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The universal program of linear elasticity

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Abstract
Universal displacements are those displacements that can be maintained, in the absence of body forces, by applying only boundary tractions for any material in a given class of materials. Therefore, equilibrium equations must be satisfied for arbitrary elastic moduli for a given anisotropy class. These conditions can be expressed as a set of partial differential equations for the displacement field that we call *universality constraints*. The classification of universal displacements in *homogeneous* linear elasticity has been completed for all the eight anisotropy classes. Here, we extend our previous work by studying universal displacements in *inhomogeneous* anisotropic linear elasticity assuming that the directions of anisotropy are known. We show that universality constraints of inhomogeneous linear elasticity include those of homogeneous linear elasticity. For each class and for its known universal displacements, we find the most general inhomogeneous elastic moduli that are consistent with the universality constrains. It is known that the larger the symmetry group, the larger the space of universal displacements. We show that the larger the symmetry group, the more severe the universality constraints are on the inhomogeneities of the elastic moduli. In particular, we show that inhomogeneous isotropic and inhomogeneous cubic linear elastic solids do not admit universal displacements and we completely characterize the universal inhomogeneities for the other six anisotropy classes.

Keywords
Universal deformation, universal displacement, linear elasticity, anisotropic solids, inhomogeneities

1. Introduction

In nonlinear elasticity, universal deformations are those deformations that are possible for a body made of any material in a given class of materials in the absence of body forces and by applying only boundary tractions [1]. Motivated by the works of Rivlin [2–4], Ericksen [5] presented the first systematic analysis of universal deformations in homogeneous compressible isotropic solids and incompressible isotropic solids [6]. In the case of compressible isotropic solids, Ericksen [5] proved that universal deformations

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must be homogeneous. Characterizing universal deformations in the incompressible case turned out to be much more complicated. Ericksen [6] found four families of universal deformations (in addition to homogeneous deformations that define family 0). In his analysis, Ericksen conjectured that a deformation with constant principal invariants must be homogeneous, which turned out to be incorrect [7]. This motivated the discovery of a fifth family of inhomogeneous universal deformations with constant principal invariants [8,9]. The known five families of universal deformations other than homogeneous deformations are the following (see Truesdell and Noll [10], Tadmor et al. [11, p. 265], and Goriely [12, p. 305] for a visualization and discussion): (1) family 1: bending, stretching, and shearing of a rectangular block; (2) family 2: straightening, stretching, and shearing of a sector of a cylindrical shell; (3) family 3: inflation, bending, torsion, extension, and shearing of a sector of an annular wedge; (4) family 4: inflation/inversion of a sector of a spherical shell; and (5) family 5: inflation, bending, extension, and azimuthal shearing of an annular wedge. The case of constant principal invariants is still an open problem. However, the conjecture is that there are no solutions other than family 5 deformations.

The study of universal deformations has been extended to anelasticity by Yavari and Goriely [13] in the compressible case and by Goodbrake et al. [14] in the incompressible case. In the literature, the study of universal deformations has been restricted to homogeneous solids. Recently, Yavari [15] extended Ericksen's analysis to inhomogeneous isotropic solids. It was observed that the universality constraints of inhomogeneous solids include those of the corresponding homogeneous solids. It was shown that inhomogeneous compressible isotropic solids do not admit universal deformations. For incompressible isotropic solids, the universal inhomogeneities were characterized for each of the six known families of universal deformations.[1] Until recently, there were only some limited studies of universal deformations in anisotropic nonlinear solids [17]. In Yavari and Goriely [18], we systematically studied universal deformations and universal material preferred directions in homogeneous compressible and incompressible transversely isotropic, orthotropic, and monoclinic solids.[2] In the case of inhomogeneous anisotropic solids, we recently studied the universal inhomogeneities [16]. This systematic analysis completed what we referred to as *the universal program of nonlinear hyperelasticity*.

The analogue of universal deformations in linear elasticity is universal displacements [19–21] and our goal here is to complete the *universal program of linear elasticity*. Universal displacements in homogeneous anisotropic linear elasticity were studied in [21]. Universal displacements were fully characterized for all the eight symmetry classes assuming that the directions of anisotropy are known. In this paper, we extend the analysis of universal displacements to inhomogeneous anisotropic linear elasticity.

This paper is organized as follows. In Section 2, we study universal displacements and inhomogeneities in isotropic linear elasticity. In Section 3, the same problem is studied for the remaining seven symmetry classes (triclinic, monoclinic, tetragonal, trigonal, orthotropic, transversely isotropic, and cubic). Conclusions are given in Section 4 section.

2. Universal displacements in inhomogeneous isotropic linear elasticity

We first extend the work of Yavari et al. [21] to characterize universal displacements in inhomogeneous isotropic linear elasticity. In a Cartesian coordinate system $\{x^a\}$, the elasticity tensor has components $\mathsf{C}_{abcd}(\mathbf{x}) = \lambda(\mathbf{x})\delta_{ab}\delta_{cd} + \mu(\mathbf{x})(\delta_{ac}\delta_{bd} + \delta_{ad}\delta_{bc})$, where δ_{ab} denotes Kronecker's delta, and λ and μ are the Lamé constants that explicitly depend on position $\mathbf{x}$. The equilibrium equations in the absence of body forces read

$$\sigma_{ab,b} = (\lambda + \mu)u_{b,ba} + \mu u_{a,bb} + \lambda_{,a}u_{b,b} + \mu_{,b}(u_{a,b} + u_{b,a}) = 0, \quad a = 1,2,3, \tag{2.1}$$

where σ_{ab} and u_a are the Cauchy stress and displacement components, respectively; $\sigma_{ab,b}$ denotes the partial derivatives of σ_{ab} with respect to x^b, and summation over repeated indices is assumed. Equation (2.1) must hold for arbitrary elastic moduli. In particular, it must hold for uniform elastic moduli. This implies that

$$u_{b,ba} = u_{a,bb} = 0, \quad \text{or} \quad \operatorname{grad} \circ \operatorname{div} \mathbf{u} = \mathbf{0},\ \Delta\mathbf{u} = \mathbf{0}, \tag{2.2}$$

which are the universality constraints of homogeneous isotropic linear elasticity that we derived previously [21]. Therefore, for inhomogeneous isotropic linear elasticity, universal displacements must be constant-divergence harmonic vector fields. We define the *universal inhomogeneities* to be those nonuniform elastic moduli that satisfy equation (2.1) for constant-divergence harmonic vector fields. In other words, the extra universality constraints of inhomogeneous isotropic linear elasticity are

$$\lambda_{,a}u_{b,b} + \mu_{,b}(u_{a,b} + u_{b,a}) = 0, \quad a = 1,2,3. \tag{2.3}$$

On $\mathbb{R}^3$, constant-divergence vector fields have the following representation [22]:

$$u_a(\mathbf{x}) = S_{ab,b}(\mathbf{x}) + \frac{c}{3}x_a + k_a, \tag{2.4}$$

where $S_{ab}(\mathbf{x}) = -S_{ba}(\mathbf{x})$, and c and k_a are constants. More specifically, in Cartesian coordinates $\mathbf{x} = (x_1, x_2, x_3)$, $\mathbf{S}$ has the following representation:

$$\mathbf{S}(\mathbf{x}) = \begin{bmatrix} 0 & \alpha(\mathbf{x}) & \beta(\mathbf{x}) \\ -\alpha(\mathbf{x}) & 0 & \gamma(\mathbf{x}) \\ -\beta(\mathbf{x}) & -\gamma(\mathbf{x}) & 0 \end{bmatrix}, \tag{2.5}$$

where α, β, and γ are arbitrary functions that must satisfy the following system of PDEs [21]:

$$\begin{cases} \Delta\alpha_{,2} + \Delta\beta_{,3} = 0, \\ \Delta\gamma_{,3} - \Delta\alpha_{,1} = 0, \\ \Delta\beta_{,1} + \Delta\gamma_{,2} = 0. \end{cases} \tag{2.6}$$

Substituting equation (2.4) into (2.3) leads to

$$\frac{c}{3}\left[3\lambda_{,a}(\mathbf{x}) + 2\mu_{,a}(\mathbf{x})\right] + [S_{am,bm}(\mathbf{x}) + S_{bm,am}(\mathbf{x})]\mu_{,b}(\mathbf{x}) = 0, \tag{2.7}$$

which must hold for arbitrary c, and hence $3\lambda_{,a}(\mathbf{x}) + 2\mu_{,a}(\mathbf{x}) = 0$, that is, $3\lambda(\mathbf{x}) + 2\mu(\mathbf{x})$ must be uniform. Therefore, the universality constraints are simplified to read

$$\begin{aligned} (S_{a1,11} + S_{a2,12} + S_{a3,13} + S_{12,a2} + S_{13,a3})\mu_{,1} = 0, \\ (S_{a1,12} + S_{a2,22} + S_{a3,23} + S_{21,a1} + S_{23,a3})\mu_{,2} = 0, \\ (S_{a1,13} + S_{a2,23} + S_{a3,33} + S_{31,a1} + S_{32,a2})\mu_{,3} = 0. \end{aligned} \tag{2.8}$$

For $a = 1$, equation (2.8) reads

$$\begin{aligned} 2(S_{12,12} + S_{13,13})\mu_{,1} = 0, \\ (-S_{12,11} + S_{12,22} + S_{13,23} + S_{23,13})\mu_{,2} = 0, \\ (S_{12,23} + S_{13,33} - S_{13,11} + S_{32,12})\mu_{,3} = 0. \end{aligned} \tag{2.9}$$

Since these conditions must be satisfied for all α, β, and γ that satisfy equation (2.6), we can choose $\alpha(\mathbf{x}) = a_0x^2$ and $\beta = \gamma = 0$ and the above constraints are simplified to read $-2a_0\mu_{,2} = 0$, which implies that $\mu_{,2} = 0$. For $\beta(\mathbf{x}) = b_0x_3^2$ and $\alpha = \gamma = 0$, which also satisfy equation (2.6), equation (2.9) is simplified to read $2b_0\mu_{,3} = 0$, which implies that $\mu_{,3} = 0$. Now the constraint (2.8) for $a = 2$ reads $(-S_{12,11} + S_{23,13} - S_{12,22} + S_{13,23})\mu_{,1} = 0$. For the choice $\alpha(\mathbf{x}) = c_0x_2^2$ and $\beta = \gamma = 0$, which satisfy equation (2.6), this constraint is simplified to read $2c_0\mu_{,1} = 0$, which implies that $\mu_{,1} = 0$. Therefore, $\mu(\mathbf{x})$ is constant. Knowing that $3\lambda(\mathbf{x}) + 2\mu(\mathbf{x})$ is also constant, one concludes that both Lamé constants must be uniform, and hence, we have proved the following result:

Proposition 2.1. *Inhomogeneous compressible isotropic linear elastic solids do not admit universal displacements.*

3. Universal displacements and inhomogeneities in anisotropic linear elasticity

Yavari et al. [21] characterized the universal displacements for all the eight anisotropy classes. Here, we extend that work to inhomogeneous anisotropic linear elasticity. Consider an inhomogeneous body made of a linear elastic solid at point $\mathbf{x}$. The elasticity tensor $\mathsf{C}_{abcd}(\mathbf{x})$ has major symmetries, $\mathsf{C}_{abcd}(\mathbf{x}) = \mathsf{C}_{cdab}(\mathbf{x})$, and minor symmetries, $\mathsf{C}_{abcd}(\mathbf{x}) = \mathsf{C}_{bacd}(\mathbf{x}) = \mathsf{C}_{abdc}(\mathbf{x})$. The constitutive equations are written as $\sigma_{ab} = \mathsf{C}_{abcd}\, u_{c,d}$, and the equilibrium equations in the absence of body forces in Cartesian coordinates read

$$\frac{\sigma_{ab}}{\partial x_b} = \mathsf{C}_{abcd}\,\frac{\partial^2 u_c}{\partial x^d \partial x^b} + \frac{\partial \mathsf{C}_{abcd}}{\partial x^b}\,\frac{\partial u_c}{\partial x_d} = 0, \quad a = 1,2,3\,. \tag{3.1}$$

For homogeneous solids, this is reduced to $\mathsf{C}_{abcd}\, u_{c,db} = 0$. For a given class of linear elastic solids, equilibrium equations must be satisfied for arbitrary elastic moduli in the given class. Using this idea, for each of the anisotropy classes—triclinic, monoclinic, tetragonal, trigonal, orthotropic, transversely isotropic, and cubic [23–26]—Yavari et al. [21] characterized the corresponding universal displacements. From equation (3.1), one observes that the universality constraints of inhomogeneous linear elastic solids include those of homogeneous isotropic solids as a particular case.[3] Therefore, for a given anisotropy class and its known universal displacements, the problem is to find the forms of the inhomogeneities of the elastic moduli that are consistent with the following extra universality constraints:

$$\frac{\partial \mathsf{C}_{abcd}}{\partial x^b}\,\frac{\partial u_c}{\partial x_d} = 0\,, \quad a = 1,2,3\,. \tag{3.2}$$

There is no obvious compact way to solve this problem and we resort to explicit computation by using the bijection $(11,22,33,23,31,12) \leftrightarrow (1,2,3,4,5,6)$ and writing the constitutive equations in Voigt notation as $\sigma_\alpha = c_{\alpha\beta}\epsilon_\beta$, where Greek indices run from 1 to 6. The advantage of this classic notation is that the tensorial problem is replaced by a linear algebra problem since the elasticity tensor is now represented by a symmetric 6×6 stiffness matrix as

$$\mathbf{C}(\mathbf{x}) = \begin{bmatrix} c_{11}(\mathbf{x}) & c_{12}(\mathbf{x}) & c_{13}(\mathbf{x}) & c_{14}(\mathbf{x}) & c_{15}(\mathbf{x}) & c_{16}(\mathbf{x}) \\ c_{12}(\mathbf{x}) & c_{22}(\mathbf{x}) & c_{23}(\mathbf{x}) & c_{24}(\mathbf{x}) & c_{25}(\mathbf{x}) & c_{26}(\mathbf{x}) \\ c_{13}(\mathbf{x}) & c_{23}(\mathbf{x}) & c_{33}(\mathbf{x}) & c_{34}(\mathbf{x}) & c_{35}(\mathbf{x}) & c_{36}(\mathbf{x}) \\ c_{14}(\mathbf{x}) & c_{24}(\mathbf{x}) & c_{34}(\mathbf{x}) & c_{44}(\mathbf{x}) & c_{45}(\mathbf{x}) & c_{46}(\mathbf{x}) \\ c_{15}(\mathbf{x}) & c_{25}(\mathbf{x}) & c_{35}(\mathbf{x}) & c_{45}(\mathbf{x}) & c_{55}(\mathbf{x}) & c_{56}(\mathbf{x}) \\ c_{16}(\mathbf{x}) & c_{26}(\mathbf{x}) & c_{36}(\mathbf{x}) & c_{46}(\mathbf{x}) & c_{56}(\mathbf{x}) & c_{66}(\mathbf{x}) \end{bmatrix}. \tag{3.3}$$

In this notation, the equilibrium equations read

$$\begin{bmatrix} \frac{\partial}{\partial x_1} & 0 & 0 & 0 & \frac{\partial}{\partial x_3} & \frac{\partial}{\partial x_2} \\ 0 & \frac{\partial}{\partial x_2} & 0 & \frac{\partial}{\partial x_3} & 0 & \frac{\partial}{\partial x_1} \\ 0 & 0 & \frac{\partial}{\partial x_3} & \frac{\partial}{\partial x_2} & \frac{\partial}{\partial x_1} & 0 \end{bmatrix} \begin{bmatrix} c_{11}(\mathbf{x}) & c_{12}(\mathbf{x}) & c_{13}(\mathbf{x}) & c_{14}(\mathbf{x}) & c_{15}(\mathbf{x}) & c_{16}(\mathbf{x}) \\ c_{12}(\mathbf{x}) & c_{22}(\mathbf{x}) & c_{23}(\mathbf{x}) & c_{24}(\mathbf{x}) & c_{25}(\mathbf{x}) & c_{26}(\mathbf{x}) \\ c_{13}(\mathbf{x}) & c_{23}(\mathbf{x}) & c_{33}(\mathbf{x}) & c_{34}(\mathbf{x}) & c_{35}(\mathbf{x}) & c_{36}(\mathbf{x}) \\ c_{14}(\mathbf{x}) & c_{24}(\mathbf{x}) & c_{34}(\mathbf{x}) & c_{44}(\mathbf{x}) & c_{45}(\mathbf{x}) & c_{46}(\mathbf{x}) \\ c_{15}(\mathbf{x}) & c_{25}(\mathbf{x}) & c_{35}(\mathbf{x}) & c_{45}(\mathbf{x}) & c_{55}(\mathbf{x}) & c_{56}(\mathbf{x}) \\ c_{16}(\mathbf{x}) & c_{26}(\mathbf{x}) & c_{36}(\mathbf{x}) & c_{46}(\mathbf{x}) & c_{56}(\mathbf{x}) & c_{66}(\mathbf{x}) \end{bmatrix} \begin{bmatrix} \frac{\partial u_1}{\partial x_1} \\ \frac{\partial u_2}{\partial x_2} \\ \frac{\partial u_3}{\partial x_3} \\ \frac{\partial u_2}{\partial x_3} + \frac{\partial u_3}{\partial x_2} \\ \frac{\partial u_1}{\partial x_3} + \frac{\partial u_3}{\partial x_1} \\ \frac{\partial u_1}{\partial x_2} + \frac{\partial u_2}{\partial x_1} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}. \tag{3.4}$$

3.1. Triclinic linear elastic solids

Triclinic solids are the least symmetric in the sense that the identity and minus identity are the only symmetry transformations. Other than positive definiteness, there are no constraints on the elastic moduli. In other words, triclinic linear elastic solids have 21 independent elastic moduli. Yavari et al. [21] showed that for triclinic linear elastic solids homogeneous displacements are the only universal displacements.

For the universal displacements (with 9 free parameters), we would like to determine the most general inhomogeneous form of the elastic moduli that are consistent with equation (3.2). The universality constraints (3.2) (for $a = 1$) give us the following six independent PDEs:[4]

$$\begin{aligned}
&\frac{\partial c_{11}}{\partial x_1} + \frac{\partial c_{16}}{\partial x_2} + \frac{\partial c_{15}}{\partial x_3} = 0\,,\\
&\frac{\partial c_{12}}{\partial x_1} + \frac{\partial c_{26}}{\partial x_2} + \frac{\partial c_{25}}{\partial x_3} = 0\,,\\
&\frac{\partial c_{13}}{\partial x_1} + \frac{\partial c_{36}}{\partial x_2} + \frac{\partial c_{35}}{\partial x_3} = 0\,,\\
&\frac{\partial c_{14}}{\partial x_1} + \frac{\partial c_{46}}{\partial x_2} + \frac{\partial c_{45}}{\partial x_3} = 0\,,\\
&\frac{\partial c_{15}}{\partial x_1} + \frac{\partial c_{56}}{\partial x_2} + \frac{\partial c_{55}}{\partial x_3} = 0\,,\\
&\frac{\partial c_{16}}{\partial x_1} + \frac{\partial c_{66}}{\partial x_2} + \frac{\partial c_{56}}{\partial x_3} = 0\,.
\end{aligned} \tag{3.5}$$

Equation (3.2) (for $a = 2$) gives the following six independent PDEs:

$$\begin{aligned}
&\frac{\partial c_{16}}{\partial x_1} + \frac{\partial c_{12}}{\partial x_2} + \frac{\partial c_{14}}{\partial x_3} = 0\,,\\
&\frac{\partial c_{26}}{\partial x_1} + \frac{\partial c_{22}}{\partial x_2} + \frac{\partial c_{24}}{\partial x_3} = 0\,,\\
&\frac{\partial c_{36}}{\partial x_1} + \frac{\partial c_{23}}{\partial x_2} + \frac{\partial c_{34}}{\partial x_3} = 0\,,\\
&\frac{\partial c_{46}}{\partial x_1} + \frac{\partial c_{24}}{\partial x_2} + \frac{\partial c_{44}}{\partial x_3} = 0\,,\\
&\frac{\partial c_{56}}{\partial x_1} + \frac{\partial c_{25}}{\partial x_2} + \frac{\partial c_{45}}{\partial x_3} = 0\,,\\
&\frac{\partial c_{66}}{\partial x_1} + \frac{\partial c_{26}}{\partial x_2} + \frac{\partial c_{46}}{\partial x_3} = 0\,,
\end{aligned} \tag{3.6}$$

and for $a = 3$ gives the following six independent PDEs:

$$\begin{aligned}
&\frac{\partial c_{15}}{\partial x_1} + \frac{\partial c_{14}}{\partial x_2} + \frac{\partial c_{13}}{\partial x_3} = 0\,,\\
&\frac{\partial c_{25}}{\partial x_1} + \frac{\partial c_{24}}{\partial x_2} + \frac{\partial c_{23}}{\partial x_3} = 0\,,\\
&\frac{\partial c_{35}}{\partial x_1} + \frac{\partial c_{34}}{\partial x_2} + \frac{\partial c_{33}}{\partial x_3} = 0\,,\\
&\frac{\partial c_{45}}{\partial x_1} + \frac{\partial c_{44}}{\partial x_2} + \frac{\partial c_{34}}{\partial x_3} = 0\,,\\
&\frac{\partial c_{55}}{\partial x_1} + \frac{\partial c_{45}}{\partial x_2} + \frac{\partial c_{35}}{\partial x_3} = 0\,,\\
&\frac{\partial c_{56}}{\partial x_1} + \frac{\partial c_{46}}{\partial x_2} + \frac{\partial c_{36}}{\partial x_3} = 0\,.
\end{aligned} \tag{3.7}$$

We first notice that $c_{11}, c_{22},$ and c_{33} each appears only once in the above PDEs, and hence

$$\begin{aligned}\frac{\partial c_{11}}{\partial x_1} &= -\frac{\partial c_{15}}{\partial x_3} - \frac{\partial c_{16}}{\partial x_2},\\ \frac{\partial c_{22}}{\partial x_2} &= -\frac{\partial c_{24}}{\partial x_3} - \frac{\partial c_{26}}{\partial x_1},\\ \frac{\partial c_{33}}{\partial x_3} &= -\frac{\partial c_{34}}{\partial x_2} - \frac{\partial c_{35}}{\partial x_1},\end{aligned} \tag{3.8}$$

and thus

$$\begin{aligned}c_{11}(x_1, x_2, x_3) &= -\int (c_{15,3} + c_{16,2})dx_1 + \hat{c}_{11}(x_2, x_3),\\ c_{22}(x_1, x_2, x_3) &= -\int (c_{24,3} + c_{26,1})\,dx_2 + \hat{c}_{22}(x_1, x_3),\\ c_{33}(x_1, x_2, x_3) &= -\int (c_{34,2} + c_{35,1})\,dx_3 + \hat{c}_{33}(x_1, x_2),\end{aligned} \tag{3.9}$$

where $\hat{c}_{11}(x_2, x_3)$, $\hat{c}_{22}(x_1, x_3)$, and $\hat{c}_{33}(x_1, x_2)$ are arbitrary functions.

The elastic moduli $c_{12}, c_{13},$ and c_{23} each appears twice:

$$\begin{aligned}\frac{\partial c_{12}}{\partial x_1} &= -\frac{\partial c_{26}}{\partial x_2} - \frac{\partial c_{25}}{\partial x_3},\\ \frac{\partial c_{12}}{\partial x_2} &= -\frac{\partial c_{16}}{\partial x_1} - \frac{\partial c_{14}}{\partial x_3},\\ \frac{\partial c_{13}}{\partial x_1} &= -\frac{\partial c_{36}}{\partial x_2} - \frac{\partial c_{35}}{\partial x_3},\\ \frac{\partial c_{13}}{\partial x_3} &= -\frac{\partial c_{15}}{\partial x_1} - \frac{\partial c_{14}}{\partial x_2},\\ \frac{\partial c_{23}}{\partial x_2} &= -\frac{\partial c_{36}}{\partial x_1} - \frac{\partial c_{34}}{\partial x_3},\\ \frac{\partial c_{23}}{\partial x_3} &= -\frac{\partial c_{25}}{\partial x_1} - \frac{\partial c_{24}}{\partial x_2}.\end{aligned} \tag{3.10}$$

From the above PDEs, $c_{12}, c_{13},$ and c_{23} are determined as long as the following three integrability conditions are satisfied:

$$\begin{aligned}\frac{\partial^2 c_{26}}{\partial x_2^2} + \frac{\partial^2 c_{25}}{\partial x_2 \partial x_3} &= \frac{\partial^2 c_{16}}{\partial x_1^2} + \frac{\partial^2 c_{14}}{\partial x_1 \partial x_3},\\ \frac{\partial^2 c_{36}}{\partial x_2 \partial x_3} + \frac{\partial^2 c_{35}}{\partial x_3^2} &= \frac{\partial^2 c_{15}}{\partial x_1^2} + \frac{\partial^2 c_{14}}{\partial x_1 \partial x_2},\\ \frac{\partial^2 c_{36}}{\partial x_1 \partial x_3} + \frac{\partial^2 c_{34}}{\partial x_3^2} &= \frac{\partial^2 c_{25}}{\partial x_1 \partial x_2} + \frac{\partial^2 c_{24}}{\partial x_2^2}.\end{aligned} \tag{3.11}$$

Thus,

$$
\begin{aligned}
c_{12}(x_1,x_2,x_3) &= -\int (c_{26,2}+c_{25,3})\,dx_1 + \hat{c}_{12}(x_2,x_3),\\
c_{13}(x_1,x_2,x_3) &= -\int (c_{15,1}+c_{14,2})dx_3 + \hat{c}_{13}(x_1,x_2),\\
c_{23}(x_1,x_2,x_3) &= -\int (c_{36,1}+c_{34,3})dx_2 + \hat{c}_{23}(x_1,x_3),
\end{aligned}
\tag{3.12}
$$

where $\hat{c}_{12}(x_2,x_3)$, $\hat{c}_{13}(x_1,x_2)$, and $\hat{c}_{23}(x_1,x_3)$ are arbitrary functions.

The remaining PDEs can be rearranged as

$$
\begin{aligned}
\frac{\partial c_{44}}{\partial x_2} &= -\frac{\partial c_{45}}{\partial x_1} - \frac{\partial c_{34}}{\partial x_3},\\
\frac{\partial c_{44}}{\partial x_3} &= -\frac{\partial c_{46}}{\partial x_1} - \frac{\partial c_{24}}{\partial x_2},\\
\frac{\partial c_{55}}{\partial x_1} &= -\frac{\partial c_{45}}{\partial x_2} - \frac{\partial c_{35}}{\partial x_3},\\
\frac{\partial c_{55}}{\partial x_3} &= -\frac{\partial c_{15}}{\partial x_1} - \frac{\partial c_{56}}{\partial x_2},\\
\frac{\partial c_{66}}{\partial x_1} &= -\frac{\partial c_{26}}{\partial x_2} - \frac{\partial c_{46}}{\partial x_3},\\
\frac{\partial c_{66}}{\partial x_2} &= -\frac{\partial c_{16}}{\partial x_1} - \frac{\partial c_{56}}{\partial x_3},
\end{aligned}
\tag{3.13}
$$

and

$$
\begin{aligned}
\frac{\partial c_{14}}{\partial x_1} + \frac{\partial c_{46}}{\partial x_2} + \frac{\partial c_{45}}{\partial x_3} &= 0,\\
\frac{\partial c_{56}}{\partial x_1} + \frac{\partial c_{25}}{\partial x_2} + \frac{\partial c_{45}}{\partial x_3} &= 0,\\
\frac{\partial c_{56}}{\partial x_1} + \frac{\partial c_{46}}{\partial x_2} + \frac{\partial c_{36}}{\partial x_3} &= 0.
\end{aligned}
\tag{3.14}
$$

The elastic moduli c_{44}, c_{55}, and c_{66} are determined from equation (3.13) as long as the following integrability conditions are satisfied:

$$
\begin{aligned}
\frac{\partial^2 c_{45}}{\partial x_1 \partial x_3} + \frac{\partial^2 c_{34}}{\partial x_3^2} &= \frac{\partial^2 c_{46}}{\partial x_1 \partial x_2} + \frac{\partial^2 c_{24}}{\partial x_2^2},\\
\frac{\partial^2 c_{45}}{\partial x_2 \partial x_3} + \frac{\partial^2 c_{35}}{\partial x_3^2} &= \frac{\partial^2 c_{15}}{\partial x_1^2} + \frac{\partial^2 c_{56}}{\partial x_1 \partial x_2},\\
\frac{\partial^2 c_{26}}{\partial x_2^2} + \frac{\partial^2 c_{46}}{\partial x_2 \partial x_3} &= \frac{\partial^2 c_{16}}{\partial x_1^2} + \frac{\partial^2 c_{56}}{\partial x_1 \partial x_3}.
\end{aligned}
\tag{3.15}
$$

Thus,

$$
\begin{aligned}
c_{44}(x_1,x_2,x_3) &= -\int(c_{46,1}+c_{24,2})dx_3+\hat{c}_{44}(x_1,x_2),\\
c_{55}(x_1,x_2,x_3) &= -\int(c_{45,2}+c_{35,3})\,dx_1+\hat{c}_{55}(x_1,x_2),\\
c_{66}(x_1,x_2,x_3) &= -\int(c_{16,1}+c_{56,3})dx_2+\hat{c}_{66}(x_1,x_3),
\end{aligned}
\tag{3.16}
$$

where $\hat{c}_{44}(x_1,x_2)$, $\hat{c}_{55}(x_1,x_2)$, and $\hat{c}_{66}(x_1,x_3)$ are arbitrary functions.

From equation (3.14), one obtains

$$
\begin{aligned}
\frac{\partial c_{46}}{\partial x_2} &= \frac{1}{2}\left(-\frac{\partial c_{14}}{\partial x_1}+\frac{\partial c_{25}}{\partial x_2}-\frac{\partial c_{36}}{\partial x_3}\right),\\
\frac{\partial c_{45}}{\partial x_3} &= \frac{1}{2}\left(-\frac{\partial c_{14}}{\partial x_1}-\frac{\partial c_{25}}{\partial x_2}+\frac{\partial c_{36}}{\partial x_3}\right),\\
\frac{\partial c_{56}}{\partial x_1} &= \frac{1}{2}\left(\frac{\partial c_{14}}{\partial x_1}-\frac{\partial c_{25}}{\partial x_2}-\frac{\partial c_{36}}{\partial x_3}\right),
\end{aligned}
\tag{3.17}
$$

and hence

$$
\begin{aligned}
c_{46}(x_1,x_2,x_3) &= \frac{1}{2}\int(-c_{14,1}+c_{25,2}-c_{36,3})dx_2+\hat{c}_{46}(x_1,x_3),\\
c_{45}(x_1,x_2,x_3) &= \frac{1}{2}\int(-c_{14,1}-c_{25,2}+c_{36,3})\,dx_3+\hat{c}_{45}(x_1,x_2),\\
c_{56}(x_1,x_2,x_3) &= \frac{1}{2}\int(c_{14,1}-c_{25,2}-c_{36,3})dx_1+\hat{c}_{56}(x_2,x_3),
\end{aligned}
\tag{3.18}
$$

where $\hat{c}_{46}(x_1,x_2)$, $\hat{c}_{45}(x_1,x_2)$, and $\hat{c}_{56}(x_1,x_3)$ are arbitrary functions. Substituting equation (3.17) into equation (3.15), one can show that the integrability conditions (3.15) are identical to (3.11). To make sense of the results in a compact form, we partition the elasticity matrix into four 3×3 submatrices:

$$
\mathsf{C}(\mathbf{x}) = \left[\begin{array}{c|c}\mathbf{A}(\mathbf{x}) & \mathbf{B}(\mathbf{x})\\ \hline \mathbf{B}(\mathbf{x}) & \mathbf{D}(\mathbf{x})\end{array}\right] = \left[\begin{array}{ccc|ccc}
c_{11}(\mathbf{x}) & c_{12}(\mathbf{x}) & c_{13}(\mathbf{x}) & c_{14}(\mathbf{x}) & c_{15}(\mathbf{x}) & c_{16}(\mathbf{x})\\
c_{12}(\mathbf{x}) & c_{22}(\mathbf{x}) & c_{23}(\mathbf{x}) & c_{24}(\mathbf{x}) & c_{25}(\mathbf{x}) & c_{26}(\mathbf{x})\\
c_{13}(\mathbf{x}) & c_{23}(\mathbf{x}) & c_{33}(\mathbf{x}) & c_{34}(\mathbf{x}) & c_{35}(\mathbf{x}) & c_{36}(\mathbf{x})\\ \hline
c_{14}(\mathbf{x}) & c_{24}(\mathbf{x}) & c_{34}(\mathbf{x}) & c_{44}(\mathbf{x}) & c_{45}(\mathbf{x}) & c_{46}(\mathbf{x})\\
c_{15}(\mathbf{x}) & c_{25}(\mathbf{x}) & c_{35}(\mathbf{x}) & c_{45}(\mathbf{x}) & c_{55}(\mathbf{x}) & c_{56}(\mathbf{x})\\
c_{16}(\mathbf{x}) & c_{26}(\mathbf{x}) & c_{36}(\mathbf{x}) & c_{46}(\mathbf{x}) & c_{56}(\mathbf{x}) & c_{66}(\mathbf{x})
\end{array}\right].
\tag{3.19}
$$

We have shown that the submatrices **A** and **D** depend on **B**. The nine elastic moduli in the submatrix **B** are constrained by the three integrability conditions (3.11). More specifically, one has

$$
\begin{aligned}
\frac{\partial^2 c_{26}}{\partial x_2^2} &= -\frac{\partial^2 c_{25}}{\partial x_2\partial x_3}+\frac{\partial^2 c_{16}}{\partial x_1^2}+\frac{\partial^2 c_{14}}{\partial x_1\partial x_3},\\
\frac{\partial^2 c_{35}}{\partial x_3^2} &= -\frac{\partial^2 c_{36}}{\partial x_2\partial x_3}+\frac{\partial^2 c_{15}}{\partial x_1^2}+\frac{\partial^2 c_{14}}{\partial x_1\partial x_2},\\
\frac{\partial^2 c_{24}}{\partial x_2^2} &= -\frac{\partial^2 c_{25}}{\partial x_1\partial x_2}+\frac{\partial^2 c_{36}}{\partial x_1\partial x_3}+\frac{\partial^2 c_{34}}{\partial x_3^2}.
\end{aligned}
\tag{3.20}
$$

Therefore, c_{26}, c_{35}, and c_{24} are functions of the other six elastic consents in **B**. We see that homogeneous displacements are universal for a large class of inhomogeneous triclinic solids. In summary, we have proved the following result:

Proposition 3.1. *For inhomogeneous triclinic linear elastic solids, all homogeneous displacements are universal as long as the elastic moduli have the universal inhomogeneities. Of the 21 elastic moduli, 6 of them (c_{14}, c_{15}, c_{16}, c_{25}, c_{34}, and c_{36}) are arbitrary functions of (x_1, x_2, x_3). The remaining 15 elastic moduli are determined using these 6 functions and certain linear PDEs.*

3.2. Monoclinic linear elastic solids

A monoclinic solid has one plane of material symmetry, which, without loss of generality, is assumed to be parallel to the x^1x^2-plane. A monoclinic linear elastic solid has 13 independent elastic moduli and its elasticity matrix has the following representation:

$$\mathsf{C}(\mathbf{x}) = \begin{bmatrix} c_{11}(\mathbf{x}) & c_{12}(\mathbf{x}) & c_{13}(\mathbf{x}) & 0 & 0 & c_{16}(\mathbf{x}) \\ c_{12}(\mathbf{x}) & c_{22}(\mathbf{x}) & c_{23}(\mathbf{x}) & 0 & 0 & c_{26}(\mathbf{x}) \\ c_{13}(\mathbf{x}) & c_{23}(\mathbf{x}) & c_{33}(\mathbf{x}) & 0 & 0 & c_{36}(\mathbf{x}) \\ 0 & 0 & 0 & c_{44}(\mathbf{x}) & c_{45}(\mathbf{x}) & 0 \\ 0 & 0 & 0 & c_{45}(\mathbf{x}) & c_{55}(\mathbf{x}) & 0 \\ c_{16}(\mathbf{x}) & c_{26}(\mathbf{x}) & c_{36}(\mathbf{x}) & 0 & 0 & c_{66}(\mathbf{x}) \end{bmatrix}. \tag{3.21}$$

Yavari et al. [21] showed that for a monoclinic linear elastic solid with planes of symmetry parallel to the x_1x_2-plane, universal displacements are the superposition of homogeneous displacements $\mathbf{F}\cdot\mathbf{x}$ ($\mathbf{F}$ is a constant matrix) and the one-parameter inhomogeneous displacement field $(cx_2x_3, -cx_1x_3, 0)$. Now for these universal displacements (with 10 free parameters), we would like to determine the most general inhomogeneous form of the elastic moduli that are consistent with equation (3.2). For $a = 1$, equation (3.2) gives us the following six independent PDEs:

$$\begin{aligned} &\frac{\partial c_{11}}{\partial x_1} + \frac{\partial c_{16}}{\partial x_2} = 0, \\ &\frac{\partial c_{12}}{\partial x_1} + \frac{\partial c_{26}}{\partial x_2} = 0, \\ &\frac{\partial c_{13}}{\partial x_1} + \frac{\partial c_{36}}{\partial x_2} = 0, \\ &\frac{\partial c_{16}}{\partial x_1} + \frac{\partial c_{66}}{\partial x_2} = 0, \\ &\frac{\partial c_{45}}{\partial x_3} = \frac{\partial c_{55}}{\partial x_3} = 0. \end{aligned} \tag{3.22}$$

For $a = 2$, equation (3.2) gives us the following five independent PDEs:

$$\begin{aligned} &\frac{\partial c_{16}}{\partial x_1} + \frac{\partial c_{12}}{\partial x_2} = 0, \\ &\frac{\partial c_{26}}{\partial x_1} + \frac{\partial c_{22}}{\partial x_2} = 0, \\ &\frac{\partial c_{36}}{\partial x_1} + \frac{\partial c_{23}}{\partial x_2} = 0, \\ &\frac{\partial c_{66}}{\partial x_1} + \frac{\partial c_{26}}{\partial x_2} = 0, \\ &\frac{\partial c_{44}}{\partial x_3} = 0. \end{aligned} \tag{3.23}$$

For $a=3$, equation (3.2) gives us the following six independent PDEs:

$$\begin{aligned}
&\frac{\partial c_{13}}{\partial x_3}=\frac{\partial c_{23}}{\partial x_3}=\frac{\partial c_{33}}{\partial x_3}=\frac{\partial c_{36}}{\partial x_3}=0,\\
&\frac{\partial c_{45}}{\partial x_1}+\frac{\partial c_{44}}{\partial x_2}=0,\\
&\frac{\partial c_{55}}{\partial x_1}+\frac{\partial c_{45}}{\partial x_2}=0.
\end{aligned}\tag{3.24}$$

Thus, from equations $(3.22)_5$, $(3.23)_5$, and $(3.24)_1$, one concludes that

$$\begin{aligned}
&c_{13}=c_{13}(x_1,x_2), \quad c_{23}=c_{23}(x_1,x_2), \quad c_{33}=c_{33}(x_1,x_2), \quad c_{36}=c_{36}(x_1,x_2),\\
&c_{44}=c_{44}(x_1,x_2), \quad c_{45}=c_{45}(x_1,x_2), \quad c_{55}=c_{55}(x_1,x_2).
\end{aligned}\tag{3.25}$$

From the last two PDEs in equation (3.24), and for an arbitrary $c_{45}(x_1,x_2)$, one has

$$\begin{aligned}
c_{44}(x_1,x_2)&=-\int c_{45,1}(x_1,x_2)\,dx_2+\hat{c}_{44}(x_1),\\
c_{55}(x_1,x_2)&=-\int c_{45,2}(x_1,x_2)\,dx_1+\hat{c}_{55}(x_2),
\end{aligned}\tag{3.26}$$

where $\hat{c}_{44}(x_1)$ and $\hat{c}_{55}(x_2)$ are arbitrary functions. Similarly, from equations $(3.22)_3$ and $(3.23)_3$, and for an arbitrary $c_{36}(x_1,x_2)$, one obtains

$$\begin{aligned}
c_{13}(x_1,x_2)&=-\int c_{36,2}(x_1,x_2)dx_1+\hat{c}_{13}(x_2),\\
c_{23}(x_1,x_2)&=-\int c_{36,1}(x_1,x_2)\,dx_2+\hat{c}_{23}(x_1),
\end{aligned}\tag{3.27}$$

where $\hat{c}_{13}(x_2)$ and $\hat{c}_{23}(x_1)$ are arbitrary functions.

The remaining PDEs are

$$\begin{aligned}
&\frac{\partial c_{11}}{\partial x_1}+\frac{\partial c_{16}}{\partial x_2}=0,\\
&\frac{\partial c_{12}}{\partial x_2}+\frac{\partial c_{16}}{\partial x_1}=0,\\
&\frac{\partial c_{66}}{\partial x_2}+\frac{\partial c_{16}}{\partial x_1}=0,\\
&\frac{\partial c_{12}}{\partial x_1}+\frac{\partial c_{26}}{\partial x_2}=0,\\
&\frac{\partial c_{22}}{\partial x_2}+\frac{\partial c_{26}}{\partial x_1}=0,\\
&\frac{\partial c_{66}}{\partial x_1}+\frac{\partial c_{26}}{\partial x_2}=0.
\end{aligned}\tag{3.28}$$

The form of the above PDEs suggests that c_{11}, c_{12}, c_{22}, and c_{66} are functions of c_{16} and c_{26}. First, note that from equations $(3.28)_2$ and $(3.28)_4$, one concludes that

$$\frac{\partial^2 c_{26}}{\partial x_2^2}=\frac{\partial^2 c_{16}}{\partial x_1^2},\tag{3.29}$$

and hence

$$c_{26}(x_1,x_2,x_3) = \iint c_{16,11}(x_1,x_2)dx_2\,dx_2 + x_2\widehat{c}_{26}(x_1,x_3) + \widetilde{c}_{26}(x_1,x_3), \tag{3.30}$$

for arbitrary functions $\widehat{c}_{26}(x_1,x_3)$, and $\widetilde{c}_{26}(x_1,x_3)$.

From the first three PDEs in equation (3.28) and for an arbitrary $c_{16}(x_1,x_2,x_3)$, one obtains

$$\begin{aligned} c_{11}(x_1,x_2,x_3) &= -\int c_{16,2}(x_1,x_2,x_3)\,dx_1 + \hat{c}_{11}(x_2,x_3), \\ c_{12}(x_1,x_2,x_3) &= -\int c_{16,1}(x_1,x_2,x_3)\,dx_2 + \hat{c}_{12}(x_1,x_3), \\ c_{66}(x_1,x_2,x_3) &= -\int c_{16,1}(x_1,x_2,x_3)\,dx_2 + \hat{c}_{66}(x_1,x_3), \end{aligned} \tag{3.31}$$

where $\hat{c}_{11}(x_2,x_3)$, $\hat{c}_{12}(x_1,x_3)$, and $\hat{c}_{66}(x_1,x_3)$ are arbitrary functions. Finally, from equation $(3.28)_5$ one concludes that

$$c_{22}(x_1,x_2,x_3) = -\int c_{26,1}(x_1,x_2,x_3)dx_2 + \hat{c}_{22}(x_1,x_3), \tag{3.32}$$

for an arbitrary function $\hat{c}_{22}(x_1,x_3)$.

Proposition 3.2. *For inhomogeneous monoclinic linear elastic solids with planes of symmetry parallel to the x_1x_2-plane, the following position-dependence of the elasticity matrix is universal*:

$$\mathbf{C}(\mathbf{x}) = \begin{bmatrix} c_{11}(x_1,x_2,x_3) & c_{12}(x_1,x_2,x_3) & c_{13}(x_1,x_2) & 0 & 0 & c_{16}(\mathbf{x}) \\ c_{12}(x_1,x_2,x_3) & c_{22}(x_1,x_2,x_3) & c_{23}(x_1,x_2) & 0 & 0 & c_{26}(\mathbf{x}) \\ c_{13}(x_1,x_2) & c_{23}(x_1,x_2) & c_{33}(x_1,x_2) & 0 & 0 & c_{36}(x_1,x_2) \\ 0 & 0 & 0 & c_{44}(x_1,x_2) & c_{45}(x_1,x_2) & 0 \\ 0 & 0 & 0 & c_{45}(x_1,x_2) & c_{55}(x_1,x_2) & 0 \\ c_{16}(\mathbf{x}) & c_{26}(\mathbf{x}) & c_{36}(x_1,x_2) & 0 & 0 & c_{66}(\mathbf{x}) \end{bmatrix}, \tag{3.33}$$

where $c_{33}(x_1,x_2)$, $c_{36}(x_1,x_2)$, $c_{45}(x_1,x_2)$, *and* $c_{16}(x_1,x_2,x_3)$ *are arbitrary functions while* $c_{13}(x_1,x_2)$, $c_{23}(x_1,x_2)$, $c_{44}(x_1,x_2)$, $c_{55}(x_1,x_2)$, $c_{26}(x_1,x_2,x_3)$, $c_{11}(x_1,x_2,x_3)$, $c_{12}(x_1,x_2,x_3)$, $c_{66}(x_1,x_2,x_3)$, *and* $c_{44}(x_1,x_2,x_3)$ *are given in equations* (3.27), (3.26), (3.30), (3.31), *and* (3.32). *For such inhomogeneous monoclinic linear elastic solids, universal displacements are the superposition of homogeneous displacement fields and the one-parameter inhomogeneous displacement field* $(cx_2x_3,\ -cx_1x_3, 0)$.

3.3. Tetragonal linear elastic solids

In a tetragonal solid, there are five planes of symmetry such that the normals of four of them are coplanar and the fifth one is normal to the other four. We assume that in the Cartesian coordinate system (x_1,x_2,x_3), the fifth normal is parallel to the x_3 axis. There are two planes of symmetry parallel to the x_1x_3 and x_2x_3-planes. The other two symmetry planes are related to the ones parallel to the x_1x_3-plane by $\pi/4$ and $3\pi/4$ rotations about the x_3 axis. Tetragonal solids have 6 independent elastic moduli with elasticity matrices of the following form:

$$\mathbf{C}(\mathbf{x}) = \begin{bmatrix} c_{11}(\mathbf{x}) & c_{12}(\mathbf{x}) & c_{13}(\mathbf{x}) & 0 & 0 & 0 \\ c_{12}(\mathbf{x}) & c_{11}(\mathbf{x}) & c_{13}(\mathbf{x}) & 0 & 0 & 0 \\ c_{13}(\mathbf{x}) & c_{13}(\mathbf{x}) & c_{33}(\mathbf{x}) & 0 & 0 & 0 \\ 0 & 0 & 0 & c_{44}(\mathbf{x}) & 0 & 0 \\ 0 & 0 & 0 & 0 & c_{44}(\mathbf{x}) & 0 \\ 0 & 0 & 0 & 0 & 0 & c_{66}(\mathbf{x}) \end{bmatrix}. \tag{3.34}$$

Yavari et al. [21] showed that in a tetragonal linear elastic solid with the tetragonal axes parallel to the x_3-axis in a Cartesian coordinate system (x_1, x_2, x_3), the universal displacements are a superposition of homogeneous displacements and the following inhomogeneous displacements:[5]

$$\begin{aligned} u_1(x_1,x_2,x_3) &= F_{11}x_1 + F_{12}x_2 + F_{13}x_3 + c_1x_2x_3 + c_2x_1x_3, \\ u_2(x_1,x_2,x_3) &= F_{21}x_1 + F_{22}x_2 + F_{23}x_3 - c_2x_2x_3 + c_3x_1x_3, \\ u_3(x_1,x_2,x_3) &= F_{31}x_1 + F_{32}x_2 + F_{33}x_3 + g(x_1,x_2), \end{aligned} \tag{3.35}$$

where c_1 and c_2 are constants, and $g = g(x_1, x_2)$ is a harmonic function.

Now for these universal displacements (with 12 free parameters and an arbitrary harmonic function), we would like to determine the most general inhomogeneous form of the elastic moduli that are consistent with equation (3.2). For $a = 1$, equation (3.2) gives us the following five independent PDEs:

$$\begin{aligned} &\frac{\partial g}{\partial x_1}\frac{\partial c_{44}}{\partial x_3} = 0, \\ &\frac{\partial c_{11}}{\partial x_1} = \frac{\partial c_{12}}{\partial x_1} = \frac{\partial c_{13}}{\partial x_1} = 0, \\ &\frac{\partial c_{66}}{\partial x_2} = 0. \end{aligned} \tag{3.36}$$

As $g(x_1, x_2)$ is an arbitrary harmonic function, from the first equation one concludes that $\partial c_{44}/\partial x_3 = 0$. Thus, $c_{44} = c_{44}(x_1, x_2)$, $c_{11} = c_{11}(x_2, x_3)$, $c_{12} = c_{12}(x_2, x_3)$, $c_{13} = c_{13}(x_2, x_3)$, and $c_{66} = c_{66}(x_1, x_3)$. For $a = 2$, equation (3.2) gives us the following four independent PDEs:

$$\frac{\partial c_{11}}{\partial x_2} = \frac{\partial c_{12}}{\partial x_2} = \frac{\partial c_{13}}{\partial x_2} = \frac{\partial c_{66}}{\partial x_1} = 0. \tag{3.37}$$

Thus, $c_{11} = c_{11}(x_3)$, $c_{12} = c_{12}(x_3)$, $c_{13} = c_{13}(x_3)$, $c_{44} = c_{44}(x_1, x_2)$, and $c_{66} = c_{66}(x_3)$. For $a = 3$, equation (3.2) gives us the following four independent PDEs:

$$\frac{\partial c_{13}}{\partial x_3} = \frac{\partial c_{33}}{\partial x_3} = \frac{\partial c_{44}}{\partial x_1} = \frac{\partial c_{44}}{\partial x_2} = 0. \tag{3.38}$$

Hence, c_{13} and c_{44} are constant, and $c_{11} = c_{11}(x_3)$, $c_{12} = c_{12}(x_3)$, $c_{66} = c_{66}(x_3)$, and $c_{33} = c_{33}(x_1, x_2)$. Therefore, we have proved the following result:

Proposition 3.3. *For a tetragonal linear elastic solid with the tetragonal axis parallel to the x_3-axis in a Cartesian coordinate system (x_1, x_2, x_3), and with the following inhomogeneous elasticity matrix*

$$\mathbf{C}(\mathbf{x}) = \begin{bmatrix} c_{11}(x_3) & c_{12}(x_3) & c_{13} & 0 & 0 & 0 \\ c_{12}(x_3) & c_{11}(x_3) & c_{13} & 0 & 0 & 0 \\ c_{13} & c_{13} & c_{33}(x_1,x_2) & 0 & 0 & 0 \\ 0 & 0 & 0 & c_{44} & 0 & 0 \\ 0 & 0 & 0 & 0 & c_{44} & 0 \\ 0 & 0 & 0 & 0 & 0 & c_{66}(x_3) \end{bmatrix}, \tag{3.39}$$

the universal displacements are the superposition of homogeneous displacement fields and the following inhomogeneous displacement field:

$$\begin{aligned} u_1^{\mathrm{inh}}(x_1,x_2,x_3) &= c_1 x_2 x_3 + c_2 x_1 x_3, \\ u_2^{\mathrm{inh}}(x_1,x_2,x_3) &= -c_2 x_1 x_3 + c_3 x_1 x_3, \\ u_3^{\mathrm{inh}}(x_1,x_2,x_3) &= g(x_1,x_2), \end{aligned} \tag{3.40}$$

where c_1 and c_2 are constants, and $g = g(x_1,x_2)$ is an arbitrary harmonic function.

3.4. Trigonal linear elastic solids

In a trigonal solid, there are three planes of symmetry whose normals lie in the same plane and are related by $\pi/3$ rotations. We assume that the trigonal axis is parallel to the x_3-axis. A trigonal solid has 6 independent elastic moduli and its elasticity matrix has the following representation:

$$\mathsf{C}(\mathbf{x}) = \begin{bmatrix} c_{11}(\mathbf{x}) & c_{12}(\mathbf{x}) & c_{13}(\mathbf{x}) & 0 & c_{15}(\mathbf{x}) & 0 \\ c_{12}(\mathbf{x}) & c_{11}(\mathbf{x}) & c_{13}(\mathbf{x}) & 0 & -c_{15}(\mathbf{x}) & 0 \\ c_{13}(\mathbf{x}) & c_{13}(\mathbf{x}) & c_{33}(\mathbf{x}) & 0 & 0 & 0 \\ 0 & 0 & 0 & c_{44}(\mathbf{x}) & 0 & -c_{15}(\mathbf{x}) \\ c_{15}(\mathbf{x}) & -c_{15}(\mathbf{x}) & 0 & 0 & c_{44}(\mathbf{x}) & 0 \\ 0 & 0 & 0 & -c_{15}(\mathbf{x}) & 0 & \frac{1}{2}(c_{11}(\mathbf{x}) - c_{12}(\mathbf{x})) \end{bmatrix}. \tag{3.41}$$

Yavari et al. [21] showed that universal displacements are a superposition of homogeneous displacements and the following inhomogeneous displacements:

$$\begin{aligned} u_1^{\mathrm{inh}}(x_1,x_2,x_3) &= a_{123} x_1 x_2 x_3 + a_{12} x_1 x_2 + a_{13} x_1 x_3 + a_{23} x_2 x_3, \\ u_2^{\mathrm{inh}}(x_1,x_2,x_3) &= \frac{1}{2}(a_{12} + a_{123} x_3)\left(x_1^2 - x_2^2\right) + b_{13} x_1 x_3 - a_{13} x_2 x_3, \\ u_3^{\mathrm{inh}}(x_1,x_2,x_3) &= -a_{123} x_1^2 x_2 - (a_{23} + b_{13}) x_1 x_2 + \frac{1}{3} a_{123} x_2^3 - a_{13}\left(x_1^2 - x_2^2\right). \end{aligned} \tag{3.42}$$

For the above universal displacements (with 14 free parameters), we would like to find the most general inhomogeneous form of the elastic moduli that are consistent with equation (3.2). For $a = 1,2,3$, equation (3.2) gives us the following PDEs:

$$\begin{aligned} &\frac{\partial c_{11}}{\partial x_2} = \frac{\partial c_{12}}{\partial x_2} = \frac{\partial c_{15}}{\partial x_2} = 0, \\ &\frac{\partial c_{13}}{\partial x_1} = \frac{\partial c_{13}}{\partial x_2} = \frac{\partial c_{33}}{\partial x_3} = 0, \\ &\frac{\partial c_{44}}{\partial x_1} = \frac{\partial c_{44}}{\partial x_2} = 0. \end{aligned} \tag{3.43}$$

Thus, $c_{11} = c_{11}(x_1,x_3)$, $c_{12} = c_{12}(x_1,x_3)$, $c_{15} = c_{15}(x_1,x_3)$, $c_{13} = c_{13}(x_3)$, $c_{33} = c_{33}(x_1,x_2)$, and $c_{44} = c_{44}(x_3)$. Substituting these back into (3.2) one obtains the following PDEs:

$$\begin{aligned} &\frac{\partial c_{15}}{\partial x_1} = \frac{\partial c_{15}}{\partial x_3} = 0, \\ &\frac{\partial c_{13}}{\partial x_3} = \frac{\partial c_{44}}{\partial x_3} = 0, \\ &\frac{\partial c_{11}}{\partial x_1} = \frac{\partial c_{12}}{\partial x_1} = 0. \end{aligned} \tag{3.44}$$

Thus, $c_{11}=c_{11}(x_3)$, $c_{12}=c_{12}(x_3)$, and $c_{33}=c_{33}(x_1,x_2)$, and c_{13}, c_{15}, and c_{44} are constant. In summary, we have proved the following result:

Proposition 3.4. *For inhomogeneous trigonal linear elastic solids whose trigonal axes are parallel to the x_3 axis and have the following inhomogeneous elastic moduli*

$$\mathsf{C}(\mathbf{x})=\begin{bmatrix} c_{11}(x_3) & c_{12}(x_3) & c_{13} & 0 & c_{15} & 0 \\ c_{12}(x_3) & c_{11}(x_3) & c_{13} & 0 & -c_{15} & 0 \\ c_{13} & c_{13} & c_{33}(x_1,x_2) & 0 & 0 & 0 \\ 0 & 0 & 0 & c_{44} & 0 & -c_{15} \\ c_{15} & -c_{15} & 0 & 0 & c_{44} & 0 \\ 0 & 0 & 0 & -c_{15} & 0 & \frac{1}{2}(c_{11}(x_3)-c_{12}(x_3)) \end{bmatrix}, \tag{3.45}$$

the universal displacements are the superposition of homogeneous displacements and the following inhomogeneous displacement fields:

$$\begin{aligned} u_1^{\text{inh}}(x_1,x_2,x_3) &= a_{123}x_1x_2x_3 + a_{12}x_1x_2 + a_{13}x_1x_3 + a_{23}x_2x_3, \\ u_2^{\text{inh}}(x_1,x_2,x_3) &= \frac{1}{2}(a_{12}+a_{123}x_3)\left(x_1^2-x_2^2\right) + b_{13}x_1x_3 - a_{13}x_2x_3, \\ u_3^{\text{inh}}(x_1,x_2,x_3) &= -a_{123}x_1^2x_2 - (a_{23}+b_{13})x_1x_2 + \frac{1}{3}a_{123}x_2^3 - a_{13}\left(x_1^2-x_2^2\right). \end{aligned} \tag{3.46}$$

3.5. Orthotropic linear elastic solids

In an orthotropic solid, there are three mutually orthogonal symmetry planes. We choose Cartesian coordinates (x_1,x_2,x_3) such that the coordinate planes are parallel to the symmetry planes. An orthotropic solid has 9 independent elastic moduli, and its elasticity matrix has the following representation:

$$\mathsf{C}(\mathbf{x})=\begin{bmatrix} c_{11}(\mathbf{x}) & c_{12}(\mathbf{x}) & c_{13}(\mathbf{x}) & 0 & 0 & 0 \\ c_{12}(\mathbf{x}) & c_{22}(\mathbf{x}) & c_{23}(\mathbf{x}) & 0 & 0 & 0 \\ c_{13}(\mathbf{x}) & c_{23}(\mathbf{x}) & c_{33}(\mathbf{x}) & 0 & 0 & 0 \\ 0 & 0 & 0 & c_{44}(\mathbf{x}) & 0 & 0 \\ 0 & 0 & 0 & 0 & c_{55}(\mathbf{x}) & 0 \\ 0 & 0 & 0 & 0 & 0 & c_{66}(\mathbf{x}) \end{bmatrix}. \tag{3.47}$$

Yavari et al. [21] showed that in an orthotropic linear elastic solid whose planes of symmetry are normal to the coordinate axes in a Cartesian coordinate system (x_1,x_2,x_3), the universal displacements are the superposition of homogeneous displacement fields and the 3-parameter inhomogeneous displacement field $(a_1x_2x_3, a_2x_1x_3, a_3x_1x_2)$.

For the above universal displacements (with 12 free parameters), the universality constraints (3.2) force the elastic moduli to have the following inhomogeneous forms:

$$c_{11}=c_{11}(x_2,x_3), \quad c_{22}=c_{22}(x_1,x_3), \quad c_{33}=c_{33}(x_1,x_2), \tag{3.48}$$

$$c_{44}=c_{44}(x_1), \qquad c_{55}=c_{55}(x_2), \qquad c_{66}=c_{66}(x_3), \tag{3.49}$$

$$c_{12}=c_{12}(x_3), \qquad c_{13}=c_{13}(x_2), \qquad c_{23}=c_{23}(x_1). \tag{3.50}$$

Therefore, we have proved the following result:

Proposition 3.5. *For orthotropic linear elastic solids with planes of symmetry normal to the coordinate axes in a Cartesian coordinate system (x_1,x_2,x_3), and with the following inhomogeneous elastic moduli*

$$\mathbf{C}(\mathbf{x}) = \begin{bmatrix} c_{11}(x_2,x_3) & c_{12}(x_3) & c_{13}(x_2) & 0 & 0 & 0 \\ c_{12}(x_3) & c_{22}(x_1,x_3) & c_{23}(x_1) & 0 & 0 & 0 \\ c_{13}(x_2) & c_{23}(x_1) & c_{33}(x_1,x_2) & 0 & 0 & 0 \\ 0 & 0 & 0 & c_{44}(x_1) & 0 & 0 \\ 0 & 0 & 0 & 0 & c_{55}(x_2) & 0 \\ 0 & 0 & 0 & 0 & 0 & c_{66}(x_3) \end{bmatrix}, \tag{3.51}$$

universal displacements are the superposition of homogeneous displacement fields and the 3-parameter inhomogeneous displacement field $(a_1x_2x_3, a_2x_1x_3, a_3x_1x_2)$.

3.6. Transversely isotropic linear elastic solids

For a transversely isotropic solid, there is an axis of symmetry such that the isotropy planes are planes normal to it. We choose Cartesian coordinates (x_1,x_2,x_3) such that the axis of transverse isotropy is parallel to the x_3-axis. A transversely isotropic solid has 5 independent elastic moduli, and its elasticity matrix has the following representation:

$$\mathbf{C}(\mathbf{x}) = \begin{bmatrix} c_{11}(\mathbf{x}) & c_{12}(\mathbf{x}) & c_{13}(\mathbf{x}) & 0 & 0 & 0 \\ c_{12}(\mathbf{x}) & c_{11}(\mathbf{x}) & c_{13}(\mathbf{x}) & 0 & 0 & 0 \\ c_{13}(\mathbf{x}) & c_{13}(\mathbf{x}) & c_{33}(\mathbf{x}) & 0 & 0 & 0 \\ 0 & 0 & 0 & c_{44}(\mathbf{x}) & 0 & 0 \\ 0 & 0 & 0 & 0 & c_{44}(\mathbf{x}) & 0 \\ 0 & 0 & 0 & 0 & 0 & \frac{1}{2}(c_{11}(\mathbf{x})-c_{12}(\mathbf{x})) \end{bmatrix}. \tag{3.52}$$

Yavari et al. [21] showed that universal deformations have the following form:

$$\begin{aligned} u_1(x_1,x_2,x_3) &= c_1x_1 + c_2x_2 + cx_2x_3 + x_3h_1(x_1,x_2) + k_1(x_1,x_2), \\ u_2(x_1,x_2,x_3) &= -c_2x_1 + c_1x_2 - cx_1x_3 + x_3h_2(x_1,x_2) + k_2(x_1,x_2), \\ u_3(x_1,x_2,x_3) &= c_3x_3 + \hat{u}_3(x_1,x_2), \end{aligned} \tag{3.53}$$

where $\xi(x_2+ix_1) = h_2(x_1,x_2) + ih_1(x_1,x_2)$ and $\eta(x_2+ix_1) = k_2(x_1,x_2) + ik_1(x_1,x_2)$[6] are holomorphic, and $\hat{u}_3(x_1,x_2)$ is harmonic. For the above universal displacements (with 4 free parameters and 5 harmonic functions), the constraints (3.2) force the elastic moduli to have the following inhomogeneous forms:

$$c_{11} = c_{11}(x_3), \quad c_{12} = c_{12}(x_3), \quad c_{33} = c_{33}(x_1,x_2), \quad c_{13}, c_{44} \text{ are constant}. \tag{3.54}$$

Therefore, we have proved the following result:

Proposition 3.6. *In a transversely isotropic linear elastic solid with the isotropy plane parallel to the* x_1x_2*-plane that has the following inhomogeneous elastic moduli*

$$\mathbf{C}(\mathbf{x}) = \begin{bmatrix} c_{11}(x_3) & c_{12}(x_3) & c_{13} & 0 & 0 & 0 \\ c_{12}(x_3) & c_{11}(x_3) & c_{13} & 0 & 0 & 0 \\ c_{13} & c_{13} & c_{33}(x_1,x_2) & 0 & 0 & 0 \\ 0 & 0 & 0 & c_{44} & 0 & 0 \\ 0 & 0 & 0 & 0 & c_{44} & 0 \\ 0 & 0 & 0 & 0 & 0 & \frac{1}{2}(c_{11}(x_3)-c_{12}(x_3)) \end{bmatrix}, \tag{3.55}$$

the universal displacements have the following form:

$$
\begin{aligned}
u_1(x_1,x_2,x_3) &= c_1x_1 + c_2x_2 + cx_2x_3 + x_3h_1(x_1,x_2) + k_1(x_1,x_2),\\
u_2(x_1,x_2,x_3) &= -c_2x_1 + c_1x_2 - cx_1x_3 + x_3h_2(x_1,x_2) + k_2(x_1,x_2),\\
u_3(x_1,x_2,x_3) &= c_3x_3 + \hat{u}_3(x_1,x_2),
\end{aligned}
\tag{3.56}
$$

where $\xi(x_2+ix_1) = h_2(x_1,x_2) + ih_1(x_1,x_2)$ and $\eta(x_2+ix_1) = k_2(x_1,x_2) + ik_1(x_1,x_2)$ are holomorphic, and $\hat{u}_3(x_1,x_2)$ is harmonic.

3.7. Cubic linear elastic solids

At every point, a cubic solid has nine planes of symmetry whose normals are parallel to the edges and face diagonals of a cube. We choose a Cartesian coordinate system (x_1,x_2,x_3) whose coordinate lines are parallel to the edges of the cube. A cubic solid has 3 independent elastic moduli and its matrix of elastic moduli reads

$$
\mathsf{C}(\mathbf{x}) = \begin{bmatrix}
c_{11}(\mathbf{x}) & c_{12}(\mathbf{x}) & c_{12}(\mathbf{x}) & 0 & 0 & 0\\
c_{12}(\mathbf{x}) & c_{11}(\mathbf{x}) & c_{12}(\mathbf{x}) & 0 & 0 & 0\\
c_{12}(\mathbf{x}) & c_{12}(\mathbf{x}) & c_{11}(\mathbf{x}) & 0 & 0 & 0\\
0 & 0 & 0 & c_{44}(\mathbf{x}) & 0 & 0\\
0 & 0 & 0 & 0 & c_{44}(\mathbf{x}) & 0\\
0 & 0 & 0 & 0 & 0 & c_{44}(\mathbf{x})
\end{bmatrix}.
\tag{3.57}
$$

Yavari et al. [21] showed that for cubic solids, universal displacements have the following form:

$$
\begin{aligned}
u_1(x_1,x_2,x_3) &= \frac{a}{2}x_1\left(x_3^2 - x_2^2\right) + c_1x_1x_3 + b_1x_1x_2 + d_1x_1 + g_1(x_2,x_3),\\
u_2(x_1,x_2,x_3) &= \frac{a}{2}x_2\left(x_1^2 - x_3^2\right) + a_1x_1x_2 - c_1x_2x_3 + d_2x_2 + g_2(x_1,x_3),\\
u_3(x_1,x_2,x_3) &= \frac{a}{2}x_3\left(x_2^2 - x_1^2\right) - a_1x_1x_3 - b_1x_2x_3 + d_3x_3 + g_3(x_1,x_2),
\end{aligned}
\tag{3.58}
$$

where g_1, g_2, and g_3 are arbitrary harmonic functions. For the above universal displacements (with seven free parameters and three arbitrary harmonic functions) the universality constraints (3.2) force the three elastic moduli to be uniform. Therefore, we have proved the following result:

Proposition 3.7. *Inhomogeneous compressible cubic linear elastic solids do not admit universal displacements.*

4. Conclusion

We studied universal displacements and inhomogeneities in linear elasticity for the eight symmetry classes (triclinic, monoclinic, tetragonal, trigonal, orthotropic, transversely isotropic, cubic, and isotropic) assuming that material preferred directions are known. We showed that equilibrium equations in the absence of body forces and for arbitrary position-dependent elastic moduli impose restrictions on both the displacement field and the inhomogeneities of the elastic moduli in the form of a system of PDEs, which we call universality constraints. We observed that the universality constraints of inhomogeneous solids include those of homogeneous solids. For each symmetry class and its known universal displacements, we characterized the corresponding universal inhomogeneities. It is known that the larger the symmetry group, the larger the space of universal displacements [21]. We showed that the larger the symmetry group, the smaller the space of universal inhomogeneities. In particular, it was shown that inhomogeneous isotropic and inhomogeneous cubic solids do not admit universal displacements. For the other six symmetry classes, there are enough freedom to allow the existence of universal displacements and we classified all the universal inhomogeneities of the other six symmetry classes. This work therefore completes the universal program of linear elasticity.

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Notes

1. There was a mistake in the case of family 0 deformations that was corrected in [16].
2. There was a mistake in the case of family 5 deformations that was corrected in [16].
3. This is the case in nonlinear elasticity as well [15,16].
4. All the symbolic computations in this paper were performed using Mathematica Version 13.0.0.0, Wolfram Research, Champaign, IL.
5. There is a typo in equation $(3.22)_2$ in [21]: $-c_2x_1x_3$ should read $-c_2x_2x_3$.
6. Note that there is a typo in [21], Proposition 3.6.

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