Local Stability and Convergence Analysis of Neural Network Controllers with Error Integral Inputs

Xingang Fu, *Member, IEEE,* Shuhui Li, *Senior Member, IEEE,* Donald C. Wunsch, *Fellow, IEEE,* and Eduardo Alonso

***Abstract*—This paper investigates the local stability and local convergence of a class of Neural Network (NN) controllers with error integrals as inputs for reference tracking. It is formally proved that if the input of the NN controller consists exclusively of error terms, the control system shows a non-zero steady-state error for any constant reference except for one specific point, for both single-layer and multi-layer NN controllers. It is further proved that adding error integrals to the input of the (single- and multi-layer) NN controller is one sufficient way to remove the steady-state error for any constant reference. Due to the nonlinearity of the NN controllers, the NN control systems are linearized at the equilibrium points. We provide proof that if all the eigenvalues of the linearized NN control system have negative real parts, local asymptotic stability and local exponential convergence are guaranteed. Two case studies were explored to verify the theoretical results: a single-layer NN controller in a one-dimensional system and a four-layer NN controller in a two-dimensional system applied to renewable energy integration. Simulations demonstrate that when NN controllers and the corresponding Generalized Proportional-Integral (PI) controllers have the same eigenvalues, all control systems exhibit almost the same responses in a small neighborhood of their respective equilibrium points.**

***Index Terms*—Neural Network Controller, Error Integral, Steady-State Error, Local Asymptotic Stability, Local Exponential Convergence, Generalized PI controller.**

I. INTRODUCTION

RECENTLY, Dynamic Programming (DP) [1] has been used extensively in the study of optimal control of nonlinear systems [2], [3], [4]. As one type of Approximate Dynamic Programming (ADP), Adaptive Critic Designs (ACD) have been adopted to approximate the optimal cost and the optimal control of a system [5], [6], [7], [2]. In [8], [9], a Neural Network (NN) was trained based on the ADP principle to control a three-phase Inductor (L) filter-based Grid-Connected Converter (GCC) system. An ADP-based NN controller of Inductor-Capacitor-Inductor (LCL) filter-based three-phase [10] and single-phase [11] GCC systems was also demonstrated to be able to yield an excellent performance compared to conventional Proportional-Integral (PI) controllers. In [10], a Recurrent Neural Network (RNN) vector controller shows a wider stability region for the system parameter change than Active Damping (AD) or Passive Damping (PD) vector controllers in LCL-based GCC systems. [12] implemented a NN vector controller with error integral inputs in a Permanent-Magnet Synchronous Motor (PMSM) to overcome the decoupling inaccuracy problem associated with conventional PI-based vector-control methods.

Even though NN controllers have a huge potential, they are considered as a black-box technique [13] and their theoretical foundations are missing. When NNs are applied to real-world problems, many issues arise concerning their stability and convergence properties. The stability problem, in particular, is critical in neural control systems [14], [15]. If the system is unstable, it can cause serious damage and financial loss. This is the main concern that curbs the application of NN controllers in real-life systems by control engineers and electric engineers. In addition, how to guarantee that the training of an NN will converge, and towards which direction training will be more effective and faster are still unsolved problems [16]. A formal analysis of such issues, as the one demonstrated in this paper, is thus much needed.

This research specifically intends to study the local stability and local convergence of NN controllers with error integral terms. The specific contributions of the paper are as follows: 1) proving that adding an error integral to the inputs of the NN controller is sufficient to remove the steady-state error of the NN controller for any constant reference; 2) establishing the condition for NN controllers to guarantee local asymptotic stability and local exponential convergence, which is that all eigenvalues of the NN control systems should have negative real parts; 3) revealing that NN controllers and generalized PI controllers that share the same eigenvalues generate almost the same responses in a small enough domain of their respective equilibrium points; and 4) performing case studies of a one-dimensional single-layer NN controller and a two-dimensional four-layer NN controller applied in renewable energy integration that verify the theoretical results experimentally.

The rest of the paper is structured as follows: Section II analyses mathematically the local stability and local convergence of the single-layer NN controller of two structures, one with only error terms and another with error terms and error integral terms. Likewise, the local stability and local convergence of multi-layer NN controllers with only error terms and with error terms and error integral terms are investigated formally in Section III. A case study for a one-dimensional single-layer NN controller is demonstrated in Section IV to verify experimentally the conclusions of Section

Xingang Fu is with the Department of Electrical Engineering and Computer Science, Texas A&M Kingsville, Kingsville, TX 78363, USA (email: Xingang.Fu@tamuk.edu).

Shuhui Li is with the Department of Electrical and Computer Engineering, The University of Alabama, Tuscaloosa, AL 35487, USA (email: sli@eng.ua.edu).

Donald C. Wunsch, is with the Department of Electrical and Computer Engineering, Missouri University of Science and Technology, Rolla, MO 65409, USA (email: dwunsch@mst.edu).

Eduardo Alonso is with the Artificial Intelligence Research Centre (CitAI), City, University of London, EC1V 0HB, UK (email: e.alonso@city.ac.uk).

II. Section V investigates a four-layer NN controller in a two-dimensional system applied to electric power applications to validate the theoretical conclusions of Section III. Finally, the paper concludes with summary remarks in Section VI.

II. Single-Layer NN Controllers

In this section, the conditions for the local stability and local convergence of single-layer NN controllers are formally proved.

A. State-Space Model

Consider the following time-invariant state-space model

$$\dot{x} = Ax + Bu \tag{1}$$

where x is the system state vector with $x \in R^n$, $\dot{x}$ denotes the derivative of the state vector x with respect to time t, u stands for the input or control vector with $u \in R^m$, and A and B are the state or system matrix and input matrix, respectively, with $A \in R^{n\times n} \neq 0$ and $B \in R^{n\times m} \neq 0$.

B. NN Controllers with Only Error Term Inputs

If a single-layer NN controller has only the error term e as input, the control vector u can be expressed as

$$u = k_a \tanh(w_p e + b) \tag{2}$$

where w_p represents the weight matrix for error terms with $w_p \in R^{m\times n}$, b is the bias vector of the NN controller with $b \in R^m$, the constant scalar k_a stands for an actuator gain with $k_a \in R$, and the error e is defined as

$$e = x_{ref} - x \tag{3}$$

with x_{ref} ($\in R^n$) representing the reference for the system state x and $e \in R^n$.

According to the definition of the error term e in (3), the following two equations hold.

$$x = x_{ref} - e \tag{4}$$

$$\dot{e} = -\dot{x} \tag{5}$$

We can now substitute (2), (4), and (5) into (1), and rewrite (1) into the closed-loop system with tracking error e as the system state, as follows:

$$\dot{e} = f(e) = A(e - x_{ref}) - k_a B \tanh(w_p e + b) \tag{6}$$

Theorem 1. *For a neural dynamic system (6), $e = 0$ is not an equilibrium point except when $x_{ref} = -\frac{1}{k_a}A^{-1}B\tanh(b)$. The system will have a non-zero steady-state error for any constant reference except for $x_{ref} = -\frac{1}{k_a}A^{-1}B\tanh(b)$.*

Proof: The equilibrium point of (6) is the root of the function $f(e)$. If we substitute $e = 0$ into (6), the function $f(e)$ equals

$$f(e) = -Ax_{ref} - k_a B \tanh(b) \neq 0. \tag{7}$$

Only when $x_{ref} = -\frac{1}{k_a}A^{-1}B\tanh(b)$, $f(e) = 0$. Thus, $e = 0$ is not an equilibrium point of system (6) except for one specific reference point.

We denote e^* to represent the root of $f(e)$, which satisfies the following equation

$$f(e^*) = A(e^* - x_{ref}) - k_a B \tanh(w_p e^* + b) = 0 \tag{8}$$

Thus e^* is the equilibrium point of (6) and $e^* \neq 0$, which also means that the system has a non-zero steady-state error. ■

Lemma 1. *For the Linear Time-Invariant system*

$$\dot{x} = Gx \tag{9}$$

where $x \in R^n$, constant system matrix $G \in R^{n\times n}$, if all eigenvalues of G have negative real parts, the equilibrium point x = 0 is globally asymptotically stable, and globally exponential convergence is also guaranteed.

Proof: The analytical solution of (9) for a given initial state $x(0)$ has the following form

$$x(t) = exp(Gt)x(0) \tag{10}$$

where exp represents the base of the natural logarithm. For any system matrix G, there exists r distinct eigenvalues λ_1, λ_2,$\cdots$,λ_r with algebraic multiplicity of n_1, n_2,$\cdots$,n_r and $n_1 + n_2 + \cdots + n_r = n$. Thus (10) can be further expressed as

$$x(t) = \sum_{i=1}^{r}\sum_{j=1}^{n_i} c_{ij} t^{j-1} e^{\lambda_i t} \tag{11}$$

where constant vector $c_{ij} \in R^n$, and can be determined by the initial state $x(0)$ and the corresponding eigenvectors of each eigenvalue.

When all eigenvalues have negative real parts, that is $Re(\lambda_i) < 0$, $\lim_{t\to\infty} x(t) = 0$, which means that the equilibrium point x = 0 is globally asymptotically stable and also globally exponentially convergent ([17], [18], [19], [20]). ■

Theorem 2. *For a neural dynamic system (6), local asymptotic stability and local exponential convergence are guaranteed if the weight matrix w_p and bias vector b of the NN controller satisfy the following condition*

$$Re\left\{\text{eig}\left(A - k_a B w_p \,\text{diag}(1 - \tanh^2(w_p e^* + b))\right)\right\} < 0 \tag{12}$$

where eig *denotes the eigenvalue operator, Re stands for the real part,* diag *represents the diagonal matrix operator, and e^* is the equilibrium point of (6).*

Proof: The equilibrium point of (6) can be shifted from e^* to 0 by defining a new variable e_n

$$e_n = e - e^* \tag{13}$$

and thus

$$\dot{e_n} = \dot{e} \tag{14}$$

If we substitute (13) and (14) into (6), the new system equation will be

$$\dot{e_n} = f(e_n) = A(e_n + e^* - x_{ref}) - k_a B \tanh(w_p(e_n + e^*) + b) \tag{15}$$

For (15), the equilibrium point of the system is $e_n = 0$.

The right-hand side of (15) are nonlinear functions. Under the definition of Lyapunov stability [18], we can use the first-order derivative to linearize the system at $e_n = 0$ and obtain the following set of linear equations

$$\dot{e_n} = (\frac{\partial f}{\partial e_n}|_{e_n=0})e_n = Ge_n \tag{16}$$

where the system matrix G equals

$$G = A - k_a B w_p \operatorname{diag}(1 - \tanh^2(w_p e^* + b)) \tag{17}$$

According to Lemma 1, as long as all the eigenvalues of G have negative real parts, that is $Re\{\text{eig}(G)\} < 0$, the system's global asymptotic stability and global exponential convergence are guaranteed. However, as the system (16) is linearized at the equilibrium point, only the local asymptotic stability and local exponential convergence can be guaranteed. ■

Remark 1. In (12), the reference x_{ref} does not exist explicitly. However, the equilibrium point e^* are the roots of (6). When all system parameters (A, B, k_a) and the NN weight w_p and bias b are kept unchanged, e^* depends on x_{ref}. So the eigenvalues of system matrix G are implicit functions of x_{ref} and thus the reference x_{ref} affects the stability of the system.

Corollary 2.1. *Consider a generalized Proportional (P) controller $u = k_a K_p e$ with the constant proportional gain matrix K_p and $K_p \in R^{m\times n}$, which can be regarded as a special case of the single-layer NN controller with a linear identity function as the activation function and no bias. Thus the steady-state error $e(\infty)$ and the equilibrium point e^* are*

$$e(\infty) = e^* = (A - k_a B K_p)^{-1} A x_{ref} \tag{18}$$

and the corresponding global stability condition is

$$Re\{\text{eig}(A - k_a B K_p)\} < 0 \tag{19}$$

The reference x_{ref} is not contained in (19) and thus does not affect the system stability.

C. NN Controllers with Error Integral Inputs

Consider a single-layer NN controller having error e and error integral s as inputs. The control vector u is expressed as

$$u = k_a \tanh(w_p e + w_i s + b) \tag{20}$$

where the error integral s is defined as

$$s = \int_0^t e(\tau) d\tau \tag{21}$$

with $s \in R^n$ and w_i represents the weight matrix for error integral terms with $w_i \in R^{m\times n}$.

If we substitute (4), (5), and (20) into (1), the system equation will be simplified as

$$\dot{e} = A(e - x_{ref}) - k_a B \tanh(w_p e + w_i s + b) \tag{22}$$

From the definition of error integral s in (21), the following equation can be derived

$$\dot{s} = e \tag{23}$$

Thus combining (22) and (23), a new augmented state-space model can be obtained

$$\begin{cases} \dot{e} = f_1(e, s) = A(e - x_{ref}) - k_a B \tanh(w_p e + w_i s + b) \\ \dot{s} = f_2(e, s) = e \end{cases} \tag{24}$$

Through this conversion, the original n-dimension NN control system (22) is converted into a $2n$-dimensional system (24).

Remark 2. This conversion is not an equivalent transformation. From (21), (23) can be derived. However, from (23), (21) is not the only solution. In general, many solutions can be obtained from (23) and the general solution is

$$s = \int_0^t e(\tau) d\tau + C \tag{25}$$

where C is one constant vector $C \in R^n$.

Theorem 3. *For a neural dynamic system (24), $e = 0$ is an equilibrium point and the system does not have a steady-state error for any constant reference x_{ref}.*

Proof: The equilibrium point of (24) are the roots of the right side function, that is

$$\begin{cases} f_1(e, s) = A(e - x_{ref}) - k_a B \tanh(w_p e + w_i s + b) = 0 \\ f_2(e, s) = e = 0 \end{cases} \tag{26}$$

From the second equation of (26), e must be 0. Thus the equilibrium point will be (0, s^*), where s^* satisfies

$$A x_{ref} + k_a B \tanh(w_i s^* + b) = 0 \tag{27}$$

The equilibrium point of (24) is (0, s^*), which means that the system error e converges to 0 whereas the error integral s converges to s^* when the time goes to infinity. When there is an error integral term s feeding into the input of the NN controller, it is guaranteed that there is no steady-state error in the system. ■

Theorem 4. *For a neural dynamic system (24), local asymptotic stability and local exponential convergence are guaranteed if the weight matrices w_p and w_i of the NN controller satisfy the following condition*

$$Re\left\{\text{eig}\left(\begin{bmatrix} G_{11} & G_{12} \\ I & 0 \end{bmatrix}\right)\right\} < 0 \tag{28}$$

where G_{11} and G_{12} equal

$$G_{11} = A - k_a B w_p \operatorname{diag}(1 - \tanh^2(w_i s^* + b)) \tag{29}$$

$$G_{12} = -k_a B w_i \operatorname{diag}(1 - \tanh^2(w_i s^* + b)) \tag{30}$$

and s^* is is the equilibrium point of (24).

Proof: The equilibrium point of (24) can be shifted from [0; s^*] to [0; 0] using the following conversion:

$$\begin{cases} e = e \\ s_n = s - s^* \end{cases} \tag{31}$$

Substituting (31) into (24), the new augmented system equation will be

$$\begin{cases} \dot{e} = f_1(e, s_n) = A(e - x_{ref}) - k_a B \tanh[w_p e + w_i(s_n + s^*) + b] \\ \dot{s_n} = f_2(e, s_n) = e \end{cases} \tag{32}$$

Under the definition of Lyapunov stability [18] and linearizing (32) at the equilibrium point [0, 0], the system equation will become

$$\begin{bmatrix} \dot{e} \\ \dot{s_n} \end{bmatrix} = \begin{bmatrix} G_{11} & G_{12} \\ G_{21} & G_{22} \end{bmatrix} \begin{bmatrix} e \\ s_n \end{bmatrix} \tag{33}$$

where G_{11}, G_{12}, G_{21}, and G_{22} are defined as

$$\begin{aligned} G_{11} &= \frac{\partial f_1(e, s_n)}{\partial e}|_{e=0, s_n=0} \\ &= A - k_a B w_p \operatorname{diag}(1 - \tanh^2(w_i s^* + b)) \end{aligned} \tag{34}$$

$$\begin{aligned} G_{12} &= \frac{\partial f_1(e, s_n)}{\partial s_n}|_{e=0, s_n=0} \\ &= -k_a B w_i \operatorname{diag}(1 - \tanh^2(w_i s^* + b)) \end{aligned} \tag{35}$$

$$G_{21} = \frac{\partial f_2(e, s_n)}{\partial e}|_{e=0, s_n=0} = I \tag{36}$$

$$G_{22} = \frac{\partial f_2(e, s_n)}{\partial s_n}|_{e=0, s_n=0} = 0 \tag{37}$$

As G_{11}, G_{12}, G_{21}, and G_{22} are all constants, according to Lemma 1, if the NN weights w_p and w_i satisfy the following condition,

$$Re\left\{\operatorname{eig}\left(\begin{bmatrix} G_{11} & G_{12} \\ I & 0 \end{bmatrix}\right)\right\} < 0 \tag{38}$$

the system's global asymptotic stability and global exponential convergence are guaranteed. However, as the system (33) is linearized at the equilibrium point, only the local asymptotic stability and local exponential convergence can be guaranteed. ■

Remark 3. Although the bias vector b of NNs is not contained in (29) and (30), b affects the location of s^* from (27) and thus affects the convergence region of the equilibrium point.

Corollary 4.1. *Consider a generalized PI controller $u = k_a(K_p e + K_i s)$, where K_p and K_i are the constant matrices representing the proportional gains and the integral gains, respectively, and $K_p, K_i \in R^{m \times n}$. This generalized PI controller can be regarded as a special case of the single-layer NN controller with a linear identity function as the activation function and no bias. Thus the equilibrium point of the system is (0, s^*) and s^* equals*

$$s^* = -\frac{1}{k_a}(BK_i)^{-1} A x_{ref} \tag{39}$$

To guarantee global stability and exponential convergence, the following condition needs to be satisfied

$$Re\left\{eig\left(\begin{bmatrix} A - k_a B K_p & -k_a B K_i \\ I & 0 \end{bmatrix}\right)\right\} < 0 \tag{40}$$

The reference x_{ref} will not affect the stability and convergence of the control system.

Remark 4. For a single-layer NN controller with only error terms (2) or with error terms and error integral terms (20), the reference x_{ref} will appear in the condition equations (8) and (27) explicitly or inexplicitly, and thus will affect the system stability. Hence, the weights and bias vector of the NN controller together with the reference will determine the local stability and the local convergence of the system.

III. Multi-layer NN Controllers

In this section, a multi-layer NN controller with a more generic function format that expands the single-layer NN controller in Section II is studied theoretically.

A. NN Controllers with Only Error Term Inputs

If a multi-layer NN controller has only the error term e as input, we use $R(e)$ to represent the NN controller and the control vector u can be expressed as

$$u = R(e) \tag{41}$$

where $R(e)$ can be any continuous and continuously differentiable functions of e, that is $R(e) \in C^1[R^n, R^m]$.

If we substitute (4), (5), and (41) into (1), we can rewrite (1) into the following equation

$$\dot{e} = f(e) = A(e - x_{ref}) - BR(e) \tag{42}$$

Theorem 5. *For a neural dynamic system (42), $e = 0$ is not an equilibrium point except when $x_{ref} = -A^{-1}BR(0)$. Such a system will have a non-zero steady-state error for any constant reference except for $x_{ref} = -A^{-1}BR(0)$.*

Proof: The equilibrium point of (42) is the root of the function $f(e)$. If we substitute $e = 0$ into (6), the function $f(e)$ equals

$$f(e) = -Ax_{ref} - BR(0) \neq 0 \tag{43}$$

The only exception is when $x_{ref} = -A^{-1}BR(0)$. Thus, $e = 0$ is not an equilibrium point of the system (42).

Denote e^* to represent the root, the following equation will be satisfied

$$f(e^*) = A(e^* - x_{ref}) - BR(e^*) = 0 \tag{44}$$

Thus e^* is the equilibrium point of (42) and $e^* \neq 0$, which also means that the system has a non-zero steady-state error. ■

Theorem 6. *For a neural dynamic system (42), local asymptotic stability and local exponential convergence are guaranteed if the weight matrix and bias vector of the NN satisfy the following condition*

$$Re\left\{\operatorname{eig}\left(A - B\frac{\partial R(e)}{\partial e}|_{e=e^*}\right)\right\} < 0 \tag{45}$$

where e^* is the equilibrium point of (42).

Proof: Define $e_n = e - e^*$ and shift the equilibrium point of (42) from e^* to 0. The new system equation will be

$$\dot{e_n} = f(e_n) = A(e_n + e^* - x_{ref}) - BR(e_n + e^*) \tag{46}$$

The right-hand side of (46) is a nonlinear function. Under the definition of Lyapunov stability [18], we use the first-order derivative to linearize the system at $e_n = 0$ and obtain the following set of linear equations

$$\dot{e_n} = (\frac{\partial f}{\partial e_n}|_{e_n=0})e_n = Ge_n \tag{47}$$

where the system matrix G is defined as

$$G = A - B\frac{\partial R(e_n + e^*)}{\partial e_n}|_{e_n=0} = A - B\frac{\partial R(e)}{\partial e}|_{e=e^*} \quad (48)$$

According to Lemma 1, as long as all the eigenvalues of G have negative real parts, that is $Re\{\text{eig}(G)\} < 0$, the system's global asymptotic stability and global exponential convergence are guaranteed. However, as the system (47) is linearized at the equilibrium point, only the local asymptotic stability and local exponential convergence can be guaranteed. ■

Remark 5. In (45), x_{ref} does not exist explicitly. However, e^* is the roots of (44) and depends on x_{ref}. So, the system matrix G and its eigenvalues are implicit functions of x_{ref} and thus the reference x_{ref} affects the stability of the system.

B. NN Controllers with Error Integral Inputs

For a multi-layer NN controller containing error term e and error integral s as the inputs, we use $R(e, s)$ to represent the NN controller and the control vector u can be expressed as

$$u = R(e, s) \quad (49)$$

where $R(e)$ can be any continuous and continuously differentiable functions of e and s, that is $R(e, s) \in C^1[R^n \times R^n, R^m]$.

Substituting (4), (5), and (49) into (1), the system equation can be simplified as

$$\dot{e} = A(e - x_{ref}) - BR(e, s) \quad (50)$$

From the definition of error integral s in (21), the following equation can be derived

$$\dot{s} = e \quad (51)$$

Thus combining (50) and (51), a new augmented state-space model can be obtained

$$\begin{cases} \dot{e} &= f_1(e, s) = A(e - x_{ref}) - BR(e, s) \\ \dot{s} &= f_2(e, s) = e \end{cases} \quad (52)$$

Through this conversion, the original n-dimension neural network control system (50) is converted into a $2n$-dimensional system (52).

Theorem 7. *For a neural dynamic system (52), $e = 0$ is an equilibrium point and the system does not have a steady-state error for any constant reference x_{ref}.*

Proof: The equilibrium point of (52) is the roots of the right side function, that is

$$\begin{cases} f_1(e, s) &= A(e - x_{ref}) - BR(e, s) = 0 \\ f_2(e, s) &= e = 0 \end{cases} \quad (53)$$

To satisfy the second equation of (53), e must be 0. Thus the equilibrium point will be (0, s^*), where s^* satisfies

$$Ax_{ref} + BR(0, s^*) = 0 \quad (54)$$

The equilibrium point of (53) is (0, s^*), which means that the system error e converges to 0 while the error integral s converges to s^* when the time goes to infinity. When there is an error integral term s feeding into the input of the NN controllers, it is guaranteed that there is no steady-state error in the system. ■

Theorem 8. *For a neural dynamic system (52), local asymptotic stability and local exponential convergence are guaranteed if the weight matrix and bias vector of the NN satisfy the following condition*

$$Re\left\{\text{eig}\left(\begin{bmatrix} G_{11} & G_{12} \\ I & 0 \end{bmatrix}\right)\right\} < 0 \quad (55)$$

where G_{11} and G_{12} equal

$$G_{11} = A - B\frac{\partial R(e, s)}{\partial e}|_{e=0, s=s^*} \quad (56)$$

$$G_{12} = -B\frac{\partial R(e, s)}{\partial s}|_{e=0, s=s^*} \quad (57)$$

Proof: The equilibrium point of (52) can be shifted from (0, s^*) to (0, 0) using the following conversion:

$$\begin{cases} e &= e \\ s_n &= s - s^* \end{cases} \quad (58)$$

Substituting (58) into (52), the new system equation will be

$$\begin{cases} \dot{e} = f_1(e, s_n) = & A(e - x_{ref}) - BR(e, s_n + s^*) \\ \dot{s_n} = f_2(e, s_n) = & e \end{cases} \quad (59)$$

Under the definition of Lyapunov stability [18] and linearizing (59) at the equilibrium point (0, 0), the system equation will become

$$\begin{bmatrix} \dot{e} \\ \dot{s_n} \end{bmatrix} = \begin{bmatrix} G_{11} & G_{12} \\ G_{21} & G_{22} \end{bmatrix}\begin{bmatrix} e \\ s_n \end{bmatrix} \quad (60)$$

in which, G_{11}, G_{12}, G_{21}, and G_{22} are defined as

$$G_{11} = \frac{\partial f_1(e, s_n)}{\partial e}|_{e=0, s_n=0} = A - B\frac{\partial R(e, s)}{\partial e}|_{e=0, s=s^*} \quad (61)$$

$$G_{12} = \frac{\partial f_1(e, s_n)}{\partial s_n}|_{e=0, s_n=0} = -B\frac{\partial R(e, s)}{\partial s}|_{e=0, s=s^*} \quad (62)$$

$$G_{21} = \frac{\partial f_2(e, s_n)}{\partial e}|_{e=0, s_n=0} = I \quad (63)$$

$$G_{22} = \frac{\partial f_2(e, s_n)}{\partial s_n}|_{e=0, s_n=0} = 0 \quad (64)$$

As G_{11}, G_{12}, G_{21}, and G_{22} are all constants, according to Lemma 1, if the NN weights and bias vector satisfy the following condition,

$$Re\left\{\text{eig}\left(\begin{bmatrix} G_{11} & G_{12} \\ I & 0 \end{bmatrix}\right)\right\} < 0 \quad (65)$$

the system's global asymptotic stability and global exponential convergence are guaranteed. However, as the system (60) is linearized at the equilibrium point, only the local asymptotic stability and local exponential convergence can be guaranteed. ■

Remark 6. In (60), x_{ref} does not exit explicitly. However, s^* is the equilibrium point of (52) depending on x_{ref}. So, the system matrices G_{11}, G_{12}, G_{21}, and G_{22} and eigenvalues are implicit functions of x_{ref} and thus the reference x_{ref} affects the stability of the system.

Remark 7. Similar to the conclusion in Remark 4, the weights and the bias vector of the multi-layer NN controller, together with the reference, will affect the local stability of the system and thus the local convergence at the equilibrium point. Therefore, to guarantee the stable operation of the system, the weights and the bias vector of the NN controller need to satisfy the stability requirement (45) or (55) for all possible references.

IV. Case Study I : One-Dimensional Single-layer NN Controllers

In this section, a single-layer NN controller in a one-dimensional state-space model is implemented to verify the theorems in Section II numerically. The performance, under the same conditions, of a conventional one-dimensional PI controller is also reported to show that the stability and convergence properties of the single-layer NN controller proved in Section II and experimentally tested in this section are comparable to a conventional one-dimensional PI controller in a small neighborhood of their respective equilibrium points. Thus, engineers have reassurances of the formal properties of the NN controllers, which are not black boxes anymore, and a practical way of checking their performance against ubiquitous PI controllers.

Consider a one-dimensional system of (1) with $A = 2$, $B = 0.5$, and $k_a = 5$.

A. Single-Layer NN Controllers with Only Error Term Inputs

A single-layer NN controller contains only an error term input and the control action can be expressed as

$$u = k_a \tanh(w_p e) = 5 \tanh(w_p e) \tag{66}$$

Without loss of generality, the bias b is selected as 0.

According to Theorem 1, the system has a steady-state error $e(\infty) = e^*$ for a step reference $x_{ref} = 1$, where e^* is the root of the following equation

$$\begin{aligned} f(e^*) =& A(e^* - x_{ref}) - k_a B \tanh(w_p e^* + b) \\ =& 2(e^* - 1) - 5 \times 0.5 \tanh(w_p e^* + 0) = 0 \end{aligned} \tag{67}$$

To guarantee local asymptotic stability and local exponential convergence, the NN weight w_p needs to satisfy the condition specified in Theorem 2. Since we are working with a one-dimensional system, the condition can be simplified further as

$$\begin{aligned} \lambda &= A - k_a B w_p \operatorname{diag}(1 - \tanh^2(w_p e^* + b)) \\ &= 2 - 5 \times 0.5 w_p [1 - \tanh^2(w_p e^* + 0)] < 0 \end{aligned} \tag{68}$$

Combining (67) and (68), the range of weight w_p can be obtained. Fig.1 shows the range of w_p for a step reference $x_{ref} = 1$. When $w_p = 9.8$, $\lambda = -0.507424234870289 < 0$, which satisfies the stability condition.

A Simulink model as shown in Fig.2 was built to verify the tracking performance and the steady-state error $e(\infty)$. Fig.3 shows the tracking error when the NN weight $w_p = 9.8$ for a step reference. As we can see in Fig.3, when $t = 20s$, $e(20s) = -0.184308205223854$, which is pretty close to $e(\infty) = e^* = -0.184308971562349$ with e^* as the root of (67) when $w_p = 9.8$.

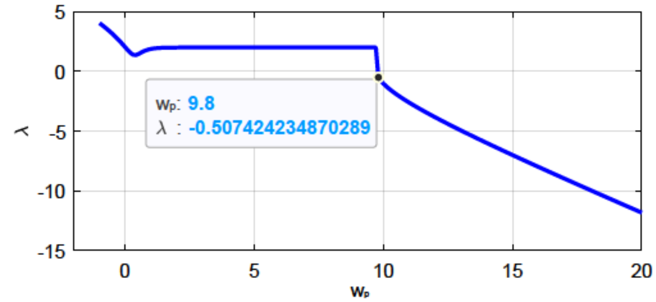

Fig. 1. The eigenvalue λ vs. the NN weight w_p for a step reference $x_{ref} = 1$.

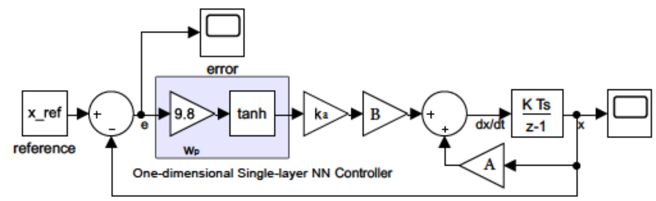

Fig. 2. The Simulink model for the one-dimensional single-layer NN controller.

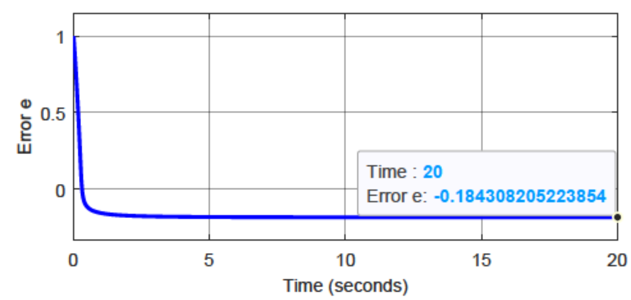

Fig. 3. The tracking error e for a step reference $x_{ref} = 1$ when $w_p = 9.8$.

B. Adding Error Integral Inputs to Remove The Steady-State Error

To remove the steady-state error, we consider adding the error integral input to the single-layer NN controller as follows

$$u = k_a \tanh(w_p e + w_i s) = 5 \tanh(9.8e + w_i s) \tag{69}$$

According to Theorem 3 and (27), the equilibrium point of the system is $[0;\ s^*]$. If w_i is selected as 1, then $s^* = -1.098612288668110$.

The eigenvalues of the NN control system according to Theorem 4 are $\lambda_1 = -6.685377840799436$ and $\lambda_2 = -0.134622159200560$, which satisfy the requirements of local asymptotic stability and local exponential convergence.

A Simulink model as shown in Fig. 4 was built to verify the tracking performance of the NN controller after adding the error integral term.

Fig. 5 shows the equilibrium point of the one-dimensional single-layer NN control system. When $t = 100s$, $e(100s) = -0.000000214389590$, which is pretty close to the theoretical equilibrium point $e(\infty) = 0$. Also, $s(100s) = -1.098610696140083$, which is also very close to theoretical equilibrium point $s(\infty) = s^* = -1.098612288668110$.

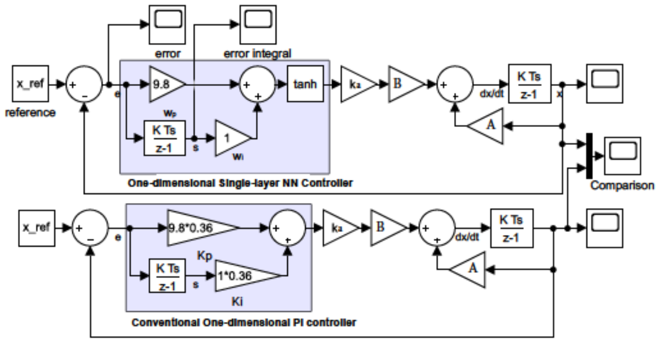

Fig. 4. The Simulink model for a one-dimensional single-layer NN controller and the corresponding conventional one-dimensional PI controller.

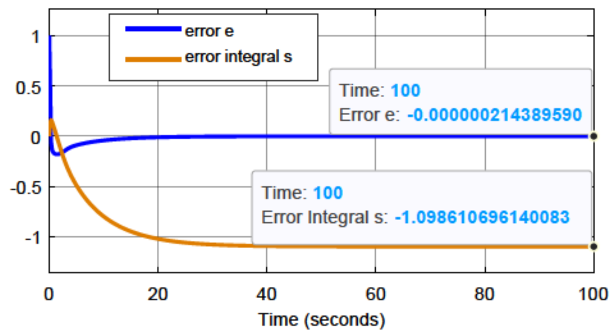

Fig. 5. The equilibrium point of the one-dimensional single-layer NN control system.

C. Corresponding Conventional One-dimensional PI Controller

The corresponding one-dimensional PI controller was added to the Simulink model in Fig. 4. To guarantee that the designed one-dimensional PI controller has the same eigenvalues as the single-layer NN controller, we compare (40) and (28)-(30) in Theorem 4, thus set K_p and K_i as

$$K_p = w_p \operatorname{diag}(1 - \tanh^2(w_i s^* + b)) = 9.8 \times 0.36 \quad (70)$$

$$K_i = w_i \operatorname{diag}(1 - \tanh^2(w_i s^* + b)) = 1 \times 0.36 \quad (71)$$

where s^* is the equilibrium point of the NN control system and $s^* = -1.098612288668110$.

D. Step Response Comparison within A Small Neighborhood of Equilibrium Points

To investigate the responses of one-dimensional single-layer NN and conventional one-dimensional PI controllers within a neighborhood of their respective equilibrium points, initial values were added to the system state x and the error integral s. As $e = 0$ is the equilibrium point of both controllers, x was set as $x(0s) = 0.95$ for both control systems, which means $e(0s) = 1 - 0.95 = 0.05$. According to (39), the equilibrium point s^* for the PI controller is

$$\begin{aligned} s^* &= -\frac{1}{k_a}(BK_i)^{-1}Ax_{ref} \\ &= -\frac{1}{5} \times (0.5 \times 0.36)^{-1} \times 2 \times 1 \\ &= -2.222222222222222 \end{aligned} \quad (72)$$

So the starting points of the error integral s for the one-dimensional single-layer NN and conventional one-dimensional controllers were set as $s(0s) = -1.098612288668110 + 0.05$ and $s(0s) = -2.222222222222222 + 0.05$, respectively.

Fig. 6 demonstrates the step response for $x_{ref} = 1$ under both one-dimensional single-layer NN and conventional one-dimensional PI controllers with starting points from a neighborhood of their respective equilibrium points. Their responses are almost the same, which is expected and can be explained by the fact that both control systems have exactly the same two eigenvalues.

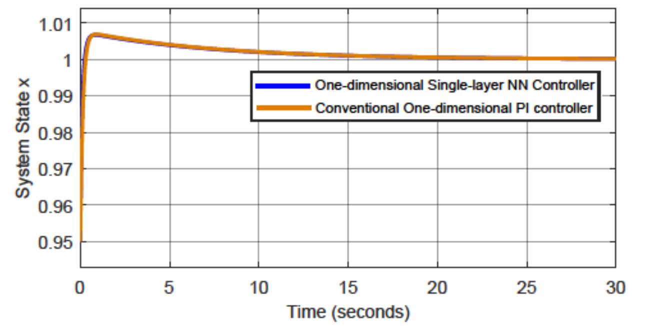

Fig. 6. Step response comparison within a small neighborhood of their respective equilibrium points.

V. Case Study II: Two-Dimensional Four-Layer NN Controllers In Electric Power Applications

In this section, a four-layer NN controller in a two-dimensional state-space model for renewable energy integration with the electric power grid is investigated to test the proofs in Section III numerically. Further, we add a simulation of the corresponding generalized PI controller and single-layer NN controller to show that the convergence and stability properties proved in Section III are not merely valid, but that all three control systems with the same eigenvalues are guaranteed to perform almost the same within a small neighborhood of their respective equilibrium points.

A. Grid-Connected Converter

A Grid-Connected Converter (GCC) is a key component that physically connects renewable energy resources such as wind turbines and solar panels to the grid [21], [22], [23], [24], [25]. Fig. 7 shows the schematic of an L filter-based GCC, in which a DC-link capacitor is on the left, and a three-phase voltage source, representing the voltage at the Point of Common Coupling (PCC) of the AC system, is on the right.

In the d-q reference frame, the state-space model of the integrated GCC and grid system ([26]) can be expressed as

$$\frac{d}{dt}\begin{bmatrix} i_d \\ i_q \end{bmatrix} = \underbrace{\begin{bmatrix} -\frac{R_g}{L_g} & \omega_s \\ -\omega_s & -\frac{R_g}{L_g} \end{bmatrix}}_{A} \underbrace{\begin{bmatrix} i_d \\ i_q \end{bmatrix}}_{i_{dq}} + \underbrace{\begin{bmatrix} -\frac{1}{L_g} & 0 \\ 0 & -\frac{1}{L_g} \end{bmatrix}}_{B} \underbrace{\begin{bmatrix} V_{d1} - V_d \\ V_{q1} - V_q \end{bmatrix}}_{u_{dq}} \quad (73)$$

where ω_s is the angular frequency of the grid voltage, and L_g and R_g represent the inductance and resistance of the grid

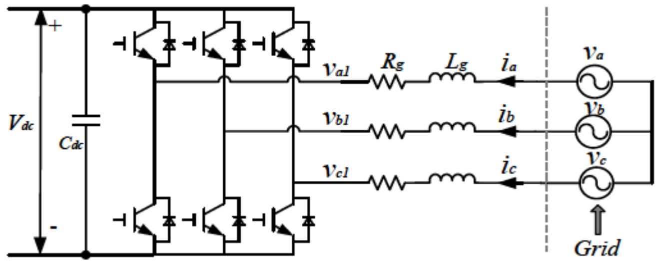

Fig. 7. A Grid-Connected Converter for renewable energy integration.

filter respectively, the system states are $i_{dq} = [i_d; i_q]$, the grid PCC voltages $V_{dq} = [V_d; V_q]$ are normally constants, $V_{dq1} = [V_{d1}; V_{q1}]$ are the converter output voltages that are specified by the current controller outputs, and the control vector is $u_{dq} = V_{dq1} - V_{dq}$.

Table I specifies all system parameters in a lab experiment setup [27]. Using the parameters from Table I, $V_{dq} = [V_g; 0] = [20; 0]$ and $k_{pwm} = \sqrt{3/2}\frac{V_{dc}}{2} = 30.618621784789724$.

TABLE I
THE L FILTER BASED GCC SYSTEM PARAMETERS

Symbol	Description	Value	Unit
V_g	test grid voltage (rms)	20	V
f	nominal grid frequency	60	Hz
ω_s	nominal grid angular frequency	120π	rad
V_{dc}	DC-link voltage	50	V
L_g	the inductance of the grid filter	25	mH
R_g	the resistance of the grid filter	0.25	Ω

B. Four-Layer NN Controller

As the ratio of the converter output voltage V_{dq1} to the outputs of the current controller is the gain of the Pulse-Width-Modulation (PWM) k_{pwm}[28], the control action u_{dq} is then expressed by

$$u_{dq} = R(e_{dq}, s_{dq}) = V_{dq1} - V_{dq} = k_{pwm} N(e_{dq}, s_{dq}, w) - V_{dq} \quad (74)$$

The structure of the four-layer NN controller ([10], [11]) is shown in Fig. 8. The function format of the four-layer NN

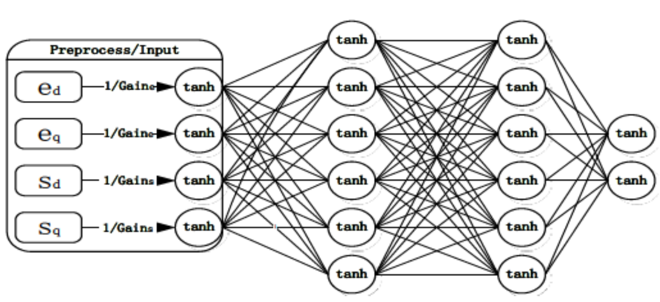

Fig. 8. The structure of the four-layer NN controller.

controller can be represented as

$$N(e_{dq}, s_{dq}, w) = \tanh\left(w_3\begin{bmatrix}\tanh\left(w_2\begin{bmatrix}\tanh\left(w_1\begin{bmatrix}\tanh\begin{bmatrix}\frac{e_{dq}}{Gain_e}\\ \frac{s_{dq}}{Gain_s}\end{bmatrix}\\ -1\end{bmatrix}\right)\\ -1\end{bmatrix}\right)\\ -1\end{bmatrix}\right) \quad (75)$$

where w_1, w_2, and w_3 represent the weights from the input layer to the first hidden layer, from the first to the second hidden layer and from the second hidden layer to the output layer respectively. The biases of each layer have been incorporated into w_1, w_2, and w_3 to simplify the weight updating process.

The four-layer NN controller was trained by the LMBP [29], [30], [31] and the FATT algorithm [32]. For the four-layer NN controller, its weight parameters $Gain_e$, $Gain_s$, w_1, w_2, and w_3 are listed in Table II.

The equilibrium point of the system is (0, s^*_{dq}). According to Theorem 7 and (54), s^*_{dq} satisfies the following function

$$\underbrace{\begin{bmatrix}-\frac{R_g}{L_g} & \omega_s\\ -\omega_s & -\frac{R_g}{L_g}\end{bmatrix}}_{A} i_{dq\_ref} + \underbrace{\begin{bmatrix}-\frac{1}{L_g} & 0\\ 0 & -\frac{1}{L_g}\end{bmatrix}}_{B}[k_{pwm}N(0, s^*_{dq}, w) - V_{dq}] = 0 \quad (76)$$

According to (56) and (57) in Theorem 8, G_{11} and G_{12} can be calculated as

$$\begin{aligned} G_{11} &= A - B\frac{\partial R(e_{dq}, s_{dq})}{\partial e_{dq}}|_{e_{dq}=0, s_{dq}=s^*_{dq}}\\ &= A - k_{pwm}B\frac{\partial N(e_{dq}, s_{dq}, w)}{\partial e_{dq}}|_{e_{dq}=0, s_{dq}=s^*_{dq}} \quad (77)\\ G_{12} &= -B\frac{\partial R(e_{dq}, s_{dq})}{\partial s_{dq}}|_{e_{dq}=0, s_{dq}=s^*_{dq}}\\ &= -k_{pwm}B\frac{\partial N(e_{dq}, s_{dq}, w)}{\partial s_{dq}}|_{e_{dq}=0, s_{dq}=s^*_{dq}} \quad (78)\end{aligned}$$

The details of calculating G_{11} and G_{12} are listed in Appendix A.

Given the current reference $i_{dq\_ref} = [1; 0]$, the corresponding four eigenvalues can be calculated and are listed in Table III.

C. Corresponding Generalized PI Controller

A generalized PI controller was designed to have the same four eigenvalues as those of the four-layer NN controller for comparison. Table III lists the target four eigenvalues for the generalized PI controller.

To guarantee the designed generalized PI controller to have the same eigenvalues as the four-layer NN controller, we compare (40) and (77) and (78), thus set K_p and K_i as

$$K_p = \frac{\partial N(e_{dq}, s_{dq}, w)}{\partial e_{dq}}|_{e_{dq}=0, s_{dq}=s^*_{dq}} \quad (79)$$

$$K_i = \frac{\partial N(e_{dq}, s_{dq}, w)}{\partial s_{dq}}|_{e_{dq}=0, s_{dq}=s^*_{dq}} \quad (80)$$

Table IV lists the values of K_p and K_i. Unlike the conventional one-dimensional PI controller with a scalar proportional

TABLE II
THE WEIGHT PARAMETERS OF THE FOUR-LAYER NN CONTROLLER

$Gain_e$	0.5
$Gain_s$	0.5
w_1	$\begin{bmatrix} 0.105118602490750 & -0.869195807768507 & 3.910726574451215 & 3.829650558215191 & 0.043137396666237 \\ 0.805253127771219 & 0.116082719739100 & 5.081202415079452 & -1.666910901747036 & 1.246185328567662 \\ 0.142117134906564 & 0.272375503071100 & 4.040974596086221 & 2.048223953909149 & 0.145066179115310 \\ -0.395272323007882 & -1.422986921530577 & 4.255131423501219 & 5.561432608781822 & 0.000251202007245 \\ -0.277224746255928 & 0.699935881635053 & 1.636989905274748 & 2.678881970530615 & -0.109536124233592 \\ 0.376127545234790 & 1.285716734931245 & -2.973060687194107 & 8.095548964772654 & 0.028991336931277 \end{bmatrix}$
w_2	$\begin{bmatrix} 1.440539114493213 & -0.272530718390058 & 0.527886890221929 & 1.371222680616433 & 2.255139286184510 & 1.394844625523901 & 0.344937425499452 \\ 3.378981724654836 & 0.198608148623109 & 3.459270721071458 & -1.911027029429807 & 0.224751908404989 & -1.002210347176314 & -0.475383054734012 \\ -0.359022430219653 & 1.217655164464906 & 3.145578151429633 & 1.863120732645271 & 3.708974043074285 & -0.096082441939513 & 1.566135015097376 \\ 4.031099888420788 & -2.685187585928909 & -2.749868864734965 & 2.748659888667571 & 2.439552173754654 & 5.660170953027147 & 0.925728746264457 \\ -0.622152942608992 & 0.732064874764135 & 4.212370496471141 & -4.081216558783956 & 1.547382976445135 & -6.456534076312040 & -0.817547511050558 \\ 1.522500050952956 & -1.036035004009775 & 1.703072013081991 & 0.534432723278869 & 0.630762934216796 & 1.093038633050528 & 0.074249030990423 \end{bmatrix}$
w_3	$\begin{bmatrix} -1.711155394435648 & -0.447196877031189 & -2.614508912856286 & -5.956955188009836 & 0.958589844957509 & 1.641209174573893 & -2.440547421725287 \\ 1.317977684822903 & 1.038607717509133 & 2.191677954355899 & -0.515283801531746 & 0.973061014722440 & 3.038069686362197 & -1.386654203297988 \end{bmatrix}$

TABLE III
EIGENVALUES

Control method	λ			
Four-layer NN	-802.233078413318 +1100.64099842807i	-802.233078413318 - 1100.64099842807i	-147.10811464909 + 54.0774179743671i	-147.10811464909 - 54.0774179743671i
Single-layer NN	-802.233078413318 +1100.64099842807i	-802.233078413318 - 1100.64099842807i	-147.10811464909 + 54.0774179743671i	-147.10811464909 - 54.0774179743671i
Generalized PI	-802.233078413318 +1100.64099842807i	-802.233078413318 - 1100.64099842807i	-147.10811464909 + 54.0774179743671i	-147.10811464909 - 54.0774179743671i

gain and a scalar integral gain, the generalized PI controller shown in Table IV has cross-coupling terms and are in a more generalized gain matrix format, which has better and stronger performance than the conventional one-dimensional PI controller.

TABLE IV
THE PARAMETERS OF THE CORRESPONDING GENERALIZED PI CONTROLLER

K_p	$\begin{bmatrix} -0.344022164281883 & 0.727142679990575 \\ -0.754209007295918 & -1.18991558063817 \end{bmatrix}$
K_i	$\begin{bmatrix} -2.3221654488264 & 196.559123343741 \\ -153.905082611539 & -54.8126346765554 \end{bmatrix}$

D. Corresponding Single-layer NN Controller

For the single-layer NN controller design, the bias vector b was selected as zeros to simplify the design process. We compare (40) and (29) - (30), and thus weights w_p and w_i can be calculated from the following two equations

$$w_p \,\mathrm{diag}(1 - \tanh^2(w_i s^* + b)) = K_p \tag{81}$$

$$w_i \,\mathrm{diag}(1 - \tanh^2(w_i s^* + b)) = K_i \tag{82}$$

where s^* is the equilibrium point of the single-layer NN controller system, and $\tanh(w_i s^* + b)$ can be obtained from (27) as

$$\tanh(w_i s^* + b) = \frac{1}{k_{pwm}} B^{-1} A i_{dq\_ref} \tag{83}$$

since the B of the GCC system is one 2×2 square matrix.

Thus weights w_p and w_i equal

$$w_p = [\mathrm{diag}(1 - (\frac{1}{k_{pwm}} B^{-1} A i_{dq\_ref})^2)]^{-1} K_p \tag{84}$$

$$w_i = [\mathrm{diag}(1 - (\frac{1}{k_{pwm}} B^{-1} A i_{dq\_ref})^2)]^{-1} K_i \tag{85}$$

Table V lists the values of w_p and w_i for the single-layer NN controller.

TABLE V
THE PARAMETERS OF THE CORRESPONDING SINGLE-LAYER NN CONTROLLER

w_p	$\begin{bmatrix} -0.589146302572011 & 0.803249086937196 \\ -1.291601222678247 & -1.314458124906575 \end{bmatrix}$
w_i	$\begin{bmatrix} -3.97676466861468 & 217.131988947592 \\ -263.566187826589 & -60.5496004678695 \end{bmatrix}$

E. Equilibrium Point Comparison

A Simulink model as shown in Fig. 9 was built to simulate all three controllers.

TABLE VI
THE EQUILIBRIUM POINTS

Control method	e^*_{dq}		s^*_{dq}	
Four-layer NN	0	0	0.000570367398365	0.000995539550846
Single-layer NN	0	0	0.000827793898875	0.003291399362787
Generalized PI	0	0	0.000394111939873	0.003538453922689

Table VI lists the equilibrium points for all three control methods. As all three control methods have the error integral

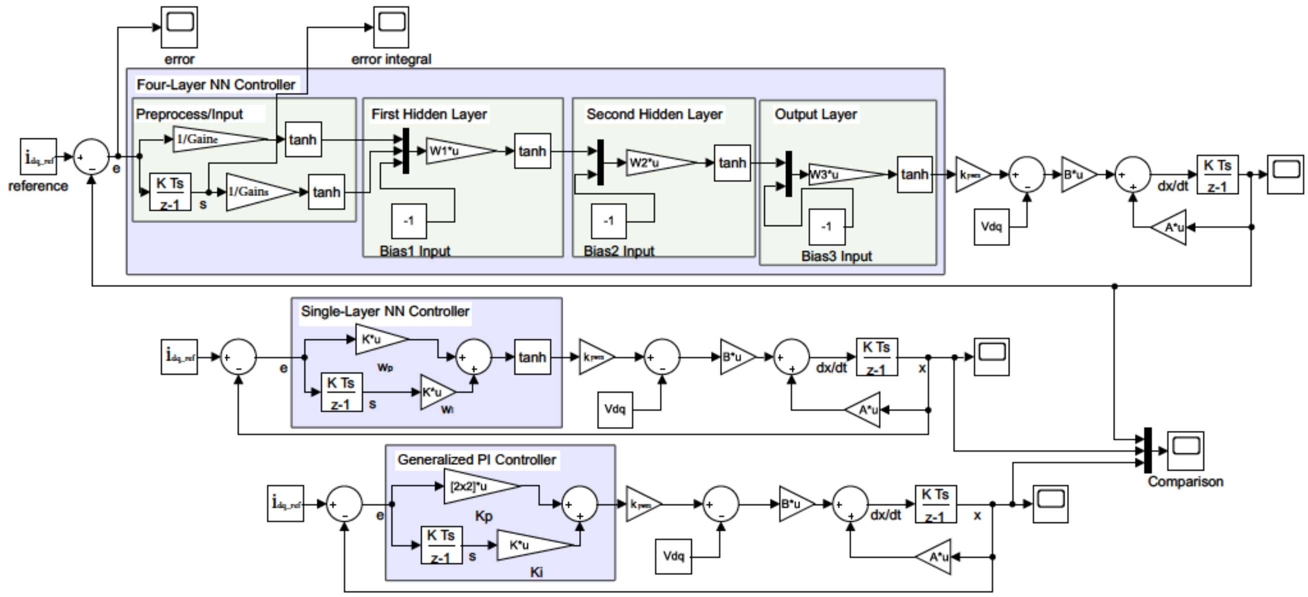

Fig. 9. The Simulink model for three controllers with the same four eigenvalues: four-layer NN controller, single-layer NN controller, and generalized PI controller.

inputs, the equilibrium points for the e_{dq} are all zeros, which means that the steady-state error $e_{dq}(\infty) = 0$. For the error integral s_{dq}, they all converge to their respective equilibrium points as each control method has different weights or parameters.

Fig.10 shows the tracking error e_{dq} of the four-layer NN controller for a step response $i_{dq\_ref} = [1; 0]$. At time $t = 0.1s$, $e_{dq} = [-1.886e-7; 6.882e-08]$, which is already very close to the equilibrium point [0;0].

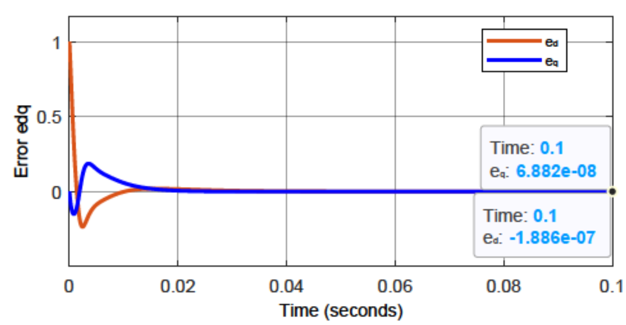

Fig. 10. The tracking error e_{dq} of the four-layer NN controller for step response $i_{dq\_ref} = [1; 0]$.

Fig.11 shows the tracking error integral s_{dq} of the four-layer NN controller for a step response of $i_{dq\_ref} = [1; 0]$. At time $t = 0.1s$, $s_{dq} = [0.000570368236706; 0.000995538938232]$, which has 8 significant bits the same as the equilibrium point [0.000570367398365;0.000995539550846] in Table VI.

F. Step Response Comparison within A Small Neighborhood of Equilibrium Points

To evaluate and compare the steady-state behaviors of all three control methods close to their equilibrium points s^*_{dq}, instead of starting from $s_{dq}(0s) = [0; 0]$, the staring points for

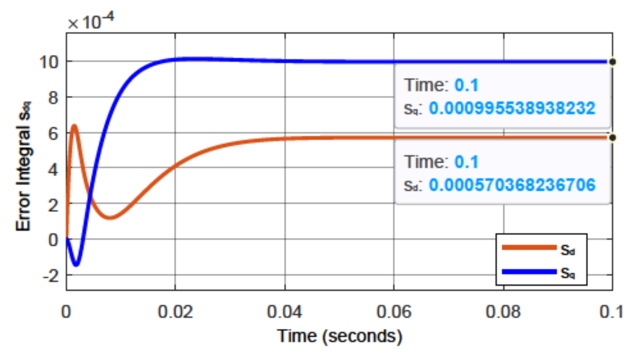

Fig. 11. The error integral s_{dq} of the four-layer NN controller for step response $i_{dq\_ref} = [1; 0]$.

s_{dq} were set as $s_{dq}(0s) = s^*_{dq} - [0.001; 0]$. The starting points of s_{dq} are listed in Table VII.

TABLE VII
STARTING POINTS OF s_{dq}

Control method	$s_{dq}(0s) = s^*_{dq} - [0.001; 0]$	
Four-layer NN	0.000570367398365 - 0.001	0.000995539550846
Single-layer NN	0.000827793898875 - 0.001	0.003291399362787
Generalized PI	0.000394111939873 - 0.001	0.003538453922689

Fig. 12 shows the step responses under this condition, which are almost the same within a small neighborhood of their equilibrium points, and verify the fact that they all have the same four eigenvalues.

Thus, it is expected that when the size of the neighborhood around the equilibrium points is small enough, all three control methods will demonstrate identical responses because they all have the same four eigenvalues.

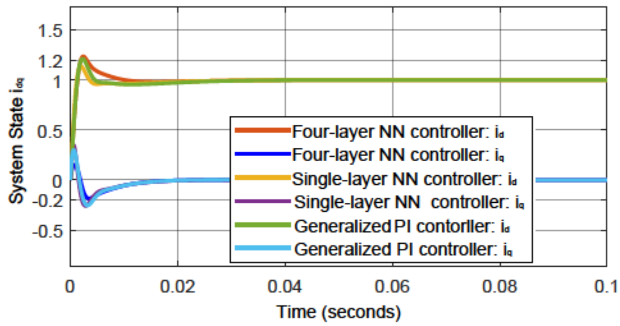

Fig. 12. Step response comparison with starting points from a neighborhood of their respective equilibrium points s^*_{dq}.

VI. Conclusion

This paper (Sections II and III) mathematically proves that if (single- and multi-) NN controllers only have error terms as inputs, the corresponding control system shows a non-zero steady-state error for any constant reference, except for one specific reference point, and that adding an error integral term to the inputs of the NN controller is sufficient to eliminate the steady-state error for any constant reference.

More importantly, we provide a simple way of using eigenvalues of the NN control system to evaluate local stability and local convergence for reference tracking. The NN controllers have almost the same responses as the corresponding generalized PI controllers with the same eigenvalues in a small neighborhood of their respective equilibrium points, as shown experimentally in Sections IV and V.

We trust that the formal analysis of the conditions under which the stability and convergence properties of NN controllers are guaranteed, along with the accompanying confirmatory empirical results will help engineers understand the functioning of NN controllers and pave the way for their applications in real-life scenarios.

We plan to carry out a theoretical comparison of the responses between NN controllers and corresponding PI controllers globally, that is with starting points anywhere far away from their equilibrium points, in our next work. We also plan to include the error derivative terms in the NN controller and investigate their impact on the control system in the future.

Appendix A
Derivation of G_{11} and G_{12} for the Four-Layer NN Controller

To simplify the derivation process, define o_e, o_s, o_1, o_2, and o_3 as follows:

$$o_e = \tanh(e_{dq}/Gain_e)|_{e_{dq}=0} = \tanh([0;0]/Gain_e) \quad (86)$$

$$o_s = \tanh(s_{dq}/Gain_s)|_{s_{dq}=s^*_{dq}} = \tanh(s^*_{dq}/Gain_s) \quad (87)$$

$$o_1 = \tanh(w_1[o_e; o_s; -1]) \quad (88)$$

$$o_2 = \tanh(w_2[o_1; -1]) \quad (89)$$

$$o_3 = N(e_{dq}, s_{dq}, w) = \tanh(w_3[o_2; -1]) \quad (90)$$

Then

$$\frac{\partial N(e_{dq},s_{dq},w)}{\partial e_{dq}}\Big|_{e_{dq}=0,s_{dq}=s^*_{dq}} = \frac{\partial o_3}{\partial o_2}\frac{\partial o_2}{\partial o_1}\frac{\partial o_1}{\partial o_e}\frac{\partial o_e}{\partial e_{dq}}\Big|_{e_{dq}=0,s_{dq}=s^*_{dq}}$$
$$= [\mathrm{diag}(1-o_3^2)w_3(:,1:6)][\mathrm{diag}(1-o_2^2)w_2(:,1:6)]$$
$$* [\mathrm{diag}(1-o_1^2)w_1(:,1:2)][\mathrm{diag}((1-o_e^2)/Gain_e)] \quad (91)$$

$$\frac{\partial N(e_{dq},s_{dq},w)}{\partial s_{dq}}\Big|_{e_{dq}=0,s_{dq}=s^*_{dq}} = \frac{\partial o_3}{\partial o_2}\frac{\partial o_2}{\partial o_1}\frac{\partial o_1}{\partial o_s}\frac{\partial o_s}{\partial s_{dq}}\Big|_{e_{dq}=0,s_{dq}=s^*_{dq}}$$
$$= [\mathrm{diag}(1-o_3^2)w_3(:,1:6)][\mathrm{diag}(1-o_2^2)w_2(:,1:6)]$$
$$* [\mathrm{diag}(1-o_1^2)w_1(:,3:4)][\mathrm{diag}((1-o_s^2)/Gain_s)] \quad (92)$$

Substitute (91) and (92) into (77) and (78). Thus G_{11} and G_{12} can be obtained.

Appendix B
The List of Symbols

Table VIII summarizes the symbols utilized in this paper.

TABLE VIII
The list of symbols and their descriptions.

Symbols	Description
A	the state or system matrix in the state-space model; $A \in R^{n\times n}$
B	the input matrix in the state-space model; $B \in R^{n\times m}$
G	the linear time-invariant system matrix; $G \in R^{n\times n}$
x, x_{ref}	the system states and the references for system states x; $x, x_{ref} \in R^n$
e	the tracking error; $e \in R^n$
s	the tracking error integral; $s \in R^n$
e^*	the equilibrium point of the state error e
s^*	the equilibrium point of the state error integral s
u	the input/control vector in the state-space model; $u \in R^m$
i_{dq}, i_{dq_ref}	the d-q currents and the references for d-q currents
e_{dq}	the tracking error in the d-q domain
s_{dq}	the tracking error integral in the d-q domain
e^*_{dq}	the equilibrium point of the state error e_{dq}
s^*_{dq}	the equilibrium point of the state error integral s_{dq}
u_{dq}	the control action in the d-q domain
k_a	the actuator gain; $k_a \in R$
k_{pwm}	the gain of the Pulse-Width-Modulation
w_p, w_i, b	weights for error terms and error integral terms, and the bias of the single-layer NN controller; $w_p, w_i \in R^{m\times n}$ and $b \in R^m$
$R(e)$	the function representation of multi-layer NN controllers with only error terms; $R(e) \in C^1[R^n, R^m]$
$R(e,s)$	the function representation of multi-layer NN controllers with error terms and error integral terms; $R(e,s) \in C^1[R^n \times R^n, R^m]$
$N(e_{dq}, s_{dq}, w)$	the function representation of the four-layer NN controller
w_1, w_2, w_3	weights of the four-layer NN controller
$Gain_e$, $Gain_s$	scaling factors for error terms and error integral terms in the input layer of the four-layer NN controller
K_p, K_i	proportional and integral gains of the generalized PI controller

REFERENCES

[1] R. E. Bellman, *Dynamic Programming*. Princeton, New Jersey, USA: Princeton University Press, 1957.

[2] S. N. Balakrishnan and V. Biega, "Adaptive-critic-based neural networks for aircraft optimal control," *Journal of Guidance, Control, and Dynamics*, vol. 19, no. 4, pp. 893–898, Jul./Aug. 1996.

[3] S. Al-Dabooni and I. Donald C. Wunsch, "Online model-free n-step hdp with stability analysis," *IEEE Transactions on Neural Networks and Learning Systems*, vol. 31, no. 4, pp. 1255–1269, Apr. 2020.

[4] ——, "An improved n-step value gradient learning adaptive dynamic programming algorithm for online learning," *IEEE Transactions on Neural Networks and Learning Systems*, vol. 31, no. 4, pp. 1155–1169, Apr. 2020.

[5] D. V. Prokhorov and D. C. Wunsch, "Adaptive critic designs," *IEEE Transactions on Neural Networks*, vol. 8, no. 5, pp. 997–1007, Sep. 1997.

[6] F. Wang, H. Zhang, and D. Liu, "Adaptive dynamic programming: An introduction," *IEEE Computational Intelligence Magazine*, vol. 4, no. 2, pp. 39–47, May 2009.

[7] G. K. Venayagamoorthy, R. G. Harley, and D. C. Wunsch, "Comparison of heuristic dynamic programming and dual heuristic programming adaptive critics for neurocontrol of a turbogenerator," *IEEE Transactions on Neural Networks*, vol. 13, no. 3, pp. 764–773, May 2002.

[8] S. Li, M. Fairbank, D. C. Wunsch, and E. Alonso, "Vector control of a grid-connected rectifier/inverter using an artificial neural network," in *Proc. IEEE World Congress on Computational Intelligence*, Brisbane, Australia, Jun. 2012, pp. 1–7.

[9] S. Li, M. Fairbank, C. Johnson, D. C. Wunsch, E. Alonso, and J. L. Proaño, "Artificial neural networks for control of a grid-connected rectifier/inverter under disturbance, dynamic and power converter switching conditions," *IEEE Transactions on Neural Networks And Learning Systems*, vol. 25, no. 4, pp. 738–750, Apr. 2014.

[10] X. Fu, S. Li, and I. Jaithwa, "Implement optimal vector control for LCL-filter-based grid-connected converters by using recurrent neural networks," *IEEE Transactions on Industrial Electronics*, vol. 62, no. 7, pp. 4443 – 4454, Jul. 2015.

[11] X. Fu and S. Li, "Control of single-phase grid-connected converters with LCL filters using recurrent neural network and conventional control methods," *IEEE Transactions on Power Electronics*, vol. 31, no. 7, pp. 5354 – 5364, Jul. 2016.

[12] S. Li, H. Won, X. Fu, M. Fairbank, D. C. Wunsch, and E. Alonso, "Neural-network vector controller for permanent-magnet synchronous motor drives: Simulated and hardware-validated results," *IEEE Transactions on Cybernetics*, vol. 50, no. 7, pp. 3218–3230, Jul. 2020.

[13] J. M. Bentez, J. L. Castro, and I. Requena, "Are artificial neural networks black boxes?" *IEEE Transactions on Neural Networks*, vol. 8, no. 5, pp. 1156–1164, Sept. 1997.

[14] N. E. Barabanov and D. V. Prokhorov, "Stability analysis of discrete-time recurrent neural networks," *IEEE Transactions on Neural Networks and Learning Systems*, vol. 13, no. 2, pp. 292–303, Mar. 2002.

[15] H. Zhang, Z. Wang, and D. Liu, "Global asymptotic stability of recurrent neural networks with multiple time-varying delays," *IEEE Transactions on Neural Networks and Learning Systems*, vol. 19, no. 5, pp. 855–873, May 2008.

[16] G.-H. Liu and E. Theodorou, "Deep learning theory review: An optimal control and dynamical systems perspective," *ArXiv*, vol. abs/1908.10920, 2019.

[17] H. K. Khalil, *Nonlinear Control*. Upper Saddle River, New Jersey: Pearson Education, 2015.

[18] H. Nijmeijer and A. van der Schaft, *Nonlinear Dynamical Control Systems*. New York City, New York: Springer, 1990.

[19] N. S. Nise, *Control Systems Engineering, 7th Edition*. Hoboken, New Jersey: John Wiley & Sons, 2014.

[20] R. C. Dorf and R. H. Bishop, *Modern Control Systems, 13th Edition*. Upper Saddle River, New Jersey: Pearson Education, 2016.

[21] G. Buticchi, D. Barater, E. Lorenzani, and G. Franceschini, "Digital control of actual grid-connected converters for ground leakage current reduction in pv transformerless systems," *IEEE Transactions on Industrial Informatics*, vol. 8, no. 3, pp. 563–572, August 2012.

[22] J. Shen, H. Jou, and J. Wu, "Novel transformerless grid-connected power converter with negative grounding for photovoltaic generation system," *IEEE Transactions on Power Electronics*, vol. 27, no. 4, pp. 1818–1829, April 2012.

[23] X. Zhou, S. Lukic, S. Bhattacharya, and A. Huang, "Design and control of grid-connected converter in bi-directional battery charger for plug-in hybrid electric vehicle application," in *Proc. IEEE Vehicle Power and Propulsion Conference (VPEC'09)*, Dearborn, Michigan, USA, September 2009, pp. 1716–1721.

[24] M. Hanif, V. Khadkikar, W. Xiao, and J. L. Kirtley, "Two degrees of freedom active damping technique for LCL filter-based grid connected pv systems," *IEEE Transactions on Industrial Electronics*, vol. 61, no. 6, pp. 2795–2803, Jun. 2014.

[25] M. Liserre, F. Blaabjerg, and S. Hansen, "Design and control of an LCL filter-based three-phase active rectifier," *IEEE Transactions on Industrial Applications*, vol. 41, no. 5, pp. 1281–1291, Oct. 2005.

[26] S. Li, M. Fairbank, C. Johnson, D. C. Wunsch, E. Alonso, and J. L. Proaño, "Artificial neural networks for control of a grid-connected rectifier/inverter under disturbance, dynamic and power converter switching conditions," *IEEE Transactions on Neural Networks And Learning Systems*, vol. 25, no. 4, pp. 738–750, Apr. 2014.

[27] X. Fu and S. Li, "A novel neural network vector control for single-phase grid-connected converters with L, LC and LCL filters," *Energies*, vol. 9, no. 5, p. 328, May 2016.

[28] N. Mohan, T. M. Undeland, and W. P. Robbins, *Power Electronics: Converters, Applications, and Design*, 3rd ed. Chichester, West Sussex, United Kingdom: John Wiley and Sons, 2002.

[29] M. T. Hagan and M. B. Menhaj, "Training feedforward networks with the marquardt algorithm," *IEEE Transactions on Neural Networks*, vol. 5, no. 6, pp. 989–993, November 1994.

[30] D. W. Marquardt, "An algorithm for least-squares estimation of non-linear parameters," *Journal of the Society for Industrial and Applied Mathematics*, vol. 11, no. 2, pp. 431–441, Jun. 1963.

[31] M. T. Hagan, H. B. Demuth, and M. H. Beale, *Neural Network Design*. Boston, USA: PWS Publishing, 2002.

[32] X. Fu, S. Li, M. Fairbank, D. C. Wunsch, and E. Alonso, "Training recurrent neural networks with the Levenberg-Marquardt algorithm for optimal control of a grid-connected converter," *IEEE Transactions on Neural Networks and Learning Systems*, vol. 26, no. 9, pp. 1900 – 1912, Sept. 2015.