Moduli of flat connections on smooth varieties

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Abstract

This paper is a companion to our paper [Poisson geometry of the moduli of local systems on smooth varieties, Publ. RIMS 57 (2021), no. 3–4, 959—991]. We study the moduli functor of flat bundles on the smooth, possibly non-proper, algebraic variety X over a field of characteristic zero. For this we introduce the notion of a *formal boundary* of X, denoted by $\widehat{\partial}X$, which is a formal analog of the boundary at ∞ of the Betti topological space associated with X. We explain how to construct two derived moduli stacks $\mathsf{Vect}^\nabla(X)$ and $\mathsf{Vect}^\nabla\big(\widehat{\partial}X\big)$ of flat bundles on X and on $\widehat{\partial}X$, respectively, as well as a restriction map $\mathsf{R}\colon \mathsf{Vect}^\nabla(X) \to \mathsf{Vect}^\nabla\big(\widehat{\partial}X\big)$. This work contains two main results. First we prove that the morphism R comes equipped with a canonical shifted Lagrangian structure. This result can be understood as the de Rham analog of the existence of Poisson structures on moduli of local systems on X. As a second statement, we prove that the geometric fibers of R are representable by *quasi-algebraic spaces*, a slight weakening of the notion of algebraic spaces.

Introduction

This work is a sequel of [PT21], in which we studied moduli of local systems on a topological space underlying a smooth non-proper complex algebraic variety X. One of the main results of [PT21] asserts that this moduli stack is a derived Artin stack endowed with a natural shifted Poisson structure whose symplectic leaves can be studied by fixing monodromies of local systems at infinity.

In this paper, we begin the study of the de Rham analog of the results of [PT21]. The content of the present work can be summarized in the statement that for a smooth variety X over a field k of characteristic zero, the derived moduli stack $\mathsf{Vect}^\nabla(X)$ of flat connections on X carries a canonical shifted Poisson structure. However, this statement needs to be qualified as $\mathsf{Vect}^\nabla(X)$ is not representable for non-proper X. Thus in order to state and prove the existence of Poisson structures in this context, we have to overcome a number of technical difficulties.

When restricting to regular connections, it is possible to approach this representability question by working on some good compactification, as is done in [Nit99]. What we propose in this

paper is slightly different, as we do not make any regularity assumptions and also propose an intrinsic construction, independent of any choice of compactification. A key ingredient for this is the notion of a *formal boundary* $\widehat{\partial}X$ of a smooth variety X. It is difficult to make sense of the formal boundary directly as a geometric object, but one can make sense of it as a non-commutative space. In particular, it is possible to define ∞-categories of vector bundles and flat bundles on $\widehat{\partial}X$. The putative object $\widehat{\partial}X$ is morally the punctured formal completion of $\mathfrak{X}$ along D, for $\mathfrak{X}$ a smooth compactification of X with $D = \mathfrak{X} - X$ a normal crossings divisor. Rigid analytic and formal versions of $\widehat{\partial}X$ have been considered previously in [BT13, Efi17, HPV16]. The novelty here is the systematic study of the corresponding de Rham theory: vector bundles with connections on $\widehat{\partial}X$ and their de Rham complexes. Some glimpses of such theory already exist the literature. In particular, when X is a curve, the de Rham theory of $\widehat{\partial}X$ was developed and analyzed by S. Raskin in [Ras15] in the context of the local geometric Langlands correspondence. In this paper, we deal with the case of a higher-dimensional X. We construct derived stacks $\mathsf{Vect}^\nabla(X)$ and $\mathsf{Vect}^\nabla(\widehat{\partial}X)$ of flat bundles on X and $\widehat{\partial}X$, together with a restriction map $\mathsf{R}\colon \mathsf{Vect}^\nabla(X) \to \mathsf{Vect}^\nabla(\widehat{\partial}X)$. We study the infinitesimal properties of these derived stacks and show, in particular, that they are formally representable at any field-valued point. The formal representability allows us to define the notion of shifted symplectic and shifted Lagrangian structures on these derived stacks, even though they are not representable. Our first main result then is the following theorem.

THEOREM A. *There exists a canonical $(3 - 2d)$-shifted Lagrangian structure on the restriction map*

$$\mathsf{R}\colon \mathsf{Vect}^\nabla(X) \longrightarrow \mathsf{Vect}^\nabla(\widehat{\partial}X).$$

At the linear level of tangent complexes, the above theorem is an incarnation of Poincaré duality in de Rham cohomology and de Rham cohomology with compact supports. The existence of the Lagrangian structure globally is itself a version of Poincaré duality relative to various derived base schemes, together with the general existence result [Toë18, Theorem 3.7]. Also, Theorem A immediately implies the existence of a $(2-2d)$-shifted Poisson structure on $\mathsf{Vect}^\nabla(X)$ thanks to the comparison result [MS18, Theorem 4.22].

Our second main result in this work is the following representability result. We fix a flat bundle at infinity $V_\infty \in \mathsf{Vect}^\nabla(\widehat{\partial}X)(k)$ and consider the fiber of R at V_∞, denoted by $\mathsf{Vect}^\nabla_{V_\infty}(X)$. Our original goal was to prove that $\mathsf{Vect}^\nabla_{V_\infty}(X)$ is representable by a derived Artin stack (even by an algebraic space if no components of X are proper) locally of finite presentation over k. Though we have not been able to prove this last statement, we prove the following weaker version.

THEOREM B. *The derived stack $\mathsf{Vect}^\nabla_{V_\infty}(X)$ is a derived quasi-algebraic space locally of finite presentation in the sense of Definition A.2.*

Derived quasi-algebraic spaces are almost algebraic spaces—they satisfy all conditions in the Artin–Lurie representability criterion (see [Lur18] and the appendix) except that they may not be locally of finite presentation as a functor. Quasi-algebraic spaces only satisfy local finite presentability generically, and the result is that these derived stacks only have a smooth atlas generically, that is, have a smooth atlas whose image is Zariski dense in an appropriate sense.

Notation and conventions

k – a field of characteristic zero

G – an affine reductive group over k

X – a smooth variety over k

$\widehat{\partial} X$ – the formal boundary of X

$\mathsf{Vect}^{\nabla}(X)$ – the derived stack of flat connections on X

$\mathsf{Vect}^{\nabla}(\widehat{\partial} X)$ – the derived stack of flat connections on $\widehat{\partial} X$

$\mathsf{R}\colon \mathsf{Vect}^{\nabla}(X) \to \mathsf{Vect}^{\nabla}(\widehat{\partial} X)$ – the restriction to the boundary map

$\mathfrak{X}$ – a smooth compactification of X with a normal crossings divisor boundary $D = \mathfrak{X} - X$

$R_G(\pi_1(X, x))$ – the G-character scheme parametrizing representations of $\pi_1(X, x)$ in G

$\mathrm{Loc}_G(X)$ – the derived moduli stack of G-local systems on X

$\mathbb{T}$ – the ∞-category of spaces or, equivalently, the ∞-category of simplicial sets

$t_0 \mathrm{Loc}_G(X)$ – the underived truncation of $\mathrm{Loc}_G(X)$

$\mathsf{commalg}_k$ – the category of commutative k-algebras

C_{λ} – the conjugacy class of a group element $\lambda \in G$

$\mathrm{Loc}_G(X, \lambda)$ – the derived moduli stack of G local systems on X with monodromy at infinity in C_{λ} (assumes that X admits a smooth compactification with a smooth connected divisior at infinity)

$G * G$ – the derived commuting variety of G

$\mathcal{D}_X$ – the sheaf of rings of differential operators on a smooth variety X

B – a connective commutative differential graded algebra (connective cdga for short)

$\mathsf{D}_{\mathrm{qcoh}}(\mathcal{D}_{X,B})$ – the dg-category of all $\mathcal{D}_X \otimes_k B$-modules whose underlying $\mathcal{O}_X \otimes_k B$-modules are quasi-coherent on $X \times \mathrm{Spec}\, B$

a symmetric monoidal dg-category – an E_∞-algebra object inside the symmetric monoidal ∞-category of locally presentable dg-categories (see [Toë12, Section 2])

$k - \mathbf{dg}$ – the ∞-category of complexes of k-modules

$k - \mathbf{dg}_\epsilon^{\mathrm{gr}}$ – the ∞-category of graded mixed k-modules

$\mathbf{DR}_X - \mathbf{dg}_\epsilon^{\mathrm{gr}}$ – dg-category of cofibrant graded mixed $\mathbf{DR}_X$-dg-modules for a smooth affine variety X over k

$\mathcal{H}$ – the group stack of autoequivalences of $B\mathbb{G}_a$

$\mathbb{D}g^{\mathrm{lp}}$ – the derived stack of locally presentable dg-categories

$\mathbb{D}g^{\mathrm{lp}}(B\mathcal{H})$ – the ∞-category of $\mathcal{H}$-equivariant locally presentable dg-categories

$\mathsf{j}_n\colon \mathfrak{X}_{(n)} := \mathrm{Spec}\left(\mathcal{O}_{\mathfrak{X}}/I_D^n\right) \to \mathfrak{X}$ – the $(n-1)$st infinitesimal thickening of D inside $\mathfrak{X}$

$\widehat{\mathsf{j}}\colon \widehat{\mathfrak{X}} \to \mathfrak{X}$ – the full formal neighborhood of D inside $\mathfrak{X}$

$\mathsf{Perf}(\widehat{\mathfrak{X}})$ – the derived stack of perfect complexes on $\widehat{\mathfrak{X}}$

$\mathsf{Perf}(\widehat{\partial} X)$ – the derived stack of perfect complexes on the formal boundary of X

$\mathsf{Perf}^{\mathrm{ex}}(\widehat{\partial} X)$ – the derived stack of extendable perfect complexes on the formal boundary of X

$\mathsf{Perf}^{\nabla}(\widehat{\mathfrak{X}})$ – the derived stack of perfect complexes of flat bundles on $\widehat{\mathfrak{X}}$

$\mathsf{Perf}^{\nabla}(\widehat{\partial} X)$ – the derived stack of perfect complexes of flat bundles on the formal boundary of X

$\mathsf{Perf}^{\nabla,\mathrm{ex}}(\widehat{\partial} X)$ – the derived stack of extendable perfect complexes of flat bundles on the formal boundary of X

1. A brief review of the Betti case

Before we plunge into the technical aspects of formal boundaries and their de Rham theory, it is useful to review the Betti case results from [PT21] which motivate the present discussion.

A precursor of this whole line of investigation is the classical story about Poisson structures on the moduli space of representations of fundamental groups of topological surfaces: if X is a compact oriented topological surface and G is a complex reductive group, then it is well known (see [FR99, GHJW97, Gol06, GR98]) that the smooth part of the moduli space of representations $\rho: \pi_1(X) \to G$ carries a canonical algebraic Poisson structure and that the symplectic leaves of this Poisson structure are moduli spaces of representations ρ whose values at the loops at infinity belong to fixed conjugacy classes in G. In [PT21], we extended this story to higher-dimensional smooth open varieties X, simultaneously refining the moduli problem for representations of the fundamental group in a way which removes the smoothness restriction on the moduli. More precisely, we proved the following.

THEOREM 1.1 ([PT21, Theorem 4.7]). *Fix a field k with $\mathsf{char}\, k = 0$. Let X be a d-dimensional smooth complex algebraic variety, and let G be a reductive algebraic group over k.*

(1) *The derived moduli stack $\mathrm{Loc}_G(X)$ of G-local systems on X has a natural $(2 - 2d)$-shifted Poisson structure.*

(2) *This shifted Poisson structure admits generalized symplectic leaves. Among those are the derived moduli of G-local systems with fixed monodromy at infinity.*

Remark 1.2. (i) When $d = 1$, the Poisson structure in Theorem 1.1(1) specializes to Goldman's Poisson structure on the moduli space of representations $\pi_1(X) \to G$.

(ii) The precise formulation of Theorem 1.1(2) is tricky since for it one needs to understand how to fix local monodromies in the derived setting. To solve this problem in [PT21], we had to deal with a couple of subtle issues:

- The fixing of the local monodromies cannot be seen solely on the underlying underived moduli stack $t_0 \mathrm{Loc}_G(X)$ and involves higher homotopy coherences.
- In higher dimension, an additional constraint called *strictness* has to be imposed on the local monodromies at infinity in order to select a symplectic leaf.

In more details, suppose that X is a finite CW complex and G a reductive group over k. The derived moduli stack $\mathrm{Loc}_G(X)$ of G-local systems (that is, locally constant principal G bundles) is a derived Artin stack locally of finite presentation over k. The truncated underived stack $t_0 \mathrm{Loc}_G(X)$ depends only on the fundamental group of X. It is the moduli stack of representations of $\pi_1(X, x)$ into G; that is,

$$t_0 \mathrm{Loc}_G(X) = [R_G(\pi_1(X, x))/G] \,,$$

where $R_G(\pi_1(X, x))$ is the *character scheme* of X, namely the affine k-scheme representing the functor

$$R_G(\pi_1(X, x)): \quad \mathsf{commalg}_k \longrightarrow \mathsf{Sets} \,,$$
$$A \longmapsto \mathrm{Hom}_{\mathrm{grp}}(\pi_1(X, x), G(A)) \,.$$

In this sense, $\mathrm{Loc}_G(X)$ refines the moduli space of representations of $\pi_1(X, x)$. Note, however (see [PT21, Section 1.2]), that in general the derived structure on $\mathrm{Loc}_G(X)$ depends on the full homotopy type of X, so it captures more information than the moduli of representations.

To explain the content of Theorem 1.1 properly, recall that the *boundary of a topological space* X is the pro-homotopy type $\partial X := \lim_{K \subset X}(X - K)$, where the limit is taken in the ∞-category $\mathbb{T}$ of spaces and over the opposite category of compact subsets $K \subset X$. In general, ∂X can be quite complicated, but if X is the underlying topological space of a smooth d-dimensional complex algebraic variety, then ∂X is equivalent to a constant pro-object in $\mathbb{T}$ which has the homotopy type of a compact oriented topological manifold of dimension $2d - 1$. This implies (see [Cal15] and [PT21, Section 4.1]) that the canonical map $\partial X \to X$ induces a restriction morphism of derived locally finite presentation Artin stacks

$$r \colon \mathrm{Loc}_G(X) \longrightarrow \mathrm{Loc}_G(\partial X)$$

which is equipped with a canonical $(2 - 2d)$-shifted Lagrangian structure with respect to the canonical shifted symplectic structure on $\mathrm{Loc}_G(\partial X)$. Therefore, by [MS18, Theorem 4.22], the map r can be viewed as a $(2 - 2d)$-shifted Poisson structure on $\mathrm{Loc}_G(X)$, which gives part (1) of Theorem 1.1.

For part (2) of Theorem 1.1, the restriction on the local monodromies at infinity has to be made precise in at least two different ways. First, we need to take into account the fact that the boundary of X has higher dimension and the local monodromy around the boundary may get twisted as we move along a connected component of the boundary. Second, we have to make sure that we have good control over the interaction of the local monodromies going around intersecting divisor components in a normal crossings compactification.

To sketch how one deals with the first issue, suppose that we have a smooth compactification $\mathfrak{X}$ of X. To simplify the discussion, assume that $D = \mathfrak{X} - X$ is a smooth connected divisor. Then ∂X has the homotopy type of an oriented circle bundle over D classified by $\alpha = c_1(N_{D/\mathfrak{X}}) \in H^2(D, \mathbb{Z})$. Now given $\lambda \in G$ with centralizer $Z < G$, the group S^1 acts on BZ (via λ) and acts naturally on the conjugation quotient $[G/G]$ so that the 1-shifted Lagrangian structure on the map $BZ \to [G/G]$ is S^1-equivariant. Twisting by α gives a 1-shifted Lagrangian morphism

$$_\alpha\widetilde{BZ} \longrightarrow {}_\alpha\widetilde{[G/G]} \tag{1.1}$$

of locally constant families of derived Artin stacks over D. Passing to global sections gives moduli stacks

$$\mathrm{Loc}_G(\partial X) = \mathsf{Map}\left(\partial X, BG\right) = \Gamma\left(D_i, {}_\alpha\widetilde{[G/G]}\right)$$
$$\mathrm{Loc}_{Z,\alpha}(D) = \Gamma\left(D, {}_\alpha\widetilde{BZ}\right)$$

of G local systems on ∂X and of α-twisted Z-local systems on D, respectively. Furthermore, since D is a compact topological manifold endowed with a canonical orientation, the map (1.1) induces a $(3 - 2d)$-shifted Lagrangian morphism of derived Artin stacks

$$\mathrm{Loc}_{Z,\alpha}(D) \longrightarrow \mathrm{Loc}_G(\partial X).$$

By the Lagrangian intersection theorem of [PTVV13, Section 2.9], the fiber product of derived stacks

$$\mathrm{Loc}_G(X, \lambda) := \mathrm{Loc}_{Z,\alpha}(D) \underset{\mathrm{Loc}_G(\partial X)}{\times} \mathrm{Loc}_G(X)$$

has a canonical $(2 - 2d)$-shifted symplectic structure.

- By construction, $\mathrm{Loc}_G(X, \lambda)$ is the derived stack of G-local systems on X whose local monodromy around D is fixed to be in the conjugacy class C_λ of λ.

- The natural map

$$\mathrm{Loc}_G(X, \lambda) \longrightarrow \mathrm{Loc}_G(X)$$

  thus realizes $\mathrm{Loc}_G(X, \lambda)$ as a generalized symplectic leaf of the $(2 - 2d)$-shifted Poisson structure on $\mathrm{Loc}_G(X)$.

This explains Theorem 1.1(2) in the case when X admits a compactification with a smooth divisor boundary.

To sketch how one deals with the second issue, start again with a smooth compactification $\mathfrak{X}$ of X, but this time assume that the divisor at infinity $D = \mathfrak{X} - X = D_1 \cup D_2$ has two smooth irreducible components meeting transversally at a smooth connected subvariety D_{12}. In this case,

$$\partial X \simeq \partial_1 X \sqcup_{\partial_{12} X} \partial_2 X \,,$$

where $\partial_i X$ is an oriented circle bundle over $D_i^o = D_i - D_{12}$ and $\partial_{12} X$ is an oriented $S^1 \times S^1$-bundle over D_{12}. Note that here each $\partial_i X$ has the homotopy type of an oriented compact manifold of dimension $2d - 1$ with boundary canonically equivalent to $\partial_{12} X$.

The problem we need to solve now is to understand what conditions on a pair of commuting elements $\lambda_1, \lambda_2 \in G$ will guarantee that prescribing the λ_i as the local monodromies around the components D_i will select a symplectic leaf in $\mathrm{Loc}_G(X)$. In [PT21], we found a natural sufficient condition called *strictness*.

DEFINITION 1.3. A pair of commuting elements $(\lambda_1, \lambda_2) \in G \times G$ is called *strict* if the morphism

$$BZ_{12} \longrightarrow [Z_1/Z_1] \times_{[G*G/G]} [Z_2/Z_2]$$

is Lagrangian (for its canonical isotropic structure).

Here Z_i denotes the centralizer subgroup of λ_i, and Z_{12} denotes the centralizer of the pair (λ_1, λ_2). Furthermore, $G * G \subset G \times G$ is the derived commuting variety of G, and $[G * G/G]$ is the stack quotient of $G * G$ by the diagonal conjugation action of G.

Remark 1.4. By definition, strictness is a group-theoretic property and in fact can be expressed in elementary group-theoretic terms. In [PT21, Proposition 4.9], we show that if (λ_1, λ_2) is a commuting pair of elements in G and $u := \mathsf{Id} - \mathsf{ad}(\lambda_1)$ and $v := \mathsf{Id} - \mathsf{ad}(\lambda_2)$ are the corresponding endomorphisms of $\mathfrak{g}$, then the pair (λ_1, λ_2) is strict if and only u is strict with respect to the kernel of v, that is, if and only if $\mathrm{Im}(v_{|\ker(u)}) = \mathrm{Im}(v) \cap \ker(u)$.

With the notion of strictness at hand, we can formulate a precise version of the statement of Theorem 1.1(2) in the case of a strict normal crossings boundary divisor with two components.

THEOREM 1.5 ([PT21, Theorem 4.7]). *Let (λ_1, λ_2) be a strict pair of commuting elements in G, and let*

$$\mathrm{Loc}_G(X, \{\lambda_1, \lambda_2\})$$

be the derived Artin stack of local systems on X whose local monodromy around D_i belongs to the conjugacy class C_{λ_i}. Then $\mathrm{Loc}_G(X, \{\lambda_1, \lambda_2\})$ comes equipped with a natural $(2 - 2d)$-shifted symplectic structure which is a symplectic leaf of the Poisson stack $\mathrm{Loc}_G(X)$.

Our main objective of the present paper is to understand and prove the de Rham versions of these Betti statements. In the next four sections, we develop the necessary framework and prove the de Rham analog of Theorem 1.1(1). The bulk of the work goes into understanding the algebraic $\mathcal{D}$-module theory of the formal boundary $\widehat{\partial}X$ of a smooth variety X, where X is viewed

as a non-commutative space. Understanding and proving the de Rham analog of Theorem 1.1(2) is more delicate. An important part of this is to show the algebraicity of the derived stack of flat bundles on X which are framed by a fixed flat bundle on the formal boundary. We prove a generic representability result for this stack of framed flat bundles in Section 6. To get a full de Rham analog of Theorem 1.1(2), we will need to study the de Rham versions of the gerbe twist and the strictness property for local monodromies at intersections of components. While the first of these is fairly straightforward, the second is quite intricate in the de Rham context, and so this discussion is left for a future work.

2. Preliminaries

In this section, we have gathered some known and folklore results about $\mathcal{D}_X$-modules on smooth varieties. We first discuss compactness/perfection in the setting of relative $\mathcal{D}$-modules and its preservation under proper push-forwards. We then recall how $\mathcal{D}$-modules can be defined as graded mixed modules over the de Rham algebra.

2.1 Perfect relative $\mathcal{D}$-modules

In this subsection, we have gathered some basic results about $\mathcal{D}$-modules in the relative setting. Most of these results are already contained in Gaitsgory–Rozenblyum's treatise [GR14], and this part does not claim originality. We include it here since we were unable to find a reference treating the algebraic situation allowing for k to not be algebraically closed and also allowing for $\mathcal{D}$-modules that are relative over bases $\operatorname{Spec} B$ with B an arbitrary connective cdga.

First, we discuss the compact generation and characterization of compact objects inside quasi-coherent relative $\mathcal{D}$-modules. Fix a smooth variety X over k and an affine derived scheme $S = \operatorname{Spec} B$. We have $\mathcal{D}_X \otimes_k B$, which is a sheaf of dg-algebras over X. We can therefore consider the dg-category of all sheaves of $\mathcal{D}_X \otimes_k B$-modules whose underlying $\mathcal{O}_X \otimes_k B$-modules are quasi-coherent on $X \times S$. We denote this category by $\mathsf{D}_{\mathrm{qcoh}}(\mathcal{D}_{X,B})$ and call it the dg-category of relative $\mathcal{D}$-modules on $X \times S$ over S. An object $E \in \mathsf{D}_{\mathrm{qcoh}}(\mathcal{D}_{X,B})$ will be called *perfect* if locally on X it is given by a perfect dg-module over the dg-algebra $\mathcal{D}_X \otimes_k B$. In the special case when B is a regular discrete k-algebra, $\mathcal{D}_X \otimes_k B$ is locally a finitely generated algebra of finite homological dimension (and thus is of finite type in the sense of [TV07, Definition 2.4]), which implies that the perfect objects are precisely the bounded coherent $\mathcal{D}_X \otimes_k B$-modules. In general, the two notions do not coincide since being perfect implies, in particular, being of finite tor dimension over B. Nevertheless, we have the following.

PROPOSITION 2.1. *The dg-category $\mathsf{D}_{\mathrm{qcoh}}(\mathcal{D}_{X,B})$ is compactly generated, and its compact objects are the perfect $\mathcal{D}_X \otimes_k B$-modules.*

Proof. There is a forgetful functor $\mathsf{D}_{\mathrm{qcoh}}(\mathcal{D}_{X,B}) \to \mathsf{D}_{\mathrm{qcoh}}(X \times S)$ to the dg-category of quasi-coherent complexes on $X \times S$. This dg-functor is conservative and continuous. Moreover, it has a left adjoint

$$\mathsf{ind} \colon \mathsf{D}_{\mathrm{qcoh}}(X \times S) \longrightarrow \mathsf{D}_{\mathrm{qcoh}}(\mathcal{D}_{X,B}),$$

which sends a quasi-coherent complex E on $X \times S$ to $\mathcal{D}_X \otimes_{\mathcal{O}_X} E$, with its natural $\mathcal{D}_X \otimes_k B$-module structure. It is well known that perfect complexes in $\mathsf{D}_{\mathrm{qcoh}}(X \times S)$ are the compact generators, and it is a formal consequence of this that ind-images of perfect complexes will be compact generators of $\mathsf{D}_{\mathrm{qcoh}}(\mathcal{D}_{X,B})$. These are obviously perfect $\mathcal{D}_X \otimes_k B$-modules. Finally, any perfect $\mathcal{D}_X \otimes_k B$-module is locally compact and thus compact by the quasi-compactness of X. $\qquad\square$

Now let $f\colon X \to Y$ be a morphism between smooth varieties over k. The usual definition gives a direct image dg-functor

$$f_{*,B}\colon \mathsf{D}_{\mathrm{qcoh}}(\mathcal{D}_{X,B}) \longrightarrow \mathsf{D}_{\mathrm{qcoh}}(\mathcal{D}_{Y,B}).$$

We will often drop the B in the notation and simply write f_*. On the level of compact generators, f_* acts as follows. Let E be a perfect complex on $X \times S$ and $\mathsf{ind}(E) = \mathcal{D}_X \otimes_{\mathcal{O}_X} E$. Then we have a canonical isomorphism $f_*(\mathsf{ind}(E)) \simeq \mathsf{ind}(f_*(E))$, where $f_*(E)$ is the direct image of E as a quasi-coherent complex on $X \times S$. In particular, when f is proper, the dg-functor f_* preserves perfect objects. It is easy to check that the formation of f_* commutes with base change: for any morphism $B \to B'$ of connective cdga, the square

$$
\begin{array}{ccc}
\mathsf{D}_{\mathrm{qcoh}}(\mathcal{D}_{X,B}) & \xrightarrow{\otimes_B B'} & \mathsf{D}_{\mathrm{qcoh}}(\mathcal{D}_{X,B'}) \\
{\scriptstyle f_{*,B}}\downarrow & & \downarrow{\scriptstyle f_{*,B'}} \\
\mathsf{D}_{\mathrm{qcoh}}(\mathcal{D}_{Y,B}) & \xrightarrow[\otimes_B B']{} & \mathsf{D}_{\mathrm{qcoh}}(\mathcal{D}_{Y,B'})
\end{array}
$$

canonically commutes. We have thus proved the following proposition, which is well known when B is itself a smooth algebra but for which we could not find any general reference.

PROPOSITION 2.2. *If f is proper, then f_* preserves perfect objects and its formation commutes with change of bases B.*

We also recall the following notion of holonomicity. First, recall that any coherent $\mathcal{D}_X \otimes_k B$-module admits a good filtration and that the support of the associated graded sheaf is a well-defined closed algebraic subset inside $T^*X \times S$.

DEFINITION 2.3. Let $E \in \mathsf{D}_{\mathrm{qcoh}}(\mathcal{D}_{X,B})$ be a quasi-coherent $\mathcal{D}_X \otimes_k B$-module. We say that E is *holonomic* if it satisfies the following two conditions:

(1) The module E is perfect.

(2) There exists a conic Lagrangian algebraic subset $\Lambda \subset T^*X$ such that the characteristic variety of E is contained in $\Lambda \times S$.

In contrast with the case of a base field, it is not true that holonomicity for relative $\mathcal{D}$-modules is preserved by all six operations. However, this holds on a dense open subset in S. Specifically, for us the following proposition will be useful.

PROPOSITION 2.4. *Suppose that B is a discrete noetherian k-algebra. Let E and F be two holonomic objects in $\mathsf{D}_{\mathrm{qcoh}}(\mathcal{D}_{X,B})$. There exists a non-empty open derived subscheme $\operatorname{Spec} B[f^{-1}] \subset \operatorname{Spec} B$ such that the tensor product $E \otimes_{\mathcal{O}} F$ is a perfect $\mathcal{D}_{X,B[f^{-1}]}$-module on $X \times \operatorname{Spec} B[f^{-1}]$.*

Proof. Write $B_0 = B_{\mathrm{red}}$ for the reduced algebra of B. Note that a given object $E \in \mathsf{D}_{\mathrm{qcoh}}(\mathcal{D}_{X,B})$ is perfect if and only if its restriction to $\mathsf{D}_{\mathrm{qcoh}}(\mathcal{D}_{X,B_0})$ is perfect. Indeed, we can use induction on the power annihilating the nil-radical of B to reduce to the case where B is a square zero extension of B_0 by an ideal I. It is easy to see that the functor sending a cdga B to the space of all quasi-coherent $\mathcal{D}_{X,B}$-module is 1-proximate in the sense of formal deformation theory (see [Lur11b]). More precisely, given a discrete noetherian k-algebra B_0, an ideal $I \subset B_0$, and a derivation $d\colon B_0 \to I[1]$, consider the square zero extension $B = B_0 \oplus_d I$ classified by d. The

square of ∞-categories

$$
\begin{array}{ccc}
\mathsf{D}_{\mathrm{qcoh}}(\mathcal{D}_{X,B}) & \longrightarrow & \mathsf{D}_{\mathrm{qcoh}}(\mathcal{D}_{X,B_0}) \\
\downarrow & & \downarrow \\
\mathsf{D}_{\mathrm{qcoh}}(\mathcal{D}_{X,B_0}) & \longrightarrow & \mathsf{D}_{\mathrm{qcoh}}(\mathcal{D}_{X,B_0\oplus I[1]})
\end{array}
$$

induces a full embedding from $\mathsf{D}_{\mathrm{qcoh}}(\mathcal{D}_{X,B})$ to the fiber product of the three other terms. This implies that for a given $E \in \mathsf{D}_{\mathrm{qcoh}}(\mathcal{D}_{X,B})$, the object E is compact if its restriction in $\mathsf{D}_{\mathrm{qcoh}}(\mathcal{D}_{X,B_0})$ is compact.

Thus it is sufficient to tackle the case where B is a reduced noetherian k-algebra. By picking a dense open subset in an irreducible component, we can even assume that B is a smooth domain. Let $K = \mathrm{Frac}(B)$ be its fraction field, and consider $X \times_k K$ as a smooth variety over K. By the standard lore of algebraic $\mathcal{D}$-modules, we know that for any smooth K-variety Z, any smooth divisor $i_Y \colon Y \hookrightarrow Z$ given by a single equation $f = 0$ on Z, and any holonomic coherent $\mathcal{D}_Z$-module M, there exists a Bernstein polynomial $b(M)$ for M with respect to the equation f. The polynomial exists as a monic polynomial over a localization $B[f^{-1}]$ of B. Replacing B with $B[f^{-1}]$, we can assume that $b(M)$ exists as a monic polynomial over B. Now the standard Bernstein–Kashiwara argument gives that the existence of $b(M)$ implies that the pull-back $i_Y^*(E)$ is a bounded coherent complex of $\mathcal{D}_{Y,B}$-modules with coherent cohomology and thus is perfect since B is smooth.

Since the statement of the proposition is local on X, we can apply the above reasoning to the diagonal $X \subset X \times X$, by writing it as a complete intersection, and to the exterior tensor product $E \boxtimes F$ of E and F, which is manifestly a holonomic $\mathcal{D}_{X \times X,B}$-module. The proposition follows. $\qquad\square$

2.2 Connections as graded mixed modules

We will use freely the formalism of graded mixed k-modules from [PTVV13, CPT$^+$17]. We denote the ∞-category of graded mixed k modules by $k - \mathbf{dg}_\epsilon^{\mathrm{gr}}$. It comes equipped with an ∞-functor

$$
|-| := \mathbb{R}\underline{\mathrm{Hom}}(k(0), -) \colon k - \mathbf{dg}_\epsilon^{\mathrm{gr}} \longrightarrow k - \mathbf{dg},
$$

where $k(0)$ denotes the unit in this category, that is, the pure weight zero graded mixed complex. Explicitly, $|-|$ sends a graded mixed complex E to $\prod_i E(i)[-2i]$ endowed with the total differential which is the sum of the cohomological differential and the mixed structure. This ∞-functor is lax symmetric monoidal and thus induces a corresponding ∞-functor on algebras, modules, etc.

Let $X = \mathrm{Spec}\, A$ be a smooth affine variety over k, and let $\mathcal{D}_X$ be the k-algebra of global differential operators on X. Consider the de Rham algebra $\mathbf{DR}_X = \mathrm{Sym}_A\left(\Omega_A^1[-1]\right)$ of X, viewed as a graded mixed cdga with its natural structure of a graded algebra and with mixed structure given by the de Rham differential (see [PTVV13, Section 1.1]). Denote by $\mathsf{D}_{\mathrm{qcoh}}(\mathcal{D}_X)$ the dg-category of complexes of left $\mathcal{D}_X$-modules with inverted quasi-isomorphisms (see Section 2.1 for more on dg-categories of $\mathcal{D}$-modules). Recall that a model for $\mathsf{D}_{\mathrm{qcoh}}(\mathcal{D}_X)$ is the dg-category of all cofibrant $\mathcal{D}_X$-dg-modules. In the same way, we denote by $\mathbf{DR}_X - \mathbf{dg}_\epsilon^{\mathrm{gr}}$ the dg-category of graded mixed $\mathbf{DR}_X$-dg-modules up to quasi-isomorphisms (again, an explicit model is the dg-category of cofibrant graded mixed dg-modules). We have a natural dg-functor

$$
\mathbf{DR} \colon \mathsf{D}_{\mathrm{qcoh}}(\mathcal{D}_X) \longrightarrow \mathbf{DR}_X - \mathbf{dg}_\epsilon^{\mathrm{gr}}
$$

from dg-modules over $\mathcal{D}_X$ to graded mixed $\mathbf{DR}_X$-dg-modules. The dg-functor $\mathbf{DR}$ is defined

by sending a (cofibrant) $\mathcal{D}_X$-dg-module E to its de Rham complex $\mathbf{DR}(E) := \mathbf{DR}_X \otimes_A E$. By definition, $\mathbf{DR}(E)$ is free as a graded module over $\mathbf{DR}_X$, and its mixed structure is induced by the connection $\nabla \colon E \to \Omega^1_A \otimes_A E$ coming from the left $\mathcal{D}_X$-module structure on E.

Proposition 2.5. *The dg-functor*

$$\mathbf{DR} \colon \mathsf{D}_{\mathrm{qcoh}}(\mathcal{D}_X) \longrightarrow \mathbf{DR}_X - \mathbf{dg}^{\mathrm{gr}}_\epsilon$$

is fully faithful. Its essential image consists of all objects that are free as graded dg-modules, that is, objects of the form $\mathbf{DR}_X \otimes_A E_0$ for some A-dg-module E_0.

Proof. To prove full faithfulness, we use the following method to compute mapping complexes inside $\mathbf{DR}_X - \mathbf{dg}^{\mathrm{gr}}_\epsilon$. Let B be a graded mixed cdga and E and F be two graded mixed B-dg-modules. We assume that E and F are cofibrant as graded B-modules. Consider the complex

$$H(E,F) := \prod_{p \geqslant 0} \underline{\mathrm{Hom}}_{B-\mathbf{dg}^{\mathrm{gr}}}(E, F(p))[-p] \,,$$

where $F(p)$ is the graded B-dg-module defined by shifting the grading by p (so $\underline{\mathrm{Hom}}_{B-\mathbf{dg}^{\mathrm{gr}}}(E, F(p))$ consists of graded maps of degree p). The complex $H(E,F)$ is endowed with a total differential D sending a family of elements $\{f_p\}_{p \geqslant 0}$ to

$$D(\{f_p\}) := \{\nabla_F f_p + f_{p-1}\nabla_E + d(f_{p+1})\}_{p \geqslant 0} \,,$$

where ∇_E and ∇_F are the mixed structures on E and F and d is the cohomogical differential. Using an explicit cofibrant model of E, one checks that the complex of k-modules $H(E,F)$ is naturally quasi-isomorphic to the complex $\underline{\mathrm{Hom}}_{B-\mathbf{dg}^{\mathrm{gr}}_\epsilon}(E,F)$. This implies that the dg-functor $\mathbf{DR}$ is fully faithful: for two $\mathcal{D}_X$-dg-modules E and F, it sends $\mathbb{R}\underline{\mathrm{Hom}}_{\mathcal{D}_X}(E,F)$ to the de Rham complex of the $\mathcal{D}_X$-module $\mathbb{R}\underline{\mathrm{Hom}}_A(E,F)$.

For the second part of the proposition, start with a graded mixed $\mathbf{DR}_X$-module E which is of the form $E_0 \otimes_A \mathbf{DR}_X$ as a graded module. We can write E_0 as a filtered colimit of perfect complexes of A-modules. As the dg-functor $\mathbf{DR}$ is continuous and fully faithful, it suffices to check the case where E_0 is perfect. By a cell decomposition induction, we can reduce to the case where $E_0 = M$ is a projective A-module of finite rank. We thus have a graded mixed $\mathbf{DR}_X$-module E whose underlying graded module is quasi-isomorphic to $M \otimes_A \mathbf{DR}_X$. We can also recover the $\mathcal{D}_X$-module structure on M simply by considering the map $M \to M \otimes_A \Omega^1_A$ induced by the mixed structure on E.

This yields a canonical morphism of graded mixed dg-modules $\mathbf{DR}(M) \to E$, which by construction is a quasi-isomorphism. $\qquad\square$

Proposition 2.5 extends by stackification to the case where X is a smooth scheme over k or even a smooth DM-stack over k. It can be stated as the existence of a full and faithful embedding of dg-categories

$$\mathrm{DR} \colon \mathsf{D}_{\mathrm{qcoh}}(\mathcal{D}_X) \hookrightarrow \mathbf{DR}_X - \mathbf{dg}^{\mathrm{gr}}_\epsilon \,,$$

where the dg-categories $\mathsf{D}_{\mathrm{qcoh}}(\mathcal{D}_X)$ and $\mathbf{DR}_X - \mathbf{dg}^{\mathrm{gr}}_\epsilon$ are defined by descent

$$\mathsf{D}_{\mathrm{qcoh}}(\mathcal{D}_X) := \lim_{U = \mathrm{Spec}\, A \to X} \mathcal{D}_U - \mathbf{dg} \,, \quad \mathbf{DR}_X - \mathbf{dg}^{\mathrm{gr}}_\epsilon := \lim_{U = \mathrm{Spec}\, A \to X} \mathbf{DR}_U - \mathbf{dg}^{\mathrm{gr}}_\epsilon \,,$$

where the limits are taken over the small étale site of X and inside the ∞-category of presentable dg-categories (see [Toë12, Section 2]). The essential image of the dg-functor DR consists of all graded mixed $\mathbf{DR}_X$ dg-modules which, as graded modules, are of the form $E \otimes_{\mathcal{O}_X} \mathbf{DR}_X$ for some quasi-coherent $\mathcal{O}_X$-module E.

It is also possible to extend the statement to the relative setting. Let B be a connective cdga and X a smooth DM-stack. Consider $\mathcal{D}_X \otimes_k B$ as a sheaf of dg-algebras and $\mathbf{DR}_X \otimes_k B$ as a sheaf of graded mixed B-linear cdga (over the small étale site of X). The full embedding DR extends to a full and faithful embedding of presentable dg-categories

$$\mathrm{DR} \colon \mathsf{D}_{\mathrm{qcoh}}(\mathcal{D}_{X,B}) \lhook\joinrel\longrightarrow (\mathbf{DR}_X \otimes_k B) - \mathbf{dg}_\epsilon^{\mathrm{gr}},$$

whose essential image consists of graded mixed modules which, as graded modules, are of the form $E \otimes_{\mathcal{O}_X} \mathbf{DR}_X$ for E a quasi-coherent $\mathcal{O}_X \otimes_k B$-dg-module.

We conclude this part by analyzing the inverse image functor for $\mathcal{D}$-modules in terms of graded mixed modules over de Rham algebras. Let $f \colon X = \operatorname{Spec} A' \to Y = \operatorname{Spec} A$ be a morphism of smooth affine k-varieties, corresponding to a morphism of smooth k-algebras $A \to A'$. We have the usual pull-back functor of $\mathcal{D}$-modules

$$f^* \colon \mathsf{D}_{\mathrm{qcoh}}(\mathcal{D}_Y) \longrightarrow \mathsf{D}_{\mathrm{qcoh}}(\mathcal{D}_X) \,.$$

By Proposition 2.5, this can be seen as a dg-functor on dg-categories of graded mixed modules which are free as graded modules. From this point of view, the functor can be described explicitly. It is the natural functor given by base change. Indeed, the morphism f induces a morphism of graded mixed cdga $\mathbf{DR}_Y \to \mathbf{DR}_X$ which, in turn, defines a base change functor on graded mixed modules. This base change is canonically equivalent to f^* when restricted to graded mixed modules which are free as in Proposition 2.5. As a final comment, note that the above discussion also makes sense without the affiness conditions on X and Y, as well as in the relative setting, as can be seen by tensoring with a connective cdga B.

2.3 Graded mixed modules and equivariant objects

We now turn to an equivalent but more conceptual description of the dg-category of $\mathcal{D}$-modules, as equivariant objects inside the dg-category of quasi-coherent modules on the shifted cotangent stack. This will be useful later as it will allow us to reduce some statements about $\mathcal{D}$-modules to statements about quasi-coherent modules.

We let $\mathcal{H} := \mathrm{aut}(B\mathbb{G}_a)$ be the group stack of autoequivalences of $B\mathbb{G}_a$. It can be described explicitly as a semi-direct product $\mathcal{H} = B\mathbb{G}_a \rtimes \mathbb{G}_m$, with $\mathbb{G}_m$ acting on $B\mathbb{G}_a$ by its natural action of weight 1 on $\mathbb{G}_a$. In this description, $\mathbb{G}_m$ acts on $B\mathbb{G}_a$ by its standard action, and $B\mathbb{G}_a$ acts on itself by translations (using the fact that $B\mathbb{G}_a$ is a commutative group stack).

Recall [Toë12, Section 2] that there is a derived stack $\mathbb{D}g^{\mathrm{lp}} \in \mathbf{dSt}_k$ of locally presentable dg-categories with descent. We have the following definition.

DEFINITION 2.6. An $\mathcal{H}$-*equivariant locally presentable dg-category* T is a morphism of derived stacks $T \colon B\mathcal{H} \to \mathbb{D}g^{\mathrm{lp}}$. Locally presentable $\mathcal{H}$-equivariant dg-categories form an ∞-category

$$\mathbb{D}g^{\mathrm{lp}}(B\mathcal{H}) := \mathbf{Map}\big(B\mathcal{H}, \mathbb{D}g^{\mathrm{lp}}\big) \,.$$

Also recall that $\mathbb{D}g^{\mathrm{lp}}$ admits a canonical extension to a derived stack of symmetric monoidal ∞-categories for the tensor product of locally presentable dg-categories of [Toë12]. Thus we can view a symmetric monoidal dg-category with a compatible $\mathcal{H}$-action as a morphism $B\mathcal{H} \to E_\infty - \mathrm{Alg}\big(\mathbb{D}g^{\mathrm{lp}}\big)$ from $B\mathcal{H}$ to the derived stack of E_∞-algebra objects in $\mathbb{D}g^{\mathrm{lp}}$. We will not spell this out, but the interested reader can easily fill in the details of this monoidal extension.

Given an $\mathcal{H}$-equivariant dg-category T, we can form its direct image (see [Toë12]) by the natural projection $p \colon B\mathcal{H} \to \operatorname{Spec} k$. We define the dg-category of $\mathcal{H}$-equivariant objects in T to

be this direct image:

$$T^{\mathcal{H}} := p_*(T) \,.$$

Now assume that, as in the previous subsection, X is a smooth DM-stack and B a connective cdga. Consider $\mathbf{DR}_X \otimes_k B$ as a sheaf of graded cdga on X, and let $(\mathbf{DR}_X \otimes_k B) - \mathbf{dg}$ be its dg-category of (non-graded, non-mixed) dg-modules. The group $\mathcal{H}$ acts on the commutative dg-algebra $(\mathbf{DR}_X \otimes_k B)$ in an obvious manner: the $\mathbb{G}_m$-action is the grading, and the $B\mathbb{G}_a$-action is the mixed structure. This is formalized by the following proposition.

PROPOSITION 2.7. *Let $\mathcal{H}$ act trivially on the dg-category $k - \mathbf{dg}$ of complexes of k-modules. Then, there are natural equivalences of symmetric monoidal dg-categories*

$$(k - \mathbf{dg})^{\mathcal{H}} \simeq \mathsf{D}_{\mathrm{qcoh}}(B\mathcal{H}) \simeq k - \mathbf{dg}_{\epsilon}^{\mathrm{gr}} \,.$$

Proof. The first equivalence holds by definition, so the content of the proposition is the existence of the second equivalence. For this, we let $\pi \colon B\mathcal{H} \to B\mathbb{G}_m$ be the natural projection. Using this morphism, we can view $B\mathcal{H}$ as an affine stack over $B\mathbb{G}_m$ whose fiber is $K(\mathbb{G}_a, 2)$. In other words, we have $B\mathcal{H} \simeq \mathrm{Spec}_{B\mathbb{G}_m} A$, where A equals $\pi_*(\mathcal{O}_{B\mathcal{H}})$ considered as an E_∞-algebra in $B\mathbb{G}_m$. This algebra simply is $A = k[u]$, where u is in cohomological degree 2 and weight 1. For any affine stack $F = \mathrm{Spec}\, A$, there is a symmetric monoidal ∞-functor

$$A - \mathrm{Mod} \longrightarrow \mathsf{D}_{\mathrm{qcoh}}(F) \,,$$

which makes $\mathsf{D}_{\mathrm{qcoh}}(F)$ into the left completion of the $A - \mathrm{Mod}$ for the natural t-structure (see [Lur11a]). This statement carries over verbatim to the relative setting over $B\mathbb{G}_m$: there is a natural symmetric monoidal ∞-functor

$$A - \mathrm{Mod}(\mathsf{D}_{\mathrm{qcoh}}(B\mathbb{G}_m)) \longrightarrow \mathsf{D}_{\mathrm{qcoh}}(B\mathcal{H}) \,,$$

which is an equivalence when restricted to objects bounded on the left for the natural t-structures on both sides. Since $A = k[u]$, we have that $\mathsf{D}_{\mathrm{qcoh}}(B\mathcal{H})$ can be identified with the left completion of the natural t-structure on the dg-category of graded $k[u]$-dg-modules. This completion is in turn identified with the dg-category of graded mixed complexes via the dg-functor

$$k - \mathbf{dg}_{\epsilon}^{\mathrm{gr}} \longrightarrow k[u] - \mathbf{dg}^{\mathrm{gr}} \,,$$

sending E to the graded $k[u]$-module whose piece of weight p is $\mathbb{R}\underline{\mathrm{Hom}}(k(p), E)$. This dg-functor is manifestly a symmetric monoidal equivalence when restricted to graded mixed complexes which are cohomologically bounded on the left. This proves the proposition. $\qquad\square$

Let X be an affine smooth variety over k and B a connective cdga. Then $\mathbf{DR}_X \otimes_k B$ is a graded mixed cdga, and thus Proposition 2.7 can be used to view $\mathbf{DR}_X \otimes_k B$ as a quasi-coherent sheaf of cdga on the stack $B\mathcal{H}$. The dg-category $(\mathbf{DR}_X \otimes_k B) - \mathbf{dg}$ can then be seen as a natural E_∞-algebra object in $\mathbb{D}g^{\mathrm{lp}}(B\mathcal{H})$ or, in other words, as an $\mathcal{H}$-equivariant symmetric monoidal dg-category.

COROLLARY 2.8. *There is a natural equivalence of symmetric monoidal dg-categories*

$$(\mathcal{D}_X \otimes_k B) - \mathbf{dg} \simeq ((\mathbf{DR}_X \otimes_k B) - \mathbf{dg})^{\mathcal{H}} \,.$$

Proof. This follows immediately from the proposition. Indeed, the equivalence of the proposition is symmetric monoidal, so preserves algebras and modules over algebras. $\qquad\square$

3. The formal boundary of a smooth variety

In this section, we discuss the notion of a formal boundary of a smooth algebraic variety X over a base field k of characteristic zero. In contrast to the Betti setting analyzed in [PT21], in this case, the formal boundary does not itself exist as an algebraic variety or stack in any form and will only be defined as a non-commutative space; that is, it will be defined through its category of perfect complexes. The requisite categories of perfect complexes have been studied recently by several authors [BT13, Efi17, HPV16]. We follow a similar approach for the case of perfect complexes endowed with integrable connections, where many statements can be reduced to the case without connections. However, the ∞-category of perfect complexes with flat connections we introduce below is a new object which cannot be recovered from the ∞-category of perfect complexes on the formal boundary. Thus the results of this section are new and do not follow formally from the results of [BT13, Efi17, HPV16].

In this section, all varieties, schemes, and stacks are defined over a base field k of characteristic zero.

3.1 Perfect complexes on the formal boundary

In this section, we recall the notion of the formal boundary $\widehat{\partial}X$ of a smooth variety X, studied in [BT13, Efi17, HPV16]. As we prefer to avoid any analytical aspects and constructions, we mainly follow the approaches in [Efi17] and [HPV16].

The setting. Let X be a smooth algebraic variety. Fix an open dense embedding $X \hookrightarrow \mathfrak{X}$, where $\mathfrak{X}$ is a smooth and proper scheme over k. We moreover assume that $\mathfrak{X}$ is chosen so that the reduced closed complement $D \subset X$ of X inside $\mathfrak{X}$ is a simple normal crossing divisor on $\mathfrak{X}$. At some point, we will need to relax the conditions of the setting and allow for $\mathfrak{X}$ to be a smooth and proper DM-stack, for which the arguments are similar. We call such an embedding $X \hookrightarrow \mathfrak{X}$ a *good compactification.*

For any affine scheme $\operatorname{Spec} A$ with an étale map $u \colon \operatorname{Spec} A \to \mathfrak{X}$, we consider the ideal $I \subset A$ of the pull-back $u^*(D) \subset \operatorname{Spec} A$ as well as the formal completion $\widehat{A} = \lim_n A/I^n$ of A along I. When u varies in the small étale site of $\mathfrak{X}$, we obtain a presheaf of commutative rings on $\mathfrak{X}_{\mathrm{et}}$, sending $u \colon \operatorname{Spec} A \to \mathfrak{X}$ to $\widehat{A}$. This presheaf of commutative rings comes equipped with a presheaf of ideals, which simply is the ideal generated by I inside $\widehat{A}$.

DEFINITION 3.1. The ∞-*category of perfect complexes on* $\widehat{\partial}X$ is defined by

$$\operatorname{Perf}\big(\widehat{\partial}X\big) := \lim_{\operatorname{Spec} A \to \mathfrak{X}} \operatorname{Perf}\big(\operatorname{Spec}\widehat{A} - V(I)\big).$$

This definition has a version with coefficients in any derived affine scheme $S = \operatorname{Spec} B$ which goes as follows. For each $u \colon \operatorname{Spec} A \to \mathfrak{X}$ in $\mathfrak{X}_{\mathrm{et}}$, we can form the cgda $\widehat{A \otimes B} := \lim_n (A/I^n \otimes_k B)$. The ideal I defines an open subset in the derived scheme $\operatorname{Spec} \widehat{A \otimes B}$ which simply is the pull-back of $\operatorname{Spec} \widehat{A} - V(I)$ by the natural projection $\operatorname{Spec} \widehat{A \otimes B} \to \operatorname{Spec} \widehat{A}$. We will write $\operatorname{Spec} \widehat{A \otimes B} - V(I)$ for this open derived subscheme. We now set

$$\operatorname{Perf}\big(\widehat{\partial}X\big)(S) := \lim_{\operatorname{Spec} A \to \mathfrak{X}} \operatorname{Perf}\big(\operatorname{Spec} \widehat{A \otimes B} - V(I)\big) \in \mathbf{dgCat}$$

and call this the ∞-*category of families of perfect complexes* on $\widehat{\partial}X$ parametrized by S. When S varies in the ∞-category of derived affine schemes $\mathbf{dAff}$, the application $S \mapsto \operatorname{Perf}\big(\widehat{\partial}X\big)(S)$

defines an ∞-functor

$$\mathsf{Perf}\big(\widehat{\partial X}\big)\colon \mathbf{dAff}^{\mathrm{op}} \longrightarrow \mathbf{dgCat}\,.$$

By [HPV16, Proposition 3.23], this ∞-functor is a derived stack for the étale topology on $\mathbf{dAff}$. In the same manner, we have the derived stack $\mathsf{Perf}(\widehat{\mathfrak{X}})$ of perfect complexes on the formal completion of $\mathfrak{X}$ along D. For $S = \operatorname{Spec} B$, the S-points can be defined as before:

$$\mathsf{Perf}\big(\widehat{\mathfrak{X}}\big)(S) = \lim_{\operatorname{Spec} A \to \mathfrak{X}} \mathsf{Perf}\big(\operatorname{Spec} \widehat{A \otimes B}\big) \in \mathbf{dgCat}\,.$$

Another equivalent description is as the derived mapping stack

$$\mathsf{Perf}\big(\widehat{\mathfrak{X}}\big) \simeq \mathbf{Map}_{\mathbf{dSt}_k}\big(\widehat{\mathfrak{X}}, \mathsf{Perf}\big)\,.$$

Here the formal scheme $\widehat{\mathfrak{X}}$ is defined as $\operatorname{colim}_n \mathfrak{X}_{(n)}$, where the colimit is taken in $\mathbf{dSt}_k$ and $\mathfrak{X}_{(n)} = \operatorname{Spec} \mathcal{O}_X/I^n \subset X$ is the $(n-1)$th infinitesimal neighborhood of D inside $\mathfrak{X}$.

DEFINITION 3.2. The object $\mathsf{Perf}\big(\widehat{\mathfrak{X}}\big) \in \mathbf{dSt}_k$ is called the *derived stack of perfect complexes on* $\widehat{\mathfrak{X}}$, while $\mathsf{Perf}\big(\widehat{\partial X}\big) \in \mathbf{dSt}_k$ is called the *derived stack of perfect complexes on* $\widehat{\partial X}$.

These derived stacks admit sheaf-theoretic interpretations. The structure sheaf $\widehat{\mathcal{O}}_D$ of $\widehat{\mathfrak{X}}$ can be considered as a sheaf of commutative $\mathcal{O}_X$-algebras, sending an étale map $\operatorname{Spec} A \to \mathfrak{X}$ to the A-algebra $\widehat{A}$. We also have $\widehat{\mathcal{O}}_D \simeq \lim_n \mathcal{O}_{\mathfrak{X}_{(n)}}$, where the limit is taken in the category of all sheaves of $\mathcal{O}_X$-algebras. Note that $\widehat{\mathcal{O}}_D$ is in general not a quasi-coherent sheaf on X. In the same manner, if $S = \operatorname{Spec} B$ is a derived affine scheme, we have a sheaf of commutative $\mathcal{O}_X$-dg-algebras $\widehat{\mathcal{O}}_{D,B}$, sending an étale map $\operatorname{Spec} A \to \mathfrak{X}$ to $\widehat{A \otimes_k B} = \lim_n \big(A/I^n \otimes_k B\big)$. Again, in general, this is not a quasi-coherent sheaf on X.

Similarly, we can define a sheaf of commutative $\mathcal{O}_X$-algebras $\widehat{\mathcal{O}}_D^o$ by locally inverting the equation of D in $\widehat{\mathcal{O}}_D$. More precisely, for a derived affine scheme $S = \operatorname{Spec} B$, we send the étale map $\operatorname{Spec} A \to \widehat{\mathfrak{X}}$ to $\Gamma\big(\operatorname{Spec}\big(\widehat{A \otimes_k B}\big) - V(I), \mathcal{O}\big)$. When $\operatorname{Spec} A \to \widehat{\mathfrak{X}}$ is small enough so that D becomes principal over $\operatorname{Spec} A$ (which we can always assume for the purpose of defining the sheaf $\widehat{\mathcal{O}}_D^o$), the value of $\widehat{\mathcal{O}}_{D,B}^o$ on $\operatorname{Spec} A \to \widehat{\mathfrak{X}}$ is the cdga $\big(\widehat{A \otimes_k B}\big)\big[t^{-1}\big]$, where $t \in A$ is a generator of the ideal $I \subset A$.

Both sheaves $\widehat{\mathcal{O}}_{D,B}$ and $\widehat{\mathcal{O}}_{D,B}^o$ of cdga on $\mathfrak{X}_{\mathrm{et}}$ are set-theoretically supported on D and can therefore be considered as sheaves of cdga on the site D_{et}. Thus it makes sense to consider the ∞-categories of sheaves on D_{et} which are perfect modules over the sheaves of cdga $\widehat{\mathcal{O}}_{D,B}$ and $\widehat{\mathcal{O}}_{D,B}^o$. Let us denote these ∞-categories by $\operatorname{Perf}\big(\widehat{\mathcal{O}}_{D,B}\big)$ and $\operatorname{Perf}\big(\widehat{\mathcal{O}}_{D,B}^o\big)$. The descent result proved in [HPV16, Proposition 3.23] precisely implies that we have natural equivalences of ∞-categories

$$\mathsf{Perf}\big(\widehat{\mathfrak{X}}\big)(S) \simeq \operatorname{Perf}\big(\widehat{\mathcal{O}}_{D,B}\big)\,, \quad \mathsf{Perf}\big(\widehat{\partial X}\big)(S) \simeq \operatorname{Perf}\big(\widehat{\mathcal{O}}_{D,B}^o\big)\,,$$

which are moreover functorial in $S = \operatorname{Spec} B$.

One aspect of Definition 3.3 is that it depends a priori on a choice of $\mathfrak{X}$. For the perfect complexes over $\widehat{\mathfrak{X}}$, this is certainly expected, but the idea is that the derived stack $\mathsf{Perf}\big(\widehat{\partial X}\big)$ should only depend on the variety X. Unfortunately, we do not know if this is the case, and we could not deduce this from the combined results of [BT13, Efi17, HPV16]. It is shown in [HPV16, Corollary A.4] (together with [BT13]) that the ∞-category $\mathsf{Perf}\big(\widehat{\partial X}\big)(k)$ of global k-points only depends on X. However, as noted in [HPV16, Appendix A], the setting of [BT13] is only for smooth varieties, and it is therefore unclear whether $\mathsf{Perf}\big(\widehat{\partial X}\big)(B)$ remains independent of the choice of $\mathfrak{X}$ for a general base cdga B (or already for a non-smooth commutative k-algebra B of

finite type). To deal with this issue, we introduce the full substack $\mathsf{Perf}^{\mathrm{ex}}(\widehat{\partial}X) \subset \mathsf{Perf}(\widehat{\partial}X)$ of *extendable perfect complexes* and use the categorical approach of [Efi17] to show that $\mathsf{Perf}^{\mathrm{ex}}(\widehat{\partial}X)$ only depends on X.

To define $\mathsf{Perf}^{\mathrm{ex}}(\widehat{\partial}X)$, consider the map of stacks in ∞-categories

$$\mathsf{Perf}(\widehat{\mathfrak{X}}) \longrightarrow \mathsf{Perf}(\widehat{\partial}X)$$

from perfect complexes on the formal completion of $\mathfrak{X}$ along D to perfect complexes on $\widehat{\partial}X$. This is a morphism of stacks in stable ∞-categories, and it therefore makes sense to define its Karoubian image. This is the substack of objects that are locally (for the étale topology) direct summands of objects in the essential image of the above map. More precisely, for any affine derived scheme $S \in \mathbf{dAff}$, we have a stable ∞-functor

$$\mathsf{Perf}(\widehat{\mathfrak{X}})(S) \longrightarrow \mathsf{Perf}(\widehat{\partial}X)(S),$$

and we denote by $\mathsf{Perf}^{\mathrm{ex,pr}}(\widehat{\partial}X)(S) \subset \mathsf{Perf}(\widehat{\partial}X)(S)$ the full sub-∞-category of objects that are retracts of objects in the essential image of $\mathsf{Perf}(\widehat{\mathfrak{X}})(S) \to \mathsf{Perf}(\widehat{\partial}X)(S)$. When S varies, this defines a full sub-prestacks $\mathsf{Perf}^{\mathrm{ex,pr}}(\widehat{\partial}X) \subset \mathsf{Perf}^{\mathrm{ex}}(\widehat{\partial}X)$.

Definition 3.3. The *derived stack of extendable perfect complexes on $\widehat{\partial}X$* is the stack associated with the prestack $\mathsf{Perf}^{\mathrm{ex,pr}}(\widehat{\partial}X)$ defined above. It is denoted by $\mathsf{Perf}^{\mathrm{ex}}(\widehat{\partial}X)$.

Note that, by definition, $\mathsf{Perf}^{\mathrm{ex}}(\widehat{\partial}X)$ is a full substack in $\mathsf{Perf}(\widehat{\partial}X)$. An important property of the stack $\mathsf{Perf}^{\mathrm{ex}}(\widehat{\partial}X)$ is that it depends only on X and not on the choice of $\mathfrak{X}$.

Proposition 3.4. *For a given $S = \operatorname{Spec} B \in \mathbf{dAff}$, the ∞-category $\mathsf{Perf}^{\mathrm{ex}}(\widehat{\partial}X)(S)$ can be reconstructed from the k-linear dg-category $\mathrm{Perf}(X)$ of perfect complexes over the variety X. Moreover, this reconstruction is functorial in B.*

Proof. This is essentially the main result of [Efi17, Theorem 3.2], which we have bootstrapped to work over an arbitrary cdga base. First, note that since $\mathsf{Perf}^{\mathrm{ex}}(\widehat{\partial}X)$ is the stack associated with the prestack $\mathsf{Perf}^{\mathrm{ex,pr}}(\widehat{\partial}X)$, it is enough to show that $\mathsf{Perf}^{\mathrm{ex}}(\widehat{\partial}X)(S)$ can be recovered from $\mathrm{Perf}(X)$. We start with the following lemma.

Lemma 3.5. *Let $\mathcal{K}(S)$ be the kernel of the ∞-functor $\mathsf{Perf}(\widehat{\mathfrak{X}})(S) \to \mathsf{Perf}^{\mathrm{ex,pr}}(\widehat{\partial}X)(S)$. The sequence of stable ∞-categories*

$$\mathcal{K}(S) \hookrightarrow \mathsf{Perf}(\widehat{\mathfrak{X}})(S) \longrightarrow \mathsf{Perf}^{\mathrm{ex,pr}}(\widehat{\partial}X)(S)$$

identifies $\mathsf{Perf}^{\mathrm{ex,pr}}(\widehat{\partial}X)(S)$ as the triangulated quotient of $\mathsf{Perf}(\widehat{\mathfrak{X}})(S)$ by $\mathcal{K}(S)$.

Proof. By descent, the ∞-functor can be written as a finite limit over an affine cover $\mathcal{U}$ of $\mathfrak{X}$,

$$\lim_{\operatorname{Spec} A \in \mathcal{U}} \mathrm{Perf}\left(\widehat{A \otimes_k B}\right) \longrightarrow \lim_{\operatorname{Spec} A \in \mathcal{U}} \mathrm{Perf}\left(\widehat{A \otimes_k B}\left[t^{-1}\right]\right),$$

where the affine cover $\mathcal{U}$ has been chosen such that D becomes principal on each $\operatorname{Spec} A$ and we have denoted by t a local equation of D in $\operatorname{Spec} A$. For a given $\operatorname{Spec} A \in \mathcal{U}$, we have an exact sequence of stable ∞-categories

$$\mathcal{K}(S) \hookrightarrow \mathrm{Perf}\left(\widehat{A \otimes_k B}\right) \longrightarrow \mathrm{Perf}\left(\widehat{A \otimes_k B}\left[t^{-1}\right]\right).$$

But for any finite diagram of full faithful stable ∞-functors $T_\alpha \hookrightarrow T'_\alpha$, the induced ∞-functor on triangulated quotients

$$\left(\lim_\alpha T'_\alpha\right)/\left(\lim_\alpha T_\alpha\right) \longrightarrow \lim_\alpha (T'_\alpha/T_\alpha)$$

is fully faithful. Therefore, the ∞-functor $\mathsf{Perf}\big(\widehat{\mathfrak{X}}\big)(S)/\mathcal{K}(S) \to \mathsf{Perf}^{\mathrm{ex,pr}}\big(\widehat{\partial X}\big)(S)$ is always fully faithful. Finally, by the definition of extendable objects, it is also essentially surjective up to direct factors, which implies that it is an equivalence. $\square$

Proof of Proposition 3.4, *continued.* Going back to the proof of the proposition, we will need a more precise description of the kernel $\mathcal{K}(S)$. For this, we choose a compact generator $K \in \mathsf{Perf}(\mathfrak{X})$ for $\mathrm{Perf}_D(\mathfrak{X}) \subset \mathrm{Perf}(\mathfrak{X})$, the sub-dg-category of perfect complexes with supports on D. The corresponding object $K \otimes_k B \in \mathrm{Perf}(\mathfrak{X}) \otimes_k B$ is a compact generator for $\mathrm{Perf}_D(\mathfrak{X}) \otimes_k B$, and this remains true after Zariski localization on $\mathfrak{X}$: for any Zariksi open $U = \mathrm{Spec}\, A \subset \mathfrak{X}$, the object $K_{|U} \otimes_k B \in \mathrm{Perf}(U)$ is a compact generator for $\mathrm{Perf}_D(U)$. Formal gluing for the affine U (see [HPV16]) tells us that we have a fibered square of dg-categories

$$
\begin{array}{ccc}
\mathrm{Perf}(A \otimes B) & \longrightarrow & \mathrm{Perf}\left(A \otimes_k B\big[t^{-1}\big]\right) \\
\downarrow & & \downarrow \\
\mathrm{Perf}\left(\widehat{A \otimes_k B}\right) & \longrightarrow & \mathrm{Perf}\left(\widehat{A \otimes_k B}\big[t^{-1}\big]\right)
\end{array}
$$

and thus an equivalence of the kernels of the horizontal ∞-functors. This kernel is precisely $\mathrm{Perf}_D(U)$ and thus generated by $K_{|U} \otimes_k B$. By descent, we now have

$$
\mathcal{K}(S) \simeq \lim_{U \in \mathfrak{X}_{\mathrm{Zar}}} \mathrm{Perf}_{D \times S}(U \times S) \simeq \mathrm{Perf}_{D \times S}(\mathfrak{X} \times S) \,.
$$

To summarize, let $C = \mathrm{End}(K)$ be the dg-algebra of endomophisms of the object K. We have an exact sequence of stable ∞-categories

$$
\mathrm{Perf}(C \otimes_k B) \longrightarrow \mathrm{Perf}\left(\widehat{\mathfrak{X} \times S}\right) \longrightarrow \mathsf{Perf}^{\mathrm{ex,pr}}\big(\widehat{\partial X}\big)(S) \,.
$$

The ∞-category $\mathrm{Perf}\left(\widehat{X \times S}\right)$ can itself be written in terms of the dg-algebra C. Again by descent, we can replace $\widehat{\mathfrak{X}}$ with an affine open subscheme $U = \mathrm{Spec}\, A$ and K with its restriction $K_{|U}$ to U. Setting $C_U := \mathrm{End}(K_{|U}))$, we immediately see that $\mathrm{Perf}\left(\widehat{A \otimes_k B}\right)$ is naturally equivalent to the dg-category $\Psi\,\mathrm{Perf}\left(C \otimes_k B\right)$ of C_U-dg-modules inside $\mathrm{Perf}(B)$ (called pseudo-perfect dg-modules relative to B; see [TV07, Definition 2.7]). Such an equivalence is produced by sending a perfect dg-module E over $A \otimes_k B$ to $\mathrm{Hom}(K_{|U}, E)$ as a dg-module over $\mathrm{End}(K_{|U})$. We refer to [Efi17] for more details.

We thus have an exact sequence of dg-categories

$$
\mathrm{Perf}(C \otimes_k B) \longrightarrow \Psi\,\mathrm{Perf}(C \otimes_k B) \longrightarrow \mathsf{Perf}^{\mathrm{ex,pr}}\big(\widehat{\partial X}\big)(S) \,.
$$

As $\mathrm{Perf}(\mathfrak{X} \times S)$ is a smooth and proper dg-category over B, we can now apply [Efi17, Theorem 3.2] to the object $K \otimes_k B \in \mathrm{Perf}(\mathfrak{X} \times S)$, which precisely states that the above quotient can be functorially reconstructed from the B-linear dg-category $\mathrm{Perf}(\mathfrak{X} \times S)/\langle K \otimes_k B \rangle \simeq \mathrm{Perf}(X \times S) \simeq \mathrm{Perf}(X) \otimes_k B$ and thus from $\mathrm{Perf}(X)$ as a dg-category over k. $\square$

COROLLARY 3.6. *The derived stack* $\mathsf{Perf}^{\mathrm{ex}}\big(\widehat{\partial X}\big)$ *does not depend on the choice of* $\mathfrak{X}$.

Corollary 3.6 can be made more precise as follows. Suppose that we have two good compactifications $\mathfrak{X}$ and $\mathfrak{X}'$ as well as a morphism $\pi \colon \mathfrak{X}' \to \mathfrak{X}$ inducing an isomorphism over X. Let $\mathsf{Perf}\big(\widehat{\partial X}\big)$ and $\mathsf{Perf}\big(\widehat{\partial X'}\big)$ be the two derived stacks constructed above for $\mathfrak{X}$ and $\mathfrak{X}'$, respectively. There is an obvious pull-back morphism $\pi^* \colon \mathsf{Perf}\big(\widehat{\partial X}\big) \to \mathsf{Perf}\big(\widehat{\partial X'}\big)$, and the corollary states that this morphism is an equivalence of derived stacks.

Moreover, for any étale affine $\operatorname{Spec} A \to \mathfrak{X}$, we have a natural morphism of schemes

$$\operatorname{Spec} \widehat{A} - V(I) \longrightarrow \operatorname{Spec} A - V(I) .$$

Similarly, for any $S = \operatorname{Spec} B \in \mathbf{dAff}$, we have a morphism of derived schemes

$$\operatorname{Spec} \widehat{A \otimes_k B} - V(I) \longrightarrow (\operatorname{Spec}(A) - V(I)) \times S .$$

When A varies in the étale site of $\mathfrak{X}$ and S inside derived affine schemes, we obtain by base change a natural restriction map

$$R \colon \mathsf{Perf}(X) \longrightarrow \mathsf{Perf}(\widehat{\partial}X) ,$$

where $\mathsf{Perf}(X) := \mathbf{Map}(X, \mathsf{Perf})$ is the derived stack of perfect complexes on X. Similarly, we get a restriction map

$$R' \colon \mathsf{Perf}(X) \longrightarrow \mathsf{Perf}(\widehat{\partial}X') .$$

Corollary 3.6 and its proof then say that we have a commutative triangle of derived stacks

with π^* an equivalence.

We do not know if the above corollary continues to hold for the bigger stack $\mathsf{Perf}(\widehat{\partial}X)$. Because of [HPV16, Theorem 7.3], the inclusion $\mathsf{Perf}^{\mathrm{ex}}(\widehat{\partial}X)(S) \subset \mathsf{Perf}(\widehat{\partial}X)(S)$ is an equivalence as soon as S is a smooth variety over k, so the restriction of $\mathsf{Perf}(\widehat{\partial}X)$ to smooth varieties does not depend on $\mathfrak{X}$. We believe that this remains true for a general derived affine scheme S, but we could not find a reference (or prove it). The question is essentially equivalent to proving the analog of the localization for coherent complexes of [HPV16] where coherent complexes are replaced by perfect complexes.

3.2 Perfect complexes with flat connections on the formal boundary

In Section 3.1, we discussed the derived stack of perfect complexes on the formal boundary of X. In this subsection, we use similar ideas to introduce the derived stack $\mathsf{Perf}^{\nabla}(\widehat{\partial}X)$ of perfect complexes on $\widehat{\partial}X$ endowed with integrable connections. When X is $\mathbb{A}^1$, the underived version of $\mathsf{Perf}^{\nabla}(\widehat{\partial}X)$ was extensively studied by S. Raskin [Ras15] in the context of the local geometric Langlands correspondence.

We keep the setup from the previous section: we fix a smooth variety X and a good compactification $X \hookrightarrow \mathfrak{X}$, with $D \subset \mathfrak{X}$ the divisor at infinity. In order to define the derived stack of perfect complexes of flat connections on $\widehat{\partial}X$, we first define certain sheaves of graded mixed cgda on the small étale site $\mathfrak{X}_{\mathrm{et}}$ of $\mathfrak{X}$ and then define perfect complexes with flat connections as graded mixed dg-modules.

Let A be a smooth commutative k-algebra of finite type. We will view the de Rham algebra $\mathbf{DR}(A)$ of A over k as a graded mixed cdga over k. Concretely, $\mathbf{DR}(A)$ equals $\operatorname{Sym}_A\left(\Omega_A^1[1]\right)$ considered as a $\mathbb{Z}$-graded cdga with zero differential and for which Ω_A^1 sits in weight 1. The

graded cdga $\mathbf{DR}(A)$ comes equipped with an extra differential, namely the de Rham differential, which we denote here by ϵ. This additional structure makes $\mathbf{DR}(A)$ into a graded mixed cdga in the sense of [PTVV13, CPT+17]. When A is equipped with an ideal $I \subset A$, we denote by $\widehat{\mathbf{DR}}(A)$ the I-adic completion of $\mathbf{DR}(A)$ which is defined by

$$\widehat{\mathbf{DR}}(A) := \lim_n \mathbf{DR}\left(A/I^n\right),$$

where the limit is taken in the category of graded mixed cdga. The underlying graded cdga of $\widehat{\mathbf{DR}}(A)$ is naturally isomorphic to $\mathrm{Sym}_{\widehat{A}}\left(\widehat{\Omega}^1_A[1]\right)$, the symmetric algebra over the completion of Ω^1_A. The mixed structure on $\mathrm{Sym}_{\widehat{A}}\left(\widehat{\Omega}^1_A[1]\right)$ simply is the canonical extension of the de Rham differential on A to its completion.

Let $\mathrm{Spec}\, A \to \mathfrak{X}$ be an étale map and $I \subset A$ be the ideal of definition of the divisor D. We have the completed de Rham graded mixed cdga $\widehat{\mathbf{DR}}(A)$. When $\mathrm{Spec}\, A \to \mathfrak{X}$ varies in the small étale site of $\mathfrak{X}$, this defines a sheaf of graded mixed cdga $\widehat{\mathbf{DR}}$ on $\mathfrak{X}_{\mathrm{et}}$. This sheaf is set-theoretically supported on D and thus defines a sheaf of graded mixed cdga on D_{et}. As before, this sheaf has a version with coefficients in a cdga B over k denoted by $\widehat{\mathbf{DR}}_B$. Its values on an étale $U = \mathrm{Spec}\, A \to \mathfrak{X}$ form the graded mixed cgda

$$\widehat{\mathbf{DR}}_B(U) := \lim_n \left(\mathbf{DR}\left(A/I^n\right) \otimes_k B\right).$$

The sheaf $\widehat{\mathbf{DR}}_B$ is now a sheaf of graded mixed B-linear cdga. Note that the weight zero part of $\widehat{\mathbf{DR}}_B$ is the sheaf $\widehat{\mathcal{O}}_{D,B}$ constructed before. We can therefore invert a local equation of the divisor D to define $\widehat{\mathbf{DR}}^o_B$, another sheaf of graded mixed B-linear cdga. For an étale map $U = \mathrm{Spec}\, A \to \mathfrak{X}$ on which the divisor D is principal with equation $t \in A$, we have $\widehat{\mathbf{DR}}^o_B(U) := \widehat{\mathbf{DR}}_B(U)\left[t^{-1}\right]$. The part of weight zero in $\widehat{\mathbf{DR}}^o_B(U)$ is of course the sheaf $\widehat{\mathcal{O}}^o_{D,B}$ defined in Section 3.1.

For $S = \mathrm{Spec}\, B \in \mathbf{dAff}_k$, we let $\mathrm{Perf}^\nabla\left(\widehat{\partial}X\right)(S)$ be the dg-category of sheaves E of graded mixed $\widehat{\mathbf{DR}}^o_B(U)$-dg-modules which are locally free of weight zero in the following sense: locally on $\mathfrak{X}_{\mathrm{et}}$, the underlying graded $\widehat{\mathbf{DR}}^o_B$-dg-module E (obtained by forgetting the mixed structure) is of the form $\widehat{\mathbf{DR}}^o_B \otimes_{\widehat{\mathcal{O}}^o_{D,B}} E(0)$ for some perfect $\widehat{\mathcal{O}}^o_{D,B}$-module $E(0)$ of weight zero. When $S = \mathrm{Spec}\, B$ varies in $\mathbf{dAff}$, the dg-categories $\mathrm{Perf}\left(\partial X\right)(S)$ define an dg-functor $\mathsf{Perf}^\nabla\left(\partial X\right)\colon \mathbf{dAff}^{\mathrm{op}} \to \mathbf{dgCat}$. There is an obvious forgetful map of derived prestacks

$$\mathsf{Perf}^\nabla\left(\widehat{\partial}X\right) \longrightarrow \mathsf{Perf}\left(\widehat{\partial}X\right)$$

sending a graded mixed dg-module to its part of weight zero.

DEFINITION 3.7. The object $\mathsf{Perf}^\nabla\left(\widehat{\partial}X\right)$ is called the *derived prestack of perfect complexes with flat connections on $\widehat{\partial}X$*. The *derived prestack of extendable perfect complexes with flat connections on $\widehat{\partial}X$* is defined to be the fiber product of derived prestacks

$$\mathsf{Perf}^{\nabla,\mathrm{ex}}\left(\widehat{\partial}X\right) \times_{\mathsf{Perf}\left(\widehat{\partial}X\right)} \mathsf{Perf}^{\mathrm{ex}}\left(\widehat{\partial}X\right).$$

By construction, $\mathsf{Perf}^{\nabla,\mathrm{ex}}\left(\widehat{\partial}X\right)$ is a full derived sub-prestack in $\mathsf{Perf}^\nabla\left(\widehat{\partial}X\right)$ defined by the local condition "the underlying perfect complex is extendable." The main result of this section is the following descent and invariance statement.

PROPOSITION 3.8. (1) *The derived prestacks $\mathsf{Perf}^\nabla\left(\widehat{\partial}X\right)$ and $\mathsf{Perf}^{\nabla,\mathrm{ex}}\left(\widehat{\partial}X\right)$ are stacks.*

(2) *The derived stack $\mathsf{Perf}^{\nabla,\mathrm{ex}}\left(\widehat{\partial}X\right)$ only depends on X.*

Proof. The key to the proof of this proposition is the interpretation of graded mixed structures as actions of the group stack $\mathcal{H} := B\mathbb{G}_a \rtimes \mathbb{G}_m$ (see Proposition 2.5). For a graded mixed cdga Ω, the group stack $\mathcal{H}$ acts on Ω by cdga automorphisms, where the $\mathbb{G}_m$-action provides the grading and the $B\mathbb{G}_a$-action induces the mixed structure. This action induces an action of $\mathcal{H}$ on the k-linear dg-category $\mathrm{Perf}(\Omega)$ of perfect dg-modules over Ω. The dg-category of graded mixed Ω-modules which are perfect as Ω-dg-modules can be recovered by taking invariants (see Proposition 2.5):

$$\mathrm{Perf}^{\mathrm{gr},\boldsymbol{\epsilon}}(\Omega) \simeq \mathrm{Perf}(\Omega)^{\mathcal{H}}.$$

This presentation of graded mixed dg-modules implies the statement of the proposition as follows.

For part (1), the derived prestack $\mathrm{Perf}^{\nabla}(\widehat{\partial}X)$ is obtained as follows. We start with the prestack $\mathrm{Perf}(\widehat{\mathbf{DR}}^o)$ of perfect $\widehat{\mathbf{DR}}^o$-dg-modules, where $\widehat{\mathbf{DR}}^o$ is simply considered as a sheaf of graded cdga. This is a derived prestack with values in $\mathcal{H}$-equivariant dg-categories. It is moreover a stack, which follows by noticing that $\widehat{\mathbf{DR}}^o$ is a cdga inside $\mathrm{Perf}(\widehat{\partial}X)$ and by using [HPV16, Proposition 3.23]. This implies that its fixed points by $\mathcal{H}$ remain a stack (because taking fixed points commutes with taking limits). This stack is denoted by $\mathrm{Perf}^{\mathrm{gr},\boldsymbol{\epsilon}}(\widehat{\mathbf{DR}}^o)$ and is the stack of graded mixed $\widehat{\mathbf{DR}}^o$-dg-modules which are perfect over $\widehat{\mathbf{DR}}^o$. But $\mathrm{Perf}^{\nabla}(\widehat{\partial}X)$ is a sub-prestack of the stack $\mathrm{Perf}^{\mathrm{gr},\boldsymbol{\epsilon}}(\widehat{\mathbf{DR}}^o)$ which is defined by a local condition and thus is a stack. The fact that $\mathrm{Perf}^{\nabla,\mathrm{ex}}(\widehat{\partial}X)$ is also a stack now follows from the fact that it is defined as a fiber product of stacks.

For part (2), we use a similar argument. The derived stack $\mathrm{Perf}^{\nabla,\mathrm{ex}}(\widehat{\partial}X)$ can be expressed as a full substack of the fixed points by $\mathcal{H}$ acting on $\widehat{\mathbf{DR}}^o$-dg-modules inside $\mathrm{Perf}^{\mathrm{ex}}(\widehat{\partial}X)$ (note that as a graded cdga, $\widehat{\mathbf{DR}}^o$ lives in $\mathrm{Perf}^{\mathrm{ex}}(\widehat{\partial}X)$). Therefore, to prove that $\mathrm{Perf}^{\nabla,\mathrm{ex}}(\widehat{\partial}X)$ is independent of the choice of $\mathfrak{X}$, we have to check that the stack of $\mathcal{H}$-equivariant dg-categories $\mathrm{Perf}(\widehat{\mathbf{DR}}^o)$ only depends on X. This reduces to the following lemma.

LEMMA 3.9. *Let $\pi\colon \mathfrak{X}' \to \mathfrak{X}$ be a morphism between two good compactifications of X. Let $D' = \pi^{-1}(D)$, so that π induces an isomorphism between $\mathfrak{X}'-D'$ and $\mathfrak{X}-D$. Let $\widehat{\mathbf{DR}}^o_{\mathfrak{X}}$ and $\widehat{\mathbf{DR}}^o_{\mathfrak{X}'}$ be the corresponding sheaves of graded mixed cdga constructed above. Then, for any $\mathrm{Spec}\, B \in \mathbf{dAff}$, we have the following:*

(1) *There is a pull-back map $f_\pi\colon \pi^{-1}(\widehat{\mathbf{DR}}^o_{\mathfrak{X},B}) \to \widehat{\mathbf{DR}}^o_{\mathfrak{X}',B}$ of sheaves of graded mixed cdga on $\mathfrak{X}'_{\mathrm{et}}$.*

(2) *The map f_π induces an equivalence of dg-categories $\pi^*\colon \mathrm{Perf}(\widehat{\mathbf{DR}}^o_{\mathfrak{X},B}) \simeq \mathrm{Perf}(\widehat{\mathbf{DR}}^o_{\mathfrak{X}',B})$.*

Before giving a proof of the lemma, let us explain how this finishes the proof of the proposition. The fact that f_π exists implies that the dg-functor π^* also exists, by simply pulling back graded mixed dg-modules. Moreover, as f_π is a morphism of graded mixed cdga, it is clear that the dg-functor π^* is naturally $\mathcal{H}$-equivariant. As it is an equivalence, it also induces an equivalence on the fixed points dg-categories, and the result follows immediately by considering the full sub-dg-categories corresponding to $\mathrm{Perf}^{\nabla,\mathrm{ex}}(\widehat{\partial}X)$.

Proof of Lemma 3.9 and therefore Proposition 3.8. The existence of the map f_π simply follows from the fact that the assignments $A \mapsto \mathbf{DR}_B(A)$, $A \mapsto \widehat{\mathbf{DR}}_B(A)$, and $A \mapsto \widehat{\mathbf{DR}}_B(A)[t^{-1}]$ are functors from smooth k-algebras of finite type to graded mixed cdga. To prove part (2), we observe that $\widehat{\mathbf{DR}}^o_{\mathfrak{X},B}$ and $\widehat{\mathbf{DR}}^o_{\mathfrak{X}',B}$, when considered as sheaves of cdga, are perfect over $\widehat{\mathcal{O}}^o_{D,B}$ and $\widehat{\mathcal{O}}^o_{D',B}$ and extendable. They can thus be considered as graded cdga inside the symmetric monoidal dg-categories $\mathrm{Perf}^{\mathrm{ex}}(\widehat{\partial}X)(B)$ and $\mathrm{Perf}^{\mathrm{ex}}(\widehat{\partial}X')(B)$. By Corollary 3.6, we know that

pull-back along π induces an equivalence of symmetric monoidal dg-categories

$$\pi^* \colon \mathsf{Perf}^{\mathrm{ex}}\big(\widehat{\partial}X\big)(B) \simeq \mathsf{Perf}^{\mathrm{ex}}\big(\widehat{\partial}X'\big)(B)\,.$$

To finish the proof, it remains to show that the symmetric monoidal equivalence π^* sends the cdga $\widehat{\mathbf{DR}}^o_{\mathfrak{X},B}$ to $\widehat{\mathbf{DR}}^o_{\mathfrak{X}',B}$. There are canonical restriction maps

$$R \colon \mathsf{Perf}(X) \longrightarrow \mathsf{Perf}\big(\widehat{\partial}X\big), \quad R' \colon \mathsf{Perf}(X) \longrightarrow \mathsf{Perf}\big(\widehat{\partial}X'\big)\,,$$

and we have $\pi^* \circ R' \simeq R$. Moreover, by construction, we have

$$\widehat{\mathbf{DR}}^o_{\mathfrak{X},B} \simeq R(\mathbf{DR}_X), \quad \widehat{\mathbf{DR}}^o_{\mathfrak{X}',B} \simeq R'(\mathbf{DR}_X)\,,$$

where $\mathbf{DR}_X = \mathrm{Sym}_{\mathcal{O}_X}\big(\Omega^1_X[1]\big)$ as a sheaf of perfect cdga over $\mathcal{O}_X$. This completes the proof of the lemma and of Proposition 3.8. $\qquad\square$

To finish this section, note that as for the case of perfect complexes, there is a restriction morphism

$$\mathsf{R} \colon \mathsf{Perf}^{\nabla}(X) \longrightarrow \mathsf{Perf}^{\nabla,\mathrm{ex}}\big(\widehat{\partial}X\big) \subset \mathsf{Perf}^{\nabla}\big(\widehat{\partial}X\big)$$

from the derived stack $\mathsf{Perf}^{\nabla}(X)$ of perfect complexes on X endowed with flat connections to the derived stack of extendable perfect complexes with flat connections on the formal boundary of X. It is defined as follows. First of all, the derived stack $\mathsf{Perf}^{\nabla}(X)$ is defined as the derived stack of graded mixed dg-modules over $\mathbf{DR}_X$, the de Rham algebra of X, which are perfect of weight zero. More precisely, if $S = \mathrm{Spec}\, B \in \mathbf{dAff}$, then $\mathsf{Perf}^{\nabla}(X)(S)$ is defined to be the ∞-category of graded mixed $\mathbf{DR}_X \otimes_k B$-dg-modules E, so that $E \simeq E(0) \otimes_{\mathcal{O}_X} \mathbf{DR}_X$ as a graded dg-modules over B, where $E(0)$ is perfect over $\mathcal{O}_X \otimes_k B$. The restriction map R is then induced by the natural morphism of sheaves of graded mixed cdga over $\mathfrak{X}_{\mathrm{et}}$

$$\mathbf{DR}_X \otimes_k B \longrightarrow \widehat{\mathbf{DR}}^o_B\,.$$

Locally on an étale affine $\mathrm{Spec}\, A \to \mathfrak{X}$ on which D is principal with equation $t \in A$, this morphism is the natural map

$$\mathbf{DR}(A \otimes_k B)\big[t^{-1}\big] \longrightarrow \widehat{\mathbf{DR}}(A \otimes_k B)\big[t^{-1}\big]$$

induced by the completion morphism $A \otimes_k B \to \lim_n \big(A/I^n \otimes_k B\big)$.

This defines a restriction map $\mathsf{R} \colon \mathsf{Perf}^{\nabla}(X) \to \mathsf{Perf}^{\nabla}\big(\widehat{\partial}X\big)$ which covers the restriction map of perfect complexes $R \colon \mathsf{Perf}(X) \to \mathsf{Perf}\big(\widehat{\partial}X\big)$. As the later map factors through extendable perfect complexes (because any perfect complex on $X \times S$ extends to $\mathfrak{X} \times S$ up to a retract), we also get a restriction map $\mathsf{Perf}^{\nabla}(X) \to \mathsf{Perf}^{\nabla,\mathrm{ex}}\big(\widehat{\partial}X\big)$.

DEFINITION 3.10. The *de Rham restriction map* is the morphism of derived stacks

$$\mathsf{R} \colon \mathsf{Perf}^{\nabla}(X) \longrightarrow \mathsf{Perf}^{\nabla,\mathrm{ex}}\big(\widehat{\partial}X\big)$$

defined above.

3.3 De Rham cohomology of the formal boundary and compactly supported cohomology

To finish this section, let us describe the Hom-complexes of the dg-category $\mathsf{Perf}^{\nabla}\big(\widehat{\partial}X\big)(B)$ in terms of the hypercohomology of complexes of sheaves on D and relate this to the notion of compactly supported de Rham cohomology. The notion of de Rham cohomology with compact supports already appeared in [BCF04], but our treatment here is new as it is based on the theory

of Tate objects and their duality (see [Hen17]), which also makes the theory available over any base cdga B. In this part, we give the constructions and definitions of compactly supported cohomology. The duality itself is studied in Section 5.2.

Again we fix a good compactification $X \hookrightarrow \mathfrak{X}$ with divisor at infinity D. For any connective cdga B and any object $E \in \mathsf{Perf}^\nabla(\widehat{\partial}X)(B)$, we define a sheaf of B-dg-modules on D_{Zar} as follows. By definition, E is a sheaf of graded mixed modules over $\widehat{\mathbf{DR}}_B^o$. We define $|E|$ to be the sheaf of B-modules $\underline{\mathrm{Hom}}_{\mathbf{dg}_\epsilon^{\mathrm{gr}}}(k, E)$ of graded mixed morphisms from the unit k to E. Note that, a priori, $|E|$ is given as an infinite product

$$|E| = \prod_{i \geqslant 0} E(i)[-2i],$$

where the differential is the sum of the cohomological differential and the mixed structure. However, in our situation this infinite product is in fact a finite product, as $E(i)$ is non-zero only for a finite number of indices i (because E is free as a graded module and $\widehat{\mathbf{DR}}_B^o$ only has weights in the interval $[0, d]$, where $d = \dim_k X$). We will call the sheaf of B-dg-modules $|E|$ the *de Rham complex of E completed along D*. With this notation we now have the following.

DEFINITION 3.11. The *de Rham cohomology of $\widehat{\partial}X$ with coefficients in E* is the B-module defined by

$$\mathbb{H}_{\mathrm{DR}}(\widehat{\partial}X, E) := \mathbb{H}(D, |E|) \in B - \mathbf{dg}.$$

Going back to the problem of computing Hom-complexes, let E and F be two objects in $\mathsf{Perf}^\nabla(\widehat{\partial}X)(B)$. The dg-category of graded mixed modules over $\widehat{\mathbf{DR}}_B^o$ has a canonical symmetric monoidal structure, for which the tensor product is given by tensoring the underlying B-dg-modules (see [PTVV13, CPT+17]). Since perfect complexes of $\widehat{\mathcal{O}}_{D,B}^o$-modules form a rigid symmetric monoidal dg-category, it follows that $\mathsf{Perf}^\nabla(\widehat{\partial}X)(B)$ is also rigid. We can then form $E^\vee \otimes_{\widehat{\mathcal{O}}_{D,B}^o} F$, which is a new graded mixed $\widehat{\mathbf{DR}}_B^o$-module and an object in $\mathsf{Perf}^\nabla(\widehat{\partial}X)(B)$. To simplify notation, we will simply write $E^\vee \otimes F$ for this object. We then have a natural quasi-isomorphism

$$\underline{\mathrm{Hom}}_{\mathsf{Perf}^\nabla(\widehat{\partial}X)(B)}(E, F) \simeq \mathbb{H}_{\mathrm{DR}}(\widehat{\partial}X, E^\vee \otimes F),$$

giving us the desired interpretation of mapping complexes of $\mathsf{Perf}^\nabla(\widehat{\partial}X)(B)$ in terms of de Rham cohomology of $\widehat{\partial}X$.

Remark 3.12. For any connective B and any object $E \in \mathsf{Perf}^\nabla(\widehat{\partial}X)(B)$, the complex of sheaves $|E|$ on D_{Zar} is built out of acyclic sheaves on affines. Therefore, the hypercohomology complex $\mathbb{H}(D, |E|)$ can be computed by a finite limit using an affine cover of D. In particular, if $|E|$ is locally perfect as a B-module on D, then $\mathbb{H}(D, |E|)$ is a perfect B-module.

We now show how the formal boundary $\widehat{\partial}X$ can be used to define a notion of cohomology with compact supports, both for perfect complexes and for perfect complexes with flat connections on X. We start with a connective cdga B and a perfect complex E on $X \times S$, where $S = \mathrm{Spec}\, B$. As explained in Section 3.2 (right before Definition 3.10), we have its restriction $R(E) \in \mathsf{Perf}(\widehat{\partial}X)(B)$ and, by functoriality, an induced map on cohomology

$$\mathbb{H}(X, E) = \underline{\mathrm{Hom}}(\mathcal{O}_X, E) \longrightarrow \mathbb{H}(\widehat{\partial}X, R(E)) = \underline{\mathrm{Hom}}(R(\mathcal{O}_X), R(E)). \tag{3.1}$$

The *cohomology of X with compact supports and with coefficients in E* is defined to be the

homotopy fiber of the map of complexes (3.1). It is denoted by

$$\mathbb{H}_c(X, E) := \mathsf{fib}\big(\mathbb{H}(X, E) \longrightarrow \mathbb{H}\big(\widehat{\partial}X, R(E)\big)\big) \in B - \mathbf{dg}\,.$$

By construction, this is a B-dg-module. This is not quite enough for our purpose, as this B-dg-module turns out to be the realization of a natural pro-object that we will now describe. This pro-structure is going to be very important for us, as it will allow us to work with compactly supported cohomology as dual of cohomology, even if the latter is infinite-dimensional. For simplicity, we assume that E extends to our fixed good compactification as a perfect complex $\mathcal{E}$ on $\mathfrak{X} \times S$. It is not always possible to find $\mathcal{E}$ in general, although it always exists if $B = k$ (because $K_{-1}(X) = 0$). Moreover, such an extension always exists up to a retract, so assuming the existence of $\mathcal{E}$ is thus not a real restriction. Also note that in all our applications, E will always come with an extension to $\mathfrak{X}$.

Using the formal gluing formalism of [HPV16], we obtain a cartesian square of B-dg-modules

$$
\begin{array}{ccc}
\mathbb{H}(\mathfrak{X}, \mathcal{E}) & \longrightarrow & \mathbb{H}(X, E) \\
\downarrow & & \downarrow \\
\mathbb{H}(\widehat{\mathfrak{X}}, \widehat{\mathcal{E}}) & \longrightarrow & \mathbb{H}\big(\widehat{\partial}X, R(E)\big)\,.
\end{array}
$$

Here $\widehat{\mathfrak{X}}$ is the formal completion of X along D, and $\mathbb{H}\big(\widehat{\mathfrak{X}}, \widehat{\mathcal{E}}\big)$ is defined by

$$\mathbb{H}\big(\widehat{\mathfrak{X}}, \widehat{\mathcal{E}}\big) := \lim_n \mathbb{H}\big(\mathfrak{X}_{(n)}, \mathsf{j}_n^* \mathcal{E}\big)\,,$$

where $\mathsf{j}_n \colon \mathfrak{X}_{(n)} := \mathrm{Spec}\big(\mathcal{O}_{\mathfrak{X}}/I_D^n\big) \to \mathfrak{X}$ is the $(n-1)$th infinitesimal thickening of D inside $\mathfrak{X}$.

From the diagram above, we have that $\mathbb{H}_c(X, E)$ can also be described as the fiber of the map

$$\mathbb{H}(\mathfrak{X}, \mathcal{E}) \longrightarrow \lim_n \mathbb{H}(\mathfrak{X}_{(n)}, \mathsf{j}_n^* \mathcal{E})\,. \tag{3.2}$$

The morphism (3.2) can itself be considered as a morphism of pro-objects in $\mathrm{Perf}(B)$. This allows us to define a pro-perfect B-module by

$$\widetilde{\mathbb{H}}_c(X, E) := \mathsf{fib}\big(\mathbb{H}(\mathfrak{X}, \mathcal{E}) \longrightarrow \text{``}\lim_n\text{''} \mathbb{H}(\mathfrak{X}_{(n)}, \mathsf{j}_n^* \mathcal{E})\big) \in \mathrm{ProPerf}(B)\,.$$

It is easy to show that this definition does not depend on the choice of either $\mathfrak{X}$ or $\mathcal{E}$, but we will not do it here. For us, this will be a consequence of Serre duality with supports, which is studied in Section 5.2, as the dual B-module turns out to be canonically equivalent to $\mathbb{H}\big(X, E^\vee \otimes_{\mathcal{O}_X} \omega_X\big)$, which only depends on X and E. For future reference, we record the following.

DEFINITION 3.13. The *refined cohomology with compact supports* of X with coefficients in E is the pro-perfect B-module $\widetilde{\mathbb{H}}_c(X, E)$ defined above.

One nice aspect of the refined version of compactly supported cohomology is that it is manifestly compatible with base changes of B. Let $B \to B'$ be any morphism of connective cdga; then the natural map

$$\widetilde{\mathbb{H}}_c(X, E)\widehat{\otimes}_B B' \longrightarrow \widetilde{\mathbb{H}}_c(X, E \otimes_B B')$$

is an equivalence of pro-perfect B'-modules. Here we have denoted by

$$\widehat{\otimes}_B B' \colon \mathrm{ProPerf}(B) \longrightarrow \mathrm{ProPerf}(B')$$

the functor induced on pro-objects by the usual base change $\otimes_B B' \colon \mathrm{Perf}(B) \to \mathrm{Perf}(B')$.

Another feature of the refined compactly supported cohomology comes from the existence of a fiber sequence of B-modules

$$\mathbb{H}_c(X,E) \longrightarrow \mathbb{H}(X,E) \longrightarrow \mathbb{H}\big(\widehat{\partial}X, R(E)\big).$$

The first map in this sequence arises from a natural morphism of ind-pro-perfect B-modules $\widetilde{\mathbb{H}}_c(X,E) \to \mathbb{H}(X,E)$, where $\mathbb{H}(X,E)$ is considered as an ind-perfect B-module via the canonical equivalence $\mathrm{IndPerf}(B) \simeq B-\mathbf{dg}$. This implies that $\mathbb{H}\big(\widehat{\partial}X, R(E)\big)$ is itself the realization of an ind-pro-perfect module $\widetilde{\mathbb{H}}\big(\widehat{\partial}X, R(E)\big) \in \mathrm{IndProPerf}(B)$, sitting in a triangle

$$\widetilde{\mathbb{H}}_c(X,E) \longrightarrow \mathbb{H}(X,E) \longrightarrow \widetilde{\mathbb{H}}\big(\widehat{\partial}X, R(E)\big).$$

By construction, the ind-pro-perfect object $\widetilde{\mathbb{H}}\big(\widehat{\partial}X, R(E)\big)$ is an extension of a pro-perfect by an ind-perfect and thus, by definition, is a Tate B-module in the sense of [Hen17].

We now turn to the case of an object $E \in \mathsf{Perf}^{\nabla}(X)(B)$. The naive de Rham cohomology of E with compact supports is again defined as

$$\mathbb{H}_{c,\mathrm{DR}}(X,E) := \mathsf{fib}\big(\mathbb{H}_{\mathrm{DR}}(X,E) \longrightarrow \mathbb{H}_{\mathrm{DR}}\big(\widehat{\partial}X, \mathsf{R}(E)\big)\big) \in B-\mathbf{dg}.$$

As before, this B-module is the realization of a natural pro-perfect B-module denoted by $\widetilde{\mathbb{H}}_{c,\mathrm{DR}}(X,E)$. We assume again that the underlying perfect complex $E(0)$ of E extends to a perfect complex $\mathcal{E}(0)$ on $\mathfrak{X} \times S$. Using this, one immediately checks that the sheaf of B-modules $|R(E)|$ on D has a natural structure of a sheaf of ind-pro B-modules. Indeed, it is of the form $\oplus_i R(E(i))[-2i]$ with a suitable differential. Each $R(E(i))$ is itself of the form $\mathcal{E}(0) \otimes_{\mathcal{O}_{\mathfrak{X}}} \Omega^i_{\mathfrak{X}} \otimes_{\mathcal{O}_{\mathfrak{X}}} \widehat{\mathcal{O}}^o_{D,B}$. As $\widehat{\mathcal{O}}^0_{X,B}$ has a canonical ind-pro structure, and since the functor $\mathcal{E}(0) \otimes_{\mathcal{O}_X} \Omega^i_X \otimes_{\mathcal{O}_X} (-)$ commutes with limits and colimits of $\mathcal{O}_X$-modules, we see that each $R(E)(i)$ is the realization of a canonical sheaf of ind-pro B-modules. Moreover, since $\widehat{\mathcal{O}}^o_{D,B}$ is ind-pro-perfect as a B-module, this endows $|E|$ with a natural structure of sheaf of Tate B-modules. This provides a canonical Tate structure on the hypercohomology of D with coefficients in $|E|$, that is, on the de Rham cohomology of $\widehat{\partial}X$ with coefficients in $\mathsf{R}(E)$. We denote this Tate B-module by $\widetilde{\mathbb{H}}_{\mathrm{DR}}\big(\widehat{\partial}X, \mathsf{R}(E)\big)$.

The restriction map R induces a morphism $\mathbb{H}_{\mathrm{DR}}(X,E) \to \widetilde{\mathbb{H}}_{\mathrm{DR}}\big(\widehat{\partial}X, \mathsf{R}(E)\big)$, which is a morphism of ind-pro-perfect B-modules if one endows the left-hand side with the canonical structure of an ind-perfect B-module. It thus lifts to a morphism of Tate B-modules

$$\widetilde{\mathbb{H}}_{\mathrm{DR}}(X,E) \longrightarrow \widetilde{\mathbb{H}}_{\mathrm{DR}}\big(\widehat{\partial}X, \mathsf{R}(E)\big).$$

With this notation, we can now formulate the following.

DEFINITION 3.14. The *refined de Rham cohomology of X with compact supports with coefficients in E* is the Tate B-module defined by

$$\widetilde{\mathbb{H}}_{c,\mathrm{DR}}(X,E) := \mathsf{fib}\big(\widetilde{\mathbb{H}}_{\mathrm{DR}}(X,E) \longrightarrow \widetilde{\mathbb{H}}_{\mathrm{DR}}\big(\widehat{\partial}X, \mathsf{R}(E)\big)\big).$$

It is instructive to note that the ind-pro-perfect B-module structure on $\widetilde{\mathbb{H}}_{c,\mathrm{DR}}(X,E)$ is in fact pro-perfect (in particular, it is a Tate B-module in the sense of [Hen17]). This can be seen by reducing to the previously treated case of perfect complexes without connections. Indeed, the complexes of sheaves $|E|$ and $|R(E)|$ are canonically filtered using their Hodge filtrations. The graded pieces of the Hodge filtration on $\widetilde{\mathbb{H}}_{c,\mathrm{DR}}(X,E)$ are $\widetilde{\mathbb{H}}_c\big(X, \Omega^i_X \otimes_{\mathcal{O}_X} E(0)\big)[-i]$ and thus are pro-perfect. Since this filtration is finite, we deduce that the ind-pro-object $\widetilde{\mathbb{H}}_{c,\mathrm{DR}}(X,E)$ is filtered with pro-perfect associated graded. This implies that $\widetilde{\mathbb{H}}_{c,\mathrm{DR}}(X,E)$ itself is pro-perfect. We thus have proven the following corollary.

COROLLARY 3.15. *The ind-pro-perfect B-module $\widetilde{\mathbb{H}}_{c,\mathrm{DR}}(X, E)$ is pro-perfect. Furthermore, the ind-pro-perfect B-module $\widetilde{\mathbb{H}}_{\mathrm{DR}}\big(\widehat{\partial}X, \mathsf{R}(E)\big)$ is a Tate B-module in the sense of* [Hen17].

As in the case of perfect complexes, the formation of $\widetilde{\mathbb{H}}_{c,\mathrm{DR}}(X, E)$ commutes with base change over B: for any map $B \to B'$ of connective cdga, the natural morphism

$$\widetilde{\mathbb{H}}_{c,\mathrm{DR}}(X, E)\widehat{\otimes}_B B' \longrightarrow \widetilde{\mathbb{H}}_{c,\mathrm{DR}}(X, E \otimes_B B')$$

is an equivalence of pro-perfect B'-modules.

4. Formal properties of moduli functors

We start by recalling some of the general formal properties of derived stacks (see [TV08, Lur18]). Let $F \in \mathbf{dAff}_k$ be a derived stack over k. For any derived affine scheme $U = \operatorname{Spec} B$ with a map $u\colon U \to F$ and any connective B-dg-module M, we can define the space of derivations of F on U with coefficients in M as the fiber at u of the restriction map

$$F(B \oplus M) \longrightarrow F(B) \,,$$

where $B \oplus M$ is the trivial square zero extension of B by M. Denote this space by $\operatorname{Der}_u(F, M) \in \mathbb{T}$. For any morphism $B' \to B$ of connective cdga and any connective B-dg-module M, we have a canonical morphism $B' \oplus M \to B \oplus M$ covering the map $B' \to B$. Therefore, for any commutative diagram of derived stacks

$$
U = \operatorname{Spec} B \xrightarrow{\quad f \quad} U' = \operatorname{Spec} B' \,,
$$
$$
\begin{array}{ccc}
& u \searrow \quad \swarrow u' & \\
& F &
\end{array}
$$

there is a natural induced morphism on the corresponding spaces of derivations

$$f^*\colon \operatorname{Der}_u(F, M) \longrightarrow \operatorname{Der}_{u'}(F, M') \,.$$

DEFINITION 4.1. (1) The derived stack F has a *cotangent complex at* $u\colon U = \operatorname{Spec} B \to F$ if there exist a possibly connective B-dg-module $\mathbb{L}_{F,u}$ and functorial equivalences

$$\operatorname{Map}_{B-\mathrm{mod}}(\mathbb{L}_{u,F}, M) \simeq \operatorname{Der}_u(F, M) \,.$$

(2) We say that F has a (global) *cotangent complex* if it has cotangent complexes at all maps $u\colon U = \operatorname{Spec} B \to F$ and if, moreover, for any commutative diagram

$$
U = \operatorname{Spec} B \xrightarrow{\quad f \quad} U' = \operatorname{Spec} B' \,,
$$
$$
\begin{array}{ccc}
& u \searrow \quad \swarrow u' & \\
& F &
\end{array}
$$

the induced morphism $\operatorname{Der}_u(F, M) \to \operatorname{Der}_{u'}(F, M')$ is an equivalence.

It is shown in [TV08, Lur18] that $\mathbb{L}_{F,u}$, if it exists, is uniquely characterized by the ∞-functor $\operatorname{Der}_u(F, -)$. Also, the condition in (2) can be reformulated as the statement that the natural morphism $\mathbb{L}_{u,F} \otimes_B B' \to \mathbb{L}_{u',F}$ is an equivalence of dg-modules.

Let $\operatorname{Spec} B \in \mathbf{dAff}_k$ be a derived affine scheme and M a connective B-module. Let $d\colon B \to M[1]$ be a k-linear derivation, which by definition means a section of $B \oplus M \to B$ inside the

∞-category of cdga over k. Recall that the square zero extension of B by M with respect to d, denoted by $B \oplus_d M$, is defined by the cartesian square of cdga (see [TV08])

$$
\begin{array}{ccc}
B \oplus_d M & \longrightarrow & B \\
\downarrow & & \downarrow 0 \\
B & \xrightarrow{\ d\ } & B \oplus M[1]\,,
\end{array}
$$

where 0 denotes the natural inclusion of B as a direct factor in the trivial square zero extension $B \oplus M[1]$.

DEFINITION 4.2. Let F be a derived stack.

(1) We say that F is *inf-cartesian* if for any B, M, and d as above, the square

$$
\begin{array}{ccc}
F(B \oplus_d M) & \longrightarrow & F(B) \\
\downarrow & & \downarrow 0 \\
F(B) & \xrightarrow{\ d\ } & F(B \oplus M[1])
\end{array}
$$

is cartesian.

(2) We say that F is *nilcomplete* if for any $\operatorname{Spec} B \in \mathbf{dAff}_k$ with Postnikov tower $\{B_{\leqslant n}\}_n$, the natural morphism

$$
F(B) \longrightarrow \lim_n F(B_{\leqslant n})
$$

is an equivalence.

Now suppose that F is a derived stack which is inf-cartesian. For any $x \colon \operatorname{Spec} B \to F$, we have an ∞-functor

$$
\mathbb{T}_{F,x} \colon B - \operatorname{Mod}^c \longrightarrow \mathbb{T}
$$

from connective B-modules to spaces that sends M to the fiber of $F(B \oplus M) \to F(B)$ at the point x. This ∞-functor restricts to the full sub-∞-category of B-modules of the form $B[i]^n$ for various $i \geqslant 0$ and various n. Because F is inf-cartesian, the ∞-functor $\mathbb{T}_{F,x}$ preserves finite products as well as the looping construction Ω_* (that is, the natural map $\mathbb{T}_{F,x}(M[-1]) \to \Omega_*(\mathbb{T}_{F,x}(M))$ is an equivalence of spaces). This implies that there exists a unique B-dg-module $\mathbb{T}_{F,x}$ such that $\mathbb{T}_{F,x}(B[i]^n) \simeq \operatorname{Map}_{B-\operatorname{Mod}}(B[-i]^n, \mathbb{T}_{F,x})$ for all $i \geqslant 0$ and n. We still denote this complex by $\mathbb{T}_{F,x}$ and call it the *tangent complex of F at x*. The following result is an easy criterion for the existence of cotangent complexes.

LEMMA 4.3. *Let F be a derived stack which is inf-cartesian and $x \colon \operatorname{Spec} B \to F$ a map. Assume that the two conditions below are satisfied.*

(1) *The ∞-functor $M \mapsto \mathbb{T}_{F,x}(M)$ commutes with arbitrary colimits.*

(2) *The B-module $\mathbb{T}_{F,x}$ is perfect.*

Then F has a cotangent complex $\mathbb{L}_{F,x}$ at x, and, moreover, $\mathbb{L}_{F,x}$ is naturally identified with $\mathbb{T}^\vee_{F,x}$, the B-linear dual of $\mathbb{T}_{F,x}$.

Proof. We consider the two ∞-functors $B - \operatorname{Mod}^c \to \mathbb{T}$ sending a B-module M to either $\operatorname{Map}_{B-\operatorname{Mod}}(B, \mathbb{T}_{F,x} \otimes_B M)$ or $\mathbb{T}_{F,x}(M)$. There is a canonical equivalence of functors

$$
\operatorname{Map}_{B-\operatorname{Mod}}(B, \mathbb{T}_{F,x} \otimes_B -) \cong \mathbb{T}_{F,x}(-) \tag{4.1}
$$

when these functors are restricted to the full sub-∞-category of objects of the form $B[i]^n$. However, these objects generate $B - \mathrm{Mod}^c$ by colimits, so by condition (1), the map (4.1) extends to an equivalence of ∞-functors defined on the whole ∞-category $B - \mathrm{Mod}^c$. In formulas, for any connective B-module M, we have a natural equivalence

$$\mathrm{Map}_{B-\mathrm{Mod}}(B, \mathbb{T}_{F,x} \otimes_B M) \simeq \mathbb{T}_{F,x}(M) \, .$$

When $\mathbb{T}_{F,x}$ is moreover perfect, this implies that $\mathbb{T}_{F,x}(M) \simeq \mathrm{Map}_{B-\mathrm{Mod}} (\mathbb{T}_{F,x}^\vee, M)$ and thus that the cotangent complex of F at x exists and is $\mathbb{L}_{F,x} = \mathbb{T}_{F,x}^\vee$. $\qquad\square$

4.1 Infinitesimal properties of Perf^∇

We now study the infinitesimal structure of the derived moduli stacks $\mathsf{Perf}^\nabla(X)$ and $\mathsf{Perf}^\nabla(\widehat{\partial X})$ constructed in Section 3. The main result is the following.

PROPOSITION 4.4. *Let X be a smooth algebraic variety over k, and let $X \hookrightarrow \mathfrak{X}$ be a good compactification. Then, the two derived moduli stacks $\mathsf{Perf}^\nabla(X)$ and $\mathsf{Perf}^\nabla(\widehat{\partial X})$ are nilcomplete and infinitesimally cartesian.*

Proof. We start with $\mathsf{Perf}^\nabla(X)$. By construction, this derived stack is a derived mapping stack and can be written in the form $\mathsf{Perf}^\nabla(X) \simeq \mathbf{Map}_{\mathbf{dSt}_k}(X_{\mathrm{DR}}, \mathsf{Perf})$, where X_{DR} is the de Rham functor associated with X (see, for example, [GR14] for the relation between $\mathcal{D}$-modules and sheaves on X_{DR}). We can write $X = \mathrm{colim}\,\mathrm{Spec}\,A_i$ as a finite colimit of affine schemes, and thus $X_{\mathrm{DR}} \simeq \mathrm{colim}(\mathrm{Spec}\,A_i)_{\mathrm{DR}}$. The derived stack $\mathsf{Perf}^\nabla(X)$ is then the limit of $\mathsf{Perf}^\nabla(\mathrm{Spec}\,A_i)$. Since a limit of nilcomplete (respectively, infinitesimally cartesian) derived stacks is again nilcomplete (respectively, infinitesimally cartesian), we have reduced the statement to the case where $X = \mathrm{Spec}\,A$ is furthermore affine. The $\mathsf{Perf}^\nabla(X)$ statement thus boils down to the following.

LEMMA 4.5. *Let F be a nilcomplete (respectively infinitesimally cartesian) derived stack over k. For any affine scheme X, the derived mapping stack $\mathbf{Map}_{\mathbf{dSt}_k}(X, F)$ is again nilcomplete (respectively, infinitesimally cartesian).*

Proof. Let $X = \mathrm{Spec}\,A$, and let B be any connective cdga. First, assume that F is nilcomplete. We consider the Postnikov tower $\{B_{\leqslant n}\}_n$ of B. The natural map

$$\mathbf{Map}_{\mathbf{dSt}_k}(X, F)(B) \longrightarrow \lim_n \mathbf{Map}_{\mathbf{dSt}_k}(X, F)(B_{\leqslant n})$$

can be written as

$$F(A \otimes_k B) \longrightarrow \lim_n F(A \otimes_k B_{\leqslant n}) \, .$$

As k is a field, A is a flat over k, and the tower $\{A \otimes_k B_{\leqslant n}\}_n$ is a Postnikov tower for $A \otimes_k B$, and thus by the assumption on F, the above morphism is an equivalence.

Let us now assume that F is infinitesimally cartesian. Let $B \oplus_d M$ be a square zero extension of B by a connective module M, given by a cartesian square

$$
\begin{array}{ccc}
B \oplus_d M & \longrightarrow & B \\
\downarrow & & \downarrow{\scriptstyle d} \\
B & \xrightarrow{\quad 0 \quad} & B \oplus M[1] \, .
\end{array}
$$

Again because A is flat over k, tensoring with A induces a pull-back diagram of connective cdga

$$
\begin{array}{ccc}
C \oplus_d M_C & \longrightarrow & C \\
\downarrow & & \downarrow d \\
C & \xrightarrow{\quad 0 \quad} & C \oplus M_C[1] \,,
\end{array}
$$

where $C := A \otimes_k B$ and $M_C := C \otimes_B M$. As F is assumed infinitesimally cartesian, the image of this diagram by F remains a pull-back. By definition, this diagram is equivalent to

$$
\begin{array}{ccc}
\mathbf{Map_{dSt_k}}(X, F)(B \oplus_d M) & \longrightarrow & \mathbf{Map_{dSt_k}}(X, F)(B) \\
\downarrow & & \downarrow \\
\mathbf{Map_{dSt_k}}(X, F)(B) & \longrightarrow & \mathbf{Map_{dSt_k}}(X, F)(B \oplus M[1]) \,.
\end{array}
$$

This shows that $\mathbf{Map_{dSt_k}}(X, F)$ is infinitesimally cartesian. $\qquad\square$

Proof of Proposition 4.4, *continued.* Next, we analyze $\mathsf{Perf}^\nabla\big(\widehat{\partial}X\big)$. The argument here is slightly different since this is not a derived mapping stack. We start by writing $\mathfrak{X} = \mathrm{colim}\,\mathrm{Spec}\,A_i$ as a colimit of open affine sub-schemes. Without loss of generality, we can assume that the divisor D is principal on each $\mathrm{Spec}\,A_i$, defined by an equation $f_i \in A_i$. By the descent result of [HPV16], we know that $\mathsf{Perf}^\nabla\big(\widehat{\partial}X\big)$ is then equivalent to a limit $\mathsf{Perf}^\nabla\big(\widehat{\partial}X\big) = \lim_i F_i$ of derived stacks. These derived stacks F_i can be described as follows. For each connective cdga B, we have the completed de Rham algebra of $A_i \otimes_k B$ defined by

$$
\widehat{\mathrm{DR}}_B(A_i) := \lim_j \big(\mathrm{DR}\big(A_i/(f_i)^j\big) \otimes_k B\big)\,.
$$

This is a B-linear graded mixed cdga for which the weight zero part is

$$
\widehat{A_i \otimes_k B} := \lim_j \big(A_i/(f_i)^j \otimes_k B\big)\,.
$$

By inverting the weight zero element f_i, we have a new graded mixed cdga

$$
\widehat{\mathrm{DR}}^o_B(A_i) := \lim_j \big(\mathrm{DR}\big(A_i/(f_i)^j\big) \otimes_k B\big)\big[f_i^{-1}\big]\,.
$$

The derived stack F_i is then the functor sending B to the space of all graded mixed $\widehat{\mathrm{DR}}^o_B(A_i)$-dg-modules which are perfect as $\widehat{A \otimes_k B}$-dg-modules. Let us drop the index i and simply write A and f for A_i and f_i. The derived stacks under consideration naturally carry structures of stacks in dg-categories and will be considered as such below.

We have a forgetful dg-functor

$$
\widehat{\mathrm{DR}}^o_B(A) - \mathbf{dg}^{\mathrm{gr}}_\epsilon \longrightarrow \widehat{\mathrm{DR}}^o_B(A) - \mathbf{dg}
$$

from graded mixed dg-modules to dg-modules. According to Corollary 2.8, this realizes the left-hand side as the dg-category of fixed points in $\widehat{\mathrm{DR}}^o_B(A) - \mathbf{dg}$ for the natural action by the group $\mathcal{H}$ on the right-hand side. Restricting to dg-modules which are perfect over $\widehat{A \otimes_k B}$ on both sides provides a similar forgetful dg-functor

$$
F(A) \longrightarrow \mathrm{Perf}\big(\widehat{\mathrm{DR}}^o_B(A)\big)\,.
$$

Our statement now reduces to the following.

LEMMA 4.6. *The ∞-functor $B \mapsto \mathrm{Perf}\big(\widehat{\mathrm{DR}}^o_B(A)\big)$ is nilcomplete and infinitesimally cartesian as a derived stack of dg-categories.*

Proof. Note that if $\{B_{\leqslant n}\}_n$ is the Postnikov tower for B, then $\{A \widehat{\otimes_k B_{\leqslant n}})\}_n$ is a Postnikov tower for $A \widehat{\otimes_k B}$. In the same manner, $A \widehat{\otimes_k (-)}$ will transform a square zero extension to a square zero extension. The lemma therefore reduces to the fact that the derived stack Perf is nilcomplete and infinitesimally cartesian. This, however, is automatic since Perf is locally geometric (see [TV07]). $\qquad\square$

Proof of Proposition 4.4, *continued.* The proof of the proposition follows from Lemma 4.6 and the fact that the operation of taking $\mathcal{H}$-fixed points preserves limits of dg-categories. $\qquad\square$

It is unclear to us whether $\mathsf{Perf}^{\nabla,\mathrm{ex}}(\widehat{\partial} X)$ is also nilcomplete and infinitesimally cartesian. Again, we believe that the inclusion $\mathsf{Perf}^{\nabla,\mathrm{ex}}(\widehat{\partial} X) \hookrightarrow \mathsf{Perf}^\nabla(\widehat{\partial} X)$ is an equivalence but are unable to prove this at the moment. Also note that we explicitly included the compactification $\mathfrak{X}$ in the statement of Proposition 4.4 as we do not know whether $\mathsf{Perf}^\nabla(\widehat{\partial} X)$ is independent of the choice of compactification (as opposed to $\mathsf{Perf}^{\nabla,\mathrm{ex}}(\widehat{\partial} X)$).

4.2 Cotangent complexes

We now turn to the study of cotangent complexes of the derived stacks $\mathsf{Perf}^\nabla(X)$ and $\mathsf{Perf}^\nabla(\widehat{\partial} X)$. In general, these cotangent complexes do not exist except when X is proper. We thus introduce the following notion.

DEFINITION 4.7. Let B be a connective cdga. We say that an object $E \in \mathsf{Perf}^\nabla(X)(B)$ or $E \in \mathsf{Perf}^\nabla(\widehat{\partial} X)(B)$ is *End-Fredholm* (or simply *Fredholm*) if the cotangent complex at E exists and is perfect.

The computation of the tangent complexes of $\mathsf{Perf}^\nabla(X)$ and $\mathsf{Perf}^{\nabla,\mathrm{ex}}(\widehat{\partial} X)$ is standard and is given by the following proposition.

PROPOSITION 4.8. *Let B be a connective cdga and $E \in \mathsf{Perf}^\nabla(X)(B)$ and object with restriction* $\mathsf{R}(E) \in \mathsf{Perf}^{\nabla,\mathrm{ex}}(\widehat{\partial} X)(B)$.

(1) *The ∞-functor $M \mapsto \mathbb{T}_{\mathsf{Perf}^\nabla(X),E}(M)$ is equivalent to $M \mapsto \mathbb{H}_{\mathrm{DR}}\big(X, E^\vee \otimes E \otimes_B M\big)[1]$.*

(2) *The ∞-functor $M \mapsto \mathbb{T}_{\mathsf{Perf}^\nabla(\widehat{\partial} X),\mathsf{R}(E)}(M)$ is equivalent to*

$$M \mapsto \mathbb{H}_{\mathrm{DR}}\big(\widehat{\partial} X, \mathsf{R}\big(E^\vee \otimes E\big) \otimes_B M\big)[1].$$

As a direct consequence of Proposition 4.8 and Definition 4.7, we have the following corollary.

COROLLARY 4.9. *Let B be any connective cdga and $E \in \mathsf{Perf}^\nabla(X)(B)$ any object.*

(1) *The object E is Fredholm if and only if $\mathbb{H}_{\mathrm{DR}}\big(X, E^\vee \otimes E\big)$ is a perfect B-module.*

(2) *The object $\mathsf{R}(E)$ is Fredholm if and only if $\mathbb{H}_{\mathrm{DR}}\big(\widehat{\partial} X, \mathsf{R}\big(E^\vee \otimes E\big)\big)$ is a perfect B-module.*

Proof. For part (1), we note that, by definition, it is enough to show that the formation of $\mathbb{H}_{\mathrm{DR}}\big(X, E^\vee \otimes E\big)$ is compatible with base changes of B. But this follows immediately from the corresponding statement for Tate B-module that was proven in Section 3.3. The proof of part (2) is similar. $\qquad\square$

We will see later that all objects are Fredholm when B is a field (see Corollary 5.9). More generally, if $E \in \mathsf{Perf}^\nabla(X)(B)$ is any object, we will see that E and $R(E)$ are Fredholm under the condition that for some good compactification $j \colon X \hookrightarrow \mathfrak{X}$, both $j_*(E)$ and $j_*(E^\vee)$ are perfect $\mathcal{D}_X \otimes_k B$-modules in the sense of Section 2.1. We refer to Corollary 5.8 for this important statement, which will be crucial in the proof of the representability theorem.

5. The Lagrangian restriction map

In this section, we construct a natural shifted Lagrangian structure on the restriction morphism

$$\mathsf{R}\colon \mathsf{Perf}^{\nabla}(X) \longrightarrow \mathsf{Perf}^{\nabla}\big(\widehat{\partial}X\big),$$

which is the de Rham analog of the Betti statements in [PT21]. However, the new feature here is that the derived stacks $\mathsf{Perf}^{\nabla}(X)$ and $\mathsf{Perf}^{\nabla}\big(\widehat{\partial}X\big)$ are not representable, and their tangent complexes can be infinite-dimensional. We thus have to be careful with the notion of Lagrangian structure itself. The definitions of closed forms and isotropic structures make sense on general derived stacks. However, the non-degeneracy condition in the definition of a Lagrangian structure causes a problem as there is a priori no direct relationship between 2-forms on a derived stack F and global sections of $\wedge^2 \mathbb{L}_F$ (even assuming that $\mathbb{L}_F$ exists).

Therefore, in our setting, non-degeneracy has to be defined pointwise, at all field-valued points. For this, we use in a crucial manner that the derived stacks $\mathsf{Perf}^{\nabla}(X)$ and $\mathsf{Perf}^{\nabla}\big(\widehat{\partial}X\big)$ are nilcomplete and infinitesimally cartesian and, moreover, that their cotangent complexes exist and are perfect at all field-valued points (see Proposition 4.4 and Corollary 5.9).

5.1 Closed forms and symplectic structures

Recall from [PTVV13] that for any derived stack F, we have a complex of p-forms $\mathcal{A}^p(F)$ and a complex of closed p-forms $\mathcal{A}^{p,\mathrm{cl}}(F)$, together with a forgetful morphism $\mathcal{A}^{p,\mathrm{cl}}(F) \to \mathcal{A}^p(F)$. When $F = \operatorname{Spec} A$ is a derived affine scheme, the complex $\mathcal{A}^p(F) \sim \wedge^p_A \mathbb{L}_A$ simply is the pth wedge power of the cotangent complex of A. In the same manner, $\mathcal{A}^{p,\mathrm{cl}}(F) \sim \operatorname{tot}\big(\prod_{i \geqslant p} (\wedge^i_A \mathbb{L}_A)[-i]\big)$ is the totalization of the completed derived truncated de Rham complex.

Suppose that F is any derived stack that possesses a cotangent complex $\mathbb{L}_F \in \mathsf{D}_{\mathrm{qcoh}}(F)$ in the sense recalled in Definition 4.1. There is a descent morphism

$$\mathbb{H}\big(F, \wedge^p_{\mathcal{O}_F} \mathbb{L}_F\big) \longrightarrow \mathcal{A}^p(F).$$

When F is a derived Artin stack, it is shown in [PTVV13, Proposition 1.14] that this morphism is a quasi-isomorphism. In general, this descent morphism has no reason to be a quasi-isomorphism. This fact creates complications when one tries to define the non-degeneracy of 2-forms [PTVV13]. In this paper, we overcome this complication by working pointwise on F, as follows.

DEFINITION 5.1. A derived stack F is *formally good* if it is infinitesimally cartesian and for any k-field L and any $x \in F(L)$, the tangent complex $\mathbb{T}_x F$ is perfect over L.

Proposition 4.4 and Corollary 5.9 show that the derived stacks $\mathsf{Perf}^{\nabla}(X)$ and $\mathsf{Perf}^{\nabla}\big(\widehat{\partial}X\big)$ are formally good in the sense of Definition 5.1.

Let F be a formally good derived stack and $x \in F(L)$ be a field valued point. We can restrict the functor F to the ∞-category of artinian local augmented L-cdga by sending such a cdga $A \in \mathbf{dgArt}^*_L$ to the fiber of $F(A) \to F(L)$ taken at x. By definition, this restriction is the formal completion of F at x; we will denote it by $\widehat{F}_x$. Since F is assumed to be infinitesimally cartesian, the ∞-functor $\widehat{F}_x$ is a formal moduli problem over L in the sense of [Lur11b]. It therefore corresponds to an L-linear dg-Lie algebra $\mathcal{L}_x$ whose underlying complex is $\mathbb{T}_x F[-1]$.

By left Kan extension from artinian cdga to connective cdga, the ∞-functor $\widehat{F}_x$ can itself be considered as a derived stack. As such, it possesses a complex of p-forms $\mathcal{A}^p(\widehat{F}_x)$. It turns out that this complex can be computed purely in terms of the dg-Lie algebra $\mathcal{L}_x$, as follows.

PROPOSITION 5.2. *Let F be a formal moduli problem over L associated with a dg-Lie algebra $\mathcal{L}$. There is a canonical quasi-isomorphism*

$$\mathcal{A}^p(F) \simeq \underline{\mathrm{Hom}}_{\mathcal{L}-\mathbf{dg}}\left(k, \wedge^p\left(\mathcal{L}^\vee[-1]\right)\right),$$

where $\mathcal{L}^\vee$ is the L-linear dual of $\mathcal{L}$ considered as a dg-module over $\mathcal{L}$ by the coadjoint action.

Proof. We first prove the statement when F is representable, that is, $F = \mathrm{Spec}\, A$ for $A \in \mathbf{dgArt}_L^*$. In this case, F has a cotangent complex $\mathbb{L}_{A/L} \in \mathsf{D}_{\mathrm{qcoh}}(F)$. By [Lur11b], there is a full embedding $\mathsf{D}_{\mathrm{qcoh}}(F) \hookrightarrow D(\mathcal{L} - \mathbf{dg})$, and the image of $\mathbb{L}_{A/L}$ is the dg-module $\mathcal{L}^\vee[-1]$, which follows immediately from the universal property of $\mathbb{L}_{A/L}$. Finally, the above full embedding also sends $\mathcal{O}$ to k, which implies the existence of the required equivalence

$$\mathcal{A}^p(F) = \underline{\mathrm{Hom}}\left(\mathcal{O}, \wedge^p \mathbb{L}_{A/L}\right) \simeq \underline{\mathrm{Hom}}_{\mathcal{L}-\mathbf{dg}}\left(k, \wedge^p\left(\mathcal{L}^\vee[-1]\right)\right).$$

This extends easily to the case where $F = \mathrm{colim}\,\mathrm{Spec}\, A_i$ is now only pro-representable by a pro-object "$\lim_i A_i$" in $\mathbf{dgArt}_L^*$.

To deduce the general case, we use the existence of smooth hyper-coverings proved in [Lur11b]. Having smooth hyper-coverings guarantees that a general formal moduli problem F can be written as a geometric realization $|F_*|$ of a simplicial object in pro-representables which moreover satisfies the smooth hyper-coverings condition. We can then use the same descent argument as done in the algebraic case in [PTVV13]. We consider the formal moduli problem $TF[-1] = \mathrm{Map}(\mathrm{Spec}(k \oplus k[1]), F)$ corresponding to the shifted tangent of F. Using a smooth hyper-covering F_*, we observe that $TF[-1]$ is again the realization of $TF_*[-1]$. Passing to the complex of functions, we find that the natural morphism

$$\underline{\mathrm{Hom}}_{\mathcal{L}-\mathbf{dg}}\left(k, \wedge^p\left(\mathcal{L}^\vee[-1]\right)\right) \longrightarrow \lim_n \underline{\mathrm{Hom}}_{\mathcal{L}_n-\mathbf{dg}}\left(k, \wedge^p\left(\mathcal{L}_n^\vee[-1]\right)\right)$$

is a quasi-isomorphism (where we have denoted by $\mathcal{L}_n$ the dg-Lie algebra corresponding to F_n). This last descent statement, together with the already treated case of pro-representable F, proves the general result. $\qquad\square$

Going back to our formally good stack F, let $x \in F(L)$ be a field-valued point. Using Proposition 5.2, we see that there is a natural restriction map

$$\mathcal{A}^p(F) \longrightarrow \mathcal{A}^p\left(\widehat{F}_x\right) \simeq \underline{\mathrm{Hom}}_{\mathcal{L}_x-\mathbf{dg}}\left(k, \wedge^p\left(\mathcal{L}_x^\vee[-1]\right)\right) \longrightarrow \wedge^p\left(\mathcal{L}_x^\vee[-1]\right),$$

where the last morphism is obtained by forgetting the $\mathcal{L}_x$-module structure.

DEFINITION 5.3. Let F be a formally good derived stack and $\omega \in H^n\left(\mathcal{A}^{2,\mathrm{cl}}(F)\right)$ be a closed 2-form of degree n on F. We say that ω *is non-degenerate* if for all field-valued points $x \in F(L)$, the image of ω by the morphism

$$\mathcal{A}^{2,\mathrm{cl}}(F) \longrightarrow A^2(F) \longrightarrow \wedge^p\left(\mathcal{L}_x^\vee[-1]\right) \simeq \wedge^2\left(\mathbb{T}_{F,x}^\vee\right)$$

is a non-degenerate pairing of degree n and induces an equivalence $\mathbb{T}_{F,x} \simeq \mathbb{T}_{F,x}^\vee[n]$.

The above definition generalizes immediately to the relative setting, as follows. Suppose that we have a morphism of formally good derived stacks $f: F \to F'$ and a closed 2-form ω of degree n on F'. Assume that we are given a homotopy to zero $h: f^*(\omega) \sim 0$ inside $\mathcal{A}^{2,\mathrm{cl}}(F)$. By what we have seen, for any field-valued point $x \in F(L)$, the form ω and the homotopy h induce an n-cocycle ω_x in $\wedge^2 \mathbb{T}_{F',x}^\vee$ as well as a null homotopy of its image $f^*(\omega_x)$ in $\wedge^2 \mathbb{T}_{F,x}^\vee$. This null homotopy induces a well-defined morphism of complexes

$$\mathbb{T}_{F,x} \longrightarrow \mathbb{T}_{F/F',x}^\vee[n-1].$$

DEFINITION 5.4. Let $f\colon F \to F'$ be a morphism of formally good derived stacks. Let ω be a closed 2-form of degree n on F' and $h\colon f^*(\omega) \sim 0$ an isotropic structure on f with respect to ω. We say that the isotropic structure is *Lagrangian* if for any field-valued point $x \in F(L)$, the induced morphism of complexes

$$\mathbb{T}_{F,x} \longrightarrow \mathbb{T}_{F/F',x}^{\vee}[n-1]$$

is a quasi-isomorphism.

5.2 Orientation on the formal boundary

In this section, we will prove that the conditions for applying the results of [Toë18] are satisfied for the restriction morphism of derived stacks

$$\mathsf{R}\colon \mathsf{Perf}^{\nabla}(X) \longrightarrow \mathsf{Perf}^{\nabla}(\widehat{\partial}X)\,.$$

The main step consists of studying Serre duality on $\widehat{\partial}X$, and the key ingredient is the construction of the integration map

$$or\colon \mathbb{H}(\widehat{\partial}X, R(\omega_X)) \longrightarrow k[1-d]\,,$$

where d is the dimension of X (for simplicity, we assume that X is connected).

Here $\omega_X := \Omega_X^d$ is the canonical sheaf of X. We pick a good compactification $j\colon X \hookrightarrow \mathfrak{X}$ once and for all. As before, we will write $\widehat{\mathfrak{X}}$ for the formal completion of $\mathfrak{X}$ along the divisor $D = \mathfrak{X} - X$, and we will write $\widehat{j}\colon \widehat{\mathfrak{X}} \to \mathfrak{X}$ for the natural map.

The formal gluing theorem of [HPV16] and the observation that $\omega_X = j^*\omega_{\mathfrak{X}}$ yield a cartesian square

$$
\begin{array}{ccc}
\mathbb{H}(\mathfrak{X}, \omega_{\mathfrak{X}}) & \longrightarrow & \mathbb{H}(X, \omega_X) \\
\downarrow & & \downarrow \\
\mathbb{H}(\widehat{\mathfrak{X}}, \widehat{j}^*\omega_{\mathfrak{X}}) & \longrightarrow & \mathbb{H}(\widehat{\partial}X, R(\omega_X))\,.
\end{array}
\tag{5.1}
$$

The boundary map for this cartesian square produces a morphism

$$u\colon \mathbb{H}(\widehat{\partial}X, R(\omega_X)) \longrightarrow \mathbb{H}(\mathfrak{X}, \omega_{\mathfrak{X}})[1]$$

of complexes over k. Composing with Grothendieck's trace isomorphism $H^d(\mathfrak{X}, \omega_{\mathfrak{X}}) \simeq k$, we get the required morphism of complexes

$$or\colon \mathbb{H}(\widehat{\partial}X, R(\omega_X)) \longrightarrow k[1-d]\,.$$

This morphism is a version of the residue map; for instance, it coincides with the usual residue of forms when X is a curve and the residues are taken at the points at infinity.

We defined the morphism *or* above as a morphism of complexes over k. However, as explained in Section 3.3, the source of this morphism is the realization of the ind-pro complex $\widetilde{\mathbb{H}}(\widehat{\partial}X, R(\omega_X))$. By construction, the formal gluing giving the cartesian square (5.1) lifts canonically to give a cartesian square of Tate complexes over k. This implies that the boundary morphism u also lifts canonically as a morphism in the ind-pro category. As a result, *or* arises as the realization of a natural morphism of Tate complexes

$$\widetilde{or}\colon \widetilde{\mathbb{H}}(\widehat{\partial}X, R(\omega_X)) \longrightarrow k[1-d]\,.$$

By base change (see Section 3.3), for every connective cdga B, we get an induced morphism

$$\widetilde{or}\colon \widetilde{\mathbb{H}}(\widehat{\partial}X, R(\omega_X) \otimes_k B) \longrightarrow B[1-d]\,.$$

Now assume that B is a connective cdga and E and F are two perfect complexes over $X \times S$, with $S = \operatorname{Spec} B$. To simplify the discussion, we assume that E and F can be extended to perfect complexes on $\mathfrak{X} \times S$ (even though this is not strictly necessary for the results below). We have a composition morphism

$$\mathbb{H}\big(\widehat{\partial}X, R(E)^\vee \otimes R(F)\big) \otimes_B \mathbb{H}\big(\widehat{\partial}X, R(F)^\vee \otimes R(E) \otimes R(\omega_X)\big) \longrightarrow \mathbb{H}\big(\widehat{\partial}X, R(E)^\vee \otimes R(E) \otimes R(\omega_X)\big),$$

which we can compose with the trace morphism $R(E)^\vee \otimes R(E) \longrightarrow R(\mathcal{O}_X)$ and with the orientation or in order to get a pairing

$$\mathbb{H}\big(\widehat{\partial}X, R(E)^\vee \otimes R(F)\big) \otimes_B \mathbb{H}\big(\widehat{\partial}X, R(F)^\vee \otimes R(E) \otimes R(\omega_X)\big) \longrightarrow B[1-d].$$

This pairing also admits a canonical lift as a pairing of Tate B-modules. Indeed, we have already seen that or has such a lift, and composition and trace are also compatible with the ind-pro structures. We have thus defined a canonical pairing of Tate B-modules

$$\widetilde{\mathbb{H}}\big(\widehat{\partial}X, R(E)^\vee \otimes R(F)\big) \widehat{\otimes}_B \widetilde{\mathbb{H}}\big(\widehat{\partial}X, R(F)^\vee \otimes R(E) \otimes R(\omega_X)\big) \longrightarrow B[1-d].$$

By rigidity, we may assume $F = \mathcal{O}_X$ without loss of generality. The pairing can then be written as

$$\widetilde{\mathbb{H}}\big(\widehat{\partial}X, R(E)^\vee\big) \widehat{\otimes}_B \widetilde{\mathbb{H}}\big(\widehat{\partial}X, R(E) \otimes R(\omega_X)\big) \longrightarrow B[1-d].$$

By construction, the orientation morphism $\widetilde{or}$ canonically vanishes on $\mathbb{H}(X, \omega_X)$, and so we get an induced pairing of Tate B-modules

$$\mathbb{H}\big(X, E^\vee\big) \widehat{\otimes}_B \widetilde{\mathbb{H}}_c(X, E \otimes \omega_X) \longrightarrow B[-d]. \tag{5.2}$$

The following result is Serre duality for cohomology with compact supports.

PROPOSITION 5.5. *The pairing* (5.2) *is non-degenerate. It induces an equivalence of Tate B-modules*

$$\widetilde{\mathbb{H}}_c(X, E \otimes \omega_X) \simeq \mathbb{H}\big(X, E^\vee\big)^\vee[-d].$$

Proof. The pairing (5.2) induces a morphism of B-modules $\alpha \colon \mathbb{H}\big(X, E^\vee\big) \to \widetilde{\mathbb{H}}_c(X, E \otimes \omega_X)^\vee[-d]$. Here, $\widetilde{\mathbb{H}}_c(X, E \otimes \omega_X)^\vee$ is the dual of $\widetilde{\mathbb{H}}_c(X, E \otimes \omega_X)$ as a Tate module. Since $\widetilde{\mathbb{H}}_c(X, E \otimes \omega_X)$ is pro-perfect, this dual is a genuine B-module. We must show that the morphism α is an equivalence. For this, we go back to examine the formal gluing cartesian square (5.1) and the definition of the pairing. We have the exact triangle of Tate B-modules

$$\widetilde{\mathbb{H}}_c(X, E \otimes \omega_X) \longrightarrow \mathbb{H}(\mathfrak{X}, E \otimes \omega_{\mathfrak{X}}) \longrightarrow \text{``}\lim_n\text{''} \, \mathbb{H}\big(\mathfrak{X}_{(n)}, j_n^*(E \otimes \omega_{\mathfrak{X}})\big).$$

The rightmost term can be written as "$\lim_n$" $\mathbb{H}\big(\mathfrak{X}_{(n)}, j_n^*(E) \otimes \omega_{\mathfrak{X}_{(n)}} \otimes \mathcal{L}_n\big)$, where $\mathcal{L}_n$ is the conormal sheaf of $j_n \colon \mathfrak{X}_{(n)} \hookrightarrow \mathfrak{X}$. By Serre duality on $\mathfrak{X}$ and $\mathfrak{X}_{(n)}$ (for each n), the restriction map

$$\mathbb{H}(\mathfrak{X}, E \otimes \omega_X) \longrightarrow \mathbb{H}\big(\mathfrak{X}_{(n)}, j_n^*(E) \otimes \omega_{\mathfrak{X}_{(n)}} \otimes \mathcal{L}_n\big)$$

is dual to the natural map $\mathbb{H}\big(\mathfrak{X}_{(n)}, j_n^*\big(E^\vee\big) \otimes \mathcal{L}_n^\vee\big)[d-1] \to \mathbb{H}(\mathfrak{X}, E^\vee)[d]$. When we pass to the colimit over n, these assemble in a natural map

$$\mathbb{H}_D\big(\mathfrak{X}, E^\vee\big)[d] \longrightarrow \mathbb{H}\big(\mathfrak{X}, E^\vee\big)[d],$$

where the source is cohomology with supports in D. The cofiber of this map is then naturally equivalent to $\mathbb{H}(X, E^\vee)[d]$. This constructs a natural equivalence of B-modules

$$\widetilde{\mathbb{H}}_c(X, E \otimes \omega_X)^\vee \simeq \mathbb{H}\big(X, E^\vee\big)[d].$$

It is straightforward to check that this equivalence is the morphism α. $\qquad\square$

Corollary 5.6. *The pairing of Tate B-modules*

$$\widetilde{\mathbb{H}}\big(\widehat{\partial}X, R(E)^\vee\big)\widehat{\otimes}_B\widetilde{\mathbb{H}}\big(\widehat{\partial}X, R(E)\otimes R(\omega_X)\big) \longrightarrow B[1-d]$$

is non-degenerate.

Proof. We have two exact triangles of Tate B-modules

$$\widetilde{\mathbb{H}}_c\big(X, E^\vee\big) \longrightarrow \mathbb{H}\big(X, E^\vee\big) \longrightarrow \widetilde{\mathbb{H}}\big(\widehat{\partial}X, R(E^\vee)\big)$$
$$\widetilde{\mathbb{H}}_c(X, E\otimes\omega_X) \longrightarrow \mathbb{H}(X, E\otimes\omega_X) \longrightarrow \widetilde{\mathbb{H}}\big(\widehat{\partial}X, R(E\otimes\omega_X)\big)\,.$$

The dual, inside Tate B-modules, of the second triangle is (up to a rotation and shift by $-d$)

$$\mathbb{H}(X, E\otimes\omega_X)^\vee[-d] \longrightarrow \widetilde{\mathbb{H}}_c(X, E\otimes\omega_X)^\vee[-d] \longrightarrow \widetilde{\mathbb{H}}\big(\widehat{\partial}X, R(E\otimes\omega_X)\big)^\vee[1-d]\,.$$

By the construction of the orientation *or*, the natural pairing produces a commutative diagram of Tate B-modules

$$\begin{array}{ccccc}
\widetilde{\mathbb{H}}_c\big(X, E^\vee\big) & \longrightarrow & \mathbb{H}\big(X, E^\vee\big) & \longrightarrow & \widetilde{\mathbb{H}}\big(\widehat{\partial}X, R(E^\vee)\big) \\
\downarrow & & \downarrow & & \downarrow \\
\mathbb{H}(X, E\otimes\omega_X)^\vee[d] & \longrightarrow & \widetilde{\mathbb{H}}_c(X, E\otimes\omega_X)^\vee[d] & \longrightarrow & \widetilde{\mathbb{H}}\big(\widehat{\partial}X, R(E\otimes\omega_X)\big)^\vee[d-1]\,.
\end{array}$$

The first two vertical morphisms on the left are equivalences by Proposition 5.5. Therefore, the third vertical morphism is also an equivalence. $\qquad\square$

The same orientation morphism can be used to prove a duality statement for de Rham cohomology with compact supports. It goes as follows. The complex of sheaves $\big|\widehat{\mathbf{DR}}^o_B\big|$ on D computing[1] $\mathbb{H}_{\mathrm{DR}}\big(\widehat{\partial}X, \mathsf{R}(\mathcal{O}_X)\big)$ is bounded of amplitude contained in $[0, d]$. Moreover, its last non-zero term is $R(\omega_X)$. Therefore, there is a canonical map

$$H^{2d-1}_{\mathrm{DR}}\big(\widehat{\partial}X, \mathsf{R}(\mathcal{O}_X)\big) \longrightarrow H^{d-1}\big(\widehat{\partial}X, R(\omega_X)\big)\,.$$

Composing with the orientation map $or\colon H^{d-1}\big(\widehat{\partial}X, R(\omega_X)\big) \to k[1-d]$, we get an orientation morphism $\mathbb{H}_{\mathrm{DR}}\big(\widehat{\partial}X, \mathsf{R}(\mathcal{O}_X)\big) \to k[1-2d]$. As before, it extends naturally as a morphism of Tate complexes over k

$$\widetilde{or}\colon \widetilde{\mathbb{H}}_{\mathrm{DR}}\big(\widehat{\partial}X, \mathsf{R}(\mathcal{O}_X)\big) \longrightarrow k[1-2d]\,.$$

For any connective cdga B and any $E \in \mathsf{Perf}^\nabla(X)(B)$, this orientation defines, as before, two pairings of Tate B-modules

$$\widetilde{\mathbb{H}}_{c,\mathrm{DR}}(X, E)\widehat{\otimes}_B\widetilde{\mathbb{H}}_{\mathrm{DR}}\big(X, E^\vee\big) \longrightarrow B[-2d] \tag{5.3}$$

$$\widetilde{\mathbb{H}}_{\mathrm{DR}}\big(\widehat{\partial}X, \mathsf{R}(E)\big)\widehat{\otimes}_B\widetilde{\mathbb{H}}_{\mathrm{DR}}\big(\widehat{\partial}X, \mathsf{R}(E)^\vee\big) \longrightarrow B[1-2d]\,. \tag{5.4}$$

We now have the following.

Proposition 5.7. *The pairings (5.3) and (5.4) are non-degenerate and induce natural equivalences of Tate B-modules*

$$\widetilde{\mathbb{H}}_{c,\mathrm{DR}}(X, E) \simeq \widetilde{\mathbb{H}}_{\mathrm{DR}}\big(X, E^\vee\big)^\vee[1-2d]\,, \quad \widetilde{\mathbb{H}}_{\mathrm{DR}}\big(\widehat{\partial}X, \mathsf{R}(E)\big) \simeq \widetilde{\mathbb{H}}_{\mathrm{DR}}\big(\widehat{\partial}X, \mathsf{R}(E)^\vee\big)^\vee[-2d]\,.$$

---

[1] Here we use a slight abuse of notation and write simply $\mathcal{O}_X$ for the trivial rank 1 flat bundle $(\mathcal{O}_X, d_{\mathrm{DR}})$ on X.

Proof. We use the Hodge filtrations on the various complexes computing these cohomology groups. In terms of graded mixed modules, these are the filtrations on $|E|$ given by $\oplus_{i \geqslant p} E(i)[-2i]$ $\subset \oplus_i E(i)[-2i]$. The associated graded of these filtrations are perfect complexes of the form $E(0) \otimes_{\mathcal{O}_X} \Omega_X^i[-i]$. The pairings of the proposition are compatible with these filtrations, and the induced pairings are those for Serre duality of perfect complexes. Therefore, the proposition follows from the Serre duality with compact supports from Proposition 5.5. $\qquad\square$

One important corollary of the previous results is the following criterion for the finiteness of the de Rham cohomology of $\widehat{\partial} X$.

COROLLARY 5.8. *Let $E \in \mathsf{Perf}^\nabla(X)(B)$ be such that $\mathbb{H}_{\mathrm{DR}}(X, E)$ and $\mathbb{H}_{\mathrm{DR}}(X, E^\vee)$ are both perfect B-modules. Then the Tate B-modules $\widetilde{\mathbb{H}}_{\mathrm{DR}}(\widehat{\partial}(X), \mathsf{R}(E))$ and $\widetilde{\mathbb{H}}_{c,\mathrm{DR}}(X, E)$ are both perfect.*

Proof. Using the exact triangle

$$\widetilde{\mathbb{H}}_{c,\mathrm{DR}}(X, E) \longrightarrow \mathbb{H}_{\mathrm{DR}}(X, E) \longrightarrow \widetilde{\mathbb{H}}_{\mathrm{DR}}(\widehat{\partial} X, \mathsf{R}(E)),$$

we see that the Tate B-module $\widetilde{\mathbb{H}}_{\mathrm{DR}}(\widehat{\partial} X, \mathsf{R}(E))$ must be pro-perfect. But Corollary 5.6 implies that its dual is also pro-perfect. This implies that it must be perfect. $\qquad\square$

One important consequence is the following.

COROLLARY 5.9. *(1) Let $E \in \mathsf{Perf}^\nabla(X)(B)$ be such that $\mathbb{H}_{\mathrm{DR}}(X, E^\vee \otimes E)$ is perfect over B. Then E and $\mathsf{R}(E)$ are both Fredholm in the sense of Definition 4.7.*
(2) If $B = k$, any $E \in \mathsf{Perf}^\nabla(X)(k)$ is Fredholm, and so is $\mathsf{R}(E)$.

Proof. Part (1) is a direct consequence of Corollary 5.8 and the fact that both $\mathbb{H}_{\mathrm{DR}}(X, E)$ and $\widetilde{\mathbb{H}}_{\mathrm{DR}}(\widehat{\partial}(X), R(E))$ are stable by base changes of B. For part (2), we have to show that for any $E \in \mathsf{Perf}^\nabla(X)(k)$, the complex $\mathbb{H}_{\mathrm{DR}}(X, E^\vee \otimes E)$ is perfect over k. But $E^\vee \otimes E$ is a bounded complex of coherent $\mathcal{D}_X$-modules with holonomic cohomologies. By Bernstein's theorem, holonomic $\mathcal{D}$-modules are stable by push-forward, and so $\mathbb{H}_{\mathrm{DR}}(X, E^\vee \otimes E)$ is a bounded complex with finite-dimensional cohomology and thus perfect. $\qquad\square$

We are now ready to construct the Lagrangian structure on the restriction morphism

$$\mathsf{R} \colon \mathsf{Perf}^\nabla(X) \longrightarrow \mathsf{Perf}^\nabla(\widehat{\partial} X).$$

For this we use the main result of [Toë18]. The derived stack $\mathsf{Perf}^\nabla(X)$ is the underlying stack of a derived stack in symmetric monoidal rigid dg-categories. According to [Toë18], in order to construct a closed 2-form ω on $\mathsf{Perf}^\nabla(\widehat{\partial} X)$, together with an homotopy $h \colon \mathsf{R}^*(\omega) \sim 0$, it is enough to

(i) construct a morphism of complexes of k-modules

$$or \colon \mathbb{H}_{\mathrm{DR}}(\widehat{\partial} X, \mathsf{R}(\mathcal{O}_X)) \longrightarrow k[1 - 2d]$$

together with a homotopy to zero of the restriction

$$\mathsf{R}^*(or) \colon \mathbb{H}_{\mathrm{DR}}(X, \mathcal{O}_X) \longrightarrow k[1 - 2d]$$

and

(ii) prove that for any connective cdga B, the induced morphisms

$$\mathbb{H}_{\mathrm{DR}}(\widehat{\partial} X, \mathsf{R}(\mathcal{O}_X)) \otimes_k B \longrightarrow \mathbb{H}_{\mathrm{DR}}(\widehat{\partial} X, \mathsf{R}(\mathcal{O}_X) \otimes_k B)$$
$$\mathbb{H}_{\mathrm{DR}}(X, \mathcal{O}_X) \otimes_k B \longrightarrow \mathbb{H}_{\mathrm{DR}}(X, \mathcal{O}_X \otimes_k B)$$

are equivalences of B-modules.

Statement (ii) holds thanks to Corollary 5.8. The map *or* is constructed at the beginning of the section. Recall that it comes from the cartesian square

$$
\begin{array}{ccc}
\mathbb{H}(\mathfrak{X}, \omega_{\mathfrak{X}}) & \longrightarrow & \mathbb{H}(X, \omega_X) \\
\downarrow & & \downarrow \\
\mathbb{H}(\widehat{\mathfrak{X}}, \widehat{\mathfrak{j}}^* \omega_{\mathfrak{X}}) & \longrightarrow & \mathbb{H}(\widehat{\partial} X, R(\omega_X))
\end{array}
$$

and the associated boundary map $H^{d-1}(\widehat{\partial} X, R(\omega_X)) \to H^d(\mathfrak{X}, \omega_{\mathfrak{X}}) \simeq k$. Precomposing with the canonical map $H^{2d-1}_{\mathrm{DR}}(\widehat{\partial} X, \mathsf{R}(\mathcal{O}_X)) \to H^{d-1}(\widehat{\partial} X, R(\omega_X))$ provides the orientation morphism

$$
or \colon \mathbb{H}_{\mathrm{DR}}(\widehat{\partial} X, \mathsf{R}(\mathcal{O}_X)) \longrightarrow k[1 - 2d].
$$

By construction, the composition $H^{d-1}(X, \omega_X) \to H^{d-1}(\widehat{\partial} X, R(\omega_X)) \to H^d(\mathfrak{X}, \omega_{\mathfrak{X}})$ is the zero map, so a null homotopy of the morphism $\mathsf{R}^*(or) \colon \mathbb{H}_{\mathrm{DR}}(X, \mathcal{O}_X) \to k[1 - 2d]$ is given by a morphism $H^{2d}_{\mathrm{DR}}(X, \mathcal{O}_X) \to k$. If X is proper, we take this map to be the natural isomorphism. If X is not proper, then $H^{2d}_{\mathrm{DR}}(X, \mathcal{O}_X) = 0$, and this map is the zero map.

By the main result of [Toë18], the derived stack $\mathsf{Perf}^\nabla(\widehat{\partial} X)$ carries a canonical closed 2-form ω of degree $3 - 2d$. Moreover, the pull-back form $\mathsf{R}^*(\omega)$ comes equipped with a natural null homotopy $h \colon \mathsf{R}^*(\omega) \sim 0$. We thus have proved the following statement.

COROLLARY 5.10. *The morphism of derived stacks* $\mathsf{R} \colon \mathsf{Perf}^\nabla(X) \to \mathsf{Perf}^\nabla(\widehat{\partial} X)$ *carries a canonical isotropic structure of degree* $2 - 2d$.

As explained in Definition 5.4, the non-degeneracy condition on an isotropic structure is imposed at all field-valued points of $\mathsf{Perf}^\nabla(X)$. Given such point $E \in \mathsf{Perf}^\nabla(X)(L)$ defined over a k-field L, after the identifications given by Proposition 4.8, the morphism

$$
\mathbb{T}_{\mathsf{Perf}^\nabla(X), E} \longrightarrow \mathbb{L}_{\mathsf{Perf}^\nabla(X)/\mathsf{Perf}^\nabla(\widehat{\partial} X), E}[2 - 2d]
$$

induced by the isotropic structure becomes equal to the duality morphism

$$
\mathbb{H}_{\mathrm{DR}}(X, E^\vee \otimes E) \longrightarrow \widetilde{\mathbb{H}}_{c, \mathrm{DR}}(X, E^\vee \otimes E)^\vee[-2d].
$$

The latter morphism is an equivalence by Proposition 5.7. This proves the following.

COROLLARY 5.11. *The isotropic structure of Corollary* 5.10 *is a Lagrangian structure in the sense of Definition* 5.4.

6. The relative representability theorem

In this section, we prove that the fibers of the restriction morphism R over field-valued points are locally representable by quasi-algebraic spaces in the sense of our appendix. We prove this statement for vector bundles endowed with flat connections. The extension to the perfect complexes setting can be reduced to this special case by truncation; we leave it to the interested reader to fill in the details.

We first consider the derived substack $\mathsf{Vect}^\nabla(X) \subset \mathsf{Perf}^\nabla(X)$ consisting of all objects whose underlying $\mathcal{O}_X$-module is a vector bundle. Explicitly, for a connective cdga B, an object $E \in \mathsf{Perf}^\nabla(X)(B)$ lies in $\mathsf{Vect}^\nabla(X)(B)$ if the $\mathcal{O}_X \otimes_k B$-module $E(0)$ is locally free of finite rank. We define similarly $\mathsf{Vect}^\nabla(\widehat{\partial} X)(B) \subset \mathsf{Perf}^\nabla(\widehat{\partial} X)(B)$ as the substack of objects E such that $E(0)$ is locally free of finite rank as a $\widehat{\mathcal{O}}^o_{D,B}$-module.

We fix once and for all $V_\infty \in \mathsf{Vect}^\nabla(\widehat{\partial}X)(k)$, a vector bundle with flat connection on the formal boundary of X. The fiber of the restriction morphism $\mathsf{R}\colon \mathsf{Vect}^\nabla(X) \to \mathsf{Vect}^\nabla(\widehat{\partial}X)$ taken at V_∞ will be denoted by $\mathsf{Vect}^\nabla_{V_\infty}(X)$. It is the derived stack of vector bundles with flat connections on X framed by V_∞ along $\widehat{\partial}X$. When no component of X is proper, the rank of V_∞ fixes the rank of all objects in $\mathsf{Vect}^\nabla_{V_\infty}(X)$. Since the proper case of the result is well understood, we will assume that X has no proper component.

THEOREM 6.1. *With the notation above, the derived stack* $\mathsf{Vect}^\nabla_{V_\infty}(X)$ *is a derived quasi-algebraic space in the sense of Definition* A.2.

Proof. We will prove the theorem by applying the version of the Artin–Lurie representability criterion by quasi-algebraic derived spaces recalled in Theorem A.3 of our appendix. By Galois descent, we may assume that k is algebraically closed. We also assume that the derived stack $\mathsf{Vect}^\nabla_{V_\infty}(X)$ is not empty or, equivalently, that V_∞ extends to a flat vector bundle V on the whole X.

By Proposition 4.4, we know that $\mathsf{Vect}^\nabla_{V_\infty}(X)$ is infinitesimally cartesian and nilcomplete since it is defined as the fiber of a morphism between two infinitesimally cartesian and nilcomplete derived stacks. Let us show, moreover, that it has a global cotangent complex. By Definition 4.7, this amounts to the following lemma.

LEMMA 6.2. *Let B be any connective cdga and $E \in \mathsf{Vect}^\nabla_{V_\infty}(X)(B)$ an object. Then the image of E in $\mathsf{Vect}^\nabla(X)(B)$ is Fredholm over B.*

Proof. This is a consequence of Corollary 4.9. Indeed, we have an exact triangle of Tate B-modules

$$\widetilde{\mathbb{H}}_{c,\mathrm{DR}}\big(X, E^\vee \otimes E\big) \longrightarrow \mathbb{H}_{\mathrm{DR}}\big(X, E^\vee \otimes E\big) \longrightarrow \widetilde{\mathbb{H}}_{\mathrm{DR}}\big(\widehat{\partial}X, R\big(E^\vee \otimes E\big)\big).$$

The rightmost module is equivalent to $\widetilde{\mathbb{H}}_{\mathrm{DR}}\big(\widehat{\partial}X, R\big(V_\infty^\vee \otimes V_\infty\big)\big) \otimes_k B$ and, by Corollary 5.9, is perfect over B. In particular, it is compact and cocompact as an ind-pro-perfect B-module. Since $\mathbb{H}_{\mathrm{DR}}\big(X, E^\vee \otimes E\big)$ is ind-perfect, it is cocompact as an ind-pro-perfect B-module. We thus have that $\widetilde{\mathbb{H}}_{c,\mathrm{DR}}\big(X, E^\vee \otimes E\big)$ is also cocompact as an ind-pro-perfect B-module. Since it is pro-perfect, it must be perfect. But this implies that $\mathbb{H}_{\mathrm{DR}}\big(X, E^\vee \otimes E\big)$ is perfect and thus that E is Fredholm by Corollary 4.9. $\square$

Proof of Theorem 6.1, *continued.* Lemma 6.2 shows that $\mathsf{Vect}^\nabla_{V_\infty}(X)$ has a global cotangent complex which is furthermore perfect. In order to apply Theorem A.3, it remains to prove that $\mathsf{Vect}^\nabla_{V_\infty}(X)$ satisfies the three conditions (2), (5), and (6). These three statements are properties of the restriction of $\mathsf{Vect}^\nabla_{V_\infty}(X)$ to underived k-algebras. Let us denote this restriction by $\mathsf{Vect}^\nabla_{V_\infty}(X)_0$.

We start by studying the diagonal morphism of $\mathsf{Vect}^\nabla_{V_\infty}(X)_0$ in order to check condition (2) of Theorem A.3.

LEMMA 6.3. *The diagonal morphism*

$$\mathsf{diag}\colon \mathsf{Vect}^\nabla_{V_\infty}(X)_0 \longrightarrow \mathsf{Vect}^\nabla_{V_\infty}(X)_0 \times \mathsf{Vect}^\nabla_{V_\infty}(X)_0$$

is representable by a scheme of finite type over k.

Proof. The statement of the lemma is equivalent to the statement that for any discrete cdga B and any two points E and F in $\mathsf{Vect}^\nabla_{V_\infty}(X)(B)$, the sheaf of isomorphisms $\mathrm{Iso}(E, F)$ is representable by a scheme of finite type over $\mathrm{Spec}\, B$. This sheaf is an open subsheaf inside the sheaf

of morphisms $\mathrm{Hom}(E, F)$ from E to F; it is therefore enough to prove that $\mathrm{Hom}(E, F)$ is representable by a scheme of finite type over B. The value of this sheaf over a B-algebra B' is given as the fiber at the identity of the restriction map

$$0 \longrightarrow \mathrm{Hom}(E, F)(B') \longrightarrow \mathbb{H}^0_{\mathrm{DR}}\big(X, E^\vee \otimes F \otimes_B B'\big) \longrightarrow \mathbb{H}^0_{\mathrm{DR}}\big(\widehat{\partial}X, V_\infty^\vee \otimes V_\infty\big) \otimes_k B'.$$

In other words, $\mathrm{Hom}(E, F)$ is the sheaf of morphisms with compact supports (that is, restrict to the identity morphism on $\widehat{\partial}X$) from E to F. Because $E^\vee \otimes F$ is automatically Fredholm, the functor sending B' to $\mathbb{H}^0_{\mathrm{DR}}\big(X, E^\vee \otimes F \otimes_B B'\big)$ is the H^0-functor of a perfect complex over B of amplitude $[0, \infty)$ and thus is representable by a scheme of finite type.

SUBLEMMA 6.4. *Let K be a perfect complex on a commutative k-algebra B, and suppose that K has amplitude contained in $[0, \infty)$. Then the functor $B' \mapsto H^0(K \otimes_B B')$ is representable by an affine scheme of finite presentation over B'.*

Proof. Because of the amplitude hypothesis, K can be presented by a bounded complex of projective modules of finite rank

$$0 \longrightarrow K^0 \longrightarrow K^1 \longrightarrow \cdots \longrightarrow K^n$$

for some integer n. The functor under consideration is then the kernel of $K^0 \to K^1$, that is, the kernel of a morphism between vector bundles over $\mathrm{Spec}\, B$, and the result follows as affine schemes of finite presentation over B are stable by fiber products. $\square$

Proof of Lemma 6.3, continued. The sublemma and the fact that $E^\vee \otimes F$ is automatically Fredholm imply that the two functors

$$B' \longmapsto \mathbb{H}^0_{\mathrm{DR}}\big(X, E^\vee \otimes F \otimes_B B'\big)$$
$$B' \longmapsto \mathbb{H}^0_{\mathrm{DR}}\big(\widehat{\partial}X, V_\infty^\vee \otimes V_\infty\big) \otimes_k B'$$

are representable by affine schemes of finite presentation over $\mathrm{Spec}\, B$. We thus get that the sheaf $\mathrm{Hom}(E, F)$ is also representable by an affine scheme of finite presentation over $\mathrm{Spec}\, B$, which completes the proof of the lemma. $\square$

Proof of Theorem 6.1, continued. Lemma 6.3 implies that condition (2) of Theorem A.3 is also satisfied. Indeed, the diagonal morphism has the property that it is nilcomplete and inf-cartesian and possesses a perfect cotangent complex, so the fact that it is representable on the level of truncations implies that it is representable (see [TV08]). Condition (1) of Theorem A.3 is also satisfied as no components of X are assumed to be proper, so for any $V \in \mathsf{Vect}^\nabla(X)(k)$, the induced morphism

$$\mathbb{H}^0_{\mathrm{DR}}\big(X, V^\vee \otimes V\big) \longrightarrow \mathbb{H}^0_{\mathrm{DR}}\big(\widehat{\partial}X, R(V)^\vee \otimes R(V)\big)$$

is injective. It thus remains to check conditions (5) and (6) of Theorem A.3.

First, we will check that condition (5) of Theorem A.3 is satisfied by $\mathsf{Vect}^\nabla_{V_\infty}(X)$. By [Moc09], we can chose a (possibly stacky) good compactification $X \hookrightarrow \mathfrak{X}$ such that the underlying bundle of V extends to a vector bundle $\mathcal{V}$ on $\mathfrak{X}$. We denote the divisor at infinity by $D \hookrightarrow \mathfrak{X}$. The connection on V can then be represented by a connection with poles

$$\mathfrak{d} \colon \mathcal{V} \longrightarrow \Omega^1_{\mathfrak{X}}(nD) \otimes_{\mathcal{O}_{\mathfrak{X}}} \mathcal{V}$$

for some integer n. The morphism $\mathfrak{d}$ can also be interpreted as a splitting of the Atiyah extension with poles along D:

$$E(\mathcal{V}, n)\colon \ 0 \longrightarrow \Omega^1_{\mathfrak{X}}(nD) \otimes_{\mathcal{O}_{\mathfrak{X}}} \mathcal{V} \longrightarrow \mathcal{P}(\mathcal{V})(nD) \longrightarrow \mathcal{V} \longrightarrow 0,$$

where $\mathcal{P}(\mathcal{V})(nD)$ is the vector bundle of principal parts of $\mathcal{V}$ possibly with poles of order at most n along D.

We consider the (underived) stack of pairs $(\mathcal{W}, \delta)$ consisting of a vector bundle $\mathcal{W}$ on $\mathfrak{X}$ and a flat connection δ on $\mathcal{W}$ with poles of order at most n along D. By definition, this stack sends a commutative k-algebra B to the groupoid of vector bundles $\mathcal{W}$ on $\mathfrak{X} \times \operatorname{Spec} B$, together with a splitting δ of the exact sequence of bundles on $\mathfrak{X} \times \operatorname{Spec} B$

$$E(\mathcal{W}, n)\colon \ 0 \longrightarrow \Omega^1_{\mathfrak{X}}(nD) \otimes_{\mathcal{O}_{\mathfrak{X}}} \mathcal{W} \longrightarrow \mathcal{P}_{\mathfrak{X},B}(\mathcal{W})(nD) \longrightarrow \mathcal{W} \longrightarrow 0$$

satisfying the integrability condition $\delta^2 = 0$ as a section of $\Omega^2_{\mathfrak{X}}(2nD) \otimes_{\mathcal{O}_{\mathfrak{X}}} \operatorname{End}(\mathcal{W})$. Here $\mathcal{P}_{\mathfrak{X},B}(\mathcal{W})(nD)$ denotes the sheaf of principal parts of $\mathcal{W}$, taken relative to the map $\mathfrak{X} \times \operatorname{Spec} B \to \operatorname{Spec} B$ and with poles of order at most n along $D \times \operatorname{Spec} B$.

Let us denote this stack by $\mathcal{F}_{\mathfrak{X}}$. This is clearly an Artin stack locally of finite type over k. In the same manner, we can define $\mathcal{F}_X := \mathsf{Vect}^{\nabla}(X)_0$, the underived stack of vector bundles with flat connections on X, as well as $\mathcal{F}_{\widehat{\mathfrak{X}}}$, the stack of vector bundles on the formal completion $\widehat{\mathfrak{X}}$ endowed with flat connections with poles of order at most n along $D \hookrightarrow \widehat{\mathfrak{X}}$. Finally, we have $\mathcal{F}_{\widehat{\partial} X} := \mathsf{Vect}^{\nabla}(\widehat{\partial} X)_0$. The formal gluing of [HPV16] again implies that there exists a cartesian square of underived stacks

$$\begin{array}{ccc} \mathcal{F}_{\mathfrak{X}} & \longrightarrow & \mathcal{F}_X \\ \downarrow & & \downarrow \\ \mathcal{F}_{\widehat{\mathfrak{X}}} & \longrightarrow & \mathcal{F}_{\widehat{\partial} X} \,. \end{array}$$

The stack $\mathcal{F}_{\widehat{\mathfrak{X}}}$ is a limit of Artin stacks locally of finite type and thus satisfies condition (5) of Theorem A.3. The stack $\mathcal{F}_{\mathfrak{X}}$ satisfies conditions (5) and (6) of Theorem A.3. This implies that the fiber of the left vertical map, taken at $\widehat{\mathsf{j}}^*(\mathcal{V}, \mathfrak{d})$, will satisfy condition (5). But, by construction, this fiber is the truncated stack $\mathsf{Vect}^{\nabla}_{V_\infty}(X)_0$. This implies that $\mathsf{Vect}^{\nabla}_{V_\infty}(X)_0$ satisfies condition (5) of Theorem A.3, as desired.

Finally we need to show that $\mathsf{Vect}^{\nabla}_{V_\infty}(X)$ satisfies condition (6) of Theorem A.3. For this, let $B = \operatorname{colim}_i B_i$ as in condition (6) and assume that each B_i, as well as B, is a noetherian ring. We consider

$$\operatorname{colim}_i \mathsf{Vect}^{\nabla}_{V_\infty}(X)(B_i) \longrightarrow \mathsf{Vect}^{\nabla}_{V_\infty}(X)(B) \,.$$

By Lemma 6.3, this map is injective, and so we need to show that it is surjective as well. Let us fix an object in $\mathsf{Vect}^{\nabla}_{V_\infty}(X)(B)$, represented by a pair (E, α) of $E \in \mathsf{Vect}^{\nabla}(X)(B)$ and $\alpha\colon \mathsf{R}(E) \simeq V_\infty \times_k B$ in $\mathsf{Vect}^{\nabla}(\widehat{\partial} X)(B)$. Since the stack $\mathsf{Vect}^{\nabla}(X)$ of flat bundles on X is locally of finite presentation, there exist an i and $E_i \in \mathsf{Vect}^{\nabla}(X)(B_i)$ such that $E_i \otimes_{B_i} B \simeq E$.

We now consider the sheaf $\mathcal{I}$ of isomorphisms between $\mathsf{R}(E_i)$ and $V_\infty \otimes_k B_i$, which is a sheaf on the big étale site of affine schemes over $S_i = \operatorname{Spec} B_i$. This sheaf is a subsheaf in $\mathcal{J}$, the sheaf of all morphisms from $\mathsf{R}(E_i)$ to $V_\infty \otimes_k B_i$.

LEMMA 6.5. *There exists a non-empty Zariski open $U_i \subset S_i = \operatorname{Spec} B_i$ such that the restriction of the sheaf $\mathcal{J}$ is representable by a scheme of finite type over U_i.*

Proof. The argument is similar to that used for Lemma 6.3. We have to prove that if we set

$$E'_i := \mathsf{R}(E_i)^{\vee} \otimes V_\infty \otimes_k B_i \in \mathsf{Vect}^{\nabla}(\widehat{\partial} X)(B_i) \,,$$

then the Tate object $\widetilde{\mathbb{H}}_{\mathrm{DR}}(\widehat{\partial} X, E'_i)[f^{-1}]$ is a perfect $B_i[f^{-1}]$-module for some non-zero localization $B_i[f^{-1}]$. For this, we use the criterion from Corollary 5.8 and Proposition 2.4.

First, let us recall some notation. We will again write $j\colon X \hookrightarrow \mathfrak{X}$ for the embedding in the good compactification. Also recall that at the beginning of the proof of Theorem 6.1, we fixed a flat bundle $V \in \mathsf{Vect}^\nabla(X)$ satisfying $\mathsf{R}(V) \simeq V_\infty$.

We first notice that $j_* E$ is a perfect $\mathcal{D}_{\mathfrak{X},B}$-module on $\mathfrak{X} \times \operatorname{Spec} B$. This is a local statement on $\mathfrak{X}$ which reduces to the following algebraic fact. Let A be a smooth k-algebra of finite type and $f \in A$. We consider $\widehat{A \otimes_k B}$, the formal completion of $A \otimes_k B$ at $f \otimes 1$. We denote by $\widehat{\mathcal{D}}_{\mathfrak{X},B}$ the ring of completed relative differential operators. As a module, it is $\widehat{A \otimes_k B}_{A \otimes_k B}(\mathcal{D}_{\mathfrak{X}} \otimes_k B)$, where the ring structure is defined naturally by making $\mathcal{D}_{\mathfrak{X}} \otimes_k B$ act on the completion $\widehat{A \otimes_k B}$ by extending derivations to the completion. In the same manner, we let $\widehat{\mathcal{D}}_{X,B}$ be $\widehat{\mathcal{D}}_{\mathfrak{X},B}\big[f^{-1}\big]$. Using the formal gluing of [Bha10], we have a cartesian square of ∞-categories

$$
\begin{array}{ccc}
\mathsf{D}_{\mathrm{qcoh}}(\mathcal{D}_{\mathfrak{X},B}) & \xrightarrow{\ j^*\ } & \mathsf{D}_{\mathrm{qcoh}}(\mathcal{D}_{X,B}) \\
\downarrow & & \downarrow \\
\mathsf{D}_{\mathrm{qcoh}}\big(\widehat{\mathcal{D}}_{\mathfrak{X},B}\big) & \xrightarrow[\ \mathfrak{J}^*\ ]{} & \mathsf{D}_{\mathrm{qcoh}}\big(\widehat{\mathcal{D}}_{X,B}\big)\,.
\end{array}
$$

The functor $\mathfrak{J}^*$ is an explicit incarnation of the pull-back functor for $\mathcal{D}$-modules from the full infinitesimal neighborhood $\widehat{\mathfrak{X}}$ of D in $\mathfrak{X}$ to the "punctured infinitesimal neighborhood" $\widehat{\partial X}$ of D in $\mathfrak{X}$. Similarly to the way the map $j\colon X \hookrightarrow \mathfrak{X}$ gives rise to the pull-back/push-forward adjoint pair of functors $j^* \dashv j_*$ acting on relative $\mathcal{D}$-modules, the functor $\mathfrak{J}^*$ possesses a right adjoint denoted by $\mathfrak{J}_*$. This right adjoint is given by considering a $\widehat{\mathcal{D}}_{X,B}$-module as a $\widehat{\mathcal{D}}_{\mathfrak{X},B}$-module via the canonical morphism of sheaves of dg-algebras $\widehat{\mathcal{D}}_{\mathfrak{X},B} \to \widehat{\mathcal{D}}_{X,B}$.

Therefore, for $j_* E \in \mathsf{D}_{\mathrm{qcoh}}(\mathcal{D}_{\mathfrak{X},B})$ to be perfect, it is enough that its restrictions as $\mathcal{D}_{X,B}$- and $\widehat{\mathcal{D}}_{\mathfrak{X},B}$-modules are both perfect. But, the first of these restrictions is E, which is perfect over $\mathcal{D}_{X,B}$, and the second of these restrictions corresponds to $\mathfrak{J}_*\big(V_\infty^\vee \otimes_k B\big)$. This is perfect because it is the restriction to $\widehat{\mathfrak{X}}$ of $j_*\big(V^\vee\big) \otimes_k B \in \mathsf{D}_{\mathrm{qcoh}}(\mathcal{D}_{\mathfrak{X},B})$, which is perfect because of Bernstein's theorem asserting that $j_*\big(V^\vee\big)$ is a coherent and holonomic complex of $\mathcal{D}_{\mathfrak{X}}$-modules.

We thus have that $j_* E$ is a perfect $\mathcal{D}_{\mathfrak{X},B}$-module. Moreover, it is also holonomic (see Proposition 2.4). Indeed, because $\mathsf{R}(E)$ is isomorphic to $V_\infty \otimes_k B$, its characteristic cycle is contained in $\Lambda \times \operatorname{Spec} B \subset T^\vee X \times \operatorname{Spec} B$, where $\Lambda = \operatorname{Char}(\mathfrak{J}_*(V_\infty))$. Since the ∞-functor sending B to perfect $\mathcal{D}_{\mathfrak{X},B}$-modules is locally of finite presentation, we can choose i and $F_i \in \mathcal{D}_{\mathsf{perf}}(\mathcal{D}_{\mathfrak{X},B})$ such that $j_* E \simeq F_i \otimes_{B_i} B$. By enlarging i if necessary, we can also assume that the characteristic variety of F_i is contained in $\Lambda \times \operatorname{Spec} B_i$ and thus that F_i is moreover holonomic. Also, we can assume that F_i and $j^* E_i^\vee$ are isomorphic as objects in $\mathsf{Vect}^\nabla(X)(B_i)$.

As now both F_i and $\mathfrak{J}_*(V_\infty) \otimes_k B_i$ are perfect and holonomic, Proposition 2.4 implies that $F_i \otimes_{\mathcal{O}} \big(\mathfrak{J}_*\big(V_\infty^\vee\big) \otimes_k B_i\big) \simeq j_*(E_i')$ remains perfect over $\mathcal{D}_{\mathfrak{X},B_i[f^{-1}]}$ for some non-zero localization of B_i. Working with $V_\infty^\vee$ and $E^\vee$ from the start, we prove in the same manner that $j_*\big((E_i')^\vee\big)$ is also perfect. By Corollary 5.8, this implies that $\widetilde{\mathbb{H}}_{\mathrm{DR}}\big(\widehat{\partial X}, E_i'\big)$ is perfect, which finishes the proof of the lemma. $\qquad\square$

Proof of Theorem 6.1, *continued.* By Lemma 6.5, the sheaf $\mathcal{J}$ is representable by a scheme of finite type. The sheaf $\mathcal{I}$ clearly is an open subsheaf of $\mathcal{J}$ and thus is also representable by a scheme of finite type over a non-empty open subscheme on $\operatorname{Spec} B_i$. The canonical isomorphism $\alpha\colon \mathsf{R}(E) \simeq V_\infty \otimes_k B$, which an element in $\mathcal{I}(B)$, is then definable over some $B_i\big[f^{-1}\big]$ for some i and non-zero localization, say $\alpha_i\colon \mathsf{R}(E_0)\big[f^{-1}\big] \simeq V_\infty \otimes_k B_i\big[f^{-1}\big]$. The pair (E_0, α_i) defines an

object in $\mathsf{Vect}^\nabla(X)_{V_\infty}\big(B_i\big[f^{-1}\big]\big)$ whose image in $\mathsf{Vect}^\nabla_{V_\infty}(X)\big(B\big[f^{-1}\big]\big)$ is the restriction of our original object E.

This finishes the proof of condition (6) of Theorem A.3, and thus of Theorem 6.1. $\qquad\square$

Unfortunately, we do not know whether Theorem 6.1 can be strengthened to the statement that $\mathsf{Vect}^\nabla_{V_\infty}(X)$ is representable by an algebraic space locally of finite type over k. The only missing condition would be that $\mathsf{Vect}^\nabla_{V_\infty}(X)$ is also locally of finite presentation, a condition that we have not been able to prove or disprove.

Appendix. Derived quasi-algebraic spaces and Artin's representability

In this section, we have gathered some definitions and results on *derived quasi-algebraic spaces* and the corresponding representability criterion. Derived quasi-algebraic spaces are slight generalizations of derived algebraic spaces for which atlases only exist generically. These derived stacks are not algebraic in general but are algebraic as soon as the functors they represent are locally of finite presentation.

To make sense of such spaces, we will need the following notion of a dominant morphism to a not necessarily algebraic derived stack F. Assume that F is a derived stack which has a perfect global cotangent complex and is nilcomplete and infinitesimally cartesian. We will also assume that F is integrable; that is, for any local complete noetherian discrete k-algebra $A = \lim_i A/\mathfrak{m}^i$, the natural morphism

$$F(A) \longrightarrow \lim_i F\big(A/\mathfrak{m}^i\big)$$

is bijective.

For any such F, any field K which is finitely generated over k, and any point $x\colon \operatorname{Spec} K \to F$, by [Lur18, Theorem 18.2.5.1], there exist a complete local noetherian cdga A with residue field K and a formally smooth morphism

$$\operatorname{Spf}(A) \longrightarrow F$$

extending the point x. This way, we get a morphism $\operatorname{Spf}(\pi_0(A)) \to F$ from its truncation and, by integrability, a well-defined morphism $\widehat{x}\colon \operatorname{Spec}(\pi_0(A)) \to F$. A morphism $\widehat{x}$ obtained this way will be called a *formally smooth lift of x*.

DEFINITION A.1. For a derived stack F as above and a derived scheme X locally of finite presentation over k, with a morphism $f\colon X \to F$, we say that f is *dominant* if for any finitely generated k-field K, any point $x\colon \operatorname{Spec} K \to F$, and any formally smooth lift $\widehat{x}\colon \operatorname{Spec}(\pi_0(A)) \to F$, the derived scheme $X \times_F \operatorname{Spec}(\pi_0(A))$ is non-empty.

Note that if F is itself representable by a derived algebraic space locally of finite presentation, then $f\colon X \to F$ is dominant in the sense above if and only if for any étale morphism $\operatorname{Spec} B \to F$, we have $X \times_F \operatorname{Spec} B \neq \varnothing$. Indeed, assume that there is an étale map $\operatorname{Spec} B \to F$ whose pull-back to X is empty. We pick a point x of $\operatorname{Spec} B$ and consider the corresponding formal completion $\widehat{B}_x$ of B. Since $\operatorname{Spec} B \to F$ is étale, the composition

$$\operatorname{Spec} \pi_0\big(\widehat{B}_x\big) \longrightarrow \operatorname{Spec} B \longrightarrow F$$

is a formally smooth lift of x. By construction, the pull-back $\operatorname{Spec} \pi_0\big(\widehat{B}_x\big) \times_F X$ is empty. This shows that the above notion of dominant map is a generalization of the notion of a morphism with Zariski dense image.

We can now give the definition of a derived quasi-algebraic space as a derived stack with dominant smooth atlases, as follows.

Definition A.2. A derived stack F is a *derived quasi-algebraic space* (locally of finite presentation with schematic diagonal of finite presentation) if it satisfies the following conditions:

(1) The diagonal of the stack $F \to F \times F$ is representable by a derived scheme of finite presentation.

(2) The derived stack F has a perfect global cotangent complex and is nil complete and infinitesimally cartesian.

(3) The derived stack F is integrable: for any local complete noetherian discrete k-algebra $A = \lim_i A/\mathfrak{m}^i$, the natural morphism $F(A) \longrightarrow \lim_i F(A/\mathfrak{m}^i)$ is bijective.

(4) There exist a family of cdga A_i of finite presentation over k and a morphism $p \colon \sqcup \operatorname{Spec} A_i \to F$ such that
   (a) for each i the morphism $\operatorname{Spec} A_i \to F$ is smooth,
   (b) the morphism p is dominant in the sense of Definition A.1 above.

A derived quasi-algebraic space is algebraic if and only if the functor F is furthermore locally of finite presentation. This follows from Artin–Lurie's representability theorem [Lur18, Theorem 18.3.0.1]. Similarly, derived quasi-algebraic spaces can be characterized by the following version of Artin's representability.

Theorem A.3. *A derived stack F is a derived quasi-algebraic space if it satisfies the following conditions:*

(1) *For any discrete cdga B, the simplicial set $F(B)$ is 0-truncated.*

(2) *The diagonal morphism of its truncation is representable by a scheme of finite presentation.*

(3) *The derived stack F has a perfect global cotangent complex.*

(4) *The derived stack F is nilcomplete and infinitesimally cartesian.*

(5) *For any discrete local k-algebra $(A, \mathfrak{m})$ essentially of finite type, with completion $\widehat{A} = \lim_i A/\mathfrak{m}^i$, the morphism $F(\widehat{A}) \to \lim_i F(A/\mathfrak{m}^i)$ is an equivalence.*

(6) *For any filtered system of noetherian discrete commutative k-algebras $B = \operatorname{colim}_i B_i$ and any $x \in F(B)$, there exist an index i and a non-empty Zariski open $U_i \subset \operatorname{Spec} B_i$ with $U = U_i \times_{\operatorname{Spec} B_i} \operatorname{Spec} B$ non-empty such that the restriction of x lies in the image of $F(U_i) \to F(U)$.*

Sketch of a proof. The proof is essentially the same as that of the usual representability theorem in [Lur18].

Consider fields K which are finitely generated over k. For any morphism $x \colon \operatorname{Spec} K \to F$, we can use [Lur18, Theorem 18.2.5.1] to find a local complete and noetherian cdga $(A, \mathfrak{m})$ with residue field $K = A/\mathfrak{m}$ and a factorization $\operatorname{Spec} K \hookrightarrow \operatorname{Spf}(\widehat{A}) \to F$, where the second map is formally smooth (that is, its relative cotangent complex is a vector bundle). We write $B = \pi_0(\widehat{A})$, which is a complete local discrete k-algebra with residue field K and consider the induced morphism on the truncation $\widehat{x} \colon \operatorname{Spf}(B) \to F$. We can use condition (4) to lift this to a factorization

$$\operatorname{Spec} K \lhook\joinrel\longrightarrow \operatorname{Spec}(B) \longrightarrow F.$$

As explained in the proof of [Lur18, Theorem 18.2.5.1], there exists a k-algebra of finite type $B' \subset B$ such that if $p = \mathfrak{m} \cap B'$, then the induced morphism on formal completions $\widehat{B'}_p \to B$ is

surjective (take A' big enough so that it contains generators for K over k as well as generators of the k-vector space $\mathfrak{m}/\mathfrak{m}^2$). We can now apply Popescu's theorem to the regular morphism $B' \to \widehat{B}'_p$ and thus write $\widehat{B}'_p = \mathrm{colim}_i B'_i$ as a filtered colimit of smooth B'-algebras. Since B is finitely presented as a $\widehat{B}'_p$-algebra, we can find an index i and a B'_i-algebra C'_i of finite presentation such that

$$C \simeq \mathrm{colim}_i \left(\widehat{B}'_p B \otimes_{B'_i} C'_i \right).$$

We let $C_i := \widehat{B}'_p B \otimes_{B'_i} C'_i$, which is a B'_i-algebra of finite presentation and thus is itself of finite presentation over k.

We now apply condition (6) to the morphism $\mathrm{Spec}\, B \to F$ and get that there exist an integer i and a Zariski open $U_i = \mathrm{Spec}\, C_i[f^{-1}] \subset \mathrm{Spec}\, C_i$ with $U = \mathrm{Spec}\, B[f^{-1}]$ non-empty which fits in a commutative diagram

LEMMA A.4. *With the notation above, and after enlarging i if necessary, we have that the morphism $p\colon U_i \to F$ constructed above is formally smooth in the underived sense: $\tau_{\leqslant -1}(\mathbb{L}_{U_i/F})$ is a vector bundle in degree 0.*

Proof. First of all, U_i being of finite type together with the fact that the diagonal of F is representable of locally of finite presentation implies that p is representable and locally of finite type in the underived sense. It thus only remains to show that p is also formally smooth in the underived sense, in other words, that its relative 1-truncated cotangent complex $\tau_{\leqslant 1}(\mathbb{L}_{U_i/F})$ is a vector bundle.

For this we first notice that $\mathbb{L}_{U_i/F}$ is almost perfect (that is, quasi-isomorphic to a complex of free modules of finite rank over $C_i[f^{-1}]$ concentrated in degree $(-\infty, 0]$). Since we are only interested in its truncation $\tau_{\leqslant 1}(\mathbb{L}_{U_i/F})$, we will be able to act as if $\mathbb{L}_{U_i/F}$ is in fact perfect (simply replace it with a perfect complex having the same cohomology in degree $[-n, 0]$ for n big enough). We start by computing the pull-back of $\mathbb{L}_{U_i}$ to $U = \mathrm{Spec}\, B[f^{-1}]$.

Consider the exact triangle of complexes of B-modules (where $\mathbb{L}_A$ stands for $\mathbb{L}_{A/k}$ for any k-algebra A)

$$\mathbb{L}_{C_i} \otimes_{C_i} B \longrightarrow \mathbb{L}_B \longrightarrow \mathbb{L}_{B/C_i}.$$

Since B is complete with respect to its maximal ideal $\mathfrak{m}$, for any connective dg-module E over B, we have its completion $\widehat{E} := \lim_i E \otimes_B B/\mathfrak{m}^i$, which is another connective B-dg-module together with a natural morphism $E \to \widehat{E}$. Moreover, when E is almost perfect, this morphism is a quasi-isomorphism. We can then complete the terms in the above triangle to get a new triangle

$$\widehat{\mathbb{L}_{C_i} \otimes_{C_i}} B \longrightarrow \widehat{\mathbb{L}_B} \longrightarrow \widehat{\mathbb{L}_{B/C_i}}.$$

As $\mathbb{L}_{C_i}$ is almost perfect, the first term is simply $\mathbb{L}_{C_i} \otimes_{C_i} B$. Moreover, by base change, $\mathbb{L}_{B/C_i}$ is naturally equivalent to $\mathbb{L}_{\widehat{B}'/C'_i} \otimes_{B'} B$. Since $B' \to B$ is a surjective local morphism, we see that the base change of $\widehat{\mathbb{L}_{B'/C'_i}}$, considered as a pro-object in connective B'-dg-modules, by the map

$\widehat{B}' \to B$ is the pro-object $\widehat{\mathbb{L}_{B/C_i}}$.

We now use that $\widehat{B}'$ is the completion of B' along the maximal ideal $\mathfrak{m} \subset B'$, and so for all i we have $\mathbb{L}_{\widehat{B}'/B'} \otimes_{B'} B'/\mathfrak{m}^i \simeq 0$. We thus have an equivalence of pro-objects

$$\widehat{\mathbb{L}_{\widehat{B}'/C_i}} \simeq \mathbb{L}_{C_i/B'} \widehat{\otimes}'_B \widehat{B}'[1].$$

Since C_i is smooth over B', we therefore conclude that the pro-object $\widehat{\mathbb{L}_{B/C_i}}$ is a vector bundle in degree 1, and so its realization as a connective B-dg-module is $\mathbb{L}_{C_i/B'} \otimes_{C_i} B[1]$. Our original triangle can therefore be written as

$$\mathbb{L}_{C_i} \otimes_{C_i} B \longrightarrow \widehat{\mathbb{L}_B} \longrightarrow V[1]$$

with V a vector bundle on $\operatorname{Spec} B$. We can now localize this triangle to the open subscheme $U = \operatorname{Spec} B[f^{-1}]$ of $\operatorname{Spec} B$ in order to get new triangle on U

$$\mathbb{L}_{C_i[f^{-1}]} \otimes_{C_i[f^{-1}]} B[f^{-1}] \longrightarrow \widehat{\mathbb{L}_B}[f^{-1}] \longrightarrow V[f^{-1}][1].$$

The morphisms $q \colon U \to U_i \to F$ induces a morphism

$$q^*(\mathbb{L}_F) \longrightarrow \mathbb{L}_{C_i[f^{-1}]} \otimes_{C_i[f^{-1}]} B[f^{-1}]$$
$$\searrow \qquad \downarrow$$
$$\mathbb{L}_B[f^{-1}],$$

which factors through completions since $q^*(\mathbb{L}_F)$ is a perfect complex by our condition (3). We get

$$q^*(\mathbb{L}_F) \longrightarrow \mathbb{L}_{C_i[f^{-1}]} \otimes_{C_i[f^{-1}]} B[f^{-1}]$$
$$\searrow \qquad \downarrow$$
$$\widehat{\mathbb{L}_B}[f^{-1}],$$

and the induced morphism on the cones sits in an exact triangle

$$\mathbb{L}_{U_i/F} \otimes_{C_i[f^{-1}]} B[f^{-1}] \longrightarrow \widehat{\mathbb{L}_{\operatorname{Spec} B/F}}[f^{-1}] \longrightarrow V[1].$$

Because $\operatorname{Spf} \to F$ was chosen to be formally smooth in the underived sense, $\tau_{\leqslant -1}(\widehat{\mathbb{L}_{\operatorname{Spec} B/F}}[f^{-1}])$ is a vector bundle in degree 0. The conclusion is that $\mathbb{L}_{U_i/F}$ is an almost perfect complex over $C_i[f^{-1}]$ such that its base change to $B[f^{-1}] = \operatorname{colim}(C_i[f^{-1}])$ has vanishing H^{-1} and a vector bundle as H^0. This implies that the same is true for $\mathbb{L}_{U_i/F} \otimes_{C_i[f^{-1}]} C_j[f^{-1}]$ for some big enough j. $\square$

Sketch of a proof of Theorem A.3, *continued.* We use [Lur18, Theorem 18.2.5.1] again, but this time for $U_i \to F$, which by Lemma A.4 can be chosen to be formally smooth in the underived sense. We can thus produce a smooth morphism $W_i \to F$, where W_i is a derived affine scheme whose truncations coincides with the given map $U_i \to F$. The derived scheme W_i is itself of finite presentation over k as its truncation is of finite type and its cotangent complex is perfect (because its maps smoothly to F).

Taking the union of all morphisms $W_i \to F$ constructed above provides the required generic atlas for F as in Definition A.2. $\square$

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