Braided Picard groups and graded extensions of braided tensor categories

Alexei Davydov [a)], Dmitri Nikshych [b)]

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[a)] Department of Mathematics, Ohio University, Athens, OH 45701, USA

[b)] Department of Mathematics and Statistics, University of New Hampshire, Durham, NH 03824, USA

Abstract

We classify various types of graded extensions of a finite braided tensor category $\mathcal{B}$ in terms of its 2-categorical Picard groups. In particular, we prove that braided extensions of $\mathcal{B}$ by a finite group A correspond to braided monoidal 2-functors from A to the braided 2-categorical Picard group of $\mathcal{B}$ (consisting of invertible central $\mathcal{B}$-module categories). Such functors can be expressed in terms of the Eilnberg-Mac Lane cohomology. We describe in detail braided 2-categorical Picard groups of symmetric fusion categories and of pointed braided fusion categories.

Contents

1 Introduction and synopsis

1.1 Extensions of tensor categories

In this paper we work over an algebraically closed field k. All tensor categories are assumed to be k-linear and finite [18].

Let $\mathcal{B}$ be a tensor category. An *extension* of $\mathcal{B}$ is an embedding of $\mathcal{B}$ into a tensor category $\mathcal{C}$ i.e. a fully faithful tensor functor $\iota : \mathcal{B} \to \mathcal{C}$. We will identify $\mathcal{B}$ with its image in $\mathcal{C}$ and use notation $\mathcal{B} \subset \mathcal{C}$ to denote an extension. An *isomorphism* between extensions $\iota_1, \iota_2 : \mathcal{B} \to \mathcal{C}$ is a tensor autoequivalence $F : \mathcal{C} \to \mathcal{C}$ such that $F \circ \iota_1 = \iota_2$.

When $\mathcal{B}$ is braided (or symmetric) there are several types of extensions reflecting different "amounts of commutativity" of $\mathcal{C}$. Namely, we say that an extension $\iota : \mathcal{B} \to \mathcal{C}$ is

- *central* if there is a lifting tensor functor $L : \mathcal{B} \to \mathcal{Z}(\mathcal{C})$ such that ι coincides with the composition $\mathcal{B} \xrightarrow{L} \mathcal{Z}(\mathcal{C}) \xrightarrow{\text{Forget}} \mathcal{C}$, where $\mathcal{Z}(\mathcal{C})$ is the center of $\mathcal{C}$ and Forget : $\mathcal{Z}(\mathcal{C}) \to \mathcal{C}$ is the forgetful functor;
- *braided* if $\mathcal{C}$ is braided;
- *symmetric* if $\mathcal{C}$ is symmetric.

The extending tensor category $\mathcal{C}$ can be viewed as a $\mathcal{B}$-module category. Furthermore, the tensor product of $\mathcal{C}$ equips it with a structure of a pseudo-monoid [12] in a monoidal 2-category $\mathbf{M}$ consisting of certain $\mathcal{B}$-modules. Commutativity properties of $\mathcal{B}$, $\mathcal{C}$ and the type of an extension $\mathcal{B} \subset \mathcal{C}$ are reflected in the choice of $\mathcal{B}$-modules in $\mathbf{M}$ and in the properties of the pseudo-monoid $\mathcal{C}$. These properties are summarized in Table 1.

In this paper, for a tensor category $\mathcal{D}$ we denote by $\mathbf{Bimod}(\mathcal{D})$ the monoidal 2-category of $\mathcal{D}$-bimodule categories [23]. For a braided tensor category $\mathcal{B}$ we denote by $\mathbf{Mod}(\mathcal{B})$ the monoidal 2-category of $\mathcal{B}$-module categories [10, 23] (it can be viewed as a monoidal 2-subcategory of $\mathbf{Bimod}(\mathcal{B})$) and $\mathbf{Mod}_{br}(\mathcal{B})$ the braided monoidal 2-category of *braided* $\mathcal{B}$-module categories. For a symmetric tensor category $\mathcal{E}$

Extension $\mathcal{B} \subset \mathcal{C}$	2-category $\mathbf{M}$ of $\mathcal{B}$-modules	$\mathcal{C}$
tensor	monoidal 2-category $\mathbf{Bimod}(\mathcal{B})$	pseudo-monoid
central	monoidal 2-category $\mathbf{Mod}(\mathcal{B})$	pseudo-monoid
braided	braided monoidal 2-category $\mathbf{Mod}_{br}(\mathcal{B})$	braided pseudo-monoid
symmetric	symmetric monoidal 2-category $\mathbf{Mod}_{sym}(\mathcal{B})$	symmetric pseudo-monoid

Table 1: Extensions $\mathcal{B} \subset \mathcal{C}$ as pseudo-monoids in a monoidal 2-category $\mathbf{M}$.

we denote *by*$\mathbf{Mod}_{sym}(\mathcal{E})$ the symmetric monoidal 2-category of *symmetric* $\mathcal{E}$-module categories[1]. By definition [21, 5, 2], a braided $\mathcal{B}$-module category is equipped with a natural collection of isomorphisms coherently extending the braiding on $\mathcal{B}$, see Definition 4.1. Braided module categories turn out to play an important role in 4d topological field theory and factorization homology. In Theorem 4.11 we show that the 2-category $\mathbf{Mod}_{br}(\mathcal{B})$ of braided module categories is braided 2-equivalent to the center (in the sense of [1]) of $\mathbf{Mod}(\mathcal{B})$:

$$\mathbf{Mod}_{br}(\mathcal{B}) \cong \mathbf{Z}(\mathbf{Mod}(\mathcal{B})). \tag{1.1}$$

1.2 Graded extensions and monoidal 2-functors to 2-categorical groups

We focus on extensions of finite braided tensor categories graded by finite groups.

Let G be a group. A *G-extension* of a tensor category $\mathcal{D}$ is an extension $\mathcal{D} \subset \mathcal{C}$ together with a faithful G-grading of $\mathcal{C}$ such that $\mathcal{D}$ is the trivial component. In other words, $\mathcal{C}$ admits a decomposition

$$\mathcal{C} = \bigoplus_{g \in G} \mathcal{C}_g \tag{1.2}$$

such that $\mathcal{C}_1 = \mathcal{D}$ and the tensor multiplication of $\mathcal{C}$ maps $\mathcal{C}_x \times \mathcal{C}_y$ to $\mathcal{C}_{xy}$ for all $x, y \in G$. An *equivalence of G-extensions* is an equivalence of extensions respecting the grading.

In [23] G-extensions of a tensor category $\mathcal{D}$ were classified by means of the Brauer-Picard 2-categorical group $\mathbf{BrPic}(\mathcal{D})$ of invertible $\mathcal{D}$-bimodule categories. Namely, it was shown that extensions (1.2) correspond to monoidal 2-functors $G \to \mathbf{BrPic}(\mathcal{D})$. As a result, equivalence classes of such extensions can be described in terms of certain cohomology groups associated to a homomorphism $G \to BrPic(\mathcal{D})$.

This paper provides a classification of various types of G-extensions (where G is an Abelian group) of a *braided* tensor category $\mathcal{B}$.

By a *2-categorical group* (respectively, *braided or symmetric 2-categorical group*) we understand a monoidal (respectively, braided or symmetric monoidal 2 category) in which every 0-cell is invertible with respect to the tensor product, every 1-cell is an equivalence, and every 2-cell is an isomorphism. For

[1] For fusion categories, these 2-categories of module categories are fusion 2-categories [16].

a monoidal 2-category $\mathbf{M}$ the set of its invertible objects is a 2-categorical group which we will denote by $\mathbf{Inv}(\mathbf{M})$.

For the monoidal 2-categories $\mathbf{Bimod}(\mathcal{B})$, $\mathbf{Mod}(\mathcal{B})$, $\mathbf{Mod}_{br}(\mathcal{B})$ and $\mathbf{Mod}_{sym}(\mathcal{B})$ (for symmetric $\mathcal{B}$) introduced above the 2-categorical groups of invertible objects

$$\begin{aligned} \mathbf{BrPic}(\mathcal{B}) &= \mathbf{Inv}(\mathbf{Bimod}(\mathcal{B})) && (1.3)\\ \mathbf{Pic}(\mathcal{B}) &= \mathbf{Inv}(\mathbf{Mod}(\mathcal{B})), && (1.4)\\ \mathbf{Pic_{br}}(\mathcal{B}) &= \mathbf{Inv}(\mathbf{Mod}_{br}(\mathcal{B})), && (1.5)\\ \mathbf{Pic_{sym}}(\mathcal{B}) &= \mathbf{Inv}(\mathbf{Mod}_{sym}(\mathcal{B})), && (1.6) \end{aligned}$$

are called the *Brauer-Picard*, *Picard*, *braided Picard*, and *symmetric Picard* 2-categorical group, respectively. These 2-categorical groups play the key role in our study of extensions of tensor categories.

The main results of this paper concerning graded extensions (see Chapter 8) can be stated as follows:

$$\left\{\begin{array}{c}\text{the groupoid of}\\ G\text{-extensions of } \mathcal{B}\\ \text{of a given type}\end{array}\right\} \quad\Longleftrightarrow\quad \left\{\begin{array}{c}\text{the groupoid of}\\ \text{corresponding monoidal 2-functors}\\ \text{between 2-categorical groups } G\to\mathbf{G}\end{array}\right\} \qquad (1.7)$$

for an appropriate 2-categorical group $\mathbf{G}$. These categorical 2-groups and the correspondence between different types of G-extensions and monoidal 2-functors $G\to\mathbf{G}$ are given in Table 2.

Extensions $\mathcal{B}\subset\mathcal{C}$	2-categorical group $\mathbf{G}$	2-functors $G\to\mathbf{G}$
tensor	2-categorical group $\mathbf{BrPic}(\mathcal{B})$	monoidal
central	2-categorical group $\mathbf{Pic}(\mathcal{B})$	monoidal
braided	braided 2-categorical group $\mathbf{Pic_{br}}(\mathcal{B})$	braided
symmetric	symmetric 2-categorical group $\mathbf{Pic}(\mathcal{B})$	symmetric

Table 2: G-extensions $\mathcal{B}\subset\mathcal{C}$ and corresponding monoidal 2-functors.

1.3 Homotopy groups and invariants of 2-categorical groups

Let $\mathbf{G}$ be a 2-categorical group with the identity object $\mathcal{I}$. We introduce its homotopy groups as follows:

$$\begin{aligned} \pi_0(\mathbf{G}) &= \text{the group of isomorphism classes of objects (0-cells) of } \mathbf{G}, && (1.8)\\ \pi_1(\mathbf{G}) &= Aut_{\mathbf{G}}(\mathcal{I}),\ \text{the group of isomorphism classes of 1-automorphisms of } \mathcal{I}, && (1.9)\\ \pi_2(\mathbf{G}) &= Aut(\mathrm{Id}_{\mathcal{I}}),\ \text{the group of 2-automorpshisms of the identity 1-automorphism of } \mathcal{I}. && (1.10) \end{aligned}$$

The multiplication of $\pi_0(\mathbf{G})$ is given by the tensor product of $\mathbf{G}$ and the multiplication of $\pi_1(\mathbf{G})$, $\pi_2(\mathbf{G})$ is the composition of automorphisms.

These homotopy groups come equipped with additional structure, which we refer to as the *standard invariants*, namely a $\pi_0(\mathbf{G})$-action on $\pi_m(\mathbf{G})$,

$$\pi_0(\mathbf{G}) \times \pi_m(\mathbf{G}) \to \pi_m(\mathbf{G}) \quad m = 0, 1, 2 \tag{1.11}$$

given by the conjugation with $\mathrm{Id}_{\mathcal{X}}$ for $\mathcal{X} \in \pi_0(\mathbf{G})$ (this action is used while making sense of the cohomology groups below) and the *first* and the *second canonical classes*

$$\alpha_{\mathbf{G}} \in H^3(\pi_0(\mathbf{G}), \pi_1(\mathbf{G})) \qquad \text{and} \qquad q_{\mathbf{G}} \in H^3_{br}(\pi_1(\mathbf{G}), \pi_2(\mathbf{G})). \tag{1.12}$$

Part of the properties of the standard invariants is that the second canonical class is invariant under the $\pi_0(\mathbf{G})$-action.

Here and in what follows we denote by

$$H^n_{br}(A, M) := H^{n+1}(A, 2; M) \quad \text{and} \quad H^n_{sym}(A, M) := H^{n+3}(A, 4; M) \tag{1.13}$$

the Eilenberg-Mac Lane cohomology [19] of level 2 and 4, respectively. Note that $H^3_{br}(A, M)$ is isomorphic to the group of quadratic functions from A to M.

For a braided 2-categorical group $\mathbf{G}$ the $\pi_0(\mathbf{G})$-action (1.11) is trivial. The canonical classes get promoted to

$$\alpha_{\mathbf{G}} \in H^3_{br}(\pi_0(\mathbf{G}), \pi_1(\mathbf{G})) \quad \text{and} \quad q_{\mathbf{G}} \in H^3_{sym}(\pi_1(\mathbf{G}), \pi_2(\mathbf{G})). \tag{1.14}$$

An additional structure is the *Whitehead products*

$$\pi_n(\mathbf{G}) \times \pi_m(\mathbf{G}) \to \pi_{n+m+1}(\mathbf{G}), \qquad n, m = 0, 1, 2. \tag{1.15}$$

Note that the product $\pi_0(\mathbf{G}) \times \pi_0(\mathbf{G}) \to \pi_1(\mathbf{G})$ is determined by the first canonical class (as the polarization of the quadratic function α).

For a symmetric 2-categorical group $\mathbf{G}$ all Whitehead products are zero and the canonical classes are

$$\alpha_{\mathbf{G}} \in H^3_{sym}(\pi_0(\mathbf{G}), \pi_1(\mathbf{G})) \quad \text{and} \quad q_{\mathbf{G}} \in H^3_{sym}(\pi_1(\mathbf{G}), \pi_2(\mathbf{G})). \tag{1.16}$$

For a tensor category $\mathcal{D}$ the homotopy groups and standard invariants of the 2-categorical group $\mathbf{BrPic}(\mathcal{D})$ were examined in [23]. One has

$$\pi_0(\mathbf{BrPic}(\mathcal{D})) = BrPic(\mathcal{D}), \quad \pi_1(\mathbf{BrPic}(\mathcal{D})) = Inv(\mathcal{Z}(\mathcal{D})), \quad \pi_2(\mathbf{BrPic}(\mathcal{D})) = k^\times.$$

It was shown there that the $BrPic(\mathcal{D})$-action on $Inv(\mathcal{Z}(\mathcal{D}))$ (i.e. the π_0-action on π_1) comes from the isomorphism $BrPic(\mathcal{D}) \simeq Aut_{br}(\mathcal{Z}(\mathcal{D}))$ and that the second canonical class is given by the quadratic

function

$$\pi_1 = Inv(\mathcal{Z}(\mathcal{D})) \to \pi_2 = k^\times : Z \mapsto c_{Z,Z},$$

where c denotes the braiding of $\mathcal{Z}(\mathcal{D})$.

The homotopy groups of 2-categorical groups introduced in (1.4) - (1.6) are

$$\begin{array}{lll} \pi_0(\mathbf{Pic}(\mathcal{B})) = Pic(\mathcal{B}), & \pi_1(\mathbf{Pic}(\mathcal{B})) = Inv(\mathcal{B}), & \pi_2(\mathbf{Pic}(\mathcal{B})) = k^\times, \\ \pi_0(\mathbf{Pic}_{br}(\mathcal{B})) = Pic_{br}(\mathcal{B}), & \pi_1(\mathbf{Pic}_{br}(\mathcal{B})) = Inv(\mathcal{Z}_{sym}(\mathcal{B})), & \pi_2(\mathbf{Pic}_{br}(\mathcal{B})) = k^\times, \\ \pi_0(\mathbf{Pic}_{sym}(\mathcal{E})) = Pic_{sym}(\mathcal{E}), & \pi_1(\mathbf{Pic}_{br}(\mathcal{E})) = Inv(\mathcal{E}), & \pi_2(\mathbf{Pic}_{br}(\mathcal{E})) = k^\times, \end{array}$$

where $\mathcal{B}$ is a braided tensor category, and $\mathcal{E}$ is a symmetric tensor category.

We investigate the standard invariants of the braided 2-categorical group $\mathbf{Pic}_{br}(\mathcal{B})$ and of the symmetric 2-categorical group $\mathbf{Pic}(\mathcal{E})$. For a braided tensor $\mathcal{B}$ we describe the Whitehead product (Proposition 5.3)

$$\pi_0 \times \pi_1 = Pic_{br}(\mathcal{B}) \times Inv(\mathcal{Z}_{sym}(\mathcal{B})) \ \to \ \pi_2 = k^\times \tag{1.17}$$

and the first canonical class (viewed as a quadratic function)

$$Q : \pi_0 = Pic_{br}(\mathcal{B}) \ \to \ \pi_1 = Inv(\mathcal{Z}_{sym}(\mathcal{B})). \tag{1.18}$$

For a symmetric tensor category $\mathcal{E}$ the first canonical class becomes a homomorphism

$$Q : Pic_{sym}(\mathcal{E}) \ \to \ Inv(\mathcal{E})_2 \tag{1.19}$$

into the 2-torsion of the group of invertible objects of $\mathcal{E}$.

1.4 Cohomological description of (braided) monoidal 2-functors

In view of the identification (1.7) it is desirable to have a good description of various types of monoidal 2-functors $G \to \mathbf{G}$. We present one in Section 2 in terms of the Eilenberg-Mac Lane cohomology.

Let $\mathbf{G}$ be a 2-categorical group (respectively, braided, symmetric 2-categorical group). Denote by $\mathbf{2\text{-}Fun}(G, \mathbf{G})$ (respectively, $\mathbf{2\text{-}Fun_{br}}(G, \mathbf{G})$, $\mathbf{2\text{-}Fun_{sym}}(G, \mathbf{G})$) the 2-groupoid of monoidal (respectively, braided, symmetric) 2-functors $G \to \mathbf{G}$. Such a functor restricts on objects to a map from $\pi_0(\mathbf{2\text{-}Fun}(G, \mathbf{G}))$ (respectively, from $\pi_0(\mathbf{2\text{-}Fun_{br}}(G, \mathbf{G}))$, $\pi_0(\mathbf{2\text{-}Fun_{sym}}(G, \mathbf{G}))$) to $\mathsf{Hom}(G, \pi_0(\mathbf{G}))$, i.e. from the set of isomorphism classes of 2-functors to the set of group homomorphisms. A homomorphism $\phi : G \to \pi_0(\mathbf{G})$ is in the image of this map (i.e. ϕ can be lifted to a monoidal (respectively, braided, symmetric) 2-functor if and only if the following two obstructions vanish.

The first obstruction is the image of ϕ under the homomorphism

$$o_3 : \mathsf{Hom}(G,\, \pi_0(\mathbf{G})) \to H^3(G, \pi_1(\mathbf{G})) \tag{1.20}$$

(respectively, $\mathsf{Hom}(G,\, \pi_0(\mathbf{G})) \to H^3_{br}(G, \pi_1(\mathbf{G}))$, $\mathsf{Hom}(G,\, \pi_0(\mathbf{G})) \to H^3_{sym}(G, \pi_1(\mathbf{G})))$, given by the pull-back along ϕ of the first canonical class $\alpha_{\mathbf{G}}$ defined in (1.12) (respectively, in (1.14), (1.16)). The obstruction $o_3(\phi)$ vanishes if and only if ϕ can be lifted to a monoidal (respectively, braided, symmetric) functor from G to the 1-categorical truncation $\Pi_{\leq 1}(\mathbf{G})$ of $\mathbf{G}$.

Suppose that a lifting $F : G \to \Pi_{\leq 1}(\mathbf{G})$ of ϕ is chosen. Then the second obstruction is the image of F under the map

$$o_4 : \mathcal{F}un(G,\, \Pi_{\leq 1}(\mathbf{G}) \to H^4(G,\, \pi_1(\mathbf{G})) \tag{1.21}$$

(respectively, $\mathcal{F}un_{br}(G,\, \Pi_{\leq 1}(\mathbf{G})) \to H^4_{br}(G,\, \pi_1(\mathbf{G}))$, $\mathcal{F}un_{sym}(G,\, \Pi_{\leq 1}(\mathbf{G})) \to H^4_{sym}(G,\, \pi_1(\mathbf{G})))$. The obstruction $o_4(F)$ measures the failure of extending F to a monoidal (respectively, braided, symmetric) 2-functor $G \to \mathbf{G}$. When $o_4(F)$ vanishes, the equivalence classes of such 2-functors extending F form a torsor over the cokernel of a certain group homomorphism $H^1(G,\, \pi_1(\mathbf{G})) \to H^3(G, \pi_2(\mathbf{G}))$ (respectively, $H^1(G,\, \pi_1(\mathbf{G})) \to H^3_{br}(G, \pi_2(\mathbf{G}))$, $H^1(G,\, \pi_1(\mathbf{G})) \to H^3_{sym}(G, \pi_2(\mathbf{G}))$) depending on F.

1.5 Computation of standard invariants and groups of extensions

For a non-degenerate braided fusion category $\mathcal{B}$ there is a monoidal 2-equivalence $\mathbf{Mod}(\mathcal{V}ect) = \mathbf{Mod}_{br}(\mathcal{B})$, see Proposition 4.17. In particular, the braided 2-categorical Picard group $\mathbf{Pic}_{br}(\mathcal{B})$ is "trivial" in this case and so (as is well known) is the extension theory: any braided graded extension of $\mathcal{B}$ splits into the tensor product of $\mathcal{B}$ and a pointed braided fusion category.

Thus, the most interesting braided Picard 2-categorical groups come from degenerate tensor categories. In Section 6 we compute the homotopy groups and standard invariants of symmetric fusion categories. For example, the homotopy groups of the braided 2-categorical group $\mathbf{Pic}_{br}(\mathcal{R}ep(G))$, where G is a finite group, are

$$\pi_0 = H^2(G,\, k^\times) \times Z(G), \qquad \pi_1 = H^1(G,\, k^\times), \qquad \pi_2 = H^0(G,\, k^\times) = k^\times,$$

where $Z(G)$ denotes the center of G. The first canonical class (1.18) is the quadratic function

$$H^2(G,\, k^\times) \times Z(G) \;\to\; H^1(G, k^\times), \qquad (\gamma, z) \mapsto \gamma_z(-) = \frac{\gamma(z,-)}{\gamma(-,z)}$$

and the second canonical class is trivial.

The Whitehead product (1.17) is

$$(H^2(G,\, k^\times) \times Z(G)) \times H^1(G,\, k^\times) \;\to\; k^\times, \qquad (\gamma, z) \times \chi \mapsto \chi(z).$$

We determine the corresponding homotopy groups and maps for a general (not necessarily Tannakian) symmetric fusion category in Section 6.3.

We show that the groupoid of symmetric A-extensions of a symmetric tensor category $\mathcal{E}$ has a structure of a symmetric 2-categorical group $\mathbf{Ex_{sym}}(A, \mathcal{E})$. We describe an exact sequence that can be used to compute $\pi_0(\mathbf{Ex_{sym}}(A, \mathcal{E}))$ in Section 2.8. We also determine the group of symmetric extensions of a symmetric fusion category in Theorem 8.26.

1.6 Organization

Section 2 contains the technical tools we need. We include a detailed description of the Eilneberg-Mac Lane cohomology [19] in low degrees and the notions of braided and symmetric monoidal 2-categories and 2-functors between them [12, 28, S]. An important observation is that the axioms of such categories and functors can be viewed as "non-commutative versions" of the higher Eilneberg-Mac Lane cocycle equations (e.g., compare equations (2.7) -(2.10) with commuting polytopes (2.26) - (2.29)). This is parallel to the pentagon axiom of a monoidal category being a non-commutative version of a 3-cocycle equation. Of a special use to us are (braided, symmetric) 2-categorical groups, characterized by invertibility of their cells with respect to the tensor product. Monoidal (braided, symmetric) 2-functors from a finite group (viewed as discrete 2-categorical group) to (braided, symmetric) 2-categorical groups can be obtained as liftings of usual (braided, symmetric) monoidal functors, provided that certain cohomological obstructions vanish. These obstructions for monoidal (respectively, braided, symmetric) 2-functors and parameterization of liftings are described in Section 2.5 (respectively, Section 2.6, Section 2.7). Symmetric monoidal 2-functors as above form a symmetric 2-categorical group. Its group of isomorphism classes of objects fits into a certain exact sequence (Theorem 2.38).

In Section 3 we recall the 2-category $\mathbf{Mod}(\mathcal{B})$ of module categories over a finite tensor category $\mathcal{B}$. When $\mathcal{B}$ is braided, $\mathbf{Mod}(\mathcal{B})$ is a monoidal 2-category. Its tensor product can be defined either by a universal property or by an explicit construction, see Remark 3.6.

Section 4 deals with braided module categories over a braided tensor category $\mathcal{B}$ introduced and studied by Enriquez [21], Brochier [5], and Ben-Zvi, Brochier, and Jordan [2]. In such categories the action of $\mathcal{B}$ has an additional symmetry compatible with the braiding of $\mathcal{B}$ (Definition 4.1). Equivalently, a module braiding on a $\mathcal{B}$-module category $\mathcal{M}$ is the same thing as a natural tensor isomorphism between the α-inductions [4] $\alpha^{\pm}_{\mathcal{M}} : \mathcal{B}^{\mathrm{op}} \to \mathcal{E}nd_{\mathcal{B}}(\mathcal{M})$ (Proposition 4.9). The 2-category $\mathbf{Mod_{br}}(\mathcal{B})$ of braided $\mathcal{B}$-module categories is 2-equivalent to the 2-center of $\mathbf{Mod}(\mathcal{B})$ (Theorem 4.11). In particular, $\mathbf{Mod_{br}}(\mathcal{B})$ is braided. The easiest examples of braided $\mathcal{B}$-module categories come from tensor automorphisms of $\mathrm{Id}_{\mathcal{B}}$, we describe these in Section 4.3. We also prove in Proposition 4.17 that $\mathbf{Mod_{br}}(\mathcal{B}) \cong \mathbf{Mod_{br}}(\mathcal{V}ect)$ when $\mathcal{B}$ is a

non-degenerate braided fusion category. Finally, module categories over a symmetric tensor category $\mathcal{E}$ can be equipped with the identity $\mathcal{E}$-module braiding and so they form a symmetric monoidal 2-category $\mathbf{Mod_{sym}}(\mathcal{E})$. We prove in Proposition 4.21 that the induction $\mathbf{Mod_{sym}}(\mathcal{Z}_{sym}(\mathcal{B})) \to \mathbf{Mod_{br}}(\mathcal{B})$ of braided module categories from the symmetric center of $\mathcal{B}$ is a braided monoidal 2-functor.

In Section 5 we describe various 2-categorical Picard groups associated to tensor categories. These are parts of the corresponding monoidal 2-categories consisting of invertible module categories. The new ones are the braided Picard 2-categorical group $\mathbf{Pic_{br}}(\mathcal{B}) = \mathbf{Inv}(\mathbf{Mod_{br}}(\mathcal{B}))$ of a braided tensor category $\mathcal{B}$ and the symmetric Picard 2-categorical group $\mathbf{Pic_{sym}}(\mathcal{E}) = \mathbf{Inv}(\mathbf{Mod_{sym}}(\mathcal{E}))$ of a symmetric tensor category $\mathcal{E}$. We describe their homotopy groups, canonical classes, and Whitehead brackets. Proposition 5.4 provides an exact sequence featuring the group $\pi_0(\mathbf{Pic_{br}}(\mathcal{B}))$ that can be seen as a sequence of homotopy groups of a certain fibration. Here we also describe Azumaya algebras in braided tensor categories, as they give a convenient description of invertible module categories.

Section 6 is dedicated to the braided 2-categorical Picard group of a symmetric fusion category $\mathcal{E}$. We recall the computation of $Pic(\mathcal{E})$ due to Carnovale [7] and use it to describe the braided categorical Picard group of $\mathcal{E}$ and its canonical classes.

In Section 7 we compute the braided categorical Picard group of a pointed braided fusion category $\mathcal{B}$. We show that in this case there is a braided monoidal equivalence of braided categorical groups $\mathcal{P}ic_{br}(\mathcal{B}) \cong \mathcal{P}ic_{br}(\mathcal{Z}_{sym}(\mathcal{B}))$, where $\mathcal{Z}_{sym}(\mathcal{B})$ is the symmetric center of $\mathcal{B}$, see Proposition 7.1.

Finally, Section 8 contains a classification of graded extensions. Tensor (respectively, central, braided, and symmetric) graded extensions are classified in Theorem 8.5 (respectively, Theorem 8.13, Theorem 8.18, and Theorem 8.24). We compute the group of symmetric extensions of a symmetric fusion category in Theorem 8.26. Here we also explain that the zesting procedure studied in [14] can be understood as a deformation of a braided monoidal functor $A \to \mathcal{P}ic_{br}(\mathcal{B})$ and compute Pontryagin-Whitehead obstructions to existence of extensions in this case.

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2 Higher categorical groups and group cohomology

2.1 Eilenberg-Mac Lane cohomology

We denote by $C^*(A, M)$ the normalized standard complex of the abelian group A with coefficients in the trivial A-module M.

Remark 2.1. We will refer to cochain complexes for the second, third, and fourth Eilenberg-Mac Lane cohomology groups as *braided*, *sylleptic*, and *symmetric*, respectively. This is justified since such cochains give rise to braided, sylleptic, and symmetric 2-categorical groups, see Sections 2.2 and 2.4. The explicit descriptions of these complexes are recalled below.

We denote by $C^*_{br}(A, M) = C^{*+1}(K(A, 2), M)$ the normalised standard complex computing the second Eilenberg-Mac Lane cohomology [19]. The first few terms of the cochain complex $C^*_{br}(A, M)$ are as follows:

$$
\begin{aligned}
&C^0_{br}(A, M) = C^0(A, M) = M, \qquad C^1_{br}(A, M) = C^1(A, M), \qquad C^2_{br}(A, M) = C^2(A, M),\\
&C^3_{br}(A, M) = C^3(A, M) \oplus C^2(A, M) = \{(a(-,-,-), a(-|-))\},\\
&C^4_{br}(A, M) = C^4(A, M) \oplus C^3(A, M) \oplus C^3(A, M) = \{(a(-,-,-,-), a(-,-|-), a(-|-,-))\},\\
&C^5_{br}(A, M) = C^5(A, M) \oplus C^4(A, M) \oplus C^4(A, M) \oplus C^4(A, M) \oplus C^3(A, M)\\
&\qquad\qquad = \{(a(-,-,-,-,-), a(-,-,-|-), a(-,-|-,-), a(-|-,-,-), a(-|-|-))\}
\end{aligned}
$$

with the differentials

$$d : C^2_{br}(A, M) \to C^3_{br}(A, M)$$

$$
\begin{aligned}
d(a)(x, y, z) &= a(y, z) - a(xy, z) + a(x, yz) - a(x, y), &(2.1)\\
d(a)(x|y) &= a(y, x) - a(x, y); &(2.2)
\end{aligned}
$$

$$d : C^3_{br}(A, M) \to C^4_{br}(A, M)$$

$$
\begin{aligned}
d(a)(x, y, z, w) &= a(y, z, w) - a(xy, z, w) + a(x, yz, w) - a(x, y, zw) + a(x, y, z), &(2.3)\\
d(a)(x, y|z) &= a(y|z) - a(xy|z) + a(x|z) + a(x, y, z) - a(x, z, y) + a(z, x, y), &(2.4)\\
d(a)(x|y, z) &= a(x|y) - a(x|yz) + a(x|z) - a(x, y, z) + a(y, x, z) - a(y, z, x), &(2.5)
\end{aligned}
$$

and

$$d : C^4_{br}(A, M) \to C^5_{br}(A, M)$$

$$d(a)(x,y,z,w,u) = a(y,z,w,u) - a(xy,z,w,u) + a(x,yz,w,u) - a(x,y,zw,u) + a(x,y,z,wu) - a(x,y,z,w), \tag{2.6}$$

$$d(a)(x|y,z,w) = a(x|z,w) - a(x|yz,w) + a(x|y,zw) - a(x|y,z) - a(x,y,z,w) + a(y,x,z,w) - a(y,z,x,w) + a(y,z,w,x), \tag{2.7}$$

$$d(a)(x,y,z|w) = a(y,z|w) - a(xy,z|w) + a(x,yz|w) - a(x,y|w) - a(x,y,z,w) + a(x,y,w,z) - a(x,w,y,z) + a(w,x,y,z), \tag{2.8}$$

$$d(a)(x,y|z,w) = a(y|z,w) - a(xy|z,w) + a(x|z,w) - a(x,y|w) + a(x,y|zw) - a(x,y|z) + a(x,y,z,w) - a(x,z,y,w) + a(z,x,y,w) + a(x,z,w,y) - a(z,x,w,y) + a(z,w,x,y), \tag{2.9}$$

$$d(a)(x|y|z) = -a(x,y|z) + a(y,x|z) - a(x|y,z) + a(x|z,y), \qquad x,y,z,w \in A. \tag{2.10}$$

Example 2.2. The first few terms of the cochain complex $C^*_{br}(\mathbb{Z}/2\mathbb{Z}, M)$ are

$$M \xrightarrow{0} M \xrightarrow{d_1} M \xrightarrow{0} M^2 \xrightarrow{d_3} M^3 \xrightarrow{d_4} M^5 \longrightarrow ...$$

where M^n is the direct sum of n copies of M and

$$\begin{aligned} d_1(m) &= 2m \\ d_3(m,l) &= (2m, 2l+m, 2l-m) \\ d_4(m,l,k) &= (0,0,2(m-l+k),0,0) \end{aligned}$$

Thus the first few braided cohomology groups are

$$\begin{aligned} H^0_{br}(\mathbb{Z}/2\mathbb{Z}, M) &= M, \\ H^1_{br}(\mathbb{Z}/2\mathbb{Z}, M) &= M_2, \\ H^2_{br}(\mathbb{Z}/2\mathbb{Z}, M) &= M/2M, \\ H^3_{br}(\mathbb{Z}/2\mathbb{Z}, M) &= M_4, \\ H^4_{br}(\mathbb{Z}/2\mathbb{Z}, M) &= M_2 \oplus M/4M. \end{aligned}$$

Here $M_s = \{m \in M |\ sm = 0\}$.

We denote by $C^*_{syl}(A, M) = C^{*+2}(K(A,3), M)$ the normalised standard complex computing the third Eilenberg-Mac Lane cohomology [19]. The first few terms of the cochain complex $C^*_{syl}(A, M)$ are as

follows:

$$
\begin{aligned}
&C^0_{syl}(A,M) = M, \quad C^1_{syl}(A,M) = C^1(A,M),\\
&C^2_{syl}(A,M) = C^2(A,M), \quad C^3_{syl}(A,M) = C^3_{br}(A,M),\\
&C^4_{syl}(A,M) = C^4_{br}(A,M) \oplus C^2(A,M) = \{(a(-,-,-,-), a(-,-|-), a(-|-,-), a(-||-))\},\\
&C^5_{syl}(A,M) = C^5_{br}(A,M) \oplus C^3(A,M) \oplus C^3(A,M) =\\
&\qquad = \{(a(-,-,-,-,-), a(-,-,-|-), a(-,-|-,-), a(-|-,-,-),\\
&\qquad\quad a(-|-|-), a(-,-||-), a(-||-,-))\}
\end{aligned}
$$

with the additional differentials

$$
\begin{aligned}
&d : C^3_{syl}(A,M) \to C^4_{syl}(A,M)\\
&d(a)(x||y) = a(x|y) + a(y|x),
\end{aligned}
\tag{2.11}
$$

and

$$
\begin{aligned}
&d : C^4_{syl}(A,M) \to C^5_{syl}(A,M)\\
&d(a)(x||y,z) = a(x|y,z) + a(y,z|x) + a(x||y) + a(x||z) - a(x||yz),
\end{aligned}
\tag{2.12}
$$

$$
d(a)(x,y||z) = a(x,y|z) + a(z|x,y) + a(x||z) + a(y||z) - a(xy||z). \tag{2.13}
$$

Example 2.3. The first few sylleptic cohomology groups of $\mathbb{Z}/2\mathbb{Z}$ are

$$
\begin{aligned}
H^0_{syl}(\mathbb{Z}/2\mathbb{Z}, M) &= M,\\
H^1_{syl}(\mathbb{Z}/2\mathbb{Z}, M) &= M_2,\\
H^2_{syl}(\mathbb{Z}/2\mathbb{Z}, M) &= M/2M,\\
H^3_{syl}(\mathbb{Z}/2\mathbb{Z}, M) &= M_2,\\
H^4_{syl}(\mathbb{Z}/2\mathbb{Z}, M) &= M_2 \oplus M/2M.
\end{aligned}
$$

We denote by $C^*_{sym}(A,M) = C^{*+3}(K(A,4),M)$ the normalised standard complex computing the fourth Eilenberg-Mac Lane cohomology [19]. The first few terms of the cochain complex $C^*_{sym}(A,M)$ are as follows:

$$
\begin{aligned}
&C^0_{sym}(A,M) = M, \qquad C^1_{sym}(A,M) = C^1(A,M), \qquad C^2_{sym}(A,M) = C^2(A,M),\\
&C^3_{sym}(A,M) = C^3_{br}(A,M), \qquad C^4_{sym}(A,M) = C^4_{syl}(A,M),\\
&C^5_{sym}(A,M) = C^5_{syl}(A,M) \oplus C^2(A,M) =\\
&\quad = \{(a(-,-,-,-,-), a(-,-,-|-), a(-,-|-,-), a(-|-,-,-),\\
&\quad\ a(-|-|-), a(-,-||-), a(-||-,-), a(-|||-))\}
\end{aligned}
$$

with the additional differential

$$d : C^4_{sym}(A, M) \to C^5_{sym}(A, M)$$

$$d(a)(x|||y) = a(x||y) - a(y||x), \qquad x, y \in A. \tag{2.14}$$

Example 2.4. The first few level 4 cohomology groups of $\mathbb{Z}/2\mathbb{Z}$ are the same as the symmetric cohomology

$$H^n_{sym}(\mathbb{Z}/2\mathbb{Z}, M) = H^n_{syl}(\mathbb{Z}/2\mathbb{Z}, M), \qquad n \leq 4.$$

Example 2.5. It is immediate from the definitions that

$$H^0_{br}(A,\, M) = H^0_{syl}(A,\, M) = H^0_{sym}(A,\, M) \cong M,$$

$$H^1_{br}(A,\, M) = H^1_{syl}(A,\, M) = H^1_{sym}(A,\, M) \cong \mathsf{Hom}(A,\, M),$$

$$H^2_{br}(A,\, M) = H^2_{syl}(A,\, M) = H^2_{sym}(A,\, M) \cong \mathsf{Ext}(A,\, M).$$

It was shown in [20] that there are isomorphisms

$$H^3_{br}(A,\, M) \cong \mathsf{Quad}(A,\, M), \qquad H^3_{syl}(A,\, M) = H^3_{sym}(A,\, M) \cong \mathsf{Hom}(A,\, M_2),$$

given by

$$(a(-,-,-), a(-|-|-)) \mapsto q, \qquad \text{where } q(x) = a(x|x),\ x \in A.$$

Here $\mathsf{Quad}(A, M)$ is the group of quadratic maps and $M_2 = \{m \in M |\ 2m = 0\}$ is the 2-torsion subgroup of M.

The 4th cohomology groups are especially important for our purposes. Let us now assume that M is divisible. The following results are from [20].

Example 2.6. There is an isomorphism

$$\theta_{sym} : H^4_{sym}(A,\, M) \xrightarrow{\sim} \mathsf{Hom}(A_2,\, M) \tag{2.15}$$

assigning to a symmetric 4-cocycle $(a(-,-,-,-), a(-,-|-), a(-|-,-), a(-||-))$ the homomorphism

$$A_2 \to M : x \mapsto a(x, x|x) - a(x|x, x) - a(x, x, x, x). \tag{2.16}$$

There is an isomorphism

$$\theta_{syl} : H^4_{syl}(A,\, M) \xrightarrow{\sim} \mathsf{Hom}(A_2,\, M) \oplus \mathsf{Hom}(\Lambda^2 A,\, M), \tag{2.17}$$

whose first component is given by (2.16) and the second component assigns to a sylleptic 4-cocycle $(a(-,-,-,-), a(-,-|-), a(-|-,-), a(-||-))$ the homomorphism

$$\Lambda^2 A \to M : x \wedge y \mapsto a(x||y) - a(y||x), \tag{2.18}$$

which is the obstruction for a sylleptic 4-cocycle to be symmetric.

Finally, there is a homomorphism

$$\theta_{br} : H^4_{br}(A,\, M) \to \mathsf{Ext}(A,\, \mathsf{Hom}(A,\, M)), \tag{2.19}$$

which is the obstruction for a braided 4-cocycle to have a sylleptic structure. It is defined as follows. Let $(a(-,-,-,-), a(-,-|-), a(-|-,-))$ be a braided 4-cocycle. For any $x \in A$ define a function $b_x \in C^2(A,\, M)$ by

$$b_x(y,z) = a(x|y,z) + a(y,z|x), \qquad x,y,z \in A.$$

It follows from formulas (2.6) -(2.10) and divisibility of M that b_x is a 2-coboundary. That is there exists a function $a(-||-) \in C^2(A,\, M)$ such that (2.12) vanishes. The function

$$A^3 \to M : (x,y,z) \mapsto a(x||y) - a(y||x) + a(x||z) - a(z||x) - a(x||yz) + a(yz||x)$$

is multiplicative in x and, hence, defines a symmetric 2-cocycle g on A with values in $\mathsf{Hom}(A,\, M)$. This 2-cocycle is cohomologically trivial if and only if the given braided 4-cocycle admits a sylleptic structure. We set $\theta_{br}(a(-,-,-,-), a(-,-|-), a(-|-,-))$ to be the class of g in $\mathsf{Ext}(A,\, \mathsf{Hom}(A,\, M))$.

The kernel of (2.19) is isomorphic to $\mathsf{Hom}(A_2,\, M)$ via (2.16).

2.2 Higher braided monoidal categories and functors

Semistrict monoidal 2-categories were defined by Kapranov and Voevodsky [28] and also by Day and Street [12] under the name of Gray monoids. It was shown that every monoidal 2-category is equivalent to a semistrict one. We refer the reader to these papers and to [33] for basic definitions. All monoidal 2-categories considered in this paper will be assumed semistrict.

Let $\mathcal{F}, \mathcal{H} : \mathbf{M} \to \mathbf{N}$ be 2-functors between 2-categories. Recall that a *pseudo-natural transformation* $P : \mathcal{F} \to \mathcal{H}$ is a collection of 1-morphisms $P_{\mathcal{M}} : \mathcal{F}(\mathcal{M}) \to \mathcal{H}(\mathcal{M})$ and invertible 2-cells

$$\begin{array}{ccc} \mathcal{F}(\mathcal{M}) & \xrightarrow{P_{\mathcal{M}}} & \mathcal{H}(\mathcal{M}) \\ {\scriptstyle \mathcal{F}(F)}\downarrow & \xRightarrow{P_F} & \downarrow{\scriptstyle \mathcal{H}(F)} \\ \mathcal{F}(\mathcal{N}) & \xrightarrow{P_{\mathcal{N}}} & \mathcal{H}(\mathcal{N}), \end{array} \tag{2.20}$$

for all objects $\mathcal{M}$ and 1-morphisms $F : \mathcal{M} \to \mathcal{N}$ in $\mathbf{M}$ such that $P_{\mathrm{Id}_{\mathcal{M}}} = \mathrm{Id}_{P_{\mathcal{M}}}$ and

$$P_{F \circ G} = P_F \circ P_G \tag{2.21}$$

for all composable 1-morphisms F and G.

Let $P, Q : \mathcal{F} \to \mathcal{H}$ be pseudo-natural transformations between 2-functors. A *modification* $\eta : P \to Q$ is a collection of 2-cells

$$\mathcal{F}(\mathcal{M}) \;\; \overset{P_{\mathcal{M}}}{\underset{Q_{\mathcal{M}}}{\Downarrow \eta_{\mathcal{M}}}} \;\; \mathcal{H}(\mathcal{M}), \tag{2.22}$$

for all objects $\mathcal{M}$ in $\mathbf{M}$, natural in 1-morphisms in $\mathbf{M}$.

Definition 2.7. A *(semistrict) braided monoidal* 2*-category* [28, 8, 1] consists of a (semistrict) monoidal 2-category $(\mathbf{M}, \boxtimes, \mathcal{I})$, where $\boxtimes$ is the tensor product, equipped with invertible 2-cells

$$\begin{array}{ccc} \mathcal{M}\boxtimes\mathcal{N} & \xrightarrow{Z\boxtimes\mathcal{N}} & \mathcal{M}'\boxtimes\mathcal{N} \\ {\scriptstyle \mathcal{M}\boxtimes W}\downarrow & \overset{\boxtimes_{Z,W}}{\Longrightarrow} & \downarrow{\scriptstyle \mathcal{M}'\boxtimes W} \\ \mathcal{M}\boxtimes\mathcal{N}' & \xrightarrow{Z\boxtimes\mathcal{N}'} & \mathcal{M}'\boxtimes\mathcal{N}', \end{array} \tag{2.23}$$

for any $Z \in \mathbf{M}(\mathcal{M}, \mathcal{M}')$, $W \in \mathbf{M}(\mathcal{N}, \mathcal{N}')$, and $\mathcal{I}$ is the unit object,

together with a pseudo-natural equivalence (*braiding*)

$$B_{\mathcal{M},\mathcal{N}} : \mathcal{M}\boxtimes\mathcal{N} \to \mathcal{N}\boxtimes\mathcal{M}, \qquad \mathcal{M},\mathcal{N} \in \mathbf{M},$$

invertible 2-cells

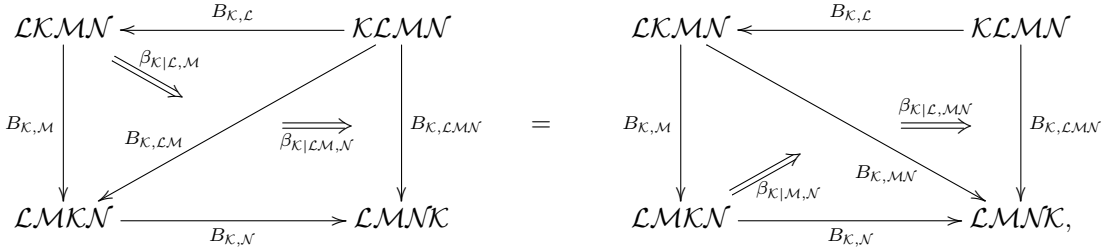
$$\begin{array}{ccc} \mathcal{M}\boxtimes\mathcal{N} & \xrightarrow{Z\boxtimes\mathcal{N}} & \mathcal{M}'\boxtimes\mathcal{N} \\ {\scriptstyle B_{\mathcal{M},\mathcal{N}}}\downarrow & \overset{B_{Z,\mathcal{N}}}{\Longrightarrow} & \downarrow{\scriptstyle B_{\mathcal{M}',\mathcal{N}}} \\ \mathcal{N}\boxtimes\mathcal{M} & \xrightarrow{\mathcal{N}\boxtimes Z} & \mathcal{N}\boxtimes\mathcal{M}', \end{array} \qquad \begin{array}{ccc} \mathcal{M}\boxtimes\mathcal{N} & \xrightarrow{\mathcal{M}\boxtimes W} & \mathcal{M}\boxtimes\mathcal{N}' \\ {\scriptstyle B_{\mathcal{M},\mathcal{N}}}\downarrow & \overset{B_{\mathcal{M},W}}{\Longrightarrow} & \downarrow{\scriptstyle B_{\mathcal{M}',\mathcal{N}}} \\ \mathcal{N}\boxtimes\mathcal{M} & \xrightarrow{W\boxtimes\mathcal{M}} & \mathcal{N}'\boxtimes\mathcal{M}, \end{array} \tag{2.24}$$

satisfying $B_{Z_1,\mathcal{N}} \circ B_{Z_2,\mathcal{N}} = B_{Z_1\otimes Z_2,\mathcal{N}}$ and $B_{\mathcal{M},W_1} \circ B_{\mathcal{M},W_2} = B_{\mathcal{M},W_1\otimes W_2}$, and two invertible modifications[2]

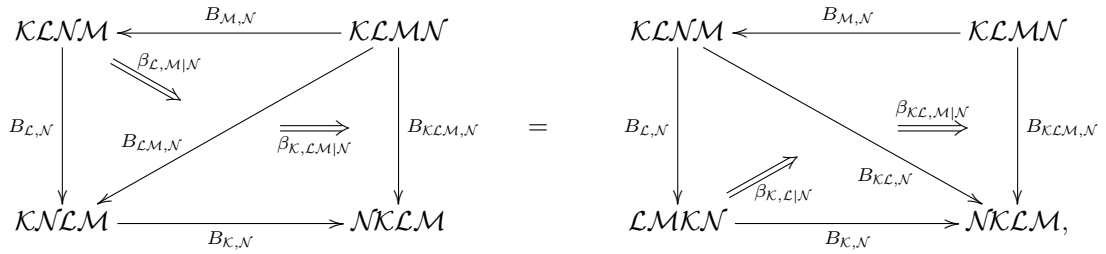
$$\begin{array}{ccc} & \mathcal{L}\mathcal{N}\mathcal{M} & \\ {\scriptstyle B_{\mathcal{M},\mathcal{N}}}\nearrow & \Downarrow \beta_{\mathcal{L},\mathcal{M}|\mathcal{N}} & \searrow{\scriptstyle B_{\mathcal{L},\mathcal{N}}} \\ \mathcal{L}\mathcal{M}\mathcal{N} & \xrightarrow[B_{\mathcal{L}\mathcal{M},\mathcal{N}}]{} & \mathcal{N}\mathcal{L}\mathcal{M} \end{array} \qquad \begin{array}{ccc} & \mathcal{M}\mathcal{L}\mathcal{N} & \\ {\scriptstyle B_{\mathcal{L},\mathcal{M}}}\nearrow & \Downarrow \beta_{\mathcal{L}|\mathcal{M},\mathcal{N}} & \searrow{\scriptstyle B_{\mathcal{L},\mathcal{N}}} \\ \mathcal{L}\mathcal{M}\mathcal{N} & \xrightarrow[B_{\mathcal{L},\mathcal{M}\mathcal{N}}]{} & \mathcal{M}\mathcal{N}\mathcal{L}, \end{array} \tag{2.25}$$

[2]Below we omit the identity functors and the tensor product symbol $\boxtimes$, so we write $\mathcal{M}\mathcal{N}$ for $\mathcal{M}\boxtimes\mathcal{N}$.

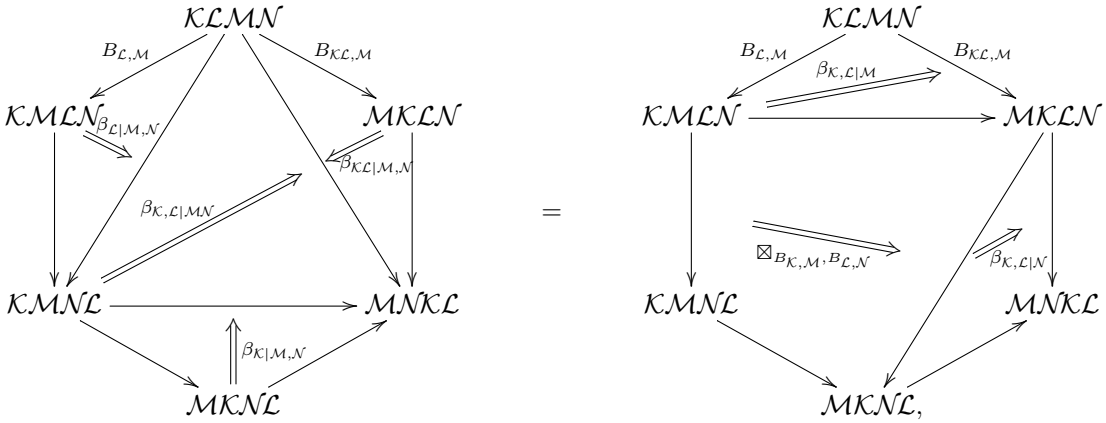
satisfying the following axioms[3]:



(2.26)

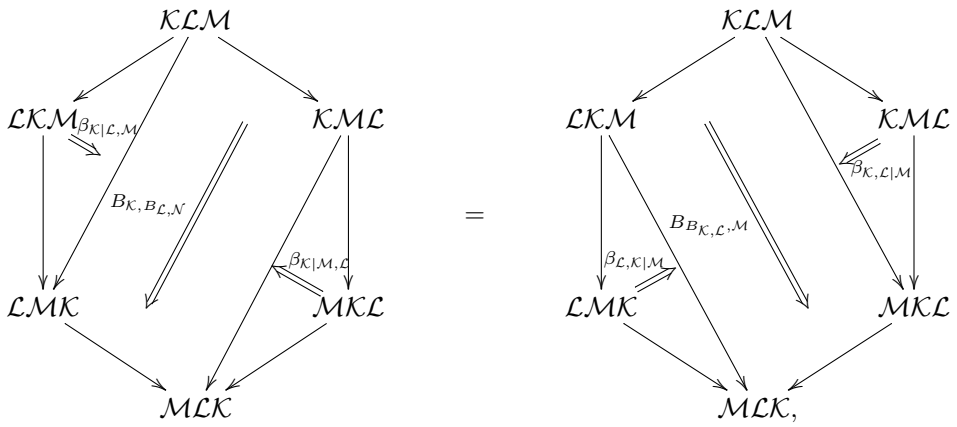


(2.27)



(2.28)

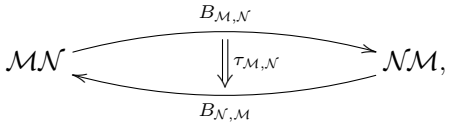
[3]Equalities of 2-cell compositions in this paper can be used to represent commuting polytopes [27]. These polytopes are recovered by gluing the diagrams on both sides of equality along the perimeter



(2.29)

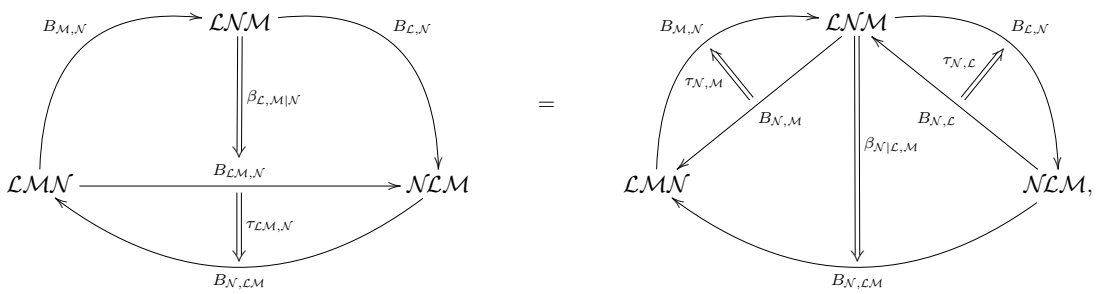
for all $\mathcal{K}$, $\mathcal{L}$, $\mathcal{M}$, $\mathcal{N} \in \mathbf{M}$.

Definition 2.8. A *sylleptic* monoidal 2-category is a braided monoidal 2-category $\mathbf{M}$ with an invertible *syllepsis* modification

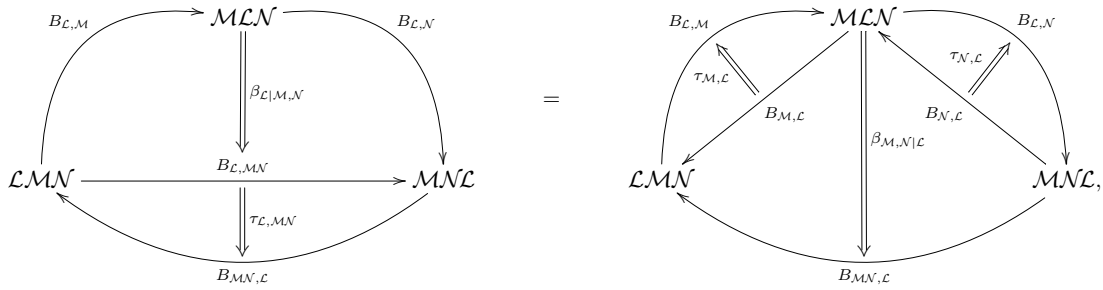


(2.30)

i.e. $\tau_{\mathcal{M},\mathcal{N}}$ is an invertible modification between $B_{\mathcal{N},\mathcal{M}}B_{\mathcal{M},\mathcal{N}}$ and $\mathrm{Id}_{\mathcal{M}\boxtimes\mathcal{N}}$ such that



(2.31)

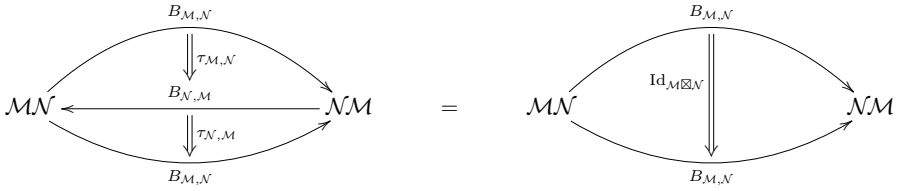


(2.32)

commute for all objects $\mathcal{L}$, $\mathcal{M}$, $\mathcal{N}$ in $\mathbf{M}$.

Definition 2.9. A sylleptic braided monoidal 2-category $\mathbf{M}$ is called *symmetric* if its syllepsis (2.30)

satisfies



(2.33)

for all objects $\mathcal{M}$, $\mathcal{N}$ in $\mathbf{M}$.

Definition 2.10. A *monoidal* 2*-functor* $\mathcal{F} : \mathbf{M} \to \mathbf{M}'$ between monoidal 2-categories is a 2-functor along with a pseudo-natural equivalence

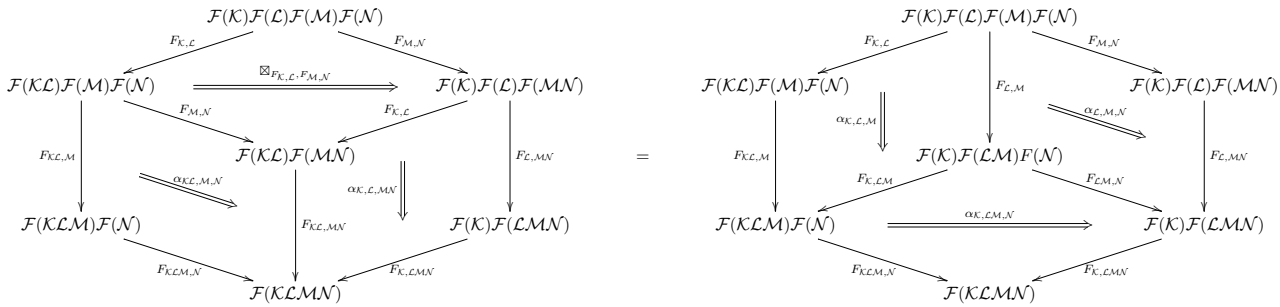
$$F_{\mathcal{M},\mathcal{N}} : \mathcal{F}(\mathcal{M})\mathcal{F}(\mathcal{N}) \to \mathcal{F}(\mathcal{M}\mathcal{N}), \tag{2.34}$$

an equivalence $U : \mathcal{F}(\mathcal{I}) \to \mathcal{I}$, and invertible modifications

$$\begin{array}{ccc} \mathcal{F}(\mathcal{L})\mathcal{F}(\mathcal{M})\mathcal{F}(\mathcal{N}) & \xrightarrow{F_{\mathcal{M},\mathcal{N}}} & \mathcal{F}(\mathcal{L})\mathcal{F}(\mathcal{M}\mathcal{N}) \\ \downarrow{\scriptstyle F_{\mathcal{L},\mathcal{M}}} & \Rightarrow{\scriptstyle \alpha_{\mathcal{L},\mathcal{M},\mathcal{N}}} & \downarrow{\scriptstyle F_{\mathcal{L},\mathcal{M}\mathcal{N}}} \\ \mathcal{F}(\mathcal{L}\mathcal{M})\mathcal{F}(\mathcal{N}) & \xrightarrow{F_{\mathcal{L}\mathcal{M},\mathcal{N}}} & \mathcal{F}(\mathcal{L}\mathcal{M}\mathcal{N}), \end{array} \tag{2.35}$$

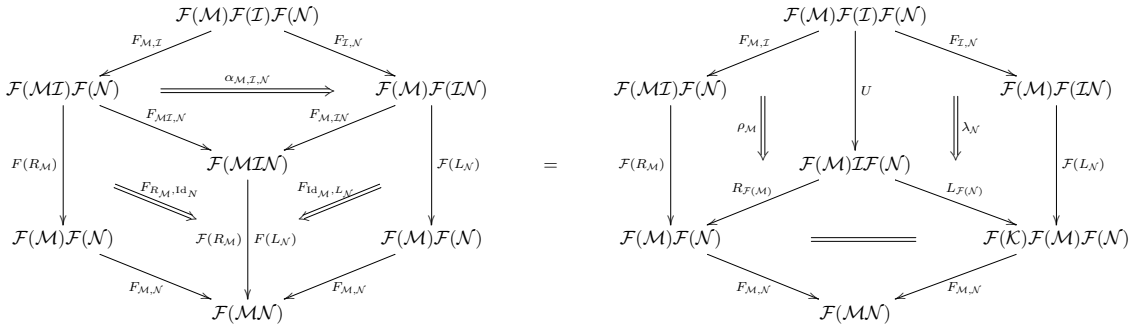
$$\begin{array}{ccc} \mathcal{F}(\mathcal{I})\mathcal{F}(\mathcal{M}) & \xrightarrow{U} & \mathcal{I}\mathcal{F}(\mathcal{M}) \\ \downarrow{\scriptstyle F_{\mathcal{I},\mathcal{M}}} & \Rightarrow{\scriptstyle \lambda_{\mathcal{M}}} & \downarrow{\scriptstyle L_{\mathcal{F}(\mathcal{M})}} \\ \mathcal{F}(\mathcal{I}\mathcal{M}) & \xrightarrow{\mathcal{F}(L_{\mathcal{M}})} & \mathcal{F}(\mathcal{M}), \end{array} \quad \text{and} \quad \begin{array}{ccc} \mathcal{F}(\mathcal{M})\mathcal{F}(\mathcal{I}) & \xrightarrow{U} & \mathcal{F}(\mathcal{M})\mathcal{I}, \\ \downarrow{\scriptstyle F_{\mathcal{M},\mathcal{I}}} & \Rightarrow{\scriptstyle \rho_{\mathcal{M}}} & \downarrow{\scriptstyle R_{\mathcal{F}(\mathcal{M})}} \\ \mathcal{F}(\mathcal{M}\mathcal{I}) & \xrightarrow{\mathcal{F}(R_{\mathcal{M}})} & \mathcal{F}(\mathcal{M}), \end{array} \tag{2.36}$$

where L and R denote the unit constraints of $\mathbf{M}$, such that



(2.37)

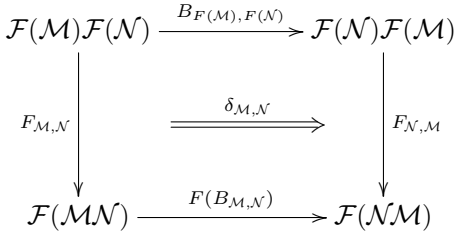
and



(2.38)

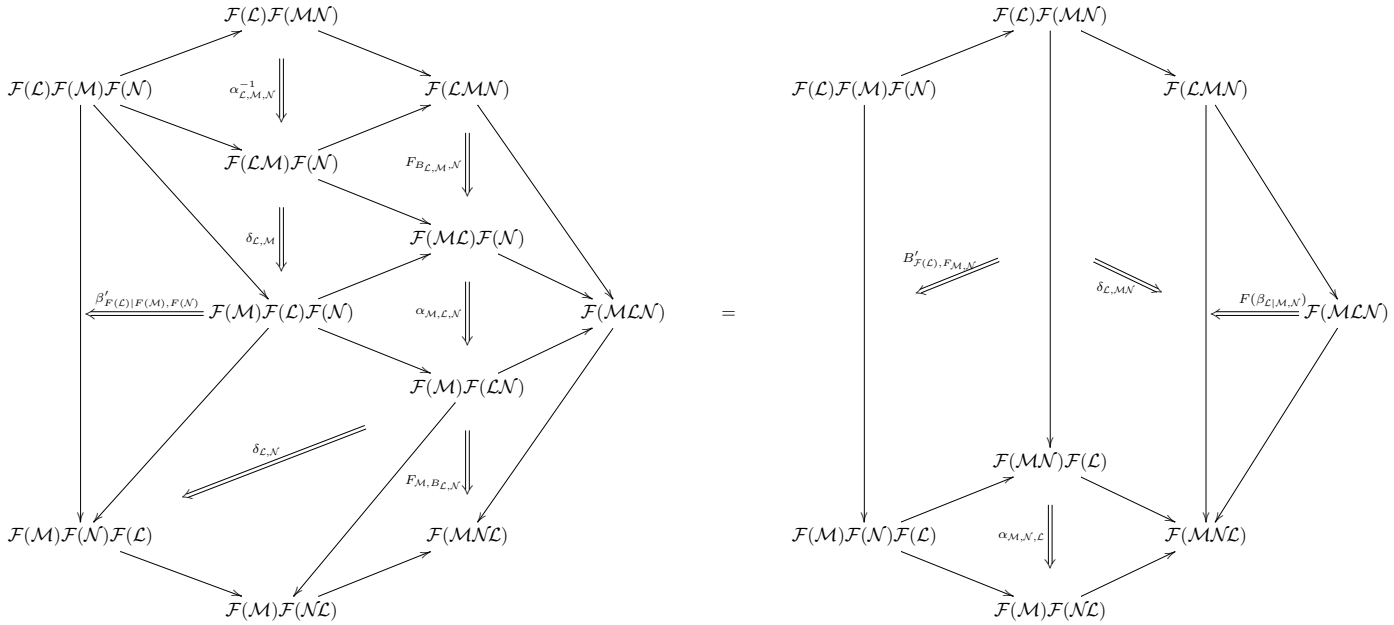
for all $\mathcal{K}, \mathcal{L}, \mathcal{M}, \mathcal{N} \in \mathbf{M}$.

Definition 2.11. A *braided monoidal* 2*-functor* $\mathcal{F} : \mathbf{M} \to \mathbf{M}'$ between braided monoidal 2-categories is a monoidal functor along with an invertible modification



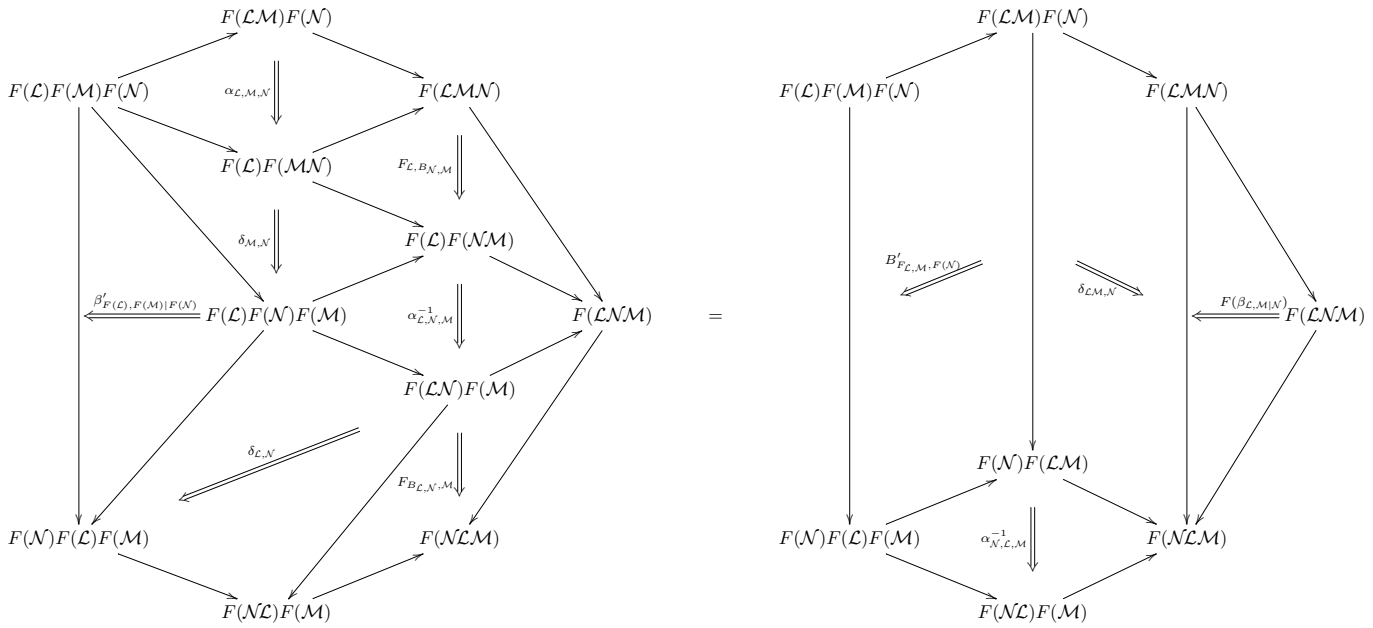
(2.39)

such that



(2.40)

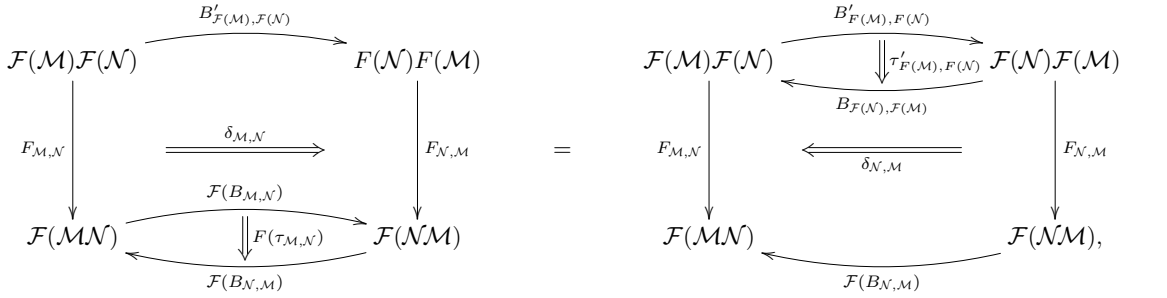
and



(2.41)

for all objects $\mathcal{L}, \mathcal{M}, \mathcal{N}$ in $\mathbf{M}$. Here B, B' denote the braidings on $\mathbf{M}$, $\mathbf{M}'$. We omit 1-cell labels to keep the diagrams readable.

Definition 2.12. A braided monoidal 2-functor $\mathcal{F} : \mathbf{M} \to \mathbf{M}'$ between sylleptic (respectively, symmetric) monoidal 2-categories $\mathbf{M}$ and $\mathbf{M}'$ is *sylleptic* (respectively, *symmetric*) if



(2.42)

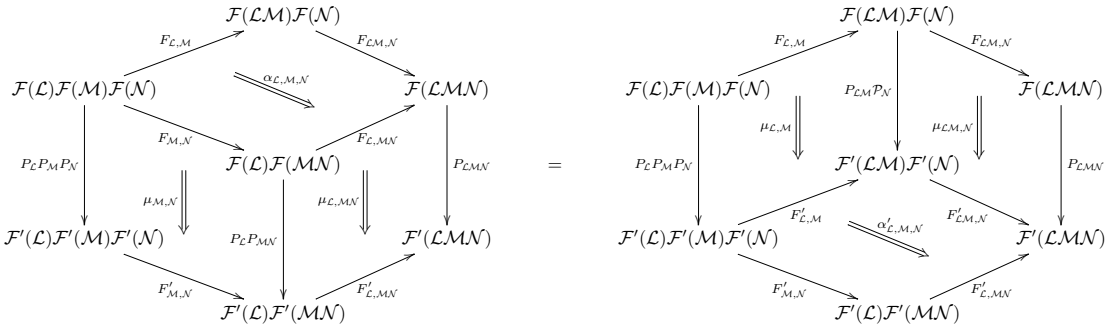
for all $\mathcal{M}$, $\mathcal{N}$ in $\mathbf{M}$. Here τ, τ' are the modification defined in (2.30).

Definition 2.13. Let $\mathcal{F}, \mathcal{F}' : \mathbf{M} \to \mathbf{M}'$ be two monoidal functors. A *monoidal pseudo-natural transfor-*

mation $P : \mathcal{F} \to \mathcal{F}'$ is a pseudo-natural transformation along with an invertible modification

$$\begin{array}{ccc} \mathcal{F}(\mathcal{L})\mathcal{F}(\mathcal{M}) & \xrightarrow{P_{\mathcal{L}}P_{\mathcal{M}}} & \mathcal{F}'(\mathcal{L})\mathcal{F}'(\mathcal{M}) \\ \downarrow{\scriptstyle F_{\mathcal{L},\mathcal{M}}} & \Longrightarrow{\scriptstyle \mu_{\mathcal{L},\mathcal{M}}} & \downarrow{\scriptstyle F'_{\mathcal{L},\mathcal{M}}} \\ \mathcal{F}(\mathcal{L}\mathcal{M}) & \xrightarrow{P_{\mathcal{L}\mathcal{M}}} & \mathcal{F}'(\mathcal{L}\mathcal{M}) \end{array} \tag{2.43}$$

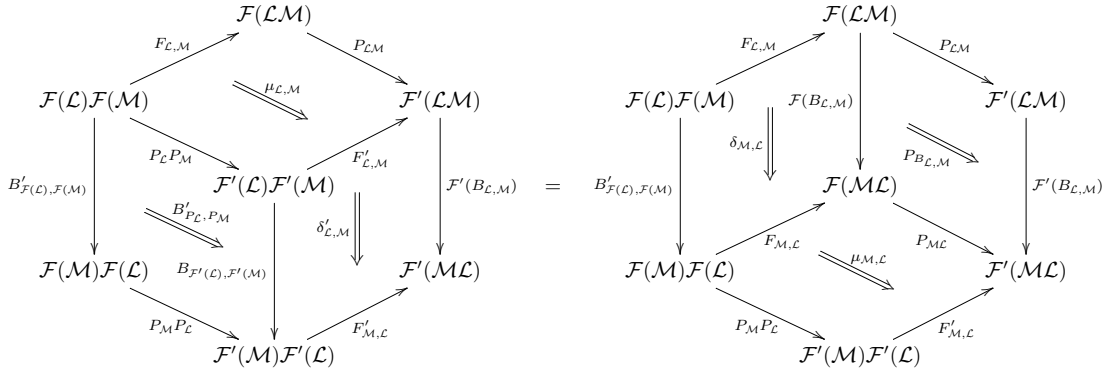
such that



(2.44)

for all $\mathcal{L}, \mathcal{M}, \mathcal{N}$ in $\mathbf{M}$. Here α and α' denote the monoidal structures of $\mathcal{F}$ and $\mathcal{F}'$.

Definition 2.14. A monoidal pseudo-natural transformation $P : \mathcal{F} \to \mathcal{F}'$ between braided monoidal 2-functors is *braided* if

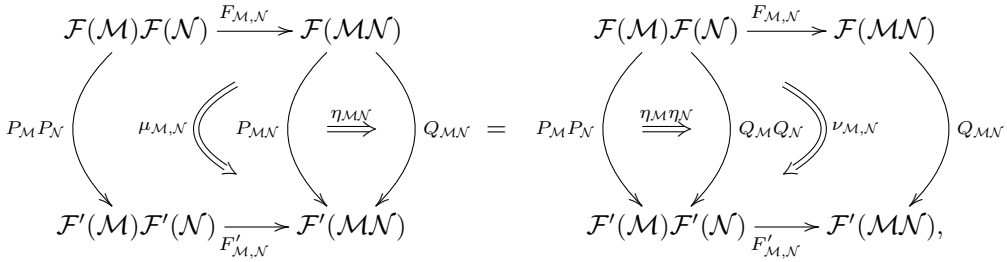


(2.45)

for all $\mathcal{L}, \mathcal{M}$ in $\mathbf{M}$. Here δ and δ' denote the braided structures on F and F'.

Definition 2.15. A modification $\eta : P \to Q$ between two monoidal pseudo-natural transformations

$P, Q : \mathcal{F} \to \mathcal{F}'$ is *monoidal* if



(2.46)

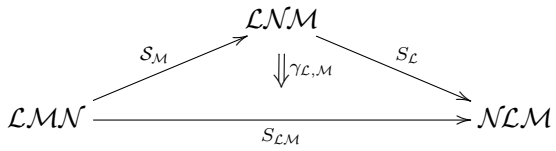
for all $\mathcal{M}, \mathcal{N} \in \mathbf{M}$, where μ and ν are modifications (2.43) defining the monoidal structures on P and Q, respectively.

2.3 The center of a monoidal 2-category

Let $\mathbf{M}$ be a braided monoidal 2-category. Its *center* $\mathbf{Z}(\mathbf{M})$ is a braided monoidal 2-category defined as follows [1, Section 3]. Objects of $\mathbf{Z}(\mathbf{M})$ are triples $(\mathcal{N}, S, \gamma)$ where $\mathcal{N}$ is an object of $\mathbf{M}$, S is a pseudo-natural collection of equivalences (called a *half-braiding*)

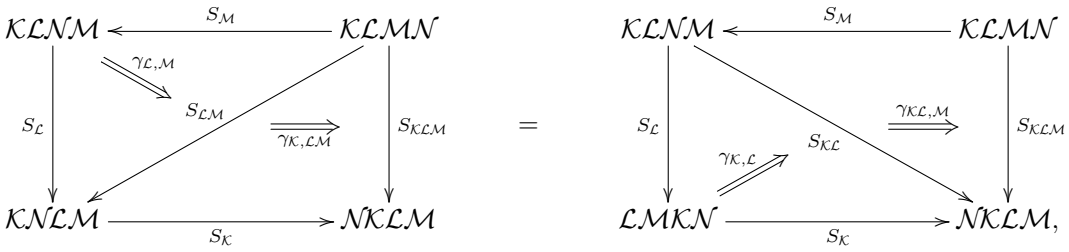
$$S_{\mathcal{M}} : \mathcal{M}\mathcal{N} \to \mathcal{N}\mathcal{M}, \qquad \mathcal{M} \in \mathbf{A},$$

and γ is an invertible modification



(2.47)

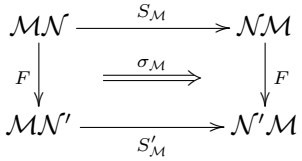
such that



(2.48)

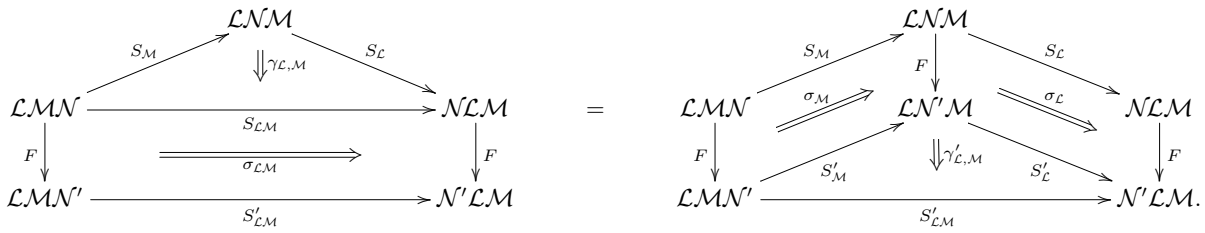
for all $\mathcal{K}$, $\mathcal{L}$, $\mathcal{M} \in \mathbf{M}$.

A morphism between $(\mathcal{N}, S, \gamma)$ and $(\mathcal{N}', S', \gamma')$ in $\mathbf{Z}(\mathbf{M})$ is pair (F, σ), where $F : \mathcal{N} \to \mathcal{N}'$ is a morphism in $\mathbf{M}$ and σ is an invertible modification



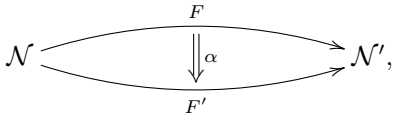
(2.49)

such that



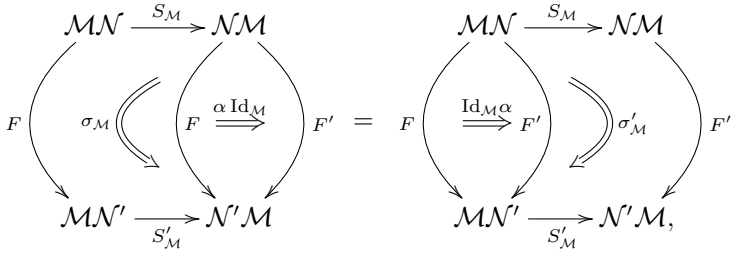
for all $\mathcal{L}$, $\mathcal{M}$ in $\mathbf{M}$.

A 2-morphism in $\mathbf{Z}(\mathbf{M})$ between (F, σ) and (F', σ') is a 2-cell



(2.50)

such that



(2.51)

for all $\mathcal{M}$ in $\mathbf{M}$.

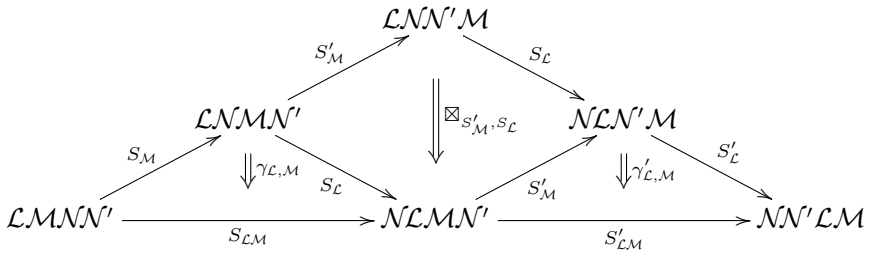
The tensor product in $\mathbf{Z}(\mathbf{M})$ is given by

$$(\mathcal{N},\, S,\, \gamma) \boxtimes (\mathcal{N}',\, S',\, \gamma') = (\mathcal{NN}',\, SS',\, \gamma\gamma'), \tag{2.52}$$

where $\mathcal{NN}'$ is the tensor product in $\mathbf{M}$, $(SS)'_{\mathcal{M}}$ is defined as the composition

$$(SS)'_{\mathcal{M}} : \mathcal{MNN}' \xrightarrow{S_{\mathcal{M}}\, N'} \mathcal{NMN}' \xrightarrow{\mathcal{N}\, S'_{\mathcal{M}}} \mathcal{NN}'\mathcal{M},$$

and $(\gamma\gamma')_{\mathcal{L},\mathcal{M}}$ is given by the following composition of 2-cells



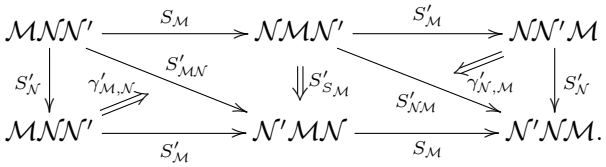
(2.53)

for all $\mathcal{L}, \mathcal{M} \in \mathbf{M}$.

The braiding between $(\mathcal{N},\, S,\, \gamma)$ and $(\mathcal{N}',\, S',\, \gamma')$ is given by

$$(S'_{\mathcal{N}},\, \Sigma) : (\mathcal{N},\, S,\, \gamma) \boxtimes (\mathcal{N}',\, S',\, \gamma') \to (\mathcal{N}',\, S',\, \gamma') \boxtimes (\mathcal{N},\, S,\, \gamma), \tag{2.54}$$

where $\Sigma_{\mathcal{M}}$ is the following composition 2-cell:



Let $\mathbf{M}$ be a braided monoidal 2 category with braiding $B_{\mathcal{M},\mathcal{N}}$ and structure modifications β and γ (2.25). There is a braided monoidal 2-functor

$$\mathcal{F} : \mathbf{M} \to \mathbf{Z}(\mathbf{M}) : \mathcal{N} \mapsto (\mathcal{N},\, B_{-,\mathcal{N}},\, \beta_{-,-|\mathcal{N}})$$

with $F_{N,N'} : \mathcal{F}(\mathcal{N})\mathcal{F}(\mathcal{N}') \to \mathcal{F}(\mathcal{N}\mathcal{N}')$ given by $(\mathrm{Id}_{\mathcal{N}\mathcal{N}'},\, \beta_{-|\mathcal{N},\mathcal{N}'})$ and identity 2-cells α (2.35) and δ (2.39). See [1, 8] for details.

2.4 2-categorical groups

Recall that a *categorical group* is a monoidal category in which every object is invertible with respect to the tensor product and each morphism is an isomorphism.

We call an object $\mathcal{P}$ of a monoidal 2-category $\mathbf{M}$ *invertible* if there is another object $\mathcal{Q}$ together with an equivalence $\mathcal{P} \boxtimes \mathcal{Q} \to \mathcal{I}$, where $\mathcal{I}$ is the unit object of $\mathbf{M}$. Note that in this case the object $\mathcal{Q}$ is unique up to an equivalence.

Note that the tensor products $-\boxtimes\mathcal{P}$ and $\mathcal{P}\boxtimes-$ with an invertible object $\mathcal{P} \in \mathbf{M}$ are 2-autoequivalences of $\mathbf{M}$. In particular, each of them defines an equivalence of monoidal categories $\mathbf{M}(\mathcal{I},\mathcal{I}) \to \mathbf{M}(\mathcal{P},\mathcal{P})$, where $\mathcal{I}$ is the unit object of $\mathbf{M}$.

Definition 2.16. A 2-*categorical group* is a monoidal 2-category whose objects are invertible with respect to the tensor product, whose 1-morphisms are equivalences, and whose 2-cells are isomorphisms.

Example 2.17. Let $\mathbf{A}$ be a 2-category. Then the monoidal 2-category $\mathbf{Aut}(\mathbf{A})$ of autoequivalences of $\mathbf{A}$ with pseudo-natural equivalences as 1-morphisms and isomorphisms as 2-morphisms is a 2-categorical group.

Example 2.18. Let $\mathbf{M}$ be a monoidal 2-category. Then the monoidal 2-category $\mathbf{Inv}(\mathbf{M})$ of invertible objects in $\mathbf{M}$ with equivalences as 1-morphisms and isomorphisms as 2-morphisms is a 2-categorical group.

Let $\mathbf{G}$ be a braided 2-categorical group with the tensor product $\boxtimes$ and unit object $\mathcal{I}$. Below we discuss some invariants of $\mathbf{G}$.

Let $\Pi_{\leq 1}(\mathbf{G})$ denote the 1-categorical truncation of $\mathbf{G}$, i.e. the categorical group whose objects are objects of $\mathbf{G}$ and morphisms are isomorphism classes of 1-cells in $\mathbf{G}$. Let $\Pi_{1\leq}(\mathbf{G}) = \mathbf{G}(\mathcal{I}, \mathcal{I})$ be the braided categorical group of autoequivalences of $\mathcal{I}$. Its braiding is given by the naturality 2-cells

$$\begin{array}{ccc} \mathcal{I}\boxtimes\mathcal{I} & \xrightarrow{\mathrm{Id}\boxtimes g} & \mathcal{I}\boxtimes\mathcal{I} \\ {\scriptstyle f\boxtimes\mathrm{Id}}\downarrow & \xRightarrow{\boxtimes_{f,g}} & \downarrow{\scriptstyle f\boxtimes\mathrm{Id}} \\ \mathcal{I}\boxtimes\mathcal{I} & \xrightarrow[\mathrm{Id}\boxtimes g]{} & \mathcal{I}\boxtimes\mathcal{I} \end{array} \tag{2.55}$$

for all $f, g \in \mathbf{G}(\mathcal{I}, \mathcal{I})$.

Definition 2.19. The *homotopy groups* of $\mathbf{G}$ are defined as follows.

- the 0th homotopy group $\pi_0(\mathbf{G})$ is the group of equivalence classes of objects of $\mathbf{G}$,
- the 1st homotopy group $\pi_1(\mathbf{G})$ is the group of isomorphism classes of autoequivalences in $\mathbf{G}(\mathcal{I}, \mathcal{I})$,
- the 2nd homotopy group $\pi_2(\mathbf{G})$ is the group of automorphisms of the identity 1-morphism $\mathrm{Id}_{\mathcal{I}}$.

Since $\Pi_{1\leq}(\mathbf{G})$ is braided, the homotopy groups $\pi_1(\mathbf{G}), \pi_2(\mathbf{G})$ are abelian.

Definition 2.20. The *first* and *second canonical classes* of $\mathbf{G}$,

$$\alpha_{\mathbf{G}} \in H^3(\pi_0(\mathbf{G}), \pi_1(\mathbf{G})) \quad \text{and} \quad q_{\mathbf{G}} \in H^3_{br}(\pi_1(\mathbf{G}), \pi_2(\mathbf{G})) \tag{2.56}$$

are, respectively, the associator of the categorical group $\Pi_{1\leq}(\mathbf{G})$ and the braided associator of the braided categorical group $\Pi_{1\leq}(\mathbf{G})$.

Proposition 2.21. *There is a monoidal functor*

$$a : \Pi_{\leq 1}(\mathbf{G}) \to \mathcal{A}ut_{br}(\Pi_{1\leq}(\mathbf{G})) \tag{2.57}$$

canonically defined up to a natural isomorphism.

Proof. For any object $\mathcal{P}$ in $\mathbf{G}$ the corresponding autoequivalence $a(\mathcal{P})$ of $\mathbf{G}(\mathcal{I}, \mathcal{I})$ is given by composing the monoidal equivalence

$$\mathbf{G}(\mathcal{I}, \mathcal{I}) \to \mathbf{G}(\mathcal{P}, \mathcal{P}) : f \mapsto f \boxtimes \mathcal{P},\ \alpha \mapsto \alpha \boxtimes \mathcal{P}$$

with the quasi-inverse of

$$\mathbf{G}(\mathcal{I}, \mathcal{I}) \to \mathbf{G}(\mathcal{P}, \mathcal{P}) : f \mapsto \mathcal{P} \boxtimes f,\ \alpha \mapsto \mathcal{P} \boxtimes \alpha.$$

That $a(\mathcal{P})$ is a braided autoequivalence and that a is a monoidal functor follow the naturality properties of $\boxtimes$. □

Remark 2.22. The action (2.57) of $\Pi_{\leq 1}(\mathbf{G})$ on $\Pi_{1\leq}(\mathbf{G})$ can also be recovered from the *adjoint action* of the 2-categorical group $\mathbf{G}$ on itself, i.e. a monoidal 2-functor $\mathbf{Ad} : \mathbf{G} \to \mathbf{Aut}(\mathbf{G})$ characterized (up to an equivalence) by a coherent collection of equivalences $\mathcal{P} \boxtimes \mathcal{X} \to \mathbf{Ad}_{\mathcal{P}}(\mathcal{X}) \boxtimes \mathcal{P}$, pseudo-natural in $\mathcal{X} \in \mathbf{G}$.

The action (2.57) yields canonical group homomorphisms

$$\pi_0(\mathbf{G}) \to Aut(\pi_1(\mathbf{G})), \quad \pi_0(\mathbf{G}) \to Aut(\pi_2(\mathbf{G})), \quad \text{and} \quad \pi_1(\mathbf{G}) \to \mathsf{Hom}(\pi_1(\mathbf{G}), \pi_2(\mathbf{G}))$$

corresponding, respectively, to the actions of objects of $\Pi_{\leq 1}(\mathbf{G})$ on objects and morphisms of $\Pi_{\leq 1}(\mathbf{G})$ and to the action of the group of automorphisms of $\mathcal{I}$ by automorphisms of $\mathrm{Id}_{\Pi_{\leq 1}(\mathbf{G})}$. We will refer to the corresponding maps

$$\pi_0(\mathbf{G}) \times \pi_1(\mathbf{G}) \to \pi_1(\mathbf{G}), \quad \pi_0(\mathbf{G}) \times \pi_2(\mathbf{G}) \to \pi_2(\mathbf{G}), \quad \text{and} \quad \pi_1(\mathbf{G}) \times \pi_1(\mathbf{G}) \to \pi_2(\mathbf{G}) \tag{2.58}$$

as the *Whitehead brackets*.

Note that the first canonical class $\alpha_{\mathbf{G}}$ is invariant with respect to the action of $\pi_0(\mathbf{G})$ and that the bimultiplicative pairing $\pi_1(\mathbf{G}) \times \pi_1(\mathbf{G}) \to \pi_2(\mathbf{G})$ is given by the polarization of the second canonical class $q_{\mathbf{G}}$.

Suppose that $\mathbf{G}$ is a braided 2-categorical group. In this case $\Pi_{\leq 1}(\mathbf{G})$ is a braided categorical group and $\Pi_{1\leq}(\mathbf{G})$ is a symmetric categorical group. Hence, the canonical classes (2.56) get promoted to

$$\alpha_{\mathbf{G}} \in H^3_{br}(\pi_0(\mathbf{G}), \pi_1(\mathbf{G})) \quad \text{and} \quad q_{\mathbf{G}} \in H^3_{sym}(\pi_1(\mathbf{G}), \pi_2(\mathbf{G})). \tag{2.59}$$

The braiding of $\mathbf{G}$ gives a trivialization of the functor (2.57) which implies that Whitehead brackets (2.58) are trivial and yields a new bilinear pairing

$$[\,,\,] : \pi_0(\mathbf{G}) \times \pi_1(\mathbf{G}) \to \pi_2(\mathbf{G}) \tag{2.60}$$

constructed as follows. For each object $\mathcal{P}$ in $\mathbf{G}$, or an element of $\pi_0(\mathbf{G})$, we have a canonical monoidal automorphism of $a(\mathcal{P}) = \mathrm{Id}_{\mathcal{I}}$, i.e. a homomorphism $\pi_1(\mathbf{G}) \to \pi_2(\mathbf{G})$. This gives a homomorphism $\pi_0(\mathbf{G}) \to \mathsf{Hom}(\pi_1(\mathbf{G}), \pi_2(\mathbf{G}))$ identified with (2.60).

For a symmetric 2-categorical group $\mathbf{G}$ the canonical classes are

$$\alpha_{\mathbf{G}} \in H^3_{sym}(\pi_0(\mathbf{G}), \pi_1(\mathbf{G})) \quad \text{and} \quad q_{\mathbf{G}} \in H^3_{sym}(\pi_1(\mathbf{G}), \pi_2(\mathbf{G})). \tag{2.61}$$

All Whitehead brackets are trivial in this case.

2.5 Monoidal 2-functors between 2-categorical groups

Let G be a group. We consider it as a 2-categorical group with identity 1- and 2-morphisms.

Let $\mathbf{G}$ be a 2-categorical group (viewed as a semistrict monoidal 2-category) with the corresponding canonical classes

$$\alpha_{\mathbf{G}} \in H^3(\pi_0(\mathbf{G}),\, \pi_1(\mathbf{G})) \quad \text{and} \quad (\omega_{\mathbf{G}},\, c_{\mathbf{G}}) \in H^3_{br}(\pi_1(\mathbf{G}),\, \pi_2(\mathbf{G})).$$

Let $C : G \to \Pi_{\leq 1}(\mathbf{G}) : x \to \mathcal{C}_x$ be a monoidal functor. This means that there are 0-cells $\mathcal{C}_x$ in $\mathbf{G}$, 1-isomorphisms $M_{x,y} : \mathcal{C}_x\mathcal{C}_y \to \mathcal{C}_{xy}$, and invertible 2-cells

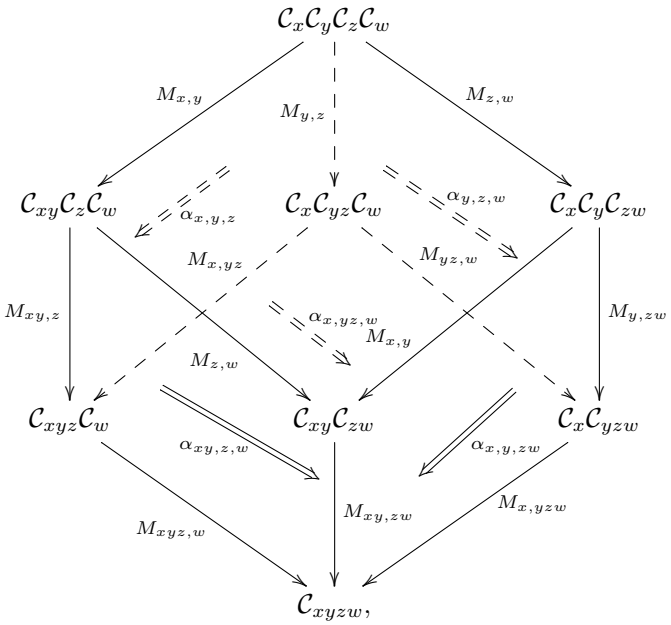
$$\begin{array}{ccc} \mathcal{C}_x\mathcal{C}_y\mathcal{C}_z & \xrightarrow{M_{y,z}} & \mathcal{C}_x\mathcal{C}_{yz} \\ {\scriptstyle M_{x,y}}\downarrow & \overset{\alpha_{x,y,z}}{\Longrightarrow} & \downarrow{\scriptstyle M_{x,yz}} \\ \mathcal{C}_{xy}\mathcal{C}_z & \xrightarrow{M_{xy,z}} & \mathcal{C}_{xyz}, \end{array} \tag{2.62}$$

for all $x, y, z \in G$.

Note that C gives rise to an action of G on $\pi_1(\mathbf{G})$ (obtained by composing the underlying group homomorphism $G \to \pi_0(\mathbf{G})$ with the action of $\pi_0(\mathbf{G})$ on $\pi_1(\mathbf{G})$). We denote this action by $(g,\, Z) \mapsto Z^g$.

Following [23] define a 4-cochain $p^0_C : G^4 \to \pi_2(\mathbf{G})$ by setting $p^0_C(x, y, z, w)$ to be the composition of faces of the following cube:

$p^0_C(x,y,z,w) =$



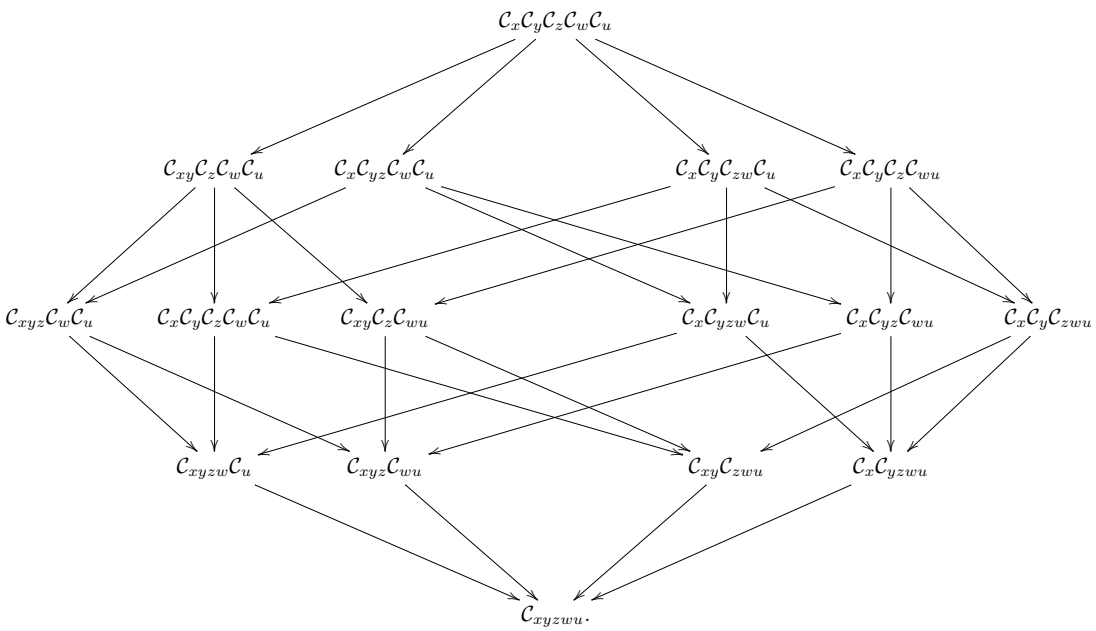
(2.63)

where the top face is given by $\boxtimes_{M_{x,y},M_{z,w}}$. That is, we view $p^0_C(x, y, z, w)$ as a 2-automorphism of the composition of morphisms between opposite corners, e.g., of $M_{xyz,w}M_{xy,z}M_{x,y}$, $x, y, z, w \in G$ (the 2-automorphisms of other compositions are conjugate to this one). We use this convention for other polytopes in this paper.

Proposition 2.23. *p_C^0 is a 4-cocycle whose cohomology class in $H^4(G, \pi_2(\mathbf{G}))$ depends only on the isomorphism class of C. A monoidal functor $C : G \to \Pi_{\leq 1}(\mathbf{G})$ extends to a monoidal 2-functor $G \to \mathbf{G}$ if and only if $p_C^0 = 0$ in $H^4(G, \pi_2(\mathbf{G}))$.*

Proof. Consider the following polytope (its planar projection is pictured):

(2.64)



The edges of this polytope are isomorphisms $M_{x,y}$. The faces are cells $\alpha_{x,y,z}$, $x, y, z \in G$, (2.62) and $\boxtimes_{f,g}$. The polytope (2.64) consists of 8 cubes (four containing the top vertex and four containing the bottom one) glued together in such a way that each of their 48 faces belongs to exactly two cubes (so that the boundary is empty). Six of these cubes are of the form (2.63); their composition is the differential of p_C^0. The remaining two cubes commute due to the naturality of the tensor product in $\mathbf{G}$. Namely, $M_{x,y}$ commutes with the 2-cell $\alpha_{z,w,u}$ and $M_{w,u}$ commutes with the 2-cell $\alpha_{x,y,z}$. Thus, $d(p_C^0) = 1$.

A different choice of 2-cells (2.62) results in multiplying p_C^0 by a 4-coboundary, so its class in $H^4(G, \pi_2(\mathbf{G}))$ is well-defined.

Finally, C extends to a monoidal 2-functor if the 2-cells (2.62) can be chosen in such a way that (2.37) is satisfied. This is equivalent to commutativity of the cube (2.63) (the latter is obtained by gluing the two sides of (2.37)), i.e. to p_C^0 being cohomologically trivial. □

For $L \in H^2(G, \pi_1(\mathbf{G}))$ the monoidal functor $L \cdot C : G \to \Pi_{\leq 1}(\mathcal{G})$ is obtained by multiplying $M_{x,y}$ by $L_{x,y}$ for all $x, y \in G$.

Let $C : G \to \Pi_{\leq 1}(\mathbf{G})$ be a monoidal functor with the monoidal structure $M_{x,y} : \mathcal{C}_x\mathcal{C}_y \to \mathcal{C}_{xy}$, $x, y \in G$.

The group $Aut(C)$ of automorphisms of C is isomorphic to $H^1(G, \pi_1(\mathbf{G}))$. Explicitly, $P \in Aut(C)$

corresponds to a collection of equivalences $P_x : \mathcal{C}_x \to \mathcal{C}_x$ such that there are invertible 2-cells

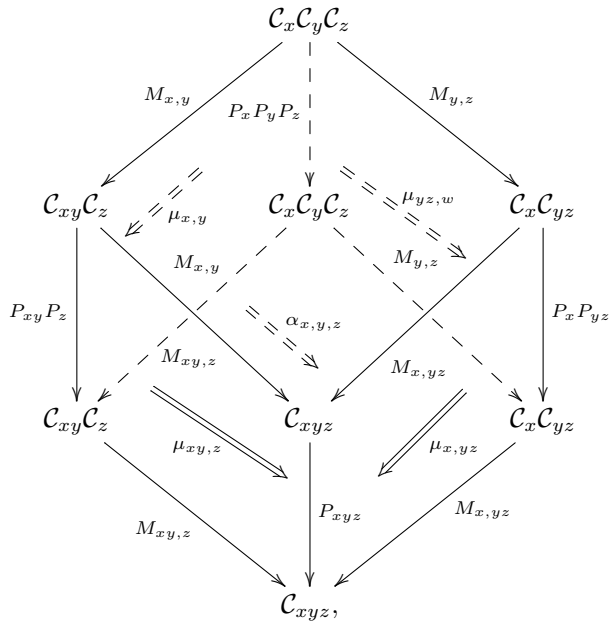
$$\begin{array}{ccc} \mathcal{C}_x\mathcal{C}_y & \xrightarrow{P_xP_y} & \mathcal{C}_x\mathcal{C}_y \\ {\scriptstyle M_{x,y}}\downarrow & \overset{\mu_{x,y}}{\Longrightarrow} & \downarrow{\scriptstyle M_{x,y}} \\ \mathcal{C}_{xy} & \xrightarrow{P_{xy}} & \mathcal{C}_{xy} \end{array} \tag{2.65}$$

for all $x, y \in G$.

Suppose that a monoidal functor $C : G \to \Pi_{\leq 1}(\mathbf{G})$ extends to a monoidal 2-functor $\mathcal{C} : G \to \mathbf{G}$. That is, there is a choice of invertible 2-cells (2.62) such that the cubes (2.63) commute, i.e. $p_C^0 = 1$. Let P be a monoidal automorphism of C.

Define a function $p_C^1(P) : G^3 \to \pi_2(\mathbf{G})$ by

$p_C^1(P)(x,y,z) =$ (2.66)



for all $x, y, z \in G$. Here the top and bottom faces are $\alpha_{x,y,z}$.

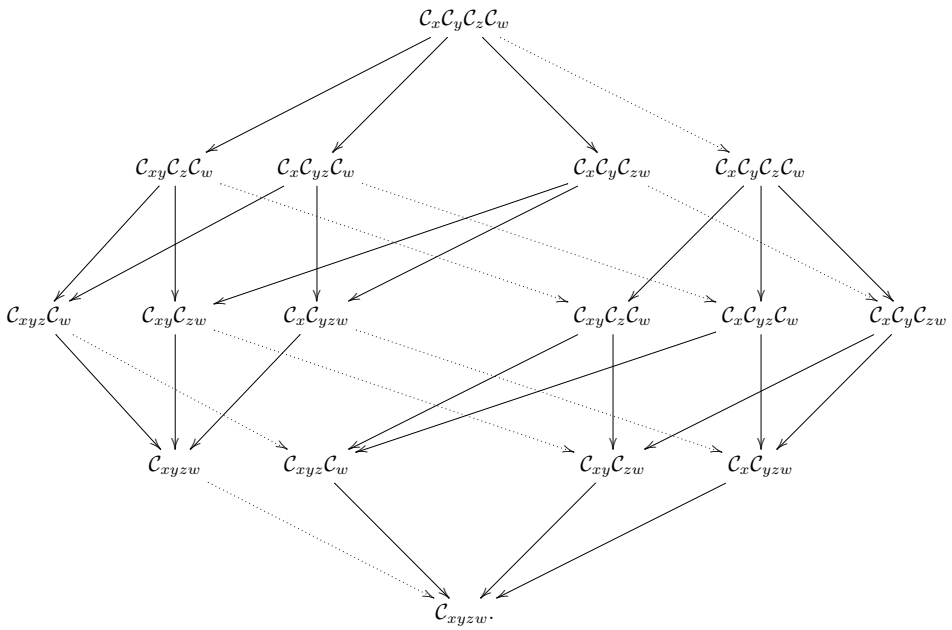
Proposition 2.24. *$p_C^1(P)$ is a 3-cocycle and the map*

$$p_C^1 : Aut(C) = H^1(G, \pi_1(\mathbf{G})) \to H^3(G, \pi_2(\mathbf{G})) : P \mapsto p_C^1(P) \tag{2.67}$$

is a well defined homomorphism. The automorphism P extends to a monoidal pseudo-natural automorphism of $\mathcal{C}$ if and only if $p_C^1(P) = 0$ in $H^3(G, \pi_2(\mathbf{G}))$.

Proof. Consider the following polytope (its planar projection is pictured):

(2.68)



The solid arrows are isomorphisms $M_{x,y}$ and the dotted ones are products of isomorphisms P_x. The faces are cells $\alpha_{x,y,z}$ (2.62), $\mu_{x,y}$ (2.65), and $\boxtimes_{x,y}$, $x, y, z \in G$.

The polytope (2.68) consists of 8 cubes (four containing the top vertex and four containing the bottom one) glued together in such a way that each of their 48 faces belongs to exactly two cubes (so that the boundary is empty). Five of these cubes are of the form (2.66); their composition is the differential of $p_C^1(P)$. Two cubes consisting of solid arrows are the cubes (2.63) and so they commute by assumption. The remaining cube commutes due to the naturality of the tensor product of $\mathbf{G}$. Thus, $d(p_C^1(P)) = 1$.

A different choice of 2-cells (2.65) results in multiplying $p_C^1(P)$ by a coboundary, so its class in $H^3(G, \pi_2(\mathbf{G}))$ is well-defined.

The equality

$$p_C^1(PQ) = p_C^1(P)p_C^1(Q), \qquad P, Q \in Aut(C)$$

is proved directly by gluing two cubes (2.66) for P and Q along the face $\alpha_{x,y,z}$.

Finally, P extends to a monoidal pseudo-natural automorphism of $\mathcal{C}$ if 2-cells (2.65) can be chosen in such a way that (2.44) is satisfied. This is equivalent to commutativity of the cube (2.66), i.e. to $p_C^1(P)$ being cohomologically trivial. □

Let $\mathcal{C} : G \to \mathbf{G}$ be a monoidal 2-functor. For any $\omega \in Z^3(G, \pi_2(\mathbf{G}))$ let $\mathcal{C}^\omega$ be a monoidal 2-functor obtained from $\mathcal{C}$ by multiplying each 2-cell $\alpha_{x,y,z}$ by $\omega(x, y, z)$, $x, y, z \in G$. The monoidal 2-equivalence class of $\mathcal{C}^\omega$ depends only on the cohomology class of ω in $H^3(G, \pi_2(\mathbf{G}))$. If $\mathcal{C}$, $\mathcal{C}'$ are extensions of the same monoidal functor $C : G \to \Pi_0(\mathbf{G})$ if and only if $\mathcal{C}' \cong \mathcal{C}^\omega$ for some ω.

Corollary 2.25. *Monoidal 2-functors $\mathcal{C}^{\omega_1}$, $\mathcal{C}^{\omega_2} : G \to \mathbf{G}$ are isomorphic if and only if $\omega_2 = p_C^1(P)\omega_1$ for some $P \in Aut(C) = H^1(G, \pi_1(\mathbf{G}))$.*

Proof. Let $\alpha_{x,y,z}$, $x, y, z \in G$, be the cells (2.62) for $\mathcal{C}$. A monoidal pseudo-natural isomorphism between $\mathcal{C}^{\omega_1}$ and $\mathcal{C}^{\omega_2}$ consists of 1-automorphisms $P_x : \mathcal{C}_x \to \mathcal{C}_x$ such that the cube (2.66) (with the top and bottom faces being, respectively, $\omega_1(x, y, z)\alpha_{x,y,z}$ and $\omega_2(x, y, z)\alpha_{x,y,z}$) commutes. This is equivalent to $\omega_2/\omega_1 = p_C^1(P)$, where $C : G \to \Pi_{\leq 1}(\mathbf{G})$ is the underlying monoidal functor of $\mathcal{C}$. □

Example 2.26. Let $I : G \to \Pi_{\leq 1}(\mathbf{G}) : x \mapsto \mathcal{I}$ denote the trivial monoidal functor. Then

$$p_I^1(P)(x, y, z) = \omega_{\mathbf{G}}(P_x, P_y, P_z), \qquad x, y, z \in G. \tag{2.69}$$

The next Corollary summarizes our description of monoidal 2-functors $G \to \mathbf{G}$.

Corollary 2.27. *Let $C : G \to \Pi_0(\mathbf{G})$ be a monoidal functor. An extension of C to a monoidal 2-functor $\mathcal{C} : G \to \mathbf{G}$ exists if and only if $p_C^0 = 0$ in $H^4(G, \pi_2(\mathbf{G}))$. Equivalence classes of such extensions of C form a torsor over $\mathsf{Coker}\left(p_C^1 : H^1(G, \pi_1(\mathbf{G})) \to H^3(G, \pi_2(\mathbf{G}))\right)$.*

2.6 Braided monoidal 2-functors between 2-categorical groups

Let A be an Abelian group. We consider it as a 2-categorical group with identity 1- and 2-morphisms.

Let $\mathbf{G}$ be a braided 2-categorical group with the corresponding canonical classes

$$\alpha_{\mathbf{G}} \in H^3_{br}(\pi_0(\mathbf{G}), \pi_1(\mathbf{G})) \quad \text{and} \quad q_{\mathbf{G}} = (\omega_{\mathbf{G}}, c_{\mathbf{G}}) \in H^3_{sym}(\pi_1(\mathbf{G}), \pi_2(\mathbf{G})).$$

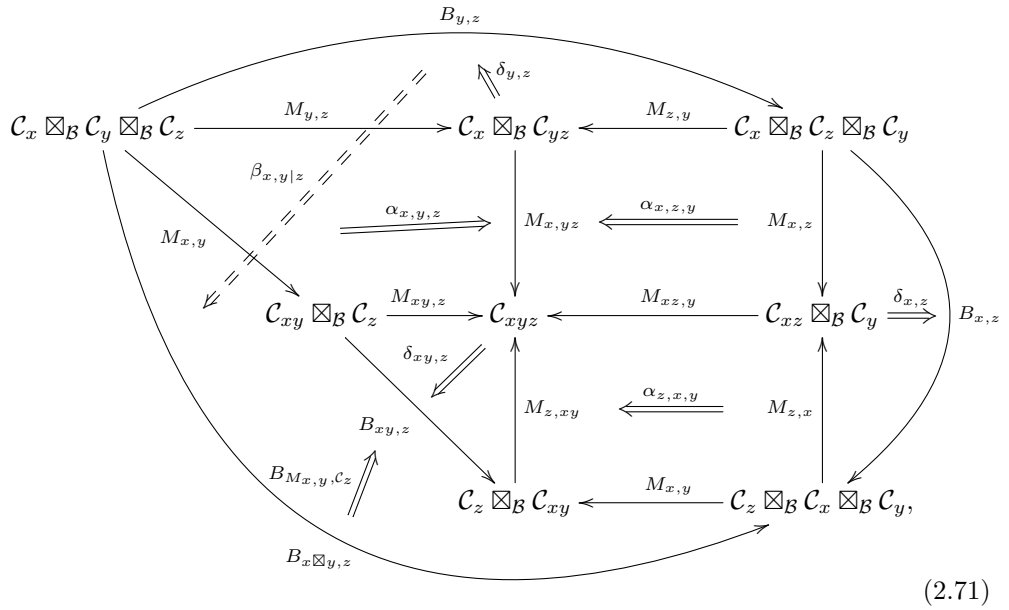
Let $C : A \to \Pi_{\leq 1}(\mathbf{G}) : x \to \mathcal{C}_x$ be a braided monoidal functor. This means that there is a 0-cell $\mathcal{C}_x$ in $\mathbf{G}$ for each $x \in G$, 1-isomorphisms $M_{x,y} : \mathcal{C}_x\mathcal{C}_y \to \mathcal{C}_{xy}$, invertible 2-cells $\alpha_{x,y,z}$ (2.62), and invertible 2-cells

$$\begin{array}{ccc} \mathcal{C}_x \boxtimes_{\mathcal{B}} \mathcal{C}_y & \xrightarrow{\quad B_{x,y} \quad} & \mathcal{C}_y \boxtimes_{\mathcal{B}} \mathcal{C}_x \\ & {\scriptstyle M_{x,y}} \searrow \quad \Uparrow \delta_{x,y} \quad \swarrow {\scriptstyle M_{y,x}} & \\ & \mathcal{C}_{xy} & \end{array} \tag{2.70}$$

for all $x, y, z \in G$. Let $\beta_{x|y,z}$ and $\beta_{x,y|z}$ denote the invertible modifications (2.25) with $\mathcal{L} = \mathcal{C}_x$, $\mathcal{M} = \mathcal{C}_y$, $\mathcal{N} = \mathcal{C}_z$.

Define a braided 4-cochain $p^0_C \in C^4_{br}(A,\, \pi_2(\mathbf{G}))$ by taking $p_0^{br}(C)(x,y,z,w)$ from (2.63),

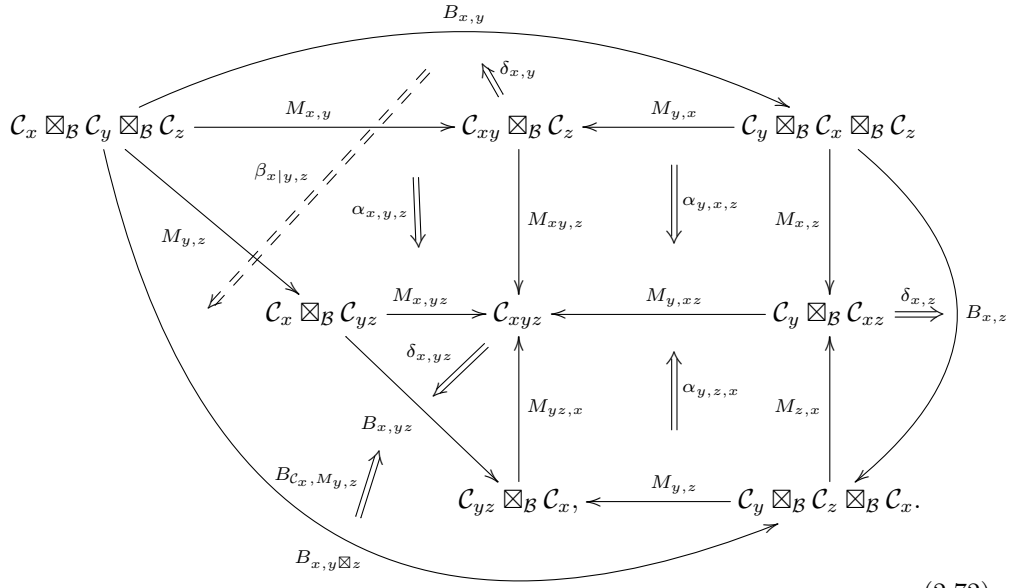
$p^0_C(x, y|z) =$



(2.71)

and

$p^0_C(x|y, z) =$



(2.72)

where the plane projections of the octahedra are pictured.

Remark 2.28. The octahedra (2.71) and (2.72) are special cases of those from the definition of a braided pseudomonoid in a braided Gray monoid [12, Definition 13].

Proposition 2.29. *p^0_C is a braided 4-cocycle whose cohomology class in $H^4_{br}(A,\, \pi_2(\mathbf{G}))$ depends only on the isomorphism class of C. A braided monoidal functor $C : A \to \Pi_{\leq 1}(\mathbf{G})$ extends to a braided monoidal 2-functor $A \to \mathbf{G}$ if and only if $p^0_C = 0$ in $H^4_{br}(A,\, \pi_2(\mathbf{G}))$.*

Proof. We need to verify vanishing of the shuffle differentials (2.6) – (2.10). That the differential (2.6) is zero follows from the construction of polytope (2.64). Vanishing of the differentials (2.7) (respectively, (2.8), (2.9), and (2.10)) is proved in a similar way. Namely, we form polytopes by gluing the octahedra (2.71), (2.72) to cubes (2.63) and to the commuting polytopes (2.26) (respectively, (2.27), (2.28), and (2.29)) so that the faces of the octahedra labelled by β's in octahedra and polytopes are glued to each other. In the resulting large polytopes each face labelled by α or δ is glued to its inverse. This implies commutativity of the polytopes, i.e. vanishing of the differentials.

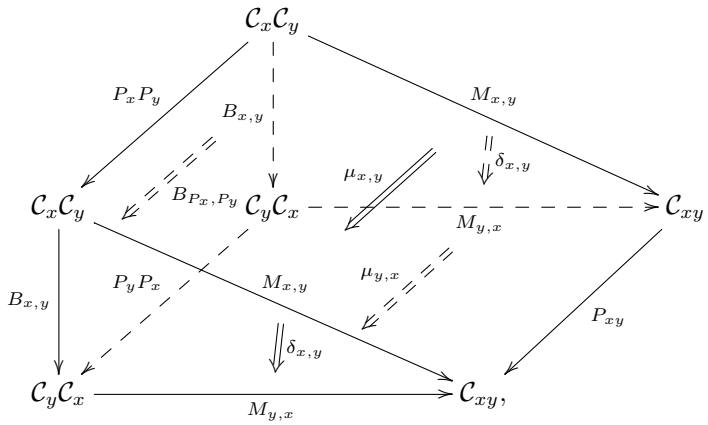
A different choice of 2-cells (2.62) and (2.70) results in multiplying p_C^0 by a braided 4-coboundary, so its class in $H^4(G,\, \pi_2(\mathbf{G}))$ is well-defined.

Note that C extends to a braided monoidal 2-functor $A \to \mathbf{G}$ if the cells $\alpha_{x,y,z}$ and $\delta_{x,y}$ can be chosen in such a way that (2.37), (2.40), and (2.41) are satisfied. This is equivalent to commutativity of cubes (2.63) and octahedra (2.71) and (2.72), i.e. to $p_C^0 = 1$. Indeed, these polytopes are obtained by gluing the two sides of (2.37), (2.40), and (2.41). □

Suppose that a braided monoidal functor $C : A \to \Pi_{\leq 1}(\mathbf{G})$ extends to a braided monoidal 2-functor $\mathcal{C} : A \to \mathbf{G}$. That is, there is a choice of invertible 2-cells (2.62) and (2.70) such that the cubes (2.63) and octahedra (2.71) and (2.72) commute, i.e. $p_C^0 = 1$. Let P be a monoidal automorphism of C.

Let $p_C^1(P)(x,y,z)$ be defined by (2.66) and let $p_C^1(P)(x\,|\,y)$ be the composition of the faces of the prism

$$p_C^1(P)(x\,|\,y) = \tag{2.73}$$



for all $x, y \in A$.

Proposition 2.30. *$p_C^1(P)$ is a braided 3-cocycle and the map*

$$p_C^1 : Aut_\otimes(C) = H^1(A,\, \pi_1(\mathbf{G})) \to H^3_{br}(A,\, \pi_2(\mathbf{G})) : P \mapsto p_C^1(P) \tag{2.74}$$

is a well defined group homomorphism.

The natural automorphism P extends to a braided monoidal pseudo-natural automorphism of $\mathcal{C}$ if and only if $p_C^1(P) = 0$ in $H^3_{br}(A,\, \pi_2(\mathbf{G}))$.

Proof. The differential (2.3) vanishes by Proposition 2.24. The vanishing of the differential (2.4) (respectively,(2.5)) is established by gluing the faces of three cubes (2.66), three prisms (2.73), and two copies of the octahedron (2.71) (respectively, (2.72)) in such a way that the result has the empty boundary.

A different choice of 2-cells (2.65) results in multiplying $\mathfrak{p}^1_C(P)$ by a braided coboundary, so its class in $H^3_{br}(G,\, \pi_2(\mathbf{G}))$ is well-defined.

The multiplicative property of $\mathfrak{p}^1_C$ is a direct consequence of the definition of prisms (2.73).

Finally, a monoidal pseudo-natural automorphism of $\mathcal{C}$ obtained by extending P is braided if 2-cells (2.65) can be chosen in such a way that the cube (2.45) is satisfied. But this cube becomes a prism (2.73) in our situation since the structural 2-cells of A, viewed as a braided 2-categorical group, are trivial. So the braided property is equivalent to the braided cohomology class of $\mathfrak{p}^1_C(P)$ being trivial. □

Let $\mathcal{C} : A \to \mathbf{G}$ be a monoidal 2-functor. For any $(\omega,\, c) \in Z^3_{br}(A,\, \pi_2(\mathbf{G}))$ let $\mathcal{C}^{(\omega,c)}$ be a monoidal 2-functor obtained from $\mathcal{C}$ by multiplying each 2-cell $\alpha_{x,y,z}$ by $\omega(x,\, y,\, z)$ and each 2-cell $\delta_{x,y}$ by $c(x, y)$ for all $x, y, z \in A$. The isomorphism class of $\mathcal{C}^{(\omega,c)}$ depends only on the class of $(\omega,\, c)$ in $H^3_{br}(A,\, \pi_2(\mathbf{G}))$. If $\mathcal{C}$, $\mathcal{C}'$ are extensions of the same braided monoidal functor $C : A \to \Pi_{\leq 1}(\mathbf{G})$ if and only if $\mathcal{C}' \cong \mathcal{C}^{(\omega,c)}$ for some $(\omega,\, c)$.

Corollary 2.31. *Braided monoidal* 2*-functors* $\mathcal{C}^{(\omega_1,c_1)}$, $\mathcal{C}^{(\omega_2,c_2)} : A \to \mathbf{G}$ *are isomorphic if and only if* $(\omega_2,\, c_2) = (\omega_1,\, c_1)\, \mathfrak{p}^1_C$ *for some* $P \in Aut(C) = H^1(A,\, \pi_1(\mathbf{G}))$.

Proof. This is similar to the proof of Corollary 2.25, where a criterion for isomorphism of monoidal 2-functors $\mathcal{C}^{\omega_1}$ and $\mathcal{C}^{\omega_2}$ was established. The braided property of such an isomorphism translates to commutativity the prism (2.73) with the front and back faces being, respectively, $c_1(x, y, z)\delta_{x,y}$ and $c_2(x, y)\delta_{x,y}$. This is equivalent to $c_2/c_1 = \mathfrak{p}^1_C(P)$, where $C : A \to \Pi_{\leq 1}(\mathbf{G})$ is the underlying monoidal functor of $\mathcal{C}$. □

The next Corollary summarizes our description of braided monoidal 2-functors $A \to \mathbf{G}$.

Corollary 2.32. *Let* $C : G \to \Pi_{\leq 1}(\mathbf{G})$ *be a braided monoidal functor. An extension of* C *to a monoidal* 2*-functor* $A \to \mathbf{G}$ *exists if and only if* $\mathfrak{p}^0_C = 0$ *in* $H^4_{br}(A,\, \pi_2(\mathbf{G}))$. *Equivalence classes of such extensions form a torsor over* $\mathsf{Coker}\left(\mathfrak{p}^1_C : H^1(A,\, \pi_1(\mathbf{G})) \to H^3_{br}(A,\, \pi_2(\mathbf{G}))\right)$.

2.7 Symmetric monoidal 2-functors between 2-categorical groups

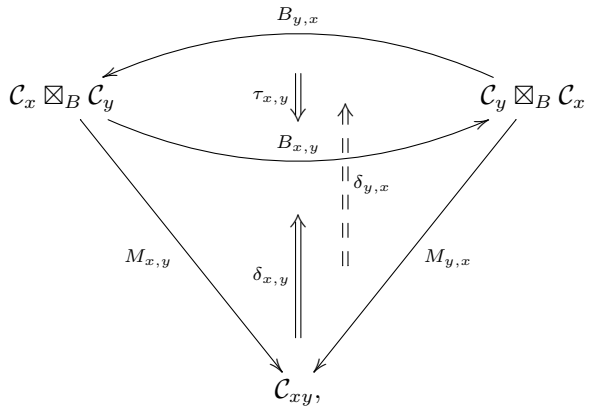
Let A be an Abelian group. Here we will consider it as a symmetric 2-categorical group with identity 1- and 2-morphisms.

Let $\mathbf{G}$ be a symmetric 2-categorical group with the corresponding canonical classes

$$\alpha_{\mathbf{G}} \in H^3_{sym}(\pi_0(\mathbf{G}),\, \pi_1(\mathbf{G})) \quad \text{and} \quad q_{\mathbf{G}} \in H^3_{sym}(\pi_1(\mathbf{G}),\, \pi_2(\mathbf{G})).$$

One can extend the obstruction theory from Section 2.6 to symmetric monoidal 2-functors $A \to \mathbf{G}$. Namely, let $C : A \to \Pi_{\leq 1}(\mathbf{G})$ be a symmetric monoidal functor. We define $p_0^C \in C^4_{sym}(A,\, \pi_2(\mathbf{G}))$ by extending the braided 4-cocycle from Proposition 2.29 as follows. The components $p_0^C(-,\, -,\, -,\, -)$, $p_0^C(-\,|\,-,\, -)$, and $p_0^C(-,\, -,\, |\,-)$ are given by (2.63) , (2.71), and (2.72), respectively, and

$p_0^C(x \,||\, y) =$



(2.75)

for all $x, y \in A$.

Proposition 2.33. *The above p_C^0 is a symmetric 4-cocycle whose cohomology class in $H^4_{sym}(A,\, \pi_2(\mathbf{G}))$ depends only on the isomorphism class of C. A symmetric monoidal functor $C : A \to \Pi_{\leq 1}(\mathbf{G})$ extends to a symmetric monoidal 2-functor $A \to \mathbf{G}$ if and only if $p_C^0 = 0$ in $H^4_{sym}(A,\, \pi_2(\mathbf{G}))$.*

Proof. We need to check vanishing of the differentials (2.12), (2.13), and (2.14). Vanishing of the differentials (2.12) and (2.13) is checked by gluing polytopes (2.71) and (2.72) along their associativity faces α and gluing their braiding faces δ to two sides of cones (2.75). The differential (2.14) vanishes thanks to axiom (2.33) of a symmetric monoidal 2-category.

Proposition 2.29 gives a criterion for $\mathcal{C}$ to have an extension to a braided monoidal 2-functor. This extension admits a symmetric monoidal 2-functor structure if and only if the cells (2.39) are chosen in such a way that the cone (2.75) commutes. This is equivalent to p_0^C being trivial in $H^4_{sym}(A,\, \pi_2(\mathbf{G}))$. □

Suppose that a symmetric monoidal functor $C : A \to \Pi_{\leq 1}(\mathbf{G})$ extends to a symmetric monoidal 2-functor $\mathcal{C} : A \to \mathbf{G}$. For any $P \in Aut_{\otimes}(C)$ the braided 3-cocycle from Proposition 2.30 is symmetric, i.e.

$$p_1^C(P)(x\,|\,y)\, p_1^C(P)(y\,|\,x) = 1$$

for all $x, y \in A$. This can be seen gluing boundaries of two prisms (2.73) and two cones (2.75).

Corollary 2.34. *Let (ω_1, c_1), $(\omega_2, c_2) \in Z^3_{sym}(A, \pi_1(\mathbf{G}))$ be symmetric 3-cocycles. Symmetric monoidal 2-functors $\mathcal{C}^{(\omega_1,c_1)}$, $\mathcal{C}^{(\omega_2,c_2)} : A \to \mathbf{G}$ are isomorphic if and only if $(\omega_2,\, c_2) = (\omega_1,\, c_1)\, p_1^C(P)$ for some $P \in Aut(C) = H^1(A,\, \pi_1(\mathbf{G}))$.*

Proof. This is the same as Corollary 2.31, since there is no difference between isomorphisms of braided and symmetric monoidal 2-functors. □

The next Corollary summarizes our description of braided monoidal 2-functors $A \to \mathbf{G}$.

Corollary 2.35. *Let $C : A \to \Pi_{\leq 1}(\mathbf{G})$ be a symmetric monoidal functor. An extension of C to a monoidal 2-functor $A \to \mathbf{G}$ exists if and only if $p_C^0 = 0$ in $H^4_{sym}(A,\, \pi_2(\mathbf{G}))$. Equivalence classes of such extensions form a torsor over $\mathsf{Coker}\left(p_C^1 : H^1(A,\, \pi_1(\mathbf{G})) \to H^3_{sym}(A,\, \pi_2(\mathbf{G}))\right)$.*

2.8 The symmetric 2-categorical group of symmetric monoidal 2-functors

Let A be a finite Abelian group and let $\mathbf{G}$ be a symmetric 2-categorical group.

Let C, $C' : A \to \Pi_{\leq 1}(\mathbf{G})$ be symmetric monoidal 2-functors, where C is given by $x \mapsto \mathcal{C}_x$ with the monoidal structure $M_{x,y} : \mathcal{C}_x \boxtimes \mathcal{C}_y \xrightarrow{\sim} \mathcal{C}_{xy}$ and C' is given by $x \mapsto \mathcal{C}'_x$ with the monoidal structure $M'_{x,y} : \mathcal{C}'_x \boxtimes \mathcal{C}'_y \xrightarrow{\sim} \mathcal{C}'_{xy}$, $x, y \in A$.

Define a symmetric monoidal functor

$$\tilde{C} := C \boxtimes C' : A \to \Pi_{\leq 1}(\mathbf{G}) : \qquad x \mapsto \mathcal{C}_x \boxtimes_{\mathcal{B}} \mathcal{C}'_x. \tag{2.76}$$

with the monoidal structure

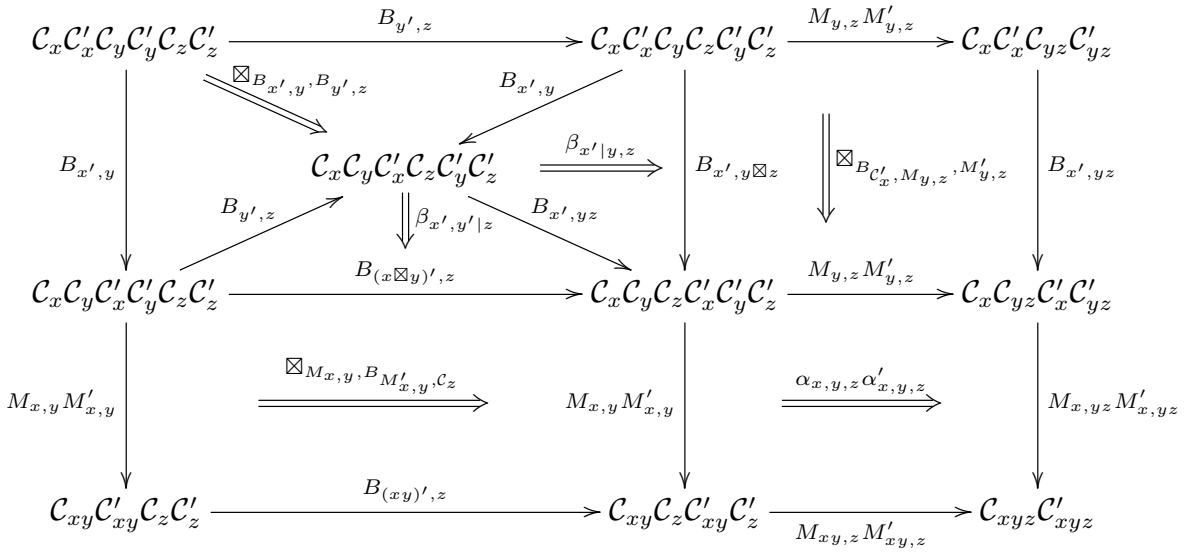
$$\tilde{M}_{x,y} : \mathcal{C}_x \boxtimes \mathcal{C}'_x \boxtimes \mathcal{C}_y \boxtimes \mathcal{C}'_y \xrightarrow{B_{x',y}} \mathcal{C}_x \boxtimes \mathcal{C}_y \boxtimes \mathcal{C}'_x \boxtimes \mathcal{C}'_y \xrightarrow{M_{x,y} \boxtimes M'_{x,y}} \mathcal{C}_{xy} \boxtimes \mathcal{C}'_{xy}, \qquad x,\, y \in A. \tag{2.77}$$

Here and below we denote $B_{x',y}$ the braiding between $\mathcal{C}'_x$ and $\mathcal{C}_y$.

Suppose that C and C' extend to symmetric monoidal 2-functors $\mathcal{C}$, $\mathcal{C}' : A \to \mathbf{G}$. The associativity and braiding 2-cells (2.35) and (2.39) for $\mathcal{C}$ and $\mathcal{C}'$ will be denoted α, α' and δ, δ', respectively.

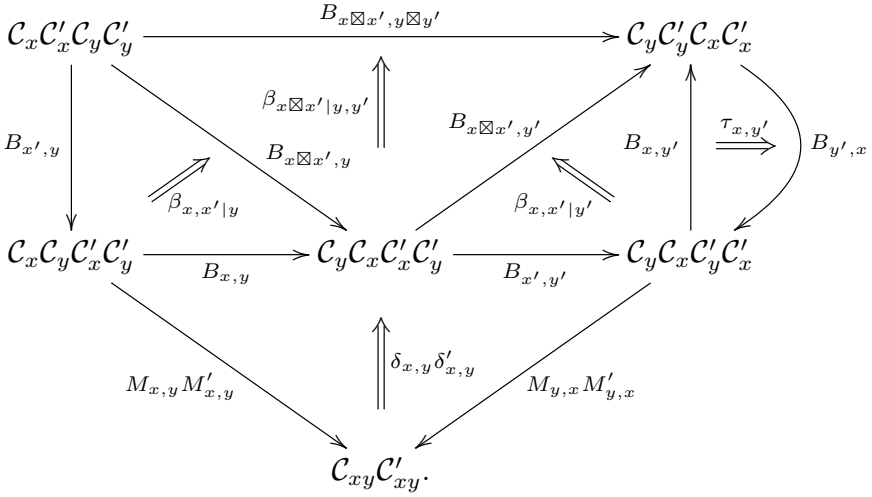
Our goal is to construct a canonical braided monoidal 2-functor $\tilde{\mathcal{C}}$ extending $\tilde{C}$.

Define the associativity 2-cells $\tilde{\alpha}_{x,y,z}\,(x,y,z\in A)$ by



(2.78)

and the braiding 2-cells $\tilde{\delta}_{x,y}\,(x,y\in A)$ by



(2.79)

Here we write $\beta_{x\boxtimes x'|y,y'}$ as a shorthand for $\beta_{\mathcal{C}_x\boxtimes\mathcal{C}'_x|\mathcal{C}_y,\mathcal{C}'_y}$ etc.

Proposition 2.36. *The* 2*-cells* (2.78) *and* (2.79) *make* $\tilde{\mathcal{C}}=\mathcal{C}\boxtimes\mathcal{C}'$ *a symmetric monoidal* 2*-functor.*

Proof. The proof is tedious but straightforward. It extends the corresponding argument for symmetric monoidal functors and consists of decomposing the commuting cube (2.63) and octahedra (2.71), (2.72) formed by 2-cells (2.78) and (2.79) into unions of commuting polytopes glued together.

For the cube (2.63) for $\tilde{\mathcal{C}}$ one gets commuting polytopes obtained by gluing both sides of (2.26) - (2.29), the polytopes commuting due to the naturality of braiding and the naturality of cells β and τ, and cubes (2.63) for $\mathcal{C}$ and $\mathcal{C}'$. For the octahedra (2.71), (2.72) one gets commuting polytopes as above,

the corresponding polytopes for $\mathcal{C}$ and $\mathcal{C}'$, and the symmetry polytopes (2.31), (2.32), and (2.33)) of $\mathbf{G}$. It follows that $\tilde{\mathcal{C}}$ is a braided monoidal 2-functor.

The cone (2.75) corresponding to the property of $\tilde{\mathcal{C}}$ being symmetric is comprised from $\delta_{x,y}$, $\delta'_{x,y}$ and $\tau_{x,y}$ for $x, y \in A$. This cone decomposes into the union of several commuting polytopes, namely the pair of corresponding cones for $\mathcal{C}$ and $\mathcal{C}'$ and the symmetry polytopes (2.31), (2.32), and (2.33). Hence, it commutes. □

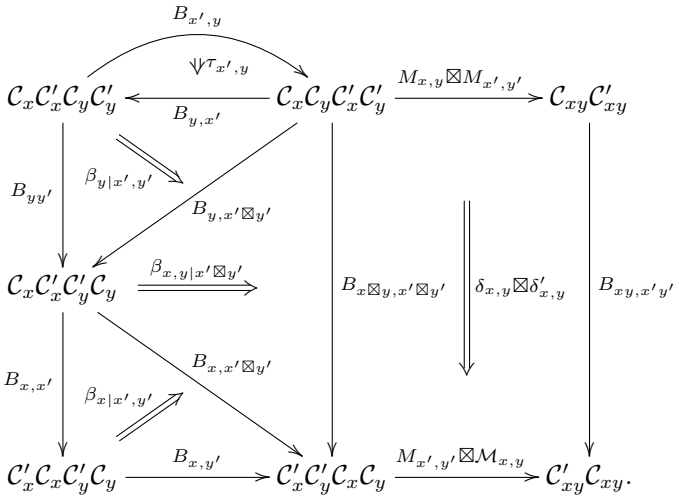
Proposition 2.37. *The above product of functors turns* $\mathbf{2\text{-}Fun_{sym}}(A, \mathbf{G})$ *into a symmetric* 2*-categorical group.*

Proof. For $\mathcal{C}$, $\mathcal{C}'$, $\mathcal{C}'' \in \mathbf{2\text{-}Fun_{sym}}(A, \mathbf{G})$ there is a pseudo-natural equivalence between $(\mathcal{C} \boxtimes \mathcal{C}') \boxtimes \mathcal{C}''$ and $\mathcal{C} \boxtimes (\mathcal{C}' \boxtimes \mathcal{C}'')$. This can be seen to be monoidal by comparing the associativities (2.78) for both 2-functors. The unit object of $\mathbf{2\text{-}Fun_{sym}}(A, \mathbf{G})$ is the trivial symmetric 2-functor (with $\mathcal{C}_x = \mathcal{I}$ for all $x \in A$ and all structure morphisms and cells being identities).

The braiding of $\mathcal{C}$ and $\mathcal{C}'$ is a pseudo-natural isomorphism given by

$$B_{\mathcal{C},\mathcal{C}'}(x) : \mathcal{C}_x \boxtimes \mathcal{C}'_x \xrightarrow{B_{x,x'}} \mathcal{C}'_x \boxtimes \mathcal{C}_x, \qquad x \in A, \tag{2.80}$$

with 2-cells (2.13) being the following compositions:



(2.81)

The 2-cells (2.25) are $\beta_{x,x'|x''}$ and $\beta_{x|x'\,x''}$, $x \in A$. One can directly verify commutativity of the cubes (2.44) and (2.45).

Finally, the symmetry 2-cell τ of $\mathbf{G}$ provides an invertible modification between $B_{\mathcal{C}',\mathcal{C}} \circ B_{\mathcal{C},\mathcal{C}'}$ and $\mathrm{Id}_{\mathcal{C}\boxtimes\mathcal{C}'}$ satisfying (2.31), (2.32), and (2.33). □

Theorem 2.38. *There is an exact sequence of group homomorphisms:*

$$H^1(A,\,\pi_1(\mathbf{G}))\to H^3_{sym}(A,\,\pi_2(\mathbf{G}))\to\pi_0(\mathbf{2\text{-}Fun_{sym}}(A,\,\mathbf{G}))\to\pi_0(\mathcal{F}un_{sym}(A,\,\Pi_{\leq 1}(\mathbf{G})))\to H^4_{sym}(A,\,\pi_2(\mathbf{G})). \tag{2.82}$$

Proof. The first three arrows are described in Section 2.7. That they are homomorphisms follows from the definition of the tensor product in $\mathbf{2\text{-}Fun_{sym}}(A,\,\mathbf{G})$.

We need to check that

$$\pi_0(\mathcal{F}un_{sym}(A,\,\Pi_{\leq 1}(\mathbf{G})))\to H^4_{sym}(A,\,\pi_2(\mathbf{G})) : C\mapsto p^0_C,$$

where the components of the symmetric 4-cocycle p^0_C are given by the values of polytopes (2.63), (2.71), (2.72), and (2.75), is a group homomorphism. This is achieved by decomposing each of these polytopes for $\mathcal{C}\boxtimes\mathcal{C}'$, where $\mathcal{C},\mathcal{C}'\in\mathbf{2\text{-}Fun_{sym}}(A,\,\mathbf{G})$, into the union of the corresponding polytopes for $\mathcal{C}$ and $\mathcal{C}'$ and commuting polytopes satisfied by the structure 2-cells of $\mathbf{G}$ as well as those of $\mathcal{C}$, $\mathcal{C}'$, glued together in such a way that the resulting boundary is empty.

The exactness of this sequence follows from Corollary 2.35. □

3 Module categories

3.1 Module categories over a tensor category

Le $\mathcal{C}$ be a tensor category with the associativity constraint $a_{X,Y,Z} : (X\otimes Y)\otimes Z\xrightarrow{\sim} X\otimes(Y\otimes Z)$. Let $\mathcal{C}^{\rm op}$ denote the tensor category with the opposite multiplication $X\otimes^{\rm op}Y = Y\otimes X$ and the associativity constraint $a^{\rm op}_{X,Y,Z} = a^{-1}_{Z,Y,X} : Z\otimes(Y\otimes X)\xrightarrow{\sim}(Z\otimes Y)\otimes X$ for $X,Y,Z\in\mathcal{C}$. Below we recall definitions from [31, 22].

Definition 3.1. A (left) *$\mathcal{C}$-module category* is a finite Abelian k-linear category $\mathcal{M}$ together with a bifunctor

$$\mathcal{C}\times\mathcal{M}\to\mathcal{M},\quad (X,M)\mapsto X*M,$$

exact in each variable, and a collection of isomorphisms (*$\mathcal{C}$-module associativity constraint*)

$$m_{X,Y,M} : (X\otimes Y)*M\xrightarrow{\sim} X*(Y*M),$$

natural in $X, Y \in \mathcal{C}, M \in M$ and such that the diagram

$$\begin{array}{ccc} & ((X \otimes Y) \otimes Z) * M & \\ {\scriptstyle a_{X,Y,Z} * \mathrm{Id}_M} \swarrow & & \searrow {\scriptstyle m_{X \otimes Y, Z, M}} \\ (X \otimes (Y \otimes Z)) * M & & (X \otimes Y) * (Z * M) \\ {\scriptstyle m_{X, Y \otimes Z, M}} \downarrow & & \downarrow {\scriptstyle m_{X,Y,Z*M}} \\ X * ((Y \otimes Z) * M) & \xrightarrow{\mathrm{Id}_X * m_{Y,Z,M}} & X * (Y * (Z * M)) \end{array} \tag{3.1}$$

commutes for all $X, Y, Z \in \mathcal{C},\ M \in \mathcal{M}$.

A *right* $\mathcal{C}$-module category is a $\mathcal{C}^{\mathrm{op}}$-module category. A $\mathcal{C}$*-bimodule* category is a $(\mathcal{C} \boxtimes \mathcal{C}^{\mathrm{op}})$-module category.

Remark 3.2. A $\mathcal{C}$-bimodule category $\mathcal{M}$ can be equivalently described as a category with both left and right $\mathcal{C}$-module structures and a collection of isomorphisms (a *middle associativity constraint*)

$$m_{X,M,Y} : (X * M) * Y \xrightarrow{\sim} X * (M * Y) \tag{3.2}$$

natural in $X, Y \in \mathcal{C},\ M \in \mathcal{M}$ compatible in a certain way [22, Definition 7.1.7].

Definition 3.3. A $\mathcal{C}$*-module* functor $F : \mathcal{M} \to \mathcal{N}$ between $\mathcal{C}$-module categories is a functor along with a collection of isomorphisms $F_{X,M} : X * F(M) \xrightarrow{\sim} F(X * M)$ natural in $X \in \mathcal{C},\ M \in \mathcal{M}$ such that the following diagram

$$\begin{array}{ccc} & (X \otimes Y) * F(M) & \\ {\scriptstyle m_{X,Y,F(M)}} \swarrow & & \searrow {\scriptstyle F_{X \otimes Y, M}} \\ X * (Y * F(M)) & & F((X \otimes Y) * M) \\ {\scriptstyle \mathrm{Id}_X * F_{Y,M}} \downarrow & & \downarrow {\scriptstyle F(m_{X,Y,M})} \\ X * F(Y * M) & \xrightarrow{F_{X, Y*M}} & F(X * (Y * M)) \end{array} \tag{3.3}$$

commutes for $X, Y \in \mathcal{C},\ , M \in \mathcal{M}$.

Definition 3.4. A natural $\mathcal{C}$*-module transformation* between $\mathcal{C}$-module functors $F, F' : \mathcal{M} \to \mathcal{N}$ is a natural transformation $\eta : F \to F'$ such that

$$\begin{array}{ccc} X * F(M) & \xrightarrow{F_{X,M}} & F(X * M) \\ {\scriptstyle \mathrm{Id}_X * \eta_M} \downarrow & & \downarrow {\scriptstyle \eta_{X*M}} \\ X * F'(M) & \xrightarrow{F'_{X,M}} & F'(X * M) \end{array} \tag{3.4}$$

commutes for all $X \in \mathcal{C},\ M \in \mathcal{M}$.

Let $F : \mathcal{L} \to \mathcal{M}$ and $F" : \mathcal{M} \to \mathcal{N}$ be $\mathcal{C}$-module functors then $F' \circ F$ has a canonical structure of $\mathcal{C}$-module functor

$$(F' \circ F)_{X,M} : X * F'(F(M)) \xrightarrow{F'_{X,F(M)}} F'(X * F(M)) \xrightarrow{F'(F_{X,M})} F'(F(X * M)), \quad X \in \mathcal{C},\ M \in \mathcal{M}. \tag{3.5}$$

Thus, $\mathcal{C}$-module categories, $\mathcal{C}$-module functors, and $\mathcal{C}$-module natural transformations form a strict 2-category $\mathbf{Mod}(\mathcal{C})$.

3.2 Tensor product of module categories

Let $\mathcal{C}$ be a tensor category, let $\mathcal{M}$ be a right $\mathcal{C}$-module category, and let $\mathcal{N}$ be a left $\mathcal{C}$-module category. The *(relative) tensor product* $\mathcal{M} \boxtimes_{\mathcal{C}} \mathcal{N}$ [23] is an abelian category $\mathcal{M} \boxtimes_{\mathcal{C}} \mathcal{N}$ along with a functor $\mathcal{M} \times \mathcal{N} \to \mathcal{M} \boxtimes_{\mathcal{C}} \mathcal{N}$ universal among $\mathcal{C}$-balanced and right exact in each variable functors from $\mathcal{M} \times \mathcal{N}$ to Abelian categories.

An explicit description is given as follows. Objects of $\mathcal{M} \boxtimes_{\mathcal{C}} \mathcal{N}$ are pairs (V, γ), where $V \in \mathcal{M} \boxtimes \mathcal{N}$ and

$$\gamma_X : V * (X \boxtimes \mathbf{1}) \xrightarrow{\sim} (\mathbf{1} \boxtimes X) * V, \tag{3.6}$$

is a *balancing isomorphism* natural in $V \in \mathcal{M} \boxtimes \mathcal{N}, X \in \mathcal{C}$ and such that the following diagram

$$\begin{array}{ccccc} V * ((X \otimes Y) \boxtimes \mathbf{1}) & & \xrightarrow{\gamma_{X \otimes Y}} & & (\mathbf{1} \boxtimes (X \otimes Y)) * V \\ \downarrow{\scriptstyle m_{V,X,Y}} & & & & \downarrow{\scriptstyle n_{X,Y,V}} \\ (V * (X \boxtimes \mathbf{1})) * (Y \boxtimes \mathbf{1}) & \xrightarrow{\gamma_X} & (\mathbf{1} \boxtimes X) * (V * (Y \boxtimes \mathbf{1})) & \xrightarrow{\gamma_Y} & (\mathbf{1} \boxtimes X) * ((\mathbf{1} \boxtimes Y) * V), \end{array} \tag{3.7}$$

commutes. Here m and n are the module associativity constraints in $\mathcal{M}$ and $\mathcal{N}$.

A morphism between $(V, \{\gamma_X\}_{X \in \mathcal{C}})$ and $(V', \{\gamma'_X\}_{X \in \mathcal{C}})$ in $\mathcal{M} \boxtimes_{\mathcal{C}} \mathcal{N}$ is a morphism $f : V \to V'$ in $\mathcal{M} \boxtimes \mathcal{N}$ such that the diagram

$$\begin{array}{ccc} V * (X \boxtimes \mathbf{1}) & \xrightarrow{f * (X \boxtimes \mathbf{1})} & V' * (X \boxtimes \mathbf{1}) \\ \downarrow{\scriptstyle \gamma_X} & & \downarrow{\scriptstyle \gamma'_X} \\ (\mathbf{1} \boxtimes X) * V & \xrightarrow{(\mathbf{1} \boxtimes X) * f} & (\mathbf{1} \boxtimes X) * V' \end{array} \tag{3.8}$$

commutes for all $X \in \mathcal{C}$.

If $\mathcal{M}$ is a $\mathcal{C}$-bimodule category then $\mathcal{M} \boxtimes_{\mathcal{C}} \mathcal{N}$ inherits the left $\mathcal{C}$-module category structure from $\mathcal{M}$:

$$Y * (V, \{\gamma_X\}) = ((Y \boxtimes \mathbf{1}) * V, \{\overline{\gamma}_X\}), \tag{3.9}$$

where

$$\begin{array}{ccc} ((Y \boxtimes \mathbf{1}) * V) * (X \boxtimes \mathbf{1}) & \xrightarrow{\overline{\gamma}_X} & (\mathbf{1} \boxtimes X) * ((Y \boxtimes \mathbf{1}) * V) \\ \downarrow{\scriptstyle m^{-1}_{Y,V,X}} & & \| \\ (Y \boxtimes \mathbf{1}) * (V * (X \boxtimes \mathbf{1})) & \xrightarrow{\gamma_X} & (Y \boxtimes \mathbf{1}) * ((\mathbf{1} \boxtimes X) * V) \end{array} \tag{3.10}$$

for all $X, Y \in \mathcal{C}$. Similarly, if $\mathcal{N}$ is a $\mathcal{C}$-bimodule category then $\mathcal{M} \boxtimes_{\mathcal{C}} \mathcal{N}$ inherits the right $\mathcal{C}$-module category structure from $\mathcal{N}$.

Thus, there is a monoidal 2-category $\mathbf{Bimod}(\mathcal{C})$ of $\mathcal{C}$-bimodule categories. Its 1-cells are $\mathcal{C}$-bimodule functors and 2-cells are natural transformations of $\mathcal{C}$-bimodule functors. The regular $\mathcal{C}$-bimodule category $\mathcal{C}$ is the identity for $\boxtimes_{\mathcal{C}}$.

3.3 From module categories over a braided tensor category to bimodule categories

Let $\mathcal{B}$ be a braided tensor category with the braiding

$$c_{X,Y} : X \otimes Y \xrightarrow{\sim} Y \otimes X, \qquad X, Y \in \mathcal{B}.$$

The braiding of $\mathcal{B}$ allows to turn a left $\mathcal{B}$-module category $\mathcal{M}$ into a $\mathcal{B}$-bimodule category as follows. Let $m_{X,Y,M} : (X \otimes Y) \otimes M \xrightarrow{\sim} X \otimes (Y \otimes M)$ denote the left $\mathcal{B}$-module associativity constraint of $\mathcal{M}$. Define the right action of $\mathcal{B}$ on $\mathcal{M}$ by $M * X := X * M$ for all $X \in \mathcal{B}$ and $M \in \mathcal{M}$. The right $\mathcal{B}$-module associativity constraint is given by the composition

$$\begin{array}{ccccc} M*(X\otimes Y) & & \xrightarrow{\quad m_{M,X,Y}\quad} & & (M*X)*Y \\ \| & & & & \| \\ (X\otimes Y)*M & \xrightarrow{c_{X,Y}} & (Y\otimes X)*M & \xrightarrow{m_{Y,X,M}} & Y*(X*M) \end{array} \tag{3.11}$$

and the middle associativity constraint is given by

$$\begin{array}{ccccccc} (X*M)*Y & & & \xrightarrow{\qquad m_{X,Y,M}\qquad} & & & X*(M*Y) \\ \| & & & & & & \| \\ Y*(X*M) & \xrightarrow{m^{-1}_{Y,X,M}} & (Y\otimes X)*M & \xrightarrow{c_{Y,X}} & (X\otimes Y)*M & \xrightarrow{m_{X,Y,M}} & X*(Y*M), \end{array} \tag{3.12}$$

for all $X, Y \in \mathcal{B}$ and $M \in \mathcal{M}$.

Remark 3.5. Since $\mathcal{B}$-module functors and their $\mathcal{B}$-module natural transformations extend to $\mathcal{B}$-bimodule functors and $\mathcal{B}$-bimodule transformations in an obvious way, there is a 2-embedding $\mathbf{Mod}(\mathcal{B}) \to \mathbf{Bimod}(\mathcal{B})$.

Using the $\mathcal{B}$-bimodule structure of $\mathcal{M}$ we define the tensor product $\mathcal{M} \boxtimes_{\mathcal{B}} \mathcal{N}$ of $\mathcal{B}$-module categories $\mathcal{M}$, $\mathcal{N}$, as in Section 3.2. It has a canonical structure of a left $\mathcal{B}$-module category. This makes $\mathbf{Mod}(\mathcal{B})$ a monoidal 2-category.

Remark 3.6. From (3.7) we see that objects of $\mathcal{M} \boxtimes_{\mathcal{B}} \mathcal{N}$ are pairs $(V, \{\gamma_X\}_{X \in \mathcal{B}})$, where $V \in \mathcal{M} \boxtimes \mathcal{N}$ and

$$\gamma_X : (X \boxtimes \mathbf{1}) * V \to (\mathbf{1} \boxtimes X) * V, \qquad V \in \mathcal{M} \boxtimes \mathcal{N}, X \in \mathcal{B},$$

is a natural balancing isomorphism satisfying

$$\begin{array}{ccccc}
((X\otimes Y)\boxtimes \mathbf{1})*V & & \xrightarrow{\quad\gamma_{X\otimes Y}\quad} & & (\mathbf{1}\boxtimes(X\otimes Y))*V \\
\downarrow{\scriptstyle c_{X,Y}} & & & & \\
((Y\otimes X)\boxtimes \mathbf{1})*V & & & & \downarrow{\scriptstyle n_{X,Y,V}} \\
\downarrow{\scriptstyle m_{Y,X,V}} & & & & \\
(Y\boxtimes \mathbf{1})\otimes((X\boxtimes \mathbf{1})*V) & \xrightarrow{\gamma_X} & (Y\boxtimes X)*V & \xrightarrow{\gamma_Y} & (\mathbf{1}\boxtimes Y)\otimes((\mathbf{1}\boxtimes X)*V).
\end{array} \tag{3.13}$$

The vertical composition on the left side is the right $\mathcal{B}$-module associativity constraint of $\mathcal{M}$.

Proposition 3.7. *Let $\mathcal{C}\subset\mathcal{B}$ be a tensor subcategory. The induction*

$$\mathbf{Mod}(\mathcal{C})\to\mathbf{Mod}(\mathcal{B}):\mathcal{N}\mapsto\mathcal{B}\boxtimes_{\mathcal{C}}\mathcal{N} \tag{3.14}$$

is a monoidal 2-functor.

Proof. The monoidal structure on the 2-functor (3.14) is given by the canonical equivalence

$$(\mathcal{B}\boxtimes_{\mathcal{C}}\mathcal{M})\boxtimes_{\mathcal{B}}(\mathcal{B}\boxtimes_{\mathcal{C}}\mathcal{N})\cong\mathcal{B}\boxtimes_{\mathcal{C}}(\mathcal{M}\boxtimes_{\mathcal{B}}\mathcal{B})\boxtimes_{\mathcal{C}}\mathcal{N}\cong\mathcal{B}\boxtimes_{\mathcal{C}}(\mathcal{M}\boxtimes_{\mathcal{C}}\mathcal{N}),\qquad \mathcal{M},\mathcal{N}\in\mathbf{Mod}(\mathcal{C}).$$

The verification of axioms is straightforward and is left to the reader. □

4 Braided module categories

4.1 Module braiding on module categories

Let $\mathcal{B}$ be a braided tensor category. The following definition appeared in [21, 5, 2].

Definition 4.1. A *braided* $\mathcal{B}$-module category is a pair $(\mathcal{M},\sigma)$, where $\mathcal{M}$ is a $\mathcal{B}$-module category and $\sigma=\{\sigma_{X,M}:X*M\to X*M\}_{X\in\mathcal{B},\,M\in\mathcal{M}}$ is a natural isomorphism (called a $\mathcal{B}$-*module braiding*) with $\sigma_{\mathbf{1},M}=1_M$ such that the diagrams

$$\begin{array}{ccc}
X*(Y*M) & \xrightarrow{\sigma_{X,Y*M}} & X*(Y*M) \\
\downarrow{\scriptstyle m^{-1}_{X,Y,M}} & & \downarrow{\scriptstyle m^{-1}_{X,Y,M}} \\
(X\otimes Y)*M & & (X\otimes Y)*M \\
\downarrow{\scriptstyle c_{X,Y}} & & \downarrow{\scriptstyle c^{-1}_{Y,X}} \\
(Y\otimes X)*M & & (Y\otimes X)*M \\
\downarrow{\scriptstyle m_{Y,X,M}} & & \downarrow{\scriptstyle m_{Y,X,M}} \\
Y*(X*M) & \xrightarrow{\sigma_{X,M}} & Y*(X*M)
\end{array}
\qquad\qquad
\begin{array}{ccc}
(X\otimes Y)*M & \xrightarrow{\sigma_{X\otimes Y,M}} & (X\otimes Y)*M \\
\downarrow{\scriptstyle c^{-1}_{Y,X}} & & \downarrow{\scriptstyle c_{X,Y}} \\
(Y\otimes X)*M & & (Y\otimes X)*M \\
\downarrow{\scriptstyle m_{Y,X,M}} & & \downarrow{\scriptstyle m_{Y,X,M}} \\
Y*(X*M) & & Y*(X*M) \\
& {\scriptstyle\sigma_{X,M}}\searrow \quad \nearrow{\scriptstyle\sigma_{Y,X*M}} & \\
& Y*(X*M) &
\end{array} \tag{4.1}$$

commute for all $X,Y\in\mathcal{B}$ and $M\in\mathcal{M}$.

Definition 4.2. A $\mathcal{B}$-module functor $F : \mathcal{M} \to \mathcal{N}$ between braided $\mathcal{B}$-module categories is *braided* if the diagram

$$\begin{array}{ccc} X * F(M) & \xrightarrow{\sigma_{X,F(M)}} & X * F(M) \\ {\scriptstyle F_{X,M}}\downarrow & & \downarrow{\scriptstyle F_{X,M}} \\ F(X * M) & \xrightarrow{F(\sigma_{X,M})} & F(X * M) \end{array} \tag{4.2}$$

commutes for all $X \in \mathcal{B}$ and $M \in \mathcal{M}$.

A morphism between braided B-module functors is a $\mathcal{B}$-module natural transformation.

Let $\mathbf{Mod_{br}}(\mathcal{B})$ denote the 2-category of braided $\mathcal{B}$-module categories.

Example 4.3. Let $\mathcal{C}$ be a braided tensor category containing $\mathcal{B}$. Then $\mathcal{C}$ is a braided $\mathcal{B}$-module category with the $\mathcal{B}$-module braiding

$$\sigma_{X,Y} = c_{Y,X}c_{X,Y}, \qquad X, Y \in \mathcal{B}, \tag{4.3}$$

where c denotes the braiding of $\mathcal{C}$. The commutativity of diagrams in Definition 4.1 follows directly from the hexagon identities and naturality of braiding.

Recall that the *symmetric center* $\mathcal{Z}_{sym}(\mathcal{B})$ of a braided tensor category $\mathcal{B}$ is the full subcategory of $\mathcal{B}$ whose objects V satisfy $c_{XV}c_{VX} = \mathrm{Id}_{X\otimes V}$ for all X in B. Clearly, $\mathcal{Z}_{sym}(\mathcal{B})$ is a symmetric tensor category.

Example 4.4. A special case of the previous example is $\mathcal{C} = \mathcal{B}$, the regular $\mathcal{B}$-module category, with the module braiding (4.3). The category of braided module endofunctors of $\mathcal{B}$ is braided equivalent to $\mathcal{Z}_{sym}(\mathcal{B})$.

Remark 4.5. The name *braided* in Definition 4.1 is justified as follows. Recall that the Artin braid group of type B is the group B_n generated by elements $\varsigma_1, \dots, \varsigma_n$ and relations

$$\begin{aligned} \varsigma_{n-1}\varsigma_n\varsigma_{n-1}\varsigma_n &= \varsigma_n\varsigma_{n-1}\varsigma_n\varsigma_{n-1}, \\ \varsigma_i\varsigma_j &= \varsigma_i\varsigma_j, \qquad |i-j| \geq 2, \\ \varsigma_i\varsigma_{i+1}\varsigma_i &= \varsigma_{i+1}\varsigma_i\varsigma_{i+1}, \qquad i = 1, \dots, n-1. \end{aligned}$$

Equivalently, B_n is the braid group of a once punctured disk.

Let σ be an element of B_n. We will use the same letter σ to denote the induced permutation in S_{n-1}. Given objects $X_1, \dots, X_{n-1}$ in a braided tensor category $\mathcal{B}$ and an object M in a braided $\mathcal{B}$-module category $\mathcal{M}$, there are isomorphisms

$$X_1 \otimes \cdots \otimes X_{n-1} * M \to X_{\sigma(1)} \otimes \dots, \otimes X_{\sigma(n-1)} * M, \qquad \sigma \in B_n,$$

compatible with the composition of braids. In particular, for any $X \in \mathcal{B}$ there is a homomorphism from the pure braid group of type B to $Aut_{\mathcal{M}}(X^{\otimes(n-1)} * M)$.

Definition 4.6. We say that a braided $\mathcal{B}$-module category is *indecomposable* if it is indecomposable as a $\mathcal{B}$-module category.

The α-*inductions* [4] for a a left $\mathcal{B}$-module category $\mathcal{M}$ are tensor functors

$$\alpha^{\pm}_{\mathcal{M}} : \mathcal{B}^{\mathrm{op}} \to \mathcal{E}nd_{\mathcal{B}}(\mathcal{M}), \qquad \alpha^{\pm}(X)(M) = X * M, \qquad X \in \mathcal{B},\ M \in \mathcal{M}\ . \tag{4.4}$$

Here $\mathcal{E}nd_{\mathcal{B}}(\mathcal{M})$ is the category of right exact $\mathcal{B}$-module endofunctors of $\mathcal{M}$, The $\mathcal{B}$-module structures on $\alpha^{\pm}(X)($ are given by the compositions

$$\begin{array}{ccc}
Y \otimes \alpha^{+}_{\mathcal{M}}(X)(M) & \xrightarrow{\alpha^{+}_{\mathcal{M}}(X)_{Y,M}} & \alpha^{+}_{\mathcal{M}}(X)(Y * M) \\
\| & & \| \\
Y * (X * M) & & X * (Y * M) \\
\downarrow{\scriptstyle m^{-1}_{Y,X,M}} & & \uparrow{\scriptstyle m_{X,Y,M}} \\
(Y \otimes X) * M & \xrightarrow{c^{-1}_{X,Y} * \mathrm{Id}_M} & (X \otimes Y) * M,
\end{array}
\qquad
\begin{array}{ccc}
Y \otimes \alpha^{-}_{\mathcal{M}}(X)(M) & \xrightarrow{\alpha^{-}_{\mathcal{M}}(X)_{Y,M}} & \alpha^{-}_{\mathcal{M}}(X)(Y * M) \\
\| & & \| \\
Y * (X * M) & & X * (Y * M) \\
\downarrow{\scriptstyle m^{-1}_{Y,X,M}} & & \uparrow{\scriptstyle m_{X,Y,M}} \\
(Y \otimes X) * M & \xrightarrow{c_{Y,X} * \mathrm{Id}_M} & (X \otimes Y) * M,
\end{array}$$

for all $X, Y \in \mathcal{C}$, $M \in \mathcal{M}$ respectively.

The monoidal structures of $\alpha^{\pm}_{\mathcal{M}}$ are

$$\alpha^{+}_{\mathcal{M}}(Y)(\alpha^{+}_{\mathcal{M}}(X)(M)) = Y * (X * M) \xrightarrow{m^{-1}_{Y,X,M}} (Y \otimes X) * M \xrightarrow{c_{Y,X}} (X \otimes Y) * M = \alpha^{+}_{\mathcal{M}}(X \otimes Y)(M),$$

$$\alpha^{-}_{\mathcal{M}}(Y)(\alpha^{-}_{\mathcal{M}}(X)(M)) = Y * (X * M) \xrightarrow{m^{-1}_{Y,X,M}} (Y \otimes X) * M \xrightarrow{c^{-1}_{X,Y}} (X \otimes Y) * M = \alpha^{-}_{\mathcal{M}}(X \otimes Y)(M).$$

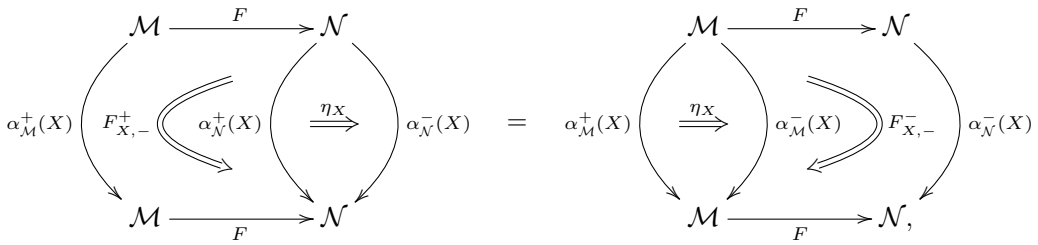
Remark 4.7. For every $\mathcal{B}$-module functor $F : \mathcal{M} \to \mathcal{N}$ there are natural transformations of $\mathcal{B}$-module functors

$$\begin{array}{ccc}
\mathcal{M} & \xrightarrow{F} & \mathcal{N} \\
{\scriptstyle \alpha^{\pm}_{\mathcal{M}}(X)}\downarrow & \xRightarrow{F^{\pm}_{X,-}} & \downarrow{\scriptstyle \alpha^{\pm}_{\mathcal{N}}(X)} \\
\mathcal{M} & \xrightarrow{F} & \mathcal{N},
\end{array} \tag{4.5}$$

for all $X \in \mathcal{B}$.

Remark 4.8. Let $\mathcal{M}$ be a $\mathcal{B}$-module category. The $\mathcal{B}$-bimodule category $\mathcal{M}$ constructed in Section 3.3 can be conveniently described by means of the functor $\alpha^{+}_{\mathcal{M}} : \mathcal{B}^{\mathrm{op}} \to \mathcal{E}nd_{\mathcal{B}}(\mathcal{M})$. Indeed, this functor turns a canonical $(\mathcal{B} \boxtimes \mathcal{E}nd_{\mathcal{B}}(\mathcal{M}))$-module category $\mathcal{M}$ into a $\mathcal{B}$-bimodule category. Note that the functor $\alpha^{-}_{\mathcal{M}}$ gives rise to a different $\mathcal{B}$-bimodule category obtained from $\mathcal{M}$ using the reverse braiding of $\mathcal{B}$.

Let $\mathbf{A}(\mathcal{B})$ denote the 2-category whose objects are pairs $(\mathcal{M}, \eta)$, where $\mathcal{M}$ is a $\mathcal{B}$-module category and $\eta : \alpha^+_{\mathcal{M}} \xrightarrow{\sim} \alpha^-_{\mathcal{M}}$ is an isomorphism of tensor functors, 1-cells are $\mathcal{B}$-module functors $F : \mathcal{M} \to \mathcal{N}$ such that



(4.6)

for all $X \in \mathcal{B}$, where $F^{\pm}_{X,-}$ are natural isomorphisms from (4.5), and 2-cells are B-module natural transformations.

Proposition 4.9. *There is canonical* 2*-equivalence* $\mathbf{Mod_{br}}(\mathcal{B}) \cong \mathbf{A}(\mathcal{B})$.

Proof. A module braiding $\sigma_{X,M}$ on $\mathcal{M}$ is the same thing as a natural isomorphism $\eta : \alpha^+_{\mathcal{M}} \xrightarrow{\sim} \alpha^-_{\mathcal{M}}$ via

$$\eta_X(M) = \sigma_{X,M} : \alpha^+_{\mathcal{M}}(X)(M) = X * M \to X * M = \alpha^-_{\mathcal{M}}(X)(M), \qquad X \in \mathcal{B},\ M \in \mathcal{M}.$$

The first diagram in (4.1) is equivalent to $\eta_X : \alpha^+_{\mathcal{M}}(X) \xrightarrow{\sim} \alpha^-_{\mathcal{M}}(X)$ being an isomorphism of left $\mathcal{B}$-module functors and the second diagram expresses the tensor property of the natural isomorphism $\eta_{\mathcal{M}}$. On the level of 1-cells, the commuting square (4.2) is equivalent to the identity (4.6). □

Remark 4.10. A version of Proposition 4.9 was proved by Safonov in [S, Proposition 2.7].

4.2 $\mathbf{Mod_{br}}(\mathcal{B})$ as a braided monoidal 2-category

Theorem 4.11. *There is a canonical* 2*-equivalence* $\mathbf{Mod_{br}}(\mathcal{B}) \cong \mathbf{Z}(\mathbf{Mod}(\mathcal{B}))$. *In particular,* $\mathbf{Mod_{br}}(\mathcal{B})$ *has a canonical structure of a braided monoidal* 2*-category.*

Proof. In view of Proposition 4.9 it suffices to construct a 2-equivalence $\mathbf{A}(\mathcal{B}) \cong \mathbf{Z}(\mathbf{Mod}(\mathcal{B}))$.

We construct a 2-functor $\mathbf{A}(\mathcal{B}) \to \mathbf{Z}(\mathbf{Mod}(\mathcal{B}))$ as follows. Let $(\mathcal{N}, \eta : \alpha^+_{\mathcal{N}} \xrightarrow{\sim} \alpha^-_{\mathcal{N}})$ be an object of $\mathbf{A}(\mathcal{B})$. Let A be an algebra in $\mathcal{B}$ and let $\mathcal{M} = \mathrm{Mod}_{\mathcal{B}}(A)$ be the category of A-modules in $\mathcal{B}$ (any $\mathcal{B}$-module category is of this form). Then $\mathcal{M} \boxtimes_{\mathcal{B}} \mathcal{N} \cong \mathrm{Mod}_{\mathcal{N}}(\alpha^+_{\mathcal{N}}(A))$, where $\alpha^+_{\mathcal{N}}(A)$ is an algebra in $\mathcal{E}nd_{\mathcal{B}}(\mathcal{N})$ and its module in $\mathcal{N}$ is an object $N \in \mathcal{N}$ along with an action $\alpha^+_{\mathcal{N}}(A) * N \to N$ satisfying usual axioms. Similarly, $\mathcal{N} \boxtimes_{\mathcal{B}} \mathcal{M} \cong \mathrm{Mod}_{\mathcal{N}}(\alpha^-_{\mathcal{N}}(A))$. Hence, the isomorphism $\eta_A : \alpha^+_{\mathcal{N}}(A) \xrightarrow{\sim} \alpha^-_{\mathcal{N}}(A)$ of algebras in $\mathcal{E}nd_{\mathcal{B}}(\mathcal{N})$ yields a pseudo-natural $\mathcal{B}$-module equivalence $S_{\mathcal{M}} : \mathcal{M} \boxtimes_{\mathcal{B}} \mathcal{N} \xrightarrow{\sim} \mathcal{N} \boxtimes_{\mathcal{B}} \mathcal{M}$.

Let $\mathcal{L} = \mathrm{Mod}_{\mathcal{B}}(A_1)$ and $\mathcal{M} = \mathrm{Mod}_{\mathcal{B}}(A_2)$. The invertible modification $\gamma_{\mathcal{L},\mathcal{M}}$ (2.47) comes from the commutative diagram of algebra isomorphisms

$$\begin{array}{ccc} \alpha^+_{\mathcal{N}}(A_1)\otimes\alpha^+_{\mathcal{N}}(A_2) & \xrightarrow{\eta_{A_1}\otimes\eta_{A_2}} & \alpha^-_{\mathcal{N}}(A_1)\otimes\alpha^-_{\mathcal{N}}(A_2) \\ \downarrow{\scriptstyle (\alpha^+_{\mathcal{N}})_{A_1\otimes A_2}} & & \downarrow{\scriptstyle (\alpha^-_{\mathcal{N}})_{A_1\otimes A_2}} \\ \alpha^+_{\mathcal{N}}(A_1\otimes A_2) & \xrightarrow{\eta_{A_1\otimes A_2}} & \alpha^-_{\mathcal{N}}(A_1\otimes A_2). \end{array} \tag{4.7}$$

Note that since $\alpha^{\pm}_{\mathcal{N}}$ is a central functor, $\alpha^{\pm}_{\mathcal{N}}(A_1)\otimes\alpha^{\pm}_{\mathcal{N}}(A_2)$ are algebras in $\mathcal{E}nd_{\mathcal{B}}(\mathcal{N})$ and $\eta_{A_1}\otimes\eta_{A_2}$ is an algebra isomorphism. The coherence condition (2.48) follows from the tensor property of η. Thus, we have an object $(\mathcal{N},\, S=\{S_{\mathcal{M}}\},\, \gamma=\{\gamma_{\mathcal{L},\mathcal{M}}\})$ of $\mathbf{Z}(\mathbf{Mod}(\mathcal{B}))$, see Section 2.3. This gives rise to a 2-functor

$$\mathbf{A}(\mathcal{B}) \to \mathbf{Z}(\mathbf{Mod}(\mathcal{B})) : (\mathcal{N},\, \eta) \mapsto (\mathcal{N},\, S,\, \gamma). \tag{4.8}$$

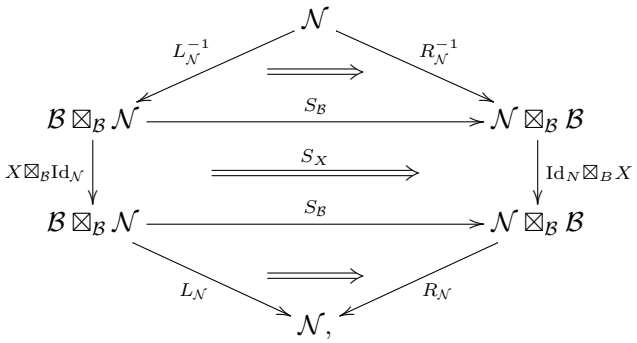
To construct a 2-functor in the opposite direction, note that for any $X \in \mathcal{E}nd_{\mathcal{B}}(\mathcal{I}_{\mathbf{Mod}(\mathcal{B})}) \cong \mathcal{B}^{\mathrm{op}}$ and $\mathcal{N} \in \mathbf{Mod}(\mathcal{B})$ the tensor functors

$$\mathcal{B}^{\mathrm{op}} \to \mathcal{E}nd_{\mathcal{B}}(\mathcal{N}) : X \mapsto L_{\mathcal{N}} \circ (X \boxtimes_{\mathcal{B}} \mathrm{Id}_{\mathcal{N}}) \circ L^{-1}_{\mathcal{N}}, \tag{4.9}$$

$$\mathcal{B}^{\mathrm{op}} \to \mathcal{E}nd_{\mathcal{B}}(\mathcal{N}) : X \mapsto R_{\mathcal{N}} \circ (\mathrm{Id}_{\mathcal{N}} \boxtimes_{\mathcal{B}} X) \circ R^{-1}_{\mathcal{N}}, \tag{4.10}$$

where $L_{\mathcal{N}}$, $R_{\mathcal{N}}$ are the unit constraint 1-cells in $\mathbf{Mod}(\mathcal{B})$, are isomorphic to $\alpha^+_{\mathcal{N}}$ and $\alpha^-_{\mathcal{N}}$, respectively.

For an object $(\mathcal{N},\, S,\, \gamma)$ in $\mathbf{Z}(\mathbf{Mod}(\mathcal{B}))$ consider the following composition of invertible 2-cells:



(4.11)

where the top and bottom triangles are canonical 2-cells coming from the unit constraints of $\mathbf{Mod}(\mathcal{B})$ and S_X is the half-braiding 2-cell. The outside compositions are (4.9) and (4.10). Thus, 2-cells (4.11) give an isomorphism of $\mathcal{B}$-module functors $\eta_X : \alpha^+_{\mathcal{N}}(X) \xrightarrow{\sim} \alpha^-_{\mathcal{N}}(X)$, $X \in \mathcal{B}$. The multiplicative property (2.21) of the pseudo-natural transformation S implies that η is an isomorphism of tensor functors.

This gives a 2-functor

$$\mathbf{Z}(\mathbf{Mod}(\mathcal{B})) \to \mathbf{A}(\mathcal{B}) : (\mathcal{N},\, S,\, \gamma) \mapsto (\mathcal{N},\, \eta_{\mathcal{N}}) \tag{4.12}$$

quasi-inverse to (4.8).

This resulting 2-equivalence $\mathbf{Mod_{br}}(\mathcal{B}) \cong \mathbf{Z}(\mathbf{Mod}(\mathcal{B}))$ obtained using Proposition 4.9 induces on $\mathbf{Mod_{br}}(\mathcal{B})$ a structure of a braided monoidal 2-category. □

For braided $\mathcal{B}$-module categories $\mathcal{M} := (\mathcal{M}, \sigma^{\mathcal{M}})$ and $\mathcal{N} := (\mathcal{N}, \sigma^{\mathcal{N}})$ the braiding

$$B_{\mathcal{M},\mathcal{N}} : \mathcal{M} \boxtimes_{\mathcal{B}} \mathcal{N} \xrightarrow{\sim} \mathcal{N} \boxtimes_{\mathcal{B}} \mathcal{M}$$

in $\mathbf{Mod_{br}}(\mathcal{B})$ is given by the half braiding of $\mathcal{N}$.

Proposition 4.12. *A braided tensor functor $F : \mathcal{B} \to \mathcal{C}$ induces a braided monoidal 2-functor*

$$\mathbf{Mod_{br}}(\mathcal{C}) \to \mathbf{Mod_{br}}(\mathcal{B}) : (\mathcal{M}, \sigma) \mapsto (\tilde{\mathcal{M}}, \tilde{\sigma}),$$

where $\tilde{\mathcal{M}} = \mathcal{M}$ with the action $X * M = F(X) * M$ and $\tilde{\sigma}_{X,M} = \sigma_{X,M}$, $X \in \mathcal{B}$, $M \in \mathcal{M}$.

Proof. This is verified directly using the $\mathcal{C}$-module braiding axioms of σ and the braided property of F. □

Remark 4.13. The braided monoidal 2-category structure on $\mathbf{Mod_{br}}(\mathcal{B})$ constructed in Theorem 4.11 can also be described explicitly as follows. Let $(\mathcal{M}, \sigma^{\mathcal{M}})$ and $(\mathcal{N}, \sigma^{\mathcal{N}})$ be braided $\mathcal{B}$-module categories. Recall that objects of $\mathcal{M} \boxtimes_{\mathcal{B}} \mathcal{N}$ are pairs (V, γ), where $V \in \mathcal{M} \boxtimes \mathcal{N}$ and

$$\gamma_X : (X \boxtimes \mathbf{1}) * V \xrightarrow{\sim} (\mathbf{1} \boxtimes X) * V,\ X \in \mathcal{B},$$

is a balancing isomorphism satisfying (3.7).

The tensor product of $\mathbf{Mod_{br}}(\mathcal{B})$ is

$$(\mathcal{M}, \sigma^{\mathcal{M}}) \boxtimes_{\mathcal{B}} (\mathcal{N}, \sigma^{\mathcal{N}}) = (\mathcal{M} \boxtimes_{\mathcal{B}} \mathcal{N}, \sigma^{\mathcal{M}\boxtimes_{\mathcal{B}}\mathcal{N}}),$$

where

$$\sigma^{\mathcal{M}\boxtimes_{\mathcal{B}}\mathcal{N}}_{X,(V,\gamma)} : (X \boxtimes \mathbf{1}) * (V, \gamma) \to (X \boxtimes \mathbf{1}) * (V, \gamma), \qquad X \in \mathcal{B},\ (V, \gamma) \in \mathcal{M} \boxtimes_{\mathcal{B}} \mathcal{N},$$

is given by the composition

$$(X \boxtimes 1) \otimes V \xrightarrow{\gamma_X} (1 \boxtimes X) \otimes V \xrightarrow{\sigma^{\mathcal{N}}_{X,V}} (1 \boxtimes X) \otimes V \xrightarrow{\gamma_X^{-1}} (X \boxtimes 1) \otimes V \xrightarrow{\sigma^{\mathcal{M}}_{X,V}} (X \boxtimes 1) \otimes V$$

The unit object for this tensor product is the regular braided $\mathcal{B}$-module category from Example 4.4.

The braiding is

$$B_{(\mathcal{M},\sigma^{\mathcal{M}}),(\mathcal{N},\sigma^{\mathcal{N}})} : (\mathcal{M}, \sigma^{\mathcal{M}}) \boxtimes_{\mathcal{B}} (\mathcal{N}, \sigma^{\mathcal{N}}) \xrightarrow{\sim} (\mathcal{N}, \sigma^{\mathcal{N}}) \boxtimes_{\mathcal{B}} (\mathcal{M}, \sigma^{\mathcal{M}}), \quad (V, \gamma) \mapsto (V^t, \tilde{\gamma}),$$

where $\mathcal{M} \boxtimes \mathcal{N} \to \mathcal{N} \boxtimes \mathcal{M} : V \mapsto V^t$ is the transposition functor, i.e. $V^t = N \boxtimes M$ for $V = M \boxtimes N$ (this extends to $\mathcal{M} \boxtimes \mathcal{N}$ thanks to the universal property of $\boxtimes$) and

$$\tilde{\gamma}_X : (X \boxtimes \mathbf{1}) * V^t \xrightarrow{\sigma^{\mathcal{N}}_{X,V}} (X \boxtimes \mathbf{1}) * V^t \xrightarrow{(\gamma^t_X)^{-1}} (\mathbf{1} \boxtimes X) * V^t, \quad X \in \mathcal{B}.$$

4.3 Examples and basic properties of braided module categories

Let $\mathcal{B}$ be a braided tensor category.

Example 4.14. The regular braided $\mathcal{B}$-module category $\mathcal{B}$ from Example 4.4 is the unit object of $\mathbf{Mod_{br}}(\mathcal{B})$. It generates a braided monoidal 2-category $\mathbf{Mod^0_{br}}(\mathcal{B})$ whose objects are direct sums of copies of $\mathcal{B}$ (identified with natural numbers), 1-cells are matrices of objects in $\mathcal{Z}_{sym}(\mathcal{B})$, and 2-cells are matrices of morphisms in $\mathcal{B}$. The tensor product is given by the Kronecker product of such matrices while the braiding 2-cells are given by the braiding of $\mathcal{Z}_{sym}(\mathcal{B})$.

Example 4.15. Note that $\mathcal{B}$ can have other $\mathcal{B}$-module braidings, in addition to one from Example 4.4. Namely, it follows from the first diagram in (4.1) that a $\mathcal{B}$-module braiding σ on $\mathcal{B}$ satisfies

$$\sigma_{X,Y} = \sigma_{X,\mathbf{1}} c_{Y,X} c_{X,Y},\ X, Y \in \mathcal{B}.$$

The second diagram in (4.1) is equivalent to $\sigma_{X,\mathbf{1}}$ being a tensor automorphism of $\mathrm{Id}_{\mathcal{B}}$. Conversely, any $\nu \in Aut_{\otimes}(\mathrm{Id}_{\mathcal{B}})$ yields a module braiding

$$\sigma^{\nu}_{X,Y} = (\nu_X \otimes \mathbf{1}) c_{Y,X} c_{X,Y},\ X, Y \in \mathcal{B}. \tag{4.13}$$

Let $\mathcal{B}^{\nu} := (\mathcal{B}, \sigma^{\nu})$ denote the corresponding braided $\mathcal{B}$-module category.

There is an exact sequence [18, 3.3.4] of groups

$$1 \to Inv(\mathcal{Z}_{sym}(\mathcal{B})) \to Inv(\mathcal{B}) \xrightarrow{\alpha} Aut(\mathrm{Id}_{\mathcal{B}}), \tag{4.14}$$

where

$$\alpha(Z)_X = c_{Z,X} c_{X,Z} \in Aut(X \otimes Z) = k^{\times} \tag{4.15}$$

for every simple object $X \in \mathcal{B}$. Two braided $\mathcal{B}$-module categories $\mathcal{B}^{\nu_1}$ and $\mathcal{B}^{\nu_2}$, $\nu_1, \nu_2 \in Aut(\mathrm{Id}_{\mathcal{B}})$, are equivalent if and only if $\nu_2 = \nu_1 \alpha(Z)$ for some $Z \in Inv(\mathcal{B})$. Thus, the group of equivalence classes of braided $\mathcal{B}$-module categories of the form $\mathcal{B}^{\nu}$ is isomorphic to $\mathsf{Coker}(Inv(\mathcal{B}) \xrightarrow{\alpha} Aut(\mathrm{Id}_{\mathcal{B}}))$.

Let $\mathbf{Mod^1_{br}}(\mathcal{B})$ denote the full braided monoidal 2-subcategory of $\mathbf{Mod_{br}}(\mathcal{B})$ generated by braided $\mathcal{B}$-module categories $\mathcal{B}^{\nu}$.

Example 4.16. Let $\mathcal{M}$ be an exact $\mathcal{B}$-module category. The 2-categorical half-braiding

$$\mathcal{M} \simeq \mathcal{M} \boxtimes_{\mathcal{B}} \mathcal{B}^{\nu} \xrightarrow{S^{\nu}_{\mathcal{M}}} \mathcal{B}^{\nu} \boxtimes_{\mathcal{B}} \mathcal{M} \simeq \mathcal{M} \tag{4.16}$$

is identified with the image of ν under the composition

$$Aut_{\otimes}(\mathrm{Id}_{\mathcal{B}}) \xrightarrow{\iota} Inv(\mathcal{Z}(\mathcal{B})) \cong Inv(\mathcal{Z}(\mathcal{End}_{\mathcal{B}}(\mathcal{M})) \to Inv(\mathcal{End}_{\mathcal{B}}(\mathcal{M})) = Aut_{\mathcal{B}}(\mathcal{M}), \tag{4.17}$$

where $\iota(\nu) = \mathbf{1}$ as an object of $\mathcal{B}$ with the half-braiding

$$\nu_X \mathrm{Id}_X : X \cong X \otimes \iota(\nu) \xrightarrow{\sim} \iota(\nu) \otimes X \cong X, \quad X \in \mathcal{B}.$$

For the trivial tensor category $\mathcal{B} = \mathcal{V}ect$ we have $\mathbf{Mod_{br}}(\mathcal{V}ect) = \mathbf{Mod^0_{br}}(\mathcal{V}ect) = \mathbf{Mod^1_{br}}(\mathcal{V}ect)$, the 2-category of 2-vector spaces. The objects of this category are natural numbers, 1-cells are matrices of vector spaces, and 2-cells are matrices of linear transformations.

The following result was established in [30] using different methods and terminology.

Proposition 4.17. *Let $\mathcal{B}$ be a non-degenerate braided fusion category. There is an equivalence of braided monoidal 2-categories*

$$\mathbf{Mod_{br}}(\mathcal{B}) \simeq \mathbf{Mod_{br}}(\mathcal{V}ect). \tag{4.18}$$

Proof. Let $\mathcal{M}$ be an indecomposable $\mathcal{B}$-module category. The tensor functors $\alpha^{\pm}_{\mathcal{M}} : \mathcal{B} \to \mathcal{E}nd_{\mathcal{B}}(\mathcal{M})$ are given by the compositions

$$\mathcal{B} \to \mathcal{Z}(\mathcal{B}) \simeq \mathcal{Z}(\mathcal{E}nd_{\mathcal{B}}(\mathcal{M})) \to \mathcal{E}nd_{\mathcal{B}}(\mathcal{M}), \tag{4.19}$$

where the first functor is the embedding of $\mathcal{B}$ (respectively, $\mathcal{B}^{\mathrm{rev}}$) into $\mathcal{Z}(\mathcal{B})$, and the last one is the forgetful functor. The images of $\mathcal{B}$ and $\mathcal{B}^{\mathrm{rev}}$ generate $\mathcal{Z}(\mathcal{E}nd_{\mathcal{B}}(\mathcal{M}))$. If $\mathcal{M}$ has a $\mathcal{B}$-module braiding, it follows from Proposition 4.9 that the full images of $\alpha^{+}_{\mathcal{M}}(\mathcal{B})$ and $\alpha^{-}_{\mathcal{M}}(\mathcal{B})$ in $\mathcal{E}nd_{\mathcal{B}}(\mathcal{M})$ coincide. Since the forgetful functor is surjective, we have $\alpha^{\pm}_{\mathcal{M}}(\mathcal{B}) = \mathcal{E}nd_{\mathcal{B}}(\mathcal{M})$. Thus, $\mathcal{M}$ is an invertible $\mathcal{B}$-module category such that the braided autoequivalence $\partial_{\mathcal{M}} := (\alpha^{+}_{\mathcal{M}})^{-1} \circ \alpha^{-}_{\mathcal{M}} \in Aut^{br}(\mathcal{B})$ is trivial. It follows from [10, 23] that $\mathcal{M} \simeq \mathcal{B}$ as a $\mathcal{B}$-module category, i.e. $\mathbf{Mod_{br}}(\mathcal{B}) \simeq \mathbf{Mod^1_{br}}(\mathcal{B})$. Since the homomorphism α from Example 4.15 is an isomorphism for a non-degenerate category $\mathcal{B}$, we have $\mathbf{Mod^1_{br}}(\mathcal{B}) \simeq \mathbf{Mod^0_{br}}(\mathcal{B})$. Since $\mathcal{Z}_{sym}(\mathcal{B}) = \mathcal{V}ect$, $\mathbf{Mod^0_{br}}(\mathcal{B})$ is 2-equivalent to $\mathbf{Mod_{br}}(\mathcal{V}ect)$, and the statement follows. □

4.4 The symmetric monoidal 2-category of symmetric module categories

Let $\mathcal{E}$ be a symmetric tensor category. For any $\mathcal{E}$-module category $\mathcal{M}$ we have $\alpha^{+}_{\mathcal{M}} = \alpha^{-}_{\mathcal{M}}$. In particular any $\mathcal{E}$-module category $\mathcal{M}$ has the identity module braiding $\mathrm{Id}_{X\otimes M}$.

Definition 4.18. A braided $\mathcal{E}$-module category $(\mathcal{M}, \sigma)$ is called *symmetric* if $\sigma_{X,M} = \mathrm{Id}_{X\otimes M}$ for all $X \in \mathcal{E}$ and $M \in \mathcal{M}$.

Example 4.19. Let $\mathcal{C}$ be a symmetric braided tensor category containing $\mathcal{E}$. Then $\mathcal{C}$ is a symmetric $\mathcal{E}$-module category.

Clearly, the tensor product of symmetric module categories is symmetric. We will denote $\mathbf{Mod_{sym}}(\mathcal{E})$ the symmetric monoidal 2-category of symmetric $\mathcal{E}$-module categories (its double braiding 2-cells (2.30)

are identities). Note that $\mathbf{Mod_{sym}}(\mathcal{E}) = \mathbf{Mod}(\mathcal{E})$ as a monoidal 2-category and can also be viewed as a braided monoidal 2-subcategory of $\mathbf{Mod_{br}}(\mathcal{E})$.

Remark 4.20. Let $\mathcal{B}$ be a braided tensor category. It follows from Example 4.16 that the braided monoidal 2-category $\mathbf{Mod^1_{br}}(\mathcal{B})$ has a symmetric structure.

Proposition 4.21. *Let $\mathcal{E}$ be a tensor subcategory of $\mathcal{Z}_{sym}(\mathcal{B})$. The induction*

$$\mathbf{Mod}(\mathcal{E}) \to \mathbf{Mod_{br}}(\mathcal{B}) : \mathcal{N} \mapsto \mathcal{B} \boxtimes_{\mathcal{E}} \mathcal{N} \tag{4.20}$$

is a braided monoidal 2-functor.

Proof. Let $\mathcal{N} = \mathrm{Mod}_{\mathcal{E}}(A) \in \mathbf{Mod}(\mathcal{E})$ for some algebra $A \in \mathcal{E}$. Then $\mathcal{B} \boxtimes_{\mathcal{E}} \mathcal{N} = \mathrm{Mod}_{\mathcal{B}}(A)$. For any $\mathcal{M} \in \mathbf{Mod}(\mathcal{B})$ the composition of $\mathcal{B}$-module equivalences

$$\mathcal{M} \boxtimes_{\mathcal{B}} (\mathcal{B} \boxtimes_{\mathcal{E}} \mathcal{N}) \cong \mathrm{Mod}_{\mathcal{M}}(\alpha^-_{\mathcal{M}}(A)) = \mathrm{Mod}_{\mathcal{M}}(\alpha^+_{\mathcal{M}}(A)) \cong (\mathcal{B} \boxtimes_{\mathcal{E}} \mathcal{N}) \boxtimes_{\mathcal{B}} \mathcal{M},$$

where the equality in the middle is due to the fact that $A \in \mathcal{Z}_{sym}(\mathcal{B})$, defines a half braiding on $\mathcal{B} \boxtimes_{\mathcal{E}} \mathcal{N}$. It follows that $\mathcal{B} \boxtimes_{\mathcal{E}} \mathcal{N}$ is a braided $\mathcal{B}$-module category and the monoidal induction 2-functor from Proposition 3.7 lifts to a braided monoidal 2-functor (4.20). □

5 2-categorical Picard groups

In this Section we describe categorical 2-groups of module categories over tensor categories in terms introduced in Section 2.4.

5.1 The 2-categorical Brauer-Picard group of a tensor category

Let $\mathcal{D}$ be a tensor category. Recall from [23, Section 4.1] that a $\mathcal{D}$-bimodule category $\mathcal{M}$ is invertible with respect to $\boxtimes_{\mathcal{D}}$ if and only if $\mathcal{M}^o \boxtimes_{\mathcal{D}} \mathcal{M} \cong \mathcal{D}$ and $\mathcal{M} \boxtimes_{\mathcal{D}} \mathcal{M}^o \cong \mathcal{D}$, where $\mathcal{M}^o$ is the opposite Abelian category of $\mathcal{M}$ with the left (respectively, right) $\mathcal{D}$-module actions of $X \in \mathcal{D}$ given by the right (respectively, left) actions of ${}^*X$ (see also [17]).

The 2-*categorical Brauer-Picard group* [23] of a tensor category $\mathcal{D}$ is

$$\mathbf{BrPic}(\mathcal{D}) = \mathbf{Inv}(\mathbf{Bimod}(\mathcal{D})). \tag{5.1}$$

Its objects are invertible $\mathcal{D}$-bimodule categories, 1-cells are $\mathcal{D}$-bimodule equivalences, and 2-cells are isomorphisms of $\mathcal{D}$-bimodule equivalences. The tensor product is $\boxtimes_{\mathcal{D}}$ and the unit object is the regular $\mathcal{D}$-bimodule category. Let $\mathcal{BrP}ic(\mathcal{D})$ denote the categorical group obtained by truncating $\mathbf{BrPic}(\mathcal{D})$ and let $BrPic(\mathcal{D})$ denote the group of isomorphism classes of objects.

The homotopy groups of $\mathbf{BrPic}(\mathcal{D})$ are

$$\pi_0(\mathbf{BrPic}(\mathcal{D})) = BrPic(\mathcal{D}) \cong Aut^{br}(\mathcal{Z}(\mathcal{D})), \tag{5.2}$$

$$\pi_1(\mathbf{BrPic}(\mathcal{D})) = Inv(\mathcal{Z}(\mathcal{D})), \tag{5.3}$$

$$\pi_2(\mathbf{BrPic}(\mathcal{D})) = k^\times. \tag{5.4}$$

The 1-categorical truncations of $\mathbf{BrPic}(\mathcal{D})$ are

$$\Pi_{\leq 1}(\mathbf{BrPic}(\mathcal{D})) = \mathcal{BrP}ic(\mathcal{D}) \cong \mathcal{A}ut^{br}(\mathcal{Z}(\mathcal{D})), \tag{5.5}$$

$$\Pi_{1\leq}(\mathbf{BrPic}(\mathcal{D})) = \mathcal{I}nv(\mathcal{Z}(\mathcal{D})). \tag{5.6}$$

The first canonical class is the associator $\alpha_{\mathbf{BrPic}(\mathcal{D})} \in H^3(Aut^{br}(\mathcal{Z}(\mathcal{D})),\, Inv(\mathcal{Z}(\mathcal{D})))$ of the categorical group $\mathcal{A}ut^{br}(\mathcal{Z}(\mathcal{D}))$. The second canonical class is the braided associator $q_{\mathbf{BrPic}(\mathcal{D})} \in H^3_{br}(Inv(\mathcal{Z}(\mathcal{D})),\, k^\times)$ of the braided categorical group $\mathcal{I}nv(\mathcal{Z}(\mathcal{D}))$, corresponding to the quadratic form

$$q_{\mathbf{BrPic}(\mathcal{D})} : Inv(\mathcal{Z}(\mathcal{D})) \to k^\times : Z \mapsto c_{Z,Z}.$$

The monoidal functor $\Pi_{\leq 1}(\mathbf{BrPic}(\mathcal{D})) \to \mathcal{A}ut^{br}(\Pi_{1\leq}(\mathbf{BrPic}(\mathcal{D})))$ coincides with the composition $\mathcal{BrP}ic(\mathcal{D}) \cong \mathcal{A}ut^{br}(\mathcal{Z}(\mathcal{D})) \to \mathcal{A}ut^{br}(\mathcal{I}nv(\mathcal{Z}(\mathcal{D})))$ [10, 23].

The non-trivial Whitehead brackets are the maps $\pi_0 \times \pi_1 \to \pi_1$ and $\pi_1 \times \pi_1 \to \pi_2$ given by

$$Aut^{br}(\mathcal{Z}(\mathcal{D})) \times Inv(\mathcal{Z}(\mathcal{D})) \to Inv(\mathcal{Z}(\mathcal{D})) : (F,\, Z) \mapsto F(Z), \tag{5.7}$$

$$Inv(\mathcal{Z}(\mathcal{D})) \times Inv(\mathcal{Z}(\mathcal{D})) \to k^\times : (Z,\, W) \mapsto c_{W,Z}c_{Z,W}. \tag{5.8}$$

Here c denotes the braiding of $\mathcal{Z}(\mathcal{D})$.

5.2 The 2-categorical Picard group of a braided tensor category

Let $\mathcal{B}$ be a braided tensor category. Recall [10, 23] that a $\mathcal{B}$-module category $\mathcal{M}$ is invertible if and only if the α-induction tensor functors $\alpha^{\pm}_{\mathcal{M}} : \mathcal{B}^{\mathrm{op}} \to \mathcal{E}nd_{\mathcal{B}}(\mathcal{M})$, see (4.4), are equivalences. Here $\mathcal{E}nd_{\mathcal{B}}(\mathcal{M})$ denotes the category of right exact $\mathcal{B}$-module endofunctors of $\mathcal{M}$.

Recall that a $\mathcal{B}$-module category $\mathcal{M}$ is *exact* [24] if for any projective object $P \in \mathcal{B}$ and any object $M \in \mathcal{M}$ the object $P \otimes M \in \mathcal{M}$ is projective. For an exact $\mathcal{M}$, the dual category $\mathcal{E}nd_{\mathcal{B}}(\mathcal{M})$ is a multitensor category. The tensor product of functors is their composition and the left and right duals of a $\mathcal{C}$-module functor $F : \mathcal{M} \to \mathcal{M}$ are its left and right adjoints.

The following result was explained to us by Victor Ostrik.

Proposition 5.1. *An invertible $\mathcal{B}$-module category is exact.*

Proof. Let $\mathcal{M}$ be an invertible $\mathcal{B}$-module category. It is equivalent to $\mathrm{Mod}_{\mathcal{B}}(A)$ for some algebra A in $\mathcal{B}$ and

$$\mathrm{Bimod}_{\mathcal{B}}(A) \cong \mathcal{E}nd_{\mathcal{B}}(\mathcal{M}) \cong \mathcal{B}^{\mathrm{op}}.$$

So the tensor product over A is exact on the category of A-bimodules. This implies that it is exact for right modules tensored with left modules (as any right A-module M can be made into a bimodule $A \otimes M$ and similarly for left modules). By [22, Proposition 7.9.7(1) and Example 7.9.8], this is equivalent to exactness of the internal Hom functor

$$\mathcal{M} \to \mathcal{B} : M \mapsto \underline{\mathrm{Hom}}(N,\, M) \qquad \text{for all } N \in \mathrm{Mod}_{\mathcal{B}}(A)$$

and, hence, to exactness of $\mathcal{M}$. □

Corollary 5.2. *Let $\mathcal{C}$ be a finite tensor category. An invertible $\mathcal{C}$-bimodule category is exact.*

Proof. A canonical monoidal 2-equivalence between $\mathbf{Pic}(\mathcal{Z}(\mathcal{C}))$ and $\mathbf{BrPic}(\mathcal{C})$ preserves exactness by [24, Theorem 3.31]. □

The 2-*categorical Picard group* [10, 23] of $\mathcal{B}$ is

$$\mathbf{Pic}(\mathcal{B}) = \mathbf{Inv}(\mathbf{Mod}(\mathcal{B})). \tag{5.9}$$

Its objects are invertible $\mathcal{B}$-module categories, 1-cells are $\mathcal{B}$-module equivalences, and 2-cells are isomorphisms of $\mathcal{B}$-module equivalences. The tensor product is $\boxtimes_{\mathcal{B}}$ and the unit object is the regular $\mathcal{B}$-module category. Let $\mathcal{P}ic(\mathcal{B})$ denote the categorical group obtained by modding out $\mathbf{Pic}(\mathcal{B})$ by 2-morphisms and let $Pic(\mathcal{B})$ denote the group of isomorphism classes of objects.

The homotopy groups of $\mathbf{Pic}(\mathcal{B})$ are

$$\pi_0(\mathbf{Pic}(\mathcal{B})) = Pic(\mathcal{B}), \tag{5.10}$$

$$\pi_1(\mathbf{Pic}(\mathcal{B})) = Inv(\mathcal{B}), \tag{5.11}$$

$$\pi_2(\mathbf{Pic}(\mathcal{B})) = k^\times. \tag{5.12}$$

The 1-categorical truncations of $\mathbf{Pic}(\mathcal{B})$ are

$$\Pi_{\leq 1}(\mathbf{Pic}(\mathcal{B})) = \mathcal{P}ic(\mathcal{B}), \tag{5.13}$$

$$\Pi_{1\leq}(\mathbf{Pic}(\mathcal{B})) = \mathcal{I}nv(\mathcal{B}). \tag{5.14}$$

The first canonical class is the associator $\alpha_{\mathbf{Pic}(\mathcal{B})} \in H^3(Pic(\mathcal{B}), Inv(\mathcal{B}))$ of the categorical group $\mathcal{P}ic(\mathcal{B})$. The second canonical class is the braided associator $q_{\mathbf{Pic}(\mathcal{B})} \in H^3_{br}(Inv(\mathcal{B}), k^\times)$ of the braided categorical group $\mathcal{I}nv(\mathcal{B})$, corresponding to the quadratic form

$$q_{\mathbf{Pic}(\mathcal{B})} : Inv(\mathcal{B}) \to k^\times : Z \mapsto c_{Z,Z}.$$

The monoidal functor $\Pi_{\leq 1}(\mathbf{Pic}(\mathcal{B})) \to Aut^{br}(\Pi_{1\leq}(\mathbf{Pic}(\mathcal{B})))$ coincides with the composition $\mathcal{P}ic(\mathcal{B}) \to \mathcal{A}ut^{br}(\mathcal{B}) \to \mathcal{A}ut^{br}(\mathcal{I}nv(\mathcal{B}))$ [10, 23].

The non-trivial Whitehead brackets are the maps $\pi_0 \times \pi_1 \to \pi_1$ and $\pi_1 \times \pi_1 \to \pi_2$ given by

$$Aut^{br}(\mathcal{B}) \times Inv(\mathcal{B}) \to Inv(\mathcal{B}) : (F,\, Z) \mapsto F(Z), \tag{5.15}$$

$$Inv(\mathcal{B}) \times Inv(\mathcal{B}) \to k^{\times} : (Z,\, W) \mapsto c_{W,Z}c_{Z,W}. \tag{5.16}$$

Here c denotes the braiding of $\mathcal{B}$.

For any tensor category $\mathcal{D}$ there is a monoidal 2-equivalence $\mathbf{BrPic}(\mathcal{D}) \cong \mathbf{Pic}(\mathcal{Z}(\mathcal{D}))$ [23, Theorem 5.2]. Thus, 2-categorical Picard groups generalize Brauer-Picard groups.

5.3 The braided 2-categorical Picard group of a braided tensor category

The *braided 2-categorical Picard group* of a braided tensor category $\mathcal{B}$ is

$$\mathbf{Pic_{br}}(\mathcal{B}) = \mathbf{Inv}(\mathbf{Mod_{br}}(\mathcal{B})) \cong \mathbf{Inv}(\mathbf{Z}(\mathbf{Mod}(\mathcal{B}))), \tag{5.17}$$

where the last 2-equivalence is by Theorem 4.11. Its objects are invertible braided $\mathcal{B}$-module categories, 1-cells are braided $\mathcal{B}$-module equivalences, and 2-cells are natural isomorphisms of $\mathcal{B}$-module equivalences. The tensor product is $\boxtimes_{\mathcal{B}}$ and the unit object is the regular braided $\mathcal{B}$-module category (see Example 4.1). Let $\mathcal{P}ic_{br}(\mathcal{B})$ denote the braided categorical group obtained by by modding out $\mathbf{Pic_{br}}(\mathcal{B})$ by 2-morphisms and let $Pic_{br}(\mathcal{B})$ denote the group of isomorphism classes of objects.

The homotopy groups of $\mathbf{Pic_{br}}(\mathcal{B})$ are

$$\pi_0(\mathbf{Pic}(\mathcal{B})) = Pic_{br}(\mathcal{B}), \tag{5.18}$$

$$\pi_1(\mathbf{Pic}(\mathcal{B})) = Inv(\mathcal{Z}_{sym}(\mathcal{B})), \tag{5.19}$$

$$\pi_2(\mathbf{Pic}(\mathcal{B})) = k^{\times}. \tag{5.20}$$

The 1-categorical truncations of $\mathbf{Pic_{br}}(\mathcal{B})$ are

$$\Pi_{\leq 1}(\mathbf{Pic}(\mathcal{B})) = \mathcal{P}ic_{br}(\mathcal{B}), \tag{5.21}$$

$$\Pi_{1\leq}(\mathbf{Pic}(\mathcal{B})) = \mathcal{I}nv(\mathcal{Z}_{sym}(\mathcal{B})). \tag{5.22}$$

The first canonical class is the braided associator $\alpha_{\mathbf{Pic_{br}}(\mathcal{B})} \in H^3_{br}(Pic_{br}(\mathcal{B}),\, Inv(\mathcal{Z}_{sym}(\mathcal{B})))$ of the braided categorical group $\mathbf{Pic_{br}}(\mathcal{B})$ corresponding to the quadratic function

$$Q_{\mathbf{Pic_{br}}(\mathcal{B})} : Pic_{br}(\mathcal{B}) \to Inv(\mathcal{Z}_{sym}(\mathcal{B})) : \mathcal{M} \mapsto B_{\mathcal{M},\mathcal{M}},$$

where B denotes the braiding of $\mathbf{Pic_{br}}(\mathcal{B})$. The second canonical class is the symmetric associator $q_{\mathbf{Pic_{br}}(\mathcal{B})} \in H^3_{sym}(Inv(\mathcal{Z}_{sym}(\mathcal{B})),\, k^\times)$ of the symmetric categorical group $\mathcal{I}nv(\mathcal{Z}_{sym}(\mathcal{B}))$ corresponding to the homomorphism

$$q_{\mathbf{Pic_{br}}(\mathcal{B})} : Inv(\mathcal{Z}_{sym}(\mathcal{B})) \to \{\pm 1\} \subset k^\times : Z \mapsto c_{Z,Z}.$$

Proposition 5.3. *Let $(\mathcal{M},\, \sigma^{\mathcal{M}})$ be an indecomposable braided $\mathcal{B}$-module category and let Z be an invertible object in $\mathcal{Z}_{sym}(\mathcal{B})$. The Whitehead bracket $[\,,\,] : \pi_0 \times \pi_1 \to \pi_2$ (2.60) of $\mathbf{Pic_{br}}(\mathcal{B})$ satisfies*

$$\sigma^{\mathcal{M}}_{Z,\,M} = [\mathcal{M},\, Z]\, Id_{Z*M} \tag{5.23}$$

for all objects $M \in \mathcal{M}$.

Proof. For any simple M in $\mathcal{M}$ we identify $\sigma^{\mathcal{M}}_{Z,M} \in Aut(Z \otimes M)$ with a non-zero scalar. It suffices to check that this scalar does not in fact depend on M. Note that

$$\sigma^{\mathcal{M}}_{Z,\,X\otimes M} = c_{Z,X} c_{X,Z} \sigma^{\mathcal{M}}_{Z,M} = \sigma^{\mathcal{M}}_{Z,M}$$

for any simple object $X \in \mathcal{B}$. Since every simple object N of $\mathcal{M}$ is contained in some $X \otimes M$ we conclude that $\sigma^{\mathcal{M}}_{Z,\,M} = \sigma^{\mathcal{M}}_{Z,\,N}$. □

Recall from [10, 23] a monoidal functor

$$\partial : \mathcal{P}ic(\mathcal{B}) \to \mathcal{A}ut^{br}(\mathcal{B}) : \mathcal{M} \mapsto (\alpha^+_{\mathcal{M}})^{-1} \circ \alpha^-_{\mathcal{M}}, \tag{5.24}$$

where $\alpha^\pm_{\mathcal{M}} : \mathcal{B}^{\mathrm{op}} \to \mathcal{E}nd_{\mathcal{B}}(\mathcal{M})$ are equivalences (4.4).

Proposition 5.4. *There is an exact sequence*

$$1 \to Inv(\mathcal{Z}_{sym}(\mathcal{B})) \to Inv(\mathcal{B}) \xrightarrow{\alpha} Aut_\otimes(Id_{\mathcal{B}}) \xrightarrow{\varepsilon} Pic_{br}(\mathcal{B}) \xrightarrow{\phi} Pic(\mathcal{B}) \xrightarrow{\partial} Aut_{br}(\mathcal{B}), \tag{5.25}$$

where α is defined in (4.15), $\varepsilon(\nu) = \mathcal{B}^\nu$ (see Example 4.16), and $\phi(\mathcal{M},\, \sigma) = \mathcal{M}$.

Proof. This is an immediate consequence of the definitions. □

Remark 5.5. By the *fiber* of the monoidal functor $F : \mathcal{G} \to \mathcal{H}$ between groupoids we mean the category of pairs (X, x), where $X \in \mathcal{G}$ and $x : F(G) \to I$ for the unit object $I \in \mathcal{H}$. It follows from Proposition 4.9 that the fiber of the monoidal functor $\mathcal{P}ic(\mathcal{B}) \to \mathcal{A}ut(\mathcal{B})$ coincides with $\mathcal{P}ic_{br}(\mathcal{B})$. The exact sequence (5.25) can be seen as the Serre exact sequence of homotopy groups of the fibration of categorical groups

$$\mathcal{P}ic_{br}(\mathcal{B}) \to \mathcal{P}ic(\mathcal{B}) \to \mathcal{A}ut(\mathcal{B}).$$

Example 5.6. Let $\mathbf{Pic}^{\mathbf{1}}_{\mathbf{br}}(\mathcal{B})$ be the braided 2-categorical subgroup of $\mathbf{Pic}_{\mathbf{br}}(\mathcal{B})$ consisting of braided $\mathcal{B}$-module categories whose underlying $\mathcal{B}$-module category is the regular $\mathcal{B}$-module category $\mathcal{B}$. That is, $\mathbf{Pic}^{\mathbf{1}}_{\mathbf{br}}(\mathcal{B}) = \mathbf{Inv}(\mathbf{Mod}^{\mathbf{1}}_{\mathbf{br}}(\mathcal{B}))$, see Example 4.15.

The objects of $\mathbf{Pic}^{\mathbf{1}}_{\mathbf{br}}(\mathcal{B})$ are braided $\mathcal{B}$-module categories $\mathcal{B}^{\nu}$, $\nu \in Aut_{\otimes}(\mathrm{Id}_{\mathcal{B}})$. The module braiding of $\mathcal{B}^{\nu}$ is

$$\sigma^{\nu}_{X,M} = (\nu_X \otimes \mathrm{Id}_M) \circ c_{M,X} c_{X,M}, \quad X, M \in \mathcal{B}. \tag{5.26}$$

The homotopy groups of $\mathbf{Pic}^{\mathbf{1}}_{\mathbf{br}}(\mathcal{B})$ are

$$\pi_0(\mathbf{Pic}^{\mathbf{1}}_{\mathbf{br}}(\mathcal{B})) = \mathrm{Coker}(Inv(\mathcal{B}) \xrightarrow{\partial} Aut_{\otimes}(\mathrm{Id}_{\mathcal{B}})), \tag{5.27}$$

$$\pi_1(\mathbf{Pic}^{\mathbf{1}}_{\mathbf{br}}(\mathcal{B})) = Inv(\mathcal{Z}_{sym}(\mathcal{B})), \tag{5.28}$$

$$\pi_2(\mathbf{Pic}^{\mathbf{1}}_{\mathbf{br}}(\mathcal{B})) = k^{\times}. \tag{5.29}$$

The following is a convenient "non-skeletal" description of $\mathbf{Pic}^{\mathbf{1}}_{\mathbf{br}}(\mathcal{B})$. The objects $\mathcal{B}^{\nu}$ correspond to elements of $Aut_{\otimes}(\mathrm{Id}_{\mathcal{B}})$, 1-morphisms are given by

$$\mathbf{Pic}^{\mathbf{1}}_{\mathbf{br}}(\mathcal{B})(\mathcal{B}^{\mu}, \mathcal{B}^{\nu}) = \{Z \in \mathcal{I}nv(\mathcal{B})) \mid c_{Z,X} \circ c_{X,Z} \circ (\mu_X \otimes \mathrm{Id}_Z) = \nu_X \otimes \mathrm{Id}_Z,\ X \in \mathcal{B}\}, \tag{5.30}$$

and 2-cells are isomorphisms between invertible objects of $\mathcal{B}$. The tensor product is $\mathcal{B}^{\mu} \boxtimes \mathcal{B}^{\nu} := \mathcal{B}^{\mu\mu}$ for all $\mu, \nu \in Aut_{\otimes}(\mathrm{Id}_{\mathcal{B}})$. The associativity and braiding 2-cells are identities and the pseudo-naturality 2-cell for the tensor product is

$$\boxtimes_{Z,W} = c_{Z,W}, \quad Z, W \in \mathcal{I}nv(\mathcal{B}),$$

where c denotes the braiding of $\mathcal{B}$. We also have

$$B_{Z,\mathcal{B}^{\nu}} = \nu_Z, \quad Z \in \mathcal{I}nv(\mathcal{B}), \quad \nu \in Aut_{\otimes}(\mathcal{B}).$$

All other structural 2-cells are identities.

5.4 The symmetric 2-categorical Picard group of a symmetric tensor category

Let $\mathcal{E}$ be a symmetric tensor category. The *symmetric 2-categorical Picard group* of $\mathcal{E}$ is

$$\mathbf{Pic}_{\mathbf{sym}}(\mathcal{E}) = \mathbf{Inv}(\mathbf{Mod}_{\mathbf{sym}}(\mathcal{E})) = \mathbf{Inv}(\mathbf{Mod}(\mathcal{E})) = \mathbf{Pic}(\mathcal{E}). \tag{5.31}$$

Its objects are invertible symmetric $\mathcal{E}$-module categories, 1-cells are $\mathcal{E}$-module equivalences, and 2-cells are natural isomorphisms of $\mathcal{E}$-module equivalences. Let $\mathcal{P}ic_{sym}(\mathcal{E})$ denote the categorical group obtained by truncating $\mathbf{Pic}_{\mathbf{sym}}(\mathcal{E})$ and let $Pic_{sym}(\mathcal{E})$ denote the group of isomorphism classes of objects.

The homotopy groups of $\mathbf{Pic_{sym}}(\mathcal{E})$ are

$$\pi_0(\mathbf{Pic}(\mathcal{E})) = Pic_{sym}(\mathcal{E}) = Pic(\mathcal{E}), \tag{5.32}$$

$$\pi_1(\mathbf{Pic}(\mathcal{E})) = Inv(\mathcal{E}), \tag{5.33}$$

$$\pi_2(\mathbf{Pic}(\mathcal{E})) = k^\times. \tag{5.34}$$

The 1-categorical truncations of $\mathbf{Pic_{sym}}(\mathcal{E})$ are

$$\Pi_{\leq 1}(\mathbf{Pic}(\mathcal{E})) = \mathcal{P}ic_{sym}(\mathcal{E}) = \mathcal{P}ic(\mathcal{E}), \tag{5.35}$$

$$\Pi_{1\leq}(\mathbf{Pic}(\mathcal{E})) = \mathcal{I}nv(\mathcal{E}). \tag{5.36}$$

The first canonical class is the symmetric associator $\alpha_{\mathbf{Pic_{sym}}(\mathcal{B})} \in H^3_{sym}(Pic(\mathcal{E}),\, Inv(\mathcal{E}))$ of the symmetric categorical group $\mathbf{Pic}(\mathcal{B})$ corresponding to the homomorphism

$$Q_{\mathbf{Pic_{sym}}(\mathcal{E})} : Pic(\mathcal{E}) \to Inv(\mathcal{E})_2 : \mathcal{M} \mapsto B_{\mathcal{M},\mathcal{M}},$$

where B denotes the braiding of $\mathbf{Pic}(\mathcal{E})$. The second canonical class is the symmetric associator $q_{\mathbf{Pic}(\mathcal{E})} \in H^3_{sym}(Inv(\mathcal{E}),\, k^\times)$ of the braided categorical group $\mathcal{I}nv(\mathcal{E})$ corresponding to the homomorphism

$$q_{\mathbf{Pic_{sym}}(\mathcal{E})} : Inv(\mathcal{E}) \to \{\pm 1\} \subset k^\times : Z \mapsto c_{Z,Z}.$$

5.5 Azumaya algebras in braided tensor categories

Let R be an algebra in a braided tensor category $\mathcal{B}$, i.e. an object together with morphisms $\mu : R \otimes R \to R$ (the *product*) and $\iota : I \to R$ (the *unit* map) satisfying the associativity and unit conditions.

Denote by $Aut_{alg}(R)$ the group of algebra automorphisms of R.

Remark 5.7. The assignment

$$Aut_\otimes(\mathrm{Id}_\mathcal{B}) \to Aut_{alg}(R) : a \mapsto a_R \tag{5.37}$$

is a group homomorphism.

Let M be a right R-module in $\mathcal{B}$ with the structural map $\rho : M \otimes R \to M$. For any $X \in \mathcal{B}$ there is an R-module structure on $X \otimes M$ defined by

$$\mathrm{Id}_X \otimes \rho : X \otimes M \otimes R \to X \otimes M.$$

Thus, the category $\mathcal{B}_R$ of right R-modules in $\mathcal{B}$ is a left $\mathcal{B}$-module category via

$$\mathcal{B} \times \mathcal{B}_R \to \mathcal{B}_R, \quad (X,\, M) \mapsto X \otimes M.$$

The α-induction functors (4.4) for $\mathcal{B}_R$ are

$$\alpha^{\pm}_{\mathcal{B}_R} : \mathcal{B} \to {}_R\mathcal{B}_R = \mathcal{E}nd_{\mathcal{B}}(\mathcal{B}_R)^{\mathrm{op}} : X \mapsto X \otimes R, \tag{5.38}$$

with the obvious right R-module structures and the left R-module structures given by

$$\begin{aligned} &R \otimes X \otimes R \xrightarrow{c_{R,X}\otimes \mathrm{Id}_R} X \otimes R \otimes R \xrightarrow{\mathrm{Id}_X \otimes \mu} X \otimes R, \\ &R \otimes X \otimes R \xrightarrow{c^{-1}_{X,R}\otimes \mathrm{Id}_A} X \otimes R \otimes R \xrightarrow{\mathrm{Id}_X \otimes \mu} X \otimes R, \qquad X \in \mathcal{B} \end{aligned}$$

for $\alpha^{+}_{\mathcal{B}_R}$ and $\alpha^{-}_{\mathcal{B}_R}$, respectively.

The tensor product $R \otimes S$ of two algebras $R, S \in \mathcal{B}$ has an algebra structure, with the multiplication map $\mu_{R\otimes S}$ defined as

$$R \otimes S \otimes R \otimes S \xrightarrow{\mathrm{Id}_R \otimes c_{S,R} \otimes \mathrm{Id}_S} R \otimes R \otimes S \otimes S \xrightarrow{\mu_R \otimes \mu_S} R \otimes S,$$

where μ_R and μ_S are multiplications of algebras R and S, respectively (here we suppress the associativity constraints in $\mathcal{B}$). We have

$$\mathcal{B}_R \boxtimes_{\mathcal{B}} \mathcal{B}_S \cong \mathcal{B}_{R\otimes S}.$$

Let $R^{op} = R$ denote the algebra with the multiplication opposite to that of R:

$$R \otimes R \xrightarrow{c_{R,R}} R \otimes R \xrightarrow{\mu} R.$$

Following [34], we say that an algebra R in a braided monoidal category $\mathcal{B}$ is *Azumaya* if the morphism

$$R \otimes R^{\mathrm{op}} \otimes R \xrightarrow{\mathrm{Id}_R \otimes c_{R,R}} R \otimes R \otimes R \xrightarrow{\mu \otimes \mathrm{Id}_R} R \otimes R \xrightarrow{\mu} R$$

induces an isomorphism $R \otimes R^{\mathrm{op}} \to R \otimes R^*$. The B-module category $\mathcal{B}_R$ is invertible in $\mathbf{Pic}(\mathcal{B})$ if and only if R is an Azumaya algebra (in which case $\alpha^{\pm}_{\mathcal{B}_R}$ are equivalences).

Thus, the 2-categorical Picard group of $\mathbf{Pic}(\mathcal{B})$ is monoidally 2-equivalent to the group of Morita equivalence classes of exact Azumaya algebras in $\mathcal{B}$ (the latter group was called in [34] the Brauer group of $\mathcal{B}$).

It was shown in [34, Theorem 3.1] that for an Azumaya algebra R the functors (**??**) are monoidal equivalences.

For an Azmaya algebra $R \in \mathcal{B}$ and an automorphism $\phi \in Aut_{alg}(R)$ let ${}_\phi R$ be the invertible R-bimodule obtained from R by twisting the right R-action by ϕ. Under the equivalence $\alpha^{\pm}_{\mathcal{B}_R}$ it corresponds to an invertible object $P_\phi \in \mathcal{B}$ and we have a group homomorphism

$$Aut_{alg}(R) \to Inv(\mathcal{B}) : \phi \mapsto P_\phi. \tag{5.39}$$

Remark 5.8. An isomorphism of R-bimodules $f : P \otimes R \to {}_\phi R$ is completely determined by the morphism $g : P \to R$ defined by $g = f(1 \otimes \iota)$. Indeed, $f = \mu(g \otimes 1)$. While the right R-module property of such f is automatic, the left R-module property amounts to the condition

$$\mu(\phi \otimes g) = \mu(g \otimes 1)c_{R,P} \ . \tag{5.40}$$

Remark 5.9. An Azumaya algebra $R \in \mathcal{B}$ gives rise to a homomorphism $\varsigma_R : Aut_\otimes(Id_\mathcal{B}) \to Inv(\mathcal{B})$, which is the composition of the homomorphisms (5.37) and (5.39).

Note that $\varsigma_{R\otimes S}(a) = \varsigma_R(a) \otimes \varsigma_S(a)$, so that we have a homomorphism

$$\varsigma : Pic(\mathcal{B}) \ \to \ \mathsf{Hom}_{gr}(Aut_\otimes(\mathrm{Id}_\mathcal{B}),\, Inv(\mathcal{B})), \tag{5.41}$$

or, equivalently, a (bimultiplicative) pairing

$$\langle -\,,\, - \rangle : Pic(\mathcal{B}) \times Aut_\otimes(\mathrm{Id}_\mathcal{B}) \to Inv(\mathcal{B}). \tag{5.42}$$

This can be interpreted in terms of module categories as follows. For any $\mathcal{M} \in \mathbf{Pic}(\mathcal{B})$ there is an isomorphism $Aut_\mathcal{B}(\mathcal{M}) \cong Inv(\mathcal{B})$ given by $\alpha^+_\mathcal{M}$. For $\nu \in Aut_\otimes(\mathrm{Id}_\mathcal{B})$ the value of $\langle \mathcal{M},\, \nu \rangle$ is the image of ν under the composition

$$Aut_\otimes(\mathrm{Id}_\mathcal{B}) \xrightarrow{\iota} Inv(\mathcal{Z}(\mathcal{B})) \cong Inv(\mathcal{Z}(\mathcal{End}_\mathcal{B}(\mathcal{M}))) \to Inv(\mathcal{End}_\mathcal{B}(\mathcal{M})) \cong Inv(\mathcal{B}). \tag{5.43}$$

Note that the object $\langle \mathcal{M},\, \nu \rangle$ coincides with the central structure of the braided module $\mathcal{B}$-category $\mathcal{B}^\nu$ (see Example 5.6), i.e. with the value of the half-braiding $\mathcal{M} \boxtimes_\mathcal{B} \mathcal{B}^\nu \to \mathcal{B}^\nu \boxtimes_\mathcal{B} \mathcal{M}$ viewed as an object of $Aut_\mathcal{B}(\mathcal{M}) \cong Inv(\mathcal{B})$.

6 The braided 2-categorical Picard group of a symmetric fusion category

Let G be a finite group and let $\mathcal{R}ep(G)$ denote the category of representations of G. It was proved by Deligne [15] that a symmetric fusion category is equivalent to the following "super" generalization of $\mathcal{R}ep(G)$. Namely, let G be a finite group and let $t \in G$ be a central element such that $t^2 = 1$. Then $\mathcal{R}ep(G)$ has a braiding defined by

$$c_{V,W} : V \otimes W \xrightarrow{\sim} W \otimes V : v \otimes w \mapsto \begin{cases} -w \otimes v & \text{if } tv = -v,\ tw = -w, \\ w \otimes v & \text{otherwise.} \end{cases} \tag{6.1}$$

The fusion category $\mathcal{R}ep(G)$ equipped with the above braiding will be denoted $\mathcal{R}ep(G,\, t)$. Any symmetric fusion category is equivalent to $\mathcal{R}ep(G,\, t)$ for a unique up to an isomorphism pair $(G,\, t)$. Under this

notation $\mathcal{Rep}(G,1)$ is nothing but $\mathcal{Rep}(G)$ with its usual transposition braiding. We call $\mathcal{Rep}(G,t)$ *Tannakian* if $t=1$ and *super-Tannakian* if $t\neq 1$.

The Picard group of $Pic(\mathcal{Rep}(G,t))$ was computed by Carnovale in [7]. We recall this description in Sections 6.2 and describe the symmetric categorical group $\mathcal{Pic}(\mathcal{Rep}(G,t))$, i.e. the homomorphism

$$Q_{Pic(\mathcal{Rep}(G,t))} : Pic(\mathcal{Rep}(G,t)) \to \widehat{G}.$$

In Sections 6.3 and 6.4 we describe the braided categorical Picard group $\mathcal{Pic}_{br}(\mathcal{Rep}(G,t))$.

6.1 The 2-categorical Picard group of a Tannakian category

Let $\mathcal{B}=\mathcal{Rep}(G)$ be the category of finite dimensional representations of a finite group G with its standard symmetric braiding. For a 2-cocycle $\gamma\in Z^2(G,k^\times)$ denote by $\mathcal{Rep}_\gamma(G)$ the category of γ-projective representations of G.

The first statement of the following Proposition is well known (see e.g. [7]).

Proposition 6.1. *The assignment*

$$H^2(G,k^\times)\to Pic(\mathcal{Rep}(G)) \ : \quad \gamma\mapsto \mathcal{Rep}_\gamma(G)$$

is an isomorphism. The homomorphism $Q_{Pic(\mathcal{Rep}(G))}: H^2(G,k^\times)\to(\widehat{G})_2$ is trivial.

Proof. Since an Azumaya algebra in $\mathcal{Rep}(G)$ is also an Azumaya algebra in $\mathcal{Vect}$ it should have the form $End_k(V)$ for some vector space V. The G-action on $End_k(V)$ corresponds to the structure of a projective G-representation on V. Its Schur multiplier (as a class in $H^2(G,k^\times)$) is the only Morita invariant of the G-algebra $End_k(V)$.

We describe $Q_{Pic(\mathcal{Rep}(G))}$ as follows. The transposition automorphism $c_{End_k(V),End_k(V)}$ of the tensor square $End_k(V)^{\otimes 2}$ of the Azumaya algebra $End_k(V)$ is inner, i.e. is given by conjugation with an invertible element $\zeta\in End_k(V)^{\otimes 2}$. The value $Q(End_k(V))$ is the character $\chi\in\widehat{G}$ defined by $g(\zeta)=\chi(g)\zeta$ for all $g\in G$. Under the isomorphism $End_k(V)^{\otimes 2}\cong End_k(V^{\otimes 2})$ the element ζ corresponds to $c_{V,V}\in End_k(V^{\otimes 2})$. Clearly, $c_{V,V}$ is G-invariant, which makes the character χ trivial. □

Proposition 6.2. *The pairing (5.42) for $\mathcal{Rep}(G)$ is given by*

$$H^2(G,k^\times)\times Z(G)\to\widehat{G} \ : \quad (\gamma,z)\mapsto\gamma_z, \qquad \gamma_z(g)=\frac{\gamma(g,z)}{\gamma(z,g)}. \tag{6.2}$$

Proof. We need to compute the invertible object $\varsigma_R(z)\in\mathcal{Rep}(G)$ for $R=End_k(V)$, where V is a projective G-representation $\rho: G\to GL(V)$ with multiplier γ. According to the remark 5.8 the invertible object corresponding to an automorphism ϕ of $End_k(V)$ is given by the (unique) character χ such that

there is an invertible $\zeta \in End_k(V)$ (the image of $g : P \to R$ on a basic vector of P) with the properties $\phi(x) = \zeta x \zeta^{-1}$ and $\chi(g)\zeta\rho(g) = \rho(g)\zeta$ for any $x \in End_k(V)$ and $g \in G$. Taking $\zeta = \rho(z)$ we get that χ is of the form γ_z. □

6.2 The 2-categorical Picard group of a super-Tannakian category

We start with the basic example.

Example 6.3. Let $G = \mathbb{Z}/2\mathbb{Z}$ and let t be the nontrivial element of G. Then $\mathcal{R}ep(G, t) = \mathcal{sV}ect$, the category of super vector spaces. It goes back to [35] that $Pic(\mathcal{sV}ect) = \mathbb{Z}/2\mathbb{Z}$. Let Π denote the non-identity simple object of $\mathcal{sV}ect$.

Let R be an Azumaya algebra in $\mathcal{sV}ect$ and let $\phi : R \to R$ be an automorphism. The equation (5.40) can be rewritten as $\phi(r)\zeta = \zeta r(-1)^{|r||\zeta|}$, where ζ is the value of ϕ on a basic element of the invertible object $P \in \mathcal{sV}ect$.

The homomorphism $Q_{Pic(\mathcal{sV}ect)} : Pic(\mathcal{sV}ect) = \mathbb{Z}/2\mathbb{Z} \to Inv(\mathcal{sV}ect) = \mathbb{Z}/2\mathbb{Z}$ is the identity map. Indeed, the Azymaya algebra $A = k\langle x | x^2 = 1\rangle = I \oplus \Pi$ (with x odd) represents the non-trivial class in $Pic(\mathcal{sV}ect)$. Its tensor square in $\mathcal{sV}ect$ is $A^{\otimes 2} = k\langle x, y | x^2 = y^2 = 1\ xy + yx = 0\rangle$. The braiding $c_{A,A}$ is the algebra automorphism τ interchanging x and y. Note that $\tau(r)(x - y) = (x - y)r(-1)^{|r|}$ for $r \in A^{\otimes 2}$. Since the element $\zeta = x - y \in A^{\otimes 2}$ is odd, the invertible object in $\mathcal{sV}ect$ corresponding to τ is the non-trivial element Π of $Inv(\mathcal{sV}ect)$.

The pairing (5.42) for $\mathcal{sV}ect$ is given by

$$Pic(\mathcal{sV}ect) \times Aut_{\otimes}(\mathrm{Id}_{\mathcal{sV}ect}) \to Inv(\mathcal{sV}ect) \qquad \langle A, \pi\rangle = \Pi\ ,$$

where π is the natural automorphism of the identity functor of $\mathcal{sV}ect$ such that $\pi_{\mathbf{1}} = \mathrm{Id}_{\mathbf{1}}$ and $\pi_\Pi = -\mathrm{Id}_\Pi$. Indeed the automorphism π_A of the Azymaya algebra A satisfies $\pi(a)x = xa(-1)^{|a|}$ for $a \in A$.

We say that $\mathcal{R}ep(G, t)$ is *split super-Tannakian* if $\langle t\rangle$ is a direct summand of G and *non-split super-Tannakian* otherwise.

The following definition was given and Theorem 6.5 below was proved by Carnovale [7]. We include the argument for the sake of completeness and to set up notation for subsequent computations.

Define the group $H^2(G, t, k^\times)$ to be the second cohomology $H^2(G, k^\times)$ as a set, with the group operation (on the level of cocycles) given by

$$(\gamma * \nu)(f, g) = (-1)^{\xi_\gamma(f)\xi_\nu(g)}\gamma(f, g)\nu(f, g) \qquad f, g \in G,\ \gamma, \nu \in H^2(G, k^\times). \tag{6.3}$$

where $\xi_\gamma : G \to \mathbb{Z}/2\mathbb{Z}$ is the homomorphism defined by

$$(-1)^{\xi_\gamma(g)} = \gamma_t(g) = \frac{\gamma(t, g)}{\gamma(g, t)}. \tag{6.4}$$

Remark 6.4. It was explained in [7] that $H^2(G,t,k^\times)$ is (non-canonically) isomorphic to $H^2(G,k^\times)$.

Theorem 6.5. *The Picard group of a split super-Tannakian symmetric fusion category is*

$$Pic(\mathcal{Rep}(G,t)) \;\cong\; H^2(G,t,k^\times)\times\mathbb{Z}/2\mathbb{Z}. \tag{6.5}$$

The Picard group of a non-split super-Tannakian symmetric fusion category is

$$Pic(\mathcal{Rep}(G,t)) \;\cong\; H^2(G,t,k^\times). \tag{6.6}$$

The homomorphism $Q_{Pic(\mathcal{Rep}(G,t))}: Pic(\mathcal{Rep}(G,t)) \to Inv(\mathcal{Rep}(G,t))_2 = (\widehat{G})_2$ *restricted to* $H^2(G,t,k^\times)$ *is given by*

$$\gamma \;\mapsto\; \gamma_t, \qquad \gamma_t(g)=\frac{\gamma(t,g)}{\gamma(g,t)}. \tag{6.7}$$

In the split case, the homomorphism $Q_{Pic(\mathcal{Rep}(G,t))}$ *restricted to* $\mathbb{Z}/2\mathbb{Z}$ *is the isomorphism* $\mathbb{Z}/2\mathbb{Z}\to\widehat{\langle t\rangle}$.

Proof. Consider the homomorphism $Pic(\mathcal{Rep}(G,t)) \to Pic(\mathcal{sVect})$ induced by the restriction functor $\mathcal{Rep}(G,t)\to\mathcal{sVect}$. We start by showing that this homomorphism is surjective if and only if $\mathcal{Rep}(G,t)$ is split. Indeed, a splitting of the restriction functor $\mathcal{Rep}(G,t)\to\mathcal{sVect}$ induces a splitting of the homomorphism $Pic(\mathcal{Rep}(G,t)) \to Pic(\mathcal{sVect})$. Conversely, an Azumaya algebra $R\in\mathcal{Rep}(G,t)$, which class is mapped to the class of $I\oplus\Pi\in\mathcal{sVect}$, has the form $(I\oplus\Pi)\otimes End_k(U)$ for some vector space U. In particular its classical center (computed in $\mathcal{Vect}$) coincides with $I\oplus\Pi$. The G-action descends from R to its center and gives a splitting $G\to Aut_{alg}(I\oplus\Pi)=\mathbb{Z}/2\mathbb{Z}$.

The restriction of the homomorphism $Q_{Pic(\mathcal{Rep}(G,t))}$ to $Pic(\mathcal{sVect})$ is described in Example 6.3.

In the following we argue that the kernel of the homomorphism $Pic(\mathcal{Rep}(G,t)) \to Pic(\mathcal{sVect})$ is isomorphic to $H^2(G,t,k^\times)$. This kernel consists of classes of Azumaya algebras of the form $End_k(V)$ for a projective G-representation V. It is straightforward to see (e.g., by computing the left center [9] in $\mathcal{Rep}(G,t)$) that $End_k(V)$ is an Azumaya algebra in $\mathcal{Rep}(G,t)$ for any projective G-representation V. Thus we have a set-theoretic bijection $H^2(G,t,k^\times)\to \mathsf{Ker}(Pic(\mathcal{Rep}(G,t)) \to Pic(\mathcal{sVect}))$ sending $\gamma\in Z^2(G,k^\times)$ to the (class of) $End_k(V)$, where V is a projective G-representation with the Schur multiplier γ. To show that this is a group isomorphism we need a few facts.

Let U and V be super vector spaces. Define a map

$$\phi: End_k(U)\otimes End_k(V)\to End_k(U\otimes V): a\otimes b\mapsto \phi_{a,b}, \tag{6.8}$$

by $\phi_{a,b}(u\otimes v)=(-1)^{|b||u|}a(u)\otimes b(v)$ for all homogeneous maps $a\in End_k(U)$, $b\in End_k(V)$ and homogeneous vectors $u\in U$, $v\in V$, where $|a|$ denotes the degree of a.

The map (6.8) is an isomorphism of algebras in $\mathcal{sVect}$, i.e. $\phi_{a,b}\circ\phi_{c,d}=(-1)^{|b||c|}\phi_{a\circ c,b\circ d}$. Indeed,

$$\begin{aligned}\phi_{a,b}(\phi_{c,d}(u\otimes v)) &= (-1)^{|d||u|}\phi_{a,b}(c(u)\otimes d(v))\\ &= (-1)^{|d||u|+|b||c(u)|}a(c(u))\otimes b(d(v))\\ &= (-1)^{|d||u|+|b||c|+||b|u|}(a\circ c)(u)\otimes(b\circ d)(v)\\ &= (-1)^{|b||c|}\phi_{a\circ c,b\circ d}(u\otimes v).\end{aligned}$$

Let now U be a projective G-representation with the Schur multiplier $\gamma\in Z^2(G,k^\times)$. Let $t\in G$ be a central involution. Denote by $U=U_0\oplus U_1$ the $\mathbb{Z}/2\mathbb{Z}$-grading corresponding to t, i.e. $u\in U$ is homogeneous of degree $|u|$ iff $t(u)=(-1)^{|u|}u$. Then the G-action is related to the grading in the following way $|g.u|=\xi_\gamma(g)+|u|$, where ξ_γ is defined in (6.4). Indeed,

$$t.(g.u)=\gamma(t,g)(tg).u=\gamma(t,g)(gt).u=\frac{\gamma(t,g)}{\gamma(g,t)}g.(t.u)=(-1)^{|u|}\frac{\gamma(t,g)}{\gamma(g,t)}g.u.$$

Now let U and V be projective G-representations with the Schur multipliers $\gamma_U,\gamma_V\in Z^2(G,k^\times)$ correspondingly. Note that $End_k(U)$ is a G-algebra with $g(a)$ defined by $g(a)(u)=g.(a(g^{-1}.u))$ (and similarly for $End_k(V)$). Then the homomorphism $\phi: End_k(U)\otimes End_k(V)\to End_k(U\otimes V)$ has the following G-equivariance property:

$$\phi_{g(a),g(b)}=\rho(g)\circ\phi_{a,b}\circ\rho(g)^{-1}\ ,$$

where $\rho(g): U\otimes V\to U\otimes V$ is given by $\rho(g)(u\otimes v)=(-1)^{\xi_\gamma(g)|v|}g.u\otimes g.v$. Indeed, the relation $\phi_{g(a),g(b)}\circ\rho(g)=\rho(g)\circ\phi_{a,b}$ can be checked directly:

$$\begin{aligned}\phi_{g(a),g(b)}(\rho(g)(u\otimes v)) &= (-1)^{\xi_\gamma(g)|v|}\phi_{g(a),g(b)}(g.u\otimes g.v)\\ &= (-1)^{\xi_\gamma(g)|v|+|g(b)||g.u|}g(a)(g.u)\otimes g(b)(g.v)\\ &= (-1)^{|b||u|+\xi_\gamma(g)|b(v)|}g.(a(u))\otimes g.(b(v))\\ &= (-1)^{|b||u|+\xi_\gamma(g)|b(v)|}g.(a(u))\otimes g.(b(v))\\ &= (-1)^{|b||u|}\rho(g)(a(u)\otimes b(v))\\ &= \rho(g)(\phi_{a,b}(u\otimes v)).\end{aligned}$$

The map $\rho: G\to GL(U\otimes V)$ is a projective representation with the Schur multiplier

$$\gamma(f,g)=(-1)^{\xi_\gamma(f)\xi_V(g)}\gamma_U(f,g)\gamma_V(f,g),\qquad f,g\in G. \tag{6.9}$$

To see this, we compute

$$\begin{aligned}\rho(fg)(u\otimes v) &= (-1)^{\xi_\gamma(fg)|v|}(fg).u\otimes(fg).v\\ &= (-1)^{\xi_\gamma(fg)|v|}\gamma_U(f,g)\gamma_V(f,g)f.(g.u)\otimes f.(g.v)\\ &= (-1)^{\xi_\gamma(g)|v|+\xi_\gamma(f)|g.v|}\gamma(f,g)f.(g.u)\otimes f.(g.v)\\ &= (-1)^{\xi_\gamma(g)|v|}\gamma(f,g)\rho(f)(g.u\otimes g.v)\\ &= \gamma(f,g)\rho(f)(\rho(g)(u\otimes v)).\end{aligned}$$

In the rest of the proof we describe the restriction of the homomorphism $Q_{\mathcal{R}ep(G,t)}$ to $H^2(G,t,k^\times)$. Let again V be a projective G-representation with the Schur multiplier $\gamma\in Z^2(G,k^\times)$. The automorphism $c_{End_k(V),End_k(V)}$ of the algebra $End_k(V)^{\otimes 2}$ in $\mathcal{R}ep(G,t)$

$$c_{End_k(V),End_k(V)}(a\otimes b) = (-1)^{|a||b|}b\otimes a,$$

transported (along ϕ) to an automorphism of the algebra $End_k(V^{\otimes 2})$, is inner. More precisely, we have

$$c_{V,V}\circ\phi_{a,b}\circ c_{V,V}^{-1} = (-1)^{|a||b|}\phi_{b,a},$$

since

$$\begin{aligned}c_{V,V}(\phi_{a,b}(u\otimes v)) &= (-1)^{|b||u|}c_{V,V}(a(u)\otimes b(v))\\ &= (-1)^{|b||u|+|a(u)||b(v)|}b(u)\otimes a(v)\\ &= (-1)^{|a||b|+|u||v|+|a||v|}b(u)\otimes a(v)\\ &= (-1)^{|a||b|+|u||v|}\phi_{b,a}(v\otimes u)\\ &= (-1)^{|a||b|}\phi_{b,a}(c_{V,V}(u\otimes v)).\end{aligned}$$

The element $c_{V,V}\in End_k(V)^{\otimes 2}$ has the following G-equivariance property: $g(c_{V,V}) = \chi_\gamma(g)c_{V,V}$, where χ_γ is defined in (6.7). That is, $\rho(g)\circ c_{V,V}\circ\rho(g)^{-1} = \chi(g)c_{V,V}$ since we have

$$\begin{aligned}\rho(g)(c_{V,V}(u\otimes v)) &= (-1)^{|u||v|}\rho(g)(v\otimes u)\\ &= (-1)^{|u||v|+\xi(g)|u|}g.v\otimes g.u\\ &= (-1)^{\xi(g)^2+\xi(g)|v|+|g.u||g.v|}g.v\otimes g.u\\ &= (-1)^{\xi(g)+\xi(g)|v|}c_{V,V}(g.u\otimes g.v)\\ &= \chi_\gamma(g)c_{V,V}(\rho(g)(u\otimes v)).\end{aligned}$$

The formula for $Q_{Pic(\mathcal{R}ep(G,t))}$ on $H^2(G,t,k^\times)$ now follows from Remark 5.8. □

Recall that the character $\xi_\gamma : G \to \mathbb{Z}/2\mathbb{Z} = \{0, 1\}$ was defined in (6.2). It depends on $\gamma \in H^2(G, k^\times)$ as well as on $t \in Z(G)$.

Proposition 6.6. *The pairing (5.42) for $\mathcal{Rep}(G, t)$ is given by*

$$\langle \gamma, z \rangle = \gamma_{zt^{\xi_\gamma(z)}} \qquad \text{in the non-split case,} \tag{6.10}$$

$$\langle (\gamma, \varepsilon), z \rangle = \gamma_{zt^{\xi_\gamma(z)}} \nu^{\xi_\gamma(z)\varepsilon} \qquad \text{in the split case,} \tag{6.11}$$

for $\gamma \in H^2(G, k^\times)$ and $z \in Z(G)$, where ν is the composition $G \to \langle t \rangle \to k^\times$ of the non-trivial character on $\langle t \rangle$ with a (chosen) splitting $G \to \langle t \rangle$ (i.e., $\nu \in \widehat{G}$ corresponds to the image of Π under a (chosen) splitting $\mathcal{sVect} \to \mathcal{Rep}(G, t)$).

Proof. First assume that the class of R is in the kernel of the homomorphism $Pic(\mathcal{Rep}(G, t)) \to Pic(\mathcal{sVect})$, i.e. $R = End_k(V)$, where V is a projective G-representation $\rho : G \to GL(V)$ with multiplier γ. According to Remark 5.8, the invertible object corresponding to the automorphism $\phi = \rho(z)(-)\rho(z)^{-1}$ of $End_k(V)$ is given by the (unique) character χ such that there is an invertible $\zeta \in End_k(V)$ with the properties $\phi(r) = \zeta r \zeta^{-1}(-1)^{|r||\zeta|}$ and $\zeta\rho(g) = \chi(g)\rho(g)\zeta$ for any $g \in G$. A (unique up to a scalar) solution is $\zeta = \rho(zt^{\xi_\gamma(z)})$, where $\xi_\gamma(z) = |\rho(z)|$ (or, equivalently, $(-1)^{\xi_\gamma(z)} = \gamma_t(z)$). Thus, $\chi = \gamma_{zt^{\xi_\gamma(z)}}$. This computes the pairing between $H^2(G, k^\times)$ and $Z(G)$ and, in particular, proves the formula (6.10).

In the split case the restriction of the pairing to $\mathcal{sVect} \subset \mathcal{Rep}(G, t)$ is $\langle (1, \varepsilon), z \rangle = \nu^{\xi_\gamma(z)\varepsilon}$ (according to Example 6.3). □

Remark 6.7. It should be noted that *both* sides of (6.11) depend on the choice of splitting $G \to \langle t \rangle$. Indeed, the identification $Pic(\mathcal{Rep}(G, t)) \cong H^2(G, k^\times) \times \mathbb{Z}/2\mathbb{Z}$ (implicit on the left hand side of (6.11)) is not canonical and depends on this choice.

6.3 The braided categorical Picard group of a Tannakian category

Theorem 6.8. *Let $\mathcal{E}$ be a symmetric tensor category. There is a group isomorphism*

$$Pic_{br}(\mathcal{E}) \cong Pic(\mathcal{E}) \times Aut_\otimes(Id_\mathcal{E}) \ . \tag{6.12}$$

The first canonical class of $Pic_{br}(\mathcal{E})$ is

$$Q_{Pic_{br}(\mathcal{E})}(\mathcal{M}, \nu) = Q_{Pic(\mathcal{E})}(\mathcal{M})\ \langle \mathcal{M}, \nu \rangle. \tag{6.13}$$

where the pairing $\langle \mathcal{M}, \nu \rangle$ is defined in (5.42).

Proof. Since every $\mathcal{E}$-module category admits an identity $\mathcal{E}$-module braiding it follows that $\mathcal{Pic}_{br}(\mathcal{E})$ is generated by symmetric categorical subgroups $\mathcal{Pic}(\mathcal{E})$ and $\mathcal{Aut}_\otimes(\mathrm{Id}_\mathcal{E})$ (the latter consists of invertible

categories in $\mathbf{Mod}^{\mathbf{1}}_{\mathbf{br}}(\mathcal{E})$). These subgroups intersect trivially, so $Pic_{br}(\mathcal{E})$ is their direct product.

For $\mathcal{M} \in Pic(\mathcal{E})$ and $\nu \in Aut_{\otimes}(\mathrm{Id}_{\mathcal{E}})$ the self-braiding $C_{(\mathcal{M},\nu),(\mathcal{M},\nu)}$ on $(\mathcal{M}, \nu) = \mathcal{M} \boxtimes_{\mathcal{B}} \mathcal{B}^{\nu}$ is the composition of (conjugates of) $C_{\mathcal{M},\mathcal{M}}, C_{\mathcal{M},\mathcal{B}^{\nu}}, C_{\mathcal{B}^{\nu},\mathcal{M}}$ and $\mathcal{C}_{\mathcal{B}^{\nu},\mathcal{B}^{\nu}}$. The self-braiding $C_{\mathcal{M},\mathcal{M}}$ is the symmetric one of $\mathcal{P}ic_{br}(\mathcal{E})$. The relation between the self-braiding $C_{\mathcal{M},\mathcal{B}^{\nu}}$ and $\langle \mathcal{M}, \nu \rangle$ is explained in Example 4.16. The braidings $C_{\mathcal{B}^{\nu},\mathcal{M}}$ and $\mathcal{C}_{\mathcal{B}^{\nu},\mathcal{B}^{\nu}}$ are in effect trivial (the first since $C_{\mathcal{B}^{\nu},\mathcal{M}}$ is the symmetric braiding $C_{\mathcal{B},\mathcal{M}}$ $\mathcal{P}ic_{br}(\mathcal{E})$, while the second is the trivial case $\mathcal{M} = \mathcal{B}$ of Example 4.16). This proves the formula for $Q_{Pic_{br}(\mathcal{E})}$. □

The Whitehead bracket $\pi_0 \times \pi_1 \to \pi_2$ (2.60) of $\mathbf{Pic_{br}}(\mathcal{E})$ is given by

$$(Pic(\mathcal{E}) \times Aut_{\otimes}(\mathrm{Id}_{\mathcal{E}})) \times Inv(\mathcal{E}) \to k^{\times} : [(\mathcal{M}, \nu), Z] = \nu_Z. \tag{6.14}$$

We can now describe the braided categorical Picard group of a Tannakian category.

Corollary 6.9. *We have*

$$\pi_0(\mathbf{Pic}_{br}(\mathcal{R}ep(G))) \cong H^2(G, k^{\times}) \times Z(G), \tag{6.15}$$

$$\pi_1(\mathbf{Pic}_{br}(\mathcal{R}ep(G))) \cong \widehat{G}, \tag{6.16}$$

where $\widehat{G}$ denotes the group of characters of G.

The first canonical class is the quadratic form

$$Q_{\mathbf{Pic}_{br}(\mathcal{R}ep(G))}(\gamma, z) = \gamma_z(-), \qquad z \in Z(G), \gamma \in H^2(G, k^{\times}). \tag{6.17}$$

The second canonical class is trivial.

The Whitehead bracket (6.14) is given by

$$[(\gamma, z), \chi] = \chi(z), \qquad \chi \in \widehat{G}, z \in Z(G). \tag{6.18}$$

Proof. Follows from Propositions 6.1, 6.2 and Theorem 6.8. □

6.4 The braided categorical Picard group of a super-Tannakian category

Here we deal with the super-Tannakian case. We start with the basic example of the symmetric fusion category $\mathcal{SV}ect$ of super vector spaces. As before, Π denotes the non-identity simple object of $\mathcal{SV}ect$.

Example 6.10. The group $\pi_0(\mathbf{Pic_{br}}(\mathcal{SV}ect)) \cong Pic_{br}(\mathcal{SV}ect) \times Aut_{\otimes}(\mathrm{Id}_{\mathcal{SV}ect}) \cong \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$ consists of pairs $(\mathcal{I}, id)$, $(\mathcal{I}, \pi)$, $(\mathcal{R}, id)$, $(\mathcal{R}, \pi)$, where $\mathcal{I}$ is the regular $\mathcal{SV}ect$-module category, $\mathcal{R} = \mathcal{V}ect$ viewed as an $\mathcal{SV}ect$-module category (i.e. $\mathcal{R} = \mathcal{SV}ect_A$, where A is the algebra from Example 6.3), π is the natural automorphism of the identity functor of $\mathcal{SV}ect$ such that $\pi_{\mathbf{1}} = \mathrm{Id}_{\mathbf{1}}$ and $\pi_{\Pi} = -\mathrm{Id}_{\Pi}$ (as in the

algebra from Example 6.3). It follows from Example 6.3 and Theorem 6.8 that the quadratic function $Q_{Pic_{br}(\mathcal{sV}ect)} : Pic_{br}(\mathcal{sV}ect) \to Inv(\mathcal{sV}ect)$ is given by

$$Q_{Pic_{br}(\mathcal{sV}ect)}(\mathcal{I}, id) = Q_{Pic_{br}(\mathcal{sV}ect)}(\mathcal{I}, \pi) = Q_{Pic_{br}(\mathcal{sV}ect)}(\mathcal{R}, \pi) = \mathcal{I}, \qquad Q_{Pic_{br}(\mathcal{sV}ect)}(\mathcal{R}, id) = \Pi.$$

Corollary 6.11.

$$\pi_0(\mathbf{Pic}_{br}(\mathcal{R}ep(G,t))) \cong \begin{cases} H^2(G,t,k^\times) \times Z(G) & \text{in the non-split case,} \\ H^2(G,t,k^\times) \times \mathbb{Z}/2\mathbb{Z} \times Z(G) & \text{in the split case.} \end{cases} \tag{6.19}$$

$$\pi_1(\mathbf{Pic}_{br}(\mathcal{R}ep(G,t))) \cong \widehat{G}. \tag{6.20}$$

The first canonical class of $\mathbf{Pic}_{br}(\mathcal{R}ep(G,t))$ *is given by the quadratic form (with values in* $\widehat{G}$*)*

$$Q_{Pic_{br}(\mathcal{R}ep(G,t))}(\gamma,\, z) = \gamma_{zt^{\xi_\gamma(z)+1}} \qquad \text{in the non-split case,} \tag{6.21}$$

$$Q_{Pic_{br}(\mathcal{R}ep(G,t))}(\gamma,\, \varepsilon,\, z) = \gamma_{zt^{\xi_\gamma(z)+1}} \nu^{\xi_\gamma(z)\varepsilon} \qquad \text{in the split case,} \tag{6.22}$$

for $\gamma \in H^2(G,\, k^\times)$, $z \in Z(G)$, *and* $x \in G$, *where* $\nu \in \widehat{G}$ *is as in Proposition 6.6.*

The second canonical class is the homomorphism

$$\widehat{G} \to \{\pm 1\} : \chi \mapsto \chi(t). \tag{6.23}$$

The Whitehead bracket is given by (6.18) *(and does not depend on* t*).*

Proof. Follows from Theorems 6.5 and 6.8 and Proposition 6.6. □

7 The braided Picard group of a pointed braided fusion category

Recall [26, 18] that a pointed braided fusion category $\mathcal{B}$ is determined by a quadratic form $q : A \to k^\times$, where A is the finite Abelian group of isomorphisms classes of simple objects of $\mathcal{B}$ and $q(x) = c_{x,x}$, $x \in A$, where c denotes the braiding of $\mathcal{B}$.

Proposition 7.1. *Let* $\mathcal{B}$ *be a pointed braided fusion category. There is an equivalence*

$$\mathcal{P}ic_{br}(\mathcal{B}) \cong \mathcal{P}ic_{br}(\mathcal{Z}_{sym}(\mathcal{B})) \tag{7.1}$$

of braided categorical groups.

Proof. The group $Pic_{br}(\mathcal{B})$ can be computed using the exact sequence (5.25). Namely, we have a short exact sequence

$$0 \to \mathrm{Coker}(Inv(\mathcal{B}) \xrightarrow{\alpha} Aut_\otimes(\mathrm{Id}_\mathcal{B})) \to Pic_{br}(\mathcal{B}) \to \mathrm{Ker}(Pic(\mathcal{B}) \xrightarrow{\partial} Aut^{br}(\mathcal{B})) \to 0. \tag{7.2}$$

By [10, Proposition 5.17], the induction

$$\mathrm{Ind} : Pic(\mathcal{Z}_{sym}(\mathcal{B})) \to Pic(\mathcal{B}) : \mathcal{M} \mapsto \mathcal{B} \boxtimes_{\mathcal{Z}_{sym}(\mathcal{B})} \mathcal{M}$$

establishes a group isomorphism $Pic(\mathcal{Z}_{sym}(\mathcal{B})) \cong \mathrm{Ker}(\partial)$. Note that (4.21) provides a splitting for (7.2). The homomorphism α is the map $A \mapsto \widehat{A}$ coming from the bilinear form on A associated to q. Its cokernel is $\widehat{A^\perp} \cong Aut_\otimes(\mathrm{Id}_{\mathcal{Z}_{sym}(\mathcal{B})})$.

Thus, the split exact sequence (7.2) yields a group isomorphism (7.1) by Proposition 6.8:

$$Pic_{br}(\mathcal{Z}_{sym}(\mathcal{B})) \cong Pic(\mathcal{Z}_{sym}(\mathcal{B})) \times Aut_\otimes(\mathrm{Id}_{\mathcal{Z}_{sym}(\mathcal{B})}) \to Pic_{br}(\mathcal{B}) : (\mathcal{M}, \nu) \mapsto \mathrm{Ind}(\mathcal{M})^\nu,$$

cf. Example 5.6. The value of the quadratic form $q_{\mathbf{Pic_{br}}(\mathcal{B})}$ on $\mathrm{Ind}(\mathcal{M})^x$ is given by the half braiding

$$\mathrm{Ind}(\mathcal{M}) \boxtimes_{\mathcal{B}} \mathcal{B}^\nu \to \mathcal{B}^\nu \boxtimes_{\mathcal{B}} \mathrm{Ind}(\mathcal{M}).$$

The latter coincides with the value of the pairing $\langle \mathcal{M}, x|_{\mathcal{Z}_{sym}(\mathcal{B})} \rangle$ (5.42), see Remark 5.8. So the result follows from Proposition 6.8. □

Note that $\tau := q|_{A^\perp}$ is an element of $\widehat{A^\perp}$ of order at most 2.

Corollary 7.2.

$$Pic_{br}(\mathcal{C}(A, q)) \cong \begin{cases} \mathsf{Hom}(\Lambda^2 \widehat{A}, k^\times) \times \widehat{A^\perp} & \text{if } \tau = 1, \\ \mathsf{Hom}(\Lambda^2 \widehat{A}, k^\times) \times \widehat{A^\perp} & \text{if } \tau \neq 1 \text{ and } A = Ker(\tau) \times \mathbb{Z}/2\mathbb{Z}, \\ \mathsf{Hom}(\Lambda^2 \widehat{A}, k^\times) \times \mathbb{Z}/2\mathbb{Z} \times \widehat{A^\perp} & \text{if } \tau \neq 1 \text{ and } A \neq Ker(\tau) \times \mathbb{Z}/2\mathbb{Z}. \end{cases} \tag{7.3}$$

Proof. This follows from Proposition 7.1 and the description of the Picard group of a symmetric fusion category, see Sections 6.1 and 6.2. □

8 Classification of graded extensions

8.1 Graded tensor extensions [23]

Let $\mathcal{D}$ be a tensor category and let G be a finite group.

Definition 8.1. A *tensor G-graded extension* (or, simply, a *G-extension*) of a tensor category $\mathcal{D}$ is a tensor category

$$\mathcal{C} = \bigoplus_{x \in G} \mathcal{C}_x, \qquad \mathcal{C}_e = \mathcal{D}, \tag{8.1}$$

such that $\mathcal{C}_x \neq 0$ and the tensor product of $\mathcal{C}$ maps $\mathcal{C}_x \times \mathcal{C}_y$ to $\mathcal{C}_{xy}$ for all $x, y \in G$.

Definition 8.2. An *equivalence* between two G-graded extensions, $\mathcal{C} = \bigoplus_{x\in G} \mathcal{C}_x$ and $\tilde{\mathcal{C}} = \bigoplus_{x\in G} \tilde{\mathcal{C}}_x$ of $\mathcal{D}$ is a tensor equivalence $F : \mathcal{C} \xrightarrow{\sim} \tilde{\mathcal{C}}$ such that $F|_{\mathcal{D}} = \mathrm{Id}_{\mathcal{D}}$ and $F(\mathcal{C}_x) = \tilde{\mathcal{C}}_x$ for all $x \in G$. An *isomorphism* between equivalences of G-extensions $F, F' : \mathcal{C} \xrightarrow{\sim} \tilde{\mathcal{C}}$ is a tensor isomorphism $\eta : F \xrightarrow{\sim} F'$ whose restriction on $F|_{\mathcal{D}} = \mathrm{Id}_{\mathcal{D}}$ is the identity isomorphism.

Thus, G-extensions of $\mathcal{D}$ form a 2-groupoid $\mathbf{Ex}(G, \mathcal{D})$ whose objects are extensions, 1-cells are equivalences of extensions, and 2-cells are isomorphisms of equivalences.

Example 8.3. G-extensions of $\mathcal{V}ect$ are precisely pointed fusion categories $\mathcal{V}ect_G^\omega$, where $\omega \in Z^3(G, k^\times)$ is a 3-cocycle. Equivalences between extensions $\mathcal{V}ect_G^\omega$ and $\mathcal{V}ect_G^{\tilde{\omega}}$ correspond to 2-cochains $\mu \in C^2(G, k^\times)$ such that $d(\mu) = \tilde{\omega}/\omega$. Thus, $\pi_0(\mathbf{Ex}(G, \mathcal{V}ect)) = H^3(G, k^\times)$.

Remark 8.4. Example 8.3 shows that there exist equivalent tensor categories that are not equivalent as extensions. Indeed, if the cohomology classes of ω and $\tilde{\omega}$ are in the same $Aut(G)$-orbit then $\mathcal{V}ect_G^\omega \cong \mathcal{V}ect_G^{\tilde{\omega}}$ as tensor categories.

The following theorem is essentially proved in [23]. We include its proof for the reader's convenience. Our arguments for central, braided, and symmetric extensions in subsequent sections will follow the same pattern.

Theorem 8.5. *There is an equivalence of* 2*-groupoids* $\mathbf{Ex}(G, \mathcal{D}) \cong \mathbf{2\text{-}Fun}(G, \mathbf{BrPic}(\mathcal{D}))$.

Proof. We construct a 2-functor

$$M : \mathbf{Ex}(G, \mathcal{D}) \to \mathbf{2\text{-}Fun}(G, \mathbf{BrPic}(\mathcal{D})) \tag{8.2}$$

as follows. Given a G-extension $\mathcal{C} = \bigoplus_{x\in G} \mathcal{C}_x$ of $\mathcal{D}$, each homogeneous component $\mathcal{C}_g$ is an invertible $\mathcal{D}$-bimodule category. The restrictions $\otimes_{x,y} : \mathcal{C}_x \times \mathcal{C}_y \to \mathcal{C}_{xy}$, $x, y \in G$, of the tensor product of $\mathcal{C}$ are $\mathcal{D}$-balanced functors and so give rise to $\mathcal{D}$-bimodule equivalences

$$M_{x,y} : \mathcal{C}_x \boxtimes_{\mathcal{D}} \mathcal{C}_y \xrightarrow{\sim} \mathcal{C}_{xy}. \tag{8.3}$$

The associativity constraints of $\mathcal{C}$ restricted to $\mathcal{C}_x \times \mathcal{C}_y \times \mathcal{C}_z$ can be viewed as natural isomorphisms of $\mathcal{D}$-balanced functors and so give rise to natural isomorphisms of $\mathcal{D}$-bimodule functors

$$\begin{array}{ccc} \mathcal{C}_x \boxtimes_{\mathcal{D}} \mathcal{C}_y \boxtimes_{\mathcal{D}} \mathcal{C}_z & \xrightarrow{M_{y,z}} & \mathcal{C}_x \boxtimes_{\mathcal{D}} \mathcal{C}_{yz} \\ {\scriptstyle M_{x,y}}\downarrow & \xRightarrow{\alpha_{x,y,z}} & \downarrow{\scriptstyle M_{x,yz}} \\ \mathcal{C}_{xy} \boxtimes_{\mathcal{D}} \mathcal{C}_z & \xrightarrow[M_{xy,z}]{} & \mathcal{C}_{xyz}, \end{array} \tag{8.4}$$

for all $x, y, z \in G$, cf. (2.35). The pentagon identity for the associativity constraints of $\mathcal{C}$ implies that (2.37) is satisfied (equivalently, the cubes (2.63) commute for all $x, y, z, w \in G$). This means that the above data

consisting of $\mathcal{D}$-bimodule categories $\mathcal{C}_x$, equivalences $M_{x,y}$, and natural isomorphisms $\alpha_{x,y,z}$, $x, y, z \in G$, determine a monoidal 2-functor $M(\mathcal{C}) : G \to \mathbf{BrPic}(\mathcal{D})$.

Suppose that there is another G-extension $\tilde{\mathcal{C}} = \bigoplus_{g\in G} \tilde{\mathcal{C}}_g$ of $\mathcal{D}$ and an equivalence of extensions $F : \mathcal{C} \to \tilde{\mathcal{C}}$. It restricts to $\mathcal{D}$-bimodule equivalences

$$F_x : \mathcal{C}_x \xrightarrow{\sim} \tilde{\mathcal{C}}_x. \tag{8.5}$$

The tensor structure of F restricted to $\mathcal{C}_x \times \mathcal{C}_y$ gives rise to an invertible 2-cell

$$\begin{array}{ccc} \mathcal{C}_x \boxtimes_{\mathcal{D}} \mathcal{C}_y & \xrightarrow{F_x \boxtimes_{\mathcal{D}} F_y} & \tilde{\mathcal{C}}_x \boxtimes_{\mathcal{D}} \tilde{\mathcal{C}}_y \\ {\scriptstyle M_{x,y}}\downarrow & \xRightarrow{\mu_{x,y}} & \downarrow{\scriptstyle \tilde{M}_{x,y}} \\ \mathcal{C}_{xy} & \xrightarrow[F_{xy}]{} & \tilde{\mathcal{C}}_{xy}, \end{array} \tag{8.6}$$

$x, y \in G$, and the coherence axiom for the tensor structure of F implies that (2.44) is satisfied (equivalently, that the cubes (2.66) commute for all $x, y, z \in G$), so that we have a pseudo-natural isomorphism $M(F) : M(\mathcal{C}) \to M(\tilde{\mathcal{C}})$.

Given an isomorphism η between a pair of equivalences F, F' of extensions $\mathcal{C}$ and $\mathcal{C}'$ its components are natural isomorphisms of $\mathcal{D}$-bimodule functors:

$$\mathcal{C}_x \underset{F'_x}{\overset{F_x}{\Rightarrow}} \tilde{\mathcal{C}}_x. \quad (\eta_x) \tag{8.7}$$

The tensor property of η implies that (2.46) is satisfied, i.e. the cylinder

$$\begin{array}{ccc} \mathcal{C}_x \boxtimes_{\mathcal{D}} \mathcal{C}_y & \overset{F_xF_y}{\Rightarrow} & \tilde{\mathcal{C}}_x \boxtimes_{\mathcal{D}} \tilde{\mathcal{C}}_y \\ {\scriptstyle M_{x,y}}\downarrow & (\eta_g\eta_h) \;\; \| \;\; F'_xF'_y & \downarrow{\scriptstyle \tilde{M}_{x,y}} \\ & \| \;\; \mu_{g,h} \;\; \| & \\ \mathcal{C}_{xy} & \overset{F_{gh}}{\Rightarrow} \;\; (\mu'_{g,h}) & \tilde{\mathcal{C}}_{gh}. \\ & \underset{F'_{xy}}{\Rightarrow} \;\; (\eta_{gh}) & \end{array} \tag{8.8}$$

commutes for all $x, y \in G$. So we get an invertible modification $M(\eta)$ between pseudo-natural isomorphisms $M(F)$ and $M(F')$. This completes the construction of a monoidal 2-functor (8.2).

A 2-functor

$$L : \mathbf{2\text{-}Fun}(G, \mathbf{BrPic}(\mathcal{D})) \to \mathbf{Ex}(G, \mathcal{D}) \tag{8.9}$$

quasi-inverse to (8.2) can be constructed by reversing the above constructions. Namely, let $C : G \to \mathbf{BrPic}(\mathcal{D}) : x \mapsto \mathcal{C}_x$ be a monoidal 2-functor. Form a $\mathcal{D}$-bimodule category $L(C) := \bigoplus_{x\in G} \mathcal{C}_x$ with

the tensor product given by composing $\mathcal{C}_x \times \mathcal{C}_y \to \mathcal{C}_x \boxtimes_{\mathcal{D}} \mathcal{C}_y$ with 1-cells (8.3) and the associativity constraints coming from 2-cells (8.4). The commuting polytopes (2.63) give the pentagon identity for the associativity constraint.

To check that $L(C)$ is rigid, note that by Corollary 5.2 it is exact as a $\mathcal{D}$-module category. Hence, the dual category $\mathcal{E}nd_{\mathcal{D}}(L(C))$ is a tensor category (i.e. is rigid). Given a homogeneous object X in $\mathcal{C}_x \subset L(C)$, $x \in G$, define a $\mathcal{D}$-module endofuctor $L(X) \in \mathcal{E}nd_{\mathcal{D}}(L(C))$ by setting $L(X) = X \otimes -$ on $\mathcal{D}$ and $L(X) = 0$ on $\mathcal{C}_g$, $g \neq 0$. Its adjoints are given by functors $X^* \otimes -$, ${}^*X \otimes - : \mathcal{C}_x \to \mathcal{D}$ for some objects $X^*, {}^*X \in \mathcal{C}_{x^{-1}}$. These objects are the duals of X. Thus, $L(C)$ is a tensor category and so it is a G-extension of $\mathcal{D}$.

From the universal property of $\boxtimes_{\mathcal{D}}$, a pseudo-natural isomorphism of functors C, $C' : G \to \mathbf{BrPic}(\mathcal{D})$ gives an equivalence of extensions with the tensor structure coming from (8.6) and that a modification of pseudo-natural isomorphisms gives a natural isomorphism of equivalences of extensions. □

Remark 8.6. The proof of Theorem 8.5 is based on the correspondences (coming from the universal property of $\boxtimes_{\mathcal{D}}$) between the structure functors and morphisms of graded tensor categories and the axioms they satisfy and the structure 1- and 2-cells of monoidal 2-functors and the commutative polytopes satisfied by them. We summarize these correspondences in Table 3 (cf. the table from [3, Section 2.3]).

Tensor G-extensions $\mathcal{C}$ of $\mathcal{D}$	Monoidal 2-functors $M : G \to \mathbf{BrPic}(\mathcal{D})$
homogeneous components $\mathcal{C}_g$	0-cells $M(g) := \mathcal{C}_g$
tensor products $\mathcal{C}_g \times \mathcal{C}_h \to \mathcal{C}_{gh}$	monoidal 1-cells $M_{g,h} : \mathcal{C}_g \boxtimes_{\mathcal{D}} \mathcal{C}_h \to \mathcal{C}_{gh}$
associativity constraints $a_{X,Y,Z}$	associativity 2-cells $\alpha_{f,g,h}$ (8.4)
commuting pentagon diagram for a	commuting cubes (2.63) for α
equivalence $F : \mathcal{C} \to \tilde{\mathcal{C}}$ of extensions	1-cells $F_g : \mathcal{C}_g \to \tilde{\mathcal{C}}_g$
tensor structure of F	monoidal 2-cells $\mu_{g,h}$ (8.6)
commuting tensor property diagram for F	commuting cubes (2.66) for μ
isomorphism $\eta : F \to F'$ of equivalences	modification 2-cells $\eta_g : F_g \to F'_g$ (8.7)
commuting tensor property diagram for η	commuting cylinders (8.8) for η

Table 3: A correspondence between tensor G-extensions and monoidal 2-functors.

We can describe G-graded extensions of $\mathcal{D}$ in terms of group cohomology. It follows from constructions of Section 2.5 that given a monoidal functor $M : G \to \mathcal{B}r\mathcal{P}ic(\mathcal{D})$ there exist a canonical cohomology class

$p_M^0 \in H^4(G, k^\times)$ and a canonical group homomorphism

$$p_M^1 : H^1(G, Inv(\mathcal{Z}(\mathcal{D}))) \to H^3(G, k^\times)$$

defined in (2.63) and (2.66), respectively.

Corollary 8.7. *A monoidal functor $M : G \to \mathcal{BrP}ic(\mathcal{D})$ gives rise to a G-graded extension of $\mathcal{D}$ if and only if $p_M^0 = 0$ in $H^4(G, k^\times)$. Equivalence classes of such extensions form a torsor over the cokernel of p_M^1.*

Proof. This follows from Theorem 8.5 and Corollary 2.27. □

Remark 8.8. In [23] the notion of an equivalence of graded extensions was not explicitly defined and extensions were parameterized by a torsor over $H^3(G, k^\times)$. We would like to point that the map p_M^1 is non-trivial in general. Here is a simple example. Let $\mathcal{D} = \mathcal{V}ect^\omega_{\mathbb{Z}/2\mathbb{Z}}$, where ω is the non-trivial element of $H^3(\mathbb{Z}/2\mathbb{Z}, k^\times)$. Then $Inv(\mathcal{Z}(\mathcal{D})) \cong \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$ with the associator $\omega \times \omega^{-1}$. Take the trivial monoidal functor $M : \mathbb{Z}/2\mathbb{Z} \to \mathcal{BrP}ic(\mathcal{D})$. The homomorphism p_M^1 is given by (2.69), i.e.

$$p_M^1 : \mathsf{Hom}(\mathbb{Z}/2\mathbb{Z}, \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}) \to H^3(\mathbb{Z}/2\mathbb{Z}, k^\times) : P \mapsto (\omega \times \omega^{-1}) \circ (P \times P \times P),$$

which is clearly non-zero. This explains the difference between our parameterization of extensions and that of [23, Theorem 1.3].

8.2 Central graded extensions

Let $\mathcal{B}$ be a braided tensor category.

Definition 8.9. A *central G-extension* of $\mathcal{B}$ is a pair $(\mathcal{C}, \iota)$, where $\mathcal{C}$ is a G-extension and $\iota : \mathcal{B} \hookrightarrow \mathcal{Z}(\mathcal{C})$ is a braided tensor functor whose composition with the forgetful functor $\mathcal{Z}(\mathcal{C}) \to \mathcal{C}$ coincides with the inclusion $\mathcal{B} \hookrightarrow \mathcal{C}$.

Definition 8.10. Let $(\mathcal{C}, \iota : \mathcal{B} \to \mathcal{Z}(\mathcal{C}))$ and $(\tilde{\mathcal{C}}, \tilde{\iota} : \mathcal{B} \to \mathcal{Z}(\tilde{\mathcal{C}}))$ be two central G-extensions of $\mathcal{B}$. An *equivalence* between these extensions is an equivalence $F : \mathcal{C} \xrightarrow{\sim} \tilde{\mathcal{C}}$ of G-extensions such that $\tilde{\iota} = \mathrm{ind}(F) \circ \iota$, where $\mathrm{ind}(F) : \mathcal{Z}(\mathcal{C}) \xrightarrow{\sim} \mathcal{Z}(\tilde{\mathcal{C}})$ is the braided equivalence induced by F.

Central G-extensions of $\mathcal{B}$ form a 2-groupoid $\mathbf{Ex_{ctr}}(G, \mathcal{B})$.

Recall that a *G-crossed braided tensor category* is a G-graded tensor category $\mathcal{C} = \bigoplus_{x \in G} \mathcal{C}_x$ equipped with the action of G on $\mathcal{C}$, i.e. a monoidal functor $G \to Aut_\otimes(\mathcal{C})$, such that $x(\mathcal{C}_y) = \mathcal{C}_{xyx^{-1}}$ and with a G-crossed braiding

$$c_{X,Y} : X \otimes Y \to g(Y) \otimes X, \qquad X \in \mathcal{C}_x,\ Y \in \mathcal{C}, \tag{8.10}$$

satisfying certain natural axioms. Note that the trivial component of the grading $\mathcal{C}_e$ is a braided tensor category. Let $\mathbf{Ex_{cr-br}}(G, \mathcal{B})$ denote the 2-groupoid of G-crossed braided fusion categories whose trivial component is $\mathcal{B}$.

The next Proposition was essentially proved in [25]. It shows that the notions of a central extension and a G-crossed extension coincide.

Proposition 8.11. *There is a* 2*-equivalence* $\mathbf{Ex_{cr-br}}(G, \mathcal{B}) \cong \mathbf{Ex_{ctr}}(G, \mathcal{B})$.

Proof. We need to explain how a G-crossed braided structure translates into a central functor and vice versa.

Let $\mathcal{C}$ be a G-crossed braided tensor category with $\mathcal{C}_e = \mathcal{B}$. The restriction of the crossed braiding (8.10) provides $X \in \mathcal{C}_e$ with the structure of a central object of $\mathcal{C}$ and

$$\iota : \mathcal{B} = \mathcal{C}_e \to \mathcal{Z}(\mathcal{C}) : X \mapsto (X, c^{-1}_{X,-})$$

is a braided tensor functor whose composition with the forgetful functor $\mathcal{Z}(\mathcal{C}) \to \mathcal{C}$ is identity.

In the opposite direction, a central G-extension $(\mathcal{C}, \iota : \mathcal{B} \to \mathcal{Z}(\mathcal{C}))$ yields a natural isomorphism

$$c_{X,Z} : X \otimes Z \xrightarrow{\sim} Z \otimes X, \qquad X \in \mathcal{B},\, Z \in \mathcal{C}, \tag{8.11}$$

satisfying the hexagon axioms. This turns each $\mathcal{C}_x$ into an invertible $\mathcal{B}$-module category. Furthermore, there are $\mathcal{B}$-module equivalences

$$\begin{aligned} \mathcal{C}_y \to \mathcal{F}un_{\mathcal{B}}(\mathcal{C}_x, \mathcal{C}_{xy}) \quad &: \quad Y \mapsto - \otimes Y, \\ \mathcal{C}_{xyx^{-1}} \to \mathcal{F}un_{\mathcal{B}}(\mathcal{C}_x, \mathcal{C}_{xy}) \quad &: \quad Y \mapsto Y \otimes -, \end{aligned}$$

for all $x, y \in G$. Here the functor categories consist of right exact $\mathcal{B}$-module functors. Combining these equivalences for a fixed $x \in G$ we obtain a tensor autoequivalence $x \in Aut_{\otimes}(\mathcal{C})$ such that $x(\mathcal{C}_y) = \mathcal{C}_{xyx^{-1}}$ and there is a natural isomorphism

$$x(Y) \otimes X \cong X \otimes Y \qquad \text{for all } X \in \mathcal{C}_x,\, Y \in \mathcal{C}.$$

The latter is a crossed braiding on $\mathcal{C}$.

These constructions are inverses of each other and are compatible with equivalences of G-crossed braided and central extensions, i.e. define a 2-equivalence between the corresponding 2-groupoids. □

Remark 8.12. Let $\mathcal{C}$ be a central G-extension of $\mathcal{B}$. The braided tensor category $\mathcal{C}^G$ obtained from $\mathcal{C}$ as the equivariantization [18, Section 4] with respect to the canonical action of G constructed in the proof of Proposition 8.11 coincides with. the centralizer of the image of $\mathcal{B}$ in $\mathcal{Z}(\mathcal{C})$.

Recall that the Picard group of $\mathcal{B}$ was introduced in Section 5.2. The following result is essentially a consequence of Proposition 8.11 and [23, Theorem 7.12]. We include the proof for the sake of completeness.

Theorem 8.13. *There is an equivalence of* 2*-groupoids* $\mathbf{Ex_{ctr}}(G, \mathcal{B}) \cong \mathbf{2\text{-}Fun}(G, \mathbf{Pic}(\mathcal{B}))$.

Proof. We adjust the proof of Theorem 8.5 to the present setting (with $\mathcal{D}$-bimodule categories, functors, and isomorphisms replaced by $\mathcal{B}$-module ones).

A central structure on a G-extension $\mathcal{C} = \bigoplus_{x\in G} \mathcal{C}_x$ of $\mathcal{B}$ consists of isomorphisms (8.11) that turn every component $\mathcal{C}_x$ into an invertible $\mathcal{B}$-module category, i.e. $\mathcal{C}_x$ belongs to $\mathbf{Pic}(\mathcal{B})$. Equivalences (8.3) coming from tensor products $\mathcal{C}_x \times \mathcal{C}_y \to \mathcal{C}_{xy}$ are $\mathcal{B}$-module equivalences in this case and natural isomorphisms (8.4) are isomorphisms of $\mathcal{B}$-module functors.

An equivalence of central G-extensions of $\mathcal{B}$ yields $\mathcal{B}$-module equivalences (8.5) between homogeneous components and isomorphisms (8.6) of $\mathcal{B}$-module functors. An isomorphism between equivalences of central G-extensions yields an isomorphism (8.7) of $\mathcal{B}$-module functors.

Diagrams (2.63), (2.66), and (8.8) commute for the same reason as in the proof of Theorem 8.5.

Thus, (8.2) becomes a 2-functor

$$\mathbf{Ex_{ctr}}(G, \mathcal{B}) \to \mathbf{2\text{-}Fun}(G, \mathbf{Pic}(\mathcal{B})). \tag{8.12}$$

Conversely, given a monoidal 2-functor $G \to \mathbf{Pic}(\mathcal{B})$, consider its composition with the inclusion $\mathbf{Pic}(\mathcal{B}) \to \mathbf{BrPic}(\mathcal{B})$. By Theorem 8.5, this yields a G-extension $\mathcal{C}$ of $\mathcal{B}$. The $\mathcal{B}$-bimodule structure of $\mathcal{C}$ comes from its left $\mathcal{B}$-module structure, so there is a natural isomorphism between the functors of left and right tensor multiplication by $X \in \mathcal{B}$:

$$c_{X,Z} : X \otimes Z \xrightarrow{\sim} Z \otimes X, \qquad Z \in \mathcal{C}. \tag{8.13}$$

The hexagon for (8.13) follows from the above definition of a $\mathcal{B}$-bimodule category structure of $\mathcal{C}$ and from the monoidal property of the 2-functor $\mathbf{Mod}(\mathcal{B}) \to \mathbf{Bimod}(\mathcal{B})$. Thus, (8.13) is a central structure on the G-extension $\mathcal{C}$ of $\mathcal{B}$ and there is a 2-functor

$$\mathbf{2\text{-}Fun}(G, \mathbf{Pic}(\mathcal{B})) \to \mathbf{Ex_{ctr}}(G, \mathcal{B}) \tag{8.14}$$

quasi-inverse to (8.12). □

Remark 8.14. It follows from Theorem 8.13 that central G-extensions of $\mathcal{B}$ can be described in terms of monoidal functors $G \to \mathcal{P}ic(\mathcal{B})$ and group cohomology analogously to Corollary 8.7.

8.3 Braided graded extensions

Let $\mathcal{B}$ be a braided tensor category and let A be an Abelian group.

Definition 8.15. A *braided A-extension* of $\mathcal{B}$ is a braided tensor category $\mathcal{C}$ that is an A-extension of $\mathcal{B}$.

Definition 8.16. An equivalence between braided A-extensions $\mathcal{C}, \tilde{\mathcal{C}}$ of $\mathcal{B}$ is an equivalence of A-extensions that is a braided functor.

Example 8.17. Braided A-extensions of $\mathcal{V}ect$ are precisely pointed braided fusion categories $\mathcal{V}ect_A^\omega$, where $\omega \in Z^3_{br}(G, k^\times)$ is an abelian 3-cocycle. Equivalences between extensions $\mathcal{V}ect_A^\omega$ and $\mathcal{V}ect_A^{\tilde{\omega}}$ correspond to abelian 2-cochains $\mu \in C^2_{br}(A, k^\times)$ such that $d(\mu) = \omega/\tilde{\omega}$. Thus, the set of isomorphism classes $Ex_{br}(A, \mathcal{V}ect)$ of braided A-extensions of $\mathcal{V}ect$ is in bijection with $H^3_{br}(A, k^\times) \cong \mathsf{Quad}(A, k^\times)$.

Let $\mathbf{2\text{-}Fun_{br}}(A, \mathbf{Pic_{br}}(\mathcal{B}))$ denote the 2-groupoid of braided monoidal 2-functors from A to $\mathbf{Pic_{br}}(\mathcal{B})$.

Theorem 8.18. *There is a 2-equivalence* $\mathbf{Ex_{br}}(A, \mathcal{B}) \xrightarrow{\sim} \mathbf{2\text{-}Fun_{br}}(A, \mathbf{Pic_{br}}(\mathcal{B}))$.

Proof. Let $\mathcal{C} = \bigoplus_{x\in A} \mathcal{C}_x$ be a braided A-extension of $\mathcal{B}$. Each homogeneous component $\mathcal{C}_y$, $y \in A$, is an invertible $\mathcal{B}$-module category. The squared braiding

$$\sigma_{X,Y} = c_{YX}c_{XY}, \qquad X \in \mathcal{B},\ Y \in \mathcal{C}_y,$$

equips it with the structure of a braided $\mathcal{B}$-module category, i.e. $\mathcal{C}_y \in \mathbf{Pic_{br}}(\mathcal{B})$. The equivalences $M_{x,y} : \mathcal{C}_x \boxtimes_{\mathcal{B}} \mathcal{C}_y \xrightarrow{\sim} \mathcal{C}_{xy}$ from (8.3) are braided module equivalences. Indeed, commutativity of the diagram (4.2) is a consequence of the identity

$$c_{Y_1\otimes Y_2,X}c_{X,Y_1\otimes Y_2} = (c_{Y_1,X}\otimes \mathrm{Id}_{Y_2})(\mathrm{Id}_{Y_1}\otimes c_{Y_2,X}c_{X,Y_2})(c_{X,Y_1}\otimes \mathrm{Id}_{Y_2}), \qquad X, Y_1, Y_2 \in \mathcal{B},$$

where we omit the associativity constraints in $\mathcal{C}$.

As in the proof of Theorem 8.5, equivalences $M_{x,y}$ along with the associativity 2-cells $\alpha_{x,y,z}$ from (8.4) define a monoidal structure on the 2-functor

$$M(\mathcal{C}) : A \to \mathbf{Pic_{br}}(\mathcal{B}) : x \mapsto \mathcal{C}_x.$$

Furthermore, the commutativity constraint of $\mathcal{C}$ gives rise to invertible 2-cells

$$\begin{array}{ccc} \mathcal{C}_x \boxtimes_{\mathcal{B}} \mathcal{C}_y & \xrightarrow{\quad B_{x,y} \quad} & \mathcal{C}_y \boxtimes_{\mathcal{B}} \mathcal{C}_x \\ & {\scriptstyle M_{x,y}} \searrow \quad \Uparrow \delta_{x,y} \quad \swarrow {\scriptstyle M_{y,x}} & \\ & \mathcal{C}_{xy} & \end{array} \tag{8.15}$$

for all $x, y \in A$. The conditions (2.40) and (2.41) in the definition of a braided monoidal 2-functor (i.e. commutativity of the octahedra (2.71) and (2.72)) follow from the hexagon axioms satisfied by the braiding of $\mathcal{C}$.

Thus, $M(\mathcal{C}) : A \to \mathbf{Pic_{br}}(\mathcal{B})$ is a braided monoidal 2-functor.

Suppose there is another braided A-extension $\tilde{\mathcal{C}} = \bigoplus_{x\in A} \tilde{\mathcal{C}}_x$ of $\mathcal{B}$ and an equivalence of braided A-extensions $F : \mathcal{C} \to \tilde{\mathcal{C}}$. The $\mathcal{B}$-module equivalences $F_x : \mathcal{C}_x \xrightarrow{\sim} \tilde{\mathcal{C}}_x$ between the homogeneous components are braided $\mathcal{B}$-module equivalences. Indeed, commutativity of diagram (4.2) is a consequence of the braided property of F. We have invertible 2-cells (8.6) satisfying (2.63) as in the proof of Theorem 8.5. The condition (2.45) (i.e. commutativity of the prism (2.73)) follows from the braided property of F. Thus, we have a pseudo-natural isomorphism $M(F) : M(\mathcal{C}) \to M(\tilde{\mathcal{C}})$ of braided monoidal 2-functors.

Given an isomorphism η between a pair of equivalences F, F' of braided extensions $\mathcal{C}$ and $\mathcal{C}'$ one constructs an invertible modification $M(\eta)$ between $M(F)$ and $M(F')$ as in (8.7).

Thus, we have a 2-functor $M : \mathbf{Ex_{br}}(A, \mathcal{B}) \xrightarrow{\sim} \mathbf{2\text{-}Fun_{br}}(A, \mathbf{Pic_{br}}(\mathcal{B}))$. In the opposite direction, the 2-functor (8.9) constructed in the proof of Theorem 8.5 carries a braided structure on a 2-functor $\mathcal{C} : A \to \mathbf{Pic_{br}}(\mathcal{B}))$ to a braiding on $L(\mathcal{C}) = \bigoplus_{x\in A} \mathcal{C}_x$. Namely, 2-cells (8.15) give rise to the braiding constraints for $L(\mathcal{C})$ while commuting octahedra (2.71), (2.72) ensure that they satisfy give the hexagon identities. □

Remark 8.19. The proof of Theorem 8.18 extends that of Theorem 8.5. So it extends the correspondences in Table 3 as follows:

Braided tensor A-extensions $\mathcal{C}$ of $\mathcal{B}$	Monoidal 2-functors $M : A \to \mathbf{Pic_{br}}(\mathcal{B})$
braiding constraints $c_{X,Y}$	braiding 2-cells $\delta_{x,y}$ (8.15)
commuting hexagon diagrams for c	commuting octahedra (2.71), (2.72)
braided property diagram for F	commuting prism (2.73)

Table 4: A correspondence between braided extensions and braided monoidal 2-functors.

We can describe A-graded extensions of $\mathcal{B}$ in terms of braided group cohomology. It follows from constructions of Section 2.6 that given a braided monoidal functor $M : A \to \mathcal{BrPic}(\mathcal{D})$ there exist a canonical braided cohomology class $\mathfrak{p}^0_M \in H^4_{br}(A, k^\times)$ and a canonical group homomorphism $\mathfrak{p}^1_M : H^1(A, Inv(\mathcal{B})) \to H^3_{br}(A, k^\times)$.

Corollary 8.20. *A braided monoidal functor $M : A \to \mathcal{Pic}_{br}(\mathcal{B})$ gives rise to an A-graded extension of $\mathcal{B}$ if and only if $\mathfrak{p}^0_M = 0$ in $H^4_{br}(A, k^\times)$. Equivalence classes of such extensions of form a torsor over the cokernel of $\mathfrak{p}^1_M$.*

Proof. This follows from Theorem 8.18 and Corollary 2.32. □

Example 8.21. Let $\mathcal{B}$ be a non-degenerate braided fusion category. By Proposition 4.17, $\mathbf{Pic_{br}}(\mathcal{B}) = \mathbf{Pic_{br}}(\mathcal{V}ect)$ and so $\mathbf{Ex_{br}}(A, \mathcal{B}) \xrightarrow{\sim} \mathbf{Ex_{br}}(A, \mathcal{V}ect)$. Thus, any braided A-extension of $\mathcal{B}$ is equivalent to one of the form $\mathcal{B} \boxtimes \mathcal{C}(A, q)$ for some $q \in \mathsf{Quad}(A, k^\times) = H^3_{br}(A, k^\times)$.

Thus, the only braided fusion categories that admit interesting extensions are degenerate ones.

8.4 Symmetric graded extensions

Let $\mathcal{E}$ be a symmetric tensor category and let A be an Abelian group.

Definition 8.22. A *symmetric A-extension* of $\mathcal{E}$ is a symmetric tensor category $\mathcal{C}$ that is an A-extension of $\mathcal{E}$.

Equivalences of symmetric A-extensions are the same as for braided A-extensions. The 2-groupoid $\mathbf{Ex_{sym}}(A, \mathcal{E})$ of symmetric A-extensions of $\mathcal{E}$ is a 2-subgroupoid of $\mathbf{Ex_{br}}(A, \mathcal{E})$.

Example 8.23. Symmetric A-extensions of $\mathcal{V}ect$ are precisely pointed braided fusion categories $\mathcal{V}ect^\omega_A$, where $\omega \in Z^3_{sym}(G, k^\times)$ is a symmetric 3-cocycle. Equivalences between extensions $\mathcal{V}ect^\omega_A$ and $\mathcal{V}ect^{\tilde{\omega}}_A$ correspond to symmetric 2-cochains $\mu \in C^2_{sym}(A, k^\times)$ such that $d(\mu) = \omega/\tilde{\omega}$. Thus, the set of isomorphism classes $Ex_{sym}(A, \mathcal{V}ect)$ of braided A-extensions of $\mathcal{V}ect$ is in bijection with $H^3_{sym}(A, k^\times) \cong \mathsf{Hom}(A, k^\times)_2 = \mathsf{Hom}(A, \mathbb{Z}_2)$.

Let $\mathcal{C} = \bigoplus_{x\in A} \mathcal{C}_x$ and $\mathcal{C}' = \bigoplus_{x\in A} \mathcal{C}'_x$ be symmetric A-extensions of $\mathcal{E}$. Then $\mathcal{C} \boxtimes_\mathcal{E} \mathcal{C}'$ is an $(A \times A)$-extension of $\mathcal{E}$. Define the tensor product of these extensions to be the diagonal subcategory of $\mathcal{C} \boxtimes_\mathcal{E} \mathcal{C}'$:

$$\mathcal{C} \odot_\mathcal{E} \mathcal{C}' = \bigoplus_{x\in A} \mathcal{C}_x \boxtimes_\mathcal{E} \mathcal{C}'_x. \tag{8.16}$$

This equips the 2-groupoid $\mathbf{Ex_{sym}}(A, \mathcal{E})$ of symmetric A-extensions of $\mathcal{E}$ with a structure of a symmetric 2-categorical group.

Recall that the symmetric 2-categorical group $\mathbf{Pic_{sym}}(\mathcal{E})$ of symmetric $\mathcal{E}$-module categories is equivalent to $\mathbf{Pic}(\mathcal{E})$, the Picard group of $\mathcal{E}$. Let $\mathbf{2\text{-}Fun_{sym}}(A, \mathbf{Pic}(\mathcal{E}))$ denote the 2-groupoid of symmetric monoidal 2-functors from A to $\mathbf{Pic}(\mathcal{E})$.

Theorem 8.24. *There is a symmetric monoidal 2-equivalence* $\mathbf{Ex_{sym}}(A, \mathcal{E}) \xrightarrow{\sim} \mathbf{2\text{-}Fun_{sym}}(A, \mathbf{Pic}(\mathcal{E}))$.

Proof. We extend Theorem 8.18 to the symmetric setting. Observe that the homogeneous components of a symmetric extension $\mathcal{C} = \oplus_{x\in A} \mathcal{C}_x$ of $\mathcal{E}$ are necessarily symmetric $\mathcal{E}$-module categories. Commutativity of the cones (2.75) is equivalent to the squared braiding of $\mathcal{C}$ being identity, i.e. to the braided monoidal 2-functor $x \mapsto \mathcal{C}_x$ being symmetric.

The monoidal structure of this 2-equivalence is established by comparing the tensor products, associativities, and braidings of $\mathbf{Ex_{sym}}(A, \mathcal{E})$ and $\mathbf{2\text{-}Fun_{sym}}(A, \mathbf{Pic}(\mathcal{E}))$. □

Corollary 8.25. *There is an exact sequence of group homomorphisms:*

$$H^1(A,\, Inv(\mathcal{E})) \to H^3_{sym}(A,\, k^\times) \to \pi_0(\mathbf{Ex_{sym}}(A,\, \mathcal{E})) \to \pi_0(\mathcal{F}un_{sym}(A,\, \mathbf{Pic}(\mathcal{E}))) \to H^4_{sym}(A,\, k^\times). \tag{8.17}$$

Proof. This follows from Theorem 2.38. □

8.5 The group of symmetric extensions of a symmetric fusion category

Let G be a finite abelian group and let $t \in G$ be a central element such that $t^2 = 1$. Let A be a finite Abelian group. In this Section we compute the group

$$Ex_{sym}(A,\, \mathcal{E}) := \pi_0(\mathbf{Ex_{sym}}(A,\, \mathcal{E}))$$

of symmetric A-extensions of $\mathcal{E} = \mathcal{R}ep(G,\, t)$.

Theorem 8.26. *There are group isomorphisms*

$$Ex_{sym}(A,\, \mathcal{R}ep(G)) \quad \cong \quad H^2(G,\, \widehat{A}) \oplus H^1(A,\, \mathbb{Z}_2), \tag{8.18}$$

$$Ex_{sym}(A,\, \mathcal{R}ep(G,\, t)) \quad \cong \quad H^2(G/\langle t\rangle,\, \widehat{A}) \quad \textit{if } t \neq 1. \tag{8.19}$$

Proof. Let us first consider symmetric A-extensions of $\mathcal{R}ep(G)$. They are of the form $\mathcal{R}ep(\tilde{G},\, \tilde{t})$, where $\tilde{G}$ is a central extension

$$1 \to \widehat{A} \to \tilde{G} \xrightarrow{\pi} G \to 1 \tag{8.20}$$

and $\tilde{t}$ is a central element of $\tilde{G}$ such that $\tilde{t}^2 = 1$ and $\pi(\tilde{t}) = 1$. Thus, every symmetric A-extension of $\mathcal{R}ep(G)$ is completely determined by the pair consisting of a cohomology class in $H^2(G,\, \widehat{A})$ corresponding to the isomorphism class of the group extension (8.20) and $\tilde{t} \in (\widehat{A})_2 = H^1(A,\, \mathbb{Z}_2)$. The corresponding map $Ex_{sym}(A,\, \mathcal{R}ep(G)) \to H^2(G,\, \widehat{A}) \oplus H^1(A,\, \mathbb{Z}_2)$ is a group isomorphism. It is clearly injective. To see that it is surjective note that the elements of $H^2(G,\, \widehat{A})$ form a subgroup $Ex_{Tan}(A,\, \mathcal{R}ep(G))$ of Tannakian A-extensions of $\mathcal{R}ep(G)$ while the elements of $H^1(A,\, \mathbb{Z}_2)$ form the subgroup of split extensions.

Now consider a symmetric A-extension $\mathcal{C}$ of $\mathcal{R}ep(G,\, t)$ with $t \neq 1$. It contains a unique maximal Tannakian subcategory $\mathcal{C}_0$ of index 2 which is a Tannakian A-extension of $\mathcal{R}ep(G/\langle t\rangle)$. We have a group homomorphism

$$f : Ex_{sym}(A,\, \mathcal{R}ep(G,\, t)) \to Ex_{Tan}(A,\, \mathcal{R}ep(G/\langle t\rangle)) = H^2(G/\langle t\rangle,\, \widehat{A}) : \mathcal{C} \mapsto \mathcal{C}_0. \tag{8.21}$$

We claim that f has an inverse given by the induction

$$g : Ex_{Tan}(A,\, \mathcal{R}ep(G/\langle t\rangle)) \to Ex_{sym}(A,\, \mathcal{R}ep(G,\, t)) : \mathcal{T} \mapsto \mathcal{R}ep(G) \boxtimes_{\mathcal{R}ep(G/\langle t\rangle)} \mathcal{T}, \tag{8.22}$$

where the tensor product of fusion categories over a symmetric fusion category is defined in [11, Section 2.5].

Indeed, we have $f \circ g = \mathrm{Id}$ since the maximal Tannakian subcategory of $\mathcal{R}ep(G) \boxtimes_{\mathcal{R}ep(G/\langle t\rangle)} \mathcal{T}$ is $\mathcal{R}ep(G/\langle t\rangle) \boxtimes_{\mathcal{R}ep(G/\langle t\rangle)} \mathcal{T} \cong \mathcal{T}$. To check that $g \circ f = \mathrm{Id}$ we observe that there is a surjective symmetric tensor functor $F : \mathcal{C}_0 \boxtimes \mathcal{R}ep(G, t) \to \mathcal{C}$ given by embedding of factors. Since the intersection of $\mathcal{C}_0$ and $\mathcal{R}ep(G, t)$ in $\mathcal{C}$ is $\mathcal{R}ep(G/\langle t\rangle)$ we see that F factors through $\mathcal{C}_0 \boxtimes_{\mathcal{R}ep(G/\langle t\rangle)} \mathcal{R}ep(G, t)$. The latter fusion category has the same Frobenius-Perron dimension as $\mathcal{C}$ so that $\mathcal{C} \cong \mathcal{C}_0 \boxtimes_{\mathcal{R}ep(G/\langle t\rangle)} \mathcal{R}ep(G, t)$. □

Corollary 8.27. $Ex_{sym}(A, \mathcal{S}vect) = 0$.

Below we describe the exact sequence (8.17) computing the group of symmetric extensions. This is meant to illustrate our obstruction theory and give an alternative proof of Theorem 8.26.

Proposition 8.28. *There are group isomorphisms*

$$\pi_0(\mathcal{F}un_{sym}(A, \mathcal{P}ic_{sym}(\mathcal{R}ep(G)))) \cong H^2(G, \widehat{A}), \tag{8.23}$$

$$\pi_0(\mathcal{F}un_{sym}(A, \mathcal{P}ic_{sym}(\mathcal{R}ep(G, t)))) \cong Ker\left(H^2(G, \widehat{A}) \xrightarrow{\Xi_t} \mathsf{Hom}(G/\langle t\rangle, (\widehat{A})_2)\right), \tag{8.24}$$

where

$$\Xi_t : H^2(G, \widehat{A}) \to \mathsf{Hom}(G/\langle t\rangle, (\widehat{A})_2) : m \mapsto m(t, -)m(-, t)^{-1}. \tag{8.25}$$

Proof. Let us consider the Tannakian case first. Let $G' = [G, G]$ and $\widehat{G} = \mathsf{Hom}(G, k^\times)$. We have a homomorphism of short exact sequences

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \mathsf{Ext}(G/G', \widehat{A}) & \longrightarrow & H^2(G, \widehat{A}) & \longrightarrow & \mathsf{Hom}(A, H^2(G, k^\times)) & \longrightarrow & 0 \\ & & \downarrow{\scriptstyle \alpha} & & \downarrow{\scriptstyle \beta} & & \| & & \\ 0 & \longrightarrow & H^2_{sym}(A, \widehat{G}) & \longrightarrow & \pi_0(\mathcal{F}un_{sym}(A, \mathcal{P}ic_{sym}(\mathcal{R}ep(G)))) & \longrightarrow & \mathsf{Hom}(A, H^2(G, k^\times)) & \longrightarrow & 0. \end{array} \tag{8.26}$$

Here α is the duality isomorphism. The homomorphism β is defined as follows. An element $m \in H^2(G, \widehat{A})$ gives rise to a central group extension

$$1 \to \widehat{A} \to \tilde{G} \xrightarrow{\pi} G \to 1. \tag{8.27}$$

The category $\mathcal{R}ep(\tilde{G})$ is a symmetric A-extension of $\mathcal{R}ep(G)$ and, therefore, yields a symmetric monoidal functor $\alpha(m) : A \to \mathcal{P}ic_{sym}(\mathcal{R}ep(G))$. The first row of (8.26) is split exact [29, Theorem 2.1.19] and the second row comes from assigning to a symmetric functor a group homomorphism. Hence, β is an isomorphism. This proves (8.23).

In the super-Tannakian case we have an exact sequence

$$0 \to H^2_{sym}(A, \widehat{G}) \to Fun_{sym}(A, \mathcal{P}ic_{sym}(\mathcal{R}ep(G, t))) \to \mathsf{Hom}(A, H^2(G, k^\times)) \xrightarrow{q^*} \mathsf{Hom}(A, (\widehat{G})_2), \tag{8.28}$$

where q^* is induced by the second canonical class of $\mathbf{Pic_{sym}}(\mathcal{R}ep(G,\,t))$,

$$q : Pic_{sym}(\mathcal{R}ep(G,\,t)) = H^2(G,\,k^\times) \to Inv(\mathcal{R}ep(G,\,t))_2 = (\widehat{G})_2 : \mu \mapsto \frac{\mu(t,\,-)}{\mu(-,\,t)}$$

see Theorem 6.5. Combining (8.28) with the commuting square

$$\begin{array}{ccc} H^2(G,\,\widehat{A}) & \xrightarrow{\quad\Xi_t\quad} & \mathsf{Hom}(A,\,(\widehat{G})_2) \\ \downarrow & & \downarrow \\ \mathsf{Hom}(A,\,H^2(G,\,k^\times)) & \xrightarrow{\quad q^*\quad} & \mathsf{Hom}(G,\,(\widehat{A})_2) \end{array} \tag{8.29}$$

we obtain (8.24). □

Recall that isomorphism (2.15) identifies $H^4_{sym}(A,\,k^\times)$ with $\mathsf{Hom}(A_2,\,k^\times) = \widehat{(A_2)}$. Combining this with the isomorphisms $H^2(\mathbb{Z}_2,\,\widehat{A}) \cong \widehat{A}/(\widehat{A})^2 \cong \widehat{(A_2)}$ we obtain

$$H^4_{sym}(A,\,k^\times) \cong H^2(\mathbb{Z}_2,\,\widehat{A}). \tag{8.30}$$

Proposition 8.29. *The obstruction homomorphism* $\pi_0(\mathcal{F}un_{sym}(A,\,\mathcal{P}ic_{sym}(\mathcal{E}))) \to H^4_{sym}(A,\,k^\times)$ *in* (8.17) *is given by*

$$Fun_{sym}(A,\,\mathcal{P}ic_{sym}(\mathcal{E})) \cong Ker(\Xi_t) \hookrightarrow H^2(G,\,\widehat{A}) \xrightarrow{res} H^2(\langle t\rangle,\,\widehat{A}) \cong H^4_{sym}(A,\,k^\times), \tag{8.31}$$

where the first isomorphism is (8.24)*, the last one is* (8.30)*,* Ξ_t *is defined in* (8.25)*, and res is the restriction map in cohomology.*

Proof. For a symmetric monoidal functor $F : A \to \mathcal{P}ic_{sym}(\mathcal{E}))$ let $a = a(F) \in H^4_{sym}(A,\,k^\times)$ be the obstruction to lifting it to a symmetric monoidal 2-functor. The isomorphism (2.15) expresses a as an element of $\widehat{(A_2)}$ in terms of its components $a(x,x,x,x)$, $a(x,x|x)$, and $a(x|x,x)$, $x \in A_2$. We have $a(x,x|x) = a(x|x,x) = 1$ while the value of $a(x,x,x,x)$ is found as follows. Let us view the $\mathcal{E}$-module equivalence $M_{x,x} : F(x) \boxtimes_{\mathcal{E}} F(x) \xrightarrow{\sim} \mathcal{E}$ coming from the monoidal functor structure of F as an element of $Inv(\mathcal{E}) = \widehat{G}$. Then $a(x,x,x,x)$ is equal to the value of the self-braiding of $M_{x,x}$, i.e. to the evaluation $M_{x,x}(t)$. Note that the map

$$A_2 \to \widehat{G} : x \mapsto M_{x,x}|_{\langle t\rangle}$$

is a homomorphism, since

$$M_{x,x}M_{y,y} = M_{xy,xy}M_{x,y}M_{y,x}, \qquad x,y \in A_2,$$

and $M_{x,y}M_{y,x}|_{\langle t\rangle} = M^2_{xy}|_{\langle t\rangle} = 1$.

By (2.15), a is identified with the homomorphism

$$A_2 \to k^\times : x \mapsto M_{x,x}(t). \tag{8.32}$$

That this map coincides with the restriction map $\mathrm{Ker}(\Xi_t) \hookrightarrow H^2(G,\,\widehat{A}) \xrightarrow{\mathrm{Res}} H^2(\langle t\rangle,\,\widehat{A})$ follows from commutativity of the following diagram

$$\begin{array}{ccccc} H^2(G,\,\widehat{A}) & \xrightarrow{\quad s\quad} & H^2_{sym}(G/[G,G],\,\widehat{A}) & \xrightarrow{\quad\sim\quad} & H^2_{sym}(A,\,\widehat{G}) \\ \downarrow{\scriptstyle \mathrm{Res}} & & & & \downarrow \\ H^2(\langle t\rangle,\,\widehat{A}) & & \xrightarrow{\qquad\qquad\sim\qquad\qquad} & & H^2_{sym}(A,\,\widehat{\langle t\rangle}), \end{array} \tag{8.33}$$

where s denotes a splitting of the first row of (8.26). □

Thus, for $t = 1$ the exact sequence (8.17) gives rise to a split short exact sequence

$$0 \to \mathsf{Hom}(A,\,\mathbb{Z}_2) \to Ex_{sym}(A,\,\mathcal{R}ep(G)) \to H^2(G,\,\widehat{A}) \to 0, \tag{8.34}$$

while for $t \neq 1$ it becomes

$$\begin{aligned} H^1(G,\,\widehat{A}) \xrightarrow{\mathrm{Res}} H^1(\langle t\rangle,\,\widehat{A}) \to & Ex_{sym}(A,\,\mathcal{R}ep(G,\,t)) \to \\ & \mathrm{Ker}\left(H^2(G,\,\widehat{A}) \xrightarrow{\Xi_t} H^1(G/\langle t\rangle,\,H^1(\langle t\rangle,\,\widehat{A}))\right) \xrightarrow{\mathrm{Res}} H^2(\langle t\rangle,\,\widehat{A}). \end{aligned} \tag{8.35}$$

The isomorphism $Ex_{sym}(A,\,\mathcal{R}ep(G,\,t)) \cong H^2(G/\langle t\rangle,\,\widehat{A})$ from (8.19) can be recovered by comparing the sequence (8.35) with the exact sequence coming from the Lyndon-Hochschild-Serre spectral sequence [13, 32]:

$$\begin{aligned} H^1(G,\,\widehat{A}) \xrightarrow{\mathrm{Res}} H^1(\langle t\rangle,\,\widehat{A}) \to & H^2(G/\langle t\rangle,\,\widehat{A}) \xrightarrow{\mathrm{Inf}} \\ & \mathrm{Ker}\left(H^2(G,\,\widehat{A}) \xrightarrow{\mathrm{Res}} H^2(\langle t\rangle,\,\widehat{A})\right) \xrightarrow{\Xi_t} H^1(G/\langle t\rangle,\,H^1(\langle t\rangle,\,\widehat{A})). \end{aligned} \tag{8.36}$$

8.6 The Pontryagin-Whitehead quadratic function and zesting

Let $\mathcal{B}$ be a braided tensor category and let A be an Abelian group. Fix a homomorphism

$$f : A \to Pic_{br}(\mathcal{B}) : x \mapsto \mathcal{C}_x \tag{8.37}$$

that extends to a braided monoidal 2-functor $A \to \mathbf{Pic_{br}}(\mathcal{B})$. That is, there is a braided extension

$$\mathcal{C} = \bigoplus_{x\in A} \mathcal{C}_x.$$

Let c denote the braiding of $\mathcal{C}$.

Let $\mathbf{Ex}^f_{br}(A,\,\mathcal{B}) \subset \mathbf{Ex}_{br}(A,\,\mathcal{B})$ be the 2-subgroupoid of extensions corresponding to f. Our goal here is to describe $\pi_0(\mathbf{Ex}^f_{br}(A,\,\mathcal{B}))$.

An extension of (8.37) to a braided monoidal functor $A \to \mathcal{P}ic_{br}(\mathcal{B})$ amounts to choosing $\mathcal{B}$-equivalences $\mathcal{C}_x \boxtimes_{\mathcal{B}} \mathcal{C}_y \xrightarrow{\sim} \mathcal{C}_{xy}$, $x, y \in A$, satisfying coherence conditions. Any two such equivalences differ by a tensor

multiplication by an invertible object $L_{x,y} \in \mathcal{Z}_{sym}(\mathcal{B})$. Hence, any extension $\tilde{\mathcal{C}} \in \mathbf{Ex}^f_{br}(A,\, \mathcal{B})$ is equal to $\mathcal{C}$ as an abelian category and has the tensor product

$$X\tilde{\otimes}Y = L_{x,y} \otimes X \otimes Y, \qquad X \in \mathcal{C}_x\, Y \in \mathcal{C}_y,\, x, y, \in A. \tag{8.38}$$

To get associativity and braiding constraints of $\tilde{\mathcal{C}}$ it is necessary to have isomorphisms

$$\xi_{x,y,z} : L_{xy,z} \otimes L_{x,y} \xrightarrow{\sim} L_{x,yz} \otimes L_{y,z}, \quad \kappa_{x,y} : L_{x,y} \xrightarrow{\sim} L_{y,x}, \qquad x, y, z \in A, \tag{8.39}$$

i.e. $L = \{L_{x,y}\}_{x,y\in A}$ must be a 2-cocycle in $Z^2_{br}(A,\, Inv(\mathcal{Z}_{sym}(\mathcal{B})))$. These constraints are given by

$$\begin{aligned}(X\tilde{\otimes}Y)\tilde{\otimes}Z = L_{xy,z} \otimes L_{x,y} \otimes X \otimes Y \otimes Z \xrightarrow{\xi_{x,y,z}} L_{x,yz} \otimes L_{y,z} \otimes X \otimes Y \otimes Z \\ \xrightarrow{c_{L_{y,z},X}} \quad L_{x,yz} \otimes X \otimes L_{y,z} \otimes Y \otimes Z = X\tilde{\otimes}(Y\tilde{\otimes}Z)\end{aligned} \tag{8.40}$$

and

$$X\tilde{\otimes}Y = L_{x,y} \otimes X \otimes Y \xrightarrow{\kappa_{x,y}} L_{y,x} \otimes X \otimes Y \xrightarrow{c_{X,Y}} L_{y,x} \otimes Y \otimes X = Y\tilde{\otimes}X \tag{8.41}$$

for all objects $X \in \mathcal{C}_x$, $Y \in \mathcal{C}_y$, $Z \in \mathcal{C}_z$, x, y, $z \in A$, where we omit the associativity constraints of $\mathcal{C}$.

This is a braided version of the construction introduced in [23, Section 8.7]. Such extensions were considered in [6] where they were called *zestings* of $\mathcal{C}$. Recently, a more general construction was studied in great detail in [14]. In Propositions 8.32 and 8.33 below we compute obstructions and give a parameterization of such extensions. Our treatment of equivalence classes of zesting extensions and obstructions seems to be different from that of [14, Section 4].

By Proposition 5.3 the Whitehead bracket

$$[-,\, -] : Pic_{br}(\mathcal{B}) \times Inv(\mathcal{Z}_{sym}(\mathcal{B})) \to k^\times$$

satisfies $c_{M,Z}c_{Z,M} = [\mathcal{M},\, Z]\mathrm{Id}_{Z\otimes M}$ for all $Z \in Inv(\mathcal{Z}_{sym}(\mathcal{B}))$ and $M \in \mathcal{M}$, where $\mathcal{M} \in Pic_{br}(\mathcal{B})$.

Define a group homomorphism

$$PW^1_{\mathcal{C}} : H^1_{br}(A,\, Inv(\mathcal{Z}_{sym}(\mathcal{B}))) \to H^3_{br}(A,\, k^\times) : Z \mapsto Q_Z, \tag{8.42}$$

where Q_Z is identified with the quadratic from

$$Q_Z(x) = [\mathcal{C}_x,\, Z(x)]\, , c_{Z(x),Z(x)}, \qquad x \in A. \tag{8.43}$$

Define a quadratic function

$$PW^2_{\mathcal{C}} : H^2_{br}(A,\, Inv(\mathcal{Z}_{sym}(\mathcal{B}))) \to H^4_{br}(A,\, k^\times), \tag{8.44}$$

by setting the components of $PW^2_{\mathcal{C}}(L)$ for a braided 2-cocycle L to be

$$PW^2_{\mathcal{C}}(L)(x, y, z, w) = c_{L_{x,y},L_{z,w}}, \tag{8.45}$$

$$PW^2_{\mathcal{C}}(L)(x, y|z) = 1, \tag{8.46}$$

$$PW^2_{\mathcal{C}}(L)(x|y, z) = [\mathcal{C}_x,\, L_{y,z}], \qquad x, y, z, w \in A. \tag{8.47}$$

Definition 8.30. We will call (8.42) and (8.44) the *first* and *second Pontryagin-Whitehead maps*, cf. [23, Section 8.7].

Remark 8.31. The maps $PW^1_{\mathcal{C}}$ and $PW^2_{\mathcal{C}}$ depend on the homomorphism $f : A \to Pic_{br}(\mathcal{B}) : x \mapsto \mathcal{C}_x$.

Proposition 8.32. *Let L be a 2-cocycle in $Z^2_{br}(A, Inv(\mathcal{Z}_{sym}(\mathcal{B})))$. One can choose isomorphisms (8.39) so that the associativity and braiding isomorphisms (8.40), (8.41) satisfy the pentagon and hexagon axioms (i.e. give rise to a tensor category) if and only if $PW^2_{\mathcal{C}}(L)$ is trivial in $H^4_{br}(A, k^\times)$.*

Proof. Define a cochain $a \in C^3_{br}(A, k^\times)$ by $a(x, y, z) = \xi_{x,y,z}$, $a(x|y) = \kappa_{x,y}$ for all $x, y, z \in A$, where ξ and κ are isomorphisms (8.39).

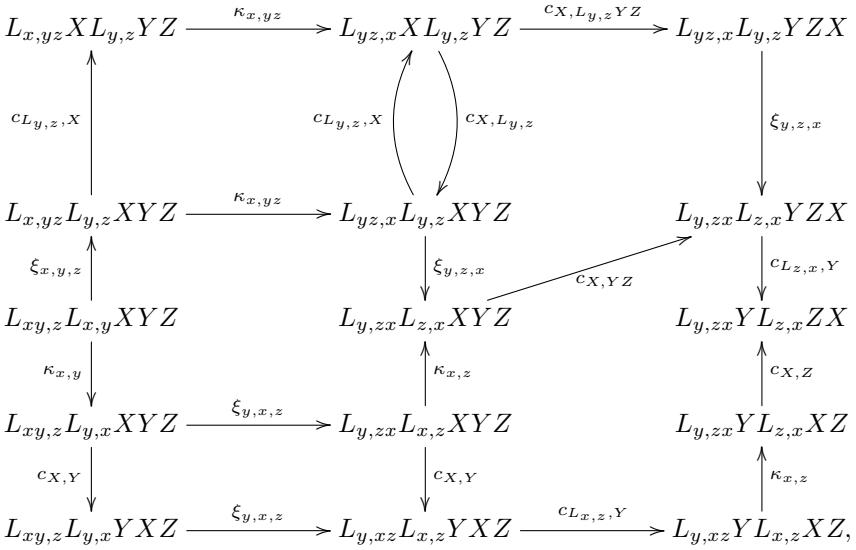
In the diagrams below we will omit the the tensor product sign and the associativity constraints of $\mathcal{C}$. The pentagon for the associativity constraint (8.40) becomes the diagram

$$\begin{array}{ccccccc}
 & & & & L_{xyz,w}L_{xy,z}L_{x,y}XYZW & \xrightarrow{\xi_{xy,z,w}} & L_{xy,zw}L_{z,w}L_{x,y}XYZW \\
 & & & \swarrow^{\xi_{x,y,z}} & & & \searrow^{c_{L_{z,w},L_{x,y}}} \\
 & & L_{xyz,w}L_{x,yz}L_{y,z}XYZW & & & & L_{xy,zw}L_{x,y}L_{z,w}XYZW \\
 & \swarrow^{c_{L_{y,z},X}} & & \searrow^{\xi_{x,yz,w}} & & \swarrow^{\xi_{x,y,zw}} & \downarrow{\scriptstyle c_{L_{z,w},XY}} \\
L_{xyz,w}L_{x,yz}XL_{y,z}YZW & & & & L_{x,yzw}L_{yz,w}L_{y,z}XYZW & \xrightarrow{\xi_{y,z,w}} & L_{x,yzw}L_{y,zw}L_{z,w}XYZW \quad L_{xy,zw}L_{x,y}XYL_{z,w}ZW \\
\downarrow{\scriptstyle \xi_{x,yz,w}} & & & \swarrow^{c_{L_{y,z},X}} & & & \downarrow{\scriptstyle c_{L_{z,w},X}} \quad \swarrow^{c_{L_{z,w},XY}} \quad \downarrow{\scriptstyle \xi_{x,y,zw}} \\
L_{x,yzw}L_{yz,w}XL_{y,z}YZW & & & & & & L_{x,yzw}L_{y,zw}XL_{z,w}YZW \quad L_{x,yzw}L_{y,zw}XYL_{z,w}ZW \\
\downarrow{\scriptstyle c_{L_{yz,w},X}} & & & & & & \downarrow{\scriptstyle c_{L_{y,zw},X}} \quad \downarrow{\scriptstyle c_{L_{y,zw},X}} \\
L_{x,yzw}XL_{yz,w}L_{y,z}YZW & & \xrightarrow{\xi_{y,z,w}} & & & & L_{x,yzw}XL_{y,zw}L_{z,w}YZW \xrightarrow{c_{L_{z,w},Y}} L_{x,yzw}XL_{y,zw}YL_{z,w}ZW,
\end{array} \tag{8.48}$$

while the hexagons are the diagrams

$$\begin{array}{ccccc}
L_{xy,z}L_{x,y}XYZ & \xrightarrow{c_{L_{x,y}XY,Z}} & L_{xy,z}ZL_{x,y}XY & \xrightarrow{\kappa_{xy,z}} & L_{z,xy}ZL_{x,y}XY \\
\uparrow{\scriptstyle \xi^{-1}_{x,y,z}} & & \downarrow{\scriptstyle c^{-1}_{L_{x,y},Z}} & & \downarrow{\scriptstyle c^{-1}_{L_{x,y},Z}} \\
L_{x,yz}L_{y,z}XYZ & & L_{xy,z}L_{x,y}ZXY & \xrightarrow{\kappa_{xy,z}} & L_{z,xy}L_{x,y}ZXY \\
\uparrow{\scriptstyle c^{-1}_{L_{y,z},X}} & \searrow^{c_{XY,Z}} & \uparrow{\scriptstyle \xi^{-1}_{x,y,z}} & & \downarrow{\scriptstyle \xi^{-1}_{z,x,y}} \\
L_{x,yz}XL_{y,z}YZ & & L_{x,yz}L_{y,z}ZXY & & L_{zx,y}L_{z,x}ZXY \\
\downarrow{\scriptstyle c_{Y,Z}} & & \downarrow{\scriptstyle \kappa_{y,z}} & & \uparrow{\scriptstyle \kappa_{x,z}} \\
L_{x,yz}XL_{y,z}ZY & & L_{x,yz}L_{y,z}ZXY & \xrightarrow{\xi^{-1}_{x,z,y}} & L_{xz,y}L_{x,z}ZXY \\
\downarrow{\scriptstyle \kappa_{y,z}} & & \uparrow{\scriptstyle c_{X,Z}} & & \uparrow{\scriptstyle c_{X,Z}} \\
L_{x,zy}XL_{z,y}ZY & \xrightarrow{c^{-1}_{L_{z,y},X}} & L_{x,yz}L_{y,z}XZY & \xrightarrow{\xi^{-1}_{x,z,y}} & L_{xz,y}L_{x,z}XZY,
\end{array} \tag{8.49}$$

and



(8.50)

for all $x, y, z, w \in A$ and $X \in \mathcal{C}_x$, $Y \in \mathcal{C}_y$, $Z \in \mathcal{C}_z$, $W \in \mathcal{C}_w$.

After cancelling internal polygons commuting by the functoriality of the tensor product of $\mathcal{C}$, naturality of c, and the Yang-Baxter equation, we see that the clockwise compositions given by the perimeters of (8.48), (8.49), and (8.50) are

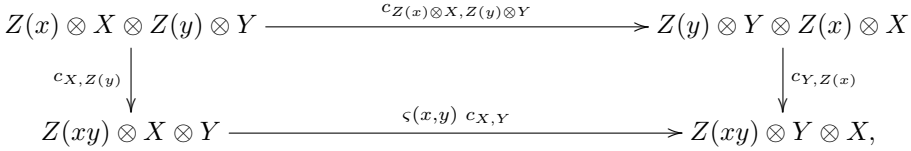
$$c_{L_{x,y},L_{z,w}}\, d(a)(x,y,z,w), \quad d(a)(x,y|z), \quad \text{and } c_{X,L_{y,z}} c_{L_{y,z},X}\, d(a)(x|y,z), \quad \text{respectively.}$$

Comparing this with the definition of $PW^2_{\mathcal{C}}(L)$ we get the result. □

Proposition 8.33. *There is a fibration $F \to \pi_0(\mathbf{Ex}^f_{br}(A, \mathcal{B})) \to B$, where the base B is the set of zeroes of $PW^2_{\mathcal{C}}$ and the fiber F is the cokernel of $PW^1_{\mathcal{C}}$.*

Proof. The assertion about the base follows from Proposition 8.32.

Let $\mathcal{C}, \tilde{\mathcal{C}}$ be A-extensions of $\mathcal{B}$ corresponding to the same braided monoidal functor $A \to \mathcal{P}ic_{br}(\mathcal{B})$. Then $\tilde{\mathcal{C}} = \mathcal{C}^{(\omega,\varsigma)}$ for some $(\omega, \varsigma) \in H^3(A, k^\times)$. An equivalence of extensions $\mathcal{C}^{(\omega,\varsigma)} \xrightarrow{\sim} \mathcal{C}$ is given on homogeneous component $\mathcal{C}_x$, $x \in A$, by $X \mapsto Z(x) \otimes X$, $X \in \mathcal{C}_x$ for $Z(x) \in Inv(\mathcal{Z}_{sym}(\mathcal{B}))$. The tensor property of this equivalence means that $Z : A \to Inv(\mathcal{Z}_{sym}(\mathcal{B}))$ is a homomorphism, while its braided property translates to commutativity of the diagram



(8.51)

for all $x, y \in A$, $X \in \mathcal{C}_x\, Y \in \mathcal{C}_y$. Here c denotes the braiding of $\mathcal{C}$.

Comparing the compositions in (8.51) we see that

$$\varsigma(x,y) = c_{Z(x),Z(y)}\, c_{Y,Z(x)} c_{Z(x),Y}, \qquad \text{for all } Y \in \mathcal{C}_y,$$

and so the corresponding quadratic form is $\varsigma(x,x) = [\mathcal{C}_x,\, Z(x)]\, c_{Z(x),Z(x)} = Q_Z(x)$, $x \in A$. □

8.7 Quasi-trivial braided extensions

Let $\mathcal{B}$ be a braided tensor category. We saw in Example 5.6 that the braided 2-categorical group $\mathbf{Pic_{br}}(\mathcal{B})$ contains a full 2-categorical subgroup $\mathbf{Pic^1_{br}}(\mathcal{B})$ consisting of braided $\mathcal{B}$-module categories $\mathcal{M}$ such that $\mathcal{M} \cong \mathcal{B}$ as a $\mathcal{B}$-module category.

Definition 8.34. Let A be a finite group. We say that a braided A-graded extension of $\mathcal{B}$ is *quasi-trivial* if it contains an invertible object in every homogeneous component.

Equivalently, an A-extension of $\mathcal{B}$ is quasi-trivial if the corresponding homomorphism $A \to Pic_{br}(\mathcal{B})$ factors through $Pic^1_{br}(\mathcal{B})$.

Remark 8.35. A quasi-trivial extension is a special type of a braided *zesting* considered in [14]. Namely, it is a zesting of $\mathcal{C}(A,\,1) \boxtimes \mathcal{B}$.

Let $\mathbf{Ex_{br-qt}}(A,\,\mathcal{B})$ denote the 2-groupoid of quasi-trivial braided A-extensions of $\mathcal{B}$. We have an equivalence of 2-groupoids

$$\mathbf{Ex_{br-qt}}(A,\,\mathcal{B}) \cong \mathbf{2\text{-}Fun_{br}}(A,\,\mathbf{Pic^1_{br}}(\mathcal{B})).$$

Since objects of $\mathbf{Pic^1_{br}}(\mathcal{B})$ are of the form $\mathcal{B}^\nu$, $\nu \in Aut_\otimes(\mathrm{Id}_\mathcal{B})$ (see Example 5.6), any braided monoidal 2-functor $A \to \mathbf{Pic^1_{br}}(\mathcal{B})$ (and any extension in $\mathbf{Ex_{br-qt}}(A,\,\mathcal{B})$ comes from a group homomorphism

$$f : A \to Aut_\otimes(\mathrm{Id}_\mathcal{B}).$$

Example 8.36. Given f as above, there is a canonical quasi-trivial A-graded braided extension $\mathcal{B}(f)$ of $\mathcal{B}$ such that $\mathcal{B}(f) = \mathcal{B} \boxtimes \mathcal{V}ect_A$ as a tensor category and its braiding is given by

$$c_{X\boxtimes x, Y\boxtimes y} = f(x)_Y\, c_{X,Y}, \qquad X, Y \in \mathcal{B},\, x, y \in A,$$

where $x \in A$ denote the simple objects of $\mathcal{V}ect_A$.

Hence,

$$\mathbf{Ex_{br-qt}}(A,\,\mathcal{B}) = \bigvee_{f \in \mathsf{Hom}(A,\, Aut_\otimes(\mathrm{Id}_\mathcal{B}))} \mathbf{Ex^f_{br-qt}}(A,\,\mathcal{B}),$$

where $\mathbf{Ex^f_{br-qt}}(A,\,\mathcal{B})$ is the 2-subgroupoid of quasi-trivial extensions corresponding to f. Furthermore, $\mathbf{Ex^{f_1}_{br-qt}}(A,\,\mathcal{B}) = \mathbf{Ex^{f_2}_{br-qt}}(A,\,\mathcal{B})$ if and only if $f_2 = f_1 \partial(Z)$ for some $Z \in Inv(\mathcal{B})$.

The Pontryagin-Whitehead maps (8.42) and (8.44) in this situation are given by

$$PW^1_{\mathcal{B}(f)}(Z)(x) = f(x)_{Z(x)}\, c_{Z(x),Z(x)}, \qquad Z \in \mathsf{Hom}(A,\, Inv(\mathcal{Z}_{sym}(\mathcal{B}))), \tag{8.52}$$

and

$$PW^2_{\mathcal{B}(f)}(L)(x,y,z,w) = c_{L_{x,y},L_{z,w}}, \tag{8.53}$$

$$PW^2_{\mathcal{B}(f)}(L)(x,y|z) = 1, \tag{8.54}$$

$$PW^2_{\mathcal{B}(f)}(L)(x|y,z) = f(x)_{L_{y,z}}, \qquad L \in H^2_{br}(A,\, Inv(\mathcal{Z}_{sym}(\mathcal{B}))), \tag{8.55}$$

for all $x, y, z, w \in A$.

Corollary 8.37. *There is a fibration $F \to \pi_0(\mathbf{Ex}^f_{\mathbf{br-qt}}(A,\, \mathcal{B})) \to B$, where the base B is the set of zeroes of $PW^2_{\mathcal{B}(f)}$ and the fiber F is the cokernel of $PW^1_{\mathcal{B}(f)}$.*

Thus, quasi-trivial A-extensions of $\mathcal{B}$ are obtained by choosing a homomorphism $f : A \to Aut_{\otimes}(\mathrm{Id}_{\mathcal{B}})$, deforming ("zesting") the tensor product and constraints of $\mathcal{B}(f)$ by means of $L \in Z^2_{br}(A,\, Inv(\mathcal{Z}_{sym}(\mathcal{B})))$ such that $PW^2_{\mathcal{B}(f)}(L) = 0$ via (8.38) - (8.41), and then twisting the result by means of a braided 3-cocycle $(\omega, \varsigma) \in Z^3_{br}(A,\, k^\times)$. Corollary 8.37 gives a description of equivalence classes of such extensions.

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