COMPUTING THE GROUP OF MINIMAL NON-DEGENERATE EXTENSIONS OF A SUPER-TANNAKIAN CATEGORY

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Abstract. We prove an analog of the Künneth formula for the groups of minimal non-degenerate extensions [26] of symmetric fusion categories. We describe in detail the structure of the group of minimal extensions of a pointed super-Tannakian fusion category. This description resembles that of the third cohomology group of a finite abelian group. We explicitly compute this group in several concrete examples.

1. Introduction

There is a notion of categorical "orthogonality" in a braided fusion category $\mathcal{C}$ [12, 29]. Namely, objects X, Y of $\mathcal{C}$ *centralize* each other if the squared braiding between them is identity, i.e., $c_{Y,X}c_{X,Y} = \mathrm{id}_{X\otimes Y}$, where c denotes the braiding of $\mathcal{C}$. For a fusion subcategory $\mathcal{B} \subset \mathcal{C}$, its *centralizer* in $\mathcal{B}$ is the fusion subcategory $\mathcal{B}' \subset \mathcal{C}$ consisting of all objects X centralizing every object Y in $\mathcal{B}$. When $\mathcal{C}$ is pointed, i.e., corresponds to a pre-metric group $(A,\, q)$, where q is a quadratic form on a finite abelian group A, fusion subcategories of $\mathcal{B}$ are in bijection with subgroups of A and the centralizers are identified with orthogonal complements. This notion allows interpreting many aspects of the theory of braided fusion categories in terms of "categorical linear algebra".

For example, a symmetric fusion subcategory $\mathcal{E} \subset \mathcal{C}$ satisfies $\mathcal{E} \subseteq \mathcal{E}'$ and so can be thought of as a categorical analog of a coisitropic subspace. When $\mathcal{C}$ is non-degenerate, one has $\mathcal{E} = \mathcal{E}'$, i.e., $\mathcal{E}$ is a *Lagrangian* subcategory, if and only if $\mathrm{FPdim}(\mathcal{E})^2 = \mathrm{FPdim}(\mathcal{C})$, where FPdim denotes the Frobenius-Perron dimension. An embedding $\mathcal{E} \hookrightarrow \mathcal{C}$ with this property will be called a *minimal non-degenerate extension* (or simply a *minimal extension*) of $\mathcal{E}$. Lan, Kong, and Wen observed in [26] that there is a natural product of minimal extensions of $\mathcal{E}$, so that the set of their equivalence classes is an abelian group $Mext(\mathcal{E})$ (in fact, minimal extensions of $\mathcal{E}$ form a symmetric 2-categorical group $\mathbf{Mext}(\mathcal{E})$, see Section 2.2).

Groups of minimal extensions were studied by many authors, including [1, 2, 3, 11, 18, 21, 26, 27, 30]. From the physics point of view, minimal extensions appear in the description of 2+1D topological orders and symmetry-protected trivial (SPT) orders [26], and symmetric invertible fermionic phases [1, 2]. It is known that for a Tannakian category $\mathcal{E} = \mathcal{R}ep(G)$, where G is a finite group, the group $Mext(\mathcal{E})$ is isomorphic to $H^3(G,\, \Bbbk^\times)$, the third cohomology group of G. For $\mathcal{E} = s\mathcal{V}ect$, the category of super-vector spaces, it is isomorphic to $\mathbb{Z}/16\mathbb{Z}$ (this statement is known in physics as Kitaev's 16-fold way [25]).

The goal of this paper is to describe the group of minimal extensions of a super-Tannakian fusion category $\mathcal{E}$ (with an emphasis on the case when $\mathcal{E}$ is pointed) and compute it in several concrete examples.

The main results of the present paper are the following.

In Section 3 we prove a version of the Künneth formula for $Mext(\mathcal{R}ep(G) \boxtimes \mathcal{E})$, where G is a finite group and $\mathcal{E}$ is a symmetric fusion category. Theorem 3.8 establishes a group isomorphism

$$Mext(\mathcal{R}ep(G) \boxtimes \mathcal{E}) \cong Mext(\mathcal{E}) \times 2\text{-}Fun(G, \mathbf{Pic}(\mathcal{E})), \tag{1}$$

where $2\text{-}Fun(G, \mathbf{Pic}(\mathcal{E}))$ is the group of monoidal 2-functors from G to the 2-categorical Picard group of $\mathcal{E}$. For $\mathcal{E} = \mathcal{R}ep(L)$ this recovers the familiar Künneth formula computing the third cohomology of the product $G \times L$.

In Section 4 we analyze the structure of the group $Mext(\mathcal{E})$ for a pointed symmetric category $\mathcal{E}$. We consider a filtration

$$Mext_{triv}(\mathcal{E}) \subset Mext_{pt}(\mathcal{E}) \subset Mext_{int}(\mathcal{E}) \subset Mext(\mathcal{E}), \tag{2}$$

consisting, respectively, of the subgroups of trivial, pointed, and integral minimal extensions of $\mathcal{E}$, and compute its composition factors in Theorem 4.18. This description of $Mext(\mathcal{E})$ generalizes that of the third cohomology group $H^3(A, \Bbbk^\times)$ of a finite abelian group A [8, 28]. A new feature is the appearance of cohomological obstructions from the theory of graded extensions [6, 17].

Finally, in Section 5 we apply our results to compute the group of minimal extensions of concrete examples of super-Tannakian categories, namely, $\mathcal{R}ep(\mathbb{Z}_{2n}^f)$ and $\mathcal{R}ep(\mathbb{Z}_2 \times \mathbb{Z}_2^f)$.

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2. Preliminaries

In this paper, we work over an algebraically closed field $\Bbbk$ of characteristic 0. We adapt the following font convention for higher categorical groups: we use italics (G) to denote ordinary groups, calligraphic ($\mathcal{G}$) for categorical groups, and boldface $\mathbf{G}$ for 2-categorical groups. We use the same name in different fonts to denote the truncations of a given 2-categorical group. For example, we write $Pic(\mathcal{E}) = \pi_0(\mathbf{Pic}(\mathcal{E}))$ and $\mathcal{P}ic(\mathcal{E}) = \pi_{\le 1}(\mathbf{Pic}(\mathcal{E}))$ for the truncations of the 2-categorical Picard group $\mathbf{Pic}(\mathcal{E})$.

2.1. **Symmetric fusion categories and their Picard groups.** We refer the reader to [13, 16] for the basics of the theory of braided fusion categories.

By Deligne's theorem [10], symmetric fusion categories are parameterized by pairs (G, t), where G is a finite group and $t \in Z(G)$ is a central element such that $t^2 = 1$. The corresponding category $\mathcal{R}ep(G, t)$ consists of finite-dimensional representations of G, with the usual tensor product and braiding given by

$$c_{V,W}(v \otimes w) = \begin{cases} -w \otimes v, & \text{if } t|_V = -1 \text{ and } t|_W = -1, \\ w \otimes v, & \text{otherwise,} \end{cases} \tag{3}$$

for all irreducible representations V, W of G, where $v \in V$, $w \in W$. This category is called *Tannakian* if $t = 1$ (so it is simply $\mathcal{R}ep(G)$) and *super-Tannakian* if $t \neq 1$. When G contains a unique automorphism orbit of central elements of order 2 we will use notation $\mathcal{R}ep(G^f)$ for $\mathcal{R}ep(G, t)$ which is common in physics. Here f stands for "fermionic". For example, $s\mathcal{V}ect = \mathcal{R}ep(\mathbb{Z}_2^f)$.

Let $\mathcal{B}$ be a braided fusion category. The 2-categorical *Picard group* $\mathbf{Pic}(\mathcal{B})$ [17] is formed by invertible $\mathcal{B}$-module categories with the tensor product $\boxtimes_{\mathcal{B}}$. Its 1-categorical truncation $\mathcal{P}ic(\mathcal{B}) := \pi_{\le 1}(\mathbf{Pic}(\mathcal{B}))$ is equivalent to the categorical group $\mathcal{A}ut(\mathcal{Z}(\mathcal{B}); \mathcal{B})$ of braided tensor autoequivalences of $\mathcal{Z}(\mathcal{B})$ trivializable on $\mathcal{B}$ [5].

The Picard group of a symmetric fusion category was determined by Carnovale in [4]:

$$Pic(\mathcal{R}ep(G, t)) = \begin{cases} H^2(G, \Bbbk^\times) & \text{if } t = 1, \\ H^2(G, t, \Bbbk^\times) & \text{if } t \neq 1 \text{ and } \langle t \rangle \text{ is not a direct summand of } G, \\ H^2(G, t, \Bbbk^\times) \times \mathbb{Z}_2 & \text{if } t \neq 1 \text{ and } \langle t \rangle \text{ is a direct summand of } G. \end{cases} \tag{4}$$

The description in [4] was given in terms of Azumaya algebras. In terms of module categories, the elements of $H^2(G, \Bbbk^\times)$ correspond to module categories $\mathcal{R}ep(\Bbbk_\mu[G])$ of projective representations of G with a fixed 2-cocycle $\mu \in Z^2(G, \Bbbk^\times)$ and the generator of $\mathbb{Z}_2$ is $\mathcal{R}ep(G_0)$, where $G = G_0 \times \langle t \rangle$. Here the action of $\mathcal{R}ep(G)$ on $\mathcal{R}ep(\Bbbk_\mu[G])$ is defined by taking the tensor product of a linear representation of G with a projective representation. The action on $\mathcal{R}ep(G_0)$ is defined by restricting a representation of G to G_0 and then taking the tensor product.

The group $H^2(G, t, \Bbbk^\times)$ is defined as follows (see [4] and also [6]). There is a canonical bilinear map

$$H^2(G, \Bbbk^\times) \times G \to \mathbb{Z}/2\mathbb{Z} : (\mu, x) \mapsto \xi_\mu(x), \tag{5}$$

where $\xi_\mu(x)$ is defined by the condition

$$(-1)^{\xi_\mu(x)} = \frac{\mu(x,\, t)}{\mu(t,\, x)}, \qquad \mu \in H^2(G,\, \Bbbk^\times),\, x \in G.$$

Introduce a new multiplication on $H^2(G,\, \Bbbk^\times)$ by

$$\mu * \nu(x,\, y) = \mu(x,\, y)\nu(x,\, y)\,(-1)^{\xi_\mu(x)\xi_\nu(y)}, \qquad x, y \in G, \tag{6}$$

on representatives μ, ν of cohomology classes in $H^2(G,\, \Bbbk^\times)$. The resulting group will be denoted $H^2(G,\, t,\, \Bbbk^\times)$. It is non-canonically isomorphic to $H^2(G,\, \Bbbk^\times)$, see [4] for details.

There is a subgroup $Pic(\mathcal{R}ep(G,\, t))_{int} \subset Pic(\mathcal{R}ep(G,\, t))$ consisting of *integral* module categories, i.e., those in which all objects have integral Frobenius-Perron dimension. We have

$$Pic(\mathcal{R}ep(G,\, t))_{int} = H^2(G,\, t,\, \Bbbk^\times). \tag{7}$$

The *braided 2-categorical Picard group* $\mathbf{Pic_{br}}(\mathcal{E})$ [6] of a symmetric fusion category $\mathcal{E}$ consists of invertible objects in the 2-center of the monoidal 2-category of $\mathcal{E}$-module categories. The underlying braided categorical group $\mathcal{P}ic_{br}(\mathcal{E})$ was described in [6, Section 6]. One has $Pic_{br}(\mathcal{E}) \cong Pic(\mathcal{E}) \times Aut_\otimes(\mathrm{id}_\mathcal{E})$ and the corresponding quadratic form

$$Q_\mathcal{E} : Pic_{br}(\mathcal{E}) \to Inv(\mathcal{E}) \tag{8}$$

is explicitly computed in [6, Proposition 6.11]. In particular, for $\mathcal{E} = \mathcal{R}ep(G,\, t)$ the integral part of $Pic_{br}(\mathcal{E})$ is $H^2(G,\, t,\, \Bbbk^\times) \times Z(G)$, where $Z(G)$ denotes the center of G. In this case, $Aut_\otimes(\mathrm{id}_\mathcal{E}) = \mathsf{Hom}(G,\, \Bbbk^\times)$ and the restriction of the quadratic form (8) on the integral part of $Pic_{br}(\mathcal{E})$ is given by

$$Q_\mathcal{E}(\mu,\, z) = \frac{\mu(zt^{\xi_\mu(z)+1},\, -)}{\mu(-,\, zt^{\xi_\mu(z)+1})} \tag{9}$$

for all $\mu \in H^2(G,\, t,\, \Bbbk^\times)$ and $z \in Z(G)$.

2.2. **The 2-categorical group of minimal extensions of a symmetric category.** We recall the definition given in [26]. Let $\mathcal{E}$ be a symmetric fusion category. An embedding $\mathcal{E} \hookrightarrow \mathcal{C}$ into a non-degenerate braided fusion category $\mathcal{C}$ is called a *minimal non-degenerate extension* (or, simply, a *minimal extension*) of $\mathcal{E}$ if the latter coincides with its centralizer in $\mathcal{C}$, i.e., $\mathcal{E} = \mathcal{E}'$, where

$$\mathcal{E}' = \{X \in \mathcal{C} \mid c_{YX} \circ c_{XY} = \mathrm{id}_{X\otimes Y} \text{ for all } Y \in \mathcal{E}\}.$$

This condition is equivalent to the equality $\mathrm{FPdim}(\mathcal{C}) = \mathrm{FPdim}(\mathcal{E})^2$, where FPdim denotes the Frobenius-Perron dimension of the category. Minimal extensions of $\mathcal{E}$ form a 2-groupoid $\mathbf{Mext}(\mathcal{E})$. An isomorphism between minimal extensions $\mathcal{E} \hookrightarrow \mathcal{C}_1$ and $\mathcal{E} \hookrightarrow \mathcal{C}_2$ is a braided equivalence $\mathcal{C}_1 \xrightarrow{\sim} \mathcal{C}_2$ that restricts to the identity on $\mathcal{E}$. A 2-isomorphism is a natural isomorphism of equivalences, again identical on $\mathcal{E}$.

There is a natural tensor product of minimal extensions of $\mathcal{E}$. Namely, let $\mathcal{E} \hookrightarrow \mathcal{C}_1$ and $\mathcal{E} \hookrightarrow \mathcal{C}_2$ be minimal extensions. Then $\mathcal{E} \boxtimes \mathcal{E}$ embeds into $\mathcal{C}_1 \boxtimes \mathcal{C}_2$. Since $\mathcal{E}$ is symmetric, the tensor product $\otimes : \mathcal{E} \boxtimes \mathcal{E} \to \mathcal{E}$ is a braided tensor functor. Its adjoint sends the unit object $\mathbf{1}$ to an étale (i.e., separable commutative) algebra $A \in \mathcal{E} \boxtimes \mathcal{E}$. The fusion category $(\mathcal{C}_1 \boxtimes \mathcal{C}_2)_A$ of A-modules in $\mathcal{C}_1 \boxtimes \mathcal{C}_2$ contains a *braided* fusion subcategory

$$\mathcal{C}_1 \boxdot \mathcal{C}_2 := (\mathcal{C}_1 \boxtimes \mathcal{C}_2)_A^0$$

of *local* modules, i.e., of A-modules $(V, \rho : A \otimes V \to V)$ with $c_{X,A} c_{A,X} \rho = \rho$. Note that $\mathcal{E} \cong (\mathcal{E} \boxtimes \mathcal{E})_A$ is embedded into $\mathcal{C}_1 \boxdot \mathcal{C}_2$. The resulting embedding $\mathcal{E} \hookrightarrow \mathcal{C}_1 \boxdot \mathcal{C}_2$ is, by definition, the tensor product of $\mathcal{E} \hookrightarrow \mathcal{C}_1$ and $\mathcal{E} \hookrightarrow \mathcal{C}_2$. The unit object for this tensor product is $\mathcal{E} \hookrightarrow \mathcal{Z}(\mathcal{E})$, the embedding of $\mathcal{E}$ into its Drinfeld center. The inverse of the embedding $\mathcal{E} \hookrightarrow \mathcal{C}$ is $\mathcal{E} \hookrightarrow \mathcal{C}^{\text{rev}}$, where $\mathcal{C}^{\text{rev}}$ coincides with $\mathcal{C}$ as a fusion category with a braiding obtained by reversing the braiding of $\mathcal{C}$, namely $c^{\text{rev}}_{X,Y} = c^{-1}_{Y,X}$, $X, Y \in \mathcal{C}$. We refer the reader to [26, Section 4.2] for details.

The associativity equivalences for the product $\boxdot$ are constructed as follows. The usual associativity constraint of $\mathcal{E}$ yields a natural isomorphism of braided tensor functors

$$\otimes \circ (\otimes \boxtimes \mathrm{id}_{\mathcal{E}}) \cong \otimes \circ (\mathrm{id}_{\mathcal{E}} \boxtimes \otimes) : \mathcal{E} \boxtimes \mathcal{E} \boxtimes \mathcal{E} \to \mathcal{E}$$

and, hence, an isomorphism of the corresponding étale algebras in $\mathcal{E} \boxtimes \mathcal{E} \boxtimes \mathcal{E}$. This, in turn, gives an equivalence between categories of local modules, and, hence, between the triple products of minimal extensions. These structures turn $\mathbf{Mext}(\mathcal{E})$ into a 2-categorical group.

As agreed above, we denote $Mext(\mathcal{E}) = \pi_0(\mathbf{Mext}(\mathcal{E}))$ and $\mathcal{M}ext(\mathcal{E}) = \pi_{\leq 1}(\mathbf{Mext}(\mathcal{E}))$, the truncations of $\mathbf{Mext}(\mathcal{E})$.

Example 2.1. Let G be a finite group. The group $Mext(\mathcal{R}ep(G))$ was computed in [26, Section 4.3]. Namely, a typical element of this group is a twisted Drinfeld double of G:

$$\mathcal{R}ep(G) \hookrightarrow \mathcal{Z}(\mathsf{Vec}_G^{\omega}), \qquad \omega \in Z^3(G, \Bbbk^\times).$$

The product of these extensions corresponds to the product of 3-cocycles. Furthermore, extensions corresponding to 3-cocycles ω_1, ω_2 are isomorphic if and only if ω_1, ω_2 are cohomologous. Thus,

$$Mext(\mathcal{R}ep(G)) \cong H^3(G, \Bbbk^\times). \tag{10}$$

This result can also be deduced from [12, Section 4.4.10] since for any minimal extension $\mathcal{R}ep(G) \hookrightarrow \mathcal{C}$, the image of $\mathcal{R}ep(G)$ is a Lagrangian subcategory of $\mathcal{C}$.

Example 2.2. It was shown independently in [3], [7, Proposition 5.14], [25], and [26, Theorem 4.25] that

$$Mext(s\mathcal{V}ect) \cong \mathbb{Z}_{16}. \tag{11}$$

This statement is known as Kitaev's 16-fold way [25]. Any *Ising* category, i.e., a non-pointed braided fusion category of dimension 4 [13, Appendix B], is a generator of this group. Other elements of $Mext(\mathcal{sVect})$ are pointed braided fusion categories coming from metric groups $(A,\, q)$ of order 4 such that there exists $u \in A$ with $q(u) = -1$.

The isomorphism (11) can be identified with

$$Mext(\mathcal{sVect}) \xrightarrow{\sim} \{\xi \in \Bbbk^\times \mid \xi^{16} = 1\} : \mathcal{C} \mapsto \xi(\mathcal{C}), \tag{12}$$

where $\xi(\mathcal{C})$ is the central charge of the category $\mathcal{C}$. Thus, the class of a minimal non-degenerate extension of $\mathcal{sVect}$ is completely determined by its central charge.

Let $\mathcal{Rep}(G,\, t)$ be a super-Tannakian category. It contains a unique maximal Tannakian subcategory $\mathcal{T} = \mathcal{Rep}(G/\langle t\rangle)$. If $\mathcal{Rep}(G,\, t) \hookrightarrow \mathcal{C}$ is a minimal extension then the de-equivariantization $\mathcal{C}^0 := \mathcal{T}' \boxtimes_{\mathcal{T}} \mathcal{Vect}$ is a minimal extension of $\mathcal{sVect}$. The assignment

$$w_{(G,t)} : Mext(\mathcal{Rep}(G,\, t)) \to Mext(\mathcal{sVect}) : \mathcal{C} \mapsto \mathcal{C}^0 \tag{13}$$

is a group homomorphism [26, Section 5.2]. As in (12), this homomorphism is identified with taking the central charge.

By [18, Corollary 4.9], $w_{(G,t)}$ is surjective if and only if $\langle t\rangle$ is a direct summand of G. In this case, the above homomorphism splits, i.e., $Mext(\mathcal{sVect})$ is a direct summand of $Mext(\mathcal{Rep}(G,\, t))$.

2.3. **Central and braided graded extensions.** Let $\mathcal{B}$ be a braided fusion category. Let G be a finite group. A *central* G-graded extension of $\mathcal{B}$ is a G-graded fusion category

$$\mathcal{C} = \bigoplus_g \mathcal{C}_g, \qquad \mathcal{C}_e = \mathcal{B}, \tag{14}$$

along with an embedding $\mathcal{B} \hookrightarrow \mathcal{Z}(\mathcal{C})$. It was shown in [22] that a central G-graded extension is the same thing as a G-crossed braided extension. By [6, 17] the 2-groupoid of such extensions is equivalent to the 2-groupoid of monoidal 2-functors $G \to \mathbf{Pic}(\mathcal{B})$. For such an extension there is canonical action of G on the trivial component given by the composition

$$G \to \mathcal{Pic}(\mathcal{B}) \to \mathcal{Aut}^{br}(\mathcal{B}), \tag{15}$$

where the first functor corresponds to the graded extension (14) and the second one is the canonical monoidal functor associated to $\mathcal{B}$ [5].

Now let A be a finite abelian group. Braided A-graded extensions of a braided fusion category $\mathcal{B}$ were classified in [6]. The 2-groupoid of such extensions is equivalent to the 2-groupoid of braided monoidal 2-functors $A \to \mathbf{Pic}_{\mathbf{br}}(\mathcal{B})$, where the latter is the braided 2-categorical group of invertible *braided* $\mathcal{B}$-module categories.

3. Central graded extensions and the Künneth formula

3.1. **The group of monoidal 2-functors to a braided 2-categorical group.** Let G be a group and let $\mathcal{G}$ be a braided categorical group. Let $C,\, C' : G \to \mathcal{G}$ be monoidal functors, where C is given by $x \mapsto \mathcal{C}_x$ with the monoidal structure $M_{x,y} : \mathcal{C}_x \otimes \mathcal{C}_y \xrightarrow{\sim} \mathcal{C}_{xy}$ and C' is given by $x \mapsto \mathcal{C}'_x$ with the monoidal structure $M'_{x,y} : \mathcal{C}'_x \otimes \mathcal{C}'_y \xrightarrow{\sim} \mathcal{C}'_{xy}$, $x, y \in G$. Clearly, such functors must factor through the commutator subgroup of G.

Define a monoidal functor

$$\tilde{C} := C \otimes C' : G \to \mathcal{G} : \qquad x \mapsto \mathcal{C}_x \otimes \mathcal{C}'_x. \tag{16}$$

with the monoidal structure

$$\tilde{M}_{x,y} : \mathcal{C}_x \otimes \mathcal{C}'_x \otimes \mathcal{C}_y \otimes \mathcal{C}'_y \xrightarrow{B_{x',y}} \mathcal{C}_x \otimes \mathcal{C}_y \otimes \mathcal{C}'_x \otimes \mathcal{C}'_y \xrightarrow{M_{x,y} \otimes M'_{x,y}} \mathcal{C}_{xy} \otimes \mathcal{C}'_{xy}, \qquad x,\, y \in G. \tag{17}$$

Here $B_{x',y}$ denotes the braiding in $\mathcal{G}$ between $\mathcal{C}'_x$ and $\mathcal{C}_y$. With this product, the isomorphism classes of such braided monoidal functors form a categorical group which we denote $\mathcal{F}un(G,\, \mathcal{G})$. The identity element of this group is the trivial functor and the inverse of $C : x \mapsto \mathcal{C}_x$ is $C^{-1} : x \mapsto \mathcal{C}_x^{-1}$. If $\mathcal{G}$ is symmetric, the underlying group $Fun(G,\, \mathcal{G})$ is abelian.

There is an obvious short exact sequence

$$0 \to H^2(G,\, \pi_1(\mathcal{G})) \to Fun(G,\, \mathcal{G}) \to \mathsf{Hom}(G,\, \pi_0(\mathcal{G})) \to 0. \tag{18}$$

Here $H^2(G,\, \pi_1(\mathcal{G}))$ is isomorphic to the group of monoidal functor structures on the trivial functor. This sequence does not split in general because the braiding of $\mathcal{G}$ may be non-trivial.

Now let $\mathbf{G}$ be a braided 2-categorical group. Let $\pi_0(\mathbf{G})$, $\pi_1(\mathbf{G})$, $\pi_2(\mathbf{G})$ denote, respectively, the group of invertible objects of $\mathbf{G}$, the group of automorphisms of the unit object $\mathbf{1_G}$, and the group of automorphisms of $\mathrm{id}_{\mathbf{1_G}}$. Let $\pi_{\leq 1}(\mathbf{G})$ denote the braided categorical group obtained by truncating $\mathbf{G}$, whose objects are objects of $\mathbf{G}$ and morphisms are isomorphism classes of 1-cells in $\mathbf{G}$. Let $\pi_{\geq 1}(\mathbf{G})$ denote the symmetric categorical group of 1-automorphisms of the unit object of $\mathbf{G}$.

There is a 2-categorical analog of the above construction of a categorical group of monoidal functors. It was explained in [6, Section 2.8] that isomorphisms classes of monoidal 2-functors from G to $\mathbf{G}$ also form a categorical group denoted *2-*$\mathcal{F}un(G,\, \mathbf{G})$. Namely, if $C,\, C' : G \to \mathbf{G}$ are such 2-functors, then the monoidal structure of the product $C \otimes C'$ is defined as above and the structural associativity 2-cells are given by [6, diagram (2.78)]. These cells involve the associativity 2-cells of C and C' and the structure 2-cells of $\mathbf{G}$.

The group 2-$Fun(G,\, \mathbf{G})$ fits into the following exact sequence [6, Theorem 2.38]:

$$H^1(G,\, \pi_1(\mathbf{G})) \xrightarrow{\alpha} H^3(G,\, \pi_2(\mathbf{G})) \xrightarrow{\beta} 2\text{-}Fun(G,\, \mathbf{G}) \xrightarrow{\gamma} Fun(G,\, \pi_{\leq 1}(\mathbf{G})) \xrightarrow{\delta} H^4(G,\, \pi_2(\mathbf{G})). \tag{19}$$

Here α assigns to a homomorphism $G \to \pi_1(\mathbf{G})$ the corresponding pullback of the associator of the categorical group $\pi_{\geq 1}(\mathbf{G})$ (the latter is an element of $H^3(\pi_1(\mathbf{G}),\, \pi_2(\mathbf{G}))$), β assigns to a third cohomology class the monoidal 2-functor structure on the trivial 2-functor, γ assigns to a monoidal 2-functor $G \to \mathbf{G}$ the underlying 1-functor to the truncation of $\mathbf{G}$, and δ gives the obstruction for a given monoidal functor to extend to a monoidal 2-functor.

Example 3.1. Let $\mathcal{E}$ be a symmetric fusion category and let $\mathbf{G} = \mathbf{Pic}(\mathcal{E})$ be the symmetric 2-categorical Picard group of $\mathcal{E}$. We have $\pi_{\leq 1}(\mathbf{Pic}(\mathcal{E})) = \mathcal{P}ic(\mathcal{E})$, the symmetric categorical Picard group of $\mathcal{E}$, and $\pi_{\geq 1}(\mathbf{Pic}(\mathcal{E})) = \mathcal{I}nv(\mathcal{E})$, the symmetric categorical group of invertible objects of $\mathcal{E}$. Since the associator of the latter is trivial, the exact sequence (19) becomes

$$0 \xrightarrow{\alpha} H^3(G,\, \Bbbk^\times) \xrightarrow{\beta} 2\text{-}Fun(G,\, \mathbf{Pic}(\mathcal{E})) \xrightarrow{\gamma} Fun(G,\, \mathcal{P}ic(\mathcal{E})) \xrightarrow{\delta} H^4(G,\, \Bbbk^\times). \tag{20}$$

Example 3.2. The restriction of the obstruction map δ from Example 3.1 to the subgroup $H^2(G,\, Inv(\mathcal{E}))$ of $Fun(G,\, \mathcal{P}ic(\mathcal{E}))$ is given by the composition

$$H^2(G,\, Inv(\mathcal{E})) \xrightarrow{q} H^2(G,\, \mathbb{Z}/2\mathbb{Z}) \xrightarrow{\cup^2} H^4(G,\, \mathbb{Z}/2\mathbb{Z}) \xrightarrow{\iota} H^4(G,\, \Bbbk^\times), \tag{21}$$

where $q : Inv(\mathcal{E}) \to \mathbb{Z}/2\mathbb{Z}$ is the quadratic homomorphism of the symmetric categorical group $Inv(\mathcal{E})$, $\cup^2 : H^2(G,\, \mathbb{Z}/2\mathbb{Z}) \to H^4(G,\, \mathbb{Z}/2\mathbb{Z})$ is the cup square in $H^*(G,\, \mathbb{Z}/2\mathbb{Z})$ (note that it is a homomorphism), and ι is induced by the inclusion of the coefficients $\mathbb{Z}/2\mathbb{Z} \to \Bbbk^\times : n \mapsto (-1)^n$. Indeed, by [17, Section 8.7] (see also [6, (8.45)]), the obstruction $\delta(L)$ for $L \in H^2(G,\, Inv(\mathcal{E}))$ is given by

$$\delta(L)(x,y,z,w) = c_{L(x,y),L(z,w)}, \qquad x,y,z,w \in G, \tag{22}$$

where c is the braiding in $Inv(\mathcal{E})$. Since $c_{X,Y} = (-1)^{q(X)q(Y)}$ for all $X, Y \in Inv(\mathcal{E})$, the formula (22) translates to (21).

For $\mathcal{E} = s\mathcal{V}ect$ formula (21) describing the obstruction map

$$H^2(G,\, Inv(\mathcal{E})) = H^2(G,\, \mathbb{Z}/2\mathbb{Z}) \to H^4(G,\, \Bbbk^\times)$$

in terms of the cup product appeared in [1, Section VI.B].

3.2. **The embedding $\mathbf{Pic}(\mathcal{E}) \hookrightarrow \mathbf{Pic}(\mathcal{Z}(\mathcal{E}))$.** Let $\mathcal{E}$ be a symmetric fusion category. The induction 2-functor

$$\mathrm{Ind} : \mathbf{Pic}(\mathcal{E}) \to \mathbf{Pic}(\mathcal{Z}(\mathcal{E})) : \mathcal{M} \mapsto \mathcal{Z}(\mathcal{E}) \boxtimes_{\mathcal{E}} \mathcal{M} \tag{23}$$

is a monoidal 2-embedding of categorical groups. On the level of 1-cells this functor embeds $Inv(\mathcal{E})$ into $Inv(\mathcal{Z}(\mathcal{E}))$.

There is an equivalence of categorical groups [6, 17]

$$\partial : \pi_{\leq 1}(\mathbf{Pic}(\mathcal{Z}(\mathcal{E})) \to \mathcal{A}ut^{br}(\mathcal{Z}(\mathcal{E})) : \mathcal{M} \mapsto \partial(\mathcal{M}), \tag{24}$$

such that $\alpha^+_{\mathcal{M}} \cong \alpha^-_{\mathcal{M}} \circ \partial(\mathcal{M})$, where $\alpha^{\pm}_{\mathcal{M}} : \mathcal{Z}(\mathcal{E}) \xrightarrow{\sim} \mathcal{F}un_{\mathcal{Z}(\mathcal{E})}(\mathcal{M}, \mathcal{M})$ are two equivalences defined, respectively, using the braiding of $\mathcal{Z}(\mathcal{E})$ and its reverse. This equivalence ∂ sends $Z \in Inv(\mathcal{Z}(\mathcal{E}))$ to $\partial(Z) \in \mathrm{id}_{\mathcal{Z}(\mathcal{E})}$ defined by $\partial(Z)_X = c_{Z,X}c_{X,Z}$ for all $X \in \mathcal{Z}(\mathcal{E})$.

It was shown in [6] that the monoidal equivalence (24) restricts to a monoidal equivalence between $\pi_{\leq 1}(\mathbf{Pic}(\mathcal{E}))$ and the categorical group $Aut^{br}(\mathcal{Z}(\mathcal{E}); \mathcal{E})$ whose objects are autoequivalences $\alpha \in Aut^{br}(\mathcal{Z}(\mathcal{E}))$ such that $\alpha|_{\mathcal{E}} = \mathrm{id}_{\mathcal{E}}$ and automorphisms of the unit object are $\nu \in \mathrm{id}_{\mathcal{Z}(\mathcal{E})}$ such that $\nu_X = 1$ for all $X \in \mathcal{E}$. This characterizes the objects and 1-morphisms of $\mathbf{Pic}(\mathcal{Z}(\mathcal{E}))$ induced from $\mathbf{Pic}(\mathcal{E})$.

For a monoidal 2-functor

$$M : G \to \mathbf{Pic}(\mathcal{Z}(\mathcal{E})) : g \mapsto \mathcal{M}(g) \tag{25}$$

we denote $\partial_M : G \to Aut^{br}(\mathcal{Z}(\mathcal{E})) : g \mapsto \partial(\mathcal{M}(g))$ the corresponding action of G.

Lemma 3.3. *Suppose that the monoidal 2-functor (25) is such that ∂_M restricts to the trivial action of G on $\mathcal{E} \subset \mathcal{Z}(\mathcal{E})$. Then there is a monoidal 2-functor $M_0 : G \to \mathbf{Pic}(\mathcal{E})$ such that (25) is isomorphic to the composition*

$$G \xrightarrow{M_0} \mathbf{Pic}(\mathcal{E}) \xrightarrow{\mathrm{Ind}} \mathbf{Pic}(\mathcal{Z}(\mathcal{E})).$$

Proof. It follows from the hypothesis that M yields a monoidal functor

$$G \to Aut^{br}(\mathcal{Z}(\mathcal{E}); \mathcal{E}) \cong \pi_{\leq 1}(\mathbf{Pic}(\mathcal{E})). \tag{26}$$

Its composition with Ind coincides with M on the level of objects and 1-morphisms. Hence, (26) gives rise to a monoidal 2-functor $M_0 : G \to \mathbf{Pic}(\mathcal{E})$, as required. □

3.3. **The group of central extensions of a symmetric fusion category.** Recall that for an étale algebra A in $\mathcal{Z}(\mathcal{C})$, where $\mathcal{C}$ is a fusion category, the category $\mathcal{C}_A$ of A-modules in $\mathcal{C}$ is a fusion category. For a braided fusion category $\mathcal{B}$, let $\mathcal{Z}_{sym}(\mathcal{B})$ denote the symmetric center of $\mathcal{B}$. If A is an étale algebra in $\mathcal{Z}_{sym}(\mathcal{B})$, then $\mathcal{B}_A$ is a braided fusion category. The tensor product over a symmetric fusion category $\mathcal{E}$ is a special case of this construction. Indeed, the tensor product $\otimes : \mathcal{E} \boxtimes \mathcal{E} \to \mathcal{E}$ is a braided tensor functor. Let $I : \mathcal{E} \to \mathcal{E} \boxtimes \mathcal{E}$ be its adjoint, then $A := I(1)$ is a canonical étale algebra in $\mathcal{E} \boxtimes \mathcal{E}$. If $\mathcal{E} \hookrightarrow \mathcal{C}_1$, $\mathcal{E} \hookrightarrow \mathcal{C}_2$ are central inclusions of $\mathcal{E}$ into fusion categories $\mathcal{C}_1, \mathcal{C}_2$ then $\mathcal{C}_1 \boxtimes_{\mathcal{E}} \mathcal{C}_2 = (\mathcal{C}_1 \boxtimes \mathcal{C}_2)_A$ is a fusion category. If $\mathcal{E} \hookrightarrow \mathcal{B}_1$, $\mathcal{E} \hookrightarrow \mathcal{B}_2$ are inclusions into symmetric centers of braided fusion categories $\mathcal{B}_1, \mathcal{B}_2$ then $\mathcal{B}_1 \boxtimes_{\mathcal{E}} \mathcal{B}_2 = (\mathcal{B}_1 \boxtimes \mathcal{B}_2)_A$ is a braided fusion category.

Lemma 3.4. *Let $\mathcal{B} \subset \mathcal{C}$ be a central extension of a braided fusion category $\mathcal{B}$ and let A be an etale algebra in $\mathcal{Z}_{sym}(\mathcal{B})$. Then $\mathcal{B}_A \subset \mathcal{C}_A$ is a central extension.*

Proof. This is straightforward since the half-braiding between objects of $\mathcal{B}$ and $\mathcal{C}$ induces the one between objects of $\mathcal{B}_A$ and $\mathcal{C}_A$. □

Let $\mathbf{Pic}(\mathcal{E})$ denote the braided 2-categorical Picard group of $\mathcal{E}$ and let $\mathbf{2\text{-}Fun}(G,\, \mathbf{Pic}(\mathcal{E}))$ denote the 2-categorical group of monoidal functors from G to $\mathbf{Pic}(\mathcal{E})$.

It was observed in [6] that the 2-groupoid $\mathbf{Ex_{ctr}}(G,\, \mathcal{E})$ of central G-graded extensions of $\mathcal{E}$ is a braided 2-categorical group. The tensor product of G-graded central extensions $\mathcal{E} \hookrightarrow \mathcal{C}^1$ and $\mathcal{E} \hookrightarrow \mathcal{C}^2$ is defined as follows. Let A be the canonical étale algebra in $\mathcal{E} \boxtimes \mathcal{E}$ defined above. Lemma 3.4 applies to the central G-graded extension $\mathcal{E} \boxtimes \mathcal{E} \hookrightarrow \bigoplus_{g\in G} \mathcal{C}^1_g \boxtimes \mathcal{C}^2_g$, so we obtain a central G-graded extension

$$\mathcal{E} \cong (\mathcal{E} \boxtimes \mathcal{E})_A \hookrightarrow \bigoplus_{g\in G} (\mathcal{C}^1_g \boxtimes \mathcal{C}^2_g)_A = \bigoplus_{g\in G} \mathcal{C}^1_g \boxtimes_{\mathcal{E}} \mathcal{C}^2_g \tag{27}$$

which is the product of extensions $\mathcal{E} \hookrightarrow \mathcal{C}^1$ and $\mathcal{E} \hookrightarrow \mathcal{C}^2$. We will denote this product by $\mathcal{E} \hookrightarrow \mathcal{C}_1 \boxdot \mathcal{C}_2$.

Theorem 3.5. *There is a monoidal 2-equivalence of 2-categorical groups*

$$\mathbf{2\text{-}Fun}(G,\, \mathbf{Pic}(\mathcal{E})) \cong \mathbf{Ex_{ctr}}(G,\, \mathcal{E}). \tag{28}$$

Proof. The 2-equivalence is established for *braided* $\mathcal{E}$ in [17, Theorem 7.12] (see also [6, Theorem 8.13]). Namely, a monoidal 2-functor $G \to Pic(\mathcal{E}) : g \mapsto \mathcal{C}_g$ gives rise to a G-graded central extension $\bigoplus_{g\in G} \mathcal{C}_g$. For symmetric $\mathcal{E}$, the monoidal structure of this 2-equivalence is evident since the tensor products in $\mathbf{2\text{-}Fun}(G,\, \mathbf{Pic}(\mathcal{E}))$ and $\mathbf{Ex_{ctr}}(G,\, \mathcal{E})$ are defined by the very same formulas (cf. (16) and (27)) and have the same associativity 2-cells. □

3.4. **The center of a central extension of a symmetric fusion category.** Let $\mathcal{E}$ be a symmetric fusion category and let

$$\mathcal{C} = \bigoplus_{g\in G} \mathcal{C}_g, \qquad \mathcal{C}_e = \mathcal{E}, \tag{29}$$

be its central G-graded extension. It was shown in [19] that $\mathcal{Z}(\mathcal{C})$ is equivalent to a G-equivariantization of the relative center $\mathcal{Z}_{\mathcal{E}}(\mathcal{C})$. The latter is equivalent to the fusion category $\mathcal{Z}(\mathcal{E}) \boxtimes_{\mathcal{E}} \mathcal{C}$. It is a central G-graded extension of $\mathcal{Z}(\mathcal{E})$ corresponding to the following monoidal 2-functor

$$G \to \mathcal{P}ic(\mathcal{E}) \xrightarrow{Ind} \mathcal{P}ic(\mathcal{Z}(\mathcal{E})), \tag{30}$$

where the first functor corresponds to the central extension (29) and the second is the induction (23). The action of G on $\mathcal{Z}(\mathcal{E})$ is obtained by composing (30) with the canonical monoidal equivalence $\mathcal{P}ic(\mathcal{Z}(\mathcal{E})) \cong \mathcal{A}ut^{br}(\mathcal{Z}(\mathcal{E}))$. It follows from [6] (see Section 2.1) that this action restricts to the trivial action on $\mathcal{E} \subset \mathcal{Z}(\mathcal{E})$. Therefore, $\mathcal{Z}(\mathcal{C}) = (\mathcal{Z}(\mathcal{E}) \boxtimes_{\mathcal{E}} \mathcal{C})^G$ is a minimal extension of the symmetric fusion category $\mathcal{E}^G = \mathcal{R}ep(G) \boxtimes \mathcal{E}$.

Proposition 3.6. *The assignment*

$$\mathbf{Ex_{ctr}}(G, \mathcal{E}) \to \mathbf{Mext}(\mathcal{R}ep(G) \boxtimes \mathcal{E}) : \mathcal{C} \mapsto \mathcal{Z}(\mathcal{C}) \tag{31}$$

is a monoidal 2-functor.

Proof. For a braided fusion category $\mathcal{B}$ containing a subcategory equivalent to $\mathcal{R}ep(G)$, let $\mathcal{B}_G = \mathcal{B} \boxtimes_{\mathcal{R}ep(G)} \mathcal{V}ect$ denote the corresponding de-equivariantization. Using definitions of the tensor products in $\mathbf{Ex_{ctr}}(G, \mathcal{E})$ and $\mathbf{Mext}(\mathcal{R}ep(G) \boxtimes \mathcal{E})$ we obtain equivalences

$$\begin{aligned}
\mathcal{Z}(\mathcal{C}^1 \boxdot \mathcal{C}^2)_G &\cong \bigoplus_{g\in G} \mathcal{Z}(\mathcal{E}) \boxtimes_{\mathcal{E}} \mathcal{C}^1_g \boxtimes_{\mathcal{E}} \mathcal{C}^2_g \\
&\cong \bigoplus_{g\in G} (\mathcal{Z}(\mathcal{E}) \boxtimes \mathcal{Z}(\mathcal{E}))^0_A \boxtimes_{\mathcal{E}} \mathcal{C}^1_g \boxtimes_{\mathcal{E}} \mathcal{C}^2_g \\
&\cong \bigoplus_{g\in G} \big((\mathcal{Z}(\mathcal{E}) \boxtimes_{\mathcal{E}} \mathcal{C}^1_g) \boxtimes (\mathcal{Z}(\mathcal{E}) \boxtimes_{\mathcal{E}} \mathcal{C}^2_g)\big)^0_A \\
&\cong \big(\mathcal{Z}(\mathcal{C}^1) \boxdot \mathcal{Z}(\mathcal{C}^2)\big)_G
\end{aligned}$$

for all G-graded central extensions $\mathcal{C}^1, \mathcal{C}^2$ of $\mathcal{E}$. Here $\mathcal{B}^0_A$ denotes the category of local A-modules in $\mathcal{B}$. The symbol $\boxdot$ stands for the tensor product in both $\mathbf{Ex_{ctr}}$ and $\mathbf{Mext}$. Taking equivariantizations we get a canonical equivalence

$$\mathcal{Z}(\mathcal{C}^1 \boxdot \mathcal{C}^2) \cong \mathcal{Z}(\mathcal{C}^1) \boxdot \mathcal{Z}(\mathcal{C}^2)$$

in $\mathbf{Mext}(\mathcal{R}ep(G) \boxtimes \mathcal{E})$ that equips the 2-functor (31) with a canonical monoidal structure. □

3.5. **The Künneth formula.** Let G, L be finite groups.

Proposition 3.7. *There is a split short exact sequence*

$$\begin{aligned}
0 \to H^3(G, \Bbbk^\times) \oplus H^3(L, \Bbbk^\times) \oplus \big(\mathsf{Hom}(G, \Bbbk^\times) \otimes \mathsf{Hom}(L, \Bbbk^\times)\big) \to H^3(G \times L, \Bbbk^\times) \to \\
\mathsf{Hom}(L, H^2(G, \Bbbk^\times)) \oplus \mathsf{Hom}(G, H^2(L, \Bbbk^\times)) \to 0.
\end{aligned} \tag{32}$$

Proof. Using the short exact sequence $0 \to \mathbb{Z} \to \mathbb{Q} \to \mathbb{Q}/\mathbb{Z} \to 0$, we obtain isomorphisms

$$H^i(G, \Bbbk^\times) \cong H^i(G, \mathbb{Q}/\mathbb{Z}) \cong H^{i+1}(G, \mathbb{Z})$$

for any finite group G and $i \geq 1$. Also, $H^1(G, \mathbb{Z}) = 0$. Therefore, the sequence in question is obtained from the usual Künneth formula for integral cohomology. □

Below we generalize the Künneth formula (32). Namely, for a finite group G and a symmetric fusion category $\mathcal{E}$, we explain how to compute the group $Mext(\mathcal{R}ep(G) \boxtimes \mathcal{E})$.

Theorem 3.8. *There is a group isomorphism*

$$Mext(\mathcal{R}ep(G) \boxtimes \mathcal{E}) \cong Mext(\mathcal{E}) \times 2\text{-}Fun(G,\, \mathbf{Pic}(\mathcal{E})) \tag{33}$$

Proof. Let $\mathcal{C}$ be a minimal extension of $\mathcal{R}ep(G) \boxtimes \mathcal{E}$. Let $\widetilde{\mathcal{C}}$ denote the centralizer of $\mathcal{R}ep(G)$ in $\mathcal{C}$. The de-equivariantization $\widetilde{\mathcal{C}} \boxtimes_{\mathcal{R}ep(G)} \mathcal{V}ect$ is a minimal extension of $\mathcal{E}$ and the assignment

$$\mathbf{Mext}(\mathcal{R}ep(G) \boxtimes \mathcal{E}) \to \mathbf{Mext}(\mathcal{E}) : \mathcal{C} \mapsto \widetilde{\mathcal{C}} \boxtimes_{\mathcal{R}ep(G)} \mathcal{V}ect \tag{34}$$

is a monoidal 2-functor between categorical groups.

The associated group homomorphism

$$Mext(\mathcal{R}ep(G) \boxtimes \mathcal{E}) \to Mext(\mathcal{E}) \tag{35}$$

is split surjective, since for any $\mathcal{D} \in Mext(\mathcal{E})$ we have a minimal extension $\mathcal{R}ep(G) \boxtimes \mathcal{E} \hookrightarrow \mathcal{Z}(\mathcal{R}ep(G)) \boxtimes \mathcal{D}$.

Let $K(G,\, \mathcal{E})$ denote the kernel of (35). It remains to show that $2\text{-}Fun(G,\, \mathbf{Pic}(\mathcal{E})) = K(G,\, \mathcal{E})$. Theorem 3.5 combined with Proposition 3.6 gives an inclusion

$$2\text{-}Fun(G,\, \mathbf{Pic}(\mathcal{E})) \cong Ex_{ctr}(G,\, \mathcal{E}) \to K(G,\, \mathcal{E}).$$

Let us show that it is surjective. A minimal extension $\mathcal{R}ep(G) \boxtimes \mathcal{E} \hookrightarrow \mathcal{C}$ is in $K(G,\, \mathcal{E})$ if and only if its de-equivariantization $\mathcal{C}_G = \mathcal{C} \boxtimes_{\mathcal{R}ep(G)} \mathcal{V}ect$ is a G-graded central extension of $\mathcal{Z}(\mathcal{E})$ such that the action of G restricts trivially to the subcategory $\mathcal{E} \subset \mathcal{Z}(\mathcal{E})$. By Lemma 3.3, this means that the corresponding monoidal 2-functor $G \to \mathbf{Pic}(\mathcal{Z}(\mathcal{E}))$ is the composition of a monoidal 2-functor $F : G \to \mathbf{Pic}(\mathcal{E})$ and the induction 2-functor (23). As explained in Section 3.4, this means that $\mathcal{C} \cong \mathcal{Z}(\mathcal{A}_F)$, where $\mathcal{A}_F$ is the central extension of $\mathcal{E}$ corresponding to F. Thus, $\mathcal{C} \in K(G,\, \mathcal{E})$, as required. □

Remark 3.9. Recall that a G-*gauging* of a braided fusion category $\mathcal{B}$ is the equivariantization of a faithful G-crossed braided (i.e., G-graded central) extension of $\mathcal{B}$. Theorem 3.8, in particular, characterizes the centers of central G-extensions of a symmetric fusion category $\mathcal{E}$ as G-gaugings of $\mathcal{Z}(\mathcal{E})$ in which the associated action of G on $\mathcal{E} \subset \mathcal{Z}(\mathcal{E})$ is trivial. For $\mathcal{E} = \mathcal{V}ect$, one recovers from this result the classification of twisted group doubles from [12, Theorem 4.64].

Example 3.10. Let L be a finite group and set $\mathcal{E} = \mathcal{R}ep(L)$ in Theorem 3.8. We recover the Künneth formula (32) as follows. In this case, $Pic(\mathcal{E}) = H^2(L,\, \Bbbk^\times)$ and sequence (18) (with $\mathcal{G} = \mathcal{P}ic(\mathcal{E})$) splits:

$$Fun(G,\, \mathcal{P}ic(\mathcal{E})) \cong \mathsf{Hom}(G,\, H^2(L,\, \Bbbk^\times)) \oplus H^2(G,\, \widehat{L}), \tag{36}$$

where $\widehat{L} = \mathsf{Hom}(L, \Bbbk^\times)$. We claim that the obstruction δ in (19) vanishes. Indeed, it suffices to check that it vanishes on both summands of (36). Vanishing on the first one follows for the existence of a Schur covering group L^* of L [24, Section 2.1], since $\mathcal{R}ep(L^*)$ is a faithful $H^2(L, \Bbbk^\times)$-graded extension of $\mathcal{R}ep(L)$. Vanishing on the second one follows from Example 3.2, since for the Tannakian category $\mathcal{R}ep(L)$ the quadratic homomorphism $q : \widehat{L} \to \mathbb{Z}/2\mathbb{Z}$ is trivial.

Therefore, the sequence (20) becomes the following short exact sequence:

$$0 \to H^3(G, \Bbbk^\times) \to 2\text{-}Fun(G, \mathbf{Pic}(\mathcal{E})) \to \mathsf{Hom}(G, H^2(L, \Bbbk^\times)) \oplus H^2(G, \widehat{L}) \to 0.$$

By the universal coefficient theorem, the last summand is further decomposed as

$$H^2(G, \widehat{L}) \cong \mathsf{Ext}(G/G', \widehat{L}) \oplus \mathsf{Hom}(L, H^2(G, \Bbbk^\times)) \cong \left(\widehat{G} \otimes \widehat{L}\right) \oplus \mathsf{Hom}(L, H^2(G, \Bbbk^\times)),$$

where G' is the commutator subgroup of G. Combining this with Theorem 3.8, we recover all the summands of $H^3(G \times L) \cong Mext(\mathcal{R}ep(G) \boxtimes \mathcal{R}ep(L))$ in the Künneth formula (32).

Example 3.11. We will deal with minimal extensions of *pointed* symmetric fusion categories in Section 4. For now, let us note that Theorem 3.8 allows calculating of their groups of minimal extensions inductively as follows. Let $r(\mathcal{E})$ denote the finite rank of the group $Inv(\mathcal{E})$, i.e., the minimal number of its generators. For $r(\mathcal{E}) = 1$, the group $Mext(\mathcal{E})$ will be computed in Section 5.1. When $r(\mathcal{E}) > 1$ we can write

$$\mathcal{E} \cong \mathcal{R}ep(\mathbb{Z}_N) \boxtimes \mathcal{E}_1, \tag{37}$$

where $r(\mathcal{E}_1) = r(\mathcal{E}) - 1$. By Theorem 3.8,

$$Mext(\mathcal{E}) \cong Mext(\mathcal{E}_1) \oplus 2\text{-}Fun(\mathbb{Z}_N, \mathcal{P}ic(\mathcal{E}_1)).$$

The last summand is computed as follows. Since $H^4(\mathbb{Z}_N, \Bbbk^\times) = 0$ and $H^2(\mathbb{Z}_N, Inv(\mathcal{E}_!)) = \mathsf{Ext}(\mathbb{Z}_N, Inv(\mathcal{E}_1))$, sequences (18) and (20) become

$$0 \to \mathsf{Ext}(\mathbb{Z}_N, Inv(\mathcal{E}_1)) \to Fun(\mathbb{Z}_N, \mathcal{P}ic(\mathcal{E}_1)) \to \mathsf{Hom}(\mathbb{Z}_N, Pic(\mathcal{E}_1)) \to 0$$

and

$$0 \to H^3(\mathbb{Z}_N, \Bbbk^\times) \to 2\text{-}Fun(\mathbb{Z}_N, \mathcal{P}ic(\mathcal{E}_1)) \to Fun(\mathbb{Z}_N, \mathcal{P}ic(\mathcal{E}_1)) \to 0.$$

Thus, the direct complement of $Mext(\mathcal{E}_1)$ in $Mext(\mathcal{E})$ has a filtration with factors

$$H^3(\mathbb{Z}_N, \Bbbk^\times) \cong \mathbb{Z}_N, \quad \mathsf{Ext}(\mathbb{Z}_N, Inv(\mathcal{E}_1)), \quad \text{and} \quad \mathsf{Hom}(\mathbb{Z}_N, Pic(\mathcal{E}_1)). \tag{38}$$

We will see in Section 5.2 that the group $2\text{-}Fun(\mathbb{Z}_N, \mathcal{P}ic(\mathcal{E}_1))$ is not, in general, a direct sum of factors (38).

Remark 3.12. Let $\mathcal{E} = \mathcal{R}ep(G) \boxtimes s\mathcal{V}ect$. Since $Mext(s\mathcal{V}ect) \cong \mathbb{Z}_{16}$, Theorem 3.8 gives

$$Mext(\mathcal{R}ep(G) \boxtimes s\mathcal{V}ect) \cong \mathbb{Z}_{16} \oplus 2\text{-}Fun(G, \mathcal{P}ic(s\mathcal{V}ect)). \tag{39}$$

We have $Inv(s\mathcal{V}ect) \cong \mathbb{Z}_2$ and $Pic(s\mathcal{V}ect) \cong \mathbb{Z}_2$, so it follows from exact sequences (18) and (20) that the group 2-$Fun(G, \mathcal{P}ic(s\mathcal{V}ect))$ has a filtration with factors

$$H^3(G, \Bbbk^\times), \quad \mathsf{Ker}\left(H^2(G, \mathbb{Z}_2) \xrightarrow{\delta} H^4(G, \Bbbk^\times)\right), \quad \text{and} \quad H^1(G, \mathbb{Z}_2) \tag{40}$$

where δ is the obstruction map (21). Explicit formulas describing the product in this group are given in [1].

Remark 3.13. Let $\mathcal{E} = \mathcal{R}ep(\tilde{G}, t)$ be a general symmetric fusion category, where the group $\tilde{G}$ fits into a (not necessarily split) short exact sequence

$$1 \to \mathbb{Z}_2 = \langle t \rangle \to \tilde{G} \to G \to 1.$$

In this case, there is a parameterization of $Mext(\mathcal{E})$ by torsors over the cohomology groups listed in (40), see [1, Section VII.C] and [2, Table I]. These papers also contain explicit formulas for products of minimal extensions (i.e., symmetric invertible fermionic topological phases) in terms of this parameterization.

4. The group of minimal extensions of a pointed symmetric fusion category

From now on we let A be a finite abelian group and let $\mathcal{E} = \mathcal{R}ep(A, t)$ be a pointed symmetric fusion category.

4.1. A canonical grading on a minimal extension.

Proposition 4.1. *Let $\mathcal{E} \hookrightarrow \mathcal{C}$ be a minimal non-degenerate extension. Then $\mathcal{C}$ is faithfully A-graded:*

$$\mathcal{C} = \bigoplus_{x \in A} \mathcal{C}_x, \qquad \mathcal{C}_e = \mathcal{E}, \tag{41}$$

where $\mathcal{C}_x = \{X \in \mathcal{C} \mid c_{X,V} c_{V,X} = x|_V \otimes id_X \text{ for all } V \in \mathcal{E}\}$, $x \in A$.

Conversely, any A-graded braided extension of this form is a minimal non-degenerate extension of $\mathcal{E}$. Two such minimal extensions are equivalent if and only if they are equivalent as A-graded braided extensions of $\mathcal{E}$.

Proof. It follows from [13, Section 3.4] that $\mathcal{E}$-module components of $\mathcal{C}$ are parameterized by characters of $K_0(\mathcal{E})$, i.e., by elements of A. Namely, the squared braiding of a simple object $X \in \mathcal{C}$ with simple objects of $\phi \in \mathcal{E}$, where $\phi \in \widehat{A} = \mathsf{Hom}(A, \Bbbk^\times)$ determines a character on $\widehat{A}$, i.e., an element $a_X \in A = \widehat{\widehat{A}}$ such that

$$c_{\phi,X} c_{x,\phi} = \phi(a_X)\, \mathrm{id}_{\phi \otimes X}.$$

It follows from the hexagon axioms that the assignment $X \mapsto a_X$ determines a grading on $\mathcal{C}$. Since $\mathcal{C}$ is non-degenerate, we must have $\mathcal{C}_x \neq 0$ for all $x \in A$, i.e., the above grading is faithful.

Conversely, we claim that an A-graded braided extension (41) is non-degenerate. For $x \neq e$, we have $\mathcal{Z}_{sym}(\mathcal{C}) \cap \mathcal{C}_x = 0$, since there is $V \in \mathcal{E}$ such that $x|_V \neq \mathrm{id}_V$, i.e.,

$$c_{X,V}c_{V,X} \neq \mathrm{id}_{V\otimes X}$$

for all non-zero $X \in \mathcal{C}_x$. If $V \in \mathcal{Z}_{sym}(\mathcal{C}) \cap \mathcal{E}$ is a non-trivial representation of A, then there is $x \in A$ such that $x|_V \neq \mathrm{id}_V$, i.e., V does not centralize $\mathcal{C}_x$. Hence, $\mathcal{Z}_{sym}(\mathcal{C}) = \mathcal{V}ect$, i.e., $\mathcal{C}$ is non-degenerate. It is a minimal extension since $\mathrm{FPdim}(\mathcal{C}) = \mathrm{FPdim}(\mathcal{E})^2$.

An equivalence of minimal extensions preserves the squared braiding and restricts to the identity on $\mathcal{E}$. Therefore it must preserve the grading (41). □

Recall [20] that a fusion category is *nilpotent* if it is obtained from $\mathcal{V}ect$ by a sequence of graded extensions.

Corollary 4.2. *Let $\mathcal{E}$ be a pointed symmetric fusion category and let $p_1, \dots, p_n$ be distinct primes dividing FPdim($\mathcal{E}$). Let $\mathcal{E} = \mathcal{E}_1 \boxtimes \cdots \boxtimes \mathcal{E}_n$ be the Sylow decomposition of $\mathcal{E}$, where FPdim($\mathcal{E}_i$) is a power of p_i, $i = 1, \dots, n$. Then any minimal extension of $\mathcal{E}$ is nilpotent and*

$$Mext(\mathcal{E}) = Mext(\mathcal{E}_1) \times \cdots \times Mext(\mathcal{E}_n). \tag{42}$$

Proof. By Proposition 4.1, a minimal extension of $\mathcal{E}$ is a graded extension of a pointed fusion category, so it is nilpotent of nilpotency class at most 2. It is shown in [12, Theorem 6.12] that any nilpotent braided fusion category $\mathcal{C}$ admits a Sylow decomposition $\mathcal{C} = \mathcal{C}_1 \times \cdots \times \mathcal{C}_n$. So if $\mathcal{E} \hookrightarrow \mathcal{C}$ is a minimal extension then $\mathcal{E}_i \hookrightarrow \mathcal{C}_i$ is a minimal extension for all $i = 1, \dots n$. This implies the statement. □

Remark 4.3. Minimal extensions of Tannakian fusion categories were classified in [26] where it was shown that $Mext(\mathcal{R}ep(G)) = H^3(G, \Bbbk^\times)$ for any finite group G. In view of Corollary 4.2, to classify extension of pointed symmetric fusion categories it remains to determine $Mext(\mathcal{R}ep(A, t))$, where A is an abelian group with $|A| = 2^n$, $n \geq 1$, i.e., A is a 2-group in the sense of the classical (not higher-categorical) group theory.

Remark 4.4. It follows from the description of the homogeneous components of the grading (41) that the groupoid $\mathcal{M}ext(\mathcal{R}ep(A, t))$ is equivalent to the groupoid of braided monoidal 2-functors $F : A \to \mathbf{Pic_{br}}(\mathcal{R}ep(A, t))$ such that the composition of group homomorphisms

$$A \xrightarrow{\pi_0(F)} Pic_{br}(\mathcal{R}ep(A, t)) = Pic(\mathcal{R}ep(A, t)) \times A \xrightarrow{p_A} A, \tag{43}$$

equals id_A. Here p_A denotes the projection on A.

4.2. **A canonical filtration of $Mext(\mathcal{E})$.** We say that a minimal extension $\mathcal{E} \hookrightarrow \mathcal{C}$ is *integral* if $\mathcal{C}$ is an integral fusion category, i.e., $\text{FPdim}(X)$ is an integer for all objects $X \in \mathcal{C}$. Such extensions are characterized by the following property: the image of the corresponding homomorphism composition

$$A \to Pic_{br}(\mathcal{E}) \to Pic(\mathcal{E}) \tag{44}$$

lies in $Pic_{int}(\mathcal{E})$, see (7).

An integral minimal extension $\mathcal{E} \hookrightarrow \mathcal{C}$ is *pointed* if $\mathcal{C}$ is a pointed category. Equivalently, this extension is *quasi-trivial* in the sense of [6, Section 8.7], i.e., the homomorphism (44) is trivial. In this case, the homomorphism $A \to Pic_{br}(\mathcal{E})$ is identified with the identity map $A \to Aut_{\otimes}(\text{id}_{\mathcal{E}}) = A$. The braided monoidal functor

$$A \to \mathcal{P}ic_{br}(\mathcal{E}) \tag{45}$$

is determined by an element $L \in H^2_{ab}(A,\, \widehat{A}) = \mathsf{Ext}(A,\, \widehat{A})$.

Here and below $H^*_{ab}(A, M) = H^{*+1}(K(A,2), M)$ denotes the abelian Eilenberg-Mac Lane cohomology group of A with coefficients in M [14]. A description of low dimensional abelian cohomology groups $H^n_{ab}(A,\, M)$, $n \leq 4$ can be found, e.g., in [6, Section 2.1], where the term "braided cohomology" was used.

Finally, a pointed minimal extension $\mathcal{E} \hookrightarrow \mathcal{C}$ is *trivial* if the monoidal functor (45) is trivial. Such extensions are easy to describe explicitly as follows. We have $\mathcal{E} = \mathcal{R}ep(A,\, t) = \mathcal{C}(\widehat{A},\, t)$, where t is viewed as a quadratic character on $\widehat{A}$. For any quadratic form $q : A \to \Bbbk^\times$ define a quadratic form $h_q : A \times \widehat{A} \to \Bbbk^\times$ by

$$h_q(a,\, \phi) = \langle at,\, \phi \rangle\, q(a), \qquad a \in A,\, \phi \in \widehat{A}. \tag{46}$$

It is easy to see that this form is non-degenerate and

$$\mathcal{E} = \mathcal{C}(\widehat{A},\, t) \hookrightarrow \mathcal{M}_q := \mathcal{C}(A \times \widehat{A},\, h_q). \tag{47}$$

is a typical trivial minimal extension of $\mathcal{E}$.

Lemma 4.5. *The set of trivial (respectively, pointed, integral) minimal extensions of $\mathcal{E}$ is closed under the tensor product.*

Proof. The statement about trivial extensions follows from their explicit description (47). Indeed, one can directly check that the assignment

$$H^3_{ab}(A,\, \Bbbk^\times) \to Mext(\mathcal{R}ep(A,\, t)) : q \mapsto \mathcal{M}_q \tag{48}$$

is a group homomorphism.

Since $\mathcal{E}$ is pointed, the tensor product of its minimal extensions is obtained by taking a de-equivariantization with respect to the diagonal Tannakian subcategory in $\mathcal{E} \boxtimes \mathcal{E}$. Clearly, a de-equivariantization of a pointed (respectively, integral) fusion category is pointed (respectively, integral). This proves the statement about pointed and integral extensions. □

Remark 4.6. Homomorphism (48) is, in general, not injective.

Thus, we have a filtration

$$\mathcal{M}ext_{triv}(\mathcal{E}) \subset \mathcal{M}ext_{pt}(\mathcal{E}) \subset \mathcal{M}ext_{int}(\mathcal{E}) \subset \mathcal{M}ext(\mathcal{E}), \tag{49}$$

where $\mathcal{M}ext_{triv}(\mathcal{E})$, $\mathcal{M}ext_{pt}(\mathcal{E})$, $\mathcal{M}ext_{int}(\mathcal{E})$ denote the categorical groups of trivial, pointed, and integral minimal extensions of $\mathcal{E}$. There is a corresponding filtration of abelian groups

$$Mext_{triv}(\mathcal{E}) \subset Mext_{pt}(\mathcal{E}) \subset Mext_{int}(\mathcal{E}) \subset Mext(\mathcal{E}), \tag{50}$$

which we are going to study next. Our goal is to determine the factors of this filtration.

4.3. **Trivial minimal extensions.** Recall that the third abelian cohomology group $H^3_{ab}(A,\, \Bbbk^\times)$ is isomorphic to the group $\mathsf{Quad}(A,\, \Bbbk^\times)$ of quadratic forms on A with values in $\Bbbk^\times$.

Proposition 4.7. *$Mext_{triv}(\mathcal{E}) \cong \mathsf{Coker}\left(H^1(A,\, \widehat{A}) \xrightarrow{\kappa^t} H^3_{ab}(A,\, \Bbbk^\times)\right)$, where*

$$\kappa^t : H^1(A,\, \widehat{A}) \to \mathsf{Quad}(A,\, \Bbbk^\times) : Z \mapsto q_Z, \qquad q_Z(x) = \langle xt,\, Z(x)\rangle,\ x \in A. \tag{51}$$

Proof. A trivial extension of $\mathcal{E} = \mathcal{R}ep(A,\, t)$ is obtained by deforming the structure constraints of the identity extension

$$\mathcal{E} \hookrightarrow \mathcal{Z}(\mathcal{E}) = \bigoplus_{a\in A} \mathcal{Z}(\mathcal{E})_a$$

by means of an abelian 3-cocycle $(\omega,\, c) \in Z^3_{ab}(A,\, \Bbbk^\times)$. This is a special case of a *zesting* procedure studied in [9]. Namely, let $a_{W,X,Y}$ and $c_{X,Y}$ denote the associativity and braiding isomorphisms in $\mathcal{Z}(\mathcal{E})$. The deformed extension $\mathcal{Z}(\mathcal{E})^{(\omega,\, c)}$ coincides with $\mathcal{Z}(\mathcal{E})$ as an abelian category and has the same tensor product, while its associativity and braiding isomorphisms are given by

$$\begin{aligned} \tilde{a}_{W,X,Y} &= \omega(\deg(X), \deg(Y), \deg(Z))\, a_{W,X,Y}, \\ \tilde{c}_{X,Y} &= \omega(\deg(X), \deg(Y))\, c_{X,Y}, \end{aligned}$$

for all homogeneous objects W, X, Y.

Isomorphisms between trivial braided extensions were classified in [6, Section 8.7]. In particular, formula (51) follows from [6, formula (8.52)], since the self braiding $c_{X,X}$ of a simple object $X \in \mathcal{R}ep(A,\, t)$ is given by the evaluation $\langle X,\, t\rangle$. □

Proposition 4.8. *Let $A(2) = \mathbb{Z}_{2^{n_1}} \times \cdots \times \mathbb{Z}_{2^{n_r}}$ be the Sylow 2-subgroup of A. Then*

$$Mext_{triv}(\mathcal{E}) \cong \begin{cases} \mathbb{Z}_4 \times \mathbb{Z}_2^{r-1} & \text{if } \langle t\rangle \text{ is a direct summand of } A, \\ \mathbb{Z}_2^r & \text{otherwise.} \end{cases} \tag{52}$$

Proof. We write $A = A(2) \times A(\text{odd})$ and note that the map κ^t from (51) respects this decomposition and is an isomorphism on $A(\text{odd})$. So we may assume that A is a 2-group. Isomorphism (52) is easy to check when $A = \mathbb{Z}_{2^n}$ is cyclic. In this case, we have

$$H^1(\mathbb{Z}_{2^n}, \mathbb{Z}_{2^n}) = \mathbb{Z}_{2^n}, \quad \mathsf{Quad}(\mathbb{Z}_{2^n}, \Bbbk^\times) \cong \mathbb{Z}_{2^{n+1}},$$

and the homomorphism $\kappa^t : \mathbb{Z}_{2^n} \to \mathbb{Z}_{2^{n+1}}$ is injective if $n > 1$ and is zero for $n = 1$.

Let $A = \langle x_1 \rangle \times \cdots \times \langle x_r \rangle$, where $|x_i| = 2^{n_i}$. We may assume that $t \in \langle x_1 \rangle$. A quadratic form $q : A \to \Bbbk^\times$ is uniquely determined by the values

$$q(x_i),\ i = 1, \dots, r \quad \text{and} \quad b(x_j, x_l) := \frac{q(x_j x_l)}{q(x_j)q(x_l)},\ 1 \leq j < l \leq r.$$

Here $q(x_i)$ is a 2^{n_i+1}th root of unity and $b(x_j, x_l)$ is a $\min\{2^{n_j}, 2^{n_l}\}$th root of unity. Any choice of such roots of unity will give a quadratic form. Let us identify $Z \in H^1(A, \widehat{A})$ with a bilinear form on A. The symmetric bilinear form associated to q_Z is

$$b_Z(x, y) := Z(x, y)\, Z(y, x), \qquad x, y \in A,$$

which can realize all possible values of $b(x_j, x_l)$. On the other hand, $q_Z(x_i) = Z(x_i, tx_i)$ can be any 2^{n_i}th root of unity if $i > 1$ and $q_Z(x_1) = Z(x_1, t)Z(x_1, x_1)$. The latter can be any 2^{n_1}th root of unity if $t \neq x_1$ (i.e., $n_1 > 1$) and equals 1 otherwise. From this, the cokernel of (51) is easily determined. □

4.4. **Pointed minimal extensions.** The group $Mext_{pt}(\mathcal{R}ep(A, t))$ can be computed using the classification of quasi-trivial graded extensions from [6, Section 8.7]. For an abelian group A we identify $\mathsf{Ext}(A, \widehat{A})$ with $H^2_{ab}(A, \widehat{A})$. There is a natural involution

$$\varepsilon : H^2_{ab}(A, \widehat{A}) \to H^2_{ab}(A, \widehat{A})$$

that sends the class of an abelian extension

$$0 \to \widehat{A} \to C \to A \to 0 \tag{53}$$

to the class of the dual extension obtained by applying the functor $\mathsf{Hom}(-, \Bbbk^\times)$ to (53):

$$0 \to \widehat{A} \to \widehat{C} \to \widehat{\widehat{A}} = A \to 0. \tag{54}$$

This ε was explicitly described in [28, Section 8]. Namely, let $L = \{L_{x,y}\}_{x,y \in A}$ be a normalized 2-cocycle in $H^2_{ab}(A, \widehat{A})$ corresponding to the extension (53). For any $z \in A$ there is a normalized 1-cochain $a_z \in C^1(A, \Bbbk^\times)$ such that

$$L_{x,y}(z) = \frac{a_z(x)a_z(y)}{a_z(xy)}, \qquad x, y, z \in A, \tag{55}$$

and $\varepsilon(L)$ is determined by

$$\varepsilon(L)_{x,y}(z) = \frac{a_x(z)a_y(z)}{a_{xy}(z)}, \qquad x, y, z \in A. \tag{56}$$

Let us denote $H^2_{ab}(A,\, \widehat{A})^\varepsilon = \{L \in H^2_{ab}(A,\, \widehat{A}) \mid \varepsilon(L) = L\}$.

By Proposition 4.1, an extension $\mathcal{C} \in Mext_{pt}(\mathcal{E})$ defines a quasi-trivial A-graded braided extension of $\mathcal{E}$. The corresponding braided monoidal functor $F_\mathcal{C} : A \to \mathcal{P}ic_{br}(\mathcal{E})$ is completely determined by an abelian 2-cocycle $L_\mathcal{C} \in H^2_{ab}(A,\, \widehat{A})$ defining the monoidal structure of $F_\mathcal{C}$. This $L_\mathcal{C}$ is precisely the 2-cocycle corresponding to the central extension $0 \to \widehat{A} \to Inv(\mathcal{C}) \to A \to 0$.

Thus, there is a homomorphism

$$\lambda : Mext_{pt}(\mathcal{E}) \to H^2_{ab}(A,\, \widehat{A}) : \mathcal{C} \mapsto L_\mathcal{C} \tag{57}$$

whose kernel is $Mext_{triv}(\mathcal{E})$. The image of λ consists of all $L \in H^2_{ab}(A,\, \widehat{A})$ such that the corresponding braided monoidal functor $A \to \mathcal{P}ic_{br}(\mathcal{E})$ admits an extension to a braided monoidal 2-functor. By [6, Section 8.7] this image is the kernel of the Pontryagin-Whitehead homomorphism

$$PW^2 : H^2_{ab}(A,\, \widehat{A}) \to H^4_{ab}(A,\, \Bbbk^\times)$$

whose components are given by formulas [6, (8.53)-(8.55)]:

$$PW^2(L)(x,y,z,w) = c_{L_{x,y}, L_{z,w}}, \qquad PW^2(L)(x,y|,z) = 1, \qquad PW^2(L)(x,|y,z) = L_{y,z}(x), \tag{58}$$

for all $x,y,z,w \in A$. Thus, there is an exact sequence of group homomorphisms

$$0 \to Mext_{triv}(\mathcal{E}) \to Mext_{pt}(\mathcal{E}) \xrightarrow{\lambda} H^2_{ab}(A,\, \widehat{A}) \xrightarrow{PW^2} H^4_{ab}(A,\, \Bbbk^\times). \tag{59}$$

For an abelian group A let us denote $A_2 = \{x \in A \mid x^2 = e\}$.

Proposition 4.9. *There is a group isomorphism*

$$Mext_{pt}(\mathcal{E})/Mext_{triv}(\mathcal{E}) \cong \mathsf{Ker}\left(H^2_{ab}(A,\, \widehat{A})^\varepsilon \xrightarrow{\theta^t} \mathsf{Hom}(A_2/\langle t\rangle,\, \Bbbk^\times)\right),$$

where

$$\theta^t : H^2_{ab}(A,\, \widehat{A})^\varepsilon \to \mathsf{Hom}(A_2/\langle t\rangle,\, \Bbbk^\times) : L \mapsto \theta^t_L, \qquad \theta^t_L(x) = L_{x,x}(xt),\, x \in A. \tag{60}$$

Proof. Given the exact sequence (59), all we need to show is that the kernels of θ^t and PW^2 coincide. It was shown in [15] (see also [6, (2.19)]) that there is an exact sequence

$$0 \to \mathsf{Hom}(A_2,\, \Bbbk^\times) \to H^4_{ab}(A,\, \Bbbk^\times) \xrightarrow{h_A} H^2_{ab}(A,\, \widehat{A}).$$

It follows from the construction described at the end of [6, Section 2.1] and formulas (55), (56) that

$$h_A(PW^2(L)) = L\,\varepsilon(L)^{-1}, \qquad L \in H^2_{ab}(A,\, \widehat{A}). \tag{61}$$

There is a canonical isomorphism

$$\iota_A : \mathsf{Ker}(h_A) \xrightarrow{\sim} \mathsf{Hom}(A_2,\, \Bbbk^\times) : \alpha \mapsto \iota_A(\alpha), \tag{62}$$

defined by

$$\iota_A(\alpha)(x) = \frac{\alpha(x,x,x,x)\alpha(x|x,x)}{\alpha(x,x|x)}, \qquad x \in A_2. \tag{63}$$

Combining formulas (58) and (63) we obtain

$$\iota_A(PW^2(L))(x) = L_{x,x}(xt), \quad x \in A. \tag{64}$$

Thus, $PW^2(L) = 0$ if and only if $L \in H^2_{ab}(A,\, \widehat{A})^\varepsilon$ and $L_{x,x}(xt) = 1$ for all $x \in A$, i.e., $L \in \mathsf{Ker}(\theta^t)$. Since the right hand side of (64) vanishes on t, we conclude that PW^2 descends to a homomorphism

$$\theta^t : H^2_{ab}(A,\, \widehat{A})^\varepsilon \to \mathsf{Hom}(A_2/\langle t\rangle,\, \Bbbk^\times)$$

defined in (60). □

Remark 4.10. The group $H^2_{ab}(A,\, \widehat{A})$ is (non-canonically) isomorphic to $\mathsf{Hom}(\widehat{A}\otimes\widehat{A},\, \Bbbk^\times)$. It was explained in [28, Lemma 8.2] that, upon this isomorphism, ε is identified with the transposition map, so

$$H^2_{ab}(A,\, \widehat{A})^\varepsilon \cong \mathsf{Sym}^2(\widehat{A}).$$

In particular, if A is cyclic, then ε is the identity map.

Proposition 4.11. *Homomorphism (60) is surjective.*

Proof. We may assume that $A = C_1 \times \cdots \times C_r$, where each C_i is a cyclic 2-group and $t \in C_1$. It suffices to check that for $C_i = \langle x \mid x^{2N_i} = e\rangle$, $i = 2, \dots, r$, the homomorphism

$$H^2_{ab}(C_i,\, \widehat{C}_i) \to \mathsf{Hom}(\mathbb{Z}_2,\, \Bbbk^\times) \cong \{\pm 1\} : L \mapsto L_{x^{N_i},\, x^{N_i}}(x^{N_i}) \tag{65}$$

is surjective. To see that, let ξ be a primitive $2N_i$-th root of 1 in $\Bbbk$ and let L be a generator of $H^2_{ab}(C_i,\, \widehat{C}_i) \cong \mathbb{Z}_{2N_i}$ explicitly defined by

$$L_{x^k,x^l}(x^m) = \begin{cases} 1 & \text{if } k + l < 2N_i \\ \xi^m & \text{if } k+l \geq 2N_i, \qquad m = 1, \dots, 2N_i. \end{cases}$$

Then (65) sends L to -1, as required. □

4.5. **Integral minimal extensions.** We continue to denote $\mathcal{E} = \mathcal{R}ep(A,\, t)$.

Lemma 4.12. *We have*

$$[Mext_{int}(\mathcal{E}) : Mext_{pt}(\mathcal{E})] = |\wedge^3 A|. \tag{66}$$

Proof. We will prove this by induction on $r(\mathcal{E})$, the finite rank of the abelian group $Inv(\mathcal{E})$. When $r(\mathcal{E}) = 1$, i.e., when A is cyclic, we have $Pic_{int}(\mathcal{E}) = 0$ and, hence, all minimal extensions of $\mathcal{E}$ are pointed, so both sides of (66) are equal to 0.

As in Example 3.11, any $\mathcal{E} = \mathcal{R}ep(A, t)$ with $r(\mathcal{E}) = r + 1$, $r > 1$, can be written as $\mathcal{E} = \mathcal{R}ep(\mathbb{Z}_N) \boxtimes \mathcal{E}_1$, where $r(\mathcal{E}_1) = r$ and $\mathcal{E} = \mathcal{R}ep(A_1, t)$. We have

(67)
$$[Mext_{int}(\mathcal{E}) : Mext_{pt}(\mathcal{E})] = [Mext_{int}(\mathcal{E}_1) : Mext_{pt}(\mathcal{E}_1)] \times \frac{|Mext_{int}(\mathcal{E})|}{|Mext_{int}(\mathcal{E}_1)|} \times \frac{|Mext_{pt}(\mathcal{E}_1)|}{|Mext_{pt}(\mathcal{E})|}.$$

The first factor in (67) is equal to $|\wedge^3 A_1|$ by the inductive assumption. By Example 3.11, the second factor is equal to

$$|H^3(\mathbb{Z}_N, \Bbbk^\times)| \times |\,\mathsf{Ext}(\mathbb{Z}_N, \widehat{A}_1)| \times |\,\mathsf{Hom}(\mathbb{Z}_N, \wedge^2 A_1)| = |\mathbb{Z}_N| \times |\mathbb{Z}_N \otimes A_1| \times |\mathbb{Z}_N \otimes \wedge^2 A_1|.$$

Using Propositions 4.8, 4.9, and 4.11, we compute the last factor in (67) as

$$\frac{|Mext_{triv}(\mathcal{E}_1)|}{|Mext_{triv}(\mathcal{E})|} \times \frac{[Mext_{pt}(\mathcal{E}_1) : Mext_{triv}(\mathcal{E}_1)]}{[Mext_{pt}(\mathcal{E}) : Mext_{triv}(\mathcal{E})]} = 2 \times \frac{|H^2_{ab}(A, \widehat{A})^\varepsilon|}{2|H^2_{ab}(A_1, \widehat{A}_1)^\varepsilon|}$$
$$= \frac{|\,\mathsf{Sym}^2(A_1)|}{|\,\mathsf{Sym}^2(A)|} = \frac{1}{|\mathbb{Z}_N| \times |\mathbb{Z}_N \otimes A_1|}.$$

Substituting these quantities into (67) we obtain

$$[Mext_{int}(\mathcal{E}) : Mext_{pt}(\mathcal{E})] = |\mathbb{Z}_N| \times |\mathbb{Z}_N \otimes A_1| \times |\wedge^3 A_1| = |\wedge^3 A|, \tag{68}$$

as required. □

Recall from (41) that a minimal extension $\mathcal{E} \hookrightarrow \mathcal{C}$ admits a canonical faithful A-grading

$$\mathcal{C} = \bigoplus_{x \in A} \mathcal{C}_x, \qquad \mathcal{C}_e = \mathcal{E}. \tag{69}$$

When this extension is integral, its components are of the form $\mathcal{C}_x = \mathcal{R}ep(\Bbbk_{\mu_x}[G])$, where

$$A \to H^2(A, t, \Bbbk^\times) = Pic_{int}(\mathcal{E}) : x \mapsto \mu_x \tag{70}$$

is a group homomorphism. It follows from the description of the group $Pic_{br}(\mathcal{E})$ in Section 2.1 and Remark 4.4 that the corresponding homomorphism $A \to Pic_{br}(\mathcal{E})$ is

$$A \to H^2(A, t, \Bbbk^\times) \times A : x \mapsto (\mu_x, x). \tag{71}$$

Since (71) comes from a braided monoidal 2-functor $A \to \mathbf{Pic_{br}}(\mathcal{E})$, it must satisfy

$$Q_{\mathcal{E}}(\mu_x, x) = 1, \quad \text{for all } x \in A, \tag{72}$$

where $Q_{\mathcal{E}} : H^2(A, t, \Bbbk^\times) \times A \to \widehat{A}$ is the quadratic form (9).

Define a map $\tau_{\mathcal{C}} : A^3 \to \Bbbk^\times$ by

$$\tau_{\mathcal{C}}(x, y, z) = (-1)^{\xi_{\mu_x}(y)\xi_{\mu_x}(z)} \frac{\mu_x(y, z)}{\mu_x(z, y)}, \tag{73}$$

where the bilinear map $\xi_\mu : H^2(A, t, \Bbbk^\times) \times A \to \mathbb{Z}/2\mathbb{Z}$, was introduced in (5).

Proposition 4.13. *The map* (73) *is a trilinear form on* A *satisfying*

$$\tau_{\mathcal{C}}(x,\, tx,\, y) = 1 \text{ and } \tau_{\mathcal{C}}(x,\, ty,\, y) = 1. \tag{74}$$

for all $x, y \in A$.

Proof. The linearity of $\tau_{\mathcal{C}}$ in the second and third arguments is clear since for each $\mu \in H^2(A,\, \Bbbk^\times)$ the map

$$A^2 \to \Bbbk^\times : (x, y) \mapsto \frac{\mu(x,\, y)}{\mu(y,\, x)}$$

is an alternating bilinear form. To check the linearity in the first argument, we compute, using the definition of the product $*$ from (6):

$$\begin{aligned}
\tau_{\mathcal{C}}(wx,\, y, z) &= (-1)^{\xi_{\mu_{wx}}(y)\, \xi_{\mu_{wx}}(z)}\ \frac{\mu_{wx}(y,z)}{\mu_{wx}(z,y)} \\
&= (-1)^{\xi_{\mu_w * \mu_x}(y)\, \xi_{\mu_w * \mu_x}(z)}\ \frac{(\mu_w * \mu_x)(y,z)}{(\mu_w * \mu_x)(z,y)} \\
&= (-1)^{\xi_{\mu_w * \mu_x}(y)\, \xi_{\mu_w * \mu_x}(z)}\ (-1)^{\xi_{\mu_x}(y)\xi_{\mu_w}(z) + \xi_{\mu_w}(y)\xi_{\mu_x}(z)}\ \frac{\mu_w(y,z)}{\mu_w(z,y)}\, \frac{\mu_x(y,z)}{\mu_x(z,y)} \\
&= (-1)^{\xi_{\mu_x}(y)\xi_{\mu_x}(z) + \xi_{\mu_w}(y)\xi_{\mu_w}(z)}\ \frac{\mu_w(y,z)}{\mu_w(z,y)}\, \frac{\mu_x(y,z)}{\mu_x(z,y)} \\
&= \tau_{\mathcal{C}}(w,\, y, z)\tau_{\mathcal{C}}(x,\, y, z),
\end{aligned}$$

for all $x, w, y, z \in A$, where we used that $\xi_{\mu * \nu} = \xi_\mu + \xi_\nu$ for all $\mu, \nu \in H^2(A,\, t,\, \Bbbk^\times)$.

The condition (72) along with the formula (9) imply that $\xi_{\mu_x}(x) = 0$, i.e., $\frac{\mu_x(x,t)}{\mu_x(t,x)} = 1$, and

$$1 = Q_{\mathcal{E}}(\mu_x,\, x)(y) = \frac{\mu_x(xt,\, y)}{\mu_x(y,\, xt)} = (-1)^{\xi_{\mu_x}(xt)\xi_{\mu_x}(y)}\ \frac{\mu_x(xt,\, y)}{\mu_x(y,\, xt)} = \tau_{\mathcal{C}}(x,\, xt,\, y),$$

for all $x, y \in A$, which is the first identity in (74). Finally,

$$\tau_{\mathcal{C}}(x,\, yt,\, y) = \frac{\mu_x(yt,\, y)}{\mu_x(y,\, yt)}\ (-1)^{\xi_{\mu_x}(yt)\xi_{\mu_x}(y)} = \frac{\mu_x(t,\, y)}{\mu_x(y,\, t)}\ (-1)^{\xi_{\mu_x}(y)^2} = (-1)^{\xi_{\mu_x}(y) + \xi_{\mu_x}(y)^2} = 1,$$

which is the second identity in (74). □

Proposition 4.14. *There is a (non-canonical) group isomorphism*

$$Mext_{int}(\mathcal{E})/Mext_{pt}(\mathcal{E}) \cong \mathsf{Hom}(\wedge^3 A,\, \Bbbk^\times) \tag{75}$$

Proof. The map (73) defines a group homomorphism

$$\tau : Mext_{int}(\mathcal{E}) \to \mathsf{Hom}(A^{\otimes 3},\, \Bbbk^\times) : \mathcal{C} \mapsto \tau_{\mathcal{C}}. \tag{76}$$

A minimal extension $\mathcal{C}$ is pointed if and only if the corresponding homomorphism (70) is trivial, hence pointed extensions belong to the kernel of τ. Conversely, if $\tau_{\mathcal{C}} = 1$ then

$$1 = \tau_{\mathcal{C}}(x,\, y,\, y) = (-1)^{\xi_{\mu_x}(y)},$$

so that $\xi_{\mu_x}(y) = 0$ for all $x, y \in A$. This implies that $\tau_{\mathcal{C}}(x,\, y,\, z) = \frac{\mu_x(y,z)}{\mu_x(z,y)} = 1$ for all $x, y, z \in A$, so that $\mu_x = 0$ in $H^2(A,\, t,\, \Bbbk^\times)$ and $\mathcal{C}$ is pointed.

Thus, $Mext_{int}(\mathcal{E})/Mext_{pt}(\mathcal{E})$ is isomorphic to the image of τ in $\mathsf{Hom}(A^{\otimes 3},\, \Bbbk^\times)$. We can choose a presentation $A = \langle e_1 \rangle \times \cdots \times \langle e_r \rangle$ such that we also have $A = \langle te_1 \rangle \times \cdots \times \langle te_r \rangle$. By Proposition 4.13, the trilinear form $\tau_{\mathcal{C}}$ is completely determined by its values $\tau_{\mathcal{C}}(e_i,\, te_j,\, e_k)$ when i, j, k are distinct. Hence, the group of such forms is embedded into $\mathsf{Hom}(\wedge^3 A,\, \Bbbk^\times)$. By Lemma 4.12, this embedding must be an isomorphism. □

Remark 4.15. For a Tannakian category $\mathcal{E} = \mathcal{Rep}(A)$, Proposition 4.14 implies a well-known fact that the alternator homomorphism $\mathrm{alt} : H^3(A,\, \Bbbk^\times) \to \mathsf{Hom}(\wedge^3 A,\, \Bbbk^\times)$ defined by

$$\mathrm{alt}(\omega)(x,\, y,\, z) = \prod_{\sigma \in S_3} \omega(\sigma(x),\, \sigma(y),\, \sigma(z))^{\mathrm{sign}(\sigma)} \tag{77}$$

for all $x, y, z \in A$, is surjective. Indeed, a typical minimal extension of $\mathcal{Rep}(A)$ is $\mathcal{Rep}(A) \hookrightarrow \mathcal{Z}(\mathsf{Vec}_A^\omega)$ for some $\omega \in H^3(A,\, \Bbbk^\times)$. In this case, the 2-cocycles μ_x, $x \in A$, in (70) are given by

$$\mu_x(y, z) = \frac{\omega(x,\, y,\, z)\omega(y,\, z,\, x)}{\omega(y,\, x,\, z)}, \qquad y, z \in G, \tag{78}$$

and

$$\tau_{\mathcal{Z}(\mathsf{Vec}_A^\omega)}(x,\, y,\, z) = \frac{\mu_x(y,\, z)}{\mu_x(z,\, y)} = \mathrm{alt}(\omega)(x,\, y,\, z).$$

Remark 4.16. An integral minimal extension of a pointed super-Tannakian category $\mathcal{E} = \mathcal{Rep}(A,\, t)$ of central charge 1 must be equivalent to $\mathcal{E} \hookrightarrow \mathcal{Z}(\mathsf{Vec}_G^\omega)$ for some ω in $H^3(G,\, \Bbbk^\times)$ [12]. But it is not always possible to choose a group G to be abelian.

To see that twisted Drinfeld doubles of A are not sufficient, note that $\mathcal{Z}(\mathsf{Vec}_A^\omega)$, where $\omega \in H^3(A,\, \Bbbk^\times)$, contains $\mathcal{E}$ as a fusion category only if (77) factors through

$$\mathsf{Hom}(\wedge^3(A/\langle t \rangle),\, \Bbbk^\times) \to \mathsf{Hom}(\wedge^3 A,\, \Bbbk^\times).$$

Indeed, suppose there is an embedding $\mathcal{E} \hookrightarrow \mathcal{Z}(\mathsf{Vec}_A^\omega)$. Let $\mathcal{T} = \mathcal{Rep}(A/\langle t \rangle) \subset \mathcal{E}$. Then $\mathcal{T}'$ contains pointed fusion subcategories $\mathcal{E}$ and $\mathcal{Rep}(A)$. So $\mathcal{T}'$ must be pointed. But this means that the t-component of the canonical braiding (41) is trivial, so $\mu_t = 1$ and $\mathrm{alt}(\omega)(t,\, -,\, -) = 1$.

An example of a minimal extension of $\mathcal{E}$ involving the twisted Drinfeld double of a non-abelian group can be constructed as follows. Let $\mathcal{I}$ be an Ising category. Consider $\mathcal{Z}(\mathcal{Z}(\mathcal{I})) = \mathcal{I} \boxtimes \mathcal{I} \boxtimes \mathcal{I}^{\mathrm{rev}} \boxtimes \mathcal{I}^{\mathrm{rev}}$. Let $\mathcal{C}$ be the de-equivariantization of the maximal integral subcategory $\mathcal{Z}(\mathcal{Z}(\mathcal{I}))_{int}$ by its symmetric center (the latter is equivalent to $\mathcal{Rep}(\mathbb{Z}_2)$). Then $\mathcal{C}$ is a minimal extension of $\mathcal{C}_{pt} \cong \mathcal{Rep}(\mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2^f)$. So $\mathcal{C}$ does not contain any pointed Lagrangian Tannakian subcategories and so is not equivalent to the center of a pointed fusion category.

4.6. **General minimal extensions.** Recall that A denotes a finite abelian group so that $\mathcal{E} = \mathcal{R}ep(A,\, t)$ is a pointed symmetric fusion category.

Proposition 4.17. (a) *Let $\mathcal{E}$ be a Tannakian or non-split super-Tannakian category. Then all minimal extensions of $\mathcal{E}$ are integral, i.e., $Mext(\mathcal{E}) = Mext_{int}(\mathcal{E})$.*

(b) *Let $\mathcal{E} = \mathcal{R}ep(A,\, t)$ be a split super-Tannakian category. Then*

$$Mext(\mathcal{E})/Mext_{int}(\mathcal{E}) \cong \mathsf{Hom}(A,\, \mathbb{Z}_2).$$

Proof. Part (a) is clear since in this case $Pic(\mathcal{E}) = Pic_{int}(\mathcal{E})$ and so all components of the grading (41) are integral.

For the part (b), let $A = A_0 \times \langle t \rangle$ so that $\mathcal{E} = \mathcal{R}ep(A_0) \boxtimes s\mathcal{V}ect$. By Theorem 3.8,

$$Mext(\mathcal{E}) = Mext(s\mathcal{V}ect) \times Ex_{ctr}(A_0,\, s\mathcal{V}ect).$$

Combining this with homomorphisms $Mext(s\mathcal{V}ect) \to Pic(s\mathcal{V}ect)$ and $Ex_{ctr}(A_0,\, s\mathcal{V}ect) \to \mathsf{Hom}(A_0, Pic(s\mathcal{V}ect))$ and using that $Pic(s\mathcal{V}ect) \cong \mathbb{Z}_2$, we obtain a group homomorphism $A \to \mathbb{Z}_2$. So we have a homomorphism

$$Mext_{int}(\mathcal{E}) \to \mathsf{Hom}(A,\, \mathbb{Z}_2). \tag{79}$$

Equivalently, this can also be described as follows. Let $F_{\mathcal{C}} : A \to \mathbf{Pic_{br}}(\mathcal{E})$ be a braided monoidal 2-functor corresponding to a minimal extension $\mathcal{E} \hookrightarrow \mathcal{C}$. The image of this extension under (79) is

$$A \xrightarrow{\pi_0(F_{\mathcal{C}})} Pic_{br}(\mathcal{E}) = Pic(\mathcal{E}) \times A \xrightarrow{p} Pic(\mathcal{E}) \to Pic(\mathcal{E})/Pic_{int}(\mathcal{E}) \cong \mathbb{Z}_2, \tag{80}$$

where p is the projection on $Pic(\mathcal{E})$. Its kernel consists of integral extensions, since $Pic_{int}(s\mathcal{V}ect) \cong \{1\}$. It remains to check that (79) is surjective. For this, it suffices to check that any homomorphism $\phi : A_0 \to Pic(s\mathcal{V}ect) \cong \mathbb{Z}_2$ gives rise to a central A_0-extension of $s\mathcal{V}ect$. Note that any Ising category is a central $\mathbb{Z}_2$-extension of $s\mathcal{V}ect$ and so gives a monoidal 2-functor $\mathbb{Z}_2 \to \mathbf{Pic}(s\mathcal{V}ect)$. Composing this with ϕ, we get a monoidal 2-functor $A_0 \to \mathbf{Pic}(s\mathcal{V}ect)$ and, hence, a central A_0-extension $s\mathcal{V}ect \hookrightarrow \mathcal{D}$. As explained in Section 3.5, this gives rise to a minimal extension of $\mathcal{E}$ by taking the center of $\mathcal{D}$. □

Theorem 4.18. *Let A be a finite abelian group and let $\mathcal{E} = \mathcal{R}ep(A,\, t)$ be a pointed super-Tannakian category. The filtration (50) of $Mext(\mathcal{E})$ has factors*

$$\begin{aligned}
Mext_{triv}(\mathcal{E}) &\cong \mathsf{Coker}\left(H^1(A,\, \widehat{A}) \xrightarrow{\kappa^t} H^3_{ab}(A,\, \Bbbk^\times)\right),\\
Mext_{pt}(\mathcal{E})/Mext_{triv}(\mathcal{E}) &\cong \mathsf{Ker}\left(H^2_{ab}(A,\, \widehat{A})^\varepsilon \xrightarrow{\theta^t} \mathsf{Hom}(A_2/\langle t\rangle,\, \Bbbk^\times)\right),\\
Mext_{int}(\mathcal{E})/Mext_{pt}(\mathcal{E}) &\cong \mathsf{Hom}(\wedge^3 A,\, \Bbbk^\times),\\
Mext(\mathcal{E})/Mext_{int}(\mathcal{E}) &\cong \begin{cases} \mathsf{Hom}(A,\, \mathbb{Z}_2) & \text{if } \mathcal{E} \text{ is split,}\\ 0 & \text{otherwise.}\end{cases}
\end{aligned}$$

Proof. This follows from Propositions 4.7, 4.9, 4.14, and 4.17. □

Remark 4.19. For $t = 1$, i.e., when $\mathcal{E}$ is Tannakian, the factors in Theorem 4.18 were computed in [28] and [8].

5. Examples

Recall that when t is a unique up to an automorphism central element of order 2 of a group G, we use notation $\mathcal{R}ep(G^f)$ instead of $\mathcal{R}ep(G, t)$.

5.1. $Mext(\mathcal{R}ep(\mathbb{Z}_{2^n}^f))$.

Proposition 5.1. *Let $A = \mathbb{Z}_m \times \mathbb{Z}_n$ and let $t \in A$ be such that $\langle t \rangle$ is not a direct factor. Then any minimal non-degenerate extension of $\mathcal{R}ep(A, t)$ is pointed.*

Proof. In this case, $\wedge^3 A = 0$, so the result follows from Proposition 4.14. □

For $n \geq 2$ let $\mathcal{E}_n = \mathcal{R}ep(\mathbb{Z}_{2^n}^f)$. By Proposition 5.1, $Mext(\mathcal{E}_n) = Mext_{pt}(\mathcal{E}_n)$. It follows from Propositions 4.7 and 4.9 that there is an exact sequence

$$0 \to H^1(\mathbb{Z}_{2^n}, \mathbb{Z}_{2^n}) \to H^3_{ab}(\mathbb{Z}_{2^n}, \Bbbk^\times) \to Mext(\mathcal{E}_n) \xrightarrow{\lambda} H^2_{ab}(\mathbb{Z}_{2^n}, \mathbb{Z}_{2^n}) \to 0, \tag{81}$$

where λ assigns to the minimal extension $\mathcal{R}ep(\mathcal{E}_n) \hookrightarrow \mathcal{C}$ the cohomology class of the extension $0 \to \mathbb{Z}_{2^n} \to Inv(\mathcal{C}) \to \mathbb{Z}_{2^n} \to 0$. Since $H^3_{ab}(\mathbb{Z}_{2^n}, \Bbbk^\times) \cong \mathbb{Z}_{2^{n+1}}$ and $H^2_{ab}(\mathbb{Z}_{2^n}, \mathbb{Z}_{2^n}) = \mathbb{Z}_{2^n}$, the above sequence becomes

$$0 \to \mathbb{Z}_2 \xrightarrow{\alpha} Mext(\mathcal{E}_n) \xrightarrow{\lambda} \mathbb{Z}_{2^n} \to 0. \tag{82}$$

Minimal non-degenerate extensions of $\mathcal{E}_n$ can be explicitly described as follows. Recall [16, 23] that a pointed braided fusion category $\mathcal{C}$ with the group $Inv(\mathcal{C}) = A$ of isomorphism classes of invertible objects is determined up to an equivalence by the quadratic form $q : A \to \Bbbk^\times$, where $q(X) = c_{X,X}$. In this case, we denote $\mathcal{C} = \mathcal{C}(A, q)$.

For each $m \geq 0$ and a primitive 2^{m+1}th root of unity ξ we define a non-degenerate quadratic form

$$q_\xi : \mathbb{Z}_{2^m} \to \Bbbk^\times, \qquad q_\xi(j) = \xi^{j^2} \quad \text{for all} \quad j \in \mathbb{Z}_{2^m}. \tag{83}$$

The non-degenerate pointed braided fusion category $\mathcal{C}(\mathbb{Z}_{2^m}, q_\xi)$ is a minimal extension of $\mathcal{E}_n$.

For each $k = 0, 1, \dots, n$ and a 2^{2n-k+1}th root of unity ζ let

$$\mathcal{M}_{k,\zeta} = \mathcal{C}(\mathbb{Z}_{2^k}, q_{-\zeta^{-2^{2(n-k)}}}) \boxtimes \mathcal{C}(\mathbb{Z}_{2^{2n-k}}, q_\zeta) \tag{84}$$

Again, this is a pointed non-degenerate braided fusion category.

Proposition 5.2. *For all k and ζ as above, there is a non-degenerate minimal extension $\mathcal{E}_n \hookrightarrow \mathcal{M}_{k,\zeta}$ given by the group homomorphism*

$$\iota_k : \mathbb{Z}_{2^n} \to \mathbb{Z}_{2^k} \times \mathbb{Z}_{2^{2n-k}} : j \mapsto (j,\, j^{2^{n-k}}). \tag{85}$$

Proof. Clearly, (85) is an injective group homomorphism. Let $q_{k,\zeta} : \mathbb{Z}_{2^k} \times \mathbb{Z}_{2^{2n-k}} \to \Bbbk^\times$ denote the quadratic form corresponding to $\mathcal{M}_{k,\zeta}$. We have

$$q_{k,\zeta}(\iota_k(1)) = -\zeta^{-2^{2(n-k)}} \cdot \zeta^{(2^{n-k})^2} = -1.$$

Viewing (85) as a homomorphism of metric groups, where $\mathbb{Z}_{2^n}$ is equipped with a quadratic character $q(l) = (-1)^l$, we obtain a braided tensor embedding $\mathcal{E}_n \hookrightarrow \mathcal{M}_{k,\zeta}$, i.e., a minimal extension of $\mathcal{E}_n$. □

Remark 5.3.
(1) By definition, $\mathcal{M}_{0,\zeta} = \mathcal{C}(\mathbb{Z}_{2^{2n}},\, q_\zeta)$ is a cyclic minimal extension of $\mathcal{E}_n$.
(2) For all k and ζ, the largest order of the root of unity that occurs as a value of $q_{k,\zeta}$ is 2^{2n-k+1}.
(3) For all $k = 0, \dots, n-1$, the square of $\mathcal{M}_{k,\zeta}$ in $Mext(\mathcal{E}_n)$ is $\mathcal{M}_{k+1,\zeta^2}$ (this is a straightforward computation using the definition of the product of minimal extensions).

Proposition 5.4. *$Mext(\mathcal{R}ep(\mathbb{Z}_{2^n}^f) \cong \mathbb{Z}_{2^{n+1}}$ with any $\mathcal{C}(\mathbb{Z}_{2^{2n}},\, q_\zeta)$ as a generator.*

Proof. We need to show that the exact sequence (82) does not split. Observe that, for any primitive 2^{n+1}th root of unity ζ, the minimal extension

$$\mathcal{E}_n \hookrightarrow \mathcal{M}_{n,\zeta} = \mathcal{C}(\mathbb{Z}_{2^n},\, q_{-\zeta^{-1}}) \boxtimes \mathcal{C}(\mathbb{Z}_{2^n},\, q_\zeta),$$

where $\mathcal{E}_n$ is embedded diagonally, is the generator of $\mathsf{Ker}(\lambda) \cong \mathbb{Z}_2$ in (82) (in particular, its class in $Mext(\mathcal{E}_n)$ does not depend on the choice of ζ). Thus, it suffices to check that this minimal extension has a square root. But this follows from Remark 5.3(3). □

Corollary 5.5. *The kernel of the homomorphism $Mext(\mathcal{R}ep(\mathbb{Z}_{2^n}^f)) \to Mext(s\mathcal{V}ect)$ is isomorphic to $\mathbb{Z}_{2^{n-2}}$.*

Remark 5.6. The minimal extensions of $\mathcal{R}ep(\mathbb{Z}_4^f)$ and $\mathcal{R}ep(\mathbb{Z}_8^f)$ were listed in in [27, Tables XIV and XV]. Our description of their groups of minimal extensions is consistent with these tables and with the results of [1] and [31].

For $n = 2$ Proposition 5.4 says that $Mext(\mathcal{R}ep(\mathbb{Z}_4^f)) \cong \mathbb{Z}_8$. This disagrees with [30, Example 7.17], where it is claimed that $|Mext(\mathcal{R}ep(\mathbb{Z}_4^f))| = 32$. Our explanation of this discrepancy is that [30] counts equivalence classes of $\mathbb{Z}_2$-crossed braided extensions of $\mathcal{Z}(\mathcal{R}ep(s\mathcal{V}ect))$ whose equivariantization is a minimal non-degenerate extension of $\mathcal{R}ep(\mathbb{Z}_4^f)$. However, all such $\mathbb{Z}_2$-crossed braided extensions lead to the same element of $Mext(\mathcal{R}ep(\mathbb{Z}_4^f)$, namely, to the identity extension $\mathcal{R}ep(\mathbb{Z}_4^f) \hookrightarrow \mathcal{Z}(\mathcal{R}ep(\mathbb{Z}_4))$.

5.2. $\mathcal{M}ext(\mathcal{R}ep(\mathbb{Z}_2 \times \mathbb{Z}_2^f))$. Let $\mathcal{E} = \mathcal{R}ep(\mathbb{Z}_2 \times \mathbb{Z}_2^f)$. Using Theorem 4.18, we see that the factors of the canonical filtration (50) of $\mathcal{M}ext(\mathcal{E})$ are

$$\begin{aligned} \mathcal{M}ext_{triv}(\mathcal{E}) &\cong \mathbb{Z}_4 \times \mathbb{Z}_2, \\ \mathcal{M}ext_{pt}(\mathcal{E})/\mathcal{M}ext_{triv}(\mathcal{E}) &= \mathbb{Z}_2^2, \\ \mathcal{M}ext_{int}(\mathcal{E})/\mathcal{M}ext_{pt}(\mathcal{E}) &= 0, \\ \mathcal{M}ext(\mathcal{E})/\mathcal{M}ext_{int}(\mathcal{E}) &= \mathbb{Z}_2^2. \end{aligned}$$

Therefore, $|\mathcal{M}ext(\mathcal{E})| = 128$. The canonical homomorphism

$$w : \mathcal{M}ext(\mathcal{E}) \to \mathcal{M}ext(s\mathcal{V}ect) \cong \mathbb{Z}_{16},$$

defined in (13), is split surjective and, hence,

$$\mathcal{M}ext(\mathcal{E}) \cong \mathbb{Z}_{16} \times \mathsf{Ker}(w), \tag{86}$$

where $|\,\mathsf{Ker}(w)| = 8$.

Example 5.7. The following categories are non-degenerate minimal extensions of $\mathcal{E} = s\mathcal{V}ect \boxtimes s\mathcal{V}ect$ lying in $\mathsf{Ker}(w)$:

$$\mathcal{M}_1(i) = \mathcal{Z}(\mathcal{C}(\mathbb{Z}_2, q_i)\boxtimes\mathcal{C}(\mathbb{Z}_2, q_i)), \qquad \mathcal{M}_2(\xi) = \mathcal{Z}(\mathcal{C}(\mathbb{Z}_4, q_\xi)), \qquad \mathcal{M}_3(\mathcal{I}) = \mathcal{Z}(\mathcal{I}), \tag{87}$$

where i and ξ are primitive 4th and 8th roots of unity, respectively, and $\mathcal{I}$ is an Ising braided fusion category. We use the notation introduced in (83). In each of these three cases, there is a unique embedding of $\mathcal{E}$ (note that $\mathcal{Z}(\mathcal{C}) \cong \mathcal{C}\boxtimes\mathcal{C}^{\text{rev}}$ for any non-degenerate braided fusion category $\mathcal{C}$).

We have the following equalities in $\mathcal{M}ext(\mathcal{E})$:

$$\mathcal{M}_1(i) = \mathcal{M}_1(-i) \quad \text{and} \quad \mathcal{M}_2(\xi) = \mathcal{M}_2(\xi') \quad \text{if and only if } \xi' = \pm\xi. \tag{88}$$

Lemma 5.8. $\mathsf{Ker}(w) \cap \mathcal{M}ext_{pt}(\mathcal{E}) \cong \mathbb{Z}_4$ *with a generator* $\mathcal{M}_2(\xi)$.

Proof. Since $\mathsf{Ker}(w)$ contains a non-integral extension $\mathcal{M}_3(\mathcal{I})$, we see that $|\,\mathsf{Ker}(w) \cap \mathcal{M}ext_{pt}(\mathcal{E})| = 4$.

The identity minimal extension $\mathcal{Z}(\mathcal{E})$ (respectively, $\mathcal{M}_1(i)$ and $\mathcal{M}_2(\xi)$) is a pointed minimal extension of $\mathcal{E}$ characterized by the property that the largest order of the root of unity that occurs as a value of the corresponding quadratic form is 2 (respectively, 4 and 8). Since for the square of $\mathcal{M}_2(\xi)$ in $\mathcal{M}ext(\mathcal{E})$ the quadratic form has values that are 4th roots of unity, it follows that $\mathcal{M}_1(i)$ is a square in $\mathsf{Ker}(w)$ and the statement follows. □

Lemma 5.9. $\mathsf{Ker}(w) \cong \mathbb{Z}_8$ *with a generator* $\mathcal{M}_3(\mathcal{I})$ *for any Ising category* $\mathcal{I}$.

Proof. For a braided Ising fusion category $\mathcal{I}$ the values of the canonical twist on its simple objects are 1, -1, and a primitive 16th root of unity ζ, see [13, Appendix B]. Therefore, the values of the twist on the simple objects of the integral part $(\mathcal{I} \boxtimes \mathcal{I})_{pt}$ of $\mathcal{I} \boxtimes \mathcal{I}$

are 1, 1, -1, -1, and ζ^2. By definition of the tensor product of minimal extensions, the tensor square of $\mathcal{Z}(\mathcal{I})$ in $Mext(\mathcal{E})$ is the de-equivariantization of $(\mathcal{I}\boxtimes\mathcal{I})_{pt}\boxtimes(\mathcal{I}^{\text{rev}}\boxtimes\mathcal{I}^{\text{rev}})_{pt}$ (viewed as a subcategory of $\mathcal{Z}(\mathcal{I})^{\boxtimes 2}$) by the diagonal Tannakian subcategory of $\mathcal{E}\boxtimes\mathcal{E}$. The result contains simple objects with the twist (quadratic form) values being primitive 8th roots of unity. This means that

$$\mathcal{Z}(\mathcal{I})^{\boxdot 2} = \mathcal{Z}(\mathcal{C}(\mathbb{Z}_4,\, q_{\zeta^2})) = \mathcal{M}(\zeta^2),$$

so that $\mathcal{M}_3(\mathcal{I})$ has order 8 in $Mext(\mathcal{E})$ by Lemma 5.8. □

Corollary 5.10. *$Mext(\mathcal{R}ep(\mathbb{Z}_2\times\mathbb{Z}_2^f))\cong\mathbb{Z}_{16}\times\mathbb{Z}_8$. The extensions $\mathcal{Z}(\mathcal{R}ep(\mathbb{Z}_2))\boxtimes\mathcal{I}_1$ and $\mathcal{Z}(\mathcal{I}_2)$, where $\mathcal{I}_1$, $\mathcal{I}_2$ are any Ising braided fusion categories, can be taken as generators of the cyclic factors.*

The central charge homomorphism (13) is identified with the projection on the first factor.

Proof. This follows from (86) and Lemma 5.9. □

Corollary 5.11. *$Mext_{pt}(\mathcal{R}ep(\mathbb{Z}_2\times\mathbb{Z}_2^f))\cong\mathbb{Z}_8\times\mathbb{Z}_4$.*

Proof. We have seen that $|Mext_{pt}(\mathcal{R}ep(\mathbb{Z}_2\times\mathbb{Z}_2^f))| = 32$. Since the minimal extensions that project on Ising categories in $Mext(s\mathcal{V}ect)$ are non-integral, we conclude that $Mext_{pt}(\mathcal{R}ep(\mathbb{Z}_2\times\mathbb{Z}_2^f))$ is a subgroup of $\mathbb{Z}_8\times\mathbb{Z}_8$, which implies the result. □

Remark 5.12. All 128 minimal extensions of $\mathcal{R}ep(\mathbb{Z}_2\times\mathbb{Z}_2^f)$ were listed in [27, Tables XVI-XIX]. Our contribution is the computation of the group structure of $Mext(\mathcal{R}ep(\mathbb{Z}_2\times\mathbb{Z}^f))$. The filtration of this group and its integral part found in this Section are consistent with these tables. A description of this group was also given recently in [1, Section V.B and Appendix I] and in [2, Table V], where the eight elements of the group $\mathsf{Ker}(w)$ were explicitly listed.

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