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ORIGINAL ARTICLE

Challenging preservice secondary mathematics teachers' conceptions of function

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Abstract
Preservice mathematics teachers, like all learners, have well documented difficulties with the concept of function. We designed an applet-based task to challenge these known difficulties with the aim of improving preservice secondary mathematics teachers' (PSMTs) conceptions of function. This cross-institutional study of 47 PSMTs examined the ways in which this task elicited and improved PSMTs' conceptions of function. The results of the study show a measurable increase in the participants' level of abstraction in their definition of function, and an increase in their attention to the univalence condition. In particular, the interaction with the specially designed applet was effective in initiating a series of dilemmas in their conception of function that resulted in the majority of the participants changing their conception of function in a positive direction.

Keywords Function · Preservice teachers · Technology

In the opening paragraph of their chapter on function in the 2017 *Compendium for Research in Mathematics Education*, Thompson and Carlson note that "[t]here is nothing that can be called 'the concept of function.' The phrase 'concept of function,' regardless of its meaning, immediately calls into question whom we envision having it" (p. 421). One's conception of function depends on their previous experiences with

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function, especially the contexts within which it was used. For example, a high school student will likely have a different, less developed, conception of function than a mathematician due to the difference in their experiences, and a mathematics teacher's conception of function will likely be different than that of a mathematician as a result of the different function definitions each uses in the contexts in which each works.

The concept of function is considered to be one of the most important underlying and unifying concepts of mathematics (e.g., Leinhardt et al. 1990; Thompson and Carlson 2017) and, yet, is one fraught with challenges for learners. There is an extensive body of research on students' understanding of function (e.g., Bardini et al. 2013; Carlson et al. 2003; Cooney et al. 2010; Dubinsky and Harel 1992; Martinez-Planell and Gaisman 2012) and much of that research reports that learners (including preservice secondary mathematics teachers) have considerable difficulty identifying functions and distinguishing them from non-functions.

Function and function behavior is an area of considerable emphasis in school mathematics throughout the world. Students are provided experiences with functions from the very earliest grades, usually as pattern exploration and covarying quantities (Blanton et al. 2015; Ellis 2011; Ng 2018; Stephens et al. 2017) up to and through high school with a formal treatment of functions as arbitrary mappings between sets (Carlson et al. 2003; Cooney et al. 2010; Dubinsky and Harel 1992). In the "Number and Algebra" strand of the Australian National Curriculum (Australian Curriculum, Assessment and Reporting Authority [ACARA] 2020), one of seven key objectives is that students will "recognise patterns and understand the concepts of variable and function." Similarly, the New Zealand National Curriculum (Ministry of Education 2007) lists "patterns and relationships" (typically function relationships) as one of twelve "key mathematical ideas." The National Curriculum of England (Department for Education 2014) "Mathematics Programmes of Study" refers explicitly to function 18 times in its document and, finally, in the Common Core State Standards for Mathematics in the USA, the study of function is given its own domain in grades 9–12 (National Governors Association for Best Practices & Council of Chief State School Officers 2010).

Given this emphasis on function in school mathematics, it is important to consider preservice secondary mathematics teachers' (PSMTs) conceptions of function. PSMTs must possess robust conceptions of function so they can plan for supporting the development of their future students' function understandings. However, the consistency of problematic understandings of function found across studies speaks to the need for an instructional intervention to specifically disrupt and refine these ideas. This led us to wonder how we might be able to provoke PSMTs to think deeply and differently about function. In doing so, we specifically focused on the first component of the essential understandings of function set out by the National Council of Teachers of Mathematics publication, Developing Essential Understandings of Functions (Cooney et al. 2010). These understandings include:

a) Functions are single-valued mappings from one set—the domain of the function—to another, its range.
b) Functions apply to a wide range of situations. They do not have to be described by any specific expression.

c) The domain and range of functions do not have to be numbers.
Cooney et al. 2010, p. 8

To address these understandings, we developed an interactive applet which uses a vending machine metaphor as a context for learning. Vending machines have an input (domain) being mapped to an output (range); they provide a context with which students are familiar, and one for which the domain and range are not numbers. The applet thus provides a context for users to explore function behavior without relying on an algebraic representation. The purpose of this study is to examine the ways in which engaging with the Vending Machine Applet challenges PSMTs' conceptions of function.

Issues relating to understanding of function are persistent across learning populations and across time. In addition, these issues tend to be focused on the privileging of algebraic relationships. Thus, the purpose of this study is to examine the ways in which engaging with the Vending Machine Applet, with its non-algebraic representation of function, challenges, and supports the refinement of PSMTs' conceptions of function.

Background

Defining function

In most secondary schools, the commonly used definition of function is a correspondence definition, often referred to as the Dirichlet-Bourbaki definition of function. This definition states that a function is a correspondence between arbitrary sets satisfying a univalence condition (i.e., each element in the domain corresponds to exactly one element in the codomain). Thompson and Carlson (2017), citing Cooney and Wilson (1993), note that a correspondence definition of function is used exclusively in most US textbooks. So, while we expect that most students (and PSMTs) who have attended US schools to have experience with a definition involving a correspondence (or mapping) between two sets with constraints on the mapping of individual elements (the univalence condition), Even (1993) notes that many students retain a "prototypic" (p.96) concept of functions as algebraically represented linear relationships and "many expect graphs of functions to be "reasonable" and functions to be representable by a formula." (p. 96).

Students' understandings of the function concept

The function concept is addressed throughout mathematics education from the earliest grades, but not usually in a coherent way that builds across those grades. Functions are typically introduced as very limited classes such as linear and quadratic, with attendant graphs and tables, with the result that students regularly consider functions to be mathematics objects solely defined by an algebraic formula (e.g., Bardini et al. 2014; Breidenbach et al. 1992; Carlson 1998).

Exposure to, and facility with, various representations of functions, what Best and Bikner-Ahsbahs (2017) call "flexible use of functions . . . within and between all kinds of representations and also between different functions" (p. 877), has been shown to be a critical component to a rich understanding of function (Best and Bikner-Ahsbahs 2017; Dubinsky and Wilson 2013; Martínez-Planell and Gaisman 2012). Yet, curricular materials often emphasize procedures and algebraic manipulations when studying functions, and research shows that students have difficulty in understanding different representations and different contexts for functions (Carlson and Oehrtman 2005; Cooney et al. 2010). Leinhardt et al. (1990), in a meta-study of research on function, and Mesa (2004), in a study of 24 middle grades textbooks from 15 countries (including 2 from Australia, 1 from Hong Kong, and 1 from Singapore), note the difficulty for students in apprehending the modern, abstract definition of function depending, as it does, on the mapping of one set of elements to another emphasizing the difference between function and relation (many-to-one acceptable, one-to-many not acceptable); whereas, the work on function in early grades builds on the intuitive notion of a 1-1 correspondence and the historical development of function rested on covarying quantities.

Researchers have found promising results when using novel contexts and non-standard representations of functions such as dynagraphs, arrow diagrams, and directed graphs (Dubinsky and Wilson 2013; Sinclair et al. 2009). Results included students being able to translate their experiences with different representations to then identify different properties of functions recognizing functions in a variety of representations (Dubinsky and Wilson 2013; Sinclair et al. 2009), and provide examples of and evaluate functions using multiple representations (Dubinsky and Wilson 2013).

At the heart of many student difficulties in understanding function may be a less than robust understanding of the definition (Ayalon et al. 2017; Bardini et al. 2013; Panaoura et al. 2017). Carlson and Oehrtman (2005) argue that students with a strong understanding of the function definition can successfully reason about more complex ideas such as composition of functions. On the other hand, students who possess an algebraic view of function and use procedural techniques to identify functions and non-functions struggle to comprehend a general mapping between sets (Carlson 1998; Thompson 1994).

Teachers' understandings of the function concept

Mathematics teachers should be aware of various representations of functions, many examples of functions and non-functions, and known areas of challenge for students when learning functions. However, research has shown that often teachers' understanding of function is quite similar to that of students, with teachers showing many of the same limitations and conceptions (Bannister 2014; Even 1990, 1993; Tabach and Nachlieli 2015; Wilson 1994). In particular, practicing teachers and PSMTs tend to privilege algebraic representations of functions and emphasize properties of graphs (e.g., vertical line test) in their descriptions of functions and non-functions (Even 1990, 1993; Wilson 1994). They also exhibit a limited repertoire of representations on which to draw in helping students understand functions (Bannister 2014; Doerr 2004; Hatisaru and Erbas 2017).

While in many instances preservice and inservice teachers may be able to provide a correct formal definition of a function, their knowledge is often lacking the depth to be able to deploy their definition to correctly identify functions and non-functions (Chesler 2012;Tabach and Nachlieli 2015), to successfully translate between multiple representations of functions (Bannister 2014), or exhibit "flexibility and expertise in interpreting and using mathematical definitions" (p. 38) as they examined the equivalence of various definitions of function.

Crucially, teachers' understanding of the function concept has been shown to impact the pedagogical choices they make during instruction. In a study of 152 PSMTs, Even (1993) found they could not justify the need for univalence and did not know why it was important to distinguish between functions and non-functions. Owing to this lack of content knowledge, the PSMTs limited the exposure of their students to various function representations and emphasized procedures such as the vertical line test in identifying functions. Building on Even's work, Hatisaru and Erbas (2017) found when a practicing teacher had a robust concept of function their students, in turn, developed a high level of content knowledge of functions and when the teacher had limitations and constraints in knowledge, the students exhibited those same limitations and constraints. In addition to having a rich understanding of the definition, Bannister (2014) suggests that PSMTs who are adept at translating between algebraic and graphical representations of functions may be better prepared to understand diverse student conceptions when they encounter them during instruction.

Conceptual framework

Concept vs. conception

When thinking about learning in terms of mathematics, it is helpful to consider the connection between knowledge and beliefs. Sfard (1991) distinguishes between a *concept, a* mathematical idea in its "official form" (p. 3) and a *conception*, "the whole cluster internal representations and associations evoked by a concept" (p. 3). A *concept* (sometimes referred to as objective knowledge) is the generally accepted structure of mathematics that has been culturally developed and shared formally among mathematicians for centuries (Pehkonen and Pietilä 2004) whereas a *conception* is a learner's individual, often, incomplete understanding of the concept. Conceptions, then, are the personal side of a concept, one's individual experiences, beliefs, attitudes, and emotions that result in personal definitions, examples, and non-examples of concepts. Pehkonen and Pietilä (2004) argue that conceptions are conscious beliefs, and form a subset of beliefs. In this distinction, the cognitive components of beliefs are stressed, rather than the affective components. In this sense, attending to someone's articulated mathematical conceptions provides a window to their understanding and learning.

Cognitive root

Given that PSMTs will be responsible for teaching others about function, it is important to address their conceptions of function that are inconsistent with the concept itself through carefully designed learning experiences. One strategy that has been suggested for mitigating common misunderstandings related to function is the use of a function machine as a cognitive root (Tall et al. 2000). The idea of a cognitive root was introduced by Tall and colleagues as an "anchoring concept which the learner finds easy to comprehend, yet forms a basis on which a theory may be built" (Tall et al. 2000, p.497). As an example of a cognitive root for function concepts, Tall et al. (2000) suggest the use of a function machine (sometimes referred to as a function box). The machine metaphor they describe is typically a "guess my rule" activity. In such activities, the inputs and associated outputs are provided, and students are challenged to determine what happened in the function machine (i.e., determine the function rule). While students are presented with a machine to embody the function concept, the rules used by the machine are algebraic in nature. In their studies, using such machines proved quite promising, yet some students still struggled with connecting different representations and determining what is and is not a function (McGowen et al. 2000).

Our intervention

Given the promise of a machine metaphor as a cognitive root for function coupled with our desire to present PSMTs with a situation in which they would need to grapple with their current conceptions of function, we set out to design a machine-based experience using representations that were unfamiliar for PSMTs as a stimulus for examining their conceptions of function. Since PSMTs come to their methods courses as adults with a wealth of previous experiences related to the function concept, we draw upon an adult learning theory to guide our design process—transformation theory (Mezirow 2000). Transformation theory is consistent with constructivist assumptions, specifically that meaning resides within each person and is constructed through experiences (Confrey 1990). According to Mezirow (2009), learning by transforming often begins with a stimulus, a *disorienting dilemma*, which requires one to question one's current understandings. A dilemma, which is provoked by a *trigger*, i.e., something that "signals dissatisfaction with current ways of thinking" (Marsick and Watkins 2001, p.29), often results in a questioning of one's understandings that is resolved by creating, enhancing, or transforming them—i.e., learning. Thus, our goal was to design an experience that capitalized on the use of the machine metaphor as a cognitive root to trigger dilemmas related to PSMTs conceptions of function in the form of an applet. As users interact with the applet and learn how it works, they would, per Drijvers (2015), develop "mental schemes that include the conceptual understanding of the mathematics at stake." (p.15). Unlike typical function machines, the applet we designed contained no numerical or algebraic expressions. Instead, it was built on the metaphor of a vending machine. Our intention was to put PSMTs in a context in which they would not be able to automatically rely on an algebraic, and often procedural, conceptions of functions (e.g., use of the vertical line test). The applet provides an environment that is self-directed, in that it is interactive and provides sufficient data for students to distinguish between function and non-function. Finally, we hoped to emphasize the essential understandings identified by Cooney et al. (2010), in particular that functions apply to a wide range of situations and their domain and range do not have to be numbers.

The Vending Machine Applet (https://ggbm.at/J3mJaU6H) consists of four pages; each page contains two to six vending machines and asks the user to identify each vending machine as a function or non-function (Fig. 1). The machines each consist of four buttons (Red Cola, Diet Blue, Silver Mist, and Green Dew). When a button is pressed it produces

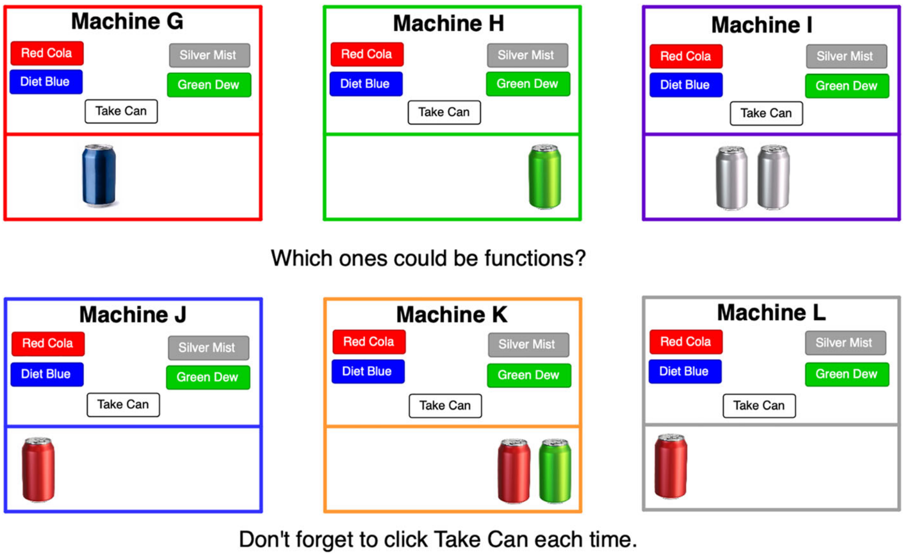

Fig. 1 Vending Machine Applet

none, one, or more than one of the four different colored cans (red, blue, silver, and green), which may or may not correspond to the color of the button pressed.

By removing numeric and algebraic representations, the applet was designed to have PSMTs attend to the nature of the input and outputs, as well as to the relationship between them. Thus, we intentionally designed so as to trigger disorienting dilemmas (Mezirow 2009), i.e., a dilemma that causes one to question their current understanding, that would focus on common conceptions from the literature. For example, researchers have shown that students, as well as teachers, exhibit difficulties identifying constant functions as functions (e.g., Carlson 1998; Rasmussen 2000); thus, there is a machine that acts as a constant function, in that every button produces the same color can. In addition, we designed several machines to give PSMTs the opportunity to grapple with different numbers of outputs. A description of each machine and the directions for each page of the applet is provided in Fig. 2. Not included in Fig. 2 is the identification of each machine as a function or non-function. This was a deliberate decision, as in many cases whether or not a machine can be classified as a function depends on how the user defines the domain, range, and codomain (e.g., Machines I and J).

Methods

The purpose of this study is to examine the ways in which engaging with the Vending Machine Applet challenges PSMTs' conceptions of function. Specifically, we aim to answer the following research questions: In what ways did PSMTs' personal definitions of function change as a result of engaging with the Vending Machine task? and what conceptions of function were challenged and refined as a result of engaging with the Vending Machine Applet?

Which one is a function?			
A	Red → Red Blue → Blue Silver → Silver Green → Green	B	Red → Red Blue → Blue Silver → Silver Green → Random
Which one is a function?			
C	Red → Blue Blue → Silver Silver → Green Green → Red	D	Red → Random Pair Blue → Blue Silver → Silver Green → Green
Which one is a function?			
E	Red → Red Blue → Blue & Random Silver → Silver Green → Green	F	Red → Red Blue → Silver Silver → Silver Green → Green
Which ones could be functions?			
G	All Random	H	Red → Green Blue → Green Silver → Green Green → Green
I	Red → 2 Silvers Blue → Green Silver → Red Green → Blue	J	Red → Red Blue → Blue Silver → No Can Green → Green
K	Red → Red Blue → Blue Silver → Silver Green → Red & Green	L	Red → Red Blue → Red Silver → Silver Green → Silver

Fig. 2 Description of each machine

Participants

The participants in this study are 47 PSMTs enrolled in a course on pedagogical methods for secondary mathematics at four different US universities, ranging from five to 18 PSMTs at each university. The PSMTs were all undergraduate mathematics and/or mathematics education majors working toward earning their secondary mathematics teaching license. The individual degree programs all required at least 36 credit hours of mathematics, and these students had all successfully completed at least a second level calculus course at the time of the study. Every PSMT at each of the four universities took part in the study ($N = 55$). However, there were some PSMTs that did not have complete data sets (e.g., video had no sound, missing artifacts); these participants were removed, leaving 47 PSMTs in this particular study.

Context

The Vending Machine task was used in each of the mathematics methods classes in the same way. At the end of a class period, PSMTs were given a sheet of paper and asked to record their personal definition of function. They were then assigned to do the Vending Machine task as homework. Specifically, they were given a worksheet that had a link to the applet at the top with instructions for the assignment. Below the instructions was a table with each machine listed in the left column and two additional columns, one that asked “Function or not a function?” and another that asked, “How do you know?” At the beginning of the next class, PSMTs were given their definitions back and asked to make any changes they would like based on their experience with the applet. Finally, in each class, there was a whole class discussion about the machines in the applet and ways in which definitions were revised and why.

Data sources

Data for this study consists of all of the PSMTs' individual work related to the Vending Machine task. Specifically, we collected PSMTs' written pre- and post-definitions of function and their written responses to the Vending Machine task worksheet. In addition, each PSMT captured a screencast of their work on the Vending Machine task as they followed a "think-aloud" protocol while working on the task.

Data analysis

To begin our analysis, we created a document for each PSMT which consisted of their pre- and post-definitions and a detailed description of the video-recorded screencast. These descriptions included a chronological record of PSMTs' engagement with the applet, verbatim transcriptions of PSMTs' verbal utterances related to their work on the task, and PSMTs' worksheet responses at the point the writing occurred. Once participant descriptions were complete, they were analyzed in two different phases. The first included analysis of the pre- and post-definitions. The second included the detailed descriptions.

Analysis of pre- and post-definitions All 94 definitions (47 pre and 47 post) were coded using a codebook which was developed in a previous study and for which reliability was established (Sherman et al. 2018). Similarly to Vinner and Dreyfus (1989), each definition was coded in terms of (1) accuracy, (2) focus, and (3) attention to output. In terms of *accuracy*, each definition was assigned a code of correct, incorrect, or close to correct. Key elements of a correct definition were (1) the definition was not limited to a specific type of function (e.g., linear or quadratic), or to a particular representation (e.g., equation), and (2) the definition addressed the idea that functions map each input to one and only one output, i.e., the univalence condition. Definitions coded as close to correct included those that indicated each input has one and only one output, but were not classified as correct because they were not general enough (e.g., the definition limited a function to a particular representation, such as an equation).

In terms of *focus*, each definition was coded regarding whether the definition indicated a function was a relationship (or mapping), an object, or neither. We referred to this set of codes as *focus*, as they indicated how the students "saw" function.[1] In general, if a student identified a function with a representation or representations (e.g., "a function is an equation…"), then the definition was assigned a code of object. If the definition referred to a function as a relationship or mapping between variables or sets, it was coded as relationship. Finally, some definitions did not identify a function as an object or a relationship, but simply described some property of a function, e.g., "a function passes the vertical line test," then the definition was coded as neither.

[1] We note that our use of the term *object* differs from its meaning in the APOS framework (Asiala et al. 1996).

Although this code was intended to be mutually exclusive, there were a few definitions that identified a function as both a relationship and an object.

Finally, definitions were coded according to whether or not they *attended to output*. In order for a definition to be coded as attending to output, the definition needed to note something special or unique about the output. For example, "an equation with an input and an output" would not be considered as attending to output, while "an equation where each input has exactly one output" would. In addition, any definition which included mention of the vertical line test was coded as attending to output.

Analysis of applet engagement descriptions The participant descriptions were uploaded to a qualitative analysis tool (i.e., Atlas.ti) and coded for evidence of the occurrence of dilemmas (i.e., Mezirow 2009), triggers for those dilemmas, articulated conceptions of function that were challenged by those triggers, and articulated refinements of those conceptions that resulted as a consequence of engaging with the task.

Dilemmas were identified based on PSMTs' verbal utterances and interactions with the applet. For example, verbal utterances such as "Ok these are two cans but they seem to be the same. Does it have to be one can of coke, or two cans can still be one output? I don't know." were coded as a dilemma as the PSMT is articulating uncertainty in applying their current personal definition of function to the situation. For the same reason, engagement with the applet in which a PSMT was working with a machine and moved to a different machine without a decision on whether the former is a function was also coded as a dilemma. Each dilemma was then assigned a trigger code. Trigger codes included both a priori triggers (i.e., those that were designed for) and emergent triggers. The final set of trigger codes included not matching colors, general applet use, two cans, no cans, and two or more inputs have the same output. Finally, each dilemma was either resolved or not resolved in the PSMTs' utterances. Those that were resolved were coded as having a change in conception since the PSMT at least changed from being uncertain about an aspect of the concept even if their understanding remained incomplete.

Given the personal nature of conceptions of function, we could only code for those that were articulated explicitly. Articulated conceptions were those that were written or explicitly stated. Conceptions included any expressed beliefs, attitudes, examples, or descriptions related to functions or non-functions. For example, when examining a particular machine, a PSMT might comment, "This is like a parabola," indicating a known example of function being drawn upon to make sense of the machine depicted in the applet. All of the quotations coded for *function conception* were then open coded using a constant comparative method to identify themes (Creswell 2014). Themes for the articulated conceptions of function are shown in Table 1.

For both the written definitions and applet engagement descriptions, our process of codebook development, team coding, and determination of reliability in our code application was guided by DeCuir-Gunby et al.'s (2011) recommendations. Specifically, once our codebook was complete, a randomly selected subset of data was then coded by two team members. This process was repeated until the codes were being applied consistently by the two researchers. Once this reliability was met, all data was coded by two researchers and any disagreements were brought to the entire team to meet consensus.

Findings

In reporting our findings, we first provide the results of our analysis of the PSMTs' pre- and post-definitions of function and classification of machines in the applet. The pre- and post-definitions provide insight to the changes in PSMTs personal definition of function resulting from engaging with the Vending Machine Applet. Then, in an effort to understand the ways in which PSMTs refined (or did not refine) their definitions as they engaged with the applet, we discuss results of the analysis of descriptions of engagement with the applet. This includes the that dilemmas were triggered through engagement with the applet, conceptions of function which were challenged, and the ways in which PSMTs' conceptions of function were refined as a result of engaging with the Vending Machine task.

PSMTs' personal definitions of function

Given the applet design goal of disrupting students' current conceptions of function, we noted how many students changed their definition from pre to post (students were also given the option of not changing their definition from pre to post). The number and percentage of definitions that were classified as correct, close to correct, or incorrect, pre- and post- engagement with the applet are shown in Table 2.

While 36 of the 47 PSMTs made a change to their definition, in many cases, the post-definition did not change in terms of accuracy from the predefinition. Of those 36 that revised their definitions, 15 PSMTs improved the accuracy of their definition from pre to post, one PSMT's definition degenerated, and the rest of the definitions did not change with respect to accuracy. All 15 PSMTs whose definition improved started with incorrect definitions; three improved to a correct definition, and the other 12 moved from incorrect to close to correct. The one PSMT whose definition declined with regard to accuracy went from close to correct to incorrect.

In terms of focus, the frequencies and percentage of definitions classified as relationship, object, both, or neither is depicted in Table 3. The notable result and very important with respect to focus is that while *object* was the most common code for the predefinitions, *relationship* was the most common code for the post-definitions. This change corresponds with the improvement in accuracy noted previously.

Finally, the classification of attention to output had the most significant change from pre- to post-definition. Sixty percent ($n = 28$) attended to the output in their predefinition and 89% ($n = 42$) attended to the output in their post-definition. All of the 28 PSMTs who attended to output in their predefinition continued to do so in their post-definition, and 14 of those who did not attend to the output in their predefinition did so in their post-definition.

Overall, the data indicate that engagement with the applet resulted in positive refinements of the PSMTs' articulated conceptions of the definition function with a large number of PSMTs improving the accuracy of their personal definition, particularly in the area of attending to output. However, to capture the more fine-grained changes in PSMTs' conceptions, we conducted a thorough examination of the screencast descriptions.

PSMTs' classification of machines

The description of each vending machine was noted earlier in Fig. 3. PSMTs' classification of each machine is displayed in Table 4. The vast majority of PSMTs agreed on the classification of Machines A through E. However, note that on these pages, the directions identified that one machine on the page was a function and one was a non-function. Starting with the page that included Machines G - L, PSMTs' were asked "which of these could be functions?" Once on this page, PSMTs had differing views on the classification of the machines. To further understand these differences, we present the findings related to dilemmas that were triggered, the ways in which PSMTs negotiated these dilemmas, and associated changes in PSMTs' function conceptions.

Triggering dilemmas and changing conceptions

From our analysis of the 47 PSMTs' screencast descriptions, we identified a total of 158 dilemmas (i.e., a little over 3 per PSMT on average), with approximately 91% (43 PSMTs) articulating at least one dilemma while engaging with the applet. It is particularly noteworthy that all but one of the PSMTs who changed their definition of function to one that was more precise as a result of engaging with the applet articulated at least one dilemma. The machines that triggered these dilemmas were those that produced two cans or no can as an output, those for which different inputs produced the same output, or those for which the color of the can (output) did not match the button pressed (input). The distribution of these triggers is shown in Table 5. Of the five PSMTs that did not experience a dilemma, four of those PSMTs were all working from a definition that was similar to "each input has one output" and worked straight through the task in an unusually short amount of time compared to the other PSMTs.

Of the 158 articulated dilemmas, 124 resulted in some sort of change in a PSMTs' conception of function. In addition, all but two PSMTs that articulated a dilemma also articulated a change in their conception of function. The two PSMTs that did not articulate a change in conception of function only articulated one dilemma, and that dilemma was related to an aspect of their conception of function that they did not need to draw upon to classify the machine they were on. As a result, they did not need to reflect on it in order to classify the machine at hand. Tables 5 and 6 show the frequencies of dilemmas that resulted in changes in conception of function per machine and conception theme.

In the following sections, patterns in the ways in which challenged conceptions of function were negotiated are reported. We have chosen to focus on the three most common conception themes: elements of the codomain, many-to-one, and continuous functions.

Conceptions of elements in the codomain The most common conception of function that was challenged was that of PSMTs considering (or not) possible ways of defining the codomain, with 41 of the 47 PSMTs (87%) articulating such a dilemma. As previously mentioned, we intentionally did not state what elements made up the domain, codomain, and range for the machines in the applet. The machines that produced two cans as an output (D, E, I, K) and no can as an output (J) were designed for this purpose. We hoped that engagement with the machines that produced two cans or no can as an output would trigger a dilemma for the PSMTs and elicit such considerations.

When encountering Machines D, E, I, J, and K, PSMTs typically articulated their dilemma by questioning whether or not no can or two cans could be outputs. For example, PSMT 2's response to Silver Mist not producing an output on Machine J, "Oh, the Silver Mist has no output. [Presses Silver Mist button many times.] That's not broken, right? So, is that ok, that Silver Mist doesn't have an output?" and PSMT 6's response to two cans as an output on Machine K, "Red is red, blue is blue, silver is silver, and green is [red and green cans appear]. That is definitely not a function, because you can't have two. I guess you could have two outputs (sigh) this is very difficult, actually, to see it, not as numbers, but as drinks." Additionally, some PSMTs went beyond just verbalizing a dilemma to explicitly discussing the possible output values. One example that typifies how PSMTs explicitly discussed defining the possible codomain elements follows from PSMT 3,

> Ok these are two cans, but they seem to be the same. That's an interesting question… That's interesting what we define as output. Does it have to be one can of coke or two cans can still be one output?

All of these PSMTs clearly articulated a dilemma related to their conception of types of elements that could possibly be in the codomain.

The first machine where these dilemmas were triggered was Machine D (Red Cola → random pair), and while 20 PSMTs articulated a dilemma on this machine, only 6 resulted in a refined conception of function. This result can be attributed to the fact that making sense of the two can output was not required to classify the machine as a function or non-function. For example, after deciding that the machine was not a function because of the random nature of the outputs, PSMT 23 stated "And, I'm still hung up on this two can thing, but I really don't know why. And I haven't been able to work through it yet. So, maybe I can explore some more and get back to that." This lingering question is clear in the PSMT's written response as well (see Fig. 3).

Most changes in conceptions of function related to defining elements of the codomain occurred on Machines I, J, and K (17, 20, and 18 respectively). As PSMTs articulated their challenged conceptions related to the nature of the codomain, they either focused on the consistency of the outputs, examples of representations of functions they were familiar with, or their personal definition of function.

Consistency of outputs Most PSMTs (23 out of 41) who articulated a change in conception with respect to codomain attended to consistency as they worked to make sense of what they observed as outputs on Machines I, J, and K. For example, as PSMT 7 engaged with Machine I, they explained,

D	Not a function	Blue, silver, and green had the same properties as machine A. But the red soda button got messy. It wasn't 1-1 because there were multiple options that you could get when you pressed the button. Two cans?

Fig. 3 PSMT 23's written response for Machine D

Table 1 Articulated conceptions of function

Theme	Description	Example
Continuous	Functions (must be / do not have to be) continuous	"I'm not sure, I do not think that would be a function because of the hole. Although, I guess we could have a non-continuous domain." PSMT 13
Many-to-one	Functions (can / cannot) be many-to-one	"Oh...interesting. They all have that output. But I'm going to say it's a function, because they all have consistently the green can as the output, there was no changing." PSMT 6
Defined codomain	Functions must map elements in the domain to elements in a defined codomain	"Um… I would say yes, it is a function because even though the red button outputs two silver, the outputs are the same. So, it would be like if you have .12 and .12 again. That one's interesting. I think yeah." PSMT 17
One to one	Functions must be one to one (with or without correct meaning of one to one)	"And so, for the same reasons it's not a one-to-one correspondence. I still do not know about this two can… I really want to say that there's nothing wrong with it but I'm not sure why." PSMT 10
Matching colors	In the context of vending machine, functions must have matching input and output colors	"Oh wait! Can it be a different color? Because it's not what you pressed. Ooooo, this is hard. This is really hard. I do not know if it is the same." PSMT 11
Not expressed	Conception challenged not expressed, sometimes because some other issue allowed them to make a decision without dealing with what was initially triggered.	"I definitely think that H and J are a function. K I do not really know." [She moves on to L.] PSMT 5

> Ok, now I'm having second thoughts about these two sodas… And like, would you have Green Dew have two different arrows? So I guess it depends on how you see your output values. Are the output values just a red soda, green soda, blue soda, silver soda? Or can they be different combinations of those? … So going off of the assumption that it's going off the same output every time, then it's a function. But since it's giving you two different drinks, is it still? Hmm, I'm questioning all my thoughts now. I guess it would depend on how you classify your outputs, so if like getting two different drinks is OK, but as long as it happens every single time that you put this input in then I think it would be OK.

Table 2 Accuracy of function pre- and post-definitions

	Correct n (%)	Close to correct n (%)	Incorrect n (%)
Pre	4 (8.5)	9 (19.1)	34 (72.4)
Post	7 (14.9)	18 (38.3)	22 (46.8)

Table 3 Focus of function pre- and post-definitions

	Focus			
	Relationship n (%)	Object n (%)	Neither n (%)	Both n (%)
Pre	17 (36%)	19 (40%)	7 (15%)	4 (9%)
Post	20 (43%)	15 (32%)	9 (19%)	3 (6%)

In this example, PSMT 7 is considering elements that might be in the codomain, going beyond noting elements of the range that have been observed with the machines so far. PSMT 7 makes the point that whether or not this is a function depends on how the codomain is defined, ultimately deciding that if it was defined to include pairs of cans, then Machine I could be a function because the results of clicking on the Red Cola button are consistently two silver cans. Similar reasoning is evident in the following PSMT's explanation,

> That's not broken, right? So, is that ok, that silver mist does not have an output? … I think its ok, because it's the same output. If it gave us something one time, then I would not be ok with that. So, even though silver does not give you anything, by giving you nothing, it is consistently giving you nothing. (PSMT 30)

In each of these responses, the PSMTs articulated changed conceptions of the codomain, typically an elaborated conception that includes two cans or no cans as elements. This change is a result of considering the importance of consistency in the relationship between input and output elements. Furthermore, all of the PSMTs that attended to consistency when deliberating about how to resolve their dilemma adapted their conception of codomain to include both two and no cans.

Table 4 PSMTs' classification of each vending machine as a function or non-function

Machine	# Classified as function	# Classified as non-function
A	47	0
B	3	44
C	45	2
D	1	46
E	2	45
F	38	9
G	6	41
H	38	9
I	32	15
J	24	23
K	24	23
L	39	8

Table 5 Trigger vs. conception challenged

	2 cans	General machine use	Two or more inputs have the same output	No cans	Input and output are different colors	Total
Continuous	0	0	0	14	0	14
Define codomain	54	3	2	26	0	85
Many-to-one	2	0	24	1	0	27
Matching colors	0	2	0	0	7	9
One to one	4	0	3	1	0	8
Not expressed	4	0	1	2	0	7
Total	64	5	31	44	7	151

Using examples of familiar functions Other PSMTs worked through dilemmas in which their conceptions of the nature of codomain were challenged by drawing upon examples of familiar representations of function. This is evident in PSMT 34's work on Machine I,

> This one is the most questionable one that I'm the least certain on. We will say that is not a function. I do not know how that would really work on a graph… I think that would basically be saying if I put in a one I would get two 2 s out of that. Which is not possible.

PSMT 34 could not imagine how two cans might be represented on a graph, resulting in a changed conception of codomain in this context that did not include elements other than single cans. PSMT 11 used similar reasoning on Machine J (no can),

Table 6 Frequency of articulated dilemmas and transformed conceptions per machine

Machine	Defined codomain (dilemma; transformed conception)	Continuous (dilemma; transformed conception)	many-to-one (dilemma; transformed conception)	One to one (dilemma; transformed conception)	Matching colors (dilemma; transformed conception)
A	1; 1	0	0	0	0
B	3; 3	0	0	0	5; 2
C	0; 0	0	0	0	4; 3
D	13; 4	0	1; 0	3; 0	1; 0
E	11; 6	0	2; 1	3; 1	0
F	3; 1	0	13; 9	2; 1	0
G	2; 1	0	0	0	0
H	1; 0	0	8; 8	0	0
I	19; 15	0	0	0	0
J	21; 12	15; 8	0	1; 1	0
K	17; 14	0	2; 1	0	0
L	1; 1	0	5; 5	2; 1	0
Total	89; 56	15; 8	30; 23	11; 4	9; 5

> Silver does not give me anything. What? … I do not know, it's just the fact that it does not give me one out, if that's the reason why I do not think it's a function or…even like a linear basic function. If you put something into it, you have to come out with something.

In this example, the PSMT is trying to imagine the situation as a known function, even a "linear basic function" but is not able to do so. After declaring that they cannot think of a function that behaves this way, the PSMT goes further to state that an input must have an output to go with it. This final claim is connecting to the PSMT's conception of the definition of function—each input should map to one output.

When PSMTs drew on familiar representations to make sense of a dilemma regarding issues with the codomain, they used the familiar representations to determine if a machine was a function or non-function. This resulted in a broader conception of the codomain in this non-algebraic context by drawing on possible codomains from algebraic contexts.

Drawing on personal definition of function Drawing upon one's personal definition was somewhat common for the PSMTs when they were thinking about elements of the range and codomain. Consider PSMT 20's explanation of Machine I,

> Immediately that red button is giving you two different cans. Which is not… They are both silvers, but I take that as still two different values even if they are the same value which you cannot have. Yeah two different cans we cannot have two cans off of one… two ys off one x. Then it's not a function. Although if they are the same can… nah I still think that's not.

Similarly, PSMT 35 working on Machine J said,

> The problem is the silver, because it does not give you anything. It's like having an x that does not go to a y. And, one of the rules of functions is that every x needs a y but not every y has to have an x. So, because silver does not give you a can, this makes it not a function.

Both of these examples are evidence of PSMTs reconsidering their conceptions of codomain in such a way that the empty set is not included based on their conceptions of the univalence requirement of the definition of function.

Conceptions of many-to-one The second most common conception challenged was that of many-to-one functions. Three machines, F (Diet Blue and Silver Mist → silver, H (All buttons → green), and L (Red Cola and Diet Blue → blue and Silver Mist and Green Dew → silver) were designed to trigger PSMTs' conceptions of many-to-one functions, though a few PSMTs articulated this conception being challenged on other machines as well. Approximately 44% (21 PSMTs) experienced a dilemma related to whether many-to-one functions represented by the vending machines were indeed functions. An example from PSMT 42 that typifies these dilemmas was "But now I'm confused because Diet Blue and Silver Mist will both give me silver. So is this going to mean….? Hmm…" This PSMT was unsure how to classify Machine F since both Diet Blue and Silver Mist had an output of a silver can.

As is shown in Table 6, the number of dilemmas triggered on Machines F, H, and L lessened as PSMTs moved from one to the next. The majority (9 out of 13) of the PSMTs whose conception of many-to-one was challenged on Machine F also articulated a change in their conception. The 4 PSMTs that did not articulate a change in conception on Machine F did on Machine H. Furthermore, the 5 PSMTs that experienced a dilemma and changed conception on Machine L had not experienced a dilemma related to many-to-oneness prior.

Machine L is the last machine in the applet; thus, it gives us a good sense of where PSMTs articulated conceptions of function, related to many-to-one, stood at the conclusion of the task. Thirty-nine classified it as a function, eight as a non-function. Of the eight that classified it as a non-function, none experienced dilemmas on Machine L. This suggests their conceptions of function include the notion that functions cannot be many-to-one. This conception was either unchallenged throughout the Vending Machine task, or was changed to this end as a result of engaging with the task. In contrast, of the 30 PSMTs who articulated a dilemma related to many-to-oneness, 27 classified Machine L as a function indicating their conception of function included that functions could be many-to-one.

When examining the ways in which PSMTs negotiated their dilemmas related to many-to-one, two themes emerge: (i) PSMTs use their conceptions of function as being a rule that is consistent and (ii) they drew on examples of functions from previous experiences to work through their dilemma.

Consistency of outputs Those PSMTs that focused on the consistency of the outputs tended to test at least one button on the machine at least three times, and often more. PSMT 30's engagement on Machine F exemplifies this,

> [Student selected Red Cola and took the can three times in a row. Then, they selected Diet Blue and took the can five times in a row.] When I hit blue it is always silver. It is not like Machine E where it was different every time. [Student then selected Silver Mist followed by Take Can five times in a row and then Green Dew followed by Take Can three times in a row.] And then green is always hitting green. So, Machine F is a function… cause each selection will give you one specific can.

In this example, the connection between the PSMT's engagement with the applet (clicking the buttons on the machine multiple times) and the PSMT's language makes clear that it is the consistency in output that led the student to deciding that this was a function, and thus a situation of more than one button resulting in the same can is acceptable for a function in this context. PSMT 18 works similarly to make sense of the dilemma articulated on Machine H,

> [Student clicks each button more than once and occasionally clicks take can.] So hum… each x…Can every?… For every input there's one output. So I'm pretty sure H is a function because there is one output for every input. But can it be the same output. I'm pretty sure it can.

In this example, PSMT 18 articulated a decision that they are satisfied that a function should map each input to the same output, and uses a test for consistency to reach that conclusion.

Relying on the consistency of the outputs helped PSMTs to make sense of a dilemma regarding issues with multiple buttons on a machine producing the same output. Doing so often resulted in a new understanding that functions can be many-to-one since the machine was consistently producing the same output.

Using familiar representations When faced with the dilemma of multiple buttons resulting in the same outputs, many of the PSMTs articulated examples of known function as they grappled with classifying the machine. There were 20 instances of algebraic examples being used to make sense of machines of this type. For example, when working on Machine F, PSMT 34 explained,

> So this one is a little bit trickier because we do see that Silver Mist comes out twice. But functions are a little bit interesting that it's ok for us to have a repeated y value. So as long as it's ok if two different x values have the same y value, we just cannot have two y values have the same x value. So this one would actually be a function. You would see something like that in a simple x squared function.

Similarly, PSMT 14 explains,

> Two different colors make the same can. Ok. So maybe Machine F is possible representing like an x squared function maybe? And, the Diet Blue can could be like negative two and the Silver Mist could be positive two, but they both equal the same thing when they are squared. So I think that Machine F is a function.

In each of these examples, the PSMTs are drawing a connection between the two buttons that result in a silver can to a quadratic function in which the two opposite values in the domain when squared will result in the same value in the range to grapple with.

When working to make sense of Machine H, many PSMTs compared the machine to a horizontal line. For example, PSMT 14 stated,

> I think that this one is a function...umm...And different x's give you the same y maybe? … yeah...or maybe it's like a horizontal line, like, which is still a function. Cuz there are different x's and the outputs are all the same. So it's like, um, a horizontal line. Which is a function. So, yeah, I'm going to say function.

In this example, the PSMT mentions more than one type of function they are familiar with until relying on one that seems to make sense for the given situation, a horizontal line. In this case, the fact that on a horizontal line each x value is paired with the same y value leads to the classification of Machine H as a function.

These PSMTs encountered a dilemma regarding if two or more inputs could produce the same output in a function. Through what they articulated on their screencasts, it is clear that they knew of specific algebraic functions that have this property and they used that knowledge of those representations to expand their conceptions of function to include many-to-oneness.

Conceptions of continuous functions The final conception challenged was PSMTs' conceptions of function related to continuity, which was triggered by Machine J (Silver □ no can). As noted previously, some PSMTs articulated dilemmas on Machine J related to the way in which elements of the codomain could be defined; however, more articulated dilemmas related to their conceptions of continuity as it relates to function. This dilemma occurred for 15 PSMTs and was the one dilemma PSMTs articulated that we did not have in mind when designing the applet. PSMT 7's response to Machine J was typical for the 15 PSMTs who articulated this dilemma,

> So, apparently the Silver Mist button is not working… So you put in a value, but you get nothing out. Well, graphically, all of these would still map to something. But your Silver Mist value would be like a hole in your graph. So, would that make this not a function?

In this example, PSMT 7 is comparing the Silver Mist button to a possible value that is restricted from the domain. Though the context of a vending machine is discrete in nature, many PSMTs, like PSMT 7, articulated that it was challenging their conceptions of continuity as it relates to functions and non-functions.

Of the 15 articulated dilemmas related to conceptions of continuity, 8 resulted in an articulated a change in their conception. The 7 PSMTs that did not articulate a change in their conception all discussed continuity, yet indicated their dilemma had not been resolved. For example, PSMT 6 explains,

> I'm thinking about zeros of functions - not zeros, but asymptotes. So I guess for this I am not sure what the exact definition of function is because, I'm trying to think...although the asymptote is not part of the function, it's like a hole, right? (sighs). Would it be a function then? The rest of it is a function without Silver Mist. Is an asymptote part of a function? So, is a hole part of a function? Does the fact that a hole or an asymptote exists, does that make the expression not a function? I think that's the question that this is: does the existence of a hole make an expression not a function?

In this example, PSMT 6 is drawing on knowledge of discontinuities in known functions to try to resolve the dilemma, but ends by stating open questions and does not articulate any decision regarding conceptions of discontinuity as it relates to this context.

Of the 7 PSMTs that did articulate a refined in conception related to continuity, all of them compared the Silver Mist button, resulting in no can, to a function with a restricted domain. For example, PSMT 34 explained "Basically the silver would just be were an asymptote would be. There are one or fewer outputs for each input and the Silver Mist would simply be a hole or an asymptote." In this example, the PSMT indicated that Machine J could be a function if Silver Mist was restricted from the domain. Several PSMTs provided similar explanations, using examples in their explanations that included piecewise functions, rational functions, and log functions. For example, PSMT 41 explained "It's kind of like a piecewise function. That's why I'm counting it as a function. Because piecewise functions not all x values have an output. So I'm going to say it's a yes." Here, the PSMT points to a specific type of function in which it is typical to restrict the domain.

While some PSMTs referenced specific types of functions in which it is typical to restrict the domain, others focused on the graphs of these functions in which either asymptotes or holes would be apparent. For example, PSMT 14 compared Machine J to a rational function,

> Maybe this function is like, like in a fraction form and silver represents zero. And in the fraction x is on the bottom and zero cannot go on the bottom, so its undefined. So maybe that's what it's trying to say? Cuz I feel like that's the kind of function it is.

Here, the PSMT is envisioning a rational function with x in the denominator as a comparison. In contrast, PSMT 16 explained,

> So Machine J when you hit Silver Mist it does not give you a can. So it does not give you an output. So there is an asymptote or a hole or something like that in the function. But there can still be a function, even though there is like a hole or something like that. There is still a function. It would still pass the vertical line test.

While it is not clear what function family PSMT 20 might be envisioning, it is clear that they are envisioning one in which the domain must be restricted resulting in a graph with an asymptote or a hole. In addition, while envisioning such a graph the PSMT argues, it is possible to have such a situation occur and the graph still pass the vertical line test.

Not all PSMTs for whom conceptions of continuity were challenged on Machine J decided that Machine J could be a function if one considered restricting the domain. One exception is PSMT 21 who explained "Silver is not giving me, so that must be there is a hole. A function has to be continuous to be considered one, so Machine J is not a function." This PSMT was the only one to articulate a change in conception of continuity with respect to function, i.e., the only one who clearly stated that if one must restrict the domain, it is not a function. However, as was noted previously, many of those PSMTs who did not articulate a changed conception were struggling with this same issue.

Discussion

At the beginning of the task, PSMTs articulated conceptions of function that we expected based on the literature. PSMTs' struggled to articulate a complete definition of function, their focus often being on objects rather than relationships or mappings. This aligns with previous research on students' understanding that functions are defined as particular algebraic representations (e.g., Breidenbach et al.1992; Carlson 1998; Even 1993). As future teachers of the function concept, perhaps the most concerning issue in PSMTs' articulated predefinition was that 59% did not note something special or unique about the output. This finding is consistent with Even's (1993) study, and suggests that prior to engaging with the applet, these PSMTs did not know why the univalence requirement is important in distinguishing functions and non-functions. However, as a result of engaging with the Vending Machine Applet, 89% attended to output in their post-definitions.

Looking beyond their definitions to the ways in which they engaged with the applet itself provided considerable insight to the PSMTs' conceptions of functions and the ways in which they were both challenged and refined. For example, the PSMTs articulated dilemmas related to their conceptions of univalence (e.g., Even 1993; Vinner and Dreyfus 1989), many to one (e.g., Carlson 1998; Rasmussen 2000), and continuity (Bezuidenhout 2001; Tall and Vinner 1981). With the exception of continuity (which we did not consider due to the discrete nature of the vending machine context), we designed the applet to trigger dilemmas related to each of these conceptions.

Using the applet proved effective in mitigating each of these misconceptions. One of our most promising findings is that while 44% of PSMTs experienced a dilemma related to many-to-oneness, by the conclusion of the Vending Machine task, 83% indicated explicitly that functions can be many-to-one. Furthermore, 41 of the 47 PSMTs (87%) experienced a dilemma related to their conceptions of codomain. Simply provoking them to critically reflect on the role of domain, range, and codomain is useful, and the fact that 76% (31 out of those 41) were able to articulate the ways in which the ways elements in the domain and codomain are defined would impact the ways in which the machines would be classified, is a significant result. We were surprised that 32% of PSMTs (15 of the 47) used continuity to make sense of the discrete context of the vending machines; however, in doing so, we gained insight to their conceptions of the relationships between continuity, restricted domains, and function/non-function.

In addition, our use of a function machine, in the form of vending machines, as a cognitive root (Tall et al. 2000) was designed to provide an accessible and meaningful context for the PSMTs.

We consider there to be two main reasons for the effectiveness of the applet: (i) the technology and (ii) the nature of the non-algebraic representation. The use of technology allowed us to create a task with which the PSMTs could interact independently and which, with the feedback of the machine outputs, allowed them to formulate conjectures as they worked, and to test those conjectures without having to wait for a class discussion or intervention from an instructor. One of the persistent problems noted in the literature is privileging algebraic representations (Even 1990, 1993; Wilson 1994), putting PSMTs in the context of the vending machines appears to have mitigated this problem to some degree. The use of the vending machine as a cognitive root (Tall et al. 2000) proved to be an accessible and meaningful contest for the PSMTs. In addition, our findings are consonant with the work of (Dubinsky and Wilson 2013; Sinclair et al. 2009) on non-standard representations of function discussed in the literature. Overall, the Vending Machine task was successful in triggering dilemmas and supporting PSMTs' refinement of their conceptions of function.

Limitations of the study

Although we achieved some promising results, there are limitations to the study. The principal limitation is the short timeline of the study. The PSMTs wrote a definition, then interacted with the applet for anything between 7 and 25 min, and then wrote a revised definition. Therefore, while there were clear improvements in the definitions, it is difficult to say how robust and long-lasting those changes will prove to be. While the PSMTs did not all express complete and correct definitions of function at the conclusion of this assignment, we believe that it did prime them for a meaningful whole class discussion focused on the function concept in which the hope is that their conceptions would be refined further.

Other limitations to this study include access to PSMTs conceptions of function and the dilemmas they may have encountered when engaging with the Vending Machine Applet. Given that conceptions are the personal side of a concept (Sfard 1991), though they are conscious (Pehkonen and Pietilä 2004), they are often internal, meaning we (the researchers) only have access to those that are explicitly articulated verbally or in writing. As such, we likely only had access to a subset of the PSMTs conceptions of function. Even so, those they did articulate explicitly provide great insight to PSMTs conceptions and the consistency among both articulated conceptions and refinements to them suggest that findings here may be generalizable—and are definitely worthy of further study.

Conclusions

Research has found that, in order to best serve their students, it is important for PSMTs have a robust conception of function, know variations in the definition of function, develop the ability to translate among different representations of functions, and know when to use each definition based on context (Bannister 2014; Hatisaru and Erbas 2017). This specialized content knowledge is needed to understand and plan for the diverse student conceptions they will encounter during instruction related to functions and concept of function. While there is a vast literature base on the limited conceptions of functions PSMTs often develop through high school and undergraduate mathematics, little is known about how to refine them after years of building on them in algebraic contexts. The results of this study indicate that by removing PSMTs from familiar function contexts, such as algebraic, and designing to trigger dilemmas based on conceptions identified in the literature, we can shift PSMTs' conceptions of function in a positive direction.

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References

Asiala, M., Brown, A., DeVries, D., Dubinsky, E., Mathews, D., & Thomas, K. (1996). A framework for research and curriculum development in undergraduate mathematics education. In A. Schoenfeld, J.

Kaput, & E. Dubinsky (Eds.), Research in collegiate mathematics education II, CBMS issues in mathematics education (pp. 1–32). American Mathematical Society.
Australian Curriculum Assessment and Reporting Authority (ACARA). (2020). The Australian Curriculum: mathematics, all curriculum elements. Retrieved from http://www.australiancurriculum.edu.au/download/f10
Ayalon, M., Watson, A., & Lerman, S. (2017). Students' conceptualisations of function revealed through definitions and examples. *Research in Mathematics Education, 19*(1), 1–19. https://doi.org/10.1080/14794802.2016.1249397.
Bannister, V. R. P. (2014). Flexible conceptions of perspectives and representations: an examination of pre-service mathematics teachers' knowledge. *International Journal of Education in Mathematics, Science and Technology, 2*(3), 223–233.
Bardini C. Pierce R. & Vincent J. (2013). First year university students' understanding of functions: over a decade after the introduction of CAS in Australian high schools what is new. In Proceedings of the 9th DELTA Conference on Teaching and Learning of Undergraduate Mathematics and Statistics (pp. 2-11).
Bardini, C., Pierce, R., Vincent, J., & King, D. (2014). Undergraduate mathematics students' understanding of the concept of function. *Indonesian Mathematical Society Journal on Mathematics Education, 5*(2), 85–107.
Best, M., & Bikner-Ahsbahs, A. (2017). The function concept at the transition to upper secondary school level: tasks for a situation of change. *ZDM, 48*, 865–880. https://doi.org/10.1007/s11858-017-0880-6.
Bezuidenhout, J. (2001). Limits and continuity: some conceptions of first-year students. *International Journal of Mathematical Education in Science and Technology, 32*, 487–500.
Blanton, M., Brizuela, B. M., Gardiner, A. M., Sawrey, K., & Newman-Owens, A. (2015). A learning trajectory in 6-year-olds' thinking about generalizing functional relationships. *Journal for Research in Mathematics Education, 46*(5), 511–558.
Breidenbach, D., Dubinsky, E., Hawks, J., & Nichols, D. (1992). Development of the process conception of function. *Educational Studies in Mathematics, 23*(3), 247–285.
Carlson, M. P. (1998). A cross-sectional investigation of the development of the function concept. In E. Dubinsky, A. H. Schoenfeld, & J. J. Kaput (Eds.), Research in collegiate mathematics education, III, Issues in mathematics education, 7(1), 115–162. American Mathematical Society.
Carlson, M., & Oehrtman, M. (2005). *Key aspects of knowing and learning the concept of function. Research Sampler Series*. Washington, DC: Mathematical Association of America.
Carlson, M. P., Smith, N., & Persson, J. (2003). Developing and connecting Calculus students' notions of rate-of change and accumulation: the fundamental theorem of Calculus. *International Group for the Psychology of Mathematics Education, 2*, 165–172.
Chesler, J. (2012). Pre-service secondary mathematics teachers making sense of definitions of functions. *Mathematics Teacher Education and Development, 14*(1), 27–40.
Cooney, T. J., Beckman, S., & Lloyd, G. M. (2010). *Developing essential understanding of functions for teaching mathematics in grades 9–12*. Reston, VA: National Council of Teachers of Mathematics.
Confrey, J. (1990). What constructivism implies for teaching. In R. B. Davis, C. A. Maher, & N. Noddings (Eds.), Journal for Research in Mathematics Education Monograph number 4: Constructivist views on the teaching and learning of mathematics (pp. 107–122). Reston, VA: National Council of Teachers of Mathematics.
Cooney, T. J., & Wilson, M. R. (1993). Teachers' thinking about functions: historical and research perspectives. In T. A. Romberg, E. Fennema, & T. P. Carpenter (Eds.), *Integrating research on the graphical representation of functions* (pp. 131–158). Hillsdale, NJ: Erlbaum.
Creswell, J. (2014). *Research design : qualitative, quantitative, and mixed methods approaches* (4th ed.). Thousand Oaks, California: SAGE Publications.
Department for Education (2014) *The national curriculum in England: complete framework for key stages 1 to 4*. Available at: https://www.gov.uk/government/publications/national-curriculum-in-england-framework-for-key-stages-1-to-4 (Accessed: 20 April 2020).
DeCuir-Gunby, J. T., Marshall, P. L., & McCulloch, A. W. (2011). Developing and using a codebook for the analysis of interview data: an example from a professional development research project. *Field Methods, 23*(2), 136–155.
Doerr, H. M. (2004). Teachers' knowledge and the teaching of algebra. In K. Stacey, H. Chick, & M. Kendal (Eds.), *The future of the teaching and learning of algebra: the 12th ICMI study* (pp. 267–290). Boston: Kluwer Academic.
Drijvers, P. (2015). Digital technology in mathematics education: why it works (or doesn't). In J. S. Cho (Ed.), *Selected regular lectures from the 12th International Congress on Mathematical Education* (pp. 135–151). Cham: Springer.

Dubinsky, E., & Harel, G. (1992). *The concept of function: aspects of epistemology and pedagogy*. Washington, D.C.: Mathematical Association of America.

Dubinsky, E., & Wilson, R. T. (2013). High school students' understanding of the function concept. *The Journal of Mathematical Behavior, 32*(1), 83–101.

Ellis, Amy B. Algebra in the middle school: developing functional relationships through quantitative reasoning. *Early algebraization*. Springer, Berlin, Heidelberg, 2011. 215–238.

Even, R. (1990). Subject matter knowledge for teaching and the case of functions. *Educational Studies in Mathematics, 21*, 521–544.

Even, R. (1993). Subject-matter knowledge and pedagogical content knowledge: prospective secondary teachers and the function concept. *Journal for Research in Mathematics Education, 24*(2), 94–116.

Hatisaru, V., & Erbas, A. K. (2017). Mathematical knowledge for teaching the function concept and student learning outcomes. *International Journal of Science and Mathematics Education, 15*, 703–722. https://doi.org/10.1007/s10763-015-9707-5.

Leinhardt, G., Zaslavsky, O., & Stein, M. K. (1990). Functions, graphs, and graphing: tasks, learning, and teaching. *Review of Educational Research, 60*(1), 1–64.

Marsick, V. J., & Watkins, K. E. (2001). Informal and incidental learning. *New Directions for Adult and Continuing Education, 89*, 25–34.

Martínez-Planell, R., & Gaisman, M. T. (2012). Students' understanding of the general notion of a function of two variables. *Educational Studies in Mathematics, 81*(3), 365–384.

McGowen, M. DeMarois, P., & Tall, D. (2000). Using the function machine as a cognitive root. In M. L. Fernandez (Ed.), Proceedings of the 22nd Annual Meeting of the North American Chapter of the International Group for the Psychology of Mathematics Education (Vol. 1, pp. 247–254).

Mesa, V. (2004). Characterizing practices associated with functions in middle school textbooks: An empirical approach. *Educational studies in mathematics, 56*(2-3), 255-286.

Mezirow, J. (2000). Learning to think like an adult: core concepts of transformation theory. In J. Merizow & Associates (Ed.), Learning as transformation: critical perspectives on a theory in progress (pp. 3–34). San Francisco, CA: Jossey-Bass.

Mezirow, J. (2009). Transformative learning theory. In J. Merizow & E. W. Taylor (Eds.), *Transformative learning in practice: insights from community, workplace, and higher education* (pp. 18–31). San Francisco: Jossey-Bass.

Ministry of Education. (2007). *The New Zealand Curriculum*. Wellington: Learning Media.

National Governors Association Center for Best Practice & Council of Chief State School Officers. (2010). *Common core state standards for mathematics*. Washington DC: Author.

Ng, S. F. (2018). Function tasks, input, output, and the predictive rule: how some Singapore primary children construct the rule. In *Teaching and Learning Algebraic Thinking with 5-to 12-Year-Olds* (pp. 167–193). Springer, Cham.

Panaoura, A., Michael-Chrysanthou, P., Gagatsis, A., Elia, I., & Philippou, A. (2017). A structural model related to the understanding of the concept of function: definition and problem solving. *International Journal of Science and Mathematics Education, 15*, 723–740. https://doi.org/10.1007/s10763-016-9714-1.

Pehkonen, E., & Pietilä, A. (2004). On relationships between beliefs and knowledge in mathematics education. In M. Mariotti (Ed.), *Proceedings of the Third Congress of European Society for Research in mathematics education (CD/ROM)*. Italy: University of Pisa Retrieved from: http://www.dm.unipi.it/~didattica/CERME3/proceedings/Groups/TG2/TG2_pehkonen_cerme3.pdf.

Rasmussen, C. L. (2000). New directions in differential equations: a framework for interpreting students' understandings and difficulties. *Journal of Mathematical Behavior, 20*, 55–87.

Sfard, A. (1991). On the dual nature of mathematical conceptions: reflections on the processes and objects as different sides of the same coin. *Educational Studies in Mathematics, 22*(1), 1–36.

Sherman, M., Lovett, J.N., McCulloch, A.W., Edgington, C.E., Dick, L.K, & Casey, S. (2018). Transforming students' definitions of function using a vending machine applet. In A. Weinberg, C. Rasmussen, J. Rabin, M. Wawro, & S. Brown (Eds.), *Proceedings of the 21st Annual Conference on Research in Undergraduate Mathematics Education* (pp. 752–760). San Diego, CA: The Special Interest Group of the Mathematical Association of America (SIGMAA) for Research in Undergraduate Mathematics Education.

Sinclair, N., Healy, L., & Sales, C. O. R. (2009). Time for telling stories: narrative thinking with dynamic geometry. *ZDM, 41*(4), 441–452.

Stephens, A. C., Fonger, N., Strachota, S., Isler, I., Blanton, M., Knuth, E., & Murphy Gardiner, A. (2017). A learning progression for elementary students' functional thinking. *Mathematical Thinking and Learning, 19*(3), 143–166.

Tabach, M., & Nachlieli, T. (2015). Classroom engagement towards using definitions for developing mathematical objects: the case of function. *Educational Studies in Mathematics, 90*, 163–187. https://doi.org/10.1007/s10649-015-9624-0.
Tall, D., McGowen, M., & DeMarois, P. (2000). The function machine as a cognitive root for the function concept. In M. L. Fernandez (Ed.), Proceedings of the 22nd Annual Meeting of the North American Chapter of the International Group for the Psychology of Mathematics Education (Vol. 1, pp. 255-261).
Tall, D., & Vinner, S. (1981). Concept image and concept definition in mathematics with particular reference to limits and continuity. *Educational Studies in Mathematics, 12*, 151–169.
Thompson, P. W. (1994). Students, functions, and the undergraduate curriculum. In E. Dubinsky, A. H. Schoenfeld, & J. J. Kaput (Eds.), Research in Collegiate Mathematics Education, 1 (Issues in Mathematics Education, Vol. 4, pp. 21–44). Providence, RI: American Mathematical Society.
Thompson, P. W., & Carlson, M. P. (2017). Variation, covariation, and functions: foundational ways of thinking mathematically. In J. Cai (Ed.), *Compendium for research in mathematics education* (pp. 421–456). Reston, VA: National Council of Teachers of Mathematics.
Vinner, S., & Dreyfus, T. (1989). Images and definitions for the concept of function. *Journal for Research in Mathematics Education, 20*(4), 356–366.
Wilson, M. R. (1994). One preservice secondary teachers' understanding of function: the impact of course integrating mathematical content and pedagogy. *Journal for Research in Mathematics Education, 25*(4), 346–370.

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