hence

$$w^{-1}v\lambda - \lambda = \mu + w^{-1}\nu = 0.$$

This implies that $\nu = -w\mu$. The weight $\nu = -w\mu$ is an extremal weight of F, so it must occur with multiplicity 1. Therefore, there is exactly one standard Whittaker module in the direct sum decomposition above with infinitesimal character χ_λ, and it is equal to $M(w\lambda, \eta)$.

The generalized $\mathcal{Z}(\mathfrak{g})$-eigenspace corresponding to χ_λ is the submodule

$$T_{[\lambda]} = \{t \in T \mid (\ker \chi_\lambda)^k \cdot t = 0 \text{ for some } k \in \mathbb{Z}\} \subset T.$$

Since $M(w\lambda, \eta)$ appears exactly once in GrT, there is some index $1 \leq i \leq n$ such that

$$T_i/T_{i-1} \simeq M(w\lambda, \eta),$$

and the quotient T/T_i is annihilated by a power of $\prod_{j=i+1}^{n} \ker \chi_{w(\lambda+\mu)+\nu_j}$ with $\chi_{w(\lambda+\mu)+\nu_j} \neq \chi_\lambda$. This implies that T/T_i is a direct sum of submodules with generalized infinitesimal characters different from χ_λ. It follows that $T_{[\lambda]} \subset T_i$.

Since T_i is annihilated by a power of $\prod_{j=1}^{i} \ker \chi_{w(\lambda+\mu)+\nu_j}$, T_i splits into a direct sum of submodules with generalized infinitesimal characters $\chi_{w(\lambda+\mu)+\nu_j}$ for $1 \leq j \leq i$. Since T_{i-1} is not annihilated by any power of $\ker \chi_\lambda$, it follows that $T_{[\lambda]}$ is a direct complement of T_{i-1} in T_i. Hence $T_{[\lambda]} \simeq M(w\lambda, \eta)$. □

Finally, we are ready to prove our desired result.

Theorem 9 *Let $\lambda \in \mathfrak{h}^*$ be antidominant, $C \in W_\Theta \backslash W$, and $\eta \in \operatorname{ch} \mathfrak{n}$ be arbitrary. Then*

$$\Gamma(X, \mathcal{M}(w^C, \lambda, \eta)) = M(w^C\lambda, \eta).$$

Proof Lemma 7 implies that for $C \in W_\Theta \backslash W$,

$$LI_{w_C}(\mathcal{M}(w^C, \lambda, \eta)) = \mathcal{M}(w_\Theta, w_C\lambda, \eta)$$

and

$$R\Gamma(LI_{w_C}(\mathcal{M}(w^C, \lambda, \eta)) = R\Gamma(\mathcal{M}(w_\Theta, w_C\lambda, \eta)).$$

If $\lambda \in \mathfrak{h}^*$ is antidominant, then by [14, Ch. 3 §3 Thm. 3.23],

$$R\Gamma(\mathcal{M}(w^C, \lambda, \eta)) = R\Gamma(\mathcal{M}(w_\Theta, w_C\lambda, \eta)),$$

and

$$H^p(X, \mathcal{M}(w^C, \lambda, \eta)) = 0 \text{ for } p > 0.$$

Therefore, by Proposition 8,

$$[\Gamma(X, \mathcal{M}(w^C, \lambda, \eta))] = [\Gamma(X, \mathcal{M}(w_\Theta, w_C\lambda, \eta))] = [M(w^C\lambda, \eta)].$$

Assume furthermore that $\lambda \in \mathfrak{h}^*$ is regular. Because $M(w^C\lambda, \eta)$ has a unique irreducible quotient and $\lambda \in \mathfrak{h}^*$ is antidominant and regular, Proposition 4 implies our result.

Now assume that $\lambda \in \mathfrak{h}^*$ is antidominant but not necessarily regular. We extend the result above to this setting using the Zuckerman translation functors of [14, Ch. 2 §2]. Let $\mu \in P(\Sigma)$ be antidominant and regular, so $\lambda + \mu$ is antidominant and regular. By definition, for any coset $C \in W_\Theta \backslash W$, $\mathcal{I}(w^C, \lambda, \eta) = \mathcal{I}(w^C, \lambda + \mu, \eta)(-\mu)$, and by dualizing, the analogous statement is also true for costandard η-twisted Harish-Chandra sheaves. Let F be the finite-dimensional irreducible $\mathfrak{g}$-module of highest weight $-\mu$. Let $\mathcal{F} = \mathcal{O}_X \otimes_{\mathbb{C}} F$. The sheaf $\mathcal{F}$ naturally has the structure of an $\mathcal{U}^\circ := \mathcal{O}_X \otimes_{\mathbb{C}} \mathcal{U}(\mathfrak{g})$-module. For any $\mathcal{U}^\circ$-module $\mathcal{V}$, we denote by $\mathcal{V}_{[\lambda]}$ the generalized $\mathcal{Z}(\mathfrak{g})$-eigensheaf corresponding to λ. (For more details on this construction, see [14, Ch. 2 §2].) Then, using the fact that $\lambda + \mu$ is antidominant and regular, we compute

$$\begin{aligned}
\Gamma(X, \mathcal{M}(w^C, \lambda, \eta)) &= \Gamma(X, \mathcal{M}(w^C, \lambda + \mu, \eta)(-\mu)) \\
&= \Gamma(X, (\mathcal{M}(w^C, \lambda + \mu, \eta) \otimes_{\mathcal{O}_X} \mathcal{F})_{[\lambda]}) \\
&= \Gamma(X, \mathcal{M}(w^C, \lambda + \mu, \eta) \otimes_{\mathcal{O}_X} \mathcal{F})_{[\lambda]} \\
&= (\Gamma(X, \mathcal{M}(w^C, \lambda + \mu, \eta)) \otimes_{\mathbb{C}} F)_{[\lambda]} \\
&= (M(w^C(\lambda + \mu), \eta) \otimes_{\mathbb{C}} F)_{[\lambda]} \\
&= M(w^C\lambda, \eta).
\end{aligned}$$

Here the second equality follows from [14, Ch. 2 §2 Lem. 2.1] and the final equality follows from Lemma 13. This completes the proof of Theorem 9. □

It is now straightforward to calculate the global sections of irreducible modules.

Theorem 10 *Let $\lambda \in \mathfrak{h}^*$ be regular antidominant. Then, for any $C \in W_\Theta \backslash W$, we have*

$$\Gamma(X, \mathcal{L}(w^C, \lambda, \eta)) = L(w^C\lambda, \eta).$$

Proof Because λ is regular antidominant, the global sections functor $\Gamma(X, -)$ is an equivalence of categories. Therefore, by Theorem 9, the unique irreducible quotient $\mathcal{L}(w^C, \lambda, \eta)$ of $\mathcal{M}(w^C, \lambda, \eta)$ must be mapped to the unique irreducible quotient $L(w^C\lambda, \eta)$ of $M(w^C\lambda, \eta)$ by $\Gamma(X, -)$. □

These results explicitly establish the connection between the category of Whittaker modules and the category of twisted Harish-Chandra sheaves and prepare us to describe the algorithm in the following section.

5 A Kazhdan–Lusztig Algorithm

This section provides an algorithm for computing composition multiplicities of standard Whittaker modules with regular integral infinitesimal character. These multiplicities are given by Whittaker Kazhdan–Lusztig polynomials which are constructed geometrically using twisted Harish-Chandra sheaves. This algorithm is the main result of this paper, and was inspired by the Kazhdan–Lusztig algorithm for Verma modules in [14, Ch. 5 §2].

To state the theorem containing the algorithm, we return to the combinatorial setting of the introduction. Let W be the Weyl group of a reduced root system Σ with simple roots $\Pi \subset \Sigma$, and let $S \subset W$ be the corresponding set of simple reflections. For a subset of simple roots $\Theta \subset \Pi$ with Weyl group $W_\Theta \subset W$, let $\mathcal{H}_\Theta$ be the free $\mathbb{Z}[q, q^{-1}]$-module with basis δ_C, $C \in W_\Theta \backslash W$. For $\alpha \in \Pi$, we define a $\mathbb{Z}[q, q^{-1}]$-module endomorphism by

$$T_\alpha(\delta_C) = \begin{cases} 0 & \text{if } Cs_\alpha = C; \\ q\delta_C + \delta_{Cs_\alpha} & \text{if } Cs_\alpha > C; \\ q^{-1}\delta_C + \delta_{Cs_\alpha} & \text{if } Cs_\alpha < C. \end{cases}$$

The order relation on cosets is the Bruhat order on longest coset representatives. This is a partial order [14, Ch. 6 §1]. The formula for T_α is inspired by formulas related to the antispherical module for the Hecke algebra appearing in [19]. We will describe explicitly the relationship between our setting and the setting of [19] in Section 6. The algorithm is given in the following theorem.

Theorem 11 *There exists a unique function $\varphi : W_\Theta \backslash W \to \mathcal{H}_\Theta$ satisfying the following properties.*

(i) *For $C \in W_\Theta \backslash W$,*

$$\varphi(C) = \delta_C + \sum_{D<C} P_{CD}\delta_D,$$

where $P_{CD} \in q\mathbb{Z}[q]$.

(ii) *For $\alpha \in \Pi$ and $C \in W_\Theta \backslash W$ such that $Cs_\alpha < C$, there exist $c_D \in \mathbb{Z}$ such that*

$$T_\alpha(\varphi(Cs_\alpha)) = \sum_{D \le C} c_D \varphi(D).$$

The function φ determines a family of polynomials P_{CD} parameterized by pairs of cosets in $W_\Theta \backslash W$. We refer to these polynomials as *Whittaker Kazhdan–Lusztig polynomials*, because, as we will see in Section 5.1, they determine composition multiplicities of standard Whittaker modules.

First we will prove uniqueness of the function $\varphi : W_\Theta \backslash W \to \mathcal{H}_\Theta$ in Theorem 11 using a straightforward combinatorial argument. Next, we prove existence of φ by appealing to geometry. Defining φ geometrically provides the critical link between the Whittaker Kazhdan–Lusztig polynomials P_{CD} of Theorem 11 and Whittaker modules. This is explained in detail in Section 5.1.

We begin by proving uniqueness of φ in a slightly stronger form. Denote by $W_\Theta \backslash W_{\le k}$ the set of cosets $C \in W_\Theta \backslash W$ such that $\ell(w^C) \le k$.

Lemma 14 *Let $k \in \mathbb{N}$. Then there exists at most one function $\varphi : W_\Theta \backslash W_{\le k} \longrightarrow \mathcal{H}_\Theta$ such that the following properties are satisfied.*

(i) *For $C \in W_\Theta \backslash W_{\le k}$,*

$$\varphi(C) = \delta_C + \sum_{D<C} P_{CD}\delta_D,$$

where $P_{CD} \in q\mathbb{Z}[q]$.

(ii) *For $\alpha \in \Pi$ and $C \in W_\Theta \backslash W_{\le k}$ such that $Cs_\alpha < C$, there exist $c_D \in \mathbb{Z}$ such that*

$$T_\alpha(\varphi(Cs_\alpha)) = \sum_{D \le C} c_D \varphi(D).$$

Proof We proceed by induction in k. By [14, Ch. 6 §1 Lem. 1.7], the unique minimal element in the coset order is W_Θ, so the base case is $k = \ell(w_\Theta)$, where w_Θ is the longest element in W_Θ. In this case, $W_\Theta\backslash W_{\leq k} = \{W_\Theta\}$. The only possible function $\varphi : W_\Theta\backslash W \longrightarrow \mathcal{H}_\Theta$ which satisfies (i) is $\varphi(W_\Theta) = \delta_{W_\Theta}$, and (ii) is void.

Assume that for $k > \ell(w_\Theta)$, there exists $\varphi : W_\Theta\backslash W_{\leq k} \longrightarrow \mathcal{H}_\Theta$ which satisfies (i) and (ii). Our induction assumption is that $\varphi|_{W_\Theta\backslash W_{\leq k-1}}$ is unique. By [14, Ch. 6 §1 Prop. 1.6], there is a coset $C \in W_\Theta\backslash W_{\leq k}$ such that $\ell(w^C) = k$. Then by [14, Ch. 6 §1 Lem. 1.7], there exists $\alpha \in \Pi$ such that $Cs_\alpha < C$. By (ii),

$$T_\alpha(\varphi(Cs_\alpha)) = \sum_{D\leq C} c_D\varphi(D).$$

Evaluating at $q = 0$ and using (i), we have

$$T_\alpha(\varphi(Cs_\alpha))(0) = \sum_{D\leq C} c_D\left(\delta_D + \sum_{E<D} P_{DE}(0)\delta_C\right) = \sum_{D\leq C} c_D\delta_D.$$

Because $\ell(w^{Cs_\alpha}) = k-1$, the induction assumption implies that the coefficients c_D in this sum are uniquely determined. On the other hand, using the definition of φ and T_α, we compute

$$\begin{aligned} T_\alpha(\varphi(Cs_\alpha)) &= T_\alpha(\delta_{Cs_\alpha}) + \sum_{D<Cs_\alpha} P_{Cs_\alpha D}T_\alpha(\delta_D) \\ &= q\delta_{Cs_\alpha} + \delta_C + \sum_{D<Cs_\alpha} P_{Cs_\alpha D}T_\alpha(\delta_D). \end{aligned}$$

Because all cosets D appearing in the sum are less than Cs_α in the coset order, $\ell(w^D) < k-1$ for any such D. In particular, δ_C does not show up in this sum. Evaluating at zero and setting this equal to our first computation, we conclude that $c_C = 1$. Therefore,

$$\varphi(C) = T(\varphi(Cs_\alpha)) - \sum_{D<C} c_D\varphi(D).$$

This shows that the Lemma holds for $W_\Theta\backslash W_{\leq k}$, and we are done by induction. □

The uniqueness of Theorem 11 follows immediately from Lemma 14. Next we establish a parity condition on solutions of Lemma 14 which will be critical in upcoming computations.

We define additive involutions i on $\mathbb{Z}[q, q^{-1}]$ and ι on $\mathcal{H}_\Theta$ by

$$\begin{aligned} i(q^m) &= (-1)^m q^m, \text{ for } m \in \mathbb{Z}, \text{ and} \\ \iota(q^m\delta_C) &= (-1)^{m+\ell(w^C)} q^m\delta_C, \text{ for } m \in \mathbb{Z} \text{ and } C \in W_\Theta\backslash W. \end{aligned}$$

A simple calculation shows that $\iota T_\alpha \iota = -T_\alpha$.

Lemma 15 *Let $k \in \mathbb{N}$. Let $\varphi : W_\Theta\backslash W_{\leq k} \longrightarrow \mathcal{H}_\Theta$ be a function satisfying properties (i) and (ii) of Lemma 14. Then*

$$P_{CD} = q^{\ell(w^C)-\ell(w^D)} Q_{CD},$$

where $Q_{CD} \in \mathbb{Z}[q^2, q^{-2}]$.

Proof Define a function $\psi : W_\Theta \backslash W_{\leq k} \to \mathcal{H}_\Theta$ by $\psi(C) = (-1)^{\ell(w^C)}\iota(\varphi(C))$. Then

$$\psi(C) = \delta_C + \sum_{D<C} (-1)^{\ell(w^C)-\ell(w^D)} i(P_{CD})\delta_D.$$

The polynomials $(-1)^{\ell(w^C)-\ell(w^D)} i(P_{CD})$ are in $q\mathbb{Z}[q]$, so ψ satisfies (i). We will show that ψ also satisfies (ii), then use Lemma 14 to conclude that $\psi = \varphi$. Let $C \in W_\Theta \backslash W_{\leq k}$ and $\alpha \in \Pi$ such that $Cs_\alpha < C$. Then

$$\begin{aligned} T_\alpha(\psi(Cs_\alpha)) &= (-1)^{\ell(w^C)} (-T_\alpha(\iota(\varphi(Cs_\alpha)))) \\ &= (-1)^{\ell(w^C)} \iota T_\alpha \iota(\iota(\varphi(Cs_\alpha))) \\ &= (-1)^{\ell(w^C)} \iota \left(\sum_{D \leq C} c_D \varphi(D) \right) \\ &= (-1)^{\ell(w^C)} \sum_{D \leq C} c_D \iota(\varphi(D)) \\ &= \sum_{D \leq C} (-1)^{\ell(w^C)-\ell(w^D)} c_D \psi(D). \end{aligned}$$

This shows that ψ satisfies (ii), so Lemma 14 implies that $\varphi = \psi$; that is, that

$$P_{CD} = (-1)^{\ell(w^C)-\ell(w^D)} i(P_{CD}).$$

This relationship implies the result. $\square$

Now we are ready to prove the existence statement of Theorem 11. Let $\mathcal{F} \in \mathcal{M}_{coh}(\mathcal{D}_X, N, \eta)$. For $w \in W$, let $i_w : C(w) \longrightarrow X$ be the canonical immersion of the corresponding Bruhat cell into the flag variety. We note the following facts.

- For any $k \in \mathbb{Z}$, $L^{-k} i_w^+(\mathcal{F})$ is an η-twisted N-equivariant connection on $C(w)$, so it is isomorphic to a direct sum of copies of $\mathcal{O}_{C(w)}$. We refer to the number of copies of $\mathcal{O}_{C(w)}$ that appear in this decomposition as the $\mathcal{O}$-dimension, and denote it $\dim_{\mathcal{O}}(L^{-k} i_w^+(\mathcal{F}))$.
- Because the dimension of $C(w)$ is $\ell(w)$, for any $k \in \mathbb{Z}$,

  $$R^{n-\ell(w)-k} i_w^!(\mathcal{F}) = L^{-k} i_w^+(\mathcal{F}).$$

  Here $n = \dim X$.

We define a function $\nu : \mathcal{M}_{coh}(\mathcal{D}_X, N, \eta) \longrightarrow \mathcal{H}_\Theta$ by

$$\nu(\mathcal{F}) = \sum_{C \in W_\Theta \backslash W} \sum_{m \in \mathbb{Z}} \dim_{\mathcal{O}}(R^m i_{w^C}^!(\mathcal{F})) q^m \delta_C. \tag{22}$$

For $C \in W_\Theta \backslash W$, let $\mathcal{I}_C := \mathcal{I}(w^C, -\rho, \eta)$ be the standard sheaf in $\mathcal{M}_{coh}(\mathcal{D}_X, N, \eta)$ corresponding to the coset C and $\mathcal{L}_C := \mathcal{L}(w^C, -\rho, \eta)$ its unique irreducible subsheaf.

Proposition 9 *Let $\varphi(C) = \nu(\mathcal{L}_C)$. Then φ satisfies conditions (i) and (ii) in Theorem 11.*

Checking that φ satisfies 11 (i) is straightforward.

Lemma 16 *Let $\varphi(C) = \nu(\mathcal{L}_C)$. Then*

$$\varphi(C) = \delta_C + \sum_{D<C} P_{CD}\delta_D,$$

where $P_{CD} \in q\mathbb{Z}[q]$.

Proof We need to show three things:

(a) If $D \not\leq C$, $\dim_{\mathcal{O}}(R^m i^!_{w^D}(\mathcal{L}_C)) = 0$ for all $m \in \mathbb{Z}$,

(b) $\dim_{\mathcal{O}}(R^m i^!_{w^C}(\mathcal{L}_C)) = \begin{cases} 1 & \text{if } m = 0 \\ 0 & \text{otherwise} \end{cases}$, and

(c) if $D < C$, $\dim_{\mathcal{O}}(R^m i^!_{w^D}(\mathcal{L}_C)) = 0$ for all $m \leq 0$.

Part (a) follows immediately from the fact that $\mathrm{supp}\mathcal{L}_C = \overline{C(w^C)}$ and $D \leq C$ in the coset order if and only if $C(w^D) \subset \overline{C(w^C)}$ [14, Prop 1.11]. To see part (b), we first observe that

$$R^0 i^!_{w^C}(\mathcal{L}_C) = R^0 i^!_{w^C}(\mathcal{I}_C) = R^0 i^!_{w^C}(i_{w^C+}(\mathcal{O}_{C(w^C)})) = \mathcal{O}_{C(w^C)}.$$

So $\dim_{\mathcal{O}}(R^0 i^!_{w^C}(\mathcal{L}_C)) = 1$. Furthermore, for $m \neq 0$,

$$R^m i^!_{w^C}(\mathcal{L}_C) = R^m i^!_{w^C}(\mathcal{I}_C) = R^m i^!_{w^C}(i_{w^C+}(\mathcal{O}_{C(w^C)})) = 0.$$

This proves (b). We end by showing (c). Let $D \in W_\Theta\backslash W$ be a coset so that $D < C$. Because i_{w^D} is an immersion, $i^!_{w^D}$ is a right derived functor, so for any $m < 0$, $R^m i^!_{w^D}(\mathcal{V}) = 0$ for any $\mathcal{D}$-module $\mathcal{V}$ on X. Thus all that remains is to show that $R^0 i^!_{w^D}(\mathcal{L}_C) = 0$. Let $X' = X - \partial C(w^D)$, and let $j_{w^D} : C(w^D) \to X'$ be the natural closed immersion, and $k_{w^D} : X' \to X$ the natural open immersion. Then we have a commutative diagram.

$$\begin{tikzcd} C(w^D) \arrow[rr, "i_{w^D}"] \arrow[dr, "j_{w^D}"'] & & X \\ & X' \arrow[ur, "k_{w^D}"'] & \end{tikzcd}$$

Using the fact that $\dim X = \dim X'$, that k_{w^D} is an open immersion, and Kashiwara's Theorem, we compute

$$\begin{aligned} R^0 j_{w^D+}(R^0 i^!_{w^D}(\mathcal{L}_C)) &= R^0 j_{w^D+}(R^0 j^!_{w^D}(R^0 k^!_{w^D}(\mathcal{L}_C))) \\ &= R^0 j_{w^D+}(R^0 j^!_{w^D}(L^0 k^+_{w^D}(\mathcal{L}_C))) \\ &= R^0 j_{w^D+}(R^0 j^!_{w^D}(\mathcal{L}_C|_{X'})) \\ &= R^0 \Gamma_{C(w^D)}(\mathcal{L}_C|_{X'}). \end{aligned}$$

From this calculation we see that $R^0 j_{w^D+}(R^0 i^!_{w^D}(\mathcal{L}_C))$ is the submodule of $\mathcal{L}_C|_{X'}$ consisting of sections supported on $C(w^D)$. However, because X' is open, $\mathcal{L}_C|_{X'}$ is irreducible, so this submodule must be zero. We conclude that $R^0 i^!_{w^D}(\mathcal{L}_D) = 0$, which completes the proof of the lemma. □

Our final step in proving Theorem 11 is establishing that φ satisfies Theorem 11(ii). Before we make this argument, we need to introduce a useful family of functors $U^k_\alpha : \mathcal{M}_{qc}(\mathcal{D}_X) \to \mathcal{M}_{qc}(\mathcal{D}_X)$ and examine their semisimplicity properties. We dedicate the next page to doing so.

Fix $\alpha \in \Pi$, and let $p_\alpha : X \longrightarrow X_\alpha$ be projection onto the flag variety of parabolic subalgebras of type α. If $P_\alpha \subset G$ is the standard parabolic of type α, then $P_\alpha = B \cup Bs_\alpha B$. Let $C(v)$ be the Bruhat cell corresponding to $v \in W$. Then we have the following facts:

- The Bruhat cell $C(v) \simeq \mathbb{C}^{\ell(v)}$, so $i_v : C(v) \longrightarrow X$ is an affine morphism.
- The image $p_\alpha(C(v))$ is an affine subvariety of X_α.
- The projection p_α is locally trivial, so $p_\alpha^{-1}(p_\alpha(C(v))$ is a smooth, affinely embedded subvariety of X.

We conclude that $p_\alpha^{-1}(p_\alpha(C(v))) = C(v) \cup C(vs_\alpha)$. One of these orbits is closed in $p_\alpha^{-1}(p_\alpha(C(v)))$ and the other is open and dense. We have two possible scenarios:

1. $\ell(vs_\alpha) = \ell(v) + 1$. Then $\dim(C(vs_\alpha)) > \dim(C(v))$, and so
   - $C(vs_\alpha)$ is open and dense in $p_\alpha^{-1}(p_\alpha(C(v)))$,
   - $C(v)$ is closed in $p_\alpha^{-1}(p_\alpha(C(v)))$, and
   - $p_\alpha : C(v) \longrightarrow p_\alpha(C(v))$ is an isomorphism.
2. $\ell(vs_\alpha) = \ell(v) - 1$. Then $\dim(C(vs_\alpha)) < \dim(C(v))$, and so
   - $C(vs_\alpha)$ is closed in $p_\alpha^{-1}(p_\alpha(C(v)))$,
   - $C(v)$ is open and dense in $p_\alpha^{-1}(p_\alpha(C(v)))$, and
   - $p_\alpha : C(v) \longrightarrow p_\alpha(C(v))$ is a fibration with fibers isomorphic to an affine line.

We define a family of functors $U_\alpha^k : \mathcal{M}_{qc}(\mathcal{D}_X) \longrightarrow \mathcal{M}_{qc}(\mathcal{D}_X)$ by

$$U_\alpha^k(\mathcal{F}) = p_\alpha^+(H^k p_{\alpha+}(\mathcal{F})).$$

Because the fibers of the projection map $p_\alpha : X \to X_\alpha$ are one-dimensional, U_α^k can be non-zero only for $k \in \{-1, 0, 1\}$. These functors are closely related to the U-functors discussed in Section 3.3. (We will make this relationship explicit in the proof of Theorem 17.) Their main utility in our argument comes from their semisimplicity properties.

Lemma 17 *Let $C \in W_\Theta \backslash W$ and $\alpha \in \Pi$ be such that $Cs_\alpha < C$. Then*

(i) $U_\alpha^k(\mathcal{L}_{Cs_\alpha}) = 0$ *for all* $k \neq 0$*, and*
(ii) $U_\alpha^0(\mathcal{L}_{Cs_\alpha})$ *is a direct sum of* $\mathcal{L}_D$ *for* $D \leq C$*.*

Proof By construction, $U_\alpha^0(\mathcal{L}_{Cs_\alpha})$ is a holonomic $(\mathcal{D}_X, N, \eta)$-module supported on $\overline{C(w^C)}$, so $U_\alpha^0(\mathcal{L}_{Cs_\alpha})$ has finite length, and its composition factors must be in the set $\{\mathcal{L}_D | D \in W_\Theta \backslash W \text{ and } D \leq C\}$. Because p_α is a locally trivial fibration with fibers isomorphic to $\mathbb{P}^1$ (in particular, it is a projective morphism of smooth quasi-projective varieties), and $\mathcal{L}_{Cs_\alpha}$ is a semisimple holonomic $\mathcal{D}$-module, the decomposition theorem [18, §1 Thm. 1.4.1] implies that $H^k p_{\alpha+}(\mathcal{L}_{Cs_\alpha})$ are semisimple. By the local triviality of p_α, this in turn implies that $U_\alpha^0(\mathcal{L}_{Cs_\alpha})$ are semisimple, which completes the proof of (ii).

To prove (i), we establish the connection between U_α^0 and the U-functors of Section 3.3. Let $Y_\alpha = X \times_{X_\alpha} X$ be the fiber product of X with itself relative to the morphism p_α with projections q_1 and q_2 onto the factors. By base change (Theorem 16),

$$U_\alpha^k(\mathcal{L}_{Cs_\alpha}) = p_\alpha^+(H^k p_{\alpha+}(\mathcal{L}_{Cs_\alpha})) = H^k q_{1+}(q_2^+(\mathcal{L}_{Cs_\alpha})).$$

Because $\mathcal{D}_X = \mathcal{D}_{-\rho}$, we have that the twist $U_\alpha^k(\mathcal{L}_{Cs_\alpha})(\alpha) = U^k(\mathcal{L}_{Cs_\alpha})$, where U^k is the functor from Section 3.3. To complete the proof, we need to show that we are in case (ii) of Theorem 7; that is, that $L^{-1}I_{s_\alpha}(\mathcal{L}_{Cs_\alpha}) = 0$. Because $Cs_\alpha < C$, we can apply Proposition 5 to the coset Cs_α and conclude that

$$LI_{s_\alpha}(\mathcal{I}(w^C s_\alpha, \lambda, \eta)) = \mathcal{I}(w^C, s_\alpha\lambda, \eta).$$

In particular, this implies that $L^{-1}I_{s_\alpha}(\mathcal{I}(w^c s_\alpha, \lambda, \eta)) = 0$, and because $\mathcal{L}_{Cs_\alpha}$ is a submodule of $\mathcal{I}(w^c s_\alpha, \lambda, \eta)$, $L^{-1}I_{s_\alpha}(\mathcal{L}_{Cs_\alpha}) = 0$ as well. □

We are working toward showing that $\varphi(C) = \nu(\mathcal{L}_C)$ satisfies (ii). We will do so by proving that for $\alpha \in \Pi$ and $C \in W_\Theta\backslash W$ such that $Cs_\alpha < C$, $T_\alpha(\varphi(Cs_\alpha)) = \nu(U_\alpha^0(\mathcal{L}_{Cs_\alpha}))$. This relationship is useful because it allows us to use Lemma 17 to decompose $\nu(U_\alpha^0(\mathcal{L}_{Cs_\alpha}))$ and obtain the desired sum in Theorem 11(ii). Before jumping into the argument, we must establish what happens if we pull back an irreducible module to a Bruhat cell which corresponds to a Weyl group element which is not a longest representative in some coset $C \in W_\Theta\backslash W$. Lemma 18 will be critical in upcoming computations.

Lemma 18 *Let $v \in W$ be a Weyl group element such that $v \neq w^C$ is not a longest coset element for any coset $C \in W_\Theta\backslash W$. Let $\mathcal{F} \in \mathcal{M}_{coh}(\mathcal{D}_X, N, \eta)$ be irreducible. Then*

$$R^k i_v^!(\mathcal{F}) = 0$$

for all $k \in \mathbb{Z}$.

Proof Let $X' = X - \partial C(v)$, and express the canonical immersion i_v as the composition of a closed immersion and an open immersion in the following way.

$$\begin{tikzcd} C(v) \arrow[r, "j_v"] \arrow[rr, bend right, "i_v"'] & X' \arrow[r, "k_v"] & X \end{tikzcd}$$

Then, if $\mathcal{F}$ is an irreducible (D_X, N, η)-module,

$$\begin{aligned} i_v^!(\mathcal{F}) &= j_v^! k_v^!(\mathcal{F}) \\ &= i_v^! k_{v+} j_{v+} j_v^! k_v^!(\mathcal{F}) \\ &= i_c^! k_{v+} R\Gamma_{C(v)}(k_v^!(\mathcal{F})) \\ &= i_v^! k_{v+} R\Gamma_{C(v)}(\mathcal{F}|_{X'}). \end{aligned}$$

Here we are using Kashiwara's theorem, the fact that $\dim X = \dim X'$, and the fact that k_v is an open immersion. Because X' is open in X and $\mathcal{F}$ is irreducible, $\mathcal{F}|_{X'}$ is irreducible as well. For all $k \in \mathbb{Z}$, $R^k\Gamma_{C(v)}\mathcal{F}|_{X'}$ is a submodule of $\mathcal{F}|_{X'}$, so either $R^k\Gamma_{C(v)}\mathcal{F}|_{X'} = 0$, or $R^k\Gamma_{C(v)}\mathcal{F}|_{X'} = \mathcal{F}|_{X'}$. In the first case, the preceding calculation implies that $R^k i_v^!(\mathcal{F}) = 0$, and we are done. In the second case, we have $\mathrm{supp}\mathcal{F}|_{X'} = \mathrm{supp}R^k\Gamma_{C(v)}\mathcal{F}|_{X'} \subseteq C(v)$. By [13, Ch. V §4 Cor. 4.2], $\mathcal{F}$ is the unique irreducible holonomic $\mathcal{D}_X$-module that restricts to $\mathcal{F}|_{X'}$, and $\mathrm{supp}\mathcal{F} = \overline{\mathrm{supp}\mathcal{F}|_{X'}} \subseteq \overline{C(v)}$. There are no irreducible objects in $\mathcal{M}_{coh}(\mathcal{D}_X, N, \eta)$ with support equal to $\overline{C(v)}$ because v is not a longest coset element, so we must have $\mathrm{supp}\mathcal{F} \subseteq \partial C(v) = \overline{C(v)} - C(v)$. But this implies that $\mathrm{supp}\mathcal{F}|_{X'} = \mathrm{supp}R^k\Gamma_{C(v)}\mathcal{F}|_{X'} = 0$, so the second case cannot happen. □

Let $C \in W_\Theta \backslash W$ and $\alpha \in \Pi$ be such that $Cs_\alpha < C$. The rest of this section is spent proving that $T_\alpha(\varphi(Cs_\alpha)) = \nu(U_\alpha^0(\mathcal{L}_{Cs_\alpha}))$. Our first step in relating these two quantities is to establish the existence of a certain long exact sequence in cohomology which will be useful in relating $\mathcal{O}$-dimensions of modules which appear in the decomposition of $\nu(U_\alpha^0(\mathcal{L}_{Cs_\alpha}))$.

Let $D \in W_\Theta \backslash W$ be a coset such that $D \leq C$, so $\ell(w^D) \leq \ell(w^C)$ and $C(w^D) \subset \overline{C(w^C)}$. By [14, Ch. 6 §1 Prop 1.6], $w^C s_\alpha$ is the longest element of Cs_α, and $\ell(w^C s_\alpha) = \ell(w) - 1$. By assumption, $C(w^C)$ is open and dense in $p_\alpha^{-1}(p_\alpha(C(w^C))) = C(w^C) \cup C(w^C s_\alpha)$, so the closure $\overline{p_\alpha^{-1}(p_\alpha(C(w^C)))} = \overline{C(w^C)}$. Because $C(w^D) \subset \overline{C(w^C)}$, the image $p_\alpha(C(w^D)) \subset p_\alpha(\overline{C(w^C)})$, so

$$C(w^D) \cup C(w^D s_\alpha) = p_\alpha^{-1}(p_\alpha(C(w^D))) \subset \overline{p_\alpha^{-1}(p_\alpha(C(w^C)))} = \overline{C(w^C)}.$$

We conclude that both $w^D s_\alpha \leq w^C$ and $w^D \leq w^C$. Because both elements are less than or equal to w^C in the Bruhat order, we can assume without loss of generality that $w^D s_\alpha \leq w^D$; i.e. $\ell(w^D s_\alpha) = \ell(w^D) - 1$ and $C(w^D)$ is open in $Z_\alpha := p_\alpha^{-1}(p_\alpha(C(w^D))) = C(w^D) \cup C(w^D s_\alpha)$.

Let $j : Z_\alpha \longrightarrow X$ and $j_D : p_\alpha(C(w^D)) \longrightarrow X_\alpha$ be natural inclusions. Let $q_\alpha : Z_\alpha \longrightarrow p_\alpha(C(w^D))$ be the restriction of p_α to Z_α. Then we have the following fiber product diagram:

$$\begin{tikzcd} Z_\alpha \arrow[r, "j"] \arrow[d, "q_\alpha"] & X \arrow[d, "p_\alpha"] \\ p_\alpha(C(w^D)) \arrow[r, "j_D"] & X_\alpha. \end{tikzcd}$$

Note that because p_α and q_α are surjective submersions, p_α^+ and q_α^+ are exact, so they both lift to functors on the respective derived categories $D^b(\mathcal{M}(\mathcal{D}_X))$ and $D^b(\mathcal{M}(\mathcal{D}_{Z_\alpha}))$. In the calculations below we denote both the functors on the derived category and the functors on modules by the same name, either p_α^+ or q_α^+. Let d be the codimension of Z_α in X. Note that the codimension of $p_\alpha(C(w^D)) = p_\alpha(Z_\alpha)$ in X_α is also d. Recall that for any immersion $i : Y \to X$ of smooth algebraic varieties, the extraordinary inverse image and the $\mathcal{D}$-module inverse image are related by $i^![\mathrm{codim}(Y)] = Li^+$. By this relationship, base change (Theorem 16), and Lemma 17, we compute

$$\begin{aligned} R^k j^!(U_\alpha^0(\mathcal{L}_{Cs_\alpha})) &= H^k(j^! p_\alpha^+ p_{\alpha+}(\mathcal{L}_{Cs_\alpha})) \\ &= H^{k+d}(Lj^+(p_\alpha^+ p_{\alpha+}(\mathcal{L}_{Cs_\alpha}))) \\ &= H^{k+d}(q_\alpha^+(Lj_D^+(p_{\alpha+}(\mathcal{L}_{Cs_\alpha})))) \\ &= H^k(q_\alpha^+ j_D^! p_{\alpha+}(\mathcal{L}_{Cs_\alpha})) \\ &= q_\alpha^+ H^k(j_D^! p_{\alpha+}(\mathcal{L}_{Cs_\alpha})) \\ &= q_\alpha^+ H^k(q_{\alpha+} j^!(\mathcal{L}_{Cs_\alpha})). \end{aligned}$$

Our next step is to analyze the complex $j^!(\mathcal{L}_{Cs_\alpha})$. Denote by $i : C(w^D) \longrightarrow Z_\alpha$ and $i' : C(w^D s_\alpha) \longrightarrow Z_\alpha$ the canonical affine immersions. Note that i is an open immersion, and i' is a closed immersion. We have the following commutative diagram.

$$\begin{tikzcd}
C(w^D) \arrow[dr, "i"] \arrow[drr, bend left, "i_{w^D}"] & & \\
& Z_\alpha \arrow[r, "j"] & X \\
C(w^D s_\alpha) \arrow[ur, "i'"] \arrow[urr, bend right, "i_{w^D s_\alpha}"'] & &
\end{tikzcd}$$

For any complex $\mathcal{F}^\cdot \in D^b(\mathcal{M}(\mathcal{D}_{Z_\alpha}))$, we have the following distinguished triangle [13, Ch. IV §9]:

$$i'_+ i'^! \mathcal{F}^\cdot \longrightarrow \mathcal{F}^\cdot \longrightarrow i_+ \mathcal{F}^\cdot|_{C(w^D)}.$$

Applying this to $\mathcal{F}^\cdot = j^!(\mathcal{L}_{Cs_\alpha})$ and using the facts that $j^!(\mathcal{L}_{Cs_\alpha})|_{C(w^D)} = i^+ j^!(\mathcal{L}_{Cs_\alpha}) = i^! j^!(\mathcal{L}_{Cs_\alpha}) = i^!_{w^D}(\mathcal{L}_{Cs_\alpha})$ because i is an open immersion and $i'^! \circ j^! = i^!_{w^D s_\alpha}$, we obtain the distinguished triangle

$$i'_+ i^!_{w^D s_\alpha}(\mathcal{L}_{Cs_\alpha}) \longrightarrow j^!(\mathcal{L}_{Cs_\alpha}) \longrightarrow i_+ i^!_{w^D}(\mathcal{L}_{Cs_\alpha}).$$

Applying the exact functor $q_{\alpha+}$ we get the following distinguished triangle in $D^b(\mathcal{M}(\mathcal{D}_{p_\alpha(C(w^D))}))$:

$$(q_\alpha \circ i')_+(i^!_{w^D s_\alpha}(\mathcal{L}_{Cs_\alpha})) \longrightarrow q_{\alpha+} j^!(\mathcal{L}_{Cs_\alpha}) \longrightarrow (q_\alpha \circ i)_+(i^!_{w^D}(\mathcal{L}_{Cs_\alpha})).$$

Because $p_\alpha(C(w^D))$ is an N-orbit in X_α and all $\mathcal{D}$-modules in the arguments above are N-equivariant, the cohomologies of the complexes in this triangle are all direct sums of copies of $\mathcal{O}_{p_\alpha(C(w^D))}$. From this final distinguished triangle, we obtain a long exact sequence in cohomology:

$$\begin{aligned}
\cdots \to\ & H^{k-1}((q_\alpha \circ i)_+(i^!_{w^D}(\mathcal{L}_{Cs_\alpha})) \to H^k((q_\alpha \circ i')_+(i^!_{w^D s_\alpha}(\mathcal{L}_{Cs_\alpha}))) \to \\
& H^k(q_{\alpha+}(j^!(\mathcal{L}_{Cs_\alpha})) \to H^k((q_\alpha \circ i)_+(i^!_{w^D}(\mathcal{L}_{Cs_\alpha})) \to \\
& H^{k+1}((q_\alpha \circ i')_+(i^!_{w^D s_\alpha}(\mathcal{L}_{Cs_\alpha}))) \to \cdots .
\end{aligned}$$

This is a sequence of $\mathcal{D}_{p_\alpha(C(w^D))}$-modules which are direct sums of copies of $\mathcal{O}_{p_\alpha(C(w^D))}$.

Note that the map

$$q_\alpha \circ i' : C(w^D s_\alpha) \longrightarrow p_\alpha(C(w^D))$$

is an isomorphism, and the map

$$q_\alpha \circ i : C(w^D) \longrightarrow p_\alpha(C(w^D))$$

is a locally trivial projection with one-dimensional fibers. This implies that

$$\dim_{\mathcal{O}} H^k((q_\alpha \circ i')_+(i^!_{w^D s_\alpha}(\mathcal{L}_{Cs_\alpha}))) = \dim_{\mathcal{O}} R^k i^!_{w^D s_\alpha}(\mathcal{L}_{Cs_\alpha}), \text{ and} \tag{23}$$

$$\dim_{\mathcal{O}} H^k((q_\alpha \circ i)_+(i^!_{w^D}(\mathcal{L}_{Cs_\alpha}))) = \dim R^{k+1} i^!_{w^D}(\mathcal{L}_{Cs_\alpha}). \tag{24}$$

Now we are ready to prove that $\varphi(C) = \nu(\mathcal{L}_C)$ satisfies 11 (ii) by induction in the length of w^C. The base case is when $w^C = w_\Theta$ and $C = W_\Theta$. In this case, for any $\alpha \in \Pi$, either

$Cs_\alpha = C$, or $Cs_\alpha > C$ because w_Θ is minimal length in the set of longest coset elements, so 11(ii) is void.

Fix $k \in \mathbb{N}$. Assume that $\varphi(C) := \nu(\mathcal{L}_C)$ satisfies 11 (ii) for $C \in W_\Theta \backslash W_{\leq k}$. This is our induction assumption. Under this assumption, we can reformulate the parity condition of Lemma 15 in the following way. Since $\varphi|_{W_\Theta \backslash W}$ satisfies conditions (i) and (ii) of Lemma 14 on $W_\Theta \backslash W_{\leq k}$, if $C \in W_\Theta \backslash W_{\leq k}$ and $D \in W_\Theta \backslash W$, then $P_{CD} = q^{\ell(w^C)-\ell(w^D)} Q_{CD}$, for some $Q_{CD} \in \mathbb{Z}[q^2, q^{-2}]$.[5] Because

$$P_{CD}(q) = \sum_{m \in \mathbb{Z}} \dim_{\mathcal{O}}(R^m i^!_{w^D}(\mathcal{L}_C)) q^m,$$

by the definition of φ, we conclude that for any $C \in W_\Theta \backslash W_{\leq k}$ and $D \in W_\Theta \backslash W$, if $m \equiv \ell(w^C) - \ell(w^D) - 1 \pmod 2$, then $R^m i^!_{w^D}(\mathcal{L}_C) = 0$. We refer to this as the inductive parity condition.

Let $C \in W_\Theta \backslash W$ be a coset such that $\ell(w^C) = k+1$ and $\alpha \in \Pi$ such that $Cs_\alpha < C$. Let $D \in W_\Theta \backslash W$ be such that $D \leq C$. Then $Cs_\alpha \in W_\Theta \backslash W_{\leq k}$, so we can apply the inductive parity condition to the cosets Cs_α and D. This yields

$$R^m i^!_{w^D}(\mathcal{L}_{Cs_\alpha}) = 0 \text{ for all } m \in \mathbb{Z} \text{ with } m \equiv \ell(w^C) - \ell(w^D) \pmod 2. \tag{25}$$

Now since we've chosen D arbitrarily, there are two possible relationships between D and α. Either $Ds_\alpha = D$ or $Ds_\alpha \neq D$. In the first case, Lemma 18 implies that for all $m \in \mathbb{Z}$, $R^m i^!_{w^D s_\alpha}(\mathcal{L}_{Cs_\alpha}) = 0$, since $w^D s_\alpha$ isn't a longest coset representative. In the second case, we can apply the inductive parity condition again to the cosets Cs_α and Ds_α to see that

$$R^m i^!_{w^D s_\alpha}(\mathcal{L}_{Cs_\alpha}) = 0 \text{ for all } m \in \mathbb{Z} \text{ with } m \equiv \ell(w^C) - \ell(w^D) + 1 \pmod 2. \tag{26}$$

Combining Eqs. 25 and 26 with Eqs. 23 and 24, we see that for any $D \leq C$ and any integer m such that $m \equiv \ell(w^C) - \ell(w^D) + 1 \pmod 2$,

$$\begin{aligned} H^m((q_\alpha \circ i)_+(i^!_{w^D}(\mathcal{L}_{Cs_\alpha}))) &= 0, \text{ and} \\ H^m((q_\alpha \circ i')_+(i^!_{w^D s_\alpha}(\mathcal{L}_{Cs_\alpha}))) &= 0. \end{aligned}$$

Using the long exact sequence in cohomology from earlier, we conclude that for any integer m such that $m \equiv \ell(w^C) - \ell(w^D) + 1 \pmod 2$,

$$H^m(q_{\alpha+} j^!(\mathcal{L}_{Cs_\alpha})) = 0.$$

The outcome of the this discussion is that the long exact sequence in cohomology associated to the cosets C and D has the form

$$\cdots \to 0 \to 0 \to 0 \to * \to * \to * \to 0 \to 0 \to 0 \to * \to * \to * \to 0 \to 0 \to 0 \to \cdots,$$

where the $*$'s represent possibly non-zero elements. Since $\mathcal{O}$-dimension sums over short exact sequences, we conclude after another application of Eqs. 23 and 24 that for any integer m such that $m \equiv \ell(w^C) + \ell(w^D) + 1 \pmod 2$,

$$\dim_{\mathcal{O}} H^m(q_{\alpha+} j^!(\mathcal{L}_{Cs_\alpha})) = \dim_{\mathcal{O}} R^m i^!_{w^D s_\alpha}(\mathcal{L}_{Cs_\alpha}) + \dim_{\mathcal{O}} R^{m+1} i^!_{w^D}(\mathcal{L}_{Cs_\alpha}).$$

By restricting this further to $C(w^D)$ and $C(w^D s_\alpha)$, we see that for any $m \in \mathbb{Z}$,

$$\dim_{\mathcal{O}} R^m i^!_{w^D}(U^0_\alpha(\mathcal{L}_{Cs_\alpha})) = \dim_{\mathcal{O}} R^{m+1} i^!_{w^D}(\mathcal{L}_{Cs_\alpha}) + \dim_{\mathcal{O}} R^m i^!_{w^D s_\alpha}(\mathcal{L}_{Cs_\alpha}), \text{ and} \tag{27}$$

$$\dim_{\mathcal{O}} R^m i^!_{w^D s_\alpha}(U^0_\alpha(\mathcal{L}_{Cs_\alpha})) = \dim_{\mathcal{O}} R^m i^!_{w^D}(\mathcal{L}_{Cs_\alpha}) + \dim_{\mathcal{O}} R^{m-1} i^!_{w^D s_\alpha}(\mathcal{L}_{Cs_\alpha}). \tag{28}$$

[5]Note that we are adopting the convention that for $D \not\leq C$, $P_{CD} = 0$, and this statement is trivially true.

In addition, if $D \in W_\Theta \backslash W$ has the property that $Ds_\alpha = D$, we can use Lemma 18 to further reduce Eqs. 27 and 28. Indeed, by Lemma 18, if $Ds_\alpha = D$,

$$\dim_{\mathcal{O}} R^{m-1} i^!_{w^D s_\alpha}(\mathcal{L}_{Cs_\alpha}) = 0, \text{ and}$$
$$\dim_{\mathcal{O}} R^m i^!_{w^D s_\alpha}(\mathcal{L}_{Cs_\alpha}) = 0$$

for all $m \in \mathbb{Z}_+$. By Lemma 17, $U^0_\alpha(\mathcal{L}_{Cs_\alpha}) = \bigoplus_{D \leq C} m_{CD}\mathcal{L}_D$ for some $m_{CD} \in \mathbb{Z}_+$, hence Lemma 18 also implies that

$$\dim_{\mathcal{O}} R^m i^!_{w^D s_\alpha}(U^0_\alpha(\mathcal{L}_{Cs_\alpha})) = 0.$$

Therefore, we conclude that for all cosets $D \leq C$ such that $Ds_\alpha = D$,

$$\dim_{\mathcal{O}} R^m i^!_{w^D}(U^0_\alpha(\mathcal{L}_{Cs_\alpha})) = 0 \tag{29}$$

for all $m \in \mathbb{Z}$.

The Eqs. 27, 28, and 29 are what we need to show that $T_\alpha(\varphi(Cs_\alpha)) = \nu(U^0_\alpha(\mathcal{L}_{Cs_\alpha}))$. The computation is as follows.

$$\begin{aligned}
\nu(U^0_\alpha(\mathcal{L}_{Cs_\alpha})) &= \sum_{D \in W_\Theta\backslash W} \sum_{m \in \mathbb{Z}} \dim_{\mathcal{O}}(R^m i^!_{w^D}(U^0_\alpha(\mathcal{L}_{Cs_\alpha})))q^m \delta_D \\
&= \sum_{Ds_\alpha > D} \sum_{m \in \mathbb{Z}} \dim_{\mathcal{O}}(R^m i^!_{w^D}(U^0_\alpha(\mathcal{L}_{Cs_\alpha})))q^m \delta_D \\
&\quad + \sum_{Ds_\alpha < D} \sum_{m \in \mathbb{Z}} \dim_{\mathcal{O}}(R^m i^!_{w^D}(U^0_\alpha(\mathcal{L}_{Cs_\alpha})))q^m \delta_D \\
&\quad + \sum_{Ds_\alpha = D} \sum_{m \in \mathbb{Z}} \dim_{\mathcal{O}}(R^m i^!_{w^D}(U^0_\alpha(\mathcal{L}_{Cs_\alpha})))q^m \delta_D \\
&= \sum_{Ds_\alpha < D} \sum_{m \in \mathbb{Z}} \dim_{\mathcal{O}}(R^m i^!_{w^D s_\alpha}(U^0_\alpha(\mathcal{L}_{Cs_\alpha})))q^m \delta_{Ds_\alpha} \\
&\quad + \sum_{Ds_\alpha < D} \sum_{m \in \mathbb{Z}} \dim_{\mathcal{O}}(R^m i^!_{w^D}(U^0_\alpha(\mathcal{L}_{Cs_\alpha})))q^m \delta_D \\
&= \sum_{Ds_\alpha < D} \sum_{m \in \mathbb{Z}} (\dim_{\mathcal{O}} R^m i^!_{w^D}(\mathcal{L}_{Cs_\alpha}) + \dim_{\mathcal{O}} R^{m-1} i^!_{w^D s_\alpha}(\mathcal{L}_{Cs_\alpha}))q^m \delta_{Ds_\alpha} \\
&\quad + \sum_{Ds_\alpha < D} \sum_{m \in \mathbb{Z}} (\dim_{\mathcal{O}} R^{m+1} i^!_{w^D}(\mathcal{L}_{Cs_\alpha}) + \dim_{\mathcal{O}} R^m i^!_{w^D s_\alpha}(\mathcal{L}_{Cs_\alpha}))q^m \delta_D \\
&= \sum_{Ds_\alpha < D} \sum_{m \in \mathbb{Z}} (\dim_{\mathcal{O}} R^{m+1} i^!_{w^D}(\mathcal{L}_{Cs_\alpha}) + \dim_{\mathcal{O}} R^m i^!_{w^D s_\alpha}(\mathcal{L}_{Cs_\alpha}))q^m (\delta_D + q\delta_{Ds_\alpha}) \\
&= \sum_{Ds_\alpha < D} \sum_{m \in \mathbb{Z}} \dim_{\mathcal{O}} R^{m+1} i^!_{w^D}(\mathcal{L}_{Cs_\alpha}) q^{m+1} (q^{-1}\delta_D + \delta_{Ds_\alpha}) \\
&\quad + \sum_{Ds_\alpha > D} \sum_{m \in \mathbb{Z}} \dim_{\mathcal{O}} R^m i^!_{w^D}(\mathcal{L}_{Cs_\alpha}))q^m (\delta_{Ds_\alpha} + q\delta_D) \\
&= T_\alpha(\nu(\mathcal{L}_{Cs_\alpha})) = T_\alpha(\varphi(Cs_\alpha)).
\end{aligned}$$

Therefore, for $C \in W_\Theta \backslash W_{\leq k+1}$ and $\alpha \in \Pi$ such that $Cs_\alpha < C$,

$$T_\alpha(\varphi(Cs_\alpha)) = \nu(U^0_\alpha(\mathcal{L}_{Cs_\alpha})) = \nu(\bigoplus_{D \leq C} c_D \mathcal{L}_D) = \sum_{D \leq C} c_D \nu(\mathcal{L}_D) = \sum_{D \leq C} c_D \varphi(D),$$

i.e. Theorem 11 (ii) holds on $W_\Theta \backslash W_{\leq k+1}$. By induction, this completes the proof of Proposition 9, which in turn completes the proof of Theorem 11.

5.1 Composition Multiplicities of Standard Whittaker Modules

We are now ready to establish the connection between Whittaker Kazhdan–Lusztig polynomials and multiplicities of irreducible Whittaker modules in standard Whittaker modules. We start with two preliminary lemmas.

Lemma 19 *The evaluation $\nu(-1)$ of the map ν at -1 factors through the Grothendieck group $K(\mathcal{M}_{coh}(\mathcal{D}_X, N, \eta))$ of $\mathcal{M}_{coh}(\mathcal{D}_X, N, \eta)$.*

Proof For an object $\mathcal{F}$ in $\mathcal{M}_{coh}(\mathcal{D}_X, N, \eta)$,

$$\nu(\mathcal{F})(-1) = \sum_{C\in W_\Theta\backslash W} \sum_{m\in\mathbb{Z}} (-1)^m \dim_{\mathcal{O}}(R^m i^!_{w^C}(\mathcal{F}))\delta_C.$$

If $0 \to \mathcal{F}_1 \to \mathcal{F}_2 \to \mathcal{F}_3 \to 0$ is a short exact sequence in $\mathcal{M}_{coh}(\mathcal{D}_X, N, \eta)$, then for each $C \in W_\Theta\backslash W$, we have a long exact sequence

$$\cdots \xrightarrow{\partial_{m-1}} R^m i^!_{w^C}(\mathcal{F}_1) \xrightarrow{f_m} R^m i^!_{w^C}(\mathcal{F}_2) \xrightarrow{g_m} R^m i^!_{w^C}(\mathcal{F}_3) \xrightarrow{\partial_m} R^{m+1} i^!_{w^C}(\mathcal{F}_1) \to \cdots$$

of N-equivariant η-twisted connections on $C(w^C)$. For each $m \in \mathbb{Z}$, we have short exact sequences

$$\begin{aligned} 0 &\to \ker f_m \to R^m i^!_{w^C}(\mathcal{F}_1) \to \operatorname{im} f_m \to 0,\\ 0 &\to \ker g_m \to R^m i^!_{w^C}(\mathcal{F}_2) \to \operatorname{im} g_m \to 0, \text{ and}\\ 0 &\to \ker \partial_m \to R^m i^!_{w^C}(\mathcal{F}_3) \to \operatorname{im} \partial_m \to 0. \end{aligned}$$

Since $\mathcal{O}$-dimension sums over short exact sequences and $\ker f_m = \operatorname{im} \partial_{m-1}$, $\ker g_m = \operatorname{im} f_m$, and $\ker \partial_m = \operatorname{im} g_m$, we have

$$\begin{aligned} \sum_{m\in\mathbb{Z}} (-1)^m \dim_{\mathcal{O}}(R^m i^!_{w^C}(\mathcal{F}_2)) &= \sum_{m\in\mathbb{Z}} (-1)^m \dim_{\mathcal{O}}(R^m i^!_{w^C}(\mathcal{F}_1))\\ &\quad - \sum_{m\in\mathbb{Z}} (-1)^m \dim_{\mathcal{O}} \ker f_m\\ &\quad + \sum_{m\in\mathbb{Z}} (-1)^m \dim_{\mathcal{O}}(R^m i^!_{w^C}(\mathcal{F}_3))\\ &\quad - \sum_{m\in\mathbb{Z}} (-1)^m \dim_{\mathcal{O}} \ker \partial_m\\ &= \sum_{m\in\mathbb{Z}} (-1)^m \dim_{\mathcal{O}}(R^m i^!_{w^C}(\mathcal{F}_1))\\ &\quad + \sum_{m\in\mathbb{Z}} (-1)^m \dim_{\mathcal{O}}(R^m i^!_{w^C}(\mathcal{F}_3)). \end{aligned}$$

This implies the result. □

Lemma 20 $\nu(\mathcal{I}_C) = \delta_C$.

Proof By definition, $\mathcal{I}_C = i_{w^C+}(\mathcal{O}_{C(w^C)})$. By Kashiwara's theorem (Theorem 15),

$$R^0 i^!_{w^C}(\mathcal{I}_C) = R^0 i^!_{w^C}(i_{w^C+}(\mathcal{O}_{C(w^C)})) = \mathcal{O}_{C(w^C)},$$

and for $m \neq 0$,

$$R^m i^!_{w^C}(\mathcal{I}_C) = R^m i^!_{w^C}(i_{w^C+}(\mathcal{O}_{C(w^C)})) = 0.$$

Let $D \neq C$ be another coset in $W_\Theta \backslash W$. Then $i^{-1}_{w^D}(C(w^C)) = 0$, so by base change (Theorem 16),

$$R^m i^!_{w^D}(\mathcal{I}_C) = R^m i^!_{w^D}(i_{w^C+}(\mathcal{O}_{C(w^C)})) = 0$$

for all $m \in \mathbb{Z}$. □

Let $\chi : \mathcal{M}_{coh}(\mathcal{D}_X, N, \eta) \to K(\mathcal{M}_{coh}(\mathcal{D}_X, N, \eta))$ be the natural map of the category $\mathcal{M}_{coh}(\mathcal{D}_X, N, \eta)$ into its Grothendieck group $K(\mathcal{M}_{coh}(\mathcal{D}_X, N, \eta))$.

Theorem 12 *Let P_{CD}, $C, D \in W_\Theta \backslash W$ be the polynomials in Theorem 11. Then*

$$\chi(\mathcal{L}_C) = \chi(\mathcal{I}_C) + \sum_{D<C} P_{CD}(-1)\chi(\mathcal{I}_D).$$

Proof By definition, $\chi(\mathcal{L}_C), C \in W_\Theta \backslash W$ form a basis for the Grothendieck group $K(\mathcal{M}_{coh}(\mathcal{D}_X, N, \eta))$. Because $\mathcal{I}_C$ contains $\mathcal{L}_C$ as a unique irreducible submodule, and the other composition factors of $\mathcal{I}_C$ are $\mathcal{L}_D$ for $D < C$, we can see that $\chi(\mathcal{I}_C), C \in W_\Theta \backslash W$ form another basis for the Grothendieck group. Therefore, there exist $\lambda_{CD} \in \mathbb{Z}$ such that

$$\chi(\mathcal{L}_C) = \sum_{D \le C} \lambda_{CD}\chi(\mathcal{I}_D).$$

By Lemma 19, $\nu(-1)$ factors through $K(\mathcal{M}_{coh}(\mathcal{D}_X, N, \eta))$ and by Lemma 20, $\nu(\mathcal{I}_D) = \delta_D$, so by comparing coefficients and using the definition of ν, we have

$$\nu(\mathcal{L}_C)(-1) = \sum_{D \le C} \lambda_{CD}\nu(\mathcal{I}_D)(-1) = \sum_{D \le C} \lambda_{CD}\delta_D.$$

By construction, $P_{CC} = 1$ for any $C \in W_\Theta \backslash W$, so $\lambda_{CC} = 1$ and $P_{CD}(-1) = \lambda_{CD}$. This proves the theorem. □

This theorem gives an algorithm for calculating the multiplicities of irreducible Whittaker modules in standard Whittaker modules. Pick a total order compatible with the partial order on $W_\Theta \backslash W$. With respect to this order, the matrix $(\lambda_{CD})_{C,D \in W_\Theta \backslash W}$ is lower triangular and has 1's on the diagonal. Here $\lambda_{CD} = P_{CD}(-1)$ as in the proof of Theorem 12. Let $(\mu_{CD})_{C,D \in W_\Theta \backslash W}$ be the inverse matrix. From Theorem 12, we have

$$\begin{aligned}
\chi(\mathcal{I}_C) &= \sum_{D \in W_\Theta \backslash W} \sum_{E \in W_\Theta \backslash W} \mu_{CE}\lambda_{ED}\chi(\mathcal{I}_D) \\
&= \sum_{E \in W_\Theta \backslash W} \mu_{CE} \left(\sum_{D \in W_\Theta \backslash W} \lambda_{ED}\chi(\mathcal{I}_D) \right) \\
&= \sum_{E \in W_\Theta \backslash W} \mu_{CE}\chi(\mathcal{L}_E) \\
&= \sum_{E \le C} \mu_{CE}\chi(\mathcal{L}_E).
\end{aligned}$$

By Theorem 9 and Theorem 10, we have established the main result of this paper.

Corollary 3 *The multiplicity of the irreducible Whittaker module $L(-w^D\rho, \eta)$ in the standard Whittaker module $M(-w^C\rho, \eta)$ is μ_{CD}.*

We can get results analogous to Theorem 12 and Corollary 3 for integral $\lambda \in \mathfrak{h}^*$ by twisting by a equivariant invertible $\mathcal{O}_X$-module.

Corollary 4 *Let $\lambda \in \mathfrak{h}^*$ be regular, integral, and antidominant. Then the multiplicity of the irreducible Whittaker module $L(w^D(\lambda - \rho), \eta)$ in the standard Whittaker module $M(w^C(\lambda - \rho), \eta)$ is μ_{CD}.*

Proof From Corollary 3, we know that in the Grothendieck group of $\mathcal{M}_{coh}(\mathcal{D}_{-\rho}, N, \eta)$,

$$[\mathcal{I}(w^C, -\rho, \eta)] = [\mathcal{M}(w^C, -\rho, \eta)] = \sum_{D \in W_\Theta \backslash W} \mu_{CD}[\mathcal{L}(w^D, -\rho, \eta)].$$

Moreover, by the projection formula (Proposition 11), we have $\mathcal{I}(w^C, -\rho, \eta)(\lambda) = \mathcal{I}(w^C, \lambda - \rho, \eta)$, which in turn implies that $\mathcal{L}(w^C, -\rho, \eta) = \mathcal{L}(w^C, \lambda - \rho, \eta)$ since the twist functor $-(\lambda)$ must send irreducible objects in $\mathcal{M}_{coh}(\mathcal{D}_{-\rho}, N, \eta)$ to irreducible objects in $\mathcal{M}_{coh}(\mathcal{D}_{\lambda-\rho}, N, \eta)$ and each standard η-twisted Harish-Chandra sheaf has a unique irreducible subsheaf. By Theorem 9 this implies the result. □

Establishing the same multiplicity results for standard Whittaker modules of arbitrary infinitesimal character requires further analysis, which we will examine in future work. It is of note that the proof of Theorem 11 immediately implies that the coefficients of the Whittaker Kazhdan–Lusztig polynomials P_{CD} are non-negative integers.

Corollary 5 *The coefficients of the polynomials P_{CD} from Theorem 11 are non-negative integers.*

Proof This follows immediately from Proposition 9 and the definition of ν. □

6 Whittaker Kazhdan–Lusztig Polynomials

This section relates the Whittaker Kazhdan–Lusztig polynomials P_{CD} of Theorem 11 to the combinatorics of Kazhdan–Lusztig polynomials appearing in [19] and [14, Ch. 5 §2 §3]. We also describe a duality between the Kazhdan–Lusztig algorithm for Whittaker modules established in Section 5 and the Kazhdan-Lusztig algorithm for generalized Verma modules established in [14, Ch. 6 §3 Thm. 3.5], following the philosophy of dual Hecke algebra modules laid out in [21, §12 §13]. To make these associations, we need to introduce the Hecke algebra into our story.

6.1 The Hecke Algebra

Let (W, S) be a Coxeter system with length function $\ell : W \to \mathbb{N}$.

Definition 7 The *Hecke algebra* $\mathcal{H} = \mathcal{H}(W, S)$ of the Coxeter system (W, S) is the associative algebra over $\mathbb{Z}[q, q^{-1}]$ with generators $\{H_s\}_{s \in S}$ satisfying the relations

(i) (quadratic)

$$(H_s + q)(H_s - q^{-1}) = 0 \text{ for all } s \in S, \text{ and}$$

(ii) (braid) for each pair $s, t \in S$,

$$H_s H_t H_s \cdots = H_t H_s H_t \cdots$$

with m_{st} elements on each side of the equality. (Here m_{st} is the order of st in W.)

All H_s for $s \in S$ are invertible with $H_s^{-1} = H_s + (q - q^{-1})$. For $w \in W$, we choose a reduced expression $rs \cdots t$ of w and define $H_w \in \mathcal{H}$ by $H_r H_s \cdots H_t$. This element is independent of choice of reduced expression. If $\ell(w) + \ell(v) = \ell(wv)$, then we have $H_w H_v = H_{wv}$. There is exactly one ring homomorphism

$$\begin{aligned} d : \mathcal{H} &\to \mathcal{H} \\ H &\mapsto \overline{H} \end{aligned}$$

such that $\overline{q} = q^{-1}$ and $\overline{H}_w = (H_{w^{-1}})^{-1}$. This is clearly an involution. We say that $H \in \mathcal{H}$ is *self-dual* if $\overline{H} = H$. For each $s \in S$, the element $C_s := H_s + q$ is self-dual. Indeed, $\overline{C_s} = (H_s)^{-1} + q^{-1} = H_s + q = C_s$.

6.2 $\mathcal{H}_\Theta$ is a Hecke Algebra Module

Now we return to the setting of Section 5. Let W be the Weyl group of a reduced root system Σ with simple roots $\Pi \subset \Sigma$ and corresponding simple reflections $S \subset W$. Then (W, S) is a Coxeter system. Let $\Theta \subset \Pi$ be a fixed subset of simple roots and let $\mathcal{H}_\Theta = \bigoplus_{C \in W_\Theta \backslash W} \mathbb{Z}[q, q^{-1}]\delta_C$ be the $\mathbb{Z}[q, q^{-1}]$-module from Theorem 11. Recall that for each $\alpha \in \Pi$ we defined a $\mathbb{Z}[q, q^{-1}]$-linear endomorphism T_α of $\mathcal{H}_\Theta$ by

$$T_\alpha(\delta_C) = \begin{cases} 0 & \text{if } Cs_\alpha = C \\ q\delta_C + \delta_{Cs_\alpha} & \text{if } Cs_\alpha > C \\ q^{-1}\delta_C + \delta_{Cs_\alpha} & \text{if } Cs_\alpha < C \end{cases}.$$

Our first observation is that the operators $\{T_\alpha\}_{\alpha \in \Pi}$ give an action of the Hecke algebra of (W, S) on $\mathcal{H}_\Theta$. Indeed, if we define $S_\alpha := T_\alpha - q$, then a computation shows that S_α satisfies both the quadratic and braid relations of the Hecke algebra, thus the map $\psi : \mathcal{H} \to \mathrm{End}_{\mathbb{Z}[q,q^{-1}]}(\mathcal{H}_\Theta)$ given by $\psi(H_{s_\alpha}) = S_\alpha$ gives $\mathcal{H}_\Theta$ the structure of a left $\mathcal{H}$-module. The map ψ sends the self-dual basis element $C_{s_\alpha} \in \mathcal{H}$ described in the previous section to the endomorphism T_α.

This extra structure will allow us to relate Theorem 11 to the results in [19, §2 §3]. Our first step is to establish a relationship between $\mathcal{H}_\Theta$ and a certain induced right $\mathcal{H}$-module (the antispherical module for the Hecke algebra) in order to extend the duality in $\mathcal{H}$ given by the involution d to a duality in $\mathcal{H}_\Theta$. If $S_\Theta \subset S$ is the subset of simple reflections corresponding to $\Theta \subset \Pi$, then the subalgebra $\mathcal{H}^\Theta$ of $\mathcal{H}$ generated by $\{H_{s_\alpha}\}$ for $\alpha \in \Theta$ is isomorphic to the Hecke algebra of the Coxeter system (W_Θ, S_Θ). The surjection $\mathcal{H}^\Theta \twoheadrightarrow \mathbb{Z}[q, q^{-1}]$ sending $H_{s_\alpha} \mapsto -q$ gives $\mathbb{Z}[q, q^{-1}]$ the structure of a $\mathcal{H}^\Theta$-bimodule, and with this bimodule structure we can form the induced right $\mathcal{H}$-module

$$\mathcal{N}^\Theta := \mathbb{Z}[q, q^{-1}] \otimes_{\mathcal{H}^\Theta} \mathcal{H}.$$

This is the *antispherical module* of the Hecke algebra $\mathcal{H}$. Note that in the special case $\Theta = \emptyset$, $\mathcal{N}^\Theta$ is the Hecke-algebra $\mathcal{H}$ as a module over itself with the right regular action. The set $\{N_w := 1 \otimes H_w\}$ for minimal coset representatives $w \in C \in W_\Theta \backslash W$ forms a basis for $\mathcal{N}^\Theta$ as a $\mathbb{Z}[q, q^{-1}]$-module.

Remark 3 By instead using the surjection $\mathcal{H}^\Theta \twoheadrightarrow \mathbb{Z}[q, q^{-1}]$ given by $H_{s_\alpha} \mapsto q^{-1}$ to form the $\mathcal{H}^\Theta$-bimodule structure on $\mathbb{Z}[q, q^{-1}]$, it is possible to construct another induced right $\mathcal{H}$-module $\mathcal{M}^\Theta := \mathbb{Z}[q, q^{-1}] \otimes_{\mathcal{H}^\Theta} \mathcal{H}$ [19, §3]. This is the *spherical module* of the Hecke algebra $\mathcal{H}$. This module also has the property that $\mathcal{M}^\emptyset = \mathcal{H}$. By an analogous argument to the one below, one can show that the Kazhdan–Lusztig combinatorics of generalized Verma modules (as described in [14, Ch. 6 §3]) is given by the spherical $\mathcal{H}$-module.

One can compute [19] that the action of C_s on $\mathcal{N}^\Theta$ for $s \in S$ is given by

$$N_w C_s = \begin{cases} 0 & \text{if } ws \in C \\ qN_w + N_{ws} & \text{if } ws > w \text{ and } ws \notin C \\ q^{-1}N_w + N_{ws} & \text{if } ws < w \text{ and } ws \notin C \end{cases}.$$

Therefore, there is a $\mathbb{Z}[q, q^{-1}]$-module isomorphism

$$\begin{aligned} \phi : \mathcal{H}_\Theta &\to \mathcal{N}^\Theta \\ \delta_C &\mapsto N_{w_\Theta w^C} \end{aligned}$$

which intertwines the left $\mathcal{H}$-action on $\mathcal{H}_\Theta$ with the right $\mathcal{H}$-action on $\mathcal{N}^\Theta$. That is, for $E \in \mathcal{H}_\Theta$, $\phi(C_{s_\alpha}E) = \phi(E)C_{s_\alpha}$. Here w_Θ is the longest element in W_Θ.

Note that in the special case $\Theta = \emptyset$, this provides an $\mathbb{Z}[q, q^{-1}]$-module isomorphism between $\mathcal{H}_\emptyset$ and the Hecke algebra $\mathcal{H}$.[6] The benefit of relating $\mathcal{H}_\Theta$ to this induced module is that it allows us to use the involution d of $\mathcal{H}$ to construct an involution of the induced module, which we can then use to define self-duality in $\mathcal{H}_\Theta$. There is a homomorphism of additive groups

$$\begin{aligned} \mathcal{N}^\Theta &\to \mathcal{N}^\Theta \\ a \otimes H &\mapsto \overline{a \otimes H} := \overline{a} \otimes \overline{H}. \end{aligned}$$

This homomorphism has the property that $\overline{N}_e = N_e$ and

$$\overline{NH} = \overline{N}\,\overline{H} \tag{30}$$

for all $N \in \mathcal{N}^\Theta$ and $H \in \mathcal{H}$. We say that an element $E \in \mathcal{H}_\Theta$ is *self-dual* if the corresponding element in $\mathcal{N}^\Theta$ is fixed under this involution; that is, if $\overline{\phi(E)} = \phi(E)$. Since $\phi(T_\alpha(E)) = \phi(E)C_{s_\alpha}$ for any $\alpha \in \Pi$ and $E \in \mathcal{H}_\Theta$ and C_{s_α} is self-dual in $\mathcal{H}$, property (30) implies that T_α preserves self-duality.

6.3 The Recursion Relation in Theorem 11 is Equivalent to Self-Duality

The main content of this section is a proof that condition (ii) in Theorem 11 is equivalent to $\varphi(C)$ being self-dual in the sense of the preceding section.

Theorem 13 *Let $\varphi : W_\Theta\backslash W \to \mathcal{H}_\Theta$ be a function satisfying*

$$\varphi(C) = \delta_C + \sum_{D<C} P_{CD}\delta_D \text{ for } P_{CD} \in q\mathbb{Z}[q] \tag{31}$$

for all $C \in W_\Theta\backslash W$. Then the following are equivalent.

[6] This justifies the notational choice in [14, Ch. 5 §2], where the $\mathbb{Z}[q, q^{-1}]$-module $\mathcal{H}_\emptyset$ is referred to as $\mathcal{H}$.

(i) *If* $\alpha \in \Pi$ *and* $C \in W_\Theta \backslash W$ *are such that* $Cs_\alpha < C$*, then there exist* $m_D \in \mathbb{Z}$ *such that*

$$T_\alpha(\varphi(Cs_\alpha)) = \sum_{D \leq C} m_D \varphi(D).$$

(ii) *All* $\varphi(C)$ *are self-dual.*

Proof Assume that (i) holds, and take C and α such that $Cs_\alpha < C$. Using the definition of T_α we compute

$$\begin{aligned} T_\alpha(\varphi(Cs_\alpha)) &= T_\alpha(\delta_{Cs_\alpha} + \sum_{E < Cs_\alpha} P_{Cs_\alpha E} \delta_E) \\ &= \delta_C + q\delta_{Cs_\alpha} + \sum_{E < Cs_\alpha} P_{Cs_\alpha E} T_\alpha(\delta_E) \\ &= \delta_C + \sum_{D < C} Q_{CD} \delta_C \end{aligned}$$

for some $Q_{CD} \in \mathbb{Z}[q]$. Therefore, $m_C = 1$. Thus, for any $\alpha \in \Pi$ such that $Cs_\alpha < C$,

$$\varphi(C) = T_\alpha(\varphi(Cs_\alpha)) - \sum_{D < C} m_D \varphi(D). \tag{32}$$

Now we show that all $\varphi(C)$ are self-dual by induction in $\ell(w^C)$. If $C = W_\Theta$, then $\varphi(W_\Theta) = \delta_{W_\Theta}$ is self-dual because $\phi(\delta_{W_\Theta}) = 1 \otimes H_e$ and $\overline{H}_e = H_e$ in $\mathcal{H}$. Assume $\varphi(D)$ is self-dual for all $D < C$. Then because T_α preserves self-duality, Eq. 32 implies that $\varphi(C)$ is self-dual. We conclude that (i) implies (ii).

Now let $\varphi : W_\Theta \backslash W \to \mathcal{H}_\Theta$ be a function satisfying Eq. 31 and condition (ii). For $C \in W_\Theta \backslash W$, choose $\alpha \in \Pi$ such that $Cs_\alpha < C$. If no such α exists, then (i) is void and we are done. If such an α does exist, we have

$$T_\alpha(\varphi(Cs_\alpha)) = \delta_C + \sum_{D < C} Q_{CD} \delta_D$$

for appropriately chosen $Q_{CD} \in \mathbb{Z}[q]$. Define

$$\widetilde{\varphi}(C) := T_\alpha(\varphi(Cs_\alpha)) - \sum_{D < C} Q_{CD}(0) \varphi(D).$$

The function $\widetilde{\varphi}$ satisfies Eq. 31 and is self-dual by the fact that T_α preserves self-duality. Next we argue that there is a unique function satisfying both Eq. 31 and condition (ii), and thus $\widetilde{\varphi} = \varphi$. First, observe that for any $E \in \sum_{C \in W_\Theta \backslash W} q\mathbb{Z}[q]\delta_C$, self-duality implies $E = 0$. Indeed, if $E = \sum_{C \in W_\Theta \backslash W} R_C \delta_C$ and we let C be maximal such that $R_C \neq 0$, then $\overline{\phi(E)} = \phi(E)$ implies that $\overline{R}_C = R_C$, which is impossible because $R_C \in q\mathbb{Z}[q]$. Therefore, if $\varphi' : W_\Theta \backslash W \to \mathcal{H}_\Theta$ and $\varphi : W_\Theta \backslash W \to \mathcal{H}_\Theta$ are two functions satisfying Eq. 31 and (ii), then $\varphi(C) - \varphi'(C) \in \sum_{C \in W_\Theta \backslash W} q\mathbb{Z}[q]\delta_C$ is self-dual, so $\varphi(C) = \varphi'(C)$.

We conclude that $\widetilde{\varphi} = \varphi$, and by rearranging we obtain

$$T_\alpha(\varphi(Cs_\alpha)) = \sum_{D \leq C} m_D \varphi(D) \text{ for } m_D = \begin{cases} Q_{CD}(0) & \text{if } D < C \\ 1 & \text{if } D = C \end{cases}.$$

Thus (ii) implies (i). □

This establishes the relationship between the results in this paper and the results in [19, §2 §3]. In particular, it establishes that Theorem 11 in this paper is equivalent to part 2 of Theorem 3.1 in [19]. This allows us to explicitly compare Whittaker Kazhdan–Lusztig polynomials P_{CD} to polynomials that have shown up elsewhere in the literature under the name "parabolic Kazhdan–Lusztig polynomials." We list these relationships now.

Remark 4 1. The Whittaker Kazhdan–Lusztig polynomials P_{CD} are equal to the polynomials $n_{y,x}$ in [19] for $x = w_\Theta w^C$ and $y = w_\Theta w^D$.
2. A normalization of P_{CD} gives the parabolic Kazhdan–Lusztig polynomials in [5]. The polynomials

$$(q^{\ell(w_\Theta w^D)} - q^{\ell(w_\Theta w^C)})P_{CD}$$

are polynomials in the variable $v := q^{-2}$, and they are precisely the polynomials $P^I_{(w_\Theta w^D)^{-1},(w_\Theta w^D)^{-1}}$ in [5] for $u = v$ and $W_\Theta = W_I$.
3. In the special case where $\Theta = \emptyset$, the polynomials

$$(q^{\ell(v)} - q^{\ell(w)})P_{wv}$$

are the Kazhdan–Lusztig polynomials as defined in [9].

6.4 Duality of Whittaker Modules and Generalized Verma Modules

We conclude this paper by relating the Whittaker Kazhdan–Lusztig polynomials P_{CD} to the polynomials arising in the Kazhdan–Lusztig algorithm for generalized Verma modules established in [14, Ch. 6 §3]. Generalized Verma modules are a class of parabolically induced highest weight modules for a Lie algebra. For details of their construction, see [14, Ch. 6]. The main results of this section are Eq. 33 which relates the algorithm in Theorem 11 to the algorithm in [14, Ch. 6 Thm. 3.5], and Proposition 10, which provides a formula relating Whittaker Kazhdan–Lusztig polynomials to Kazhdan–Lusztig polynomials. By Theorem 13, Proposition 10 is a special case of [19, Prop. 3.4], but our proof is new, and independent of results in [19]. Equation 33 also recovers the Kazhdan–Lusztig inversion formulas of [9] as a special case.

In [14, Ch. 6 §3], Miličić establishes a Kazhdan–Lusztig algorithm for generalized Verma modules. We review his results here to establish their relationship with the Whittaker Kazhdan–Lusztig algorithm of this paper. Let $\mathcal{H}_\Theta = \bigoplus_{C \in W_\Theta \backslash W} \mathbb{Z}[q, q^{-1}]\delta_C$ be the $\mathbb{Z}[q, q^{-1}]$-module from the preceding section. We can realize $\mathcal{H}_\Theta$ as a $\mathbb{Z}[q, q^{-1}]$-submodule of the $\mathbb{Z}[q, q^{-1}]$-module $\mathcal{H}_\emptyset = \bigoplus_{w \in W} \mathbb{Z}[q, q^{-1}]\delta_w$ by setting

$$\delta_C = \sum_{v \in W_\Theta} q^{\ell(v)} \delta_{vw^C}.$$

For $\alpha \in \Pi$, let $T_\alpha^\emptyset : \mathcal{H}_\emptyset \to \mathcal{H}_\emptyset$ be the endomorphism defined by

$$T_\alpha^\emptyset(\delta_w) = \begin{cases} q\delta_w + \delta_{ws_\alpha} & \text{if } ws_\alpha > w \\ q^{-1}\delta_w + \delta_{ws_\alpha} & \text{if } ws_\alpha < w \end{cases},$$

as in Section 6.2. We introduce $\emptyset$ into the notation here to emphasize that $T_\alpha^\emptyset$ is an endomorphism of $\mathcal{H}_\emptyset$. A computation shows that the endomorphism $T_\alpha^\emptyset$ transforms δ_C in the following way:

$$T_\alpha^\emptyset(\delta_C) = \begin{cases} (q + q^{-1})\delta_C & \text{if } Cs_\alpha = C; \\ q\delta_C + \delta_{Cs_\alpha} & \text{if } Cs_\alpha < C; \\ q^{-1}\delta_C + \delta_{Cs_\alpha} & \text{if } Cs_\alpha > C. \end{cases}$$

It follows that $\mathcal{H}_\Theta$ is stable under $T_\alpha^\emptyset$, so $\mathcal{H}_\Theta$ is an $\mathcal{H}$-submodule of $\mathcal{H}_\emptyset$. In [14, Ch. 6 §3], Miličić proves the following Kazhdan-Lusztig algorithm for generalized Verma modules.

Theorem 14 [14, Ch. 6 §3 Thm. 3.5] *There exists a unique function $\varphi' : W_\Theta \backslash W \to \mathcal{H}_\Theta$ satisfying the following.*

(i) *For $C \in W_\Theta \backslash W$,*

$$\varphi'(C) = \delta_C + \sum_{D<C} P'_{CD}\delta_D$$

for $P'_{CD} \in q\mathbb{Z}[q]$, and

(ii) *for $\alpha \in \Pi$ such that $Cs_\alpha < C$, there exist integers m'_D such that*

$$T_\alpha^{\emptyset}(\varphi'(Cs_\alpha)) = \sum_{D \leq C} m'_D \varphi'(D).$$

Furthermore, the polynomials P'_{CD} are given by the Kazhdan–Lusztig polynomials for (W, S) by

$$P'_{CD} = P_{w^C w^D}.$$

Since Theorem 11 specializes to the Kazdhan–Lusztig algorithm for Verma modules [14, Ch. 5 §2 Thm. 2.1] when $\Theta = \emptyset$, one can see from Miličić's proof of Theorem 14 that the unique function $\varphi' : W_\Theta \backslash W \to \mathcal{H}_\Theta$ satisfying Theorem 14 is the function $\varphi'(D) := \varphi_\emptyset(w^D)$, where $\varphi_\emptyset : W \to \mathcal{H}_\emptyset$ is the unique function guaranteed by Theorem 11 in the special case $\Theta = \emptyset$. The Kazhdan–Lusztig polynomials P'_{CD} of Theorem 14 describe the multiplicities of irreducible highest weight modules in generalized Verma modules [14, Ch. 6 §3 Cor. 3.7].

For arbitrary $\Theta \subset \Pi$, the Whittaker Kazhdan–Lusztig polynomials are inverse to the polynomials appearing in Theorem 14 in the following sense.

$$\sum_{E \in W_\Theta \backslash W} (-1)^{\ell(w^E)+\ell(w^C)} P'_{Cw_0Ew_0} P_{DE} = \begin{cases} 1 & \text{if } C = D \\ 0 & \text{if } C \neq D \end{cases}. \tag{33}$$

This relationship appears as Proposition 3.9 in [19], where it is originally attribued to Douglass [6]. If we specialize to $\Theta = \emptyset$, then $W_\Theta \backslash W = W$, and Eq. 33 recovers the Kazhdan–Lusztig inversion formulas.

$$\sum_{u \in W} (-1)^{\ell(u)+\ell(w)} P_{wu} P_{vw_0uw_0} = \begin{cases} 1 & \text{if } v = w \\ 0 & \text{if } v \neq w \end{cases}. \tag{34}$$

We complete this section by describing the relationship between the Whittaker Kazhdan–Lusztig polynomials P_{CD} and the Kazhdan–Lusztig polynomials in [14]. If $\Theta = \emptyset$, Theorem 11 specializes the algorithm in [14, Ch. 5 §2 Thm. 2.1], and the polynomials P_{wv} are the Kazhdan–Lusztig polynomials as defined in [14]. Note that these polynomials differ in normalization from the Kazhdan–Lusztig polynomials appearing in [9]; see Remark 4. The following formula relates Whittaker Kazhdan–Lusztig polynomials for general Θ to Kazhdan–Lusztig polynomials.

Proposition 10 *For $\Theta \subset \Pi$ arbitrary,*

$$P_{CD} = \sum_{v \in W_\Theta} (-q)^{\ell(v)} P_{w_\Theta w^C v w_\Theta w^D}.$$

Proof Fix an arbitrary $\Theta \subset \Pi$, and pick a total order compatible with the partial order on $W_\Theta \backslash W$. From Theorem 14 we see that $P'_{CD} = 0$ for $D > C$ and $P'_{CD} = 1$ if $C = D$, so the matrix $P = (P'_{CD})$ of polynomials with respect to our total order is lower triangular with 1's on the diagonal and coefficients in $\mathbb{Z}[q]$. The inverse matrix $Q = (Q_{CD})$ is also lower triangular with 1's on the diagonal and coefficients in $\mathbb{Z}[q]$. From Eq. 33 we see that the coefficients Q_{CD} of the inverse matrix are related to Whittaker Kazhdan–Lusztig polynomials in the following way:

$$Q_{CD} = (-1)^{\ell(w^C)+\ell(w^D)} P_{Dw_0Cw_0}. \tag{35}$$

Then, if $\varphi_\emptyset : W \to \mathcal{H}_\emptyset$ is the unique function from Theorem 11 corresponding to the subset $\Theta = \emptyset$, we have

$$\begin{aligned}\sum_{D\in W_\Theta\backslash W} Q_{CD}\varphi_\emptyset(w^D) &= \sum_{D\in W_\Theta\backslash W} Q_{CD}\left(\sum_{E\in W_\Theta\backslash W} P'_{DE}\delta_E\right)\\ &= \sum_{E\in W_\Theta\backslash W}\left(\sum_{D\in W_\Theta\backslash W} Q_{CD}P'_{DE}\right)\delta_E\\ &= \delta_C.\end{aligned}$$

Here the polynomials Q_{CD} correspond to our arbitrary fixed Θ, and only the function $\varphi_\emptyset$ is specific to the special case $\Theta = \emptyset$. Now, if we specialize further to the case that our fixed Θ is $\Theta = \emptyset$, the computation above implies

$$\sum_{v\in W} Q_{wv}\varphi(v) = \delta_w. \tag{36}$$

Then, because

$$\delta_C = \sum_{v\in W_\Theta} q^{\ell(v)}\delta_{vw^C},$$

we have the following relationship:

$$\begin{aligned}\sum_{D\in W_\Theta\backslash W} Q_{CD}\varphi(w^D) &= \sum_{v\in W_\Theta} q^{\ell(v)}\delta_{vw^C}\\ &= \sum_{v\in W_\Theta} q^{\ell(v)}\left(\sum_{u\in W} Q_{vw^C u}\varphi(u)\right)\\ &= \sum_{u\in W}\left(\sum_{v\in W_\Theta} q^{\ell(v)}Q_{vw^C u}\right)\varphi(u).\end{aligned}$$

Here the second equality follows from Eq. 36. Since $\{\varphi(u) : u \in W\}$ form a basis for $\mathcal{H}_\emptyset$ by Theorem 11, this implies that

$$Q_{CD} = \sum_{v\in W_\Theta} q^{\ell(v)}Q_{vw^C w^D}.$$

Thus, since $\ell(vw^C) = \ell(w^C) - \ell(v)$ for $v \in W_\Theta$ by [14, Ch. 6 §1 Lem. 1.8], an application of Eq. 35 for the special case $\Theta = \emptyset$ results in the following formula:

$$Q_{CD} = (-1)^{\ell(w^C)+\ell(w^D)} \sum_{v\in W_\Theta} (-1)^{\ell(v)}q^{\ell(v)}P_{w^D w_0 v w^C w_0}. \tag{37}$$

The element $w^C w_0$ is the shortest element of the coset Cw_0, so it is equal to $w_\Theta w^{Cw_0}$ by [14, Ch. 6 §1 Thm. 1.4]. The proposition then follows by combining Eq. 37 with Eq. 35. $\square$

Acknowledgements This project grew from the work of my PhD advisor Dragan Miličić and his collaborator Wolfgang Soergel, and it owes its existence to the detailed mathematical foundation established in their joint work. I am grateful for the mentorship and guidance provided by Dragan Miličić that led to this paper. I also thank Peter Trapa for suggesting that I introduce the Hecke algebra into this story, and I thank Geordie Williamson for directing me to the combinatorial existence argument in [19] and pointing out the connection with the antispherical category. I thank Emily Cliff and Christopher Leonard for providing very useful feedback on preliminary versions of this paper.

Appendix: Geometric Preliminaries

In this appendix we record some some fundamental results about functors between categories of modules over twisted sheaves of differential operators which play a critical role in the arguments of Sections 4 and 5. For a detailed treatment of this subject, see [7, 14, 15].

A.1 Twisted Sheaves of Differential Operators

Let X be a smooth complex algebraic variety of dimension n. Denote by $\mathcal{O}_X$ the structure sheaf of X, $\mathcal{D}_X$ the sheaf of differential operators on X, $\mathcal{T}_x$ the tangent sheaf on X, Ω_X the cotangent sheaf on X, and ω_X the invertible $\mathcal{O}_X$-module of differential n-forms on X. Denote by $i_X : \mathcal{O}_X \to \mathcal{D}_X$ the natural inclusion. A *twisted sheaf of differential operators on* X is a pair $(\mathcal{D}, i)$ of a sheaf $\mathcal{D}$ of associative $\mathbb{C}$-algebras with identity on X and a homomorphism $i : \mathcal{O}_X \to \mathcal{D}$ of sheaves of $\mathbb{C}$-algebras with identity that is locally isomorphic to the pair $(\mathcal{D}_X, i_X)$.

For $f : Y \to X$ a morphism of smooth algebraic varieties and $\mathcal{D}$ a twisted sheaf of differential operators on X, we define

$$\mathcal{D}_{Y\to X} = \mathcal{O}_Y \otimes_{f^{-1}\mathcal{O}_X} f^{-1}\mathcal{D}.$$

Then $\mathcal{D}_{Y\to X}$ is a left $\mathcal{O}_Y$-module for left multiplication and a right $f^{-1}\mathcal{D}$-module for right multiplication on the second factor. Denote by $\mathcal{D}^f$ the sheaf of differential $\mathcal{O}_Y$-module endomorphisms of $\mathcal{D}_{Y\to X}$ which are also $f^{-1}\mathcal{D}$-module endomorphisms. There is a natural morphism of sheaves of algebras $i_f : \mathcal{O}_Y \to \mathcal{D}^f$, and the pair $(\mathcal{D}^f, i_f)$ is a twisted sheaf of differential operators on Y.

Let $\mathcal{D}$ be a twisted sheaf of differential operators on X and $\mathcal{L}$ an invertible $\mathcal{O}_X$-module. The *twist* of $\mathcal{D}$ by $\mathcal{L}$ is the sheaf $\mathcal{D}^{\mathcal{L}}$ of differential $\mathcal{O}_X$-module endomorphisms of $\mathcal{L} \otimes_{\mathcal{O}_X} \mathcal{D}$ that commute with the right $\mathcal{D}$-action. Because $\mathcal{L} \otimes_{\mathcal{O}_X} \mathcal{D}$ is an $\mathcal{O}_X$-module for left multiplication, there is a natural homomorphism $i_{\mathcal{L}} : \mathcal{O}_X \to \mathcal{D}^{\mathcal{L}}$, and $(\mathcal{D}^{\mathcal{L}}, i_{\mathcal{L}})$ is a twisted sheaf of differential operators on X. If $f : Y \to X$ is a morphism of smooth algebraic varieties as above, $(\mathcal{D}^{\mathcal{L}})^f = (\mathcal{D}^f)^{f^*(\mathcal{L})}$.

If X is a homogeneous space for a group G with Lie algebra $\mathfrak{g}$, then a *homogeneous twisted sheaf of differential operators* on X is a triple $(\mathcal{D}, \gamma, \alpha)$, where $\mathcal{D}$ is a twisted sheaf of differential operators on X, γ is the algebraic action of G on X, and $\alpha : \mathcal{U}(\mathfrak{g}) \to \Gamma(X, \mathcal{D})$ is a morphism of algebras such that the following three conditions are satisfied:

(i) the multiplication in $\mathcal{D}$ is G-equivariant;
(ii) the differential of the G-action on $\mathcal{D}$ agrees with the action $T \mapsto [\alpha(\xi), T]$ for $\xi \in \mathfrak{g}$ and $T \in \mathcal{D}$; and

(iii) the map $\alpha : \mathcal{U}(\mathfrak{g}) \to \Gamma(X, \mathcal{D})$ is a morphism of G-modules.

For $x \in X$, denote by B_x the stabilizer of x in G and $\mathfrak{b}_x$ its Lie algebra. For each B_x-invariant linear form $\lambda \in \mathfrak{b}_x^*$ one can construct a homogeneous twisted sheaf of differential operators $\mathcal{D}_{X,\lambda}$ [7, App. A §1] and all homogeneous twisted sheaves of differential operators on X occur in this occur in this way.

If $\mathcal{A}$ is a sheaf of $\mathbb{C}$-algebras on X, we denote by $\mathcal{A}^\circ$ the opposite sheaf of $\mathbb{C}$-algebras on X. Then if $(\mathcal{D}, i)$ is a twisted sheaf of differential operators on a smooth algebraic variety X, $(\mathcal{D}^\circ, i)$ is also a twisted sheaf of differential operators on X. In particular, the pair $(\mathcal{D}_X^\circ, i_X)$ is a twisted sheaf of differential operators, and it is naturally isomorphic to $(\mathcal{D}_X^{\omega_X}, i_{\omega_X})$. If X is a homogeneous space and δ is the B_x-invariant linear form which is the differential of the representation of B_x on the top exterior power of the cotangent space at x, then $(\mathcal{D}_{X,\lambda})^\circ$ is naturally isomorphic to $\mathcal{D}_{X,-\lambda+\delta}$.

A.2 Modules over Twisted Sheaves of Differential Operators

Let $\mathcal{D}$ be a twisted sheaf of differential operators on a smooth complex algebraic variety X. For a category $\mathcal{M}(\mathcal{D})$ of $\mathcal{D}$-modules, we denote $\mathcal{M}_{qc}(\mathcal{D})$ (resp. $\mathcal{M}_{coh}(\mathcal{D})$) the corresponding category of quasicoherent (resp. coherent) $\mathcal{D}$-modules. We can view left $\mathcal{D}$-modules as right right $\mathcal{D}^\circ$-modules and vice-versa. In other words, the category $\mathcal{M}_{qc}^L(\mathcal{D})$ of quasicoherent left $\mathcal{D}$-modules on X is isomorphic to the category the category $\mathcal{M}_{qc}^R(\mathcal{D}^\circ)$ of quasicoherent right $\mathcal{D}^\circ$-modules on X. This relationship allows us to freely use right or left modules depending on the particular situation, and because of this, we frequently drop the exponents 'L' and 'R' from our notation.

For a coherent $\mathcal{D}$-module $\mathcal{V}$, we can define the *characteristic variety* Ch$\mathcal{V}$ of $\mathcal{V}$ in the same way as the non-twisted case [13, Ch. III §3]. Because this construction is local, the results in the non-twisted case carry over to our setting. In particular, we have the following structure:

(i) Ch$\mathcal{V}$ is a conical subvariety of the cotangent bungle $T^*(X)$.
(ii) $\dim(\mathrm{Ch}\mathcal{V}) \geq \dim(X)$.

If $\dim(\mathrm{Ch}\mathcal{V}) = \dim(X)$, we say that $\mathcal{V}$ is a *holonomic* $\mathcal{D}$-module. Holonomic $\mathcal{D}$-modules form a thick subcategory $\mathcal{M}_{hol}(\mathcal{D})$ of $\mathcal{M}_{coh}(\mathcal{D})$. If $\mathcal{V}$ is coherent as an $\mathcal{O}_X$-module, we call $\mathcal{V}$ a *connection*. Connections are locally free as $\mathcal{O}_X$-modules and their characteristic variety is the zero section of $T^*(X)$, so they are holonomic.

For an invertible $\mathcal{O}_X$-module $\mathcal{L}$ and a twisted sheaf $\mathcal{D}$ of differential operators on X, we define the *twist functor* from $\mathcal{M}_{qc}^L(\mathcal{D}^{\mathcal{L}})$ by

$$\mathcal{V} \mapsto (\mathcal{L} \otimes_{\mathcal{O}_X} \mathcal{D}) \otimes_{\mathcal{D}} \mathcal{V}$$

for $\mathcal{V} \in \mathcal{M}_{qc}^L(\mathcal{D})$. The twist functor is an equivalence of categories.

For an abelian category $\mathcal{C}$, we use the notation $D(\mathcal{C})$ and $D^b(\mathcal{C})$ to refer to the derived category and bounded derived category of $\mathcal{C}$, respectively. We identify $\mathcal{C}$ with its image in $D(\mathcal{C})$ (resp. $D^b(\mathcal{C})$) under the natural embedding.

For a morphism $f : Y \to X$ of smooth algebraic varieties and a twisted sheaf $\mathcal{D}$ of differential operators on X, we define the *inverse image functor* $f^+ : \mathcal{M}_{qc}^L(\mathcal{D}) \to \mathcal{M}_{qc}^L(\mathcal{D}^f)$ by

$$f^+(\mathcal{V}) = \mathcal{D}_{Y\to X} \otimes_{f^{-1}\mathcal{D}} f^{-1}\mathcal{V}$$

for $\mathcal{V} \in \mathcal{M}^L_{qc}(\mathcal{D})$. In general f^+ is right exact with left derived functor Lf^+. If f is an open immersion, then f^+ is exact and $f^+(\mathcal{V}) = \mathcal{V}|_Y$. If f is a submersion, then f^+ is exact. We define the *extraordinary inverse image functor* $f^! : D^b(\mathcal{M}^L_{qc}(\mathcal{D})) \to D^b(\mathcal{M}^L_{qc}(\mathcal{D}^f))$ by

$$f^! = Lf^+ \circ [\dim Y - \dim X].$$

If f is an immersion then $f^!$ is the right derived functor of the left exact functor $L^{\dim Y - \dim X} f^+ : \mathcal{M}^L_{qc}(\mathcal{D}) \to \mathcal{M}^L_{qc}(\mathcal{D}^f)$. In this setting, we refer to the functor $L^{\dim Y - \dim X} f^+$ as $f^!$, and for $\mathcal{V} \in \mathcal{M}_{qc}(\mathcal{D})$, we refer to the k^{th}-cohomology modules $H^k f^!(\mathcal{V})$ as $R^k f^!(\mathcal{V})$.

We define the *direct image functor* $f_+ : D^b(\mathcal{M}^R_{qc}(\mathcal{D}^f)) \to D^b(\mathcal{M}^R_{qc}(\mathcal{D}))$ by

$$f_+(\mathcal{W}^{\cdot}) = Rf_\bullet(\mathcal{W}^{\cdot} \otimes^L_{\mathcal{D}^f} \mathcal{D}_{Y \to X}),$$

for $\mathcal{W}^{\cdot} \in D^b(\mathcal{M}^R(\mathcal{D}^f))$. Here $Rf_\bullet$ is the right derived functor of the sheaf-theoretic direct image functor $f_\bullet$. If f is an immersion, f_+ is the right derived functor of the left exact functor $H^0 \circ f_+ \circ D : \mathcal{M}^R_{qc}(\mathcal{D}^f) \to \mathcal{M}^R_{qc}(\mathcal{D})$, where D is the natural embedding of $\mathcal{M}^R_{qc}(\mathcal{D}^f)$ into the derived category $D(\mathcal{M}^R_{qc}(\mathcal{D}^f))$. In this setting, we refer to $H^0 \circ f_+ \circ D$ by f_+. If f is an open immersion, then $f_+ = Rf_\bullet$ is the sheaf-theoretic direct image. If f is affine, then f_+ is exact.

The relationship between the twist functor and the direct image functor is the following.

Proposition 11 *(Projection Formula) Let $f : Y \to X$ be a morphism of smooth complex algebraic varieties, $\mathcal{D}$ a twisted sheaf of differential operators on X, and $\mathcal{L}$ an invertible $\mathcal{O}_X$-module. Then the following diagram commutes.*

$$\begin{tikzcd} D(\mathcal{M}(\mathcal{D}^f)) \arrow[r, "f_+"] \arrow[d, "f^*(\mathcal{L}) \otimes_{\mathcal{O}_Y} -"'] & D(\mathcal{M}(\mathcal{D})) \arrow[d, "\mathcal{L} \otimes_{\mathcal{O}_X} -"] \\ D(\mathcal{M}((\mathcal{D}^{\mathcal{L}})^f)) \arrow[r, "f_+"] & D(\mathcal{M}(\mathcal{D}^{\mathcal{L}})) \end{tikzcd}$$

For a module $\mathcal{V} \in \mathcal{M}^R_{qc}(\mathcal{D})$, and a smooth subvariety $Y \subset X$, denote by $\Gamma_Y(\mathcal{V})$ the $\mathcal{D}$-module of local sections Y. The functor $\Gamma_Y : \mathcal{M}^R_{qc}(\mathcal{D}) \to \mathcal{M}^R_{qc}(\mathcal{D})$ is a left-exact functor, and we denote by $R\Gamma_Y : D^b(\mathcal{M}^R_{qc}(\mathcal{D})) \to D^b(\mathcal{M}^R_{qc}(\mathcal{D}))$ its right derived functor. The following equivalence of categories is very useful in computations.

Theorem 15 (Kashiwara) *If Y is a closed smooth subvariety of a smooth algebraic variety X, $i : Y \to X$ the natural immersion, and $\mathcal{D}$ a twisted sheaf of differential operators on X, then the functor*

$$i_+ : \mathcal{M}^R_{qc}(\mathcal{D}^i) \to \mathcal{M}^R_{qc}(\mathcal{D})$$

establishes an equivalence of categories between $\mathcal{M}^R_{qc}(\mathcal{D}^i)$ and the full subcategory $\mathcal{M}^R_{qc,Y}(\mathcal{D})$ of supported in Y. The quasiinverse of i_+ is $i^!$. In particular, if $\mathcal{V}$ is a quasicoherent $\mathcal{D}^i$-module, then $i^!(i_+(\mathcal{V})) = \mathcal{V}$, and if $\mathcal{U}$ is a $\mathcal{D}^i$-module, then $i^!(i_+(\mathcal{V})) = \mathcal{V}$, and if $\mathcal{U}$ is a quasicoherent $\mathcal{D}$-module, then $i_+(i^!(\mathcal{U})) = \Gamma_Y(\mathcal{U})$.

Let $i : Y \to X$ be the immersion of a closed subvariety. If $\mathcal{J}_Y$ is the ideal of $\mathcal{O}_X$ consisting of germs vanishing on Y, we can define an filtration of $\mathcal{D}_{Y\to X}$ by (left $\mathcal{D}^i$, right $i^{-1}\mathcal{O}_X$)-modules by

$$F_p\mathcal{D}_{Y\to X} = \{T \in \mathcal{D}_{Y\to X} | T\varphi = 0 \text{ for } \varphi \in (\mathcal{J}_Y)^{p+1}\},$$

for $p \in \mathbb{Z}_+$. We call this filtration the filtration by *normal degree*. By Kashiwara's theorem, it induces a natural $\mathcal{O}_X$-module filtration on supported on Y. Namely, if $\mathcal{W} \in \mathcal{M}^R_{qc}(\mathcal{D}^i)$,

$$F_p i_+(\mathcal{W}) = i_\bullet(\mathcal{W} \otimes_{\mathcal{D}^i} F_p\mathcal{D}_{Y\to X}).$$

The associated graded module has the form

$$Gr i_+(\mathcal{W}) = i_\bullet(\mathcal{W} \otimes_{\mathcal{O}_Y} S(\mathcal{N}_{X|Y})), \tag{38}$$

where $\mathcal{N}_{X|Y} = i^*(\mathcal{T}_X)/\mathcal{T}_Y$ denotes the normal sheaf of Y, and $S(\mathcal{N}_{X|Y})$ is the corresponding sheaf of symmetric algebras [7, App. A §3.3].

The interaction between $\mathcal{D}$-module functors and fiber products is captured by base change.

Theorem 16 (Base Change Formula) *Let $f : X \to Z$ and $g : Y \to Z$ be morphisms of smooth complex algebraic varieties such that the fiber product $X \times_Z Y$ is a smooth algebraic variety, and let $\mathcal{D}$ be a twisted sheaf of differential operators on Z. Then the commutative diagram*

$$\begin{tikzcd} X \times_Z Y \arrow[r, "q"] \arrow[d, "p"'] & Y \arrow[d, "g"] \\ X \arrow[r, "f"] & Z \end{tikzcd}$$

determines an isomorphism

$$g^! \circ f_+ = q_+ \circ p^!$$

of functors from $D^b(\mathcal{M}(\mathcal{D}^f))$ to $D^b(\mathcal{M}(\mathcal{D}^g))$.

A.3 Beilinson–Bernstein Localization

A key ingredient in this story is the localization theory of Beilinson and Bernstein, which we briefly review here. Full details can be found in [2, 14]. For the remainder of this appendix, let $\mathfrak{g}$ be a complex reductive Lie algebra, $\mathfrak{h}$ the abstract Cartan subalgebra of $\mathfrak{g}$ [15, §2], and X the flag variety of $\mathfrak{g}$. Fix $\lambda \in \mathfrak{h}^*$, and let θ be the Weyl group orbit of λ in $\mathfrak{h}^*$. In [2], Beilinson and Bernstein construct a twisted sheaf of differential operators $\mathcal{D}_\lambda$ on X for each $\lambda \in \mathfrak{h}^*$. (In the notation of Section Appendix A.1, $\mathcal{D}_\lambda = \mathcal{D}_{X,\lambda+\rho}$.) They show that for any μ in the Weyl group orbit θ of λ, the global sections $\Gamma(X, \mathcal{D}_\mu)$ of $\mathcal{D}_\mu$ are equal to $\mathcal{U}_\theta$, which is the quotient of $\mathcal{U}(\mathfrak{g})$ by the ideal in $\mathcal{Z}(\mathfrak{g})$ corresponding to θ under the Harish-Chandra homomorphism. This implies that the global sections functor Γ maps quasicoherent $\mathcal{D}_\lambda$-modules into $\mathcal{U}(\mathfrak{g})$-modules with infinitesimal character χ_λ; that is, there is a left exact functor

$$\Gamma : \mathcal{M}_{qc}(\mathcal{D}_\lambda) \to \mathcal{M}(\mathcal{U}_\theta).$$

Beilinson and Bernstein define a *localization functor*

$$\Delta_\lambda : \mathcal{M}(\mathcal{U}_\theta) \to \mathcal{M}_{qc}(\mathcal{D}_\lambda)$$

by $\Delta_\lambda(V) = \mathcal{D}_\lambda \otimes_{\mathcal{U}_\theta} V$ for $V \in \mathcal{M}(\mathcal{U}_\theta)$. The localization functor is right exact and is a left adjoint to Γ. In [2] it is shown that for antidominant regular $\lambda \in \mathfrak{h}^*$, Δ_λ is an equivalence of categories, and its quasi-inverse is Γ.

A.4 Translation Functors

Fix $\lambda \in \mathfrak{h}^*$, and let $\mathcal{D}_\lambda$ be the corresponding homogeneous twisted sheaf of differential operators. Any μ in the weight lattice $P(\Sigma) = \{\lambda \in \mathfrak{h}^* | \alpha^\vee(\lambda) \in \mathbb{Z} \text{ for all } \alpha \in \Sigma\}$ naturally determines a $G = \operatorname{Int}\mathfrak{g}$-equivariant invertible $\mathcal{O}_X$-module $\mathcal{O}(\mu)$ on X. Twisting by $\mathcal{O}(\mu)$ defines a functor

$$-(\mu) : \mathcal{M}(\mathcal{D}_\lambda) \to \mathcal{M}(\mathcal{D}_{\lambda+\mu})$$

by $\mathcal{V}(\mu) = \mathcal{O}(\mu) \otimes_{\mathcal{O}_X} \mathcal{V}$ for $\mathcal{V} \in \mathcal{M}(\mathcal{D}_\lambda)$. We call this functor the *geometric translation functor*. It is evidently an equivalence of categories, and it also induces an equivalence of categories on $\mathcal{M}_{qc}(\mathcal{D}_\lambda)$ (resp. $\mathcal{M}_{coh}(\mathcal{D}_\lambda)$) with $\mathcal{M}_{coh}(\mathcal{D}_\lambda)$ (resp. $\mathcal{M}_{coh}(\mathcal{D}_{\lambda+\mu})$).

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