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CONTRAVARIANT FORMS ON WHITTAKER MODULES

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Abstract. Let $\mathfrak{g}$ be a complex semisimple Lie algebra. We give a classification of contravariant forms on the nondegenerate Whittaker $\mathfrak{g}$-modules $Y(\chi,\eta)$ introduced by Kostant. We prove that the set of all contravariant forms on $Y(\chi,\eta)$ forms a vector space whose dimension is given by the cardinality of the Weyl group of $\mathfrak{g}$. We also describe a procedure for parabolically inducing contravariant forms. As a corollary, we deduce the existence of the Shapovalov form on a Verma module, and provide a formula for the dimension of the space of contravariant forms on the degenerate Whittaker modules $M(\chi,\eta)$ introduced by McDowell.

1. Introduction

This paper concerns a classical tool in the study of representations of Lie algebras: contravariant forms. Contravariant forms are certain symmetric bilinear forms on modules over a Lie algebra which are invariant under an antiautomorphism of the Lie algebra (Definition 3.1). Many well-studied classes of Lie algebra modules, such as Verma modules, finite-dimensional irreducible modules, and, more generally, highest weight modules in Bernstein–Gelfand–Gelfand's category $\mathcal{O}$, admit a unique contravariant form up to a multiplier. In this paper we study a class of Lie algebra modules for which this is not the case.

Let $\mathfrak{g}$ be a complex semisimple Lie algebra with a fixed Borel subalgebra $\mathfrak{b} \subset \mathfrak{g}$ and Cartan subalgebra $\mathfrak{h} \subset \mathfrak{b}$. Let $\mathfrak{n} = [\mathfrak{b}, \mathfrak{b}]$ be the nilpotent radical of $\mathfrak{b}$, and W the Weyl group of $\mathfrak{g}$. Denote by $U(\mathfrak{g})$ the universal enveloping algebra of $\mathfrak{g}$ and by $Z(\mathfrak{g})$ its center. In [Kos78], Kostant introduced a family of Whittaker $\mathfrak{g}$-modules

$$Y(\chi,\eta) := U(\mathfrak{g}) \otimes_{Z(\mathfrak{g})\otimes_{\mathbb{C}} U(\mathfrak{n})} \mathbb{C}_{\chi,\eta},$$

where $\mathbb{C}_{\chi,\eta}$ is the one-dimensional $Z(\mathfrak{g}) \otimes_{\mathbb{C}} U(\mathfrak{n})$-module determined by the characters $\chi : Z(\mathfrak{g}) \to \mathbb{C}$ and $\eta : \mathfrak{n} \to \mathbb{C}$ (Definition 2.2). Each module $Y(\chi,\eta)$ is cyclically generated by a Whittaker vector $w = 1 \otimes 1 \in Y(\chi,\eta)$ on which $\mathfrak{n}$ acts by a nondegenerate Lie algebra morphism $\eta : \mathfrak{n} \to \mathbb{C}$ (Definition 2.1). The modules $Y(\chi,\eta)$ are infinite-dimensional and irreducible. The main result of this paper is a classification of contravariant forms on $Y(\chi,\eta)$.

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Theorem 1.1 (Theorem 3.14)**.** *The set of contravariant forms on the Whittaker module $Y(\chi, \eta)$ is a finite-dimensional vector space whose dimension is given by the cardinality of the Weyl group of $\mathfrak{g}$.*

To prove Theorem 1.1, we construct a vector space isomorphism between the space of contravariant forms on $Y(\chi, \eta)$ and the space of Weyl group coinvariants in the symmetric algebra $S(\mathfrak{h})$. By classical results in invariant theory, the space of W-coinvariants in $S(\mathfrak{h})$ is isomorphic to the regular representation of W, so this isomorphism lets us conclude that the space of contravariant forms is $|W|$-dimensional.

The reason we can construct such an isomorphism has to do with the fact that the modules $Y(\chi, \eta)$ have an infinitesimal character; that is, the center $Z(\mathfrak{g})$ acts on $Y(\chi, \eta)$ by $\chi : Z(\mathfrak{g}) \to \mathbb{C}$. Contravariant forms on cyclic $U(\mathfrak{g})$-modules are closely related to linear functionals on $U(\mathfrak{g})$ which vanish on the annihilator of a generating vector (Proposition 3.2). The annihilator in $U(\mathfrak{g})$ of the generating Whittaker vector $w \in Y(\chi, \eta)$ is generated by $\ker \eta \subset U(\mathfrak{n})$ and $\ker \chi \subset Z(\mathfrak{g})$ (Proposition 3.4). Hence to determine the dimension of the space of contravariant forms on $Y(\chi, \eta)$, it suffices to determine the dimension of a vector space complement in $U(\mathfrak{g})$ to the subspace spanned by $\mathrm{Ann}_{U(\mathfrak{g})} w$ and its image under the antiautomorphism of Definition 3.1 (Proposition 3.5, Lemma 3.9). Computing this codimension reduces to determining a complement in $U(\mathfrak{h})$ to the image of the ideal generated by $\ker \chi$ under an η-twisted version of the Harish-Chandra homomorphism (Definition 3.6). The bulk of our argument in Section 3 is dedicated to showing that this complement can be realized as the space of W-coinvariants.

As a secondary result, we establish a procedure for parabolically inducing contravariant forms from nondegenerate Whittaker modules for a reductive subalgebra $\mathfrak{l} \subset \mathfrak{g}$ to degenerate Whittaker modules for $\mathfrak{g}$.

Theorem 1.2 (Theorem 4.2)**.** *Let $\eta : \mathfrak{n} \to \mathbb{C}$ be a Lie algebra morphism, and $\mathfrak{l}_\eta = \overline{\mathfrak{n}}_\eta \oplus \mathfrak{h} \oplus \mathfrak{n}_\eta \subset \mathfrak{g}$ the corresponding reductive Lie subalgebra generated by the simple root spaces on which η does not vanish. Let $\mathfrak{p}_\eta$ be the standard parabolic subalgebra with Levi factor $\mathfrak{l}_\eta \subset \mathfrak{p}_\eta$. Let V be an irreducible finitely generated $U(\mathfrak{l}_\eta)$-module with the property that for each $v \in V$, there exists $k \in \mathbb{Z}_{>0}$ such that the $U(\mathfrak{n}_\eta)$ action satisfies*

$$(x - \eta(x))^k v = 0$$

for all $x \in \mathfrak{n}_\eta$. Then the vector space Ψ_V of contravariant forms on the $\mathfrak{l}_\eta$-module V is isomorphic to the vector space $\Psi_{\mathrm{Ind}_{\mathfrak{l}_\eta}^{\mathfrak{g}} V}$ of contravariant forms on the parabolically induced $\mathfrak{g}$-module $\mathrm{Ind}_{\mathfrak{l}_\eta}^{\mathfrak{g}} V := U(\mathfrak{g}) \otimes_{U(\mathfrak{p}_\eta)} V$.

If $\eta = 0$, we have $\mathfrak{l}_\eta = \mathfrak{h}$ and $\mathfrak{p}_\eta = \mathfrak{b}$. Then for $\lambda \in \mathfrak{h}^*$, an application of Theorem 1.2 to the one-dimensional $U(\mathfrak{h})$-module $\mathbb{C}_\lambda$ implies that the space of contravariant forms on the Verma module

$$M(\lambda) := U(\mathfrak{g}) \otimes_{U(\mathfrak{b})} \mathbb{C}_\lambda$$

is one-dimensional. In particular, this implies the existence and uniqueness (up to scaling) of the Shapovalov form on a Verma module [Sha72].

For partially degenerate η, an application of Theorem 1.2 to the $\mathfrak{l}_\eta$-module $Y(\chi, \eta)$ implies that the dimension of the space of contravariant forms on the Whittaker module $M(\chi, \eta)$ (equation (4.4)) introduced by McDowell in [McD85] is given by the cardinality of the Weyl group of $\mathfrak{l}_\eta$.

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This paper is organized in the following way. In Section 2, we establish our conventions and definitions. In Section 3, we prove Theorem 1.1. In Section 4 we prove Theorem 1.2 and two corollaries. In Section 5 we provide a detailed $\mathfrak{sl}_2(\mathbb{C})$ example to illustrate the main arguments of Section 3 more explicitly.

2. Preliminaries and notation

Let $\mathfrak{g}$ be a complex semisimple Lie algebra. Fix a Cartan subalgebra $\mathfrak{h}$ contained in a Borel subalgebra $\mathfrak{b}$. Let $\Pi \subset \Sigma^+ \subset \Sigma \subset \mathfrak{h}^*$ be the corresponding sets of simple and positive roots in the root system of $\mathfrak{g}$. Let W be the Weyl group associated to this root system, and let $\rho = \frac{1}{2}\sum_{\alpha\in\Sigma^+}\alpha$. For $\alpha \in \Sigma$, let $\mathfrak{g}_\alpha = \{x \in \mathfrak{g} \mid [h, x] = \alpha(h)x\}$ be the α-root space of $\mathfrak{g}$, and choose a Chevalley basis $\{y_\alpha, x_\alpha\}_{\alpha\in\Sigma^+} \cup \{h_\alpha\}_{\alpha\in\Pi}$ of $\mathfrak{g}$ with $x_\alpha \in \mathfrak{g}_\alpha$, $y_\alpha \in \mathfrak{g}_{-\alpha}$, and $h_\alpha \in \mathfrak{h}$ such that $[x_\alpha, y_\alpha] = h_\alpha$. Let $\mathfrak{n} = [\mathfrak{b}, \mathfrak{b}] = \bigoplus_{\alpha\in\Sigma^+}\mathfrak{g}_\alpha$ be the nilpotent radical of $\mathfrak{b}$, and $\overline{\mathfrak{n}} = \bigoplus_{\alpha\in\Sigma^+}\mathfrak{g}_{-\alpha}$.

For a Lie algebra $\mathfrak{a}$, we denote by $U(\mathfrak{a})$ the universal enveloping algebra of $\mathfrak{a}$ with center $Z(\mathfrak{a}) \subset U(\mathfrak{a})$. We call an algebra homomorphism $\chi : Z(\mathfrak{a}) \to \mathbb{C}$ an *infinitesimal character.*

We use the symbol η to refer to Lie algebra morphisms $\eta : \mathfrak{n} \to \mathbb{C}$. Such a Lie algebra morphism $\eta : \mathfrak{n} \to \mathbb{C}$ can be extended to an algebra homomorphism $\eta : U(\mathfrak{n}) \to \mathbb{C}$ which we will call by the same name. Any Lie algebra morphism $\eta : \mathfrak{n} \to \mathbb{C}$ determines a subset of simple roots

$$\Pi_\eta := \{\alpha \in \Pi \mid \eta|_{\mathfrak{g}_\alpha} \neq 0\}.$$

Definition 2.1. We say that a Lie algebra morphism $\eta : \mathfrak{n} \to \mathbb{C}$ is *nondegenerate* if $\Pi_\eta = \Pi$.

Fix a nondegenerate Lie algebra morphism $\eta : \mathfrak{n} \to \mathbb{C}$ and an infinitesimal character $\chi : Z(\mathfrak{g}) \to \mathbb{C}$. Let $\mathbb{C}_{\chi,\eta}$ be the one-dimensional $Z(\mathfrak{g}) \otimes_{\mathbb{C}} U(\mathfrak{n})$-module defined by

$$z \otimes x \cdot v = \chi(z)\eta(x)v$$

for $z \in Z(\mathfrak{g})$, $x \in U(\mathfrak{n})$, and $v \in \mathbb{C}$. Kostant introduced the following class of $U(\mathfrak{g})$-modules in [Kos78].

Definition 2.2. Let

$$Y(\chi, \eta) := U(\mathfrak{g}) \otimes_{Z(\mathfrak{g})\otimes_{\mathbb{C}}U(\mathfrak{n})} \mathbb{C}_{\chi,\eta}$$

be the $U(\mathfrak{g})$-module given by left multiplication.

The modules $Y(\chi, \eta)$ are generated by the vector $w = 1 \otimes 1$. The nilpotent radical acts on w by η; that is,

$$x \cdot w = \eta(x)w$$

for all $x \in \mathfrak{n}$.

In a $U(\mathfrak{g})$-module V, a *Whittaker vector* is a vector $v \in V$ with the property that for all $x \in \mathfrak{n}$, $x \cdot v = \eta(x)v$ for some Lie algebra morphism $\eta : \mathfrak{n} \to \mathbb{C}$. A $U(\mathfrak{g})$-module which is cyclically generated by a Whittaker vector is called a *Whittaker module*. Hence the modules $Y(\chi, \eta)$ are *nondegenerate Whittaker modules*.

Kostant showed that the modules $Y(\chi, \eta)$ are irreducible [Kos78, Thm. 3.6.1].

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3. Classification of contravariant forms on nondegenerate Whittaker modules

In this section we classify contravariant forms on the nondegenerate Whittaker modules $Y(\chi, \eta)$ introduced in Section 2. We show in Theorem 3.14 that the set of contravariant forms on $Y(\chi, \eta)$ is a finite-dimensional vector space whose dimension is given by the cardinality of the Weyl group of $\mathfrak{g}$.

Definition 3.1. Let $\tau : U(\mathfrak{g}) \to U(\mathfrak{g})$ be the antiautomorphism defined by $\tau(x_\alpha) = y_\alpha$, $\tau(y_\alpha) = x_\alpha$, and $\tau(h_\alpha) = h_\alpha$. A *contravariant form* on a $U(\mathfrak{g})$-module X is a symmetric bilinear form $\langle \cdot, \cdot \rangle : X \times X \to \mathbb{C}$ such that

$$\langle uv, w \rangle = \langle v, \tau(u)w \rangle$$

for all $u \in U(\mathfrak{g})$ and $v, w \in X$.

Contravariant forms on $U(\mathfrak{g})$-modules are closely related to τ-invariant linear functionals on $U(\mathfrak{g})$. In fact, we can reformulate the classification of contravariant forms on a cyclic $U(\mathfrak{g})$-module to the classification of τ-invariant linear functionals on $U(\mathfrak{g})$ which vanish on the annihilator of a generating vector of the module.

Proposition 3.2. *The vector space of contravariant forms on a cyclic $U(\mathfrak{g})$-module $X = U(\mathfrak{g})v$ is isomorphic to the vector space of linear functionals $\varphi : U(\mathfrak{g}) \to \mathbb{C}$ satisfying the following conditions:*

(a) $\varphi\left(\mathrm{Ann}_{U(\mathfrak{g})}\, v\right) = 0$, *and*
(b) $\varphi(u) = \varphi(\tau(u))$ *for all* $u \in U(\mathfrak{g})$.

Proof. Given a contravariant form $\langle \cdot, \cdot \rangle : X \times X \to \mathbb{C}$, define $\varphi : U(\mathfrak{g}) \to \mathbb{C}$ by

$$\varphi(u) = \langle uv, v \rangle.$$

The linear functional φ satisfies conditions (a) and (b). Conversely, given $\varphi : U(\mathfrak{g}) \to \mathbb{C}$ satisfying (a) and (b), define a bilinear form on X by

$$\langle x, y \rangle = \langle uv, u'v \rangle = \varphi(\tau(u')u) \tag{3.1}$$

for $x = uv, y = u'v \in X$. It is straightforward to check that this form is symmetric, bilinear, and contravariant. However, the choices of $u, u' \in U(\mathfrak{g})$ such that $x = uv$ and $y = u'v$ are not always unique, so it is not immediately apparent that the form is well-defined. However, if $x = tv$ and $y = t'v$, with $t \in U(\mathfrak{g})$ and $t' \in U(\mathfrak{g})$, then

$$\begin{aligned} \langle uv, u'v \rangle - \langle tv, t'v \rangle &= \langle (u-t)v, u'v \rangle + \langle tv, (u'-t')v \rangle \\ &= \varphi(\tau(u')(u-t)) + \varphi(\tau(t)(u'-t')) \\ &= 0, \end{aligned}$$

so equation (3.1) does indeed define a contravariant form. Here the second equality follows from condition (b) and the third equality follows from (a), since $u - t$ and $u' - t'$ are in the annihilator of v. □

Notation 3.3. For a $U(\mathfrak{g})$-module X, denote by Ψ_X the vector space of contravariant forms on X. If X is a cyclic $U(\mathfrak{g})$-module with generating vector v, denote by Φ_X the vector space of τ-invariant linear functionals on $U(\mathfrak{g})$ which vanish on $\mathrm{Ann}_{U(\mathfrak{g})}\, v$. By Proposition 3.2, for a cyclic $U(\mathfrak{g})$-module X, $\Psi_X \simeq \Phi_X$.

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Now we restrict our attention to the cyclic $U(\mathfrak{g})$-module $Y(\chi,\eta)$ with generating vector $w = 1 \otimes 1$ (Definition 2.2). By Proposition 3.2, to study the vector space of contravariant forms on $Y(\chi,\eta)$, we need to understand the annihilator of w. Kostant described this annihilator in [Kos78]. Recall that any Lie algebra morphism $\eta : \mathfrak{n} \to \mathbb{C}$ can be extended to an algebra homomorphism $\eta : U(\mathfrak{n}) \to \mathbb{C}$. More precisely, on a Poincaré–Birkhoff–Witt basis $\{x_{\alpha_1}^{k_1} \cdots x_{\alpha_n}^{k_n} \mid k_i \in \mathbb{Z}_{\geq 0}\}$ of $U(\mathfrak{n})$, we define

$$\eta(x_{\alpha_1}^{k_1} \cdots x_{\alpha_n}^{k_n}) := \eta(x_{\alpha_1})^{k_1} \cdots \eta(x_{\alpha_n})^{k_n} \text{ and for } c \in \mathbb{C}, \eta(c) := c.$$

Let $\ker \eta \subset U(\mathfrak{n})$ refer to the kernel of the extended map $\eta : U(\mathfrak{n}) \to \mathbb{C}$.

Proposition 3.4 ([Kos78, Thm. 3.1])**.** *Fix a nondegenerate Lie algebra morphism $\eta : \mathfrak{n} \to \mathbb{C}$ and infinitesimal character $\chi : Z(\mathfrak{g}) \to \mathbb{C}$. Let $w = 1 \otimes 1$ be the generating Whittaker vector in $Y(\chi,\eta)$. Then*

$$\operatorname{Ann}_{U(\mathfrak{g})} w = U(\mathfrak{g}) \ker \eta + U(\mathfrak{g}) \ker \chi.$$

Linear functionals in $\Phi_{Y(\chi,\eta)}$ (Notation 3.3) must vanish on $\operatorname{Ann}_{U(\mathfrak{g})} w + \tau(\operatorname{Ann}_{U(\mathfrak{g})} w)$. Accordingly, to determine the dimension of $\Phi_{Y(\chi,\eta)}$, it will be helpful to determine a vector space complement to $\operatorname{Ann}_{U(\mathfrak{g})} w + \tau(\operatorname{Ann}_{U(\mathfrak{g})} w)$ in $U(\mathfrak{g})$. The following proposition is a first step.

Proposition 3.5. *Let $\eta : \mathfrak{n} \to \mathbb{C}$ be a Lie algebra morphism. There is a direct sum decomposition*

$$U(\mathfrak{g}) = U(\mathfrak{h}) \oplus (\tau(U(\mathfrak{g}) \ker \eta) + U(\mathfrak{g}) \ker \eta). \tag{3.2}$$

Proof. Choose an order on the set of roots so that

$$\{y^{\overline{I}} h^J x^K := y_{\alpha_n}^{i_n} \cdots y_{\alpha_1}^{i_1} h_{\alpha_1}^{j_1} \cdots h_{\alpha_r}^{j_r} x_{\alpha_1}^{k_1} \cdots x_{\alpha_n}^{k_n} \mid i_l, j_l, k_l \in \mathbb{Z}_{\geq 0}\}$$

forms a Poincaré–Birkhoff–Witt basis of $U(\mathfrak{g})$. Here $I = (i_1, \ldots, i_n)$, $J = (j_1, \ldots, j_r)$ and $K = (k_1, \ldots, k_n)$ are multi-indices, $\overline{I} = (i_n, \ldots, i_1)$, and $y = (y_{\alpha_n}, \ldots, y_{\alpha_1})$, $h = (h_{\alpha_1}, \ldots, h_{\alpha_r})$, $x = (x_{\alpha_1}, \ldots, x_{\alpha_n})$. Then we can write $y^{\overline{I}} h^J x^K$ in the following way:

$$\begin{aligned}
y^{\overline{I}} h^J x^K &= y^{\overline{I}} h^J \left(x^K - \eta(x^K)\right) + \eta(x^K) y^{\overline{I}} h^J \\
&= y^{\overline{I}} h^J \left(x^K - \eta(x^K)\right) + \eta(x^K) \tau(h^J x^I) \\
&= y^{\overline{I}} h^J \left(x^K - \eta(x^K)\right) + \eta(x^K) \tau\left(h^J \left(x^I - \eta(x^I)\right) + \eta(x^I) h^J\right) \\
&= y^{\overline{I}} h^J \left(x^K - \eta(x^K)\right) + \eta(x^K) \tau\left(h^J \left(x^I - \eta(x^I)\right)\right) + \eta(x^K)\eta(x^I) h^J \\
&= \boxed{y^{\overline{I}} h^J \left(x^K - \eta(x^K)\right) + \tau\left(\eta(x^K) h^J \left(x^I - \eta(x^I)\right)\right)} + \boxed{\eta(x^K)\eta(x^I) h^J.}
\end{aligned} \tag{3.3}$$

The first box is in $U(\mathfrak{g}) \ker \eta + \tau(U(\mathfrak{g}) \ker \eta)$ and the second box is in $U(\mathfrak{h})$. By extending linearly, we can write any vector of $U(\mathfrak{g})$ as a sum of a vector in $U(\mathfrak{g}) \ker \eta + \tau(U(\mathfrak{g}) \ker \eta)$ and a vector in $U(\mathfrak{h})$. The intersection $U(\mathfrak{h}) \cap (U(\mathfrak{g}) \ker \eta + \tau(U(\mathfrak{g}) \ker \eta)) = 0$, so the sum is direct. ☐

Definition 3.6. Let

$$p_\eta : U(\mathfrak{g}) = U(\mathfrak{h}) \oplus (\tau(U(\mathfrak{g}) \ker \eta) + U(\mathfrak{g}) \ker \eta) \to U(\mathfrak{h})$$

be projection onto the first coordinate. We refer to p_η as the *η-twisted Harish-Chandra projection.*

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Remark 3.7. Let

$$U(\mathfrak{g})_0 = \{x \in U(\mathfrak{g}) \mid [h, x] = 0 \text{ for all } h \in \mathfrak{h}\}. \tag{3.4}$$

Note that if $\eta = 0$, the restriction of the projection p_0 in Definition 3.6 to $U(\mathfrak{g})_0$ is exactly the Harish-Chandra homomorphism [Bou05, Ch. VIII, §6.4]. This justifies our choice of terminology in Definition 3.6.

Example 3.8. Let $\mathfrak{g} = \mathfrak{sl}_2(\mathbb{C})$, and let

$$y = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, h = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, x = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$$

be the standard basis. The universal enveloping algebra $U(\mathfrak{g})$ has a basis consisting of monomials $\{y^i h^j x^k \mid i, j, k \in \mathbb{Z}_{\geq 0}\}$. Let $\eta : \mathfrak{n} \to \mathbb{C}$ be the Lie algebra morphism sending $x \mapsto 1$. We can express the vector $y \in U(\mathfrak{g})$ as

$$y = \tau(x) = \tau(x-1) + 1 = \tau(x - \eta(x)) + \eta(x),$$

so $p_\eta(y) = \eta(x)$.

For the remainder of this section, we fix a nondegenerate Lie algebra morphism $\eta : \mathfrak{n} \to \mathbb{C}$. Recall that our goal is to determine the dimension of the space $\Phi_{Y(\chi,\eta)}$ (Notation 3.3). Because $\tau(U(\mathfrak{g}) \ker \eta) + U(\mathfrak{g}) \ker \eta \subset \tau(\mathrm{Ann}_{U(\mathfrak{g})} w) + \mathrm{Ann}_{U(\mathfrak{g})} w$, any linear functional $\varphi \in \Phi_{Y(\chi,\eta)}$ must vanish on $\tau(U(\mathfrak{g}) \ker \eta) + U(\mathfrak{g}) \ker \eta$. Hence for any $\varphi \in \Phi_{Y(\chi,\eta)}$ and $u \in U(\mathfrak{g})$,

$$\varphi(u) = \varphi(p_\eta(u)). \tag{3.5}$$

Moreover, by Kostant's description of the annihilator in Proposition 3.4, φ must also vanish on $U(\mathfrak{g}) \ker \chi$, so for $v \in U(\mathfrak{g}) \ker \chi$,

$$\varphi(v) = \varphi(p_\eta(v)) = 0 \tag{3.6}$$

as well. Conversely, the following lemma shows that any linear functional in $U(\mathfrak{g})^*$ satisfying (3.5) and (3.6) is in $\Phi_{Y(\chi,\eta)}$.

Lemma 3.9. *Let $Q = U(\mathfrak{h})/p_\eta(U(\mathfrak{g}) \ker \chi)$. As vector spaces,*

$$\Phi_{Y(\chi,\eta)} \cong Q^*.$$

Proof. First we note that Q^* is canonically isomorphic to the space of linear functionals on $U(\mathfrak{h})$ which vanish on $p_\eta(U(\mathfrak{g}) \ker \chi)$. We will show that restriction of linear functionals from $U(\mathfrak{g})$ to $U(\mathfrak{h})$ defines an isomorphism of $\Phi_{Y(\chi,\eta)}$ with Q^*:

$$\begin{aligned} \mathrm{res}_{U(\mathfrak{h})} : \Phi_{Y(\chi,\eta)} &\xrightarrow{\sim} Q^* \\ \varphi &\mapsto \varphi|_{U(\mathfrak{h})}. \end{aligned}$$

For any $\varphi \in \Phi_{Y(\chi,\eta)}$, $\varphi|_{U(\mathfrak{h})}$ vanishes on $p_\eta(U(\mathfrak{g}) \ker \chi)$ by (3.6), so $\mathrm{res}_{U(\mathfrak{h})}$ is well-defined. The inverse of the restriction map is given by

$$\begin{aligned} Q^* &\to \Phi_{Y(\chi,\eta)} \\ \phi &\mapsto \phi \circ p_\eta. \end{aligned}$$

To see that this inverse map is well-defined, we must show that $\phi \circ p_\eta$ vanishes on the annihilator $\mathrm{Ann}_{U(\mathfrak{g})} w$ and is τ-invariant. We can write $a \in \mathrm{Ann}_{U(\mathfrak{g})} w$ as $a = n + u$ with $n \in U(\mathfrak{g}) \ker \eta$ and $u \in U(\mathfrak{g}) \ker \chi$ by Proposition 3.4, and

$$\phi \circ p_\eta(a) = \phi(p_\eta(n)) + \phi(p_\eta(u)) = 0$$

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because ϕ is assumed to vanish on $p_\eta(U(\mathfrak{g})\ker\chi)$ and $p_\eta(n) = 0$. To see that $\phi \circ p_\eta$ is τ-invariant, we write any $u \in U(\mathfrak{g})$ as $u = h + m$, where $h \in U(\mathfrak{h})$ and $m \in \tau(U(\mathfrak{g})\ker\eta) + U(\mathfrak{g})\ker\eta$ using (3.2). Then,

$$\phi\circ p_\eta(u) = \phi\circ p_\eta(h+m) = \phi\circ p_\eta(h) = \phi\circ p_\eta(\tau(h)) = \phi\circ p_\eta(\tau(m)+\tau(h)) = \phi\circ p_\eta(\tau(u))$$

since $p_\eta(\tau(m)) = 0$.

The function $\varphi \mapsto \varphi|_{U(\mathfrak{h})}$ for $\varphi \in \Phi_{Y(\chi,\eta)}$ and the function $\phi \mapsto \phi \circ p_\eta$ for $\phi \in Q^*$ are inverse functions by (3.5). □

Lemma 3.9 reduces the study of $\Phi_{Y(\chi,\eta)}$ to the study of the space of linear functionals on $U(\mathfrak{h})$ which vanish on $p_\eta(U(\mathfrak{g})\ker\chi)$. To determine the dimension of this space, we will identify a vector space complement to $p_\eta(U(\mathfrak{g})\ker\chi)$ in $U(\mathfrak{h})$ which is isomorphic to the regular representation of W, and hence must be $|W|$-dimensional. Before making this identification, we need to establish two technical lemmas.

Let $U(\mathfrak{g})_0$ be as in (3.4). The following lemma will be needed in the proof of Lemma 3.12.

Lemma 3.10. $U(\mathfrak{g})_0 \cap (\tau(U(\mathfrak{g})\ker\eta) + U(\mathfrak{g})\ker\eta) \subset U(\mathfrak{g})\ker\eta$.

Proof. We return to the multi-index notation from the proof of Proposition 3.5. The vector space $U(\mathfrak{g})_0$ is spanned by monomials of the form

$$y^{\overline{I}}h^J x^I := y_{\alpha_n}^{i_n}\cdots y_{\alpha_1}^{i_1} h_{\alpha_1}^{j_1}\cdots h_{\alpha_r}^{j_r} x_{\alpha_1}^{i_1}\cdots x_{\alpha_n}^{i_n}.$$

By (3.3), for such a monomial,

$$p_\eta(y^{\overline{I}}h^J x^I) = \eta(x^I)\eta(x^I)h^J.$$

Hence a monomial $y^{\overline{I}}h^J x^I \in U(\mathfrak{g})_0$ is in $(\tau(U(\mathfrak{g})\ker\eta) + U(\mathfrak{g})\ker\eta)$ exactly when $x^I \in \ker\eta$. We conclude that if $y^{\overline{I}}h^J x^I \in U(\mathfrak{g})_0 \cap (\tau(U(\mathfrak{g})\ker\eta) + U(\mathfrak{g})\ker\eta)$, then $y^{\overline{I}}h^J x^I \in U(\mathfrak{g})\ker\eta$. □

In particular, Lemma 3.10 implies that

$$Z(\mathfrak{g}) \subset U(\mathfrak{g})_0 \subset U(\mathfrak{h}) \oplus U(\mathfrak{g})\ker\eta. \tag{3.7}$$

By (3.7), any element $z \in Z(\mathfrak{g})$ can be expressed as a sum $z = p_\eta(z) + n$ for $n \in U(\mathfrak{g})\ker\eta$, and for any $h \in U(\mathfrak{h})$,

$$p_\eta(hz) = p_\eta(hp_\eta(z) + hn) = hp_\eta(z). \tag{3.8}$$

In what follows, we identify $U(\mathfrak{h})$ with $S(\mathfrak{h})$, and consider it as a representation of W in the natural way. Let

$$S = \langle S(\mathfrak{h})_+^W \rangle \tag{3.9}$$

be the ideal in $S(\mathfrak{h})$ generated by the W-invariant homogeneous polynomials with positive degree. Clearly S is W-stable, and by [Bou02, Ch.V, §5.2, Thm. 2(i)], it admits a W-stable complement $C \subset S(\mathfrak{h})$ such that

$$S(\mathfrak{h}) = C \oplus S \tag{3.10}$$

is a graded, W-stable decomposition. Moreover, for any such W-stable complement $C \subset S(\mathfrak{h})$, the representation of W in C is isomorphic to the regular representation $\mathbb{C}[W]$ of W [Bou02, Ch.V, §5.2, Theorem 2(ii)]. In particular, $\dim C = |W|$. Fix such a W-stable complement C. We refer to C as the space of W-*coinvariants*.

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Remark 3.11. There are many possible W-stable complements to S in $S(\mathfrak{h})$. For example, one such complement is the space of W-harmonic polynomials[1] which arise as solutions to a certain system of partial differential equations determined by a generating set of $S(\mathfrak{h})^W$. For more details on this perspective, see [Ber09, §8.2]. For our purposes, it does not matter which complement we choose, as we are only interested in its dimension.

The following technical lemma will be needed for induction arguments in the proof of Lemma 3.13.

Lemma 3.12. *Let S be as in (3.9), and let p_η be the η-twisted Harish-Chandra projection (Definition 3.6).*

(a) *If $s \in S$, then there exist an element $r \in p_\eta(U(\mathfrak{g})\ker\chi)$ and an element $e \in U(\mathfrak{h})$ such that*

$$s = r + e \text{ and } \deg(e) < \deg(s).$$

(b) *Additionally, if $r' \in p_\eta(U(\mathfrak{g})\ker\chi)$, then there exist an element $s' \in S$ and an element $f \in U(\mathfrak{h})$ such that*

$$r' = s' + f \text{ and } \deg(f) < \deg(r').$$

Proof. Let t_ρ be the algebra automorphism of $U(\mathfrak{h})$ induced by the ρ-twisting map $h \mapsto h - \rho(h)$ for $h \in \mathfrak{h}$. The composition of the Harish-Chandra homomorphism p_0 (Definition 3.6, Remark 3.7) with t_ρ provides an algebra isomorphism

$$t_\rho \circ p_0 : Z(\mathfrak{g}) \xrightarrow{\sim} S(\mathfrak{h})^W$$

[Bou05, Ch. VIII, §8.5, Thm. 2].

The ideal S is generated by $S(\mathfrak{h})^W_+$, so any element of S can be expressed as a sum of elements of the form

$$ht_\rho(p_0(z))$$

for various $z \in Z(\mathfrak{g})$ and $h \in S(\mathfrak{h}) = U(\mathfrak{h})$. Our first step in the proof is to show that any element of the form $ht_\rho(p_0(z))$ satisfies (a).

Because $Z(\mathfrak{g}) \subset U(\mathfrak{g})_0$, any $z \in Z(\mathfrak{g})$ is a linear combination of Poincaré–Birkhoff–Witt basis elements of the form

$$y^{\overline{I}}h^J x^I := y^{i_n}_{\alpha_n} \cdots y^{i_1}_{\alpha_1} h^{j_1}_{\alpha_1} \cdots h^{j_r}_{\alpha_r} x^{i_1}_{\alpha_1} \cdots x^{i_n}_{\alpha_n}$$

(see proof of Lemma 3.10). Hence we can express z as a sum

$$z = p_0(z) + \sum_{I,J} a_{I,J} y^{\overline{I}} h^J x^I, \tag{3.11}$$

where $a_{I,J} \in \mathbb{C}$ and $a_{I,J} = 0$ if $I = (0, \dots, 0)$. Because the composition $t_\rho \circ p_0 : Z(\mathfrak{g}) \to S(\mathfrak{h})^W$ induces an isomorphism between the corresponding graded objects (with the grading of $Z(\mathfrak{g})$ induced by the natural filtration of $U(\mathfrak{g})$ by $\mathfrak{g}$) [Bou05, Ch.VIII §8.5 proof of Thm. 2], we have

$$\deg(h^J) < \deg(p_0(z)) \text{ for all } J \text{ such that } a_{I,J} \neq 0. \tag{3.12}$$

By applying p_η to (3.11) and using equation (3.3), we obtain

$$p_\eta(z) = p_0(z) + \sum_{I,J} a_{I,J} \eta(\tau(y^{\overline{I}})x^I) h^J. \tag{3.13}$$

[1] Under a suitable choice of orthonormal basis of $\mathfrak{h}$, these W-harmonic polynomials are solutions to Laplace's equation, so they are harmonic in the usual sense of the word.

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Hence the image of $z \in Z(\mathfrak{g})$ under the η-twisted Harish-Chandra projection p_η and the image of z under the Harish-Chandra homomorphism p_0 agree up to lower degree terms. To increase readability in the arguments below, we will introduce some notation to describe this phenomenon in general. Write LDP for an element in $U(\mathfrak{h})$ with degree strictly lower than the element immediately preceding it in an expression.[2] For example, by (3.12), we can rewrite (3.13) as

$$p_\eta(z) = p_0(z) + LDP.$$

Similarly, for all $h \in U(\mathfrak{h})$, $t_\rho(h) = h + LDP$. Therefore, we have

$$ht_\rho(p_0(z)) = ht_\rho(p_\eta(z) + LDP) = hp_\eta(z) + LDP. \tag{3.14}$$

By (3.8) and the linearity of p_η, we have

$$hp_\eta(z) = p_\eta(hz) = p_\eta(h(z - \chi(z))) + h\chi(z). \tag{3.15}$$

Combining (3.14) and (3.15), we obtain

$$ht_\rho(p_0(z)) = p_\eta(h(z - \chi(z))) + LDP. \tag{3.16}$$

We conclude that any element of S which is equal to $ht_\rho(p_0(z))$ for some $h \in U(\mathfrak{h})$ and $z \in Z(\mathfrak{g})$ satisfies (a).

An arbitrary element $s \in S$ is a sum of elements of the form $ht_\rho(p_0(z))$ for various $h \in U(\mathfrak{h})$ and $z \in Z(\mathfrak{g})$, so by the linearity of p_η, there exists $k \in U(\mathfrak{g}) \ker \chi$ such that

$$s = p_\eta(k) + LDP.$$

This proves (a).

Part (b) follows from an analogous argument. Any $r' \in p_\eta(U(\mathfrak{g}) \ker \chi)$ is equal to a sum of elements of the form

$$p_\eta(u(z - \chi(z)))$$

for various $u \in U(\mathfrak{g})$ and $z \in Z(\mathfrak{g})$. We claim that

$$p_\eta(u(z - \chi(z))) = p_\eta(p_\eta(u)(z - \chi(z))).$$

Indeed, using equality (3.2) to express u as a sum $u = p_\eta(u) + m + n$ for $m \in \tau(U(\mathfrak{g}) \ker \eta)$ and $n \in U(\mathfrak{g}) \ker \eta$, we have

$$\begin{aligned} p_\eta(u(z - \chi(z))) &= p_\eta((p_\eta(u) + m + n)(z - \chi(z))) \\ &= p_\eta(p_\eta(u)(z - \chi(z)) + m(z - \chi(z)) + (z - \chi(z))n) \\ &= p_\eta(p_\eta(u)(z - \chi(z))). \end{aligned}$$

By (3.16), we conclude that

$$p_\eta(u(z - \chi(z))) = p_\eta(u)t_\rho(p_0(z)) + LDP.$$

For each $z \in Z(\mathfrak{g})$, $t_\rho(p_0(z)) \in S(\mathfrak{h})^W$, and therefore is equal to the sum of an element of S and a constant polynomial. Hence for each $z \in Z(\mathfrak{g})$ and $u \in U(\mathfrak{g})$, there exists $s \in S$ such that

$$p_\eta(u(z - \chi(z))) = s + LDP.$$

Because each $r' \in p_\eta(U(\mathfrak{g}) \ker \chi)$ is equal to a sum of elements of the form $p_\eta(u(z - \chi(z)))$, there exists $s' \in S$ such that $r' = s' + LDP$, which proves (b). □

[2] Here LDP stands for "lower degree polynomial".

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Our final step in establishing the dimension of the space $\Phi_{Y(\chi,\eta)}$ is to show that the $|W|$-dimensional vector space C, which is defined by the decomposition (3.10) to be a vector space complement to S in $U(\mathfrak{h})$, also forms a vector space complement to $p_\eta(U(\mathfrak{g})\ker\chi)$ in $U(\mathfrak{h})$. Because we can realize linear functionals in $\Phi_{Y(\chi,\eta)}$ as linear functionals on $U(\mathfrak{h})$ which vanish on $p_\eta(U(\mathfrak{g})\ker\chi)$ by Lemma 3.9, the dimension of a complement of $p_\eta(U(\mathfrak{g})\ker\chi)$ in $U(\mathfrak{h})$ determines the dimension of $\Phi_{Y(\chi,\eta)}$.

Lemma 3.13. *Let C be the W-stable complement to $S = \langle S(\mathfrak{h})_+^W\rangle$ in equation (3.10). As vector spaces,*

$$U(\mathfrak{h}) = C \oplus p_\eta(U(\mathfrak{g})\ker\chi).$$

Proof. We begin with the graded decomposition

$$U(\mathfrak{h}) = C \oplus S$$

and proceed by induction on degree. The base case is trivial, as $p_\eta(U(\mathfrak{g})\ker\chi)$ contains no nonzero constant polynomials. Let $U(\mathfrak{h})_i$ denote the set of polynomials with degree less than or equal to i. Assume $U(\mathfrak{h})_j = C_j \oplus p_\eta(U(\mathfrak{g})\ker\chi)_j$ for all $j \le i$. Let $h \in U(\mathfrak{h})_{i+1}$. Then

$$h = c + s,$$

where $c \in C_{i+1}$ and $s \in S_{i+1}$. By Lemma 3.12, we can write s as

$$s = r + e$$

with $r \in p_\eta(U(\mathfrak{g})\ker\chi)$ and $\deg(e) < \deg(s) \le i+1$. Therefore

$$h = c + r + e.$$

By the induction assumption, e can be written uniquely as $e = c' + r'$, with $c' \in C_j$ and $r' \in p_\eta(U(\mathfrak{g})\ker\chi)_j$ for some $j \le i$. So we have a decomposition of h given by

$$h = (c + c') + (r + r').$$

Moreover, since $\deg(c'), \deg(r') < i+1$, we have that $(c+c') \in C_{i+1}$ and $(r+r') \in p_\eta(U(\mathfrak{g})\ker\chi)_{i+1}$.

To complete the proof, it remains to show that $C_{i+1} \cap p_\eta(U(\mathfrak{g})\ker\chi)_{i+1} = 0$. Assume $x \in C_{i+1} \cap p_\eta(U(\mathfrak{g})\ker\chi)_{i+1}$. By Lemma 3.12, $x = s + f$, where $f \in U(\mathfrak{h})$ has lower degree than x and $s \in S$. We can decompose f as $f = c + s'$ with $c \in C_i$ and $s \in S_i$. So $x = s + c + s'$. Therefore $s + s' \in S \cap C$, which implies that $s + s' = 0$ and $x = c$. But $\deg c \le i$, so $x \in C_i \cap p_\eta(U(\mathfrak{g})\ker\chi)_i$. Hence x must be 0 by the induction hypothesis. □

Lemma 3.13 implies our main result.

Theorem 3.14. *Let $\Psi_{Y(\chi,\eta)}$ be the vector space of contravariant forms on the nondegenerate Whittaker module $Y(\chi,\eta)$. Then*

$$\dim \Psi_{Y(\chi,\eta)} = |W|.$$

Proof. The vector space C has dimension $\dim C = |W|$ because C is isomorphic to $\mathbb{C}[W]$ [Bou02, Ch. V, §5.2, Theorem 2(ii)]. Both Q^* and C^* are isomorphic to the space of linear functionals on $U(\mathfrak{h})$ which vanish on $p_\eta(U(\mathfrak{g})\ker\chi)$. Therefore $Q^* \cong C^*$. By Lemma 3.9, and Proposition 3.2 we conclude that

$$\Psi_{Y(\chi,\eta)} \cong C^*.$$

Hence $\dim \Psi_{Y(\chi,\eta)} = |W|$. □

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4. Induction of contravariant forms

In this section, we prove that the induction functor from the category of nondegenerate Whittaker modules for a Levi factor $\mathfrak{l} \subset \mathfrak{g}$ to degenerate Whittaker modules for $\mathfrak{g}$ induces an isomorphism of the corresponding spaces of contravariant forms. In particular, this implies the existence of the Shapovalov form on a Verma module as a corollary to Theorem 3.14. It also gives a formula for the dimension of the space of contravariant forms on the standard degenerate Whittaker modules introduced by McDowell in [McD85].

For a reductive Lie algebra $\mathfrak{a}$ with a fixed Cartan subalgebra $\mathfrak{c} \subset \mathfrak{a}$ and triangular decomposition $\mathfrak{a} = \overline{\mathfrak{m}} \oplus \mathfrak{c} \oplus \mathfrak{m}$, let $\mathcal{N}(\mathfrak{a})$ be the category of finitely generated $U(\mathfrak{a})$-modules which are locally $Z(\mathfrak{a})$-finite and locally $U(\mathfrak{m})$-finite. This category was introduced by Miličić–Soergel in [MS97] as a natural category containing both Kostant's nondegenerate Whittaker modules $Y(\chi, \eta)$ from Section 3 and all modules in Bernstein–Gelfand–Gelfand's category $\mathcal{O}$ [BGG76]. For a fixed Lie algebra morphism $\eta : \mathfrak{m} \to \mathbb{C}$, the collection of modules $V \in \mathcal{N}(\mathfrak{a})$ with the property that for each $v \in V$, there exists $k \in \mathbb{Z}_{>0}$ such that for all $x \in \mathfrak{m}$, $(x - \eta(x))^k v = 0$, forms a full subcategory $\mathcal{N}(\mathfrak{a})_\eta \subset \mathcal{N}(\mathfrak{a})$. For η nondegenerate (Definition 2.1), the category $\mathcal{N}(\mathfrak{a})_\eta$ contains the $\mathfrak{a}$-modules $Y(\chi, \eta)$ for all infinitesimal characters $\chi : Z(\mathfrak{a}) \to \mathbb{C}$. The categories $\mathcal{N}(\mathfrak{a})_\eta$ for η nondegenerate are very simple: each such $\mathcal{N}(\mathfrak{a})_\eta$ is equivalent to the category of finite-dimensional $Z(\mathfrak{a})$-modules,[3] and the $Y(\chi, \eta)$ exhaust the irreducible objects in $\mathcal{N}(\mathfrak{a})_\eta$.

Now let $\mathfrak{g} = \overline{\mathfrak{n}} \oplus \mathfrak{h} \oplus \mathfrak{n}$ be a semisimple Lie algebra and return to the setting of Section 2. Fix a Lie algebra morphism $\eta : \mathfrak{n} \to \mathbb{C}$, and as in Section 2, let Π_η be the set of simple roots with the property that $\eta \neq 0$ on the corresponding root subspace of $\mathfrak{g}$. Let $\Sigma_\eta \subset \mathfrak{h}^*$ be the root system generated by Π_η and $W_\eta \subset W$ the corresponding Weyl group. The morphism η determines several Lie subalgebras of $\mathfrak{g}$. In particular, we name

$$\mathfrak{l}_\eta = \mathfrak{h} \oplus \bigoplus_{\alpha \in \Sigma_\eta} \mathfrak{g}_\alpha, \mathfrak{n}_\eta = \bigoplus_{\alpha \in \Sigma_\eta^+} \mathfrak{g}_\alpha, \mathfrak{n}^\eta = \bigoplus_{\alpha \in \Sigma^+ - \Sigma_\eta^+} \mathfrak{g}_\alpha, \mathfrak{p}_\eta = \mathfrak{l}_\eta \oplus \mathfrak{n}^\eta,$$

and define $\overline{\mathfrak{n}}_\eta$ and $\overline{\mathfrak{n}}^\eta$ in the obvious way. Then $\mathfrak{l}_\eta$ is a reductive Lie subalgebra of $\mathfrak{g}$, and $\eta|_{\mathfrak{n}_\eta}$ is nondegenerate.

There is an induction functor from the category of nondegenerate Whittaker modules for the Levi factor $\mathfrak{l}_\eta$ to the category of degenerate Whittaker modules for all of $\mathfrak{g}$.

Definition 4.1. Define the *induction functor*

$$\mathrm{Ind}_{\mathfrak{l}_\eta}^{\mathfrak{g}} : \mathcal{N}(\mathfrak{l}_\eta)_\eta \to \mathcal{N}(\mathfrak{g})_\eta$$

by $\mathrm{Ind}_{\mathfrak{l}_\eta}^{\mathfrak{g}}(V) = U(\mathfrak{g}) \otimes_{U(\mathfrak{p}_\eta)} V$ for a module $V \in \mathcal{N}(\mathfrak{l}_\eta)_\eta$.

When applied to irreducible modules, the functor $\mathrm{Ind}_{\mathfrak{l}_\eta}^{\mathfrak{g}}$ induces an isomorphism on the space of contravariant forms.

Theorem 4.2. *Fix a Lie algebra morphism $\eta : \mathfrak{n} \to \mathbb{C}$. For any irreducible module $V \in \mathcal{N}(\mathfrak{l}_\eta)_\eta$, there is a vector space isomorphism*

$$\Psi_V \simeq \Psi_{\mathrm{Ind}_{\mathfrak{l}_\eta}^{\mathfrak{g}} V}$$

[3] This fact is originally due to Kostant [Kos78], a proof in language more closely aligned with this paper can be found in [MS14, Thm. 5.9].

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between the vector space Ψ_V of contravariant forms on the $\mathfrak{l}_\eta$-module V and the vector space $\Psi_{\mathrm{Ind}_{\mathfrak{l}_\eta}^{\mathfrak{g}} V}$ of contravariant forms on the $\mathfrak{g}$-module $\mathrm{Ind}_{\mathfrak{l}_\eta}^{\mathfrak{g}} V$.

Proof. All irreducible modules in $\mathcal{N}(\mathfrak{l}_\eta)_\eta$ are of the form $Y(\chi,\eta)$ for some infinitesimal character $\chi : Z(\mathfrak{l}_\eta) \to \mathbb{C}$. Fix an $\mathfrak{l}_\eta$-module $Y(\chi,\eta)$. Because $Y(\chi,\eta)$ is cyclically generated by the vector $w = 1 \otimes 1$, Proposition 3.2 implies that

$$\Psi_{Y(\chi,\eta)} \simeq \Phi_{Y(\chi,\eta)},$$

where $\Phi_{Y(\chi,\eta)}$ is the vector space of τ-invariant linear functionals on $U(\mathfrak{l}_\eta)$ vanishing on $\mathrm{Ann}_{U(\mathfrak{l}_\eta)} w$, as in Notation 3.3. The $\mathfrak{g}$-module $\mathrm{Ind}_{\mathfrak{l}_\eta}^{\mathfrak{g}} Y(\chi,\eta)$ is cyclically generated by the vector $\overline{w} = 1 \otimes w$, so again by Proposition 3.2, the vector space of contravariant forms on $\mathrm{Ind}_{\mathfrak{l}_\eta}^{\mathfrak{g}} Y(\chi,\eta)$ is isomorphic to the vector space of linear functionals on $U(\mathfrak{g})$ vanishing on $\mathrm{Ann}_{U(\mathfrak{g})} \overline{w}$:

$$\Psi_{\mathrm{Ind}_{\mathfrak{l}_\eta}^{\mathfrak{g}} Y(\chi,\eta)} \simeq \Phi_{\mathrm{Ind}_{\mathfrak{l}_\eta}^{\mathfrak{g}} Y(\chi,\eta)}.$$

Hence the result follows from the following proposition.

Proposition 4.3. *The restriction map $\varphi \mapsto \varphi|_{U(\mathfrak{l}_\eta)}$ induces an isomorphism*

$$\mathrm{res}_{U(\mathfrak{l}_\eta)} : \Phi_{\mathrm{Ind}_{\mathfrak{l}_\eta}^{\mathfrak{g}} Y(\chi,\eta)} \xrightarrow{\sim} \Phi_{Y(\chi,\eta)}.$$

Proof. Let $\varphi \in \Phi_{\mathrm{Ind}_{\mathfrak{l}_\eta}^{\mathfrak{g}} Y(\chi,\eta)}$. The $U(\mathfrak{l}_\eta)$-module $Y(\chi,\eta)$ is naturally embedded in the induced module $\mathrm{Ind}_{\mathfrak{l}_\eta}^{\mathfrak{g}} Y(\chi,\eta)$, and this embedding maps $w \in Y(\chi,\eta)$ to $\overline{w} \in \mathrm{Ind}_{\mathfrak{l}_\eta}^{\mathfrak{g}} Y(\chi,\eta)$. Hence $\mathrm{Ann}_{U(\mathfrak{l}_\eta)} w \subset \mathrm{Ann}_{U(\mathfrak{g})} \overline{w}$.

Because φ vanishes on $\mathrm{Ann}_{U(\mathfrak{g})} \overline{w}$, we have that $\varphi\left(\mathrm{Ann}_{U(\mathfrak{l}_\eta)} w\right) = 0$. Moreover, by the τ-invariance of φ, we have

$$\varphi(u) = \varphi(\tau(u)) \text{ for all } u \in U(\mathfrak{l}_\eta) \subset U(\mathfrak{g}).$$

Therefore, the image of $\mathrm{res}_{U(\mathfrak{l}_\eta)}$ is contained in $\Phi_{Y(\chi,\eta)}$ as claimed. We complete the proof by constructing an inverse to $\mathrm{res}_{U(\mathfrak{l}_\eta)}$.

Using the Poincaré–Birkhoff–Witt theorem, we can decompose $U(\mathfrak{g})$ as

$$U(\mathfrak{g}) = U(\mathfrak{l}_\eta) \oplus (\overline{\mathfrak{n}}^\eta U(\mathfrak{g}) + U(\mathfrak{g})\mathfrak{n}^\eta). \tag{4.1}$$

Let π_η denote the corresponding projection map from $U(\mathfrak{g})$ to $U(\mathfrak{l}_\eta)$. Note that

$$\pi_\eta(\tau(u)) = \tau(\pi_\eta(u)) \tag{4.2}$$

for any $u \in U(\mathfrak{g})$.

The remainder of the proof will be devoted to showing that the map

$$\begin{aligned} \mathrm{ext}_{\pi_\eta} : \Phi_{Y(\chi,\eta)} &\to \Phi_{\mathrm{Ind}_{\mathfrak{l}_\eta}^{\mathfrak{g}} Y(\chi,\eta)} \\ \phi &\mapsto \phi \circ \pi_\eta \end{aligned}$$

is well-defined and is the inverse of $\mathrm{res}_{U(\mathfrak{l}_\eta)}$. The τ-invariance of $\phi \circ \pi_\eta$ follows immediately from the τ-invariance of ϕ and equation (4.2). To show that $\phi \circ \pi_\eta(\mathrm{Ann}_{U(\mathfrak{g})} \overline{w}) = 0$, and therefore $\mathrm{ext}_{\pi_\eta}\phi \in \Phi_{\mathrm{Ind}_{\mathfrak{l}_\eta}^{\mathfrak{g}} Y(\chi,\eta)}$, we will establish the following equality:

$$U(\mathfrak{l}_\eta)\overline{w} \cap \overline{\mathfrak{n}}^\eta U(\mathfrak{g})\overline{w} = 0. \tag{4.3}$$

To prove equation (4.3), we recall some facts about the structure of the modules $\mathrm{Ind}_{\mathfrak{l}_\eta}^{\mathfrak{g}} Y(\chi,\eta)$ which were established in [McD85]. Let $\mathfrak{z} = \mathrm{rad}\,\mathfrak{l}_\eta$, and let $\omega \in \mathfrak{z}^*$ be the restriction of $\chi : Z(\mathfrak{l}_\eta) \to \mathbb{C}$ to $\mathfrak{z} \subset Z(\mathfrak{l}_\eta)$. By [McD85, Prop. 2.4], $\mathfrak{z}$ acts

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semisimply on $\mathrm{Ind}_{\mathfrak{l}_\eta}^{\mathfrak{g}} Y(\chi,\eta)$, and there is a partial order on the $\mathfrak{z}$-weights which index the irreducible factors in this decomposition. The unique maximal nonzero $\mathfrak{z}$-weight space of $\mathrm{Ind}_{\mathfrak{l}_\eta}^{\mathfrak{g}} Y(\chi,\eta)$ with respect to this partial order has weight ω, and is equal to the $U(\mathfrak{l}_\eta)$-span of $\overline{w}$:

$$\left(\mathrm{Ind}_{\mathfrak{l}_\eta}^{\mathfrak{g}} Y(\chi,\eta)\right)_\omega = U(\mathfrak{l}_\eta)\overline{w}.$$

Because $\overline{w}$ generates $\mathrm{Ind}_{\mathfrak{l}_\eta}^{\mathfrak{g}} Y(\chi,\eta)$ as a $U(\mathfrak{g})$-module,

$$\bar{\mathfrak{n}}^\eta U(\mathfrak{g})\overline{w} = \bar{\mathfrak{n}}^\eta \left(\mathrm{Ind}_{\mathfrak{l}_\eta}^{\mathfrak{g}} Y(\chi,\eta)\right).$$

By [McD85, Prop. 1.8(c)],

$$\bar{\mathfrak{n}}^\eta \left(\mathrm{Ind}_{\mathfrak{l}_\eta}^{\mathfrak{g}} Y(\chi,\eta)\right) \subset \bigoplus_{\mu<\omega} \left(\mathrm{Ind}_{\mathfrak{l}_\eta}^{\mathfrak{g}} Y(\chi,\eta)\right)_\mu.$$

Therefore, $(\bar{\mathfrak{n}}^\eta U(\mathfrak{g})\overline{w})_\omega = 0$, and $U(\mathfrak{l}_\eta)\overline{w} \cap (\bar{\mathfrak{n}}^\eta U(\mathfrak{g}))\overline{w} = 0$. This establishes equation (4.3), which we now use to show that $\phi \circ \pi_\eta(\mathrm{Ann}_{U(\mathfrak{g})}\,\overline{w}) = 0$.

Let $a \in \mathrm{Ann}_{U(\mathfrak{g})}\,\overline{w}$ and use (4.1) to write $a = \pi_\eta(a) + n + m$ for $n \in \bar{\mathfrak{n}}^\eta U(\mathfrak{g})$, $m \in U(\mathfrak{g})\mathfrak{n}^\eta$. Then

$$a\overline{w} = (\pi_\eta(a) + n + m)\overline{w} = \pi_\eta(a)\overline{w} + n\overline{w} = 0,$$

because $m \in \mathrm{Ann}_{U(\mathfrak{g})}\,\overline{w}$. Hence by (4.3)

$$\pi_\eta(a)\overline{w} = -n\overline{w} = 0,$$

and $\pi_\eta(a) \in \mathrm{Ann}_{U(\mathfrak{l}_\eta)}\,\overline{w} = \mathrm{Ann}_{U(\mathfrak{l}_\eta)}\,w$. It follows that $\phi \circ \pi_\eta(\mathrm{Ann}_{U(\mathfrak{g})}\,\overline{w}) = 0$, and therefore ext_{π_η} is well-defined. It is then clear that ext_{π_η} and $\mathrm{res}_{U(\mathfrak{l}_\eta)}$ are inverse maps, which concludes the proof. □

Theorem 4.2 now follows immediately from Proposition 4.3. □

Corollary 4.4. *Let $\lambda \in \mathfrak{h}^*$, and let $M(\lambda) = U(\mathfrak{g}) \otimes_{U(\mathfrak{b})} \mathbb{C}_\lambda$ be the corresponding Verma module. There exists a unique contravariant form on $M(\lambda)$ up to scaling.*

Proof. Let $\eta : \mathfrak{n} \to \mathbb{C}$ be the Lie algebra morphism sending $x \mapsto 0$ for all $x \in \mathfrak{n}$. Then $\mathfrak{l}_\eta = \mathfrak{h}$, and $\mathcal{N}(\mathfrak{h})_\eta$ is the category of finite-dimensional $U(\mathfrak{h})$-modules. Applying Theorem 4.2 to the one-dimensional $U(\mathfrak{h})$-module $\mathbb{C}_\lambda$ proves the corollary. □

Remark 4.5. Shapovalov defined a contravariant form on a Verma module $M(\lambda)$ by the recipe

$$\langle u \cdot 1 \otimes 1, v \cdot 1 \otimes 1\rangle := \lambda(p_0(\tau(v)u)),$$

where $u, v \in U(\mathfrak{g})$, $1 \otimes 1 \in M(\lambda)$ is the generating highest weight vector, and $p_0 : U(\mathfrak{g}) = U(\mathfrak{h}) \oplus (\overline{\mathfrak{n}}U(\mathfrak{g}) + U(\mathfrak{g})\mathfrak{n}) \to U(\mathfrak{h})$ is the Harish-Chandra projection (Definition 3.6, Remark 3.7). Shapovalov showed that this is the unique contravariant form on $M(\lambda)$ with the property that $\langle 1 \otimes 1, 1 \otimes 1\rangle = 1$ [Sha72]. Corollary 4.4 implies the existence of the Shapovalov form.

In [McD85], McDowell defines a class of $U(\mathfrak{g})$-modules which include both the Verma modules $M(\lambda)$ and the Whittaker modules $Y(\chi,\eta)$ from Section 3. These modules are constructed by applying the parabolic induction functor of Definition 4.1 to irreducible modules in $\mathcal{N}(\mathfrak{l}_\eta)$ as η varies. Specifically, for a Lie algebra

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morphism $\eta : \mathfrak{n} \to \mathbb{C}$ and an infinitesimal character $\chi : Z(\mathfrak{l}_\eta) \to \mathbb{C}$, McDowell defines a $U(\mathfrak{g})$-module

$$M(\chi, \eta) := \mathrm{Ind}_{\mathfrak{l}_\eta}^{\mathfrak{g}} Y(\chi, \eta). \tag{4.4}$$

Remark 4.6. There is an alternate definition of $M(\chi, \eta)$ in terms of generators and relations. The $U(\mathfrak{g})$-module $M(\chi, \eta)$ is the $U(\mathfrak{g})$-module generated by w subject to the relations

(1) $x \cdot w = \eta(x)w$ for all $x \in \mathfrak{n}$, and
(2) $z \cdot w = \chi(z)w$ for all $z \in Z(\mathfrak{l}_\eta)$.

An immediate consequence of Theorem 3.15 is the following.

Corollary 4.7. *The vector space of contravariant forms on $M(\chi, \eta)$ has dimension $|W_\eta|$.*

This corollary generalizes Shapovalov's results to McDowell's modules $M(\chi, \eta)$.

5. Example

In this section, we illustrate the arguments of Section 3 more concretely by showing that the vector space of contravariant forms on a nondegenerate $\mathfrak{sl}(2, \mathbb{C})$-Whittaker module $Y(\chi, \eta)$ is isomorphic to the vector space $\mathbb{C}^2$. For the remainder of this section, set $\mathfrak{g} = \mathfrak{sl}(2, \mathbb{C})$. Let

$$y = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, h = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, x = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$$

be the standard basis of $\mathfrak{g}$, and $\Omega = \frac{1}{2}h^2 + h + 2yx \in Z(\mathfrak{g})$ the Casimir element, which generates $Z(\mathfrak{g})$. The antiautomorphism $\tau : U(\mathfrak{g}) \to U(\mathfrak{g})$ maps $\tau(x) = y, \tau(h) = h$, and $\tau(y) = x$.

Let $\eta \in \mathrm{ch}\,\mathfrak{n}$ be the Lie algebra morphism sending $x \mapsto 1$. Since $\Pi = \Sigma^+ = \{\alpha\}$ consists of a single root and $\mathfrak{g}_\alpha = \mathbb{C}x$, this choice of η is nondegenerate. Let $\chi : Z(\mathfrak{g}) \to \mathbb{C}$ be the algebra homomorphism sending $\Omega \mapsto 0$. Let

$$Y(\chi, \eta) = U(\mathfrak{g}) \otimes_{Z(\mathfrak{g}) \otimes U(\mathfrak{n})} \mathbb{C}_{\chi,\eta},$$

as in Section 2. $Y(\chi, \eta)$ is an irreducible $\mathfrak{g}$-module generated by the Whittaker vector $w = 1 \otimes 1$.

Let $\Phi_{Y(\chi,\eta)}$ be the vector space of τ-invariant linear functionals on $U(\mathfrak{g})$ which vanish on $\mathrm{Ann}_{U(\mathfrak{g})} w$. By Proposition 3.2, $\Phi_{Y(\chi,\eta)}$ is isomorphic to the vector space of contravariant forms on $Y(\chi, \eta)$. Define a map $\psi : \mathbb{C}^2 \to \Phi_{Y(\chi,\eta)}$ sending $(c_0, c_1) \mapsto \varphi_{c_0,c_1} =: \varphi$ as follows.

- On the Poincaré–Birkhoff–Witt basis element $y^r h^s x^t \in U(\mathfrak{g})$, $r, s, t \in \mathbb{Z}_{\geq 0}$,
$$\varphi(y^r h^s x^t) = \eta(x^{r+t})\varphi(h^s) = \varphi(h^s).$$
- Define $\varphi(h^s)$ inductively: set $\varphi(1) = c_0$, $\varphi(h) = c_1$, and for $s \geq 2$, define
$$\varphi(h^s) = -\varphi(2h^{s-1} + 4h^{s-2}yx).$$
This is well-defined because rewriting $h^{s-2}yx$ in terms of the Poincaré–Birkhoff–Witt basis results in a sum whose terms only include powers of h which are equal to or lower than $s - 2$. (For example, $h^2yx = yh^2x - 4yhx + 4yx$.) Hence inductively, φ is already defined on $2h^{s-1} + 4h^{s-2}yx$.
- Extend linearly to define φ on all of $U(\mathfrak{g})$.

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It is clear from this construction that if $(c_0, c_1) \neq (d_0, d_1) \in \mathbb{C}^2$, then $\varphi_{c_0,c_1} \neq \varphi_{d_0,d_1}$, so φ defines an injection

$$\psi : \mathbb{C}^2 \hookrightarrow U(\mathfrak{g})^*.$$

Proposition 5.1. *$\psi : \mathbb{C}^2 \to \Phi_{Y(\chi,\eta)}$ is an isomorphism of vector spaces.*

Proof. By construction, ψ is linear and injective, so it remains to show that $\mathrm{im}\psi = \Phi_{Y(\chi,\eta)}$. We need to show that $\varphi := \varphi_{c_0,c_1}$ is contained in $\Phi_{Y(\chi,\eta)}$ by showing that it satisfies: (1) $\varphi(\tau(u)) = \varphi(u)$, and (2) $\varphi(\mathrm{Ann}_{U(\mathfrak{g})}(w)) = 0$. We begin by noting two consequences of the definition of φ.

- For any $u \in U(\mathfrak{g})$, $\varphi(ux) = \eta(x)\varphi(u) = \varphi(u)$.
- For any $u \in U(\mathfrak{g})$, $\varphi(yu) = \eta(x)\varphi(u) = \varphi(u)$.

First we check that φ satisfies (1) on the Poincaré–Birkhoff–Witt basis element $y^r h^s x^t$:

$$\varphi(y^r h^s x^t) = \varphi(h^s) = \varphi(y^t h^s x^r) = \varphi(\tau(y^r h^s x^t)).$$

Extending linearly, we conclude that φ satisfies (1) for any $u \in U(\mathfrak{g})$.

Next we check (2). Note that $\mathrm{Ann}(w) = U(\mathfrak{g})\Omega + U(\mathfrak{g})(x - \eta(x))$ (Proposition 3.4), so a generic element of the annihilator is a sum of elements of the form $y^r h^s x^t \Omega + y^i h^j x^k (x - \eta(x))$. Then

$$\begin{aligned}
&\varphi(y^r h^s x^t \Omega + y^i h^j x^k (x - \eta(x))) \\
&\quad = \big(\eta(x^r)\varphi(h^s \Omega x^t)\big) + \big(\varphi(y^i h^j x^k x) - \eta(x)\varphi(y^i h^j x^k)\big) \\
&\quad = \eta(x^{r+t})\varphi\left(h^s\left(\frac{1}{2}h^2 + h + 2yx\right)\right) \\
&\quad = \frac{1}{2}\varphi(h^{s+2}) + \varphi(h^{s+1} + 2h^s yx) \\
&\quad = -\frac{1}{2}\varphi(2h^{s+1} + 4h^s yx) + \varphi(h^{s+1} + 2h^s yx) \\
&\quad = 0.
\end{aligned}$$

By extending linearly, we see that for any $a \in \mathrm{Ann}(w)$, $\varphi(a) = 0$. This proves that $\mathrm{im}\psi \subseteq \Phi_{Y(\chi,\eta)}$. To show that $\mathrm{im}\psi = \Phi_{Y(\chi,\eta)}$, we make the following observations about any $\varphi \in \Phi_{Y(\chi,\eta)}$.

- The value of $\varphi(y^r h^s x^t)$ is completely determined by $\varphi(h^s)$. This follows from three facts: first, any Poincaré–Birkhoff–Witt basis element $y^r h^s x^t$ can be expressed as $u(x - \eta(x)) + a y^r h^s$ for some $u \in U(\mathfrak{g})$ and $a \in \mathbb{C}$ by "peeling off" x's (i.e., rewriting $y^r h^s x^t = y^r h^s x^{t-1}(x - \eta(x)) + \eta(x) y^r h^s x^{t-1}$); second, φ vanishes on the annihilator of w; and third, φ is τ-invariant.
- The value of $\varphi(h^s)$ is completely determined by $\varphi(1)$ and $\varphi(h)$. This follows from the fact that we can rewrite $h^s = 2h^{s-2}\Omega - 2h^{s-1} - 4h^{s-2}yx$, and $\Omega \in \mathrm{Ann}(w)$.

Therefore, a choice of $\varphi(1)$ and $\varphi(h)$ in $\mathbb{C}$ completely determines a linear functional $\varphi \in \Phi_{Y(\chi,\eta)}$, so $\psi : \mathbb{C}^2 \to \Phi_{Y(\chi,\eta)}$ is surjective. □

Hence the vector space of contravariant forms on the $\mathfrak{sl}_2(\mathbb{C})$-module $Y(\chi, \eta)$ is two-dimensional. Note that this is also the cardinality of $W = \mathbb{Z}/2\mathbb{Z}$.

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