Transformation Groups ©Springer Science+Business Media New York (2019)
Vol. 25, No. 3, 2020, pp. 859–886

AN ORBIT MODEL FOR THE SPECTRA OF NILPOTENT GELFAND PAIRS

HOLLEY FRIEDLANDER

Department of
Mathematics and Computer Science
Dickinson College
Carlisle, PA 17013, USA
friedlah@dickinson.edu

WILLIAM GRODZICKI

Department of
Mathematics
St. Olaf College
Northfield, MN 55057, USA
grodzi1@stolaf.edu

WAYNE JOHNSON

Department of
Mathematics and Physics
Westminster College
Fulton, MO 65251, USA
wayne.johnson@westminster-mo.edu

GAIL RATCLIFF

Department of
Mathematics
East Carolina University
Greenville, NC 27858, USA
ratcliffg@ecu.edu

ANNA ROMANOV*

School of
Mathematics and Statistics
University of Sydney
Sydney, NSW 2042, Australia
anna.romanov@sydney.edu.au

BENJAMIN STRASSER

Department of
Mathematics
University of Minnesota
Minneapolis, MN 55455, USA
stras087@math.umn.edu

BRENT WESSEL

Department of
Mathematics
St. Louis University
St. Louis, MO 63103, USA
brent.wessel@slu.edu

Abstract. Let N be a connected and simply connected nilpotent Lie group, and let K be a subgroup of the automorphism group of N. We say that the pair (K, N) is a *nilpotent Gelfand pair* if $L^1_K(N)$ is an abelian algebra under convolution. In this

DOI: 10.1007/s00031-019-09541-8
*Supported by the National Science Foundation Award No. 1803059.
Received April 4, 2018. Accepted December 9, 2018.
Published online August 2, 2019.
Corresponding Author: Anna Romanov, e-mail: anna.romanov@sydney.edu.au

document we establish a geometric model for the Gelfand spectra of nilpotent Gelfand pairs (K, N) where the K-orbits in the center of N have a one-parameter cross section and satisfy a certain non-degeneracy condition. More specifically, we show that the one-to-one correspondence between the set $\Delta(K, N)$ of bounded K-spherical functions on N and the set $\mathcal{A}(K, N)$ of K-orbits in the dual $\mathfrak{n}^*$ of the Lie algebra for N established in [BR08] is a homeomorphism for this class of nilpotent Gelfand pairs. This result had previously been shown for N a free group and N a Heisenberg group, and was conjectured to hold for all nilpotent Gelfand pairs in [BR08].

1. Introduction

A *Gelfand pair* (G, K) consists of a locally compact topological group G and a compact subgroup $K \subset G$ such that the space $L^1(G/\!/K)$ of integrable K-bi-invariant functions on G is commutative. Such pairs arise naturally in harmonic analysis and representation theory of Lie groups, with perhaps the best known examples emerging in the case of a connected semisimple Lie group G with finite center and maximal compact subgroup K. These pairs have played a critical role in understanding the representation theory of semisimple Lie groups, and have been studied extensively in the past 50 years [GV88], [Hel84]. We are interested in a class of Gelfand pairs which arise in analysis on nilpotent Lie groups. Let N be a connected and simply connected nilpotent Lie group, and let K be a compact subgroup of the automorphism group of N. We say (K, N) is a *nilpotent Gelfand pair* if $L^1_K(N)$ is an abelian algebra under convolution. In this setting, the pair $(K \ltimes N, K)$ is a Gelfand pair by our initial definition. By [BJR90, Thm. A], any such N is two-step (or abelian), with center Z and top step $V := N/Z$. Nilpotent Gelfand pairs have been classified in [Vin03], [Yak05], [Yak06]. In the case where N is a Heisenberg group, such pairs have been extensively studied, and several interesting topological models for their spectra exist in the literature [BJR92], [BJRW96], [BR13]. In this paper, we develop a topological model for the spectra of a more general class of nilpotent Gelfand pairs.

For a nilpotent Gelfand pair (K, N), let $\mathfrak{n} = \operatorname{Lie} N$ and write $\mathfrak{n} = \mathcal{V} \oplus \mathfrak{z}$, where $\mathfrak{z} = \operatorname{Lie} Z$ is the center of $\mathfrak{n}$, $\mathcal{V}$ is K-invariant, and $[\mathcal{V}, \mathcal{V}] \subseteq \mathfrak{z}$. Let $\mathbb{D}_K(N)$ be the algebra of differential operators on N that are simultaneously invariant under the action of K and left multiplication by N. The algebra $\mathbb{D}_K(N)$ is freely generated by a finite set of differential operators $\{D_0, \ldots, D_d\}$, obtained from a generating set $\{p_0, \ldots, p_d\}$ of the algebra of K-invariant polynomials on N via quantization. When (K, N) is a nilpotent Gelfand pair, it is known that $\mathbb{D}_K(N)$ is abelian. In this setting, a smooth function $\phi : N \longrightarrow \mathbb{C}$ is called *K-spherical* if

- ϕ is K-invariant,
- ϕ is a simultaneous eigenfunction for all $D \in \mathbb{D}_K(N)$, and
- $\phi(e) = 1$, where e is the identity element in N.

Our object of interest is the set $\Delta(K, N)$ of bounded K-spherical functions on N. By integration against spherical functions, $\Delta(K, N)$ can be identified with the spectrum of the commutative Banach $\star$-algebra $L^1_K(N)$. Because of this identification, we will refer to $\Delta(K, N)$ as the *Gelfand space* (or *Gelfand spectrum*) of (K, N), where the topology of uniform convergence on compact sets on $\Delta(K, N)$ coincides with the weak-$*$ topology on the spectrum of $L^1_K(N)$. There is an established topological model for $\Delta(K,N)$ in terms of eigenvalues of the differential operators in $\mathbb{D}_K(N)$. This technique was first used in [Wol06] to embed the spectrum of any Gelfand pair into an infinite-dimensional Euclidean space using all $D \in \mathbb{D}_K(N)$. It was modified by Ferrari Ruffino in [FR07] to the precise topological description in Theorem 1, which we will refer to as the "eigenvalue model."

For a differential operator $D \in \mathbb{D}_K(N)$ and a K-spherical function ϕ, we denote the corresponding eigenvalue by $\widehat{D}(\phi)$; that is,

$$D \cdot \phi = \widehat{D}(\phi)\phi.$$

Theorem 1 ([FR07])**.** *For any self adjoint generating set* $(D_0, \ldots, D_d)$ *of* $\mathbb{D}_K(N)$, *the map* $\Phi : \Delta(K,N) \to \mathbb{R}^d$ *defined by* $\phi \mapsto (\widehat{D}_0(\phi), \ldots, \widehat{D}_d(\phi))$ *is a homeomorphism onto its image.*

Let $\mathcal{E}(K,N)$ denote the image of $\Delta(K,N)$ under Φ. In this paper, we develop a different topological model for $\Delta(K,N)$ (the promised "orbit model"), and the existence of the eigenvalue model plays a critical role in proving our desired convergence results. The eigenvalue model has proven to be a useful tool in the analysis of nilpotent Gelfand pairs, as demonstrated by the sequence of papers [FR07], [FRY12], [FRY18] in which V. Fischer, F. Ricci, and O. Yakimova use the eigenvalue model to study K-invariant Schwartz functions on N. They have shown, for certain classes of Gelfand pairs (K,N), that these functions correspond to the restriction of Schwartz functions on $\mathbb{R}^d$ to the Gelfand space.

In [BR08], the authors establish a bijection between $\Delta(K,N)$ and a set $\mathcal{A}(K,N)$ of K-orbits in the dual $\mathfrak{n}^*$ of $\mathfrak{n}$, which we refer to as K-spherical orbits (Definition 1). We describe this bijection precisely below. It is conjectured in [BR08] that this bijection is a homeomorphism, and the goal of this paper is to prove the conjecture for a certain class of nilpotent Gelfand pairs (Definitions 2 and 3). The topological correspondence $\Delta(K,N) \leftrightarrow \mathcal{A}(K,N)$ is motivated by the "orbit model" philosophy of representation theory, which asserts that the irreducible unitary representations of a Lie group should correspond to coadjoint orbits in the dual of its Lie algebra. For nilpotent and exponential solvable groups, orbit methods have been established, and these methods provide a homeomorphism between the unitary dual and the space of coadjoint orbits [Bro73], [LL94].

To describe such a model for nilpotent Gelfand pairs, let $G = K \ltimes N$, let $\widehat{G}$ denote the unitary dual of G, and let

$$\widehat{G}_K = \{\rho \in \widehat{G} \mid \rho \text{ has a 1-dimensional space of } K\text{-fixed vectors}\}.$$

For each $\rho \in \widehat{G}$, the *orbit method* of Lipsman [Lip80], [Lip82] and Pukanszky [Puk78] produces a well-defined coadjoint orbit $\mathcal{O}(\rho) \subset \mathfrak{g}^*$, and the orbit mapping

$$\widehat{G} \to \mathfrak{g}^*/\mathrm{Ad}^*(G), \quad \rho \mapsto \mathcal{O}(\rho)$$

is finite-to-one. If (G,K) is a Gelfand pair, then on $\widehat{G}_K$, the correspondence is one-to-one. In [BR08] it is shown that for each $\rho \in \widehat{G}_K$, the intersection $\mathcal{K}(\rho) := \mathcal{O}(\rho) \cap \mathfrak{n}^*$ is a single K-orbit in $\mathfrak{n}^*$.

Definition 1. Let $\mathcal{A}(K,N)$ be the set of K-orbits in $\mathfrak{n}^*$ given by

$$\mathcal{A}(K,N) := \{\mathcal{K}(\rho) \mid \rho \in \widehat{G}_K\}.$$

We call $\mathcal{A}(K,N)$ the set of K*-spherical orbits* for the Gelfand pair (K,N).

A key result in [BR08] is that K-spherical orbits are in bijection with the collection of irreducible unitary representations of G with K-fixed vectors.

Theorem 2 ([BR08])**.** *The map $\mathcal{K} : \widehat{G}_K \to \mathcal{A}(K,N)$ is a bijection.*

The spherical functions for (K,N) correspond with $\widehat{G}_K$. Indeed, to each representation of G with a K-fixed vector, one can obtain a spherical function ϕ by forming the matrix coefficient for a K-fixed vector of unit length. This allows us to lift the map $\mathcal{K}$ to a well-defined map

$$\Psi : \Delta(K,N) \to \mathfrak{n}^*/K$$

sending $\phi \mapsto \mathcal{K}(\rho^\phi)$, where $\rho^\phi \in \widehat{G}_K$ is the K-spherical representation of G that gives us ϕ. There is an alternate description of Ψ which is preferable for calculations. The bounded spherical functions $\phi \in \Delta(K,N)$ are parameterized by pairs (π,α), where π and α are irreducible unitary representations of N and the stabilizer K_π of π in K, respectively. These are the *Mackey parameters* described in Section 3. For the coadjoint orbit $\mathcal{O} = \mathcal{O}^N(\pi) \subset \mathfrak{n}^*$ associated to π, one can define a *moment map* $\tau_\mathcal{O} : \mathcal{O} \to \mathfrak{k}_\pi^*$ in such a way that the image of $\tau_\mathcal{O}$ includes the $\mathrm{Ad}^*(K_\pi)$-orbit $\mathcal{O}^{K_\pi}(\alpha)$ associated to the representation $\alpha \in \widehat{K_\pi}$ (see Section 4). Moreover, one can choose a *spherical point* $\ell_{\pi,\alpha} \in \mathcal{O}$ (Definition 4) with $\tau_\mathcal{O}(\ell_{\pi,\alpha}) \in \mathcal{O}^{K_\pi}(\alpha)$ so that

$$\Psi(\phi_{\pi,\alpha}) = K \cdot \ell_{\pi,\alpha}.$$

This is the realization of Ψ that we will use in future arguments. A corollary of Theorem 2 is the following.

Corollary 3 ([BR08])**.** *The map $\Psi : \Delta(K,N) \to \mathcal{A}(K,N)$ is a bijection.*

The compact-open topology on $\Delta(K,N)$ corresponds to the Fell topology on $\widehat{G}_K$ [BR08]. We give $\mathcal{A}(K,N)$ the subspace topology from $\mathfrak{n}^*/K$. Note that $\mathfrak{n}^*/K$ is metrizable since K is compact. In [BR08], the authors prove that Ψ is a bijection for all nilpotent Gelfand pairs and a homeomorphism whenever N is a Heisenberg group or a free group. Further, it is conjectured that Ψ is a homeomorphism for all nilpotent Gelfand pairs. In this document, we show that this conjecture holds for a certain class of nilpotent Gelfand pairs. To describe exactly which (K,N) our result applies to, we first establish some terminology.

Definition 2. Let (K,N) be a nilpotent Gelfand pair. Let $\mathfrak{z}$ be the center of the Lie algebra $\mathfrak{n}$ of N. We say that (K,N) has *spherical central orbits*[1] if generic orbits of the restricted action of K on $\mathfrak{z}$ are of codimension one.

[1]Nilpotent Gelfand pairs with this property are sometimes said to have "rank one center."

In Section 1 we endow $\mathfrak{n} = \mathcal{V} \oplus \mathfrak{z}$ with an inner product $\langle \cdot\,, \cdot \rangle$ such that $\mathcal{V} = \mathfrak{z}^{\perp}$. If (K, N) has spherical central orbits, then $\mathfrak{z}$ is K-irreducible. Since $[\mathfrak{n}, \mathfrak{n}]$ is a K-invariant subspace of $\mathfrak{z}$, this implies that for such pairs, $\mathfrak{z} = [\mathfrak{n}, \mathfrak{n}]$. If (K, N) has spherical central orbits, then we can fix a unit base point $A \in \mathfrak{z}$, and define a skew-symmetric form $(v, w) \mapsto \langle [v, w], A \rangle$ on $\mathcal{V}$.

Definition 3. We say that a nilpotent Gelfand pair (K, N) is *non-degenerate on* $\mathcal{V}$ if the skew-symmetric form

$$(v, w) \mapsto \langle [v, w], A \rangle$$

is non-degenerate on $\mathcal{V}$. Here $v, w \in \mathcal{V}$ and $A \in \mathfrak{z}$ is the fixed unit base point.

The main result of this paper is the following.

Theorem 4. *Let (K, N) be a nilpotent Gelfand pair with spherical central orbits that is non-degenerate on $\mathcal{V}$. Then the map Ψ is a homeomorphism.*

We conclude this introduction with a discussion on our motivations for introducing Definitions 2 and 3 and a discussion on the class of nilpotent Gelfand pairs to which Theorem 4 applies. The assumption that central K-orbits are spheres implies that all generic, or Type I representations, are related by the K-action. These representations have the same stabilizer in K, and factor through a common Heisenberg group. There are uniform descriptions of eigenvalues across the Type I and Type II orbits, which enable us to account for sequences of Type I points in the Gelfand space converging to Type II points.

In [Vin03], Vinberg classifies all nilpotent Gelfand pairs (K, N) satisfying the following three properties: the commutator subgroup of N coincides with its center, the representation of K on $\mathcal{V}$ is irreducible, and the pair cannot be obtained by a central reduction of a larger nilpotent Gelfand pair. In [BR08], [BR13], it is shown through a case-by-case analysis that Ψ is a homeomorphism for nilpotent Gelfand pairs with N abelian, N a Heisenberg group, or N a free group. This proves the conjecture of [BR08] for all pairs on Vinberg's list [Vin03] with $\dim \mathfrak{z} = 1$. For two-step nilpotent groups other than Heisenberg groups (that is, pairs with higher dimensional center $\mathfrak{z}$), the action of K on $\mathfrak{z}$ is no longer trivial, so the next step up in complexity is to consider nilpotent Gelfand pairs where the K-orbits in $\mathfrak{z}$ are of codimension 1.

Table 1 in [FRY12] and Vinberg's list [Vin03]) classify nilpotent Gelfand pairs with spherical central orbits and higher dimensional center. There are ten of these cases which satify our conditions. However, both this list and the Vinberg list from which it is derived impose the extra requirement that the action of K on $\mathcal{V}$ is irreducible. Our arguments do not depend on this irreducibility, which implies that Theorem 4 also holds for some nilpotent Gelfand pairs that are not found on these lists. An example is given in Section 6.

This document is organized in the following way. Section 1 establishes notation and conventions. Section 2 reviews the orbit model for Heisenberg Gelfand pairs. We include this information because certain parts of the proof of our main result reduce to this setting. Section 3 describes the representation theory of the semi-direct product group $K \ltimes N$ to give a parametrization of the Gelfand space of

(K, N). In Section 4 we define a moment map on coadjoint orbits in order to calculate spherical points. The proof of our main result can be found in Section 5. Section 6 provides a detailed example.

Acknowledgments. This project was initiated at an AMS Mathematics Research Community in Snowbird, Utah in June 2016. This material is based upon work supported by the National Science Foundation under Grant Number DMS 1321794.

1. Preliminaries and notation

- Throughout this document, N is a connected and simply connected 2-step nilpotent Lie group, K is a (possibly disconnected) compact Lie subgroup of the automorphism group of N, and (K, N) is a nilpotent Gelfand pair. In Sections 3 and 4 we start by describing results which apply to all nilpotent Gelfand pairs, then in the second half of each section restrict our attention to nilpotent Gelfand pairs satisfying Definitions 2 and 3. In statements of theorems, we always indicate the restrictions we are making on (K, N).
- Let $G = K \ltimes N$ be the semidirect product of K and N, with group multiplication
$$(k, x)(k', x') = (kk', x(k \cdot x')).$$
- We denote Lie groups by capital Roman letters, and their corresponding Lie algebras by lowercase letters in fraktur font. We identify N with its Lie algebra $\mathfrak{n}$ via the exponential map. We denote the derived action of $\mathfrak{k}$ on $\mathfrak{n}$ by $A \cdot X$ for $A \in \mathfrak{k}$ and $X \in \mathfrak{n}$.
- We denote by $\widehat{H}$ the unitary dual of a Lie group H. We identify representations that are unitarily equivalent, and we do not distinguish notationally between a representation and its equivalence class.
- We denote the coadjoint actions of a Lie group H and its Lie algebra $\mathfrak{h}$ on $\mathfrak{h}^*$ by
$$\begin{gathered}\mathrm{Ad}^*(h)\varphi = \varphi \circ \mathrm{Ad}(h^{-1}) \quad \text{and} \\ \mathrm{ad}^*(X)\varphi(Y) = \varphi \circ \mathrm{ad}(-X)(Y) = -\varphi([X, Y])\end{gathered}$$
for $h \in H$, $\varphi \in \mathfrak{h}^*$, and $X, Y \in \mathfrak{h}$.
- We use the symbol $\mathcal{O}$ to denote a coadjoint orbit in $\mathfrak{n}^*$. If $x \in \mathfrak{n}^*$, $\mathcal{O}_x$ denotes the coadjoint orbit containing x. If such an orbit is determined by a parameter λ, we sometimes refer to the orbit as $\mathcal{O}_\lambda$.
- We fix a K-invariant positive definite inner product $\langle \cdot, \cdot \rangle$ on $\mathfrak{n}$, and let $\mathcal{V} = \mathfrak{z}^\perp$ so that $\mathfrak{n} = \mathcal{V} \oplus \mathfrak{z}$. Here $\mathfrak{z} = \mathrm{Lie}\, Z$ is the center of $\mathfrak{n}$. We identify $\mathfrak{n}$ and $\mathfrak{n}^*$ via this inner product.
- We fix a unit base point $A \in \mathfrak{z}$, and use it to define a form
$$(v, w) \mapsto \langle [v, w], A \rangle$$
on $\mathcal{V}$. The letter A will be used throughout the document to refer to this fixed unit base point.

2. Heisenberg Gelfand pairs

In this section we will describe the orbit model for Gelfand pairs of the form (K, H_a) where H_a is a Heisenberg group. As noted in the introduction, a more complete discussion of the situation reviewed here can be found in [BJR92], [BR13], [BJRW96]. We cover this case because our proof of Theorem 4 uses the proof of this same result when N is a Heisenberg group.

1. Spherical functions and the eigenvalue model

In the rest of this subsection, as well as the next one, let $H_V = V \oplus \mathbb{R}$, where V is a complex vector space. Let $\langle \cdot, \cdot \rangle$ be a positive definite Hermitian inner product on V. The Lie algebra, $\mathfrak{h}_V$, of H_V can be expressed as

$$\mathfrak{h}_V = V \oplus \mathbb{R} \quad \text{with Lie bracket} \quad [(v,t),(v',t')] = (0, -\Im\langle v, v'\rangle).$$

Here $\Im\langle v, v'\rangle$ denotes the imaginary part of $\langle v, v'\rangle$. Let K be a compact subgroup of the unitary group $U(V)$ for V such that (K, H_V) is a nilpotent Gelfand pair. This is equivalent [BJR90, Thm. B] to the fact that the action of K on V is a linear multiplicity-free action [BR04, Def. 1.9.4][2]. The group K acts on H_V via

$$k \cdot (v,t) = (kv, t).$$

We start by reviewing some facts about the multiplicity-free action $K : V$. Let $T \subset K$ be a maximal torus in K with Lie algebra $\mathfrak{t} \subset \mathfrak{k}$, and let $\mathfrak{h} := \mathfrak{t}_{\mathbb{C}}$ be the corresponding Cartan subalgebra in $\mathfrak{k}_{\mathbb{C}}$. Denote by H the corresponding subgroup of the complexified group $K_{\mathbb{C}}$, and let B be a fixed Borel subgroup of $K_{\mathbb{C}}$ containing H. Let $\Lambda \subset \mathfrak{h}^*$ be the set of highest B-weights for irreducible representations of $K_{\mathbb{C}}$ (or, equivalently, K) occurring in $\mathbb{C}[V]$, and denote by

$$\mathbb{C}[V] = \bigoplus_{\alpha \in \Lambda} P_\alpha$$

the decomposition of $\mathbb{C}[V]$ into irreducible subrepresentations of $K_{\mathbb{C}}$ (and K). The subspaces P_α consist of homogeneous polynomials of a fixed degree. For $\alpha \in \Lambda$, denote by $|\alpha|$ the degree of homogeneity of polynomials in P_α, denote by d_α the dimension of P_α, and fix a B-highest weight vector $h_\alpha \in P_\alpha$ (unique modulo $\mathbb{C}^\times$). If $\mathcal{P}_m(V)$ is the space of holomorphic polynomials on V of homogeneous degree m, then $P_\alpha \subset \mathcal{P}_{|\alpha|}(V)$. The highest weights in Λ are freely generated by a finite set of fundamental weights $\{\alpha_1, \ldots, \alpha_r\}$ [BR13], which have the property that the corresponding polynomial $h_i := h_{\alpha_i}$ is irreducible. It follows that the highest weight vector h_α in each invariant subspace P_α has the form

$$h_{\mathbf{m}} = h_1^{m_1} \cdots h_r^{m_r} \quad \text{with highest weight} \quad \alpha_{\mathbf{m}} = m_1\alpha_1 + \cdots + m_r\alpha_r.$$

To understand the orbit model for $\Delta(K, H_V)$, we review the representation theory of $K \ltimes H_V$. The irreducible unitary representations of H_V are classified by

[2]Here, we are adopting the conventions of [BR04], where $K : V$ is said to be a *multiplicity-free action* if the representation ρ of K on $\mathbb{C}[V]$ given by $(\rho(k)f)(v) = f(k^{-1} \cdot v)$ for $k \in K, f \in \mathbb{C}[V], v \in V$ is multiplicity-free.

Kirillov's "orbit method" [Kir04][3]. One sees that the irreducible unitary representations of H_V naturally split into two types according to their parametrization by the coadjoint orbits in $\mathfrak{h}_V^*$. In particular, the "type I" representations ρ_λ are indexed by real numbers $\lambda \neq 0$, with associated coadjoint orbit (which we refer to as a "type I" orbit)

$$\mathcal{O}_\lambda = V \oplus \{\lambda\},$$

and the "type II" representations χ_b correspond to one-point orbits $\mathcal{O}_b = \{(b, 0)\}$ for $b \in V$.

A type I representation $\rho_\lambda \in \widehat{H}_V$ can be realized in the Fock space $\mathcal{F}_V$, which is the L^2- closure of $\mathbb{C}[V]$ endowed with a Gaussian measure. For $k \in K$, we have $\rho_\lambda \circ k \cong \rho_\lambda$ and the intertwining map is the natural action of K on $\mathbb{C}[V]$. From the type I representation $\rho_\lambda \in \widehat{H}_V$, a type I spherical function can be constructed by taking the K-averaged matrix coefficient for ρ_λ on some P_α with $\alpha = \alpha_{\mathbf{m}}$ for some $\mathbf{m}$. The type I spherical functions are denoted $\phi_{\lambda,\mathbf{m}}$ with $\lambda \neq 0$ and $\mathbf{m} \in (\mathbb{Z}_{\geq 0})^r$.

Type II representations are one-dimensional characters $\phi_b(v,t) = e^{i\langle b,v\rangle}$. From such a type II representation, a spherical function ϕ_b can be constructed by the K-average

$$\phi_b(v) := \int_K e^{i\langle b, k\cdot v\rangle} dk,$$

so that the Type II spherical function ϕ_b only depends on the k-orbit through b. From this we have the following parametrization of the Gelfand space:

$$\Delta(K, H_V) \leftrightarrow \{(\lambda, \mathbf{m}) \mid \lambda \neq 0,\, \mathbf{m} \in (\mathbb{Z}_{\geq 0})^r\} \cup (V/K). \tag{1}$$

We now demonstrate how to calculate $\Phi(\phi)$, where $\Phi : \Delta(K, H_a) \to \mathcal{E}(K, H_a)$ as in Section 1 and $\phi \in \Delta(K, H_a)$. Aligning notation with [BR13], we denote by $\mathbb{C}[V_\mathbb{R}]^K$ the algebra of K-invariant polynomials on the underlying real vector space for V, and $\mathcal{PD}(V)^K$ the space of K-invariant polynomial coefficient differential operators on V. By Schur's lemma, each differential operator $D \in \mathcal{PD}(V)^K$ acts on the K-irreducible subspace $P_{\alpha_{\mathbf{m}}}$ by a scalar. In [BR08], it is shown that

$$\rho_\lambda(D) h_{\mathbf{m}} = \widehat{D}(\phi_{\lambda,\mathbf{m}}) h_{\mathbf{m}}.$$

In [BR04, §7], canonical bases for $\mathbb{C}[V_\mathbb{R}]^K$ and $\mathcal{PD}(V)^{K_\mathbb{C}}$ are constructed in the following way. For each irreducible subspace P_α fix an orthonormal basis $\{q_j \mid 1 \leq j \leq d_\alpha\}$ with respect to the Fock inner product on $\mathbb{C}[V]$. Then define

$$p_\alpha(v) := \sum_{j=1}^{d_\alpha} q_j(v)\overline{q}_j(\overline{v}).$$

The polynomial p_α is homogeneous of degree $2|\alpha|$. Write $p_\alpha(v, \overline{v})$ for the polynomial $p_\alpha(v)$, and construct a differential operator $p_\alpha(v, \partial) \in \mathcal{PD}(V)^K$ by substituting ∂_i for $\overline{v_i}$ and letting derivatives act to the right of multiplication. This differential

[3]An explicit review of the orbit method is provided in Section 3 for two-step nilpotent groups.

operator $p(v,\partial)$ is homogeneous of degree $|\alpha|$. Then the sets $\{p_\alpha : \alpha \in \Lambda\}$ and $\{p_\alpha(z,\partial)\}$ form bases for $\mathbb{C}[V_\mathbb{R}]^K$ and $\mathcal{PD}(V)^K$ respectively. Note that $\mathbb{C}[V_\mathbb{R}]^K$ is generated by the set $\{p_{\alpha_1},\dots,p_{\alpha_r}\}$, where $\{\alpha_1,\dots,\alpha_r\}$ are the fundamental weights introduced at the beginning of this section.

We use the bases for $\mathbb{C}[V_\mathbb{R}]^K$ and $\mathcal{PD}(V)^K$ above to construct a generating set for the algebra $\mathbb{D}_K(H_V)$. We can consider a K-invariant polynomial p on V to be a K-invariant polynomial on H_V by letting p act trivially on the center. For any K-invariant homogeneous polynomial $p(v,\overline{v})$ of even degree s_p on V, we construct a differential operator $D_p \in \mathbb{D}_K(H_V)$ so that for a type I representation $\rho \in \widehat{H}_a$ realized on Fock space,

$$\rho_\lambda(D_p) = (-2\lambda)^{s_p/2} p(v,\partial). \tag{2}$$

Define the K-invariant polynomial p_0 on H_V by $p_0(v,t) = -it$, and let $p_i := p_{\alpha_i}$ be the generating elements for $\mathbb{C}[V_\mathbb{R}]^K$ constructed above, considered as K-invariant polynomials on H_V. We choose labeling so that $p_1(v) = |v|^2$. The set $\{p_0,\dots,p_r\}$ forms a homogeneous generating set for the K-invariant polynomials on N, where the degree of homogeneity of p_i is $2|\alpha_i|$ for $1 \le i \le r$. From p_0 we construct the differential operator $D_0 = -i\partial/\partial t \in \mathbb{D}_K(H_V)$, and from p_i construct $D_i \in \mathbb{D}_K(H_V)$ in the process outlined above. The resulting set $\{D_0,\dots,D_r\}$ forms a homogeneous generating set for $\mathbb{D}_K(H_V)$, where the degree of homogeneity of D_i is $2|\alpha_i|$ for $1 \le i \le r$. Note that $\rho_\lambda(D_i)$ is a differential operator of degree $|\alpha_i|$ acting on $\mathbb{C}[V]$.

The bounded K-spherical functions in $\Delta(K,H_V)$ are eigenfunctions for operators in the algebra $\mathbb{D}_K(H_V)$, and our next step is to examine the corresponding eigenvalues. We know from [BJR92] that the type I spherical function $\phi_{\lambda,\mathbf{m}}$ is $\phi_{\lambda,\mathbf{m}}(v,t) = e^{i\lambda t} q_\alpha(\sqrt{|\lambda|}v)$[4], so for any homogeneous K-invariant polynomial p (of even degree s_p) on V,

$$\widehat{D}_p(\phi_{\lambda,\mathbf{m}}) = |\lambda|^{s_p/2}\widehat{D}_p(\phi_{1,\mathbf{m}}). \tag{3}$$

To compute the eigenvalue of an operator $D_p \in \mathbb{D}_K(H_a)$ on $\phi_{1,\mathbf{m}}$, one computes the action of D_p on the highest weight vector $h_\mathbf{m}$ in the corresponding irreducible K-subspace $P_{\alpha_\mathbf{m}}$. (See [BR98, §4] for more details on this calculation.) We have

$$\rho(D_p)h_\mathbf{m}(v) = (-2)^{s_p/2} p(v,\partial) h_\mathbf{m}(v) = \widehat{D}_p(\phi_{1,\mathbf{m}}) h_\mathbf{m}(v).$$

Therefore the eigenvalues $\widehat{D}_p(\phi_{1,\mathbf{m}})$ are polynomials (not necessarily homogeneous) in $\mathbf{m} = (m_1,\dots,m_r)$ of degree $s_p/2$. To emphasize this, we define

$$\widetilde{p}(\mathbf{m}) := \widehat{D}_p(\phi_{1,\mathbf{m}})$$

to be this (degree $s_p/2$) polynomial. Note that the differential operator D_p can be defined by other methods (quantizations) and that this only affects the lower order terms in the eigenvalue polynomial $\widetilde{p}$.

We end this subsection by calculating $\widehat{D}_0(\phi_{\lambda,m})$ and $\widehat{D}_1(\phi_{1,m})$. These eigenvalues are explicitly used in the proof of Theorem 4 in Section 5.

[4]Here $q_\alpha \in \mathbb{C}[V_\mathbb{R}]^K$ is a certain K-invariant polynomial on $V_\mathbb{R}$ whose top homogeneous term is equal to $(1/d_\alpha)p_{\alpha_\mathbf{m}}$. See [BJR92] for details of this construction.

Example 1. Applying the operator $D_0 = -i\partial/\partial t$ to $\phi_{\lambda,\mathbf{m}}(v,t) = e^{i\lambda t}q_\alpha(\sqrt{|\lambda|}v)$ we compute

$$\widehat{D}_0(\phi_{\lambda,\mathbf{m}}) = \lambda.$$

To compute $\widehat{D}_1(\phi_{1,\mathbf{m}})$, we notice that since $p_1(v,\overline{v}) = |v|^2$, $p_1(v,\partial) = \sum v_i\partial_i$ is the degree operator. Hence,

$$\widehat{D}_1(\phi_{1,\mathbf{m}}) = -2|\lambda||\mathbf{m}|,$$

where $|\mathbf{m}| := |\alpha_{\mathbf{m}}|$ is the degree of the polynomials in $P_{\alpha_{\mathbf{m}}}$.

2. The orbit model

In this section, we describe the orbit model for Heisenberg Gelfand pairs. These results first appeared in [BR08], and were reviewed in [BR13]. We begin by introducing some general terminology involving moment maps, then specialize to the Heisenberg setting and define moment maps on coadjoint orbits.

Let V be a Hermitian vector space and $K \subset U(V)$. We define the moment map $\tau : V \to \mathfrak{k}^*$ by

$$\tau(v)(Z) = -\tfrac{1}{2}\langle v, Z\cdot v\rangle$$

for $v \in V$ and $Z \in \mathfrak{k}$. Then τ is K-equivariant, and it is known that the action of $K : V$ is multiplicity-free if and only if τ is one-to-one on K-orbits. We make this assumption, and let

$$\mathbb{C}[V] = \sum_{\alpha\in\Lambda} P_\alpha$$

be the multiplicity-free decomposition. Following [BR13, §2.4], we associate a coadjoint orbit $\mathcal{O}_\alpha$ in $\mathfrak{k}^*$ to each irreducible subspace P_α by first extending $\alpha \in \Lambda \subset \mathfrak{t}^*$ to a real-valued linear functional on $\mathfrak{k}$ by

$$\alpha_{\mathfrak{k}}(Z) = \begin{cases} -i\alpha(Z) \text{ if } Z \in \mathfrak{t}, \\ 0 \text{ if } Z \in \mathfrak{t}^\perp, \end{cases}$$

and setting $\mathcal{O}_\alpha = \mathrm{Ad}^*(K)\alpha_{\mathfrak{k}}$. Here $\mathfrak{t}^\perp$ is the orthogonal complement of $\mathfrak{t}$ with respect to a fixed $\mathrm{Ad}(K)$-invariant inner product on $\mathfrak{k}$. By [BJLR97, Prop. 4.1], each coadjoint orbit $\mathcal{O}_\alpha$, $\alpha \in \Lambda$, lies in the image of τ.

Definition 4. If $\alpha \in \Lambda$ is a positive weight of the K action on $P(V)$, then *a spherical point of type* α is any point $v_\alpha \in V$ such that $\tau(v_\alpha) = \alpha_{\mathfrak{k}}$.

Since the action of $K_{\mathbb{C}}$ on V is multiplicity-free, there is an open Borel orbit in V. Let B be a Borel subgroup of $K_{\mathbb{C}}$ with Lie algebra $\mathfrak{b}$ as defined in Section 1, and let v_α be a spherical point of type α. Then the highest weight vector $h_\alpha \in P_\alpha$ is a weight vector for $\mathfrak{b}$. That is, we can extend α to $\mathfrak{b}$ so that $Z \cdot h_\alpha = \alpha(Z)h_\alpha$ for all $Z \in \mathfrak{b}$.

Definition 5. We say that the spherical point v_α is *well-adapted* if

$$h_\alpha(v_\alpha) \neq 0 \quad \text{and if} \quad 2\partial_i h_\alpha(v_\alpha) = \overline{(v_\alpha)}_i h_\alpha(v_\alpha).$$

We provide some motivation for this definition by showing that any v_α in the B-open orbit is well-adapted. We note that this will be true for generic α, but is not necessarily true for the generators of Λ. Since h_α is a semi-invariant for B, we must have $h_\alpha(v_\alpha) \neq 0$. For $Z \in \mathfrak{b}$, we have

$$Z \cdot h_\alpha = \alpha(Z)h_\alpha = \tau(v_\alpha)(Z)h_\alpha = -\tfrac{1}{2}\langle v_\alpha, Z \cdot v_\alpha\rangle.$$

Additionally, we have

$$Z \cdot h_\alpha(v_\alpha) = \left.\frac{d}{dt}\right|_0 h_\alpha(\exp(-tZ) \cdot v_\alpha) = -\partial_{Z\cdot v_\alpha} h_\alpha(v_\alpha).$$

By openness, any derivative is of the form $\partial_{Z\cdot v_\alpha}$, including those in the basis directions. Thus we have

$$2\partial_i h_\alpha(v_\alpha) = \overline{(v_\alpha)}_i h_\alpha(v_\alpha).$$

It has been shown that all multiplicity-free actions have well-adapted spherical points.

Now we apply these results to the Heisenberg Gelfand pair (K, H_a) to calculate the spherical points and define the map Ψ described in Section 1. The space V is a Hermitian vector space with form $\langle\cdot\,,\cdot\rangle$, and $K \subset U(V)$, so by the preceding paragraphs, we have a moment map $\tau : V \to \mathfrak{k}^*$ defined as above. We use this moment map to define moment maps on coadjoint orbits. For a type I orbit $\mathcal{O}_\lambda = V \oplus \{\lambda\}$, the moment map $\tau_\lambda : \mathcal{O}_\lambda \to \mathfrak{k}^*$ is given by

$$\tau_\lambda(v, \lambda) = \frac{1}{\lambda}\tau(v). \tag{4}$$

This relationship lets us compute the spherical points in a type I coadjoint orbit $\mathcal{O}_\lambda = V \oplus \{\lambda\}$ for the moment map τ_λ, which we will use to define our orbit model. For each $\alpha_{\mathbf{m}} \in \Lambda$ (the set of highest weights of the representation of K on $\mathbb{C}[V]$), choose $v_{\mathbf{m}} \in V$ with $\tau(v_{\mathbf{m}}) = \alpha_{\mathbf{m}\mathfrak{k}}$. Note that all choices for $v_{\mathbf{m}}$ will be in the same K-orbit. Then the point $(\sqrt{\lambda}v_{\mathbf{m}}, \lambda) \in \mathcal{O}_\lambda$ is a spherical point of τ_λ of type $\alpha_{\mathbf{m}}$.

Next we relate the values of K-invariant polynomials on these spherical points to the eigenvalues described in Section 1. This relationship is the key observation that drives our main argument in Section 5. Recall from Section 2 that $\widetilde{p}(\mathbf{m}) = \widehat{D}_p(\phi_{1,\mathbf{m}})$ is a degree $s_p/2$ polynomial on V. Write $\operatorname{top}\widetilde{p}(\mathbf{m})$ for the highest order homogeneous term in $\mathbf{m}$.

Lemma 5 ([BR13]). *Let $p(v, \overline{v})$ be a K-invariant polynomial on V, homogeneous of degree s_p. Given a well-adapted spherical point $v_{\mathbf{m}} \in V$ of type $\alpha_{\mathbf{m}}$ for the moment map $\tau : V \to \mathfrak{k}^*$, we have*

$$\operatorname{top}\widetilde{p}(\mathbf{m}) = (-1)^{s_p/2} p(v_{\mathbf{m}}, \overline{v}_{\mathbf{m}}).$$

Proof. Consider the following:

$$2\partial_i h_{\mathbf{m}} = 2\partial_i (h_1^{m_1} \cdots h_r^{m_r}) = 2\left(m_1 \frac{\partial_i h_1}{h_1} + \cdots + m_r \frac{\partial_i h_r}{h_r} \right) h_{\mathbf{m}}.$$

We define a vector $\eta(\mathbf{m}, v)$ with entries

$$\eta_i(\mathbf{m}, v) = m_1 \frac{\partial_i h_1}{h_1} + \cdots + m_r \frac{\partial_i h_r}{h_r} = \frac{\partial_i h_{\mathbf{m}}}{h_{\mathbf{m}}}.$$

For any partial derivative $\partial^{\mathbf{a}} = \prod (\partial_i)^{a_i}$, up to lower-order terms in $\mathbf{m}$, we have

$$(2\partial)^{\mathbf{a}} h_{\mathbf{m}} = (2\eta)^{\mathbf{a}} h_{\mathbf{m}} + LOT(\mathbf{m}).$$

Thus

$$(2\partial)^{\mathbf{a}} h_{\mathbf{m}}(v_{\mathbf{m}}) = (\overline{v}_{\mathbf{m}})^{\mathbf{a}} h_{\mathbf{m}}(v_{\mathbf{m}}).$$

Let $p(v, \overline{v})$ be a K-invariant polynomial.

$$p(-v, 2\partial) h_{\mathbf{m}} = p(-v_{\mathbf{m}}, 2\eta(\mathbf{m}, v_{\mathbf{m}})) h_{\mathbf{m}} + LOT(\mathbf{m}).$$

Then $p(-v, 2\partial)$ acts on $h_{\mathbf{m}}$ by a scalar, and the highest order term for the eigenvalue is

$$\operatorname{top} \widetilde{p}(\mathbf{m}) = p(-v_{\mathbf{m}}, 2\eta(\mathbf{m}, v)),$$

independent of the choice of v. If we use a well-adapted spherical point $v_{\mathbf{m}}$, then $2\eta(\mathbf{m}, v_{\mathbf{m}}) = \overline{v}_{\mathbf{m}}$, so we get

$$\operatorname{top} \widetilde{p}(\mathbf{m}) = p(-v_{\mathbf{m}}, \overline{v}_{\mathbf{m}}) = (-1)^{s_p/2} p(v_{\mathbf{m}}, \overline{v}_{\mathbf{m}}). \qquad \square$$

Using equation (3), we immediately obtain the following corollary.

Corollary 6. *For a type* I *spherical function $\phi_{\lambda,\mathbf{m}} \in \Delta(K, H_a)$, a spherical point $v_{\mathbf{m}} \in V$ of τ, and a homogeneous K-invariant polynomial p on V, we have*

$$\operatorname{top} \widehat{D}_p(\phi_{\lambda,\mathbf{m}}) = (-|\lambda|)^{s_p/2} p(v_{\mathbf{m}}).$$

For type II spherical functions, the eigenvalues are obtained by evaluation on spherical points, as can be seen in the following lemma.

Lemma 7. *Let $\phi_b(v) := \int_K e^{i\langle b, k\cdot v\rangle} dk$ be a spherical function associated to a type* II *representation, let p be a K-invariant polynomial on V, and let D_p be the corresponding differential operator. Then*

$$\widehat{D}_p(\phi_b) = p(ib).$$

Proof. Let $\{e_j\}_{1\le j\le 2a}$ be a (real) basis for V which is orthonormal with respect to the real inner product $\langle\ ,\ \rangle$. Then the corresponding vector fields E_j on H_a act as follows:

$$\begin{aligned} E_j \cdot e^{i\langle b,v\rangle} &= \frac{d}{dt}\bigg|_{t=0} e^{i\langle b,v+te_j\rangle} \\ &= \frac{d}{dt}\bigg|_{t=0} e^{i\langle b,te_j\rangle} e^{i\langle b,v\rangle} \\ &= ib_j e^{i\langle b,v\rangle}. \end{aligned}$$

For a K-invariant poynomial $p(v_1,\dots,v_{2a})$ on V, we have

$$\begin{aligned} D_p\phi_b(v) &= p(E_1,\dots,E_{2a}) \int_K e^{i\langle b,kv\rangle} dk \\ &= \int_K p(E_1,\dots,E_{2a}) e^{i\langle k^{-1}b,v\rangle} dk \\ &= \int_K p(ik^{-1}b) e^{i\langle k^{-1}b,v\rangle} dk \\ &= \int_K p(ib) e^{i\langle b,kv\rangle} dk = p(ib)\phi_b(v). \qquad \square \end{aligned}$$

Finally, using Lemma 5, Corollary 6, and Lemma 7, one can prove that the map Ψ is a homeomorphism, and hence prove Theorem 4, for the Heisenberg Gelfand pair (K, H_a). This was shown in [BR13] and we refer the reader to the proof of Theorem 1.2 in that paper.

Theorem 8 ([BR13])**.** *The map* $\Psi : \Delta(K, H_a) \to \mathcal{A}(K, H_a)$ *given by*

$$\Psi(\phi_{\lambda,\mathbf{m}}) = K \cdot (\sqrt{\lambda} v_{\mathbf{m}}, \lambda)$$

in the type I *case, and*

$$\Psi(\chi_b) = K \cdot (b, 0)$$

in the type II *case, is a homeomorphism.*

3. Representation theory of $K \ltimes N$

In this section we recall the representation theory of the group $G = K \ltimes N$, where (K, N) is a nilpotent Gelfand pair, following the treatment in [BR08]. Then we give a more detailed description of these results for the specific class of nilpotent Gelfand pairs we are interested in — those satisfying the conditions in Definitions 2 and 3. We conclude the section by describing the eigenvalue model for the Gelfand space of such nilpotent Gelfand pairs.

As a first step in this process, we review the representation theory of the nilpotent group N. Representations of simply connnected, real nilpotent Lie groups are classified by Kirollov's "orbit method" [Kir04], which proceeds as follows. Given an element $\ell \in \mathfrak{n}^*$, one selects a subalgebra $\mathfrak{m} \subseteq \mathfrak{n}$ which is maximal isotropic (i.e., subject to the condition $\ell([\mathfrak{m},\mathfrak{m}])=0$). One then defines a character χ_ℓ of $M=\exp\mathfrak{m}$ by $\chi_\ell(\exp X)=e^{i\ell(X)}$ and constructs the representation $\pi_\ell := \mathrm{ind}_M^N \chi_\ell$ of N. From Kirillov, we know that each irreducible, unitary representation of N is of the form π_ℓ for some ℓ, and $\pi_\ell \sim \pi_{\ell'}$ if and only if ℓ and ℓ' are in the same coadjoint orbit in $\mathfrak{n}^*$. That is, the association $N\cdot\ell \mapsto \pi_\ell$ yields a bijection between coadjoint orbits in $\mathfrak{n}^*$ and irreducible unitary representations of N.

In our setting, N is a two-step nilpotent Lie group. This extra structure allows us to make a canonical choice of an "aligned point" in each coadjoint orbit. We describe this process now. Recall that the Lie algebra of N is $\mathfrak{n}=\mathcal{V}\oplus\mathfrak{z}$, where $\mathfrak{z}$ and $\mathcal{V}$ are orthogonal with respect to the inner product $\langle\cdot,\cdot\rangle$ (see Section 1). For a coadjoint orbit $\mathcal{O}\subset\mathfrak{n}^*$, we choose $\ell\in\mathcal{O}$ so that $\mathcal{O}=\mathrm{Ad}^*(N)\ell$, then we define a bilinear form on $\mathfrak{n}$ by

$$B_{\mathcal{O}}(X,Y)=\ell([X,Y]).$$

Let $\mathfrak{a}_{\mathcal{O}}=\{v\in\mathcal{V}\mid \ell([v,\mathfrak{n}])=0\}$. Since N is two-step nilpotent, $B_{\mathcal{O}}$ only depends on $\ell|_{\mathfrak{z}}$, and hence both $B_{\mathcal{O}}$ and $\mathfrak{a}_{\mathcal{O}}$ do not depend on our choice of $\ell\in\mathcal{O}$. For each coadjoint orbit $\mathcal{O}$, this process gives us a decomposition

$$\mathfrak{n}=\mathfrak{a}_{\mathcal{O}}\oplus\mathfrak{w}_{\mathcal{O}}\oplus\mathfrak{z},$$

where $\mathfrak{w}_{\mathcal{O}}=\mathfrak{a}_{\mathcal{O}}^{\perp}\cap\mathcal{V}$, and $B_{\mathcal{O}}$ is non-degenerate on $\mathfrak{w}_{\mathcal{O}}$. We can define a map $\mathfrak{w}_{\mathcal{O}}\to\mathcal{O}$ by

$$X\mapsto \mathrm{Ad}^*(X)\ell=\ell-\ell\circ[X,-]. \tag{5}$$

Since N is two-step nilpotent, this map is a homeomorphism [BR08], and thus gives us an identification of $\mathfrak{w}_{\mathcal{O}}$ with $\mathcal{O}$. Note that this identification *does depend on the choice of* ℓ. However, in [BR08] it is shown that there is a canonical choice of ℓ in the following sense.

Definition 6. A point $\ell\in\mathcal{O}$ is called an *aligned point* if $\ell|_{\mathfrak{w}_{\mathcal{O}}}=0$.

This gives us a canonical identification $\mathfrak{w}_{\mathcal{O}}\simeq\mathcal{O}$. Furthermore, the action of K on $\mathfrak{n}^*$ sends aligned points to aligned points, which implies that the stabilizer $K_{\mathcal{O}}=\{k\in K\mid k\cdot\mathcal{O}=\mathcal{O}\}$ of a coadjoint orbit coincides with the stabilizer $K_\ell=\{k\in K\mid k\cdot\ell=\ell\}$ of its aligned point. (See [BR08, Sect. 3.2] for a full discussion.)

Next we recall the process for describing $\widehat{G}$ in terms of representations of N and subgroups of K. This is the *Mackey machine.* There is a natural action of K on $\widehat{N}$ by

$$k\cdot\pi=\pi\circ k^{-1},$$

where $k\in K$ and $\pi\in\widehat{N}$. Let π be an irreducible unitary representation of N corresponding to a coadjoint orbit $\mathcal{O}\subset\mathfrak{n}^*$ as described above. Denote the stabilizer of π under the K-action by

$$K_\pi=\{k\in K\mid k\cdot\pi\simeq\pi\}.$$

Here $\simeq$ denotes unitary equivalence. Note that by the discussion above, $K_\pi=K_{\mathcal{O}}$. By Lemma 2.3 of [BJR99], there is a (non-projective) unitary representation W_π of K_π given by

$$k\cdot\pi(x)=W_\pi(k)^{-1}\pi(x)W_\pi(k).$$

Mackey theory establishes that we can use such representations W_π to build all irreducible unitary representations of G.

Theorem 9 ([BR08]). *Let (K,N) be any nilpotent Gelfand pair. Given any irreducible, unitary representation α of K_π, the representation*

$$\rho_{\pi,\alpha} := \operatorname{ind}_{K_\pi \ltimes N}^{K\ltimes N} ((k,x) \mapsto \alpha(k) \otimes \pi(x) W_\pi(k))$$

is an irreducible representation of G. The representation $\rho_{\pi,\alpha}$ is completely determined by the parameters $\pi \in \widehat{N}$ and $\alpha \in \widehat{K}_\pi$. All irreducible, unitary representations of G are of this form, and $\rho_{\pi,\alpha} \cong \rho_{\pi',\alpha'}$ if and only if the pairs (π,α) and (π',α') are related by the K-action.

We say that $\rho = \rho_{\pi,\alpha}$ has *Mackey parameters* (π,α). For a coadjoint orbit $\mathcal{O} \subset \mathfrak{n}^*$ with aligned point $\ell \in \mathcal{O}$, the corresponding representation $\pi \in \widehat{N}$ factors through

$$N_{\mathcal{O}} = \exp(\mathfrak{n}/\ker(\ell|_{\mathfrak{z}})).$$

The group $N_{\mathcal{O}}$ is the product of a Heisenberg group H and the (possibly trivial) abelian group $\mathfrak{a}_{\mathcal{O}}$. The inner product $\langle\cdot,\cdot\rangle$ can be used to construct an explicit isomorphism φ from H to the standard Heisenberg group $H_a := V \oplus \mathbb{R}$, where V is a unitary K_π space (see [BR08, Sect. 5.1]). This construction allows us to realize π as the standard representation of H_V in the Fock space $\mathcal{F}_V$ on V, and thus realize W_π as the restriction to K_π of the standard representation of $U(V)$ on $\mathcal{F}_V$.

Now we specialize to the case of nilpotent Gelfand pairs satisfying the conditions in Definitions 2 and 3.

As in Section 1, we fix $A \in \mathfrak{z}$ to be a unit base point. For any $\ell = (b,B) \in \mathfrak{n}^*$ with $B \neq 0$, we have $B = \lambda A$ with $\lambda > 0$. The form $(v,w) \mapsto \langle [v,w], A\rangle$ is non-degenerate on $\mathcal{V}$, and hence the orbit through ℓ is $\mathcal{O} = \mathcal{V} \oplus \lambda A$ with aligned point $(0,\lambda A)$.

For $\ell = (b,0)$, the coadjoint orbit through ℓ is a single point. We conclude that we have two types of coadjoint orbits:

- *Type* I *orbits*: When the parameter $\lambda > 0$, we have an aligned point of the form $\ell = (0,\lambda A)$. We call the corresponding coadjoint orbits *type* I *orbits* and the corresponding representations $\pi_\lambda \in \widehat{N}$ *type* I *representations*. Since these orbits depend only on the parameter $\lambda \in \mathbb{R}^+$, we denote them $\mathcal{O}_{\lambda A}$.
- *Type* II *orbits*: When $\lambda = 0$, the corresponding coadjoint orbits contain only the aligned point $\ell = (b,0)$, where $b \in \mathcal{V}$. We call such coadjoint orbits *type* II *orbits* and the corresponding representations $\chi_b \in \widehat{N}$ *type* II *representations*. Since these orbits depend only on the parameter $b \in \mathcal{V}$, we denote them $\mathcal{O}_b$.

Consider a type I coadjoint orbit $\mathcal{O}_{\lambda A}$ with aligned point $\ell = (0,\lambda A)$ and corresponding type I representation $\pi_\lambda \in \widehat{N}$. The coadjoint orbit is of the form

$$\mathcal{O}_{\lambda A} = \mathcal{V} \oplus \lambda A.$$

The representation π_λ has codimension 1 kernel in $\mathfrak{z}$, and factors through

$$N_{\mathcal{O}_{\lambda A}} = \exp(\mathfrak{n}/\ker(\ell|_{\mathfrak{z}})).$$

where $N_{\mathcal{O}_{\lambda A}} = H_A := \mathcal{V} \oplus \mathbb{R}A$ is a Heisenberg group. On H_A, our representation is a type I representation ρ_λ of the standard Heisenberg group, which can be realized on Fock space (see Section 1 for a construction of such representations). We can make this explicit by describing the map φ between H_A and the standard Heisenberg group, which we will use to relate our nilpotent Gelfand pairs to the established results in the Heisenberg setting.

The stabilizer K_{π_λ} of π_λ in K is equal to the stabilizer K_A of A. The skew-symmetric form $(v, w) \mapsto \langle [v, w], A\rangle$ defines a linear transformation $J_z : \mathcal{V} \to \mathcal{V}$ by mapping $v \in \mathcal{V}$ to the element $J_A v \in \mathcal{V}$ defined by

$$\langle J_A v, w\rangle = \langle A, [v, w]\rangle.$$

Since the form $(\cdot\,,\cdot)$ is non-degenerate, J_A is invertible. Since J_A is also skew-symmetric, we can decompose $\mathcal{V} = \sum \mathcal{V}_\mu$, where J_A on $\mathcal{V}_\mu$ is of the form μJ where $J = \begin{pmatrix} 0 & I_m \\ -I_m & 0 \end{pmatrix}$ and I_m is the $m \times m$ identity matrix for some $m \in \mathbb{N}$. We define a K_A-equivariant group isomorphism

$$\varphi : H_A = \mathcal{V} \oplus \mathbb{R}A \to H_{\mathcal{V}},$$

where $H_{\mathcal{V}} = \mathcal{V} \oplus \mathbb{R}$ is as in Section 1, in the following way. For $v = \sum v_\mu \in \mathcal{V}$, let

$$\varphi(v) = \sum \frac{1}{\sqrt{\mu}} v_\mu,$$

and for $(v, tA) \in H_A$, let

$$\varphi(v, tA) = (\varphi(v), t).$$

This gives us a the precise relationship between type I representations $\pi_\lambda \in \widehat{N}$ and type I representations $\rho_\lambda \in \widehat{H_{\mathcal{V}}}$:

$$\pi_\lambda = \rho_\lambda \circ \varphi.$$

Type II orbits are of the form

$$\mathcal{O}_b = \{(b, 0)\}$$

for $b \in \mathcal{V}$. They correspond to 1-dimensional representations

$$\psi_b(v, z) = e^{i\langle v, b\rangle}$$

for $v \in \mathcal{V}$ and $z \in \mathfrak{z}$. The stabilizer K_{χ_b} of a type II representation χ_b is the stabilizer K_b of b in K, and the representation W_{χ_b} is the trivial one-dimensional representation 1_{K_b} of K_b, so the second Mackey parameter for type II representations is trivial.

From this discussion, we have the following corollary to Theorem 9.

Corollary 10. *Suppose that the nilpotent Gelfand pair (K,N) satisfies the conditions in Definitions 2 and 3. Fix $A \in \mathfrak{z}$ with norm 1. Then each representation $\rho \in \widehat{G}$ is unitarily equivalent to one of the form $\rho_{\pi_\lambda,\alpha}$ for $\lambda \in \mathbb{R}^+$ and $\alpha \in \widehat{K}_A$, or one of the form $\rho_{\chi_b} = \mathrm{ind}_{K_b \ltimes N}^{K\ltimes N}(1 \otimes \chi_b)$.*

We complete this section with a parametrization of the Gelfand space and a description of the eigenvalue model of the Gelfand space. By the discussion above, type I representations π_λ of N are parameterized by $\lambda > 0$ and they factor through a unique type I representation ρ_λ of $H_{\mathcal{V}}$. Therefore, the corresponding type I spherical functions in $\Delta(K,N)$ are parameterized by pairs $(\lambda, \mathbf{m})$, where $\lambda > 0$ and $\mathbf{m} = (m_1, \dots m_r)$, where $\alpha_{\mathbf{m}} = m_1\alpha_1 + \cdots + m_r\alpha_r$ is a highest weight of an irreducible subrepresentation of K_A on $\mathbb{C}[\mathcal{V}]$, as in Section 1. We denote the type I spherical function in $\Delta(K,N)$ corresponding to the parameters $(\lambda,\mathbf{m})$ by $\psi_{\lambda,\mathbf{m}}$. From a type II representation χ_b, one constructs a spherical function ψ_b by

$$\psi_b(v,z) := \int_K e^{i\langle b, k\cdot v\rangle} dk.$$

Thus, we have the following parametrization of the Gelfand space.

$$\Delta(K,N) \leftrightarrow \{(\lambda,\mathbf{m}) \mid \lambda > 0, \mathbf{m} \in (\mathbb{Z}_{\geq 0})^r\} \cup (\mathcal{V}/K).$$

The eigenvalue model mentioned in Section 1 gives us a useful geometric model of $\Delta(K,N)$.

Theorem 11 ([FR07]). *There is a homeomorphism*

$$\Phi : \Delta(K,N) \to \mathcal{E}(K,N)$$

given by $\Phi(\psi) = (\widehat{D}_0(\psi), \dots, \widehat{D}_r(\psi))$, where D_i are invariant differential operators obtained from a generating set $\{p_0, \dots, p_r\}$ of K-invariant polynomials on $\mathfrak{n}$.[5]

Without loss of generality, we may choose this generating set $\{p_0, \dots, p_r\}$ to consist of homogeneous polynomials. (Indeed, because the K-action is linear, it will preserve homogeneous terms of each degree, so each homogeneous term is itself an invariant polynomial.) In addition, we choose our enumeration so that the first two polynomials are the invariants $p_0(v,Y) = |Y|^2$ and $p_1(v,Y) = |v|^2$.

4. Moment map and spherical points

In this section we define moment maps on coadjoint $\mathrm{Ad}^*(N)$-orbits for a general nilpotent Gelfand pair (K,N), following [BR08]. We then specialize to the setting of our class of nilpotent Gelfand pairs and calculate the spherical points that we use to define the map Ψ discussed in Section 1.

[5]We remind the reader that we are identifying N and $\mathfrak{n}$ via the exponential map, so this theorem is equivalent to Theorem 1 stated in Section 1.

Definition 7. Let $\mathcal{O} \subset \mathfrak{n}^*$ be a coadjoint orbit for N, $K_{\mathcal{O}}$ the stabilizer of $\mathcal{O}$ in K and $\mathfrak{k}_{\mathcal{O}}$ its Lie algebra. The *moment map* $\tau_{\mathcal{O}} : \mathcal{O} \to \mathfrak{k}^*$ is defined via

$$\tau_{\mathcal{O}}(\mathrm{Ad}^*(X)\ell_{\mathcal{O}})(Z) = -\tfrac{1}{2}B_{\mathcal{O}}(X, Z \cdot X) = -\tfrac{1}{2}\ell_{\mathcal{O}}[X, Z \cdot X]$$

for $Z \in \mathfrak{k}_{\mathcal{O}}$, $X \in \mathfrak{n}$. Here $\ell_{\mathcal{O}}$ is the unique aligned point in $\mathcal{O}$.

Now we specialize to nilpotent Gelfand pairs (K, N) satisfying Definitions 2 and 3 and identify the spherical points in $\mathcal{A}(K, N)$. We start by analyzing the moment map $\tau_{\mathcal{O}}$ of the preceding definition in more detail for the type I orbit $\mathcal{O}_A$ with aligned point $\ell_A = (0, A)$. Let $\tau_A := \tau_{\mathcal{O}_A}$ be the moment map defined above. Then

$$\ell_A([v, Z \cdot v]) = \langle A, [v, Z \cdot v]\rangle.$$

For $v, w \in \mathcal{V}$,

$$\mathrm{Ad}^*(v)\ell_A(w) = \ell_A(w - [v, w]) = -\langle A, [v, w]\rangle = -\langle J_A v, w\rangle,$$

since $\mathcal{V}$ and $\mathfrak{z}$ are orthogonal with respect to $\langle\cdot\,,\cdot\rangle$. So if we identify $\mathcal{O}_A$ with $\mathcal{V}$ in the sense of equation (5), we have $\mathrm{Ad}^*(v)\ell_A = -J_A v$, and

$$\tau(J_A v)(Z) = \tfrac{1}{2}\langle A, [v, Z \cdot v]\rangle = \tau_A(\mathrm{Ad}^*(v)\ell_A)(Z).$$

Here $\tau : \mathcal{V} \to \mathfrak{k}_A^*$ is the moment map $\tau(v)(Z) = -\frac{1}{2}\langle v, Z \cdot v\rangle$ defined in Section 2. Recall that by equation (4), the relationship between this moment map τ and the moment map $\tau_\lambda : \mathcal{O}_\lambda \to \mathfrak{k}$ on a type I coadjoint orbit $\mathcal{O}_\lambda \subset \mathfrak{h}^*$ of the Heisenberg group is given by $\frac{1}{\lambda}\tau(v) = \tau_\lambda(v, \lambda)$.

The moment maps on all other type I orbits $\mathcal{O}_{\lambda A}$ can be obtained from τ_A by scaling. Indeed, if $\tau_{\lambda A} := \tau_{\mathcal{O}_{\lambda A}}$ is the moment map on a type I orbit $\mathcal{O}_{\lambda A}$ with aligned point $\ell_{\lambda A} = (0, \lambda A)$, then

$$\tau_{\lambda A}(\mathrm{Ad}^*(X)\ell_{\lambda A})(Z) = -\tfrac{1}{2}\langle \lambda A, [v, Z \cdot v]\rangle = \lambda\tau_A(\mathrm{Ad}^*(v)\ell_A)(Z),$$

for $v \in \mathcal{V}$ and $Z \in \mathfrak{k}_A^*$. This gives us the following relationship between moment maps:

$$\tau_{\lambda A}(\mathrm{Ad}^*(v)\ell_{\lambda A}) = \lambda\tau_A(\mathrm{Ad}^*(v)\ell_A) = \lambda\tau(J_A v) = \lambda^2\tau_\lambda(J_A v, \lambda). \tag{6}$$

This relationship allows us to compute type I spherical points of $\tau_{\lambda A}$ using type I spherical points of τ_λ, which we established in Section 2 are $(\sqrt{\lambda}v_{\mathbf{m}}, \lambda) \in \mathcal{O}_\lambda$ for $v_{\mathbf{m}} \in \mathcal{V}$ a spherical point of τ of type $\alpha_{\mathbf{m}}$. Under our association of $\mathfrak{n}$ and $\mathfrak{n}^*$ via $\langle\cdot\,,\cdot\rangle$ (see Section 1), one can compute the coadjoint action on aligned points $(0, \lambda A)$. For $v \in \mathcal{V}$,

$$\mathrm{Ad}^*(v)(0, \lambda A) = (\lambda J_A v, \lambda A).$$

Using this action and (6), we compute

$$\begin{aligned}\tau_{\lambda A}(\sqrt{\lambda}v_{\mathbf{m}}, \lambda A)(Z) &= \tau_{\lambda A}\Big(\mathrm{Ad}^*\Big(\frac{1}{\sqrt{\lambda}}J_A^{-1}v_{\mathbf{m}}\Big)\ell_{\lambda A}\Big)(Z)\\ &= \lambda\tau\Big(\frac{1}{\sqrt{\lambda}}v_{\mathbf{m}}\Big)(Z)\\ &= -\frac{\lambda}{2}\Big\langle\frac{1}{\sqrt{\lambda}}v_{\mathbf{m}}, Z\cdot\frac{1}{\sqrt{\lambda}}v_{\mathbf{m}}\Big\rangle\\ &= -\tfrac{1}{2}\langle v_{\mathbf{m}}, Z\cdot v_{\mathbf{m}}\rangle\\ &= \tau(v_{\mathbf{m}})\\ &= \alpha_{\mathbf{m}}.\end{aligned}$$

This proves the following lemma.

Lemma 12. *Let $\mathcal{O}_{\lambda A} = \mathcal{V}\oplus\lambda A$ be a type* I *orbit in $\mathfrak{n}^*$. The type* I *spherical points contained in $\mathcal{O}_{\lambda A}$ are $(\sqrt{\lambda}v_{\mathbf{m}}, \lambda A)$, where $(\sqrt{\lambda}v_{\mathbf{m}}, \lambda)$ is a spherical point in the associated Heisenberg coadjoint orbit $\mathcal{O}_\lambda \subset \mathfrak{h}^*$.*

Since type II orbits $\mathcal{O}_b$ contain a single point $(b, 0)$ and the moment map $\tau_{\mathcal{O}_b}$: $\mathcal{O}_b \to \mathfrak{k}^*$ is the zero map, the point $(b, 0)$ is a spherical point of $\tau_{\mathcal{O}_b}$.

Next we will relate invariant polynomials on $\mathfrak{n}$ to invariant polynomials on $H_{\mathcal{V}}$. Let p be a K-invariant polynomial on $\mathfrak{n}$. Then we can define a K_A-invariant polynomial p_A on $H_{\mathcal{V}}$ by

$$p_A(\varphi(v), t) = p(v, tA), \tag{7}$$

where $\varphi : H_A \to H_{\mathcal{V}}$ is the map from Section 3.

Let D_{p_A} and D_p denote the corresponding differential operators on $H_{\mathcal{V}}$ and $\mathfrak{n}$, respectively. Recall that for a type I representation $\pi_\lambda \in \widehat{N}$ and associated type I representation $\rho_\lambda \in \widehat{H_{\mathcal{V}}}$, we have the relationship $\pi_\lambda = \rho_\lambda\circ\varphi$ on $H_{\mathcal{V}}$. This implies that

$$\rho_\lambda(D_{p_A}) = \pi_\lambda(D_p).$$

This tells us that

$$\widehat{D}_{p_A}(\phi_{\lambda,\mathbf{m}}) = \widehat{D}_p(\psi_{\lambda,\mathbf{m}}). \tag{8}$$

Here, $\phi_{\lambda,\mathbf{m}}$ is a type I spherical function in $\Delta(K, H_a)$ and $\psi_{\lambda,\mathbf{m}}$ is a type I spherical function in $\Delta(K, N)$. (Note that type I spherical functions for (K, N) have the same parametrization as type I spherical functions for (K, H_V), so we distinguish between the two by using ϕ to refer to functions in $\Delta(K, H_V)$ and ψ for functions in $\Delta(K, N)$.)

Similarly, if ϕ_b is a type II spherical function with corresponding orbit $\mathcal{O}_b = \{(b, 0)\}\subset\mathfrak{n}^*$ and representation $\pi_b\in\widehat{N}$, then we still have the relationship $\pi_b = \rho_b\circ\varphi$, where $\rho_b\in\widehat{H_{\mathcal{V}}}$ is the corresponding type II representation of the Heisenberg group. This implies that, as in the type I case, for any K-invariant polynomial p on $\mathfrak{n}$,

$$\widehat{D}_{p_A}(\phi_{\varphi(b)}) = \widehat{D}_p(\psi_b).$$

Therefore, using Lemma 7,

$$\widehat{D}_p(\psi_b) = \widehat{D}_{p_A}(\phi_{\varphi(b)}) = p_A(i\varphi(b), 0) = p(ib, 0). \tag{9}$$

Now we can explicitly define Ψ.

Definition 8. Define the map $\Psi : \Delta(K,N) \to \mathcal{A}(K,N)$ by

$$\Psi(\psi_{\lambda,\mathbf{m}}) = K \cdot (\sqrt{\lambda} v_{\mathbf{m}}, \lambda A)$$

for type I spherical functions, and

$$\Psi(\psi_b) = K \cdot (b, 0)$$

for type II spherical functions. Here $v_{\mathbf{m}}$ is a spherical point of the moment map $\tau : \mathcal{V} \to \mathfrak{k}^*$ defined by $\tau(v)(Z) = -\frac{1}{2}\langle v, Z \cdot v\rangle$, as described in Section 2.

This is the same map defined in Section 1 (see [BR08, Prop. 5.3]). Our main result is that Ψ is a homeomorphism. The following section is dedicated to the proof of this fact.

5. The orbit model

In this section we establish the orbit model for nilpotent Gelfand pairs which have spherical central orbits and are nondegenerate on $\mathcal{V}$, namely those satisfying the conditions in Definitions 2 and 3. Again, we assume (K,N) is one such Gelfand pair. The goal of this section is to show that Ψ is a homeomorphism. Before starting the proof, we need two more tools.

Lemma 13. *Let p be the K-invariant polynomial on $\mathcal{V}$ given by $p(v) = |v|^2$. Then for a spherical point $v_{\mathbf{m}}$ of $\tau : \mathcal{V} \to \mathfrak{k}^*$, $\widehat{D}_p(\psi_{\lambda,\mathbf{m}}) = -\lambda|v_{\mathbf{m}}|^2 = -2\lambda|\mathbf{m}|$ and $\widehat{D}_p(\psi_b) = -|b|^2$.*

Proof. We have $\pi_\lambda(D_p) = -2\lambda \sum_j v_j \partial/\partial v_j$ acts on $P_{\alpha_{\mathbf{m}}}$ by the degree $|\mathbf{m}|$. Thus $\widehat{D}_p(\phi_{\lambda,\mathbf{m}}) = -2\lambda|\mathbf{m}| = -\lambda|v_{\mathbf{m}}|^2$. □

The following fact from invariant theory is vital.

Theorem 14 ([OV12])**.** *The orbits of a compact linear group acting in a real vector space are separated by the invariant polynomials.*

Now we are ready to prove Theorem 4. From Theorem 11, we know that a sequence $\{\psi(n)\}$ of spherical functions converges in $\Delta(K,N)$ to ψ if and only if the corresponding sequence of eigenvalues $\{\widehat{D}_p(\psi(n))\}$ converges to $\widehat{D}_p(\psi)$ for all K-invariant polynomials p. Therefore, to prove Theorem 4, it is enough to show that a sequence in $\mathcal{A}(K,N)$ converges if and only if the corresponding sequence in $\Delta(K,N)$ or $\mathcal{E}(K,N)$ converges.

Without loss of generality, we can assume that any convergent sequence in $\mathcal{A}(K,N)$ consists of orbits corresponding entirely to type I representations or entirely to type II representations, and any convergent sequence in $\Delta(K,N)$ or $\mathcal{E}(K,N)$ has the same property. We refer to elements of $\mathcal{E}(K,N)$ which correspond to type I (type II) representations as "type I (type II) eigenvalues," and elements of $\mathcal{A}(K,N)$ which correspond type I (type II) representations as "type I (type II) orbits."

In either model, $\mathcal{E}(K,N)$ or $\mathcal{A}(K,N)$, a type II sequence can only converge to another type II element. Indeed, if $\{(D_0(\psi_{b(n)}), \ldots, D_r(\psi_{b(n)}))\}$ is a convergent sequence of type II eigenvalues, then $D_0(\psi_{b(n)}) = p_0(b, 0) = 0$ by equation (9). This implies that the limit of the sequence of eigenvalues must be a type II eigenvalue. Similarly, if $\{K \cdot (b(n), 0)\}$ is a convergent sequence of type II orbits in $\mathcal{A}(K, N)$, then the limit of the sequence of orbits must be a type II orbit since $K \cdot (b(n), 0) \subset \mathcal{V}$. In contrast to this, type I sequences in either model can converge to either a type I element or a type II element. In the following arguments we treat each of these three possibilities separately.

We begin with the type II case. Since we know type II sequences converge to type II elements in either model, we only need to show that these type II elements correspond to each other under the map Ψ. By equation (9), the eigenvalues of a type II spherical function are exactly the values of the invariant polynomials on the spherical point. This implies that for a sequence $\{b(n)\}$ of parameters of type II spherical functions, $\widehat{D}_p(\psi_{b(n)}) \to \widehat{D}_p(\psi_b)$ if and only if $p(b(n)) \to p(b)$ for all invariant polynomials p on $\mathcal{V}$. Since invariant polynomials separate points (Lemma 14), this happens exactly when $K \cdot (b(n), 0) \to K \cdot (b, 0)$.

Next we address convergent type I sequences. Note that a type I sequence in either model cannot be convergent unless the sequence $\{\lambda(n)\} \subset \mathbb{R}_{>0}$ is convergent. Indeed, let $\{K \cdot (\sqrt{\lambda(n)} v_{\mathbf{m}(n)}, \lambda(n)A)\}$ be a convergent sequence of type I orbits. Since the action of K on $\mathfrak{z}$ is unitary, the norm on $\mathfrak{z}$ is an invariant polynomial: it is the polynomial p_0 in our enumeration in Section 3. By Theorem 14, the values of p_0 on the sequence of K-orbits must converge, so $\lambda(n) \to \lambda$ for some $\lambda \geq 0$. Now let $\{(D_0(\psi_{\lambda(n),\mathbf{m}(n)}), \ldots, D_r(\psi_{\lambda(n),\mathbf{m}(n)}))\}$ be a convergent sequence of type I eigenvalues. By the reductions in Section 4 and Example 1, we have eigenvalues $\widehat{D}_0(\psi_{\lambda(n),\mathbf{m}(n)}) = \lambda(n)$. Since Φ is an isomorphism, $\lambda(n) \to \lambda$, for $\lambda \geq 0$. In either model, if the limit point λ is strictly greater than zero, the sequence converges to a type I element. If the limit point $\lambda = 0$, then the sequence converges to a type II element. We now address each of these two cases.

Let $\{K \cdot (\sqrt{\lambda(n)} v_{\mathbf{m}(n)}, \lambda(n)A)\}$ be a sequence of type I orbits that converges to the type I orbit $K \cdot (\sqrt{\lambda} v_{\mathbf{m}}, \lambda A)$, where $\lambda > 0$. By moving to subsequences, we can assume that the sequence $\{(\sqrt{\lambda(n)} v_{\mathbf{m}(n)}, \lambda(n)A)\}$ of spherical points converges, with $\lambda(n) \to \lambda$. By Theorem 14, the convergence of orbits implies that the sequence $\{p_1(\sqrt{\lambda(n)} v_{\mathbf{m}(n)}) = \lambda(n)|\mathbf{m}(n)|\}$ converges to $p_1(\sqrt{\lambda} v_{\mathbf{m}}) = \lambda|\mathbf{m}|$. This implies that $\{\mathbf{m}(n)\}$ is bounded. Since $\{\mathbf{m}(n)\}$ lies on a discrete lattice and $\{\mathbf{m}(n)\}$ is bounded, the sequence $\{\mathbf{m}(n)\}$ must eventually be constant, so we can assume without loss of generality that $\mathbf{m}(n) = \mathbf{m}$ for all n. By equation (8), for any invariant polynomial p on $\mathfrak{n}$, there is an invariant polynomial p_A on $H_{\mathcal{V}}$ such that

$$\widehat{D}_p(\psi_{\lambda(n),\mathbf{m}}) = \widehat{D}_{p_A}(\phi_{\lambda(n),\mathbf{m}}).$$

Now the sequence of K_A-orbits $K_A \cdot (\sqrt{\lambda(n)} v_{\mathbf{m}}, \lambda(n))$ converges to $K_A \cdot (\lambda v_{\mathbf{m}}, \lambda)$ in $\mathfrak{h}_{\mathcal{V}}^*$, so by the corresponding result for the Heisenberg group, we have

$$\widehat{D}_{p_A}(\phi_{\lambda(n),\mathbf{m}}) \to \widehat{D}_{p_A}(\phi_{\lambda,\mathbf{m}}),$$

and therefore

$$\widehat{D}_p(\psi_{\lambda(n),\mathbf{m}}) \to \widehat{D}_p(\psi_{\lambda,\mathbf{m}}).$$

Conversely, suppose that $\{\psi_{\lambda(n),\mathbf{m}(n)}\}$ is a sequence of type I spherical functions which converges to the type I spherical function $\psi_{\lambda,\mathbf{m}}$. Then by Theorem 11, for all invariant polynomials p on $\mathfrak{n}$, $\widehat{D}_p(\psi_{\lambda(n),\mathbf{m}(n)}) \to \widehat{D}_p(\psi_{\lambda,\mathbf{m}})$. As observed above, this implies that $\lambda(n) \to \lambda$. By Lemma 13, the sequence

$$\{\lambda|v_{\mathbf{m}(n)}|^2 = 2\lambda(n)|\mathbf{m}(n)|\}$$

is convergent. Since $\{\mathbf{m}(n)\}$ is a discrete set, this convergence is only possible if $\mathbf{m}(n)$ is eventually constant, so we can assume $\mathbf{m}(n) = \mathbf{m}$. Then, the corresponding sequence of spherical points

$$\{(\sqrt{\lambda(n)}v_{\mathbf{m}}, \lambda(n)A)\}$$

converges to $(\sqrt{\lambda}v_{\mathbf{m}}, \lambda A)$ in $\mathfrak{n}^*$, and so $K\cdot(\sqrt{\lambda(n)}v_{\mathbf{m}}, \lambda(n)A) \to K\cdot(\sqrt{\lambda}v_{\mathbf{m}}, \lambda A)$.

Our final step is to address the case of a type I sequence converging to a type II element. Assume that $\{K\cdot(\sqrt{\lambda(n)}v_{\mathbf{m}(n)}, \lambda(n)A)\}$ is a sequence of type I orbits converging to the type II orbit $K\cdot(b,0)$. By moving to subsequences, we can assume that $\lambda(n) \to 0$ and $\sqrt{\lambda(n)}v_{\mathbf{m}(n)} \to b' \in K\cdot b$.

The map φ defined in Section 3 sends spherical points to spherical points, so $(\varphi(\sqrt{\lambda(n)}v_{\mathbf{m}(n)}), \lambda(n))$ is a spherical point in $\mathfrak{h}_{\mathcal{V}}^*$ for each n, and we have the convergence $(\varphi(\sqrt{\lambda(n)}v_{\mathbf{m}(n)}), \lambda(n)) \to (\varphi(b), 0)$ in $\mathfrak{h}_{\mathcal{V}}^*$. This implies that the corresponding K_A-orbits for the Heisenberg setting converge:

$$K_A\cdot(\varphi(\sqrt{\lambda(n)}v_{\mathbf{m}(n)}), \lambda) \to K_A\cdot(\varphi(b), 0)$$

in $\mathfrak{h}_{\mathcal{V}}^*/K_A$. Let p be a K-invariant polynomial on $\mathfrak{n}$, and let p_A be the corresponding K_A-invariant polynomial on $H_{\mathcal{V}}$. By the theorem for Heisenberg groups ([BR08, Prop. 7.1]), equation (8) and Lemma 7, we have

$$\widehat{D}_p(\psi_{\lambda(n),\mathbf{m}(n)}) = \widehat{D}_{p_A}(\phi_{\lambda(n),\mathbf{m}(n)}) \to \widehat{D}_{p_A}(\phi_{\varphi(b)}) = p_A(\varphi(b),0) = p(b,0) = \widehat{D}_p(\psi_{\varphi(b)}).$$

Conversely, suppose that type I eigenvalues $\{\widehat{D}_p(\psi_{\lambda(n),\mathbf{m}(n)})\}$ converge to the type II eigenvalue $\widehat{D}_p(\psi_b)$ for all K-invariant polynomials p on $\mathfrak{n}$. Then $\lambda(n) \to 0$, and for $p_1(v,Y) = |v|^2$,

$$\widehat{D}_1(\psi_{\lambda(n),\mathbf{m}(n)}) = -\lambda(n)|v_{\mathbf{m}(n)}|^2 = -2\lambda(n)|\mathbf{m}(n)| \to -|b|^2 = \widehat{D}_1(\psi_b)$$

by Lemma 13. Thus the sequences $\{\lambda(n)v_{\mathbf{m}(n)}\}$ and $\{\lambda(n)v_{\mathbf{m}(n)}\}$ are bounded. If necessary, we can go to convergent subsequences.

Let p be a homogeneous, K_A-invariant polynomial on $\mathcal{V}$. Then on $H_{\mathcal{V}}$,

$$\widehat{D}_p(\phi_{\lambda(n),\mathbf{m}(n)}) = (-\lambda(n))^{s_p/2}\widetilde{p}(\mathbf{m}(n)),$$

where $\widetilde{p}(\mathbf{m})$ is a polynomial of degree $s_p/2$. Since the sequence $\{\lambda(n)\mathbf{m}(n)\}$ is bounded, the sequence $\{(\lambda(n))^{s_p/2}\,\mathrm{top}\,\widetilde{p}(\mathbf{m}(n))\}$ will be bounded, and the lower order terms will go to zero. Hence

$$\lim_{n\to\infty}\widehat{D}_p(\phi_{\lambda(n),\mathbf{m}(n)}) = \lim_{n\to\infty}(-\lambda(n))^{s_p/2}\mathrm{top}\,\widetilde{p}(\mathbf{m}(n)) = (-1)^{s_p/2}\lim_{n\to\infty}p(\sqrt{\lambda(n)}v_{\mathbf{m}(n)}).$$

Thus if p is a K-invariant polynomial on $\mathcal{V}$ of degree s_p, then

$$\widehat{D}_p(\psi_{\lambda(n),\mathbf{m}(n)}) = \widehat{D}_{p_A}(\phi_{\lambda(n),\mathbf{m}(n)}) \to (-1)^{s_p/2}p(b,0),$$

and therefore $p(\sqrt{\lambda(n)}v_{\mathbf{m}(n)}) \to p(b,0)$.

If p is a mixed invariant on $\mathcal{V}\oplus\mathfrak{z}$, homogeneous of degree s_p on $\mathcal{V}$ and $z_p \neq 0$ on $\mathfrak{z}$, then

$$p_A(\varphi(v),t) = p(v,tA) = t^{z_p}p(v,A),$$

and so with the K_A-invariant polynomial $q(v) = p(v,A)$, we have

$$\widehat{D}_p(\psi_{\lambda(n),\mathbf{m}(n)}) = \widehat{D}_{p_A}(\phi_{\lambda(n),\mathbf{m}(n)}) = (i\lambda(n))^{z_p}\widehat{D}_q(\phi_{\lambda(n),\mathbf{m}(n)}) \to p(ib,0) = 0.$$

On the other hand,

$$\lim_{n\to\infty}\widehat{D}_q(\phi_{\lambda(n),\mathbf{m}(n)}) = \lim_{n\to\infty} q(\sqrt{\lambda(n)}v_{\mathbf{m}(n)}),$$

and hence

$$\lim_{n\to\infty} p(\sqrt{\lambda(n)}v_{\mathbf{m}(n)},\lambda(n)A) = \lim_{n\to\infty}\lambda(n)^{z_p}q(\sqrt{\lambda(n)}v_{\mathbf{m}(n)}) = 0 = p(b,0).$$

Thus for all K-invariant polynomials p on $\mathfrak{n}$, we have

$$p(\sqrt{\lambda(n)}v_{\mathbf{m}(n)},\lambda(n)A) \to p(b,0),$$

and so every limit point of the sequence $(\sqrt{\lambda(n)}v_{\mathbf{m}(n)},\lambda(n)A)$ is in $K\cdot(b,0)$, and therefore $K\cdot(\sqrt{\lambda(n)}v_{\mathbf{m}(n)},\lambda(n)A) \to K\cdot(b,0)$. This completes the proof of the theorem.

6. Example

In this section we provide a detailed description of the orbit model of the specific nilpotent Gelfand pair (K,N) where $K = \mathrm{U}_2\times\mathrm{SU}_2$ and $N = \mathcal{V}\oplus\mathfrak{z}$, with $\mathcal{V} = \mathbb{C}^2\otimes\mathbb{C}^2$ and $\mathfrak{z} = \mathfrak{su}_2(\mathbb{C})$ is the center. Along the way, we provide explicit calculations of the relevant objects described in greater generality in the previous sections.

We define the bracket on $\mathfrak{n}$ by

$$[(u,A),(v,B)] = uv^* - vu^* - \tfrac{1}{2}\mathrm{tr}(uv^* - vu^*) \in \mathfrak{z}.$$

The action of $k = (k_1,k_2)\in K$ on $x = (u,A)\in\mathfrak{n}$ is given by

$$k\cdot x = (k_1uk_2^*, k_1Ak_1^*),$$

where $u\in\mathcal{V}$ is a 2×2 complex matrix.

If one considers $K = U_2\times T_2$ acting on N as above, then we also have a Gelfand pair which satisfies our hypotheses, but $\mathcal{V}$ is no longer irreducible.

We can identify $\mathcal{V}^*$ with $\mathcal{V}$ via the real inner product $\langle w,v\rangle_{\mathcal{V}} = \mathrm{tr}(wv^*)$. Additionally, we can identify $\mathfrak{z}^*$ with $\mathfrak{z}$ via the real inner product $\langle A,B\rangle_{\mathfrak{z}} =$

$-\frac{1}{2}\Re(\mathrm{tr}(AB))$, where $\Re(\mathrm{tr}(AB))$ denotes the real part of $\mathrm{tr}(AB)$. This allows us to identify $\mathfrak{n}^*$ with $\mathfrak{n}$ via the inner product $\langle (w, A), (v, B)\rangle_{\mathfrak{n}} = \langle w, v\rangle_{\mathcal{V}} + \langle A, B\rangle_{\mathfrak{z}}$.

We use the orbit method to construct the representations of N, so we construct the coadjoint orbits of N. Let $\ell \in \mathfrak{n}^*$ be given by pairing with the element $(w, A) \in \mathfrak{n}$. If $X = (u, B) \in \mathfrak{n}$ and $Y = (v, C) \in \mathfrak{n}$, we have the action

$$\begin{aligned} \mathrm{Ad}^*(X)\ell(Y) &= \ell(Y - [X, Y]) \\ &= \langle v, w\rangle_{\mathcal{V}} + \langle A, C\rangle_{\mathfrak{z}} + \langle A, [u, v]\rangle_{\mathfrak{z}}. \end{aligned}$$

One readily computes that $\langle A, [u, v]\rangle = -\langle Au, v\rangle$ where Au is the usual matrix multiplication. In particular,

$$\mathrm{Ad}^*(X)\ell(Y) = \langle (w + Au, A), Y\rangle.$$

Thus, we see that our representations of N are broken into the type I and type II orbits described in Section 3, according to $A \neq 0$ or $A = 0$.

Since each matrix in $\mathfrak{su}_2(\mathbb{C})$ can be unitarily diagonalized, each non-zero K-orbit in $\mathfrak{z}$ has a representative of the form

$$\begin{pmatrix} \lambda i & 0 \\ 0 & -\lambda i \end{pmatrix}$$

for $\lambda \in \mathbb{R}^+$. Let $A = \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix}$ be the fixed unit base point in $\mathfrak{z}$, and consider the type I representation $\pi = \pi_{(0,A)}$. In this case, we see that the stabilizer K_π of the isomorphism class of π in K is

$$K_\pi = (\mathrm{U}_1 \times \mathrm{U}_1) \times \mathrm{SU}_2.$$

As in Section 1, we realize π in Fock space $P(\mathcal{V})$. Regarding $x \in \mathcal{V}$ as a 2×2 complex matrix, the elements of $P(\mathcal{V})$ are holomorphic polynomials in the coordinates x_{11}, x_{12}, x_{21} and x_{22}. As noted in [HU91], under the action of $\mathrm{U}(2) \times \mathrm{U}(2)$, the space $\mathbb{C}[\mathcal{V}]$ has highest weight vectors generated by $g_1(x) = x_{11}$ and $g_2(x) = \det(x)$. Since the action of K on $\mathbb{C}[\mathcal{V}]$ is multiplicity-free, we have the decomposition

$$\mathbb{C}[\mathcal{V}] = \sum_\alpha V_\alpha \otimes V_\alpha^*,$$

where the highest weight vector of $V_\alpha \otimes V_\alpha^*$ is a monomial in g_1 and g_2, and α corresponds to a two-rowed Young diagram.

Restricting to K_π amounts to restricting from $\mathrm{U}(2) \times \mathrm{U}(2)$ to $K_\pi = (\mathrm{U}(1) \times \mathrm{U}(1)) \times \mathrm{SU}(2)$. The left V_α's split into one-dimensional subspaces, and the highest weight vectors of the representation of K_π on $\mathbb{C}[\mathcal{V}]$ are monomials in $h_1(x) = x_{11}$, $h_2(x) = x_{21}$, and $h_3(x) = \det(x)$. Then the above decomposition becomes

$$\mathbb{C}[\mathcal{V}] = \bigoplus_{\mathbf{m}} V_{\mathbf{m}}$$

where $\mathbf{m} = (m_1, m_2, m_3) \in \Lambda \simeq (\mathbb{Z}_{\geq 0})^3$ is the monoid of all appearing highest weights, and $V_{\mathbf{m}}$ is the irreducible subspace with highest weight $h^{\mathbf{m}} = h_1^{m_1} h_2^{m_2} h_3^{m_3}$.

One computes the corresponding highest weights α_1, α_2, and α_3 explicitly by computing the action of K_π on $\{h_1, h_2, h_3\}$. Let $k = (t^{-1}, s) \in K_\pi$ with $t^{-1} = (t_1, t_2)$ an element of the torus of U(2) and $s = (s_1, s_1^{-1})$ an element of the torus of SU(2), with $X = (x_{ij}) \in \mathcal{V}$. Then

$$k \cdot h_1(X) = h_1(tXs^*) = t_1 \overline{s}_1 h_1(X),$$

and hence $\alpha_1 = (1, 0, 1)$. Similar computations show us that $\alpha_2 = (0, 1, 1)$ and $\alpha_3 = (1, 1, 0)$. Thus, if $h = h_1^{m_1} h_2^{m_2} h_3^{m_3}$ is the highest weight vector corresponding to the irreducible representation V of K_π, then the highest weight of V is $(m_1 + m_3, m_2 + m_3, m_1 + m_2)$. Note that since type II representations $\pi \in \widehat{N}$ are one-dimensional characters, the action of K_π is trivial, and the representation of K_π is the trivial one-dimensional representation. Therefore, the only highest weight vector in a type II representation space is the (unique) unit vector with corresponding highest weight 0.

We use the generators for K-invariant polynomials found in [FRY12]:

$$\begin{aligned}
&p_1(v, z) = |z|^2, \\
&p_2(v, z) = |v|^2, \\
&p_3(v, z) = |\det(v)|^2 = \det(v_{ij}) \det(\overline{v_{ij}}), \text{ and} \\
&p_4(v, z) = i\mathrm{tr}(v^* z v),
\end{aligned}$$

with corresponding differential operators D_1, D_2, D_3, D_4 in $\mathbb{D}_K(N)$.

Our choice of quantization produces the following operators on Fock space:

$$\begin{aligned}
&\pi_\lambda(D_1) = \lambda^2, \\
&\pi_\lambda(D_2) = \sum v_{ij} \frac{\partial}{\partial v_{ij}}, \\
&\pi_\lambda(D_3) = \det(v_{ij}) \left[\frac{\partial}{\partial v_{11}} \frac{\partial}{\partial v_{22}} - \frac{\partial}{\partial v_{12}} \frac{\partial}{\partial v_{21}} \right], \\
&\pi_\lambda(D_4) = \lambda \left[v_{11} \frac{\partial}{\partial v_{11}} - v_{21} \frac{\partial}{\partial v_{21}} + v_{12} \frac{\partial}{\partial v_{12}} - v_{22} \frac{\partial}{\partial v_{22}} \right].
\end{aligned}$$

We then compute the eigenvalues of all type I spherical functions by applying these operators to the highest weight vectors $h_1^{m_1} h_2^{m_2} h_3^{m_3}$, obtaining:

$$\begin{aligned}
&\widehat{D}_1(\psi_{\pi,\alpha}) = \lambda^2, \\
&\widehat{D}_2(\psi_{\pi,\alpha}) = \lambda(m_1 + m_2 + 2m_3), \\
&\widehat{D}_3(\psi_{\pi,\alpha}) = \lambda^2 m_3(1 + m_1 + m_2 + m_3), \\
&\widehat{D}_4(\psi_{\pi,\alpha}) = \lambda^2(m_1 - m_2).
\end{aligned}$$

Similarly, the type II eigenvalues can be directly computed as

$$\begin{aligned}
\widehat{D}_1(\chi_b) &= 0, \\
\widehat{D}_2(\chi_b) &= |b|^2, \\
\widehat{D}_3(\chi_b) &= |\det(b)|^2, \\
\widehat{D}_4(\chi_b) &= 0.
\end{aligned}$$

Recall from Section 5 that we construct the orbit model by using a moment map to identify the spherical points in $\mathcal{A}(K, N)$. We define the moment map $\tau_{\mathcal{O}} : \mathcal{O} \to \mathfrak{k}^*_{\mathcal{O}}$ on a coadjoint orbit $\mathcal{O} \subset \mathfrak{n}^*$ with aligned point $\ell = (0, A)$ as in Section 4. Let

$$u = \begin{bmatrix} u_{11} & u_{12} \\ u_{21} & u_{22} \end{bmatrix} \in \mathcal{V}, \gamma = \begin{bmatrix} \gamma_1 & 0 \\ 0 & \gamma_2 \end{bmatrix} \in \mathfrak{u}(1) \times \mathfrak{u}(1), \delta = \begin{bmatrix} \delta_{11} & \delta_{12} \\ -\bar{\delta}_{12} & -\delta_{11} \end{bmatrix} \in \mathfrak{su}_2(\mathbb{C}).$$

Note that $\tau(u)$ must be diagonal if u is to map to a weight of K. We directly compute that

$$\begin{aligned}
\tau(u) = \gamma_1(|u_{11}|^2|u_{12}|^2) &+ \gamma_2(|u_{21}|^2 + |u_{22}|^2) + \bar{\delta}_{11}(|u_{11}|^2 - |u_{12}|^2 + |u_{21}|^2 - |u_{22}|^2) \\
&- \delta_{12}(u_{11}\bar{u}_{12} + u_{21}\bar{u}_{22}) + \bar{\delta}_{12}(\bar{u}_{11}u_{12} + \bar{u}_{21}u_{22})
\end{aligned}$$

In particular, $\tau(u)$ is diagonal if and only if $u_{11}\bar{u}_{12} = -u_{21}\bar{u}_{22}$ or equivalently, when u has orthogonal columns. Comparing coefficients, the integrality conditions imply we must have

$$\begin{aligned}
|u_{11}|^2 + |u_{12}|^2 &= m_1 + m_3, \\
|u_{21}|^2 + |u_{22}|^2 &= m_2 + m_3, \\
|u_{11}|^2 + |u_{21}|^2 - |u_{12}|^2 - |u_{22}|^2 &= m_1 + m_2.
\end{aligned}$$

Evaluating our invariant polynomials on type I spherical points $(\sqrt{\lambda}u, \lambda A)$ gives

$$\begin{aligned}
p_1(\sqrt{\lambda}u, \lambda A) &= \lambda, \\
p_2(\sqrt{\lambda}u, \lambda A) &= \lambda(m_1 + m_2 + 2m_3), \\
p_3(\sqrt{\lambda}u, \lambda A) &= \lambda^2 m_3(m_1 + m_2 + m_3), \\
p_4(\sqrt{\lambda}u, \lambda A) &= \lambda^2(m_1 - m_2).
\end{aligned}$$

If $(w, 0)$ is a spherical point corresponding to a type II representation, one can directly compute that

$$\begin{aligned}
p_1(w, 0) &= 0, \\
p_2(w, 0) &= |w|^2, \\
p_3(w, 0) &= |\det(w)|^2, \\
p_4(w, 0) &= 0.
\end{aligned}$$

This example illustrates the behavior of type I spherical points as $\lambda \to 0$, with $\lambda\mathbf{m}$ bounded. For example, the lower order term (in $\mathbf{m}$) of the eigenvalues $\widehat{D}_3(\psi_{\pi,\alpha}) = \lambda^2 m_3(1 + m_1 + m_2 + m_3)$ go to zero, thus approaching the value of the invariant $p_3(\sqrt{\lambda}u, \lambda A) = \lambda^2 m_3(m_1 + m_2 + m_3)$ on spherical points. In addition, the eigenvalues $\widehat{D}_4(\psi_{\pi,\alpha}) = \lambda^2(m_1 - m_2)$ for the mixed invariant p_4 go to zero.

References

[BJLR97] C. Benson, J. Jenkins, R. Lipsman, G. Ratcliff, *A geometric criterion for Gelfand pairs associated with the Heisenberg group*, Pacific J. Math. **178** (1997), no. 1, 1–36.

[BJR92] C. Benson, J. Jenkins, G. Ratcliff, *Bounded K-spherical functions on Heisenberg groups*, J. Funct. Analysis **105** (1992), no. 2, 409–443.

[BJR90] C. Benson, J. Jenkins, G. Ratcliff, *On Gelfand pairs associated with solvable Lie groups*, Trans. Amer. Math. Soc. **321** (1990), no. 1, 85–116.

[BJR99] C. Benson, J. Jenkins, G. Ratcliff, *The orbit method and Gelfand pairs associated with nilpotent Lie groups*, J Geom. Analysis **9** (1999), no. 4, 569–582.

[BJRW96] C. Benson, J. Jenkins, G. Ratcliff, T. Worku, *Spectra for Gelfand pairs associated with the Heisenberg group*, Colloq. Math. **71** (1996), no. 2, 305–328.

[BR98] C. Benson, G. Ratcliff, *Combinatorics and spherical functions on the Heisenberg group*, Represent. Theory **2** (1998), 79–105 (electronic).

[BR13] C. Benson, G. Ratcliff, *Geometric models for the spectra of certain Gelfand pairs associated with Heisenberg groups*, Annali di Mat. Pura Appl. **192** (2013), no. 4, 719–740.

[BR04] C. Benson, G. Ratcliff, *On multiplicity-free actions*, in: *Representations of Real and p-Adic Groups*, Lecture Notes Series, Institute for Mathematical Sciences, Vol. 2, National University of Singapore, Singapore Univ. Press, Singapore, 2004, pp. 221–304.

[BR08] C. Benson, G. Ratcliff, *The space of bounded spherical functions on the free 2-step nilpotent Lie group*, Transform. Groups **13** (2008), no. 2, 243–281.

[Bro73] I. D. Brown, *Dual topology of a nilpotent lie group*, Ann. Sci. de l'École Norm. Sup. **6** (1973), no. 3, 407–411.

[FR07] F. Ferrari Ruffino, *The topology of the spectrum for Gelfand pairs on Lie groups*, Boll. dell'Unione Mat. Ital. **10** (2007), 569–579.

[FRY12] V. Fischer, F. Ricci, O. Yakimova, *Nilpotent Gelfand pairs and spherical transforms of Schwartz functions* I: *Rank-one actions on the centre*, Math. Z. **271** (2012), no. 1, 221–255.

[FRY18] V. Fischer, F. Ricci, O. Yakimova, *Nilpotent Gelfand pairs and Schwartz extensions of spherical transforms via quotient pairs*, J. Funct. Analysis **274** (2018), no. 4, 1076–1128.

[GV88] R. Gangolli, V. C. Varadarajan, *Harmonic Analysis of Spherical Functions on Real Reductive Groups*, Ergebnisse der Mathematik und ihrer Grenzgebiete, Bd. 101, Springer-Verlag, Berlin, 1988.

[Hel84] S. Helgason, *Groups and Geometric Analysis. Integral Geometry, Invariant Differential Operators, and Spherical Functions*, Pure and Applied Mathematics, Vol. 113, Academic Press, Orlando, FL, 1984.

[HU91] R. Howe, T. Umeda, *The Capelli identity, the double commutant theorem, and multiplicity-free actions*, Math. Annalen **290** (1991), no. 1, 565–619.

[Kir04] A. A. Kirillov, *Lectures on the Orbit Method*, Graduate Studies in Mathematics, Vol. 64, American Mathematical Society, Providence, RI, 2004.

[LL94] H. Leptin, J. Ludwig, *Unitary Representation Theory of Exponential Lie Groups*, De Gruyter Expositions in Mathematics, Vol. 18, Walter de Gruyter and Co., Berlin, 1994.

[Lip80] R. Lipsman, *Orbit theory and harmonic analysis on Lie groups with co-compact nilradical*, J. Math. Pures et Appl. **59** (1980), 337–374.

[Lip82] R. Lipsman, *Orbit theory and representations of Lie groups with co-compact radical*, J. Math. Pures et Appl. **61** (1982), 17–39.

[OV12] A. Onishchik, E. B. Vinberg, *Lie Groups and Algebraic Groups*, Springer Science and Business Media, Berlin, 2012.

[Puk78] L. Pukanszky, *Unitary representations on Lie groups with co-compact radical and applications*, Trans, Amer. Math. Society **236** (1978), 1–49.

[Ric85] F. Ricci, *Commutative algebras of invariant functions on groups of Heisenberg type*, J. London Math. Soc. **32** (1985), no. 2, 265–271.

[Vin03] Э. Б. Винберг, *Коммутативные однородные пространства гейзенбергова типа*, Труды ММО **64** (2003), 54–89. Engl. transl.: E. B. Vinberg, *Communative homogeneous spaces of Heisenberg type*, Trans. Moscow Math. Soc. **64** (2003), 47–80.

[Wol06] J. Wolf, *Spherical functions on Euclidean space*, J. Funct. Analysis **239** (2006), 127–136.

[Yak05] O. Yakimova, *Gelfand Pairs*, Dissertation, Rheinischen Friedrich-Wilhelms-Universität Bonn, 2004, Bonner Mathematische Schriften **374** (2005).

[Yak06] O. Yakimova, *Principal Gelfand pairs*, Transform. Groups **11** (2006), no. 2, 305–335.