

Inverse Scattering Transform for the Defocusing Manakov System with Non-Parallel Boundary Conditions at Infinity

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Abstract. The inverse scattering transform (IST) for the defocusing Manakov system is developed with non-zero boundary conditions at infinity comprising non-parallel boundary conditions — i.e., asymptotic polarization vectors. The formalism uses a uniformization variable to map two copies of the spectral plane into a single copy of the complex plane, thereby eliminating square root branching. The “adjoint” Lax pair is also used to overcome the problem of non-analyticity of some of the Jost eigenfunctions. The inverse problem is formulated in term of a suitable matrix Riemann-Hilbert problem (RHP). The most significant difference in the IST compared to the case of parallel boundary conditions is the asymptotic behavior of the scattering coefficients, which affects the normalization of the eigenfunctions and the sectionally meromorphic matrix in the RHP. When the asymptotic polarization vectors are not orthogonal, two different methods are presented to convert the RHP into a set of linear algebraic-integral equations. When the asymptotic polarization vectors are orthogonal, however, only one of these methods is applicable. Finally, it is shown that, both in the case of orthogonal and non-orthogonal polarization vectors, no reflectionless potentials can exist, which implies that the problem does not admit pure soliton solutions.

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1. Introduction

This work is concerned with the defocusing Manakov system — i.e. the two-component defocusing nonlinear Schrödinger equation, written as

$$i\mathbf{q}_t + \mathbf{q}_{xx} - 2\nu\|\mathbf{q}\|^2\mathbf{q} = \mathbf{0} \quad (1.1)$$

with $\nu = 1$ and non-zero boundary conditions (NZBC) at infinity, namely

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$$\lim_{x \rightarrow \pm\infty} \mathbf{q}(x, t) = \mathbf{q}_{\pm}. \quad (1.2)$$

Here, $\mathbf{q} = \mathbf{q}(x, t) = (q_1, q_2)^T$ is a two-component vector, $\|\cdot\|$ is the standard Euclidean norm, $\|\mathbf{q}_{\pm}\| = q_o > 0$, and subscripts x and t denote partial differentiation. Throughout, asterisk denotes complex conjugation, and superscripts T and $\dagger$ denote, respectively, matrix transpose and Hermitian conjugate — i.e., conjugate transpose. The trivial space-independent phase rotation $\mathbf{q}'(x, t) = \mathbf{q}(x, t) e^{2i\nu q_o^2 t}$ maps (1.1) into

$$i\mathbf{q}'_t + \mathbf{q}'_{xx} - 2\nu(\|\mathbf{q}'\|^2 - q_o^2)\mathbf{q}' = \mathbf{0}. \quad (1.3)$$

The asymptotic values $\mathbf{q}'_{\pm} = \mathbf{q}_{\pm} e^{2i\nu q_o^2 t}$ are then independent of time, as long as $\mathbf{q}'_{xx}$ vanishes as $x \rightarrow \pm\infty$. Hereafter we will work with (1.3), but we will omit primes for brevity.

The scalar — i.e., one-component reduction of (1.3) is of course the celebrated nonlinear Schrödinger (NLS) equation. The NLS is a fundamental physical system, since it is a universal model governing the time evolution of quasi-monochromatic, weakly nonlinear dispersive wave trains [9, 17]. As such, it arises in a bewildering variety of physical applications, ranging from deep water waves, to optics, acoustics, condensed matter (Bose-Einstein condensates) and beyond [5, 6, 30, 39]. The Manakov system (1.3) is also a physically relevant model, governing the time evolution of coupled quasi-monochromatic waves in optics as well as Bose-Einstein condensates.

It is well known that the scalar NLS equation is a completely integrable system, as shown by Zakharov and Shabat [49] (see also [3, 5, 38]). Shortly afterwards, it was shown in [37] that the two-component generalization of the NLS equation, namely the system (1.3), is also integrable, and that the initial-value problem can be solved by way of the inverse scattering transform (IST). However, only the case of localized initial conditions, namely $q_o = 0$, was studied initially. The IST for the scalar defocusing nonlinear Schrödinger equation with NZBC at infinity — i.e., $q_o \neq 0$, was also formulated early on by Zakharov and Shabat [50] (see also [15, 22, 23, 25]), but the generalization to the Manakov system remained an open problem for many decades.

Following some earlier results [27], a full formulation of the IST for the defocusing Manakov system with NZBC was finally presented in [40]. The problem was then revisited in [12] and generalized to defocusing coupled nonlinear Schrödinger systems with more than two components in [13, 41]. The IST for the focusing two-component case with NZBC was also formulated in [35].

In all of the above works, however, including those on the defocusing Manakov system, only the special case in which $\mathbf{q}_{\pm} = \mathbf{q}_o e^{i\theta_{\pm}}$, with $\theta_{\pm} \in \mathbb{R}$ was solved. We refer to this as the case of parallel NZBC. In this work we generalize the IST (and in particular the inverse problem) to the case of non-parallel NZBC, namely arbitrary vectors $\mathbf{q}_{\pm}$ subject to the only constraint $\|\mathbf{q}_{\pm}\| = q_o$. (To avoid confusion, we should emphasize that a large part of the formulation of the direct problem in [12, 35, 40] also carries over directly to the more general non-parallel case. However, the formulation of the inverse problem in those works does not, and indeed the requirement that $\mathbf{q}_+$ and $\mathbf{q}_-$ are parallel plays a key role in the normalization of the Riemann-Hilbert problem in those works.)

The motivation for considering non-parallel NZBC is twofold. On one hand, as we will see, lifting the constraint that $\mathbf{q}_+$ and $\mathbf{q}_-$ are parallel is nontrivial, and introduces various difficulties in the formulation of the IST. This is especially true in what we refer to as the case of orthogonal NZBC, namely $\mathbf{q}_-^\dagger \mathbf{q}_+ = 0$. (To distinguish it from the case of orthogonal NZBC, the case of generic non-parallel NZBC will be referred to as non-orthogonal.) Indeed, we will see that in both cases (meaning, both that of orthogonal and non-orthogonal non-parallel NZBC) it is necessary to introduce a different formulation of the inverse problem compared to that in [12, 40]. We will also see that, while two different formulations are possible in the non-orthogonal case, only one of them can also be extended to the orthogonal case.

Moreover, and most importantly, the above are not merely technical complications, but are instead a reflection of actual differences in the behavior of solutions between the parallel and non-parallel cases. In particular, we will show that no pure reflectionless solutions can exist if $\mathbf{q}_-$ and $\mathbf{q}_+$ are not parallel. This is similar to what happens for the scalar focusing and defocusing NLS equations with asymmetric NZBC [11, 16, 42]. As in those cases, the absence of pure soliton solutions has dramatic consequences on the dynamics. In particular, it means that no analogue of the soliton resolution conjecture can hold for these systems. Roughly speaking, the conjecture asserts that any reasonable — e.g., bounded energy-solution to such equations eventually resolves into a nonlinear superposition of a radiation component (behaving like a solution to the linear Schrödinger equation) plus a finite number of nonlinear bound states or solitons. This conjecture is known to be true in many perturbative cases — e.g., when the solution is close to a special solution, such as the vacuum state or a ground state — as well as in defocusing cases with zero boundary conditions (in which no non-trivial bound states or solitons exist), but it is still essentially an open question whether it holds in non-perturbative situations (in which the solution is large and not close to a special solution) with at least one bound state [43–45].

The second reason for considering the Manakov system (1.3) with non-parallel NZBC is that this enables one to study physically interesting scenarios. The unit-magnitude complex two-component vector $\mathbf{p}(x, t) = \mathbf{q}(x, t)/\|\mathbf{q}(x, t)\|$ encodes the instantaneous state of polarization of the system governed by (1.3). (More precisely, $\mathbf{p}(x, t)$ defines the so-called Jones vector. One can equivalently describe the state of polarization using the associated real three-component Stokes vector [19, 28].) Therefore, solutions in which $\mathbf{q}_+$ and $\mathbf{q}_-$ are not parallel describe system configurations containing a transition between non-parallel asymptotic states of polarization. Such configurations can easily be realized experimentally. In optical fibers, for example, it is relatively straightforward to use polarization controllers to switch the state of polarization of an input beam from one value to another. Similarly, one can consider situations in which a two-component Bose-Einstein condensate is spatially divided into halves with different properties. Indeed, some of these situations have been the subject of recent studies [31–33, 36], where polarization waves were analyzed using Whitham modulation theory. Therefore, we expect that the results of this work provide a key tool that will enable researchers to quantitatively characterize the resulting dynamics.

This work is organized as follows. Sections 2 and 3 present the direct problem. In

particular, in Section 2 we introduce the Jost eigenfunctions, the scattering matrix and we discuss their properties, while in Section 3 we formulate the direct problem and discuss the behavior of eigenfunctions and scattering coefficients at the branch points. Section 4 presents the inverse problem. Specifically, in Sections 4.1 and 4.2 we formulate the Riemann-Hilbert problem in the non-orthogonal and orthogonal cases respectively, and we show that, in both cases, no reflectionless potentials exist. Finally, in Section 5 we offer some concluding remarks.

2. Direct Problem: Jost Solutions, Scattering Data and Their Properties

We now begin presenting the formulation of the IST for the Manakov system (1.3) in the case of non-parallel NZBC, namely (1.2) with

$$0 \leq |\mathbf{q}_+^\dagger \mathbf{q}_-|/q_o^2 < 1$$

(as opposed to the parallel case treated in [12, 40], for which $|\mathbf{q}_+^\dagger \mathbf{q}_-| = q_o^2$). The case of orthogonal NZBC corresponds to $\mathbf{q}_+^\dagger \mathbf{q}_- = 0$. Specifically, in this section we begin discussing the direct problem. The formulation of the direct problem largely follows that in [12, 40], with one notable exception: The normalization of the Jost eigenfunctions, as discussed in detail in Section 2.2.

2.1. Preliminaries: Lax pair and uniformization variable

The system (1.3) is associated with the following Lax pair:

$$\phi_x = \mathbf{X} \phi, \quad \phi_t = \mathbf{T} \phi, \quad (2.1)$$

where

$$\begin{aligned} \mathbf{X}(x, t, k) &= -ik\mathbf{J} + \mathbf{Q}, \\ \mathbf{T}(x, t, k) &= 2ik^2\mathbf{J} - i\mathbf{J}(\mathbf{Q}_x - \mathbf{Q}^2 + q_o^2) - 2k\mathbf{Q}, \\ \mathbf{J} &= \begin{pmatrix} 1 & \mathbf{0}^T \\ \mathbf{0} & -\mathbf{I}_2 \end{pmatrix}, \quad \mathbf{Q}(x, t) = \begin{pmatrix} 0 & \mathbf{q}^\dagger \\ \mathbf{q} & \mathbf{0}_2 \end{pmatrix}, \end{aligned}$$

$\mathbf{I}_n$ and $\mathbf{0}_n$ are the $n \times n$ identity matrix and zero matrix, respectively. That is, (1.3) is the compatibility condition for which $\phi_{xt} = \phi_{tx}$ (as is easily verified by direct calculation and noting that $\mathbf{J}$ and $\mathbf{Q}$ anticommute, namely, $\mathbf{J}\mathbf{Q} + \mathbf{Q}\mathbf{J} = 0$). As usual, the first half of (2.1) is referred to as the scattering problem. In the development of the IST, we take $\phi(x, t, k)$ to be a 3×3 matrix. Moreover, unlike previous works on the IST for defocusing NLS and Manakov system with NZBC, we formulate the IST in a way that allows the reduction $q_o \rightarrow 0$ to be taken explicitly throughout.

The asymptotic scattering problem as $x \rightarrow \pm\infty$ is

$$\phi_x = \mathbf{X}_\pm \phi, \quad (2.2)$$

where $\mathbf{X}_\pm = -ik\mathbf{J} + \mathbf{Q}_\pm = \lim_{x \rightarrow \pm\infty} \mathbf{X}$. The eigenvalues of $\mathbf{X}_\pm$ are ik and $\pm i\lambda$, where

$$\lambda(k) = (k^2 - q_o^2)^{\frac{1}{2}}. \quad (2.3)$$

Like in the scalar case [50], these eigenvalues have branching. To deal with this, as in [12, 25, 40], we introduce the two-sheeted Riemann surface defined by (2.3). The branch points are the values of k for which $\lambda(k) = 0$, i.e. $k = \pm q_o$. Similarly to [12, 40], we take the branch cut on $(-\infty, -q_o] \cup [q_o, \infty)$, and we define $\lambda(k)$ so that $\Im \lambda \geq 0$ on sheet I and $\Im \lambda(k) \leq 0$ on sheet II (see [40] for further details). Next, we introduce the uniformization variable as in [25] via the conformal map (Joukowski transformation) $z = k + \lambda$, whose inverse is

$$k = \frac{1}{2} \left(z + \frac{q_o^2}{z} \right), \quad \lambda = \frac{1}{2} \left(z - \frac{q_o^2}{z} \right).$$

We can then express all k -dependence of eigenfunctions and scattering data (including the one resulting from λ) in terms of z , thereby eliminating all square roots. The branch cuts on the two sheets of the Riemann surface are mapped onto the real z -axis; $\mathbb{C}_I$ is mapped onto the upper half plane of the complex z -plane; $\mathbb{C}_{II}$ is mapped onto the lower half plane of the complex z -plane; $z(\infty_I) = \infty$ if $\Im k > 0$ and $z(\infty_I) = 0$ if $\Im k < 0$; $z(\infty_{II}) = 0$ if $\Im k > 0$ and $z(\infty_{II}) = \infty$ if $\Im k < 0$; $z(k, \lambda_I)z(k, \lambda_{II}) = q_o^2$; $|k| \rightarrow \infty$ in the upper half plane of $\mathbb{C}_I$ corresponds to $z \rightarrow \infty$ in the upper half z -plane; $|k| \rightarrow \infty$ in the lower half plane of $\mathbb{C}_{II}$ corresponds to $z \rightarrow \infty$ in the lower half z -plane; $|k| \rightarrow \infty$ in the lower half plane of $\mathbb{C}_I$ corresponds to $z \rightarrow 0$ in the upper half z -plane; and $|k| \rightarrow \infty$ in the upper half plane of $\mathbb{C}_{II}$ corresponds to $z \rightarrow 0$ in the lower half z -plane. Finally, the segments $k \in [-q_o, q_o]$ on each sheet correspond, respectively, to the upper half and lower half of the circle C_o of radius q_o centered at the origin in the complex z -plane. Throughout this work, subscripts $\pm$ will denote normalization as $x \rightarrow -\infty$ or as $x \rightarrow \infty$, respectively, whereas superscripts $\pm$ will denote analyticity (or, more in general, meromorphicity) in the upper or lower half of the z -plane, respectively.

2.2. Jost solutions and scattering matrix

As usual, the continuous spectrum of the scattering problem consists of all values of k (in either sheet) such that $\lambda(k) \in \mathbb{R}$. That is, the continuous spectrum is the set $k \in \mathbb{R} \setminus (-q_o, q_o)$. Because of the branching structure of $\lambda(k)$, the corresponding set in the complex z -plane is the whole real z -axis. (That is, one has two copies of the continuous spectrum, corresponding to the limiting values of λ from above and below the cut, which are respectively being mapped to the sets $z \in (-\infty, -q_o) \cup (q_o, \infty)$ and $z \in (-q_o, q_o)$).

For any two-component complex-valued vector $\mathbf{v} = (v_1, v_2)^T$, we define its orthogonal vector as $\mathbf{v}^\perp = (v_2, -v_1)^\dagger$, so that $\mathbf{v}^\dagger \mathbf{v}^\perp = (\mathbf{v}^\perp)^\dagger \mathbf{v} = 0$, as in [12]. (This definition differs from that of Ref. [40].) We may then write the eigenvalue and the eigenvector matrices of the asymptotic scattering problem (2.2) respectively as

$$i\Lambda(z) = \text{diag}(-i\lambda, ik, i\lambda), \quad \mathbf{E}_\pm(z) = \gamma(z) \begin{pmatrix} 1 & 0 & -iq_o/z \\ iq_\pm/z & \mathbf{q}_\pm^\perp/(\gamma(z)q_o) & \mathbf{q}_\pm/q_o \end{pmatrix} \quad (2.4)$$

with

$$\gamma(z) = \frac{z^2}{z^2 - q_o^2}, \quad (2.5)$$

so that

$$\mathbf{X}_\pm \mathbf{E}_\pm = \mathbf{E}_\pm i\Lambda.$$

Note this normalization differs from the one used in [12] for the parallel case. More precisely, the matrix $\mathbf{E}_\pm(z)$ defined by (2.4) is related to the eigenvector matrix in [12] as

$$\mathbf{E}_\pm(x, t, z) = \mathbf{E}_\pm^{[\text{bk}]}(x, t, z) \mathbf{G}(z), \quad (2.6)$$

where $\mathbf{G}(z) = \text{diag}(z/2\lambda, 1, z/2\lambda)$ and $\mathbf{E}_\pm^{[\text{bk}]}(x, t, z)$ denotes the asymptotic eigenvector matrix in [12].

The reason for employing a different normalization here is that the matrix $\mathbf{E}_\pm^{[\text{bk}]}$ has a pole at $z = 0$ and as a consequence some of the reflection coefficients diverge as $z \rightarrow \infty$ and/or as $z \rightarrow 0$ when $\mathbf{q}_+$ and $\mathbf{q}_-$ are not parallel. The new normalization allows us to better handle this issue, although the price to pay is that the new eigenvector matrices have poles at $z = \pm q_o$.

One could employ the invariances of the Manakov system to fix the asymptotic polarization vectors $\mathbf{q}_\pm/q_o$ so as to obtain a simpler eigenvector matrix. (The transformation of the Jost solutions and scattering matrix under each of the invariances of the Manakov system is discussed in [12, Appendix].) However, it is not necessary to do so. For future reference, we note that

$$\det \mathbf{E}_\pm(z) = \gamma(z), \quad \mathbf{E}_\pm^{-1}(z) = \begin{pmatrix} 1 & i\mathbf{q}_\pm^\dagger/z \\ 0 & (\mathbf{q}_\pm^\perp)^\dagger/q_o \\ -iq_o/z & \mathbf{q}_\pm^\dagger/q_o \end{pmatrix}$$

with $\gamma(z)$ defined in (2.5), and since $\det \mathbf{E}_\pm(z)$ has a (double) zero at $z = 0$, the asymptotic eigenvectors become linearly dependent at $z = 0$. As noted above, the eigenfunctions have poles at the branch points $z = \pm q_o$. This is another important difference with respect to the previously used normalizations, for which the asymptotic eigenvectors were finite but linearly dependent at the branch points.

Let us now discuss the asymptotic time dependence. As $x \rightarrow \pm\infty$, we expect that the time evolution of the solutions of the Lax pair will be asymptotic to

$$\phi_t = \mathbf{T}_\pm \phi,$$

where $\mathbf{T}_\pm = 2ik^2\mathbf{J} + \mathbf{H}_\pm$ and $\mathbf{H}_\pm = i\mathbf{J}\mathbf{Q}_\pm^2 - iq_o^2\mathbf{J} - 2k\mathbf{Q}_\pm$. The eigenvalues of $\mathbf{T}_\pm$ are $-i(k^2 + \lambda^2)$ and $\pm 2ik\lambda$. Since the boundary conditions are constant, the consistency of the Lax pair (2.1) implies $[\mathbf{X}_\pm, \mathbf{T}_\pm] = \mathbf{0}_3$, so $\mathbf{X}_\pm$ and $\mathbf{T}_\pm$ admit common eigenvectors. Namely,

$$\mathbf{T}_\pm \mathbf{E}_\pm = -i\mathbf{E}_\pm \Omega,$$

where $\Omega(z) = \text{diag}(-2k\lambda, k^2 + \lambda^2, 2k\lambda)$. Then for all $z \in \mathbb{R}$, we can define the Jost solutions $\phi_{\pm}(x, t, z)$ as the simultaneous solutions of both parts of the Lax pair satisfying

$$\phi_{\pm}(x, t, z) = \mathbf{E}_{\pm}(z)e^{i\Theta(x, t, z)} + o(1), \quad x \rightarrow \pm\infty, \quad (2.7)$$

where $\Theta(x, t, z)$ is the 3×3 diagonal matrix

$$\Theta(x, t, z) = \Lambda(z)x - \Omega(z)t = \text{diag}(\theta_1(x, t, z), \theta_2(x, t, z), -\theta_1(x, t, z)) \quad (2.8)$$

with $\theta_1(x, t, z) = -\lambda x + 2k\lambda t$ and $\theta_2(x, t, z) = kx - (k^2 + \lambda^2)t$. As usual, the advantage of introducing simultaneous solutions of both parts of the Lax pair is that the scattering coefficients will be independent of time.

Because the eigenvector matrix $\mathbf{E}_{\pm}(x, t, z)$ here differs from that in [12], the matrix Jost solutions $\phi_{\pm}(x, t, z)$ defined by (2.7) differ from those in [12] as well. Explicitly,

$$\phi_{\pm}(x, t, z) = \phi_{\pm}^{[\text{bk}]}(x, t, z)\mathbf{G}(z)$$

with $\mathbf{G}(z)$ as above. As in [12], to make the definitions of the Jost solutions rigorous one can rewrite the first part of the Lax pair (2.1) as

$$\partial_x \phi_{\pm} = \mathbf{X}_{\pm} \phi_{\pm} + \Delta \mathbf{Q}_{\pm} \phi_{\pm},$$

where $\Delta \mathbf{Q}_{\pm} = \mathbf{Q} - \mathbf{Q}_{\pm}$. We remove the asymptotic exponential oscillations and introduce modified eigenfunctions

$$\mu_{\pm}(x, t, z) = \phi_{\pm}(x, t, z)e^{-i\Theta(x, t, z)}, \quad (2.9)$$

so that

$$\lim_{x \rightarrow \pm\infty} \mu_{\pm}(x, t, z) = \mathbf{E}_{\pm}(z).$$

One can then formally integrate the ODE for $\mu_{\pm}(x, t, z)$ to obtain

$$\begin{aligned} \mu_{-}(x, t, z) &= \mathbf{E}_{-}(z) \\ &+ \int_{-\infty}^x \mathbf{E}_{-}(z)e^{i(x-y)\Lambda(z)}\mathbf{E}_{-}^{-1}(z)\Delta \mathbf{Q}_{-}(y, t)\mu_{-}(y, t, z)e^{-i(x-y)\Lambda(z)}dy, \end{aligned} \quad (2.10)$$

$$\begin{aligned} \mu_{+}(x, t, z) &= \mathbf{E}_{+}(z) \\ &- \int_x^{\infty} \mathbf{E}_{+}(z)e^{i(x-y)\Lambda(z)}\mathbf{E}_{+}^{-1}(z)\Delta \mathbf{Q}_{+}(y, t)\mu_{+}(y, t, z)e^{-i(x-y)\Lambda(z)}dy. \end{aligned} \quad (2.11)$$

Even though $\mathbf{E}_{\pm}^{-1}(z)$ have a simple pole at $z = 0$, one can verify from a direct computation that as $z \rightarrow 0$ from the real z -axis $\mathbf{E}_{\pm}(z)e^{\pm i(x-y)\Lambda(z)}\mathbf{E}_{\pm}^{-1}(z)$ is bounded for any $x, y \in \mathbb{R}$. Therefore, the kernel of the integral equations (2.10) and (2.11) is bounded when z approaches 0 from the real z -axis. One can rigorously define the Jost eigenfunctions as the solutions of the integral equations (2.10)-(2.11), and using standard Neumann iteration prove the following:

Theorem 2.1. *If $\mathbf{q}(x, t) - \mathbf{q}_- \in L^1(\mathbb{R}_x^-)$ or, correspondingly, $\mathbf{q}(x, t) - \mathbf{q}_+ \in L^1(\mathbb{R}_x^+)$, the modified eigenfunctions $\mu_{\pm}(x, t, z)$ are continuous functions of z for $z \in \mathbb{R} \setminus \{\pm q_0\}$. Moreover, the following columns of $\mu_-(x, t, z)$ and $\mu_+(x, t, z)$ can be analytically extended onto the corresponding regions of the complex z -plane:*

$$\mu_{-,1}(x, t, z), \mu_{+,3}(x, t, z) : \Im z > 0, \quad \mu_{-,3}(x, t, z), \mu_{+,1}(x, t, z) : \Im z < 0.$$

Eq. (2.9) implies that the same analyticity properties also hold for the columns of $\phi_{\pm}(x, t, z)$. Hereafter we will always implicitly assume that both conditions in Theorem 2.1 are satisfied, namely that $\mathbf{q}(x, t) - \mathbf{q}_{\pm} \in L^1(\mathbb{R}_x^{\pm})$.

We now introduce the scattering matrix. Eq. (2.1) implies that $\det(\phi_{\pm}(x, t, z)e^{-i\Theta(x, t, z)})$ is independent of x and t , and then (2.7) implies

$$\det \phi_{\pm}(x, t, z) = \gamma(z)e^{i\theta_2(x, t, z)}, \quad (x, t) \in \mathbb{R}^2, \quad z \in \mathbb{R} \setminus \{\pm q_0\}. \quad (2.12)$$

Therefore, $\phi_-(x, t, z)$ and $\phi_+(x, t, z)$ are two fundamental matrix solutions of the Lax pair for any $z \in \mathbb{R} \setminus \{0, \pm q_0\}$, so there exists a 3×3 matrix $\mathbf{A}(z)$ such that

$$\phi_-(x, t, z) = \phi_+(x, t, z)\mathbf{A}(z), \quad z \in \mathbb{R} \setminus \{0, \pm q_0\}. \quad (2.13)$$

As usual, $\mathbf{A}(z) = (a_{ij}(z))$ is referred to as the scattering matrix. Note that the extra term in (1.3) and our normalizations of the Jost solutions imply that $\mathbf{A}(z)$ is independent of time. Moreover, (2.12) and (2.13) imply

$$\det \mathbf{A}(z) = 1, \quad z \in \mathbb{R} \setminus \{0, \pm q_0\}.$$

For comparison purposes, note that the definition of the scattering matrix in this work differs from that in [12]. More precisely,

$$\begin{aligned} \mathbf{A}(z) &= \mathbf{G}^{-1}(z)\mathbf{A}^{[\text{bk}]}(z)\mathbf{G}(z), \quad z \in \mathbb{R} \setminus \{0, \pm q_0\}, \\ \mathbf{B}(z) &= \mathbf{G}^{-1}(z)\mathbf{B}^{[\text{bk}]}(z)\mathbf{G}(z), \quad z \in \mathbb{R} \setminus \{0, \pm q_0\}, \end{aligned}$$

where $\mathbf{B}(z) = \mathbf{A}^{-1}(z)$ and $\mathbf{G}(z)$ as in (2.6).

Theorem 2.2. *Under the same hypotheses as in Theorem 2.1, the following scattering coefficients can be analytically extended off the real z -axis in the following regions:*

$$a_{11}(z), b_{33}(z) : \Im z > 0, \quad a_{33}(z), b_{11}(z) : \Im z < 0.$$

The proofs of the above results are identical to the equivalent ones in [12] apart from the change in normalization, and are therefore omitted for brevity. However, we should remark on another difference between the formalism of the present work and that of [12]. There, it was possible to also continuously extend the Jost solutions to the branch points as long as the scattering potential satisfied additional decay requirements as $x \rightarrow \pm\infty$. Here, however, this is not possible because the eigenvector matrix has poles at $z = \pm q_0$.

2.3. Adjoint problem and auxiliary eigenfunctions

A complete set of analytic eigenfunctions is needed to solve the inverse problem, but $\phi_{\pm,2}$ are nowhere analytic in general. To circumvent this problem, as in [12, 40], we use the so-called “adjoint” Lax pair (following the terminology of [34]):

$$\tilde{\phi}_x = \tilde{X} \tilde{\phi}, \quad \tilde{\phi}_t = \tilde{T} \tilde{\phi}, \quad (2.14)$$

where $\tilde{X} = ikJ + Q^*$ and $\tilde{T} = -2ik^2J + iJQ_x^* - iJ(Q^*)^2 + iq_0^2J - 2kQ^*$. Hereafter tildes denote quantities defined for the adjoint problem (2.14) instead of the original one (2.1), and $*$ denotes complex conjugation. Note that $\tilde{X} = X^*$ and $\tilde{T} = T^*$ for all $z \in \mathbb{R}$. Then one has:

Lemma 2.1. *If $\tilde{v}(x, t, z)$ and $\tilde{w}(x, t, z)$ are two arbitrary solutions of the adjoint problem (2.14), then $u(x, t, z) = e^{i\theta_2(x, t, z)} J[\tilde{v}(x, t, z) \times \tilde{w}(x, t, z)]$ is a solution of the Lax pair (2.1).*

Here and below, “ $\times$ ” denotes the usual cross product. As in [12, 40] we use this result to construct two additional analytic eigenfunctions. The eigenvalues of $\tilde{X}_{\pm}$ are $-ik$ and $\pm i\lambda$. We then define the Jost solutions of the adjoint problem as the simultaneous solutions $\tilde{\phi}_{\pm}$ of (2.14) such that

$$\tilde{\phi}_{\pm}(x, t, z) = E_{\pm}(z) e^{-i\Theta(x, t, z)} + o(1), \quad x \rightarrow \pm\infty, \quad z \in \mathbb{R} \setminus \{\pm q_0\},$$

where $\tilde{E}_{\pm}(z) = E_{\pm}^*(z^*)$ for $z \in \mathbb{C} \setminus \{\pm q_0\}$. Introducing modified eigenfunctions $\tilde{\mu}_{\pm}(x, t, z) = \tilde{\phi}_{\pm}(x, t, z) e^{i\Theta(x, t, z)}$ as before, one can find that some of the columns of $\tilde{\mu}_{\pm}(x, t, z)$ can be extended into the complex plane: specifically, $\tilde{\mu}_{-,3}(x, t, z)$ and $\tilde{\mu}_{+,1}(x, t, z)$ to $\Im z > 0$, $\tilde{\mu}_{-,1}(x, t, z)$ and $\tilde{\mu}_{+,3}(x, t, z)$ to $\Im z < 0$. As before, $\tilde{\phi}_{\pm}$ are both fundamental matrix solutions of the same problem, and therefore, there exists an invertible 3×3 matrix $\tilde{A}(z)$ such that

$$\tilde{\phi}_{-}(x, t, z) = \tilde{\phi}_{+}(x, t, z) \tilde{A}(z), \quad z \in \mathbb{R} \setminus \{0, \pm q_0\}.$$

The same techniques used for the original scattering matrix show that, for suitable potentials, some of the coefficients can be analytically extended into the complex plane: specifically, $\tilde{a}_{11}(z)$ and $\tilde{b}_{33}(z)$ to $\Im z < 0$, and $\tilde{a}_{33}(z)$ and $\tilde{b}_{11}(z)$ to $\Im z > 0$, where $\tilde{B}(z) = \tilde{A}^{-1}(z)$. In light of these results, we can define two new solutions of the original Lax pair (2.1)

$$\begin{aligned} \bar{\chi}(x, t, z) &= -e^{i\theta_2(x, t, z)} J[\tilde{\phi}_{-,1}(x, t, z) \times \tilde{\phi}_{+,3}(x, t, z)] / \gamma(z), \quad z \in \mathbb{C}^-, \\ \chi(x, t, z) &= -e^{i\theta_2(x, t, z)} J[\tilde{\phi}_{-,3}(x, t, z) \times \tilde{\phi}_{+,1}(x, t, z)] / \gamma(z), \quad z \in \mathbb{C}^+. \end{aligned} \quad (2.15)$$

Note that although $1/\gamma(z)$ has a double pole at $z = 0$, as we will show in Section 2.5 both χ and $\bar{\chi}$ are $\mathcal{O}(1)$ at $z = 0$. By construction, $\bar{\chi}(x, t, z)$ is analytic for $\Im z < 0$, while $\chi(x, t, z)$ is analytic for $\Im z > 0$. Furthermore, as in [12, 40], we have the following relation between the adjoint Jost eigenfunctions and the eigenfunctions of the original Lax pair (2.1).

Lemma 2.2. *For $z \in \mathbb{R} \setminus \{0, \pm q_0\}$ and for all cyclic indices j, ℓ , and m ,*

$$\begin{aligned} \phi_{\pm,j}(x, t, z) &= -e^{i\theta_2(x, t, z)} J[\tilde{\phi}_{\pm,\ell}(x, t, z) \times \tilde{\phi}_{\pm,m}(x, t, z)] / \gamma_j(z), \\ \tilde{\phi}_{\pm,j}(x, t, z) &= -e^{-i\theta_2(x, t, z)} J[\phi_{\pm,\ell}(x, t, z) \times \phi_{\pm,m}(x, t, z)] / \gamma_j(z), \end{aligned} \quad (2.16)$$

where $\gamma_1(z) = -1$, $\gamma_2(z) = \gamma(z)$ and $\gamma_3(z) = 1$.

Corollary 2.1. *The scattering matrices $\mathbf{A}(z)$ and $\tilde{\mathbf{A}}(z)$ are related by*

$$\tilde{\mathbf{A}}^T(z) = \Gamma^{-1}(z) \mathbf{A}^{-1}(z) \Gamma(z), \quad z \in \mathbb{R} \setminus \{0, \pm q_o\},$$

where $\Gamma(z) = \text{diag}(-1, \gamma(z), 1)$.

As a result, we have the following decompositions for the non-analytic Jost eigenfunctions:

$$\begin{aligned} \phi_{-,2}(x, t, z) &= \frac{1}{a_{33}(z)} [a_{32}(z) \phi_{-,3}(x, t, z) - \tilde{\chi}(x, t, z)] \\ &= \frac{1}{a_{11}(z)} [a_{12}(z) \phi_{-,1}(x, t, z) + \chi(x, t, z)], \quad z \in \mathbb{R} \setminus \{0, \pm q_o\}, \end{aligned} \quad (2.17)$$

$$\begin{aligned} \phi_{+,2}(x, t, z) &= \frac{1}{b_{11}(z)} [b_{12}(z) \phi_{+,1}(x, t, z) - \tilde{\chi}(x, t, z)] \\ &= \frac{1}{b_{33}(z)} [b_{32}(z) \phi_{+,3}(x, t, z) + \chi(x, t, z)], \quad z \in \mathbb{R} \setminus \{0, \pm q_o\}. \end{aligned} \quad (2.18)$$

Note that using the asymptotic behaviors of eigenfunctions and scattering coefficients derived in Section 2.5 one can show that the above equations also hold as $z \rightarrow 0$ from the real axis.

2.4. Symmetries

As in [12, 40], the symmetries are complicated by the presence of a Riemann surface. Correspondingly, the problem admits two symmetries.

First symmetry. Consider the transformation $z \mapsto z^*$ (UHP/LHP), implying $(k, \lambda) \mapsto (k^*, \lambda^*)$.

Lemma 2.3. *If $\phi(x, t, z)$ is a non-singular solution of the scattering problem, so is $\mathbf{w}(x, t, z) = \mathbf{J}(\phi^\dagger(x, t, z^*))^{-1}$.*

Using Lemma 2.3, as in [12] one can show:

Lemma 2.4. *The analytic Jost eigenfunctions obey the following symmetry relations:*

$$\begin{aligned} \phi_{-,1}^*(x, t, z^*) &= -\frac{1}{a_{33}(z)} \mathbf{J} [\tilde{\chi}(x, t, z) \times \phi_{-,3}(x, t, z)] e^{-i\theta_2(x, t, z)}, \quad \Im z < 0, \\ \phi_{+,1}^*(x, t, z^*) &= \frac{1}{b_{33}(z)} \mathbf{J} [\chi(x, t, z) \times \phi_{+,3}(x, t, z)] e^{-i\theta_2(x, t, z)}, \quad \Im z > 0, \\ \phi_{-,3}^*(x, t, z^*) &= \frac{1}{a_{11}(z)} \mathbf{J} [\chi(x, t, z) \times \phi_{-,1}(x, t, z)] e^{-i\theta_2(x, t, z)}, \quad \Im z > 0, \\ \phi_{+,3}^*(x, t, z^*) &= -\frac{1}{b_{11}(z)} \mathbf{J} [\tilde{\chi}(x, t, z) \times \phi_{+,1}(x, t, z)] e^{-i\theta_2(x, t, z)}, \quad \Im z < 0. \end{aligned}$$

Recalling (2.13), we conclude that for all $z \in \mathbb{R} \setminus \{0, \pm q_o\}$,

$$\mathbf{A}^{-1}(z) = \mathbf{\Gamma}(z) \mathbf{A}^\dagger(z) \mathbf{\Gamma}^{-1}(z) = \begin{pmatrix} a_{11}^*(z) & -a_{21}^*(z)/\gamma(z) & -a_{31}^*(z) \\ -\gamma(z)a_{12}^*(z) & a_{22}^*(z) & \gamma(z)a_{32}^*(z) \\ -a_{13}^*(z) & a_{23}^*(z)/\gamma(z) & a_{33}^*(z) \end{pmatrix}. \quad (2.19)$$

The Schwarz reflection principle then allows us to conclude that

$$\begin{aligned} b_{11}(z) &= a_{11}^*(z^*), & \Im z < 0, \\ b_{33}(z) &= a_{33}^*(z^*), & \Im z > 0. \end{aligned}$$

We also have:

Corollary 2.2. *The auxiliary analytic eigenfunctions satisfy the following symmetry relations:*

$$\begin{aligned} \bar{\chi}(x, t, z) &= -e^{i\theta_2(x, t, z)} \mathbf{J} \left[\phi_{-,1}^*(x, t, z^*) \times \phi_{+,3}^*(x, t, z^*) \right] / \gamma(z), & \Im z < 0, \\ \chi(x, t, z) &= -e^{i\theta_2(x, t, z)} \mathbf{J} \left[\phi_{-,3}^*(x, t, z^*) \times \phi_{+,1}^*(x, t, z^*) \right] / \gamma(z), & \Im z > 0. \end{aligned} \quad (2.20)$$

In addition, by (2.16),

$$\phi_{\pm,j}^*(x, t, z) = -e^{-i\theta_2(x, t, z)} \mathbf{J} \left[\phi_{\pm,\ell}(x, t, z) \times \phi_{\pm,m}(x, t, z) \right] / \gamma_j(z), \quad (2.21)$$

where j, ℓ , and m are cyclic indices and $z \in \mathbb{R} \setminus \{0, \pm q_o\}$. As before, using the asymptotic behaviors of eigenfunctions and scattering coefficients derived in Section 2.5, one can show that (2.21) also hold as $z \rightarrow 0$ from the real axis.

Second symmetry. Consider the transformation $z \mapsto q_o^2/z$ (outside/inside the circle of radius q_o centered at 0), implying $(k, \lambda) \mapsto (k, -\lambda)$. We use this symmetry to relate the values of the eigenfunctions on the two sheets (particularly, across the cuts), where k is arbitrary but fixed (on either sheet).

It is straightforward to see that if $\phi(x, t, z)$ is a solution of the scattering problem, then so is $\varphi(x, t, z) = \phi(x, t, q_o^2/z)$. Consequently, as in [12] it follows that

$$\phi_\pm(x, t, z) = \phi_\pm(x, t, q_o^2/z) \mathbf{\Pi}(z), \quad (2.22)$$

where

$$\mathbf{\Pi}(z) = \begin{pmatrix} 0 & 0 & iz/q_o \\ 0 & 1 & 0 \\ -iz/q_o & 0 & 0 \end{pmatrix}.$$

As a consequence of (2.22) and the analyticity of the eigenfunctions, one then has

$$\begin{aligned} \phi_{\pm,3}(x, t, z) &= \frac{iz}{q_o} \phi_{\pm,1}(x, t, q_o^2/z), & \Im z \gtrless 0, \\ \phi_{\pm,2}(x, t, z) &= \phi_{\pm,2}(x, t, q_o^2/z), & z \in \mathbb{R} \setminus \{0, \pm q_o\}. \end{aligned} \quad (2.23)$$

We again use (2.13) to conclude that for all $z \in \mathbb{R} \setminus \{0, \pm q_o\}$

$$\begin{aligned} \mathbf{A}(z) &= \mathbf{\Pi}^{-1}(z) \mathbf{A}(q_o^2/z) \mathbf{\Pi}(z) \\ &= \begin{pmatrix} a_{33}(q_o^2/z) & (iq_o/z) a_{32}(q_o^2/z) & -a_{31}(q_o^2/z) \\ -(iz/q_o) a_{23}(q_o^2/z) & a_{22}(q_o^2/z) & (iz/q_o) a_{21}(q_o^2/z) \\ -a_{13}(q_o^2/z) & (-iq_o/z) a_{12}(q_o^2/z) & a_{11}(q_o^2/z) \end{pmatrix}. \end{aligned} \quad (2.24)$$

The analyticity of the scattering matrix entries then allows us to conclude

$$\begin{aligned} a_{11}(z) &= a_{33}(q_o^2/z), \quad \Im z > 0, \\ a_{33}(z) &= a_{11}(q_o^2/z), \quad \Im z < 0. \end{aligned}$$

Finally, we can combine (2.23), (2.17), and (2.24) to conclude

$$\chi(x, t, z) = -\bar{\chi}(x, t, q_o^2/z), \quad \Im z > 0.$$

The following reflection coefficients will appear in the inverse problem:

$$\begin{aligned} \rho_1(z) &= \frac{b_{13}(z)}{b_{11}(z)} = -\frac{a_{31}^*(z)}{a_{11}^*(z)}, \\ \rho_2(z) &= \frac{a_{21}(z)}{a_{11}(z)} = -\gamma(z) \frac{b_{12}^*(z)}{b_{11}^*(z)}, \quad z \in \mathbb{R} \setminus \{0, \pm q_o\}. \end{aligned} \quad (2.25)$$

Using the symmetries of the scattering coefficients, we can also express the reflection coefficients as

$$\begin{aligned} \rho_1(q_o^2/z) &= -\frac{b_{31}(z)}{b_{33}(z)} = \frac{a_{13}^*(z)}{a_{33}^*(z)}, \\ \rho_2(q_o^2/z) &= \frac{q_o}{iz} \frac{a_{23}(z)}{a_{33}(z)} = \gamma(z) \frac{q_o}{iz} \frac{b_{32}^*(z)}{b_{33}^*(z)}, \quad z \in \mathbb{R} \setminus \{0, \pm q_o\}. \end{aligned} \quad (2.26)$$

It should be noted that in the scalar (defocusing NLS) case, there is a simple symmetry for the reflection coefficient. One can indeed check that, in the scalar reduction of the Manakov system, in which for example the first component of $\mathbf{q}(x, t)$ is identically zero, one indeed recovers the symmetry of the scalar case, namely $\rho_1(q_o^2/z) = \rho_1^*(z)$ (while $\rho_2(z) \equiv 0$). In the general case of the Manakov system, however, the symmetries of the scattering coefficients involve 2×2 minors of the scattering matrix. Because of this, the above symmetries of the reflection coefficients do not yield expressions similar to those in the scalar case.

2.5. Asymptotic behavior of eigenfunctions and scattering data

To normalize the Riemann-Hilbert problem (RHP), it will be necessary to examine the asymptotic behavior both as $z \rightarrow \infty$ and as $z \rightarrow 0$. Taking into account (2.11), we consider the following formal expansion for the modified eigenfunction $\mu_+(x, t, z)$:

$$\mu_+(x, t, z) = \sum_{n=0}^{\infty} \mu_n(x, t, z), \quad (2.27)$$

where

$$\mu_{n+1}(x, t, z) = - \int_x^\infty \mathbf{E}_+(z) e^{i(x-y)\Lambda(z)} \mathbf{E}_+^{-1}(z) \Delta \mathbf{Q}_+(y, t) \mu_n(y, t, z) e^{-i(x-y)\Lambda(z)} dy.$$

Denoting by $C^1(\mathbb{R}_x^\pm)$ the space of continuously differentiable (with respect to x) complex-valued functions defined on $\mathbb{R}^\pm$, we can prove the following:

Lemma 2.5. *Suppose $\mathbf{q}(x, t) - \mathbf{q}_\pm \in L^1(\mathbb{R}_x^\pm) \cap C^1(\mathbb{R}_x^\pm)$. Then, for all $n \geq 0$, (2.27) provides an asymptotic expansion for the columns of $\mu_+(x, t, z)$ as $z \rightarrow \infty$ in the appropriate region of the complex z -plane, with*

$$\begin{aligned} [\mu_{2n}]_{bd} &= \mathcal{O}\left(\frac{1}{z^n}\right), & [\mu_{2n}]_{bo} &= \mathcal{O}\left(\frac{1}{z^{n+1}}\right), \\ [\mu_{2n+1}]_{bd} &= \mathcal{O}\left(\frac{1}{z^{n+1}}\right), & [\mu_{2n+1}]_{bo} &= \mathcal{O}\left(\frac{1}{z^{n+1}}\right). \end{aligned}$$

Here and in the following, $\mathbf{A}_{bd}$ and $\mathbf{A}_{bo}$ denote respectively the block diagonal and block off-diagonal parts of a 3×3 matrix $\mathbf{A}$, as in [12].

Lemma 2.6. *Suppose $\mathbf{q}(x, t) - \mathbf{q}_\pm \in L^1(\mathbb{R}_x^\pm) \cap C^1(\mathbb{R}_x^\pm)$. Then, for all $n \geq 0$, (2.27) provides an asymptotic expansion for the columns of $\mu_+(x, t, z)$ as $z \rightarrow 0$ in the appropriate region of the complex z -plane, with*

$$\begin{aligned} [\mu_{2n}]_{bd} &= \mathcal{O}(z^n), & [\mu_{2n}]_{bo} &= \mathcal{O}(z^{n-1}), \\ [\mu_{2n+1}]_{bd} &= \mathcal{O}(z^n), & [\mu_{2n+1}]_{bo} &= \mathcal{O}(z^n). \end{aligned}$$

Then, evaluating explicitly the first few terms in (2.27), we obtain

Corollary 2.3. *As $z \rightarrow \infty$ in the appropriate regions of the complex plane*

$$\mu_{\pm,1}(x, t, z) = \begin{pmatrix} 1 \mp (i/z)h_\pm(x, t) \\ (i/z)\mathbf{q}(x, t) \end{pmatrix} + \mathcal{O}\left(\frac{1}{z^2}\right), \quad (2.28)$$

$$\mu_{\pm,3}(x, t, z) = \begin{pmatrix} -(i/q_0 z)\mathbf{q}^\dagger(x, t)\mathbf{q}_\pm \\ (1/q_0)\mathbf{q}_\pm \pm (i\mathbf{q}_\pm^\dagger/zq_0)f_\pm(x, t) \pm (i\mathbf{q}_\pm/zq_0)g_\pm(x, t) \end{pmatrix} + \mathcal{O}\left(\frac{1}{z^2}\right), \quad (2.29)$$

where

$$\begin{aligned} f_+(x, t) &= \frac{1}{q_0^2} \int_x^\infty ((\mathbf{q}_+^\perp)^\dagger \mathbf{q})(\mathbf{q}^\dagger \mathbf{q}_+) dy, & f_-(x, t) &= \frac{1}{q_0^2} \int_{-\infty}^x ((\mathbf{q}_-^\perp)^\dagger \mathbf{q})(\mathbf{q}^\dagger \mathbf{q}_-) dy, \\ g_+(x, t) &= \int_x^\infty \left(\frac{|\mathbf{q}^\dagger \mathbf{q}_+|^2}{q_0^2} - q_0^2 \right) dy, & g_-(x, t) &= \int_{-\infty}^x \left(\frac{|\mathbf{q}^\dagger \mathbf{q}_-|^2}{q_0^2} - q_0^2 \right) dy, \\ h_+(x, t) &= \int_x^\infty (|\mathbf{q}^\dagger \mathbf{q}_+^\perp|^2 + |\mathbf{q}^\dagger \mathbf{q}_+|^2 - q_0^4) dy, & h_-(x, t) &= \int_{-\infty}^x (|\mathbf{q}^\dagger \mathbf{q}_-^\perp|^2 + |\mathbf{q}^\dagger \mathbf{q}_-|^2 - q_0^4) dy. \end{aligned}$$

Similarly, as $z \rightarrow 0$ in the appropriate region of the complex plane

$$\mu_{\pm,1}(x, t, z) = \begin{pmatrix} 0 \\ -(iz/q_0^2)\mathbf{q}_\pm \end{pmatrix} + \mathcal{O}(z^2), \quad \mu_{\pm,3}(x, t, z) = \begin{pmatrix} iz/q_0 \\ \mathbf{0} \end{pmatrix} + \mathcal{O}(z^2).$$

Next, we need the asymptotic behavior of χ and $\bar{\chi}$. It will be helpful to remove the exponential oscillations of $\bar{\chi}$ and χ (as we did with $\phi_{\pm}$), so with this goal in mind, we define

$$\bar{m}(x, t, z) = \bar{\chi}(x, t, z)e^{-i\theta_2(x, t, z)}, \quad m(x, t, z) = \chi(x, t, z)e^{-i\theta_2(x, t, z)}. \quad (2.30)$$

Combining the above asymptotic with (2.20) we obtain:

Lemma 2.7. *As $z \rightarrow \infty$ in the appropriate regions of the complex plane*

$$\begin{aligned} m(x, t, z) &= \begin{pmatrix} (-i/zq_o)\mathbf{q}^\dagger(x, t)\mathbf{q}_-^\perp \\ (1/q_o)\mathbf{q}_-^\perp + (i/zq_o)(g_-^*(x, t)\mathbf{q}_-^\perp - f_-^*(x, t)\mathbf{q}_- + h_+^*(x, t)\mathbf{q}_-^\perp) \end{pmatrix} + \mathcal{O}\left(\frac{1}{z^2}\right), \\ m(x, t, z) &= \begin{pmatrix} (i/zq_o)\mathbf{q}^\dagger(x, t)\mathbf{q}_+^\perp \\ (-1/q_o)\mathbf{q}_+^\perp + (i/zq_o)(g_+^*(x, t)\mathbf{q}_+^\perp - f_+^*(x, t)\mathbf{q}_+ + h_-^*(x, t)\mathbf{q}_+^\perp) \end{pmatrix} + \mathcal{O}\left(\frac{1}{z^2}\right). \end{aligned}$$

Similarly, as $z \rightarrow 0$ in the appropriate regions of the complex plane

$$m(x, t, z) = \begin{pmatrix} 0 \\ (1/q_o)\mathbf{q}_-^\perp \end{pmatrix} + \mathcal{O}(z), \quad \bar{m}(x, t, z) = \begin{pmatrix} 0 \\ (-1/q_o)\mathbf{q}_-^\perp \end{pmatrix} + \mathcal{O}(z).$$

Next, we find the asymptotic behavior of the scattering matrix entries.

Corollary 2.4. *As $z \rightarrow \infty$ in the appropriate regions of the complex plane*

$$\begin{aligned} a_{11}(z) &= 1 + \mathcal{O}\left(\frac{1}{z}\right), & b_{33}(z) &= \frac{1}{q_o^2}\mathbf{q}_-^\dagger\mathbf{q}_+ + \mathcal{O}\left(\frac{1}{z}\right), \\ a_{33}(z) &= \frac{1}{q_o^2}\mathbf{q}_+^\dagger\mathbf{q}_- + \mathcal{O}\left(\frac{1}{z}\right), & b_{11}(z) &= 1 + \mathcal{O}\left(\frac{1}{z}\right). \end{aligned} \quad (2.31)$$

Similarly, as $z \rightarrow 0$ in the appropriate regions of the complex plane

$$\begin{aligned} a_{11}(z) &= \frac{1}{q_o^2}\mathbf{q}_+^\dagger\mathbf{q}_- + \mathcal{O}(z), & b_{33}(z) &= 1 + \mathcal{O}(z), \\ a_{33}(z) &= 1 + \mathcal{O}(z), & b_{11}(z) &= \frac{1}{q_o^2}\mathbf{q}_-^\dagger\mathbf{q}_+ + \mathcal{O}(z). \end{aligned}$$

Below we determine the asymptotic behavior of the off-diagonal scattering matrix entries.

Corollary 2.5. *As $z \rightarrow \infty$ on the real z -axis*

$$\begin{aligned} a_{12}(z) &= b_{12}(z) = \mathcal{O}\left(\frac{1}{z}\right), & a_{13}(z) &= b_{13}(z) = \mathcal{O}\left(\frac{1}{z}\right), \\ a_{21}(z) &= b_{21}(z) = \mathcal{O}\left(\frac{1}{z}\right), & a_{22}(z) &= \frac{1}{q_o^2}\mathbf{q}_-^\dagger\mathbf{q}_+ + \mathcal{O}\left(\frac{1}{z}\right), \end{aligned}$$

$$\begin{aligned}
a_{23}(z) &= -\frac{1}{q_o^2} (\mathbf{q}_-^\perp)^\dagger \mathbf{q}_+ + \mathcal{O}\left(\frac{1}{z}\right), & a_{31}(z) &= b_{31}(z) = \mathcal{O}\left(\frac{1}{z}\right), \\
a_{32}(z) &= \frac{1}{q_o^2} \mathbf{q}_+^\dagger \mathbf{q}_-^\perp + \mathcal{O}\left(\frac{1}{z}\right), & b_{22}(z) &= \frac{1}{q_o^2} \mathbf{q}_+^\dagger \mathbf{q}_- + \mathcal{O}\left(\frac{1}{z}\right), \\
b_{23}(z) &= -\frac{1}{q_o^2} (\mathbf{q}_+^\perp)^\dagger \mathbf{q}_- + \mathcal{O}\left(\frac{1}{z}\right), & b_{32}(z) &= \frac{1}{q_o^2} \mathbf{q}_-^\dagger \mathbf{q}_+^\perp + \mathcal{O}\left(\frac{1}{z}\right).
\end{aligned}$$

Similarly, as $z \rightarrow 0$ on the real z -axis

$$\begin{aligned}
a_{13}(z) &= b_{13}(z) = \mathcal{O}(1), & a_{21}(z) &= b_{21}(z) = \mathcal{O}(1), & a_{31}(z) &= b_{31}(z) = \mathcal{O}(1), \\
a_{32}(z) &= b_{32}(z) = \mathcal{O}(1), & a_{23}(z) &= b_{23}(z) = \mathcal{O}(1), & a_{12}(z) &= \frac{i}{z q_o} \mathbf{q}_+^\dagger \mathbf{q}_-^\perp + \mathcal{O}(1), \\
b_{12}(z) &= \frac{i}{z q_o} \mathbf{q}_-^\dagger \mathbf{q}_+^\perp + \mathcal{O}(1), & a_{22}(z) &= \frac{1}{q_o^2} \mathbf{q}_-^\dagger \mathbf{q}_+ + \mathcal{O}(z), & b_{22}(z) &= \frac{1}{q_o^2} \mathbf{q}_+^\dagger \mathbf{q}_- + \mathcal{O}(z).
\end{aligned}$$

Note that, unlike what happens in the scalar case and in the case of zero boundary conditions, not all off-diagonal entries of the scattering matrix vanish as $z \rightarrow \infty$. As we will see next, however, this does not complicate the inverse problem. Importantly, however, in the orthogonal case the leading-order term in the asymptotic behavior of $a_{33}(z)$ and $b_{33}(z)$ as $z \rightarrow \infty$ and $a_{11}(z)$ and $b_{11}(z)$ as $z \rightarrow 0$ is zero. Next, we therefore compute the non-zero leading-order contribution in the asymptotic behavior of those scattering coefficients in the orthogonal case. Namely, $a_{33}(z)$, $b_{33}(z)$ when $z \rightarrow \infty$ and $a_{11}(z)$, $b_{11}(z)$ as $z \rightarrow 0$. This can be done using the scattering relation (2.13), Corollary 2.5 with the asymptotic behavior of Jost eigenfunctions in Corollary 2.3 and Lemma 2.7.

Corollary 2.6. Suppose $\mathbf{q}_+^\dagger \mathbf{q}_- = 0$. Without loss of generality, we can write $\mathbf{q}_+ = \mathbf{q}_-^\perp e^{-i\beta}$ for some $\beta \in \mathbb{R}$. Then, as $z \rightarrow \infty$ in the appropriate regions of the complex plane,

$$a_{33}(z) = \frac{i}{z q_o^2} \alpha + \mathcal{O}\left(\frac{1}{z^2}\right), \quad b_{33}(z) = -\frac{i}{z q_o^2} \alpha^* + \mathcal{O}\left(\frac{1}{z^2}\right),$$

where

$$\alpha := \frac{\mathbf{q}_-^\dagger \mathbf{q}_+^\perp}{q_o^2} \int_{\mathbb{R}} ((\mathbf{q}_-^\perp)^\dagger \mathbf{q}(x, t)) (\mathbf{q}^\dagger(x, t) \mathbf{q}_-) dx = \int_{\mathbb{R}} (\mathbf{q}_+^\dagger \mathbf{q}(x, t)) (\mathbf{q}^\dagger(x, t) \mathbf{q}_-) dx.$$

Similarly, as $z \rightarrow 0$ in the appropriate regions of the complex plane,

$$a_{11}(z) = \frac{iz}{q_o^4} \alpha + \mathcal{O}(z^2), \quad b_{11}(z) = -\frac{iz}{q_o^4} \alpha^* + \mathcal{O}(z^2).$$

Corollary 2.6 is obtained in a similar way as Corollary 2.4. Generically, one will have $\alpha \neq 0$. In that case, even though $\lim_{z \rightarrow \infty} b_{33}(z) = 0$ (and similarly for $a_{33}(z)$), using the asymptotic expansions in Corollary 2.6 it is easy to show that, for sufficiently large R , $b_{33}(z) \neq 0$ for all $|z| > R$ (and similarly for $a_{33}(z)$).

In the formulation of the inverse problem, it will be necessary to use the asymptotic behavior of the reflection coefficients. Such behavior is obtained in a straightforward way from the behavior of the scattering coefficients:

Corollary 2.7. *In the non-orthogonal case, along the real z -axis*

$$\rho_1(z) = \mathcal{O}\left(\frac{1}{z}\right), \quad \rho_2(z) = \mathcal{O}\left(\frac{1}{z}\right), \quad z \rightarrow \infty,$$

$$\hat{\rho}_1(z) := \rho_1(q_o^2/z) = \mathcal{O}\left(\frac{1}{z}\right), \quad \hat{\rho}_2(z) := \rho_2(q_o^2/z) = \frac{q_o}{iz} \left(-\frac{(\mathbf{q}_-^\perp)^\dagger \mathbf{q}_+}{\mathbf{q}_+^\dagger \mathbf{q}_-} \right) + \mathcal{O}\left(\frac{1}{z^2}\right), \quad z \rightarrow \infty.$$

Similarly, along the real z -axis

$$\begin{aligned} \hat{\rho}_1(z) &= \mathcal{O}(z), \quad \hat{\rho}_2(z) = \mathcal{O}(z), & z \rightarrow 0, \\ \rho_1(z) &= \mathcal{O}(z), \quad \rho_2(z) = \frac{iz(\mathbf{q}_-^\perp)^\dagger \mathbf{q}_+ + \mathcal{O}(z^2)}{q_o(\mathbf{q}_+^\dagger \mathbf{q}_-) + \mathcal{O}(z)} = \mathcal{O}(z), & z \rightarrow 0. \end{aligned} \quad (2.32)$$

In the orthogonal case,

$$\begin{aligned} \rho_1(z) &= \mathcal{O}\left(\frac{1}{z}\right), \quad \rho_2(z) = \mathcal{O}\left(\frac{1}{z}\right), & z \rightarrow \infty, \\ \hat{\rho}_1(z) &= \mathcal{O}(1), \quad \hat{\rho}_2(z) = \frac{q_o}{\alpha} (\mathbf{q}_-^\perp)^\dagger \mathbf{q}_+ + \mathcal{O}\left(\frac{1}{z}\right) = \mathcal{O}(1), & z \rightarrow \infty. \end{aligned}$$

Similarly,

$$\begin{aligned} \hat{\rho}_1(z) &= \mathcal{O}(z), \quad \hat{\rho}_2(z) = \mathcal{O}(z), & z \rightarrow 0, \\ \rho_1(z) &= \mathcal{O}(1), \quad \rho_2(z) = \frac{q_o}{\alpha} (\mathbf{q}_-^\perp)^\dagger \mathbf{q}_+ + \mathcal{O}(z) = \mathcal{O}(1), & z \rightarrow 0. \end{aligned}$$

Note that the reflection coefficients in [12] had different asymptotic behavior. In particular, $\hat{\rho}_2^{[\text{bk}]}(z) = \mathcal{O}(z)$ as $z \rightarrow \infty$ and $\rho_2^{[\text{bk}]}(z) = \mathcal{O}(1/z)$ as $z \rightarrow 0$. This was not an obstacle when $\mathbf{q}_+ \parallel \mathbf{q}_-$, but it would become a problem in the non-parallel case because the corresponding sectionally meromorphic matrices in the RHP would diverge as $z \rightarrow 0, \infty$. This is the reason why we choose a different normalization for the Jost solutions in (2.7).

3. Discrete Spectrum and Behavior at Branch Points

3.1. Discrete spectrum

Recall that in the 2×2 scattering problem for the NLS equation with NZBC, there is a one-to-one correspondence between zeros of the analytic scattering coefficients and discrete eigenvalues, each of which corresponds to a bound state for the eigenfunctions. Moreover, in the defocusing case the self-adjointness of the scattering problem implies that such discrete eigenvalues correspond to real values of the scattering parameter k , and one can show that no discrete eigenvalues can arise inside the continuous spectrum, so eigenvalues lie in the segment $(-q_o, q_o)$. Correspondingly, in the z -plane the discrete eigenvalues are confined to the circle C_o . The scattering problem in (2.1) for the Manakov system is also self-adjoint, and indeed a similar constraint exists for the proper eigenvalues of the scattering problem:

Lemma 3.1. *Let $\mathbf{v}(x, t, z)$ be a nontrivial solution of the scattering problem in (2.1). If $\mathbf{v}(x, t, z) \in L^2(\mathbb{R})$, then $z \in C_o$.*

Nonetheless, it was shown in [40] that to characterize the discrete spectrum one needs to consider zeros of the analytic scattering coefficients both on and off the circle C_o . This is consistent with Lemma 3.1, since (as shown in [40] and reviewed below) the zeros of the analytic scattering coefficients off C_o do not lead to bound states — i.e. eigenfunctions in L^2 .

Away from C_o , there are two possibilities: zeros outside the circle and zeros inside the circle. We will see that neither case gives us a bound state. In particular, the latter correspond to eigenfunctions which do not decay either as $x \rightarrow -\infty$ or as $x \rightarrow \infty$, and the former correspond to eigenfunctions which are singular (and as a result cannot be square-integrable).

Recall that $\phi_{-,1}$, $\phi_{+,3}$, and χ are analytic for $\Im z > 0$, while $\phi_{-,3}$, $\phi_{+,1}$, and $\bar{\chi}$ are analytic for $\Im z < 0$. Moreover, $\phi_{\pm} = \mathbf{E}_{\pm} e^{i\Theta} + o(1)$ as $x \rightarrow \pm\infty$ for all z in the region of analyticity. To characterize the discrete spectrum, it is convenient to introduce the following 3×3 matrices:

$$\begin{aligned}\Phi^+(x, t, z) &= (\phi_{-,1}(x, t, z), \chi(x, t, z), \phi_{+,3}(x, t, z)), \\ \Phi^-(x, t, z) &= (\phi_{+,1}(x, t, z), -\bar{\chi}(x, t, z), \phi_{-,3}(x, t, z)),\end{aligned}\tag{3.1}$$

which are analytic for $\Im z > 0$ and $\Im z < 0$, respectively. Recalling (2.17) and (2.18), we obtain

$$\begin{aligned}\det \Phi^+(x, t, z) &= a_{11}(z)b_{33}(z)\gamma(z)e^{i\theta_2(x, t, z)}, \quad \Im z > 0, \\ \det \Phi^-(x, t, z) &= a_{33}(z)b_{11}(z)\gamma(z)e^{i\theta_2(x, t, z)}, \quad \Im z < 0.\end{aligned}\tag{3.2}$$

Thus, the corresponding eigenfunctions become linearly dependent at the zeros of $a_{11}(z)$ and $b_{33}(z)$ in the upper-half plane (or, equivalently, $a_{33}(z)$ and $b_{11}(z)$ in the lower-half plane). Such zeros will play the role of discrete eigenvalues.

Another important difference between the scalar case — i.e., the NLS equation — and the Manakov system is that, for the latter, one cannot exclude the possibility of zeros along the continuous spectrum. Indeed, a direct calculation shows that, $\forall z \in \mathbb{R} \setminus \{0, \pm q_o\}$,

$$\det \mathbf{A}(z) = b_{11}(z)a_{11}(z) + b_{12}(z)a_{21}(z) + b_{13}(z)a_{31}(z),$$

which, applying (2.19), yields

$$|a_{11}(z)|^2 = 1 + \frac{1}{\gamma(z)}|a_{21}(z)|^2 + |a_{31}(z)|^2.\tag{3.3}$$

Since the second term on the right hand side of (3.3) is negative for $z \in (-q_o, q_o)$, we cannot exclude possible zeros of $a_{11}(z)$ in the interval $(-q_o, q_o)$. Similar results follow for the zeros of $a_{33}(z)$, $b_{11}(z)$, and $b_{33}(z)$.

The following lemmas are instrumental in the characterization of the discrete spectrum:

Lemma 3.2. *Suppose $a_{11}(z)$ has a zero z_n in the upper half z -plane. Then*

$$a_{11}(z_n) = 0 \Leftrightarrow b_{11}(z_n^*) = 0 \Leftrightarrow a_{33}(q_o^2/z_n) = 0 \Leftrightarrow b_{33}(q_o^2/z_n^*) = 0.$$

In particular, Lemma 3.2 implies that discrete eigenvalues ζ_n lying on the circle C_o appear in complex conjugate pairs $\{\zeta_n, \zeta_n^*\}$, whereas discrete eigenvalues z_n off C_o appear in symmetric quartets $\{z_n, z_n^*, q_o^2/z_n, q_o^2/z_n^*\}$.

Lemma 3.3. *Suppose $\Im z_o > 0$ and $|z_o| \neq q_o$. Then $\chi(x, t, z_o) \neq 0$.*

Lemma 3.4. *Suppose $\Im z_o > 0$. Then the following statements are equivalent:*

- (i) $\chi(x, t, z_o) = 0$,
- (ii) $\bar{\chi}(x, t, q_o^2/z_o) = 0$,
- (iii) $\chi(x, t, q_o^2/z_o^*) = 0$,
- (iv) $\bar{\chi}(x, t, z_o^*) = 0$,
- (v) *There exists a constant b_o such that $\phi_{-,3}(x, t, z_o^*) = b_o \phi_{+,1}(x, t, z_o^*)$,*
- (vi) *There exists a constant $\tilde{b}_o$ such that $\phi_{-,1}(x, t, q_o^2/z_o^*) = \tilde{b}_o \phi_{+,3}(x, t, q_o^2/z_o^*)$,*
- (vii) *There exists a constant $\hat{b}_o$ such that $\phi_{-,1}(x, t, z_o) = \hat{b}_o \phi_{+,3}(x, t, z_o)$,*
- (viii) *There exists a constant $\check{b}_o$ such that $\phi_{-,3}(x, t, q_o^2/z_o) = \check{b}_o \phi_{+,1}(x, t, q_o^2/z_o)$.*

Lemma 3.5. *If $a_{11}(\zeta_n) = 0$, where $\Im \zeta_n > 0$ and $|\zeta_n| = q_o$, then $\chi(x, t, \zeta_n) = \bar{\chi}(x, t, \zeta_n^*) = 0$. As a result, there exist constants c_n and $\bar{c}_n$ such that*

$$\phi_{-,1}(x, t, \zeta_n) = c_n \phi_{+,3}(x, t, \zeta_n), \quad \phi_{-,3}(x, t, \zeta_n^*) = \bar{c}_n \phi_{+,1}(x, t, \zeta_n^*). \quad (3.4)$$

Lemma 3.6. *Let z_n be a zero of $a_{11}(z)$ in the upper half plane with $|z_n| \neq q_o$. Then there exist constants $d_n, \check{d}_n, \hat{d}_n$, and $\bar{d}_n$ such that*

$$\begin{aligned} \phi_{-,1}(x, t, z_n) &= d_n \chi(x, t, z_n), & \phi_{-,3}(x, t, q_o^2/z_n) &= \check{d}_n \bar{\chi}(x, t, q_o^2/z_n), \\ \chi(x, t, q_o^2/z_n^*) &= \hat{d}_n \phi_{+,3}(x, t, q_o^2/z_n^*), & \bar{\chi}(x, t, z_n^*) &= \bar{d}_n \phi_{+,1}(x, t, z_n^*). \end{aligned} \quad (3.5)$$

Lemma 3.7. *If z_n is a zero of $a_{11}(z)$ in the upper half plane with $|z_n| \neq q_o$, then $|z_n| < q_o$ and $b_{33}(z_n) \neq 0$.*

The proof of Lemma 3.7 is similar to [12]. As in [12], the constraint $|z_n| < q_o$ in Lemma 3.7 is a consequence of the fact that a hypothetical zero z_o of $a_{11}(z)$ located outside the circle C_o of radius q_o would correspond to a bound state. However, because the differential operator associated with the scattering problem is self-adjoint in $L^2(\mathbb{R})$, such bound states can only occur when $|z_o| = q_o$. In other words, the discrete spectrum of the scattering problem is not the same as the discrete spectrum of the associated differential operator. By looking at the detailed behavior of the eigenfunctions, it was shown in [12], and it remains true here, that zeros z_o of $a_{11}(z)$ inside C_o do not correspond to bound states, and therefore are consistent with self-adjointness.

Now let us discuss the symmetries of the constants introduced in Lemmas 3.5 and 3.6.

Lemma 3.8. Assume that $a_{11}(z)$ has simple zeros $\{\zeta_n\}_{n=1}^{N_1}$ on C_o . Then the constants in (3.4) obey the following symmetry relations:

$$\bar{c}_n = -c_n, \quad c_n^* = \frac{b'_{11}(\zeta_n^*)}{a'_{33}(\zeta_n^*)} \bar{c}_n, \quad n = 1, \dots, N_1.$$

Lemma 3.9. Assume that $a_{11}(z)$ has zeros $\{z_n\}_{n=1}^{N_2}$ off the circle C_o . (Note that now it is not necessary to assume that such zeros are simple.) Then the constants in (3.5) obey the following relations:

$$\check{d}_n = -\frac{iq_o}{z_n} d_n, \quad \bar{d}_n = -\left[\frac{b_{33}(z_n)}{\gamma(z_n)} \right]^* d_n^*, \quad \hat{d}_n = -\frac{iz_n^*}{q_o} \left[\frac{b_{33}(z_n)}{\gamma(z_n)} \right]^* d_n^*, \quad n = 1, \dots, N_2.$$

3.2. Behavior at the branch points

We now discuss the behavior of the Jost eigenfunctions and the scattering matrix at the branch points $z = \pm q_o$. The complication there is due to the fact that $\lambda(\pm q_o) = 0$, and therefore, at $z = \pm q_o$, $\mathbf{E}_\pm(z)$ have a pole and the two exponentials $e^{\pm i\lambda x}$ reduce to the identity. Correspondingly, at $z = \pm q_o$, the matrices $\mathbf{E}_\pm^{-1}(z)$ are degenerate. It is convenient to introduce the weighted Sobolev spaces $L^{1,j}(\mathbb{R}_x^\pm) := \{f : \mathbb{R} \rightarrow \mathbb{C} \mid (1 + |x|)^j f \in L^1(\mathbb{R}_x^\pm)\}$. From (2.10) we have that

$$\mu_+(x, t, z) = \mathbf{E}_+(z) - \int_x^\infty \mathbf{E}_+(z) e^{i(x-y)\Lambda(z)} \mathbf{E}_+^{-1}(z) \Delta \mathbf{Q}_+(y, t) \mu_+(y, t, z) e^{-i(x-y)\Lambda(z)} dy.$$

Introducing $\nu_+(x, t, z) = \mu_+(x, t, z) \mathbf{E}_+^{-1}(z)$, we have

$$\nu_+(x, t, z) = \mathbf{I} - \int_x^\infty \mathbf{K}_+(x - y, z) \Delta \mathbf{Q}_+(y, t) \nu_+(y, t, z) \mathbf{K}_+^{-1}(x - y, z) dy, \quad (3.6)$$

where

$$\mathbf{K}_\pm(x - y, z) := \mathbf{E}_\pm(z) e^{i(x-y)\Lambda(z)} \mathbf{E}_\pm^{-1}(z).$$

Notice that

$$\begin{aligned} \lim_{z \rightarrow \pm q_o} \mathbf{K}_\pm(\xi, z) &= \lim_{z \rightarrow \pm q_o} \mathbf{E}_\pm(z) e^{i\xi\Lambda(z)} \mathbf{E}_\pm^{-1}(z) = \begin{pmatrix} 1 \mp iq_o\xi & \xi \mathbf{q}_\pm^\dagger \\ \xi \mathbf{q}_\pm & (1/q_o^2) \mathbf{U}_\pm(\xi) \end{pmatrix}, \\ \lim_{z \rightarrow \pm q_o} \mathbf{K}_\pm^{-1}(\xi, z) &= \lim_{z \rightarrow \pm q_o} \mathbf{E}_\pm(z) e^{-i\xi\Lambda(z)} \mathbf{E}_\pm^{-1}(z) = \begin{pmatrix} 1 \pm iq_o\xi & -\xi \mathbf{q}_\pm^\dagger \\ -\xi \mathbf{q}_\pm & (1/q_o^2) \mathbf{V}_\pm(\xi) \end{pmatrix}, \end{aligned}$$

where $\xi := x - y$, $\mathbf{U}_\pm(\xi) := (1 \pm iq_o\xi) \mathbf{q}_\pm \mathbf{q}_\pm^\dagger + e^{\pm iq_o\xi} \mathbf{q}_\pm^\perp (\mathbf{q}_\pm^\perp)^\dagger$ and $\mathbf{V}_\pm(\xi) := (1 \mp iq_o\xi) \mathbf{q}_\pm \mathbf{q}_\pm^\dagger + e^{\mp iq_o\xi} \mathbf{q}_\pm^\perp (\mathbf{q}_\pm^\perp)^\dagger$. Thus, if $\mathbf{q}(x, t) - \mathbf{q}_+ \in L^{1,1}(\mathbb{R}_x^+)$, the integral in (3.6) is convergent at $z = \pm q_o$. Moreover, $\nu_+(x, t, z)$ is well-defined and continuous at the branch points $z = \pm q_o$ approached from the real z -axis.

Therefore we have

$$v_+(x, t, z) = u_\pm(x, t) + o(1), \quad z \rightarrow \pm q_o,$$

where $u_\pm(x, t) = v_+(x, t, \pm q_o)$. Since $v_+(x, t, z) = \mu_+(x, t, z)\mathbf{E}_+^{-1}(z)$, for $z \in \mathbb{R} \setminus \{0, \pm q_o\}$, we have

$$\mu_{+,1}(x, t, z) = \frac{1}{2\lambda} \left(z v_{+,1}(x, t, z) + i q_{+,1} v_{+,2}(x, t, z) + i q_{+,2} v_{+,3}(x, t, z) \right), \quad (3.7)$$

where $q_{+,j}$ for $j = 1, 2$ denotes the j -th component of $\mathbf{q}_+$. Therefore, it follows that

$$\mu_{+,1}(x, t, z) = \frac{1}{z \mp q_o} \check{u}_{\pm,1}(x, t) + o\left(\frac{1}{z \mp q_o}\right), \quad z \rightarrow \pm q_o, \quad (3.8)$$

where $\check{u}_{\pm,1}(x, t)$ is obtained from (3.7) in terms of $u_\pm(x, t) = v_+(x, t, \pm q_o)$. Similarly, we have

$$\begin{aligned} \mu_{+,2}(x, t, z) &= \check{u}_{\pm,2}(x, t) + o(1), & z \rightarrow \pm q_o, \\ \mu_{+,3}(x, t, z) &= \frac{1}{z \mp q_o} \check{u}_{\pm,3}(x, t) + o\left(\frac{1}{z \mp q_o}\right), & z \rightarrow \pm q_o, \end{aligned} \quad (3.9)$$

and $\check{u}_{\pm,j}(x, t)$ for $j = 2, 3$ are obtained analogously.

Assuming $\mathbf{q}(x, t) - \mathbf{q}_- \in L^{1,1}(\mathbb{R}_x^-)$ and using similar analysis, one can show that

$$\begin{aligned} \mu_{-,1}(x, t, z) &= \frac{1}{z \mp q_o} \hat{u}_{\pm,1}(x, t) + o\left(\frac{1}{z \mp q_o}\right), & z \rightarrow \pm q_o, \\ \mu_{-,2}(x, t, z) &= \hat{u}_{\pm,2}(x, t) + o(1), & z \rightarrow \pm q_o, \\ \mu_{-,3}(x, t, z) &= \frac{1}{z \mp q_o} \hat{u}_{\pm,3}(x, t) + o\left(\frac{1}{z \mp q_o}\right), & z \rightarrow \pm q_o. \end{aligned} \quad (3.10)$$

Note that these limits are taken along the real z -axis.

Higher order terms in the behavior of the eigenfunctions at the branch points can be obtained by assuming faster decay of $\mathbf{q}(x, t) - \mathbf{q}_\pm$ as $x \rightarrow \pm\infty$. For instance, differentiating (3.6) with respect to z formally we have

$$\begin{aligned} \frac{\partial v_+(x, t, z)}{\partial z} &= - \int_x^\infty \frac{\partial \mathbf{K}_+(x-y, z)}{\partial z} \Delta \mathbf{Q}_+(y, t) v_+(y, t, z) \mathbf{K}_+^{-1}(x-y, z) dy \\ &\quad - \int_x^\infty \mathbf{K}_+(x-y, z) \Delta \mathbf{Q}_+(y, t) \frac{\partial v_+(y, t, z)}{\partial z} \mathbf{K}_+^{-1}(x-y, z) dy \\ &\quad - \int_x^\infty \mathbf{K}_+(x-y, z) \Delta \mathbf{Q}_+(y, t) v_+(y, t, z) \frac{\partial \mathbf{K}_+^{-1}(x-y, z)}{\partial z} dy. \end{aligned}$$

Now note that

$$\lim_{z \rightarrow \pm q_o} \frac{\partial \mathbf{K}_\pm(\xi, z)}{\partial z} = \lim_{z \rightarrow \pm q_o} \frac{\partial \mathbf{K}_\pm^{-1}(\xi, z)}{\partial z} = \mathbf{0}_{3 \times 3}.$$

Then if $\mathbf{q}(x, t) - \mathbf{q}_+ \in L^{1,2}(\mathbb{R}_x^+)$, it follows that $\partial v_+(x, t, z)/\partial z$ is well-defined and continuous as $z \rightarrow \pm q_0$ from the real z -axis. Therefore we have

$$v_+(x, t, z) = u_{\pm}(x, t) + v_{\pm}(x, t)(z \mp q_0) + o(z \mp q_0), \quad z \rightarrow \pm q_0,$$

where

$$u_{\pm}(x, t) = v_+(x, t, \pm q_0), \quad v_{\pm}(x, t) = \left. \frac{\partial v_+(x, t, z)}{\partial z} \right|_{z=\pm q_0}.$$

Since $v_+(x, t, z) = \mu_+(x, t, z)\mathbf{E}_+^{-1}(z)$, for $z \in \mathbb{R} \setminus \{0, \pm q_0\}$, (3.7) yields

$$\mu_{+,1}(x, t, z) = \frac{1}{z \mp q_0} \check{u}_{\pm,1}(x, t) + \check{v}_{\pm,1}(x, t) + o(1), \quad z \rightarrow \pm q_0, \quad (3.11)$$

where $\check{v}_{\pm,1}(x, t)$ is obtained from (3.7) in terms of $v_{\pm}(x, t) = v'_+(x, t, \pm q_0)$, with prime denoting derivative with respect to z . Similarly,

$$\begin{aligned} \mu_{+,2}(x, t, z) &= \check{u}_{\pm,2}(x, t) + \check{v}_{\pm,2}(x, t)(z \mp q_0) + o(z \mp q_0), \quad z \rightarrow \pm q_0, \\ \mu_{+,3}(x, t, z) &= \frac{1}{z \mp q_0} \check{u}_{\pm,3}(x, t) + \check{v}_{\pm,3}(x, t) + o(1), \quad z \rightarrow \pm q_0. \end{aligned}$$

Using similar analysis for $\mu_-(x, t, z)$ we see that, if $\mathbf{q}(x, t) - \mathbf{q}_- \in L^{1,2}(\mathbb{R}_x^-)$ we have,

$$\begin{aligned} \mu_{-,1}(x, t, z) &= \frac{1}{z \mp q_0} \hat{u}_{\pm,1}(x, t) + \hat{v}_{\pm,1}(x, t) + o(1), \quad z \rightarrow \pm q_0, \\ \mu_{-,2}(x, t, z) &= \hat{u}_{\pm,2}(x, t) + \hat{v}_{\pm,2}(x, t)(z \mp q_0) + o(z \mp q_0), \\ \mu_{-,3}(x, t, z) &= \frac{1}{z \mp q_0} \hat{u}_{\pm,3}(x, t) + \hat{v}_{\pm,3}(x, t) + o(1). \end{aligned}$$

Next, we discuss the behavior of $m(x, t, z)$ and $\tilde{m}(x, t, z)$ at the branch points $z = \pm q_0$. Recall that using definitions (2.15), (2.30) and (2.20) we have

$$\begin{aligned} \tilde{m}(x, t, z) &= \mathbf{J}[\mu_{+,1}^*(x, t, z^*) \times \mu_{-,3}^*(x, t, z^*)]/\gamma(z), \\ m(x, t, z) &= \mathbf{J}[\mu_{+,3}^*(x, t, z^*) \times \mu_{-,1}^*(x, t, z^*)]/\gamma(z). \end{aligned}$$

If $\mathbf{q}(x, t) - \mathbf{q}_{\pm} \in L^{1,1}(\mathbb{R}_x^{\pm})$ then by using (3.8)-(3.10) we have along the real z -axis

$$\begin{aligned} & \left[\mu_{+,3}^*(x, t, z) \times \mu_{-,1}^*(x, t, z) \right] \\ &= \frac{1}{(z \mp q_0)^2} \left[(\check{u}_{\pm,3}^*(x, t) + o(1)) \times (\hat{u}_{\pm,1}^*(x, t) + o(1)) \right] \\ &= \frac{1}{(z \mp q_0)^2} \left([\check{u}_{\pm,3}^* \times \hat{u}_{\pm,1}^*](x, t) + o(1) \right), \quad z \rightarrow \pm q_0, \end{aligned}$$

and consequently

$$\begin{aligned} m(x, t, z) &= \mathbf{J}[\mu_{+,3}^*(x, t, z) \times \mu_{-,1}^*(x, t, z)]/\gamma(z) \\ &= \frac{(z - q_0)(z + q_0)}{z^2} \frac{1}{(z \mp q_0)^2} \mathbf{J}([\check{u}_{\pm,3}^* \times \hat{u}_{\pm,1}^*](x, t) + o(1)), \end{aligned}$$

i.e.

$$m(x, t, z) = \pm \frac{2}{q_o} \frac{1}{z \mp q_o} m_{\pm}^{(-1)}(x, t) + o\left(\frac{1}{z \mp q_o}\right), \quad z \rightarrow \pm q_o, \quad (3.12)$$

where $m_{\pm}^{(-1)}(x, t) = \mathbf{J}[\check{u}_{\pm,3}^* \times \hat{u}_{\pm,1}^*](x, t)$.

Similarly,

$$\begin{aligned} \tilde{m}(x, t, z) &= \mathbf{J}[\mu_{+,1}^*(x, t, z) \times \mu_{-,3}^*(x, t, z)] / \gamma(z), \\ &= \pm \frac{2}{q_o} \frac{1}{z \mp q_o} \tilde{m}_{\pm}^{(-1)}(x, t) + o\left(\frac{1}{z \mp q_o}\right), \quad z \rightarrow \pm q_o, \end{aligned} \quad (3.13)$$

where $\tilde{m}_{\pm}^{(-1)}(x, t) = \mathbf{J}[\check{u}_{\pm,1}^* \times \hat{u}_{\pm,3}^*](x, t)$.

To increase the order of the expansion, let us assume $\mathbf{q}(x, t) - \mathbf{q}_{\pm} \in L^{1,2}(\mathbb{R}_x^{\pm})$. By using (3.11) we have

$$\begin{aligned} &[\mu_{+,3}^*(x, t, z) \times \mu_{-,1}^*(x, t, z)] \\ &= \frac{1}{(z \mp q_o)^2} \left[(\check{u}_{\pm,3}^*(x, t) + (z \mp q_o) \check{v}_{\pm,3}^*(x, t) + o(z \mp q_o)) \right. \\ &\quad \left. \times (\hat{u}_{\pm,1}^*(x, t) + (z \mp q_o) \hat{v}_{\pm,1}^*(x, t) + o(z \mp q_o)) \right], \end{aligned}$$

i.e.

$$\begin{aligned} &\mathbf{J}[\mu_{+,3}^*(x, t, z) \times \mu_{-,1}^*(x, t, z)] \\ &= \frac{1}{(z \mp q_o)^2} \left(m_{\pm}^{(-1)}(x, t) + m_{\pm}^{(0)}(x, t)(z \mp q_o) + o(z \mp q_o) \right), \quad z \rightarrow \pm q_o. \end{aligned}$$

Thus

$$\begin{aligned} m(x, t, z) &= \mathbf{J}[\mu_{+,3}^*(x, t, z) \times \mu_{-,1}^*(x, t, z)] / \gamma(z) \\ &= \frac{(z - q_o)(z + q_o)}{z^2} \frac{1}{(z \mp q_o)^2} \left(m_{\pm}^{(-1)}(x, t) + m_{\pm}^{(0)}(x, t)(z \mp q_o) + o(z \mp q_o) \right), \\ m(x, t, z) &= \pm \frac{2}{q_o} \frac{1}{z \mp q_o} m_{\pm}^{(-1)}(x, t) \pm \frac{2}{q_o} m_{\pm}^{(0)}(x, t) + o(1), \quad z \rightarrow \pm q_o \end{aligned}$$

with $m_{\pm}^{(-1)}(x, t)$ defined as in Eq. (3.12). Using a similar argument one can obtain

$$\tilde{m}(x, t, z) = \pm \frac{2}{q_o} \frac{1}{z \mp q_o} \tilde{m}_{\pm}^{(-1)}(x, t) \pm \frac{2}{q_o} \tilde{m}_{\pm}^{(0)}(x, t) + o(1), \quad z \rightarrow \pm q_o$$

with $\tilde{m}_{\pm}^{(-1)}(x, t)$ defined as in Eq. (3.13). Higher order expansions can be found similarly by placing further restrictions on the potential and looking at higher order z -derivatives.

Observe that from Eq. (2.4),

$$\lim_{z \rightarrow q_o} \mathbf{E}_{\pm,1}(z) = i \lim_{z \rightarrow q_o} \mathbf{E}_{\pm,3}(z), \quad \lim_{z \rightarrow -q_o} \mathbf{E}_{\pm,1}(z) = -i \lim_{z \rightarrow -q_o} \mathbf{E}_{\pm,3}(z).$$

Using the asymptotic behavior of $\phi_{\pm}(x, t, z)$ when $x \rightarrow \pm\infty$ and the fact that $\theta_1(x, t, z) = 0$ when $z \rightarrow \pm q_o$, one can show that,

$$\phi_{\pm,1}(x, t, q_o) = i\phi_{\pm,3}(x, t, q_o), \quad \phi_{\pm,1}(x, t, -q_o) = -i\phi_{\pm,3}(x, t, -q_o). \quad (3.14)$$

Next, we characterize the limiting behavior of the scattering matrix near the branch points. It is easy to express all entries of the scattering matrix $\mathbf{A}(z)$ as Wronskians

$$a_{j\ell}(z) = \frac{z^2 - q_o^2}{z^2} \text{wr}_{j\ell}(z),$$

where

$$\begin{aligned} \text{wr}_{j\ell}(z) &= \text{wr}(\phi_{-, \ell}(0, 0, z), \phi_{+, j+1}(0, 0, z), \phi_{+, j+2}(0, 0, z)) \\ &= \text{wr}(\mu_{-, \ell}(0, 0, z), \mu_{+, j+1}(0, 0, z), \mu_{+, j+2}(0, 0, z)) \end{aligned}$$

with $j+1$ and $j+2$ calculated modulo 3.

For example let us find the branching behavior of $a_{11}(z)$ and $a_{21}(z)$. If $\mathbf{q}(x, t) - \mathbf{q}_{\pm} \in L^{1,1}(\mathbb{R}_x^{\pm})$ then by using (3.8)-(3.10) we have

$$\begin{aligned} \text{wr}_{11}(z) &= \text{wr}(\mu_{-,1}(0, 0, z), \mu_{+,2}(0, 0, z), \mu_{+,3}(0, 0, z)) \\ &= \frac{1}{(z \mp q_o)^2} (\text{wr}(\hat{u}_{\pm,1}, \check{u}_{\pm,2}, \check{u}_{\pm,3}) + o(1)), \quad z \rightarrow \pm q_o, \end{aligned}$$

and

$$\begin{aligned} \text{wr}_{12}(z) &= \text{wr}(\mu_{-,2}(0, 0, z), \mu_{+,2}(0, 0, z), \mu_{+,3}(0, 0, z)) \\ &= \frac{1}{(z \mp q_o)} (\text{wr}(\hat{u}_{\pm,2}, \check{u}_{\pm,2}, \check{u}_{\pm,3}) + o(1)), \quad z \rightarrow \pm q_o, \end{aligned}$$

from which it follows that

$$\begin{aligned} a_{11}(z) &= \pm \frac{2}{q_o} \frac{1}{z \mp q_o} \text{wr}(\hat{u}_{\pm,1}, \check{u}_{\pm,2}, \check{u}_{\pm,3}) + o\left(\frac{1}{z \mp q_o}\right), \quad z \rightarrow \pm q_o, \\ a_{12}(z) &= \pm \frac{2}{q_o} \text{wr}(\hat{u}_{\pm,2}, \check{u}_{\pm,2}, \check{u}_{\pm,3}) + o(1), \quad z \rightarrow \pm q_o, \end{aligned}$$

here $\hat{u}_{\pm, \ell} = \hat{u}_{\pm, \ell}(0, 0)$, $\check{u}_{\pm, j} = \check{u}_{\pm, j}(0, 0)$.

Similarly, assuming $\mathbf{q}(x, t) - \mathbf{q}_{\pm} \in L^{1,2}(\mathbb{R}_x^{\pm})$ and using (3.11) we have

$$\begin{aligned} \text{wr}_{11}(z) &= \text{wr}(\mu_{-,1}(0, 0, z), \mu_{+,2}(0, 0, z), \mu_{+,3}(0, 0, z)) \\ &= \frac{1}{(z \mp q_o)^2} \left[\text{wr}(\hat{u}_{\pm,1} + \hat{v}_{\pm,1}(z \mp q_o), \check{u}_{\pm,2} + \check{v}_{\pm,2}(z \mp q_o), \check{u}_{\pm,3} + \check{v}_{\pm,3}(z \mp q_o)) \right. \\ &\quad \left. + o(z \mp q_o) \right], \quad z \rightarrow \pm q_o, \end{aligned}$$

i.e.,

$$\text{wr}_{11}(z) = \frac{1}{(z \mp q_o)^2} \left(\text{wr}_{11,\pm}^{(-1)} + \text{wr}_{11,\pm}^{(0)}(z \mp q_o) + o(z \mp q_o) \right), \quad z \rightarrow \pm q_o,$$

and

$$\begin{aligned} \text{wr}_{12}(z) &= \text{wr}(\mu_{-,2}(0,0,z), \mu_{+,2}(0,0,z), \mu_{+,3}(0,0,z)) \\ &= \frac{1}{(z \mp q_o)} \left[\text{wr}(\hat{u}_{\pm,2} + \hat{v}_{\pm,2}(z \mp q_o), \check{u}_{\pm,2} + \check{v}_{\pm,2}(z \mp q_o), \check{u}_{\pm,3} + \check{v}_{\pm,3}(z \mp q_o)) \right. \\ &\quad \left. + o(z \mp q_o) \right], \quad z \rightarrow \pm q_o, \end{aligned}$$

i.e.,

$$\text{wr}_{12}(z) = \frac{1}{(z \mp q_o)} \left(\text{wr}_{12,\pm}^{(0)} + \text{wr}_{12,\pm}^{(1)}(z \mp q_o) + o(z \mp q_o) \right), \quad z \rightarrow \pm q_o.$$

Thus,

$$\begin{aligned} a_{11}(z) &= \pm \frac{2}{q_o} \frac{1}{z \mp q_o} \text{wr}_{11,\pm}^{(-1)} \pm \frac{2}{q_o} \text{wr}_{11,\pm}^{(0)} + o(1), \quad z \rightarrow \pm q_o, \\ a_{12}(z) &= \pm \frac{2}{q_o} \text{wr}_{12,\pm}^{(0)} \pm \frac{2}{q_o} \text{wr}_{12,\pm}^{(1)}(z \mp q_o) + o(z \mp q_o), \quad z \rightarrow \pm q_o, \end{aligned}$$

where $\text{wr}_{11,\pm}^{(-1)} = \text{wr}(\hat{u}_{\pm,1}, \check{u}_{\pm,2}, \check{u}_{\pm,3})$ and $\text{wr}_{12,\pm}^{(0)} = \text{wr}(\hat{u}_{\pm,2}, \check{u}_{\pm,2}, \check{u}_{\pm,3})$. Note that using (3.14) we obtain $a_{21}(z) = a_{22}(z) = a_{23}(z) = 0$, when $z \rightarrow \pm q_o$. Furthermore, the asymptotic behavior of $\mathbf{A}(z)$ in neighborhood of the branch points can be written as

$$\mathbf{A}(z) = \pm \frac{2}{q_o} \frac{1}{z \mp q_o} \mathbf{A}_{\pm}^{(-1)} \pm \frac{2}{q_o} \mathbf{A}_{\pm}^{(0)} + o(1), \quad (3.15)$$

where

$$\mathbf{A}_{\pm}^{(-1)} = \text{wr}_{11,\pm}^{(-1)} \begin{pmatrix} 1 & 0 & \mp i \\ 0 & 0 & 0 \\ \mp i & 0 & -1 \end{pmatrix}, \quad \mathbf{A}_{\pm}^{(0)} = \text{wr}_{12,\pm}^{(0)} \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & \mp i & 0 \end{pmatrix}.$$

One could continue this analysis by placing further restrictions on the potential if higher order terms in the expansion of $\text{wr}_{j\ell}$ are needed.

Finally, we discuss the limiting behavior of the reflection coefficients near the branch points. Eqs. (2.25) and (2.26) imply that $\rho_1(z)$ [resp. $\rho_2(z)$] and $\hat{\rho}_1(z)$ [resp. $\hat{\rho}_2(z)$] have the same behavior near the branch points. Therefore it is sufficient to consider the branch point behavior of $\rho_1(z)$ and $\rho_2(z)$.

First suppose $\mathbf{q}(x, t) - \mathbf{q}_{\pm} \in L^{1,1}(\mathbb{R}_x^{\pm})$. We know that $\rho_1(z) = -a_{31}^*(z)/a_{11}^*(z)$, and $\rho_2(z) = a_{21}(z)/a_{11}(z)$. As a consequence of (3.15), we obtain

$$\lim_{z \rightarrow \pm q_o} \rho_1(z) = \mp i, \quad \lim_{z \rightarrow \pm q_o} \rho_2(z) = 0.$$

3.3. Non-existence of reflectionless potentials

An important novelty of the Manakov system with non-parallel NZBC is that, unlike the case of the scalar defocusing NLS with NZBC [22] and that of the Manakov system with parallel NZBC [12], in the non-parallel case — i.e., when the asymptotic polarization vectors $\mathbf{q}_\pm$ are not parallel — no reflectionless potentials exist. That is:

Theorem 3.1. *There are no solutions $\mathbf{q}(x, t)$ of the Manakov system (1.3) satisfying the boundary conditions (1.2) with $\mathbf{q}_+ \nparallel \mathbf{q}_-$ and $\mathbf{q} - \mathbf{q}_\pm \in L^1(\mathbb{R})$ for which $\rho_1(z) \equiv \rho_2(z) \equiv 0$ for all $z \in \mathbb{R}$.*

The proof of Theorem 3.1 proceeds by contradiction. Let us assume that $\rho_1(z) \equiv \rho_2(z) \equiv 0$, for all $z \in \mathbb{R}$. We can use the first half of (2.18) to eliminate $\phi_{+,2}$ in the scattering relation (2.13), obtaining

$$\phi_{+,3}(x, t, z) = \frac{\phi_{-,3}(x, t, z)}{a_{33}(z)}. \quad (3.16)$$

Next we need to treat the orthogonal and non-orthogonal cases separately. In the non-orthogonal case, considering the asymptotic behavior as $z \rightarrow \infty$, comparing (2.29) and (2.31) gives

$$\begin{pmatrix} 0 \\ (1/q_o)\mathbf{q}_+ \end{pmatrix} = \begin{pmatrix} 0 \\ (q_o/\mathbf{q}_+^\dagger \mathbf{q}_-)\mathbf{q}_- \end{pmatrix}$$

and this would imply $\mathbf{q}_+ = (q_o^2/\mathbf{q}_+^\dagger \mathbf{q}_-)\mathbf{q}_-$, which is not possible in the non-parallel case.

The above proof does not apply in the orthogonal case, since the quantity $\mathbf{q}_+^\dagger \mathbf{q}_-$ appearing in the denominator is zero. Nonetheless, even in the orthogonal case, no reflectionless potential exist. To see this, suppose again that $\rho_1(z) \equiv \rho_2(z) \equiv 0$ for all $z \in \mathbb{R}$. Corollary 2.6 implies that $\lim_{z \rightarrow \infty} z a_{33}(z)$ is non-zero and finite. Using Eq. (3.16) one can obtain

$$\frac{\phi_{+,3}(x, t, z)}{z} = \frac{\phi_{-,3}(x, t, z)}{z a_{33}(z)}.$$

After taking the limit $z \rightarrow \infty$, comparing (2.29) we have $\mathbf{q}_- = \mathbf{0}$, contradicting the starting assumption that $\|\mathbf{q}_-\| = q_o > 0$.

4. Inverse Problem

As usual, the inverse scattering problem is formulated in terms of an appropriate RHP. To this end, we need a suitable jump condition that expresses eigenfunctions that are meromorphic in the upper-half z -plane in terms of eigenfunctions that are meromorphic in the lower-half z -plane. The desired eigenfunctions are the columns of $\Phi^\pm(x, t, z)$ in (3.1), and, as in the scalar case, the jump condition is provided by the scattering relation (2.13). The calculations, however, are considerably more involved than in the scalar case. We will also see that the construction of the RHP breaks down in the special case of orthogonal NZBC because some of the quantities become singular in that case, and as a result the orthogonal case requires a different treatment.

4.1. Riemann-Hilbert problem: non-orthogonal boundary conditions

Since some of the Jost eigenfunctions are not analytic in general, the RHP is formulated in terms of the fundamental analytic eigenfunctions $\Phi^\pm(x, t, z)$ defined in (3.1). We therefore start by eliminating the non-analytic eigenfunctions $\phi_{\pm,2}(x, t, z)$ from (2.13) using (2.17) and (2.18). We then use (2.24) to write the resulting expressions in terms of reflection coefficients defined in (2.25) and (2.26)

$$\begin{aligned}\phi_{+,3}(x, t, z) &= \left[\frac{iz}{q_o \gamma(z)} \hat{\rho}_2(z) \rho_2^*(z) - \hat{\rho}_1^*(z) \right] \phi_{+,1}(x, t, z) \\ &\quad - \frac{iz}{q_o} \hat{\rho}_2(z) \left[-\frac{\bar{\chi}(x, t, z)}{b_{11}(z)} \right] + \frac{\phi_{-,3}(x, t, z)}{a_{33}(z)}, \\ \frac{\phi_{-,1}(x, t, z)}{a_{11}(z)} &= \left[1 - \frac{|\rho_2(z)|^2}{\gamma(z)} \right] \phi_{+,1}(x, t, z) + \rho_2(z) \left[-\frac{\bar{\chi}(x, t, z)}{b_{11}(z)} \right] \\ &\quad - \rho_1^*(z) \phi_{+,3}(x, t, z), \\ \frac{\chi(x, t, z)}{b_{33}(z)} &= \frac{-\rho_2^*(z)}{\gamma(z)} \phi_{+,1}(x, t, z) + \left[-\frac{\bar{\chi}(x, t, z)}{b_{11}(z)} \right] + \frac{iz}{q_o \gamma(z)} \hat{\rho}_2^*(z) \phi_{+,3}(x, t, z).\end{aligned}\tag{4.1}$$

The jump conditions are obtained by combining the above expression for $\phi_{+,3}(x, t, z)$ with the other two equations.

Lemma 4.1. *The sectionally meromorphic matrix $\mathbf{M}(x, t, z)$, defined as*

$$\mathbf{M}(x, t, z) = \begin{cases} \Phi^+ e^{-i\Theta} \text{diag} \left(\frac{1}{a_{11}}, \frac{1}{b_{33}}, \frac{1}{\gamma} \right) = \left(\frac{\mu_{-,1}}{a_{11}}, \frac{m}{b_{33}}, \frac{\mu_{+,3}}{\gamma} \right), & \Im z > 0, \\ \Phi^- e^{-i\Theta} \text{diag} \left(\frac{1}{\gamma}, \frac{1}{b_{11}}, \frac{1}{a_{33}} \right) = \left(\frac{\mu_{+,1}}{\gamma}, -\frac{\bar{m}}{b_{11}}, \frac{\mu_{-,3}}{a_{33}} \right), & \Im z < 0, \end{cases}\tag{4.2}$$

satisfies the jump condition

$$\mathbf{M}^+(x, t, z) = \mathbf{M}^-(x, t, z) e^{i\Theta(x, t, z)} \mathbf{L}(z) e^{-i\Theta(x, t, z)}, \quad z \in \mathbb{R} \setminus \{0, \pm q_o\},\tag{4.3}$$

where $\mathbf{M}^\pm(x, t, z)$ denote the projection of $\mathbf{M}(x, t, z)$ to the real z -axis from above/below, and

$$\mathbf{L}(z) = \begin{pmatrix} L_1 & L_2 & L_3 \end{pmatrix},$$

where

$$\begin{aligned}L_1 &= \begin{pmatrix} 1 + (q_o^2/z^2 - q_o^2) - |\rho_2|^2 - \rho_1^* [(iz/q_o) \hat{\rho}_2 \rho_2^* - \gamma \hat{\rho}_1^*] \\ \rho_2 + (iz/q_o) \rho_1^* \hat{\rho}_2 \\ -\rho_1^* \end{pmatrix}, \\ L_2 &= \begin{pmatrix} -\rho_2^* - [(z^2/q_o^2 \gamma) \rho_2^* |\hat{\rho}_2|^2 + (iz/q_o) \hat{\rho}_2^* \hat{\rho}_1^*] \\ 1 + (z^2/q_o^2 \gamma) |\hat{\rho}_2|^2 \\ (iz/q_o \gamma) \hat{\rho}_2^* \end{pmatrix}, \\ L_3 &= \begin{pmatrix} -\hat{\rho}_1^* + (iz/q_o \gamma) \hat{\rho}_2 \rho_2^* \\ (-iz/q_o \gamma) \hat{\rho}_2 \\ 1 - (q_o^2/z^2) \end{pmatrix}.\end{aligned}$$

Using Corollary 2.7 one can show that $\mathbf{L}(z) = \mathcal{O}(1)$ as $z \rightarrow \infty$ on the real z -axis. Note that $\gamma(z) = \mathcal{O}(z^2)$ when $z \rightarrow 0$, therefore $\mathbf{L}(z)$ has a double pole when z approaches zero. However, Corollaries 2.7, 2.3 and the Eq. (4.1) yield that $\mathbf{M}^-(x, t, z) e^{i\Theta(x, t, z)} \mathbf{L}(z) e^{-i\Theta(x, t, z)}$ has a simple pole when $z \rightarrow 0$ on the real z -axis. Moreover, using Eq. (3.2) we have $\det \mathbf{M}(x, t, z) = 1$ when $\Im z \neq 0$.

To solve (4.3), we need to take into account its normalization. In other words, we consider the leading order asymptotic behavior of $\mathbf{M}^\pm$ as $z \rightarrow \infty$ and $z \rightarrow 0$. Using the information from Sections 2.5, 3.2 together with the definitions in (4.2), we have the following:

Lemma 4.2. *If $\mathbf{q}(x, t) - \mathbf{q}_\pm \in L^{1,2}(\mathbb{R}_x^\pm)$, the matrices $\mathbf{M}^\pm(x, t, z)$ defined in (4.2) have the following behavior:*

$$\begin{aligned} \mathbf{M}^\pm(x, t, z) &= \mathbf{M}_\infty^\pm + \mathcal{O}\left(\frac{1}{z}\right) \quad \text{as } z \rightarrow \infty, \\ \mathbf{M}^\pm(x, t, z) &= \frac{i}{z} \mathbf{M}_0^\pm + \mathcal{O}(1) \quad \text{as } z \rightarrow 0 \end{aligned} \quad (4.4)$$

with

$$\begin{aligned} \mathbf{L}(z) &= \mathbf{L}_\infty + \mathcal{O}\left(\frac{1}{z}\right) \quad \text{as } z \rightarrow \infty, \\ \mathbf{L}(z) &= -\frac{q_o^2}{z^2} \mathbf{L}_0^{(-2)} + \mathcal{O}(1) \quad \text{as } z \rightarrow 0, \end{aligned}$$

where

$$\begin{aligned} \mathbf{M}_\infty^+ &= \begin{pmatrix} 1 & 0 & 0 \\ 0 & (q_o/\mathbf{q}_+^\dagger \mathbf{q}_+) \mathbf{q}_-^\perp & \mathbf{q}_+/q_o \end{pmatrix}, \quad \mathbf{M}_\infty^- = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \mathbf{q}_+^\perp/q_o & (q_o/\mathbf{q}_+^\dagger \mathbf{q}_-) \mathbf{q}_- \end{pmatrix}, \\ \mathbf{M}_0^+ &= \begin{pmatrix} 0 & 0 & -q_o \\ 0 & 0 & 0 \end{pmatrix}, \quad \mathbf{M}_0^- = \begin{pmatrix} 0 & 0 & 0 \\ \mathbf{q}_+ & 0 & 0 \end{pmatrix}, \\ \mathbf{L}_\infty &= \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 + |((\mathbf{q}_-^\perp)^\dagger \mathbf{q}_+)/(\mathbf{q}_+^\dagger \mathbf{q}_-)|^2 & ((\mathbf{q}_-^\perp)^\dagger \mathbf{q}_+)/(\mathbf{q}_+^\dagger \mathbf{q}_-) \\ 0 & (\mathbf{q}_+^\dagger \mathbf{q}_-^\perp)/(\mathbf{q}_-^\dagger \mathbf{q}_+) & 1 \end{pmatrix}, \quad \mathbf{L}_0^{(-2)} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}. \end{aligned}$$

The definition (4.2) of the sectionally meromorphic matrix $\mathbf{M}$ differs from the one used in [12]. Note that, even though the Jost eigenfunctions have poles as z approaches the branch points $\pm q_o$ from the appropriate regions of the complex plane, $\mathbf{M}(x, t, z)$ and $\mathbf{L}(z)$ remain finite in these limits. Furthermore, one can show that

$$\lim_{z \rightarrow 0} \mathbf{M}^-(x, t, z) e^{-i\Theta(x, t, z)} \mathbf{L}(z) e^{i\Theta(x, t, z)} = \frac{i}{z} (\mathbf{M}_0^+ - \mathbf{M}_0^-).$$

The next step in the formulation of the inverse problem is the conversion of the RHP into a set of linear algebraic-integral equations, which allows one to obtain a reconstruction formula for the potential. This is typically done by subtracting the behavior at infinity and the poles from the discrete spectrum, and then applying Cauchy projectors. In the present

case, however, the analysis is complicated by the fact that, unlike the case of the scalar NLS equation and that of the Manakov system with parallel NZBC, $\mathbf{M}_\infty^+ \neq \mathbf{M}_\infty^-$. (Note that in the parallel case one could write $\mathbf{q}_\pm = \mathbf{q}_o e^{\pm i\alpha}$ for some appropriate real constant α without loss of generality, in which case the matrices $\mathbf{M}_\infty^+$ and $\mathbf{M}_\infty^-$ defined above do indeed coincide.) Next we present two different methods by which one can overcome the difficulty of having different asymptotic limits for $\mathbf{M}$ as $z \rightarrow \infty$.

4.1.1. First method: subtract the asymptotic matrices from the jump condition

As usual, in order to complete the formulation of the RHP one must specify suitable residue conditions at the poles coming from the discrete eigenvalues. These are:

Lemma 4.3. *The meromorphic matrices defined in Lemma 4.1 satisfy the following residue conditions:*

$$\begin{aligned}
 \text{Res}_{z=\zeta_n} \mathbf{M}(x, t, z) &= C_n(\mu_{+,3}(\zeta_n), \mathbf{0}, \mathbf{0}) = \mathbf{M}(x, t, \zeta_n) \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ \gamma(\zeta_n) C_n & 0 & 0 \end{pmatrix}, \\
 \text{Res}_{z=\zeta_n^*} \mathbf{M}(x, t, z) &= \bar{C}_n(\mathbf{0}, \mathbf{0}, \mu_{+,1}(\zeta_n^*)) = \mathbf{M}(x, t, \zeta_n^*) \begin{pmatrix} 0 & 0 & \bar{C}_n \gamma(\zeta_n^*) \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \\
 \text{Res}_{z=z_n} \mathbf{M}(x, t, z) &= D_n(m(z_n), \mathbf{0}, \mathbf{0}) = \mathbf{M}(x, t, z_n) \begin{pmatrix} 0 & 0 & 0 \\ D_n b_{33}(z_n) & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \\
 \text{Res}_{z=z_n^*} \mathbf{M}(x, t, z) &= -\bar{D}_n(\mathbf{0}, \mu_{+,1}(z_n^*), \mathbf{0}) = \mathbf{M}(x, t, z_n^*) \begin{pmatrix} 0 & -\bar{D}_n \gamma(z_n^*) & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \\
 \text{Res}_{z=q_o^2/z_n} \mathbf{M}(x, t, z) &= \check{D}_n(\mathbf{0}, \mathbf{0}, \bar{m}(q_o^2/z_n)) = \mathbf{M}(x, t, q_o^2/z_n) \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -\check{D}_n b_{11}(q_o^2/z_n) \\ 0 & 0 & 0 \end{pmatrix}, \\
 \text{Res}_{z=q_o^2/z_n^*} \mathbf{M}(x, t, z) &= \hat{D}_n(\mathbf{0}, \mu_{+,3}(q_o^2/z_n^*), \mathbf{0}) = \mathbf{M}(x, t, q_o^2/z_n^*) \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & \hat{D}_n \gamma(q_o^2/z_n^*) & 0 \end{pmatrix},
 \end{aligned} \tag{4.5}$$

with

$$\begin{aligned}
 C_n(x, t) &= \frac{c_n}{a'_{11}(\zeta_n)} e^{-2i\theta_1(x, t, \zeta_n)}, & \bar{C}_n(x, t) &= \frac{\bar{c}_n}{a'_{33}(\zeta_n^*)} e^{-2i\theta_1(x, t, \zeta_n)}, \\
 D_n(x, t) &= \frac{d_n}{a'_{11}(z_n)} e^{-i(\theta_1 - \theta_2)(z_n)}, & \check{D}_n(x, t) &= \frac{\check{d}_n}{a'_{33}(q_o^2/z_n)} e^{-i(\theta_1 - \theta_2)(z_n)}, \\
 \hat{D}_n(x, t) &= \frac{\hat{d}_n}{b'_{33}(q_o^2/z_n^*)} e^{i(\theta_1 - \theta_2)(z_n^*)}, & \bar{D}_n(x, t) &= \frac{\bar{d}_n}{b'_{11}(z_n^*)} e^{i(\theta_1 - \theta_2)(z_n^*)}.
 \end{aligned} \tag{4.6}$$

Eqs. (4.5) are obtained using (3.5) and (3.6). Next, note the following symmetries (which can be found using (2.19) and (2.24)):

$$\begin{aligned} a'_{11}(z)|_{z=z_n} &= \left[b'_{11}(z)|_{z=z_n^*} \right]^*, \\ a'_{33}(z)|_{z=q_0^2/z_n} &= \left[b'_{33}(z)|_{z=q_0^2/z_n^*} \right]^*, \\ a'_{11}(z)|_{z=z_n} &= -\frac{q_0^2}{z_n^2} a'_{33}(z)|_{z=q_0^2/z_n}. \end{aligned}$$

Using (4.6) and the above symmetries, one obtains the following symmetries for the norming constants:

Lemma 4.4 (Symmetries of the residues). *The norming constants $C_n, \bar{C}_n, D_n, \check{D}_n, \hat{D}_n$, and $\bar{D}_n$ defined in Lemma 4.3 obey the following symmetry relations:*

$$\begin{aligned} \bar{C}_n(x, t) &= e^{-2i \arg(\zeta_n)} C_n(x, t), \quad \bar{C}_n(x, t) = C_n^*(x, t), \\ \check{D}_n(x, t) &= \frac{iq_0^3}{z_n^3} D_n(x, t), \quad \bar{D}_n(x, t) = -\left[\frac{b_{33}(z_n)}{\gamma(z_n)} \right]^* D_n^*(x, t), \\ \hat{D}_n(x, t) &= \frac{iq_0}{z_n^*} \left[\frac{b_{33}(z_n)}{\gamma(z_n)} \right]^* D_n^*(x, t). \end{aligned}$$

Now that the residue conditions have been specified, we are ready to derive the formal solution of the RHP and reconstruction formula for the solution of the Manakov system. To solve (4.3), we subtract from both sides of (4.3) the asymptotic behavior of $\mathbf{M}$ as $z \rightarrow \infty$ and $z \rightarrow 0$ as well as the residue contributions from the poles inside and on the circle of radius q_0 . For simplicity, suppose first that there is no discrete spectrum. After subtracting $\mathbf{M}_\infty^+$ and $\frac{i}{z} \mathbf{M}_0^+$ from both sides of Eq. (4.3) and rewriting $\mathbf{L} = \mathbf{I} + \mathbf{K}$ we have

$$\begin{aligned} &\mathbf{M}^+(x, t, z) - \mathbf{M}_\infty^+ - \frac{i}{z} \mathbf{M}_0^+ \\ &= \mathbf{M}^-(x, t, z) - \mathbf{M}_\infty^- - \frac{i}{z} \mathbf{M}_0^- + \mathbf{M}_\infty^- - \mathbf{M}_\infty^+ + \frac{i}{z} (\mathbf{M}_0^- - \mathbf{M}_0^+) \\ &\quad + \mathbf{M}^-(x, t, z) e^{i\Theta(x, t, z)} \mathbf{K}(z) e^{-i\Theta(x, t, z)}. \end{aligned}$$

The left hand side of the resulting, regularized RHP is analytic in the upper half z -plane. Also, the right hand side is analytic in the lower half z -plane. Now recall Plemelj's formulae

$$(P_\pm f)(s) = \frac{1}{2\pi i} \lim_{\epsilon \rightarrow 0^+} \int_{\mathbb{R}} \frac{f(\zeta)}{\zeta - (s \pm i\epsilon)} d\zeta, \quad s \in \mathbb{C} \setminus \mathbb{R}.$$

If $f_\pm$ are analytic in the upper half (respectively lower half) of the z -plane, and if $f_\pm = \mathcal{O}(1/z)$ as $z \rightarrow \infty$ in the appropriate half plane, then $P_\pm f_\pm = \pm f_\pm$ and $P_+ f_- = P_- f_+ = 0$. Now, allowing for the possible presence of a discrete spectrum, and recalling (4.4), (4.5), we subtract the following quantity from both sides of (4.3):

$$\mathbf{M}_\infty^+ + \frac{i}{z} \mathbf{M}_0^+ + \sum_{j=1}^{N_1} \left(\frac{\text{Res}_{z=\zeta_j} \mathbf{M}(z)}{z - \zeta_j} + \frac{\text{Res}_{z=\zeta_j^*} \mathbf{M}(z)}{z - \zeta_j^*} \right) + \sum_{j=1}^{N_2} \left(\frac{\text{Res}_{z=z_j^*} \mathbf{M}(z)}{z - z_j^*} + \frac{\text{Res}_{z=z_j} \mathbf{M}(z)}{z - z_j} \right)$$

$$+ \sum_{j=1}^{N_2} \left(\frac{\text{Res}_{z=q_o^2/z_j} \mathbf{M}(z)}{z - q_o^2/z_j} + \frac{\text{Res}_{z=q_o^2/z_j^*} \mathbf{M}(z)}{z - q_o^2/z_j^*} \right).$$

Applying $P_{\pm}$ to the regularized RHP, we then obtain

Theorem 4.1. *For all $z \notin \mathbb{R}$, the solution of the RHP Lemmas 4.1-4.3 is given by*

$$\begin{aligned} \mathbf{M}(x, t, z) = & \mathbf{M}_{\infty}(z) \\ & + \frac{i}{z} \mathbf{M}_0(z) + \frac{1}{2\pi i} \int_{\mathbb{R}} \left(\Delta \mathbf{M}_{\infty} + \frac{i}{\zeta} \Delta \mathbf{M}_0 + \mathbf{M}^{-}(x, t, \zeta) e^{i\Theta(\zeta)} \mathbf{K}(\zeta) e^{-i\Theta(\zeta)} \right) \frac{d\zeta}{\zeta - z} \\ & + \sum_{j=1}^{N_1} \left(\frac{\text{Res}_{z=\zeta_j} \mathbf{M}(z)}{z - \zeta_j} + \frac{\text{Res}_{z=\zeta_j^*} \mathbf{M}(z)}{z - \zeta_j^*} \right) + \sum_{j=1}^{N_2} \left(\frac{\text{Res}_{z=z_j^*} \mathbf{M}(z)}{z - z_j^*} + \frac{\text{Res}_{z=z_j} \mathbf{M}(z)}{z - z_j} \right) \\ & + \sum_{j=1}^{N_2} \left(\frac{\text{Res}_{z=q_o^2/z_j} \mathbf{M}(z)}{z - q_o^2/z_j} + \frac{\text{Res}_{z=q_o^2/z_j^*} \mathbf{M}(z)}{z - q_o^2/z_j^*} \right), \end{aligned} \quad (4.7)$$

where $\mathbf{K} = \mathbf{L} - \mathbf{I}$, $\mathbf{M}_{\infty}(z) = \mathbf{M}_{\infty}^{\pm}$ and $\mathbf{M}_0(z) = \mathbf{M}_0^{\pm}$ for $z \in \mathbb{C}^{\pm}$ with $\Delta \mathbf{M}_{\infty} = \mathbf{M}_{\infty}^{-} - \mathbf{M}_{\infty}^{+}$ and $\Delta \mathbf{M}_0 = \mathbf{M}_0^{-} - \mathbf{M}_0^{+}$. Moreover, the relevant columns for the residue conditions in Lemma 4.3 are given by

$$\begin{aligned} \mathbf{M}_1(x, t, w) = & \begin{pmatrix} 1 \\ i\mathbf{q}_+/w \end{pmatrix} \\ & + \frac{1}{2\pi i} \int_{\mathbb{R}} \left(\Delta \mathbf{M}_{\infty,1} + \frac{i}{\zeta} \Delta \mathbf{M}_{0,1} + (\mathbf{M}^{-}(x, t, \zeta) e^{i\Theta(\zeta)} \mathbf{K}(\zeta) e^{-i\Theta(\zeta)})_1 \right) \frac{d\zeta}{\zeta - w} \\ & + \sum_{j=1}^{N_1} \left[\frac{C_j}{w - \zeta_j} \gamma(\zeta_j) \mathbf{M}_3(\zeta_j) \right] + \sum_{j=1}^{N_2} \left[\frac{D_j}{w - z_j} b_{33}(z_j) \mathbf{M}_2(z_j) \right], \quad w = \zeta_n^*, z_n^*, \\ \mathbf{M}_3(x, t, w) = & \begin{pmatrix} -iq_o/w \\ \mathbf{q}_+/q_o \end{pmatrix} \\ & + \frac{1}{2\pi i} \int_{\mathbb{R}} \left(\Delta \mathbf{M}_{\infty,3} + \frac{i}{\zeta} \Delta \mathbf{M}_{0,3} + (\mathbf{M}^{-}(x, t, \zeta) e^{i\Theta(\zeta)} \mathbf{K}(\zeta) e^{-i\Theta(\zeta)})_3 \right) \frac{d\zeta}{\zeta - w} \\ & + \sum_{j=1}^{N_1} \left[\frac{\bar{C}_j}{w - \zeta_j^*} \gamma(\zeta_j^*) \mathbf{M}_1(\zeta_j^*) \right] - \sum_{j=1}^{N_2} \left[\frac{\check{D}_j}{w - q_o^2/z_j} b_{11} \frac{q_o^2}{z_j} \mathbf{M}_2 \frac{q_o^2}{z_j} \right], \quad w = \zeta_n, q_o^2/z_n^*, \\ \mathbf{M}_2(x, t, q_o^2/z_{\ell}) = & \begin{pmatrix} 0 \\ \mathbf{q}_+/q_o \end{pmatrix} \\ & + \frac{1}{2\pi i} \int_{\mathbb{R}} \left(\Delta \mathbf{M}_{\infty,2} + \frac{i}{\zeta} \Delta \mathbf{M}_{0,2} + (\mathbf{M}^{-}(x, t, \zeta) e^{i\Theta(\zeta)} \mathbf{K}(\zeta) e^{-i\Theta(\zeta)})_2 \right) \frac{d\zeta}{\zeta - q_o^2/z_{\ell}} \\ & - \sum_{j=1}^{N_2} \left[\frac{\bar{D}_j}{q_o^2/z_{\ell} - z_j^*} \gamma(z_j^*) \mathbf{M}_1(z_j^*) \right] + \sum_{j=1}^{N_2} \left[\frac{\hat{D}_j}{q_o^2/z_{\ell} - q_o^2/z_j^*} \gamma \frac{q_o^2}{z_j^*} \mathbf{M}_3 \frac{q_o^2}{z_j^*} \right], \end{aligned}$$

$$\begin{aligned} \mathbf{M}_2(x, t, z_\ell) = & \begin{pmatrix} 0 \\ (q_o / \mathbf{q}_-^\dagger \mathbf{q}_+) \mathbf{q}_-^\perp \end{pmatrix} \\ & + \frac{1}{2\pi i} \int_{\mathbb{R}} \left(\Delta \mathbf{M}_{\infty, 2} + \frac{i}{\zeta} \Delta \mathbf{M}_{0, 2} + (\mathbf{M}^-(x, t, \zeta) e^{i\Theta(\zeta)} \mathbf{K}(\zeta) e^{-i\Theta(\zeta)})_2 \right) \frac{d\zeta}{\zeta - w} \\ & - \sum_{j=1}^{N_2} \left[\frac{\bar{D}_j}{z_\ell - z_j^*} \gamma(z_j^*) \mathbf{M}_1(z_j^*) \right] + \sum_{j=1}^{N_2} \left[\frac{\hat{D}_j}{z_\ell - q_o^2 / z_j^*} \gamma \frac{q_o^2}{z_j^*} \mathbf{M}_3 \frac{q_o^2}{z_j^*} \right], \end{aligned}$$

where for brevity we suppressed the x and t dependence when doing so did not lead to confusion. Hereafter we denote j -th column of a matrix A by A_j .

The asymptotics (2.28) imply

$$q_j(x, t) = -i \lim_{z \rightarrow \infty} (z \mathbf{M}_{(j+1)1}(x, t, z)), \quad j = 1, 2. \quad (4.8)$$

We take $\mathbf{M}(x, t, z)$ when $z \in \mathbb{C}^-$ in (4.7) and compare its 2, 1 and 3, 1 elements in the limit as $z \rightarrow \infty$ with the corresponding elements in (2.28) to obtain the following:

Theorem 4.2 (Reconstruction formula). *Let $\mathbf{M}(x, t, z)$ be the solution of the Riemann-Hilbert problem in Theorem 4.1. The corresponding solution $\mathbf{q}(x, t) = (q_1(x, t), q_2(x, t))^T$ of the defocusing Manakov system with non-orthogonal boundary conditions (1.2) is given by*

$$\begin{aligned} q_\ell(x, t) = & q_{+, \ell} + \frac{1}{2\pi} \int_{\mathbb{R}} \left(\frac{i}{\zeta} q_{+, \ell} + (\mathbf{M}^-(x, t, \zeta) e^{i\Theta(\zeta)} \mathbf{K}(\zeta) e^{-i\Theta(\zeta)})_{(\ell+1)1} \right) d\zeta \\ & - i \sum_{j=1}^{N_1} C_j(x, t) \gamma(\zeta_j) \mathbf{M}_{(\ell+1)3}(x, t, \zeta_j) \\ & - i \sum_{j=1}^{N_2} D_j(x, t) b_{33}(z_j) \mathbf{M}_{(\ell+1)2}(x, t, z_j), \quad \ell = 1, 2. \end{aligned}$$

Note above we took into account that the first column of $\Delta \mathbf{M}_\infty$ is identically zero.

4.1.2. Second method: multiply by the inverse of the asymptotic matrices

In the non-orthogonal case, one can also use an alternative method to convert the RHP into a set of linear integral equations: redefine the sectional meromorphic matrices, multiplying them (in each sector) by the inverse of their limiting value as $z \rightarrow \infty$.

Lemma 4.5. *The modified meromorphic matrices $\tilde{\mathbf{M}}^\pm(x, t, z)$, defined as*

$$\tilde{\mathbf{M}}(x, t, z) = \begin{cases} \mathbf{M}^+(x, t, z) (\mathbf{M}_\infty^+)^{-1}, & \Im z > 0, \\ \mathbf{M}^-(x, t, z) (\mathbf{M}_\infty^-)^{-1}, & \Im z < 0, \end{cases} \quad (4.9)$$

satisfy the new jump condition

$$\tilde{\mathbf{M}}^+(x, t, z) = \tilde{\mathbf{M}}^-(x, t, z) e^{i\Delta(x, t, z)} \tilde{\mathbf{L}}(x, t, z) e^{-i\Delta(x, t, z)}, \quad z \in \mathbb{R} \setminus \{0, \pm q_o\}, \quad (4.10)$$

where $\tilde{\mathbf{M}}^\pm(x, t, z)$ denote the projection of $\tilde{\mathbf{M}}(x, t, z)$ onto the real z -axis from above/below,

$$\tilde{\mathbf{L}}(x, t, z) = \mathbf{M}_\infty^- e^{-i\Sigma(x, t, z)} \mathbf{L}(z) e^{i\Sigma(x, t, z)} (\mathbf{M}_\infty^+)^{-1}, \quad (4.11)$$

and

$$\begin{aligned} \Sigma(x, t, z) &= \text{diag}(0, -\theta_1(x, t, z), \theta_2(x, t, z)), \\ \Delta(x, t, z) &= \Sigma(x, t, z) + \Theta(x, t, z) \\ &= \text{diag}(\theta_1(x, t, z), \theta_2(x, t, z) - \theta_1(x, t, z), \theta_2(x, t, z) - \theta_1(x, t, z)). \end{aligned}$$

Lemma 4.5 can be verified by direct computation. The exponentials in (4.10) and (4.11) arise when taking into account the new normalizations $\tilde{\mathbf{M}}_\infty^\pm$ in the new jump matrix. One can now verify by direct computation that $\tilde{\mathbf{L}} \rightarrow \mathbf{I}$ as $z \rightarrow \infty$. In other words,

$$\lim_{z \rightarrow \infty} \mathbf{M}_\infty^- e^{-i\Sigma(x, t, z)} \mathbf{L}(z) e^{i\Sigma(x, t, z)} (\mathbf{M}_\infty^+)^{-1} = \mathbf{I}.$$

Lemma 4.6. *The matrices $\tilde{\mathbf{M}}^\pm(x, t, z)$ defined in (4.9) have the following asymptotic behavior:*

$$\begin{aligned} \tilde{\mathbf{M}}^\pm(x, t, z) &= \mathbf{I} + \mathcal{O}\left(\frac{1}{z}\right) \quad \text{as } z \rightarrow \infty, \\ \tilde{\mathbf{M}}^\pm(x, t, z) &= \frac{i}{z} \tilde{\mathbf{M}}_0^\pm + \mathcal{O}(1) \quad \text{as } z \rightarrow 0, \end{aligned} \quad (4.12)$$

where

$$\begin{aligned} \tilde{\mathbf{M}}_0^+ &= \mathbf{M}_0^+ (\mathbf{M}_\infty^+)^{-1} = \begin{pmatrix} 0 & (-q_o^2/\mathbf{q}_-^\dagger \mathbf{q}_+^\dagger) \mathbf{q}_-^\dagger \\ 0 & \mathbf{0}_{2 \times 2} \end{pmatrix}, \\ \tilde{\mathbf{M}}_0^- &= \mathbf{M}_0^- (\mathbf{M}_\infty^-)^{-1} = \begin{pmatrix} 0 & \mathbf{0}^T \\ \mathbf{q}_+ & \mathbf{0}_{2 \times 2} \end{pmatrix}. \end{aligned}$$

As before, in order to complete the formulation of the RHP one must specify suitable residue conditions. Using the same approach as before, one obtains:

Lemma 4.7. *The modified meromorphic matrices defined in Lemma 4.5 satisfy the following residue conditions:*

$$\text{Res}_{z=\zeta_n} \tilde{\mathbf{M}}(x, t, z) = C_n(\mu_{+,3}(\zeta_n), \mathbf{0}, \mathbf{0}) = \tilde{\mathbf{M}}(x, t, \zeta_n) \mathbf{M}_\infty^+ \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ \gamma(\zeta_n) C_n & 0 & 0 \end{pmatrix}, \quad (4.13a)$$

$$\begin{aligned} \text{Res}_{z=\zeta_n^*} \tilde{\mathbf{M}}(x, t, z) &= \overline{C}_n(\mathbf{0}, \mathbf{0}, \mu_{+,1}(\zeta_n^*)) (\mathbf{M}_\infty^-)^{-1} \\ &= \tilde{\mathbf{M}}(x, t, \zeta_n^*) \mathbf{M}_\infty^- \begin{pmatrix} 0 & 0 & \gamma(\zeta_n^*) \overline{C}_n \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} (\mathbf{M}_\infty^-)^{-1}, \end{aligned} \quad (4.13b)$$

$$\operatorname{Res}_{z=z_n} \tilde{\mathbf{M}}(x, t, z) = D_n(m(z_n), \mathbf{0}, \mathbf{0}) = \tilde{\mathbf{M}}(x, t, z_n) \mathbf{M}_\infty^+ \begin{pmatrix} 0 & 0 & 0 \\ D_n b_{33}(z_n) & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad (4.13c)$$

$$\begin{aligned} \operatorname{Res}_{z=z_n^*} \tilde{\mathbf{M}}(x, t, z) &= -\overline{D}_n(\mathbf{0}, \mu_{+,1}(z_n^*), \mathbf{0}) (\mathbf{M}_\infty^-)^{-1} \\ &= \tilde{\mathbf{M}}(x, t, z_n^*) \mathbf{M}_\infty^- \begin{pmatrix} 0 & -\overline{D}_n \gamma(z_n^*) & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} (\mathbf{M}_\infty^-)^{-1}, \end{aligned} \quad (4.13d)$$

$$\begin{aligned} \operatorname{Res}_{z=q_o^2/z_n} \tilde{\mathbf{M}}(x, t, z) &= \check{D}_n(\mathbf{0}, \mathbf{0}, \overline{m}(q_o^2/z_n)) (\mathbf{M}_\infty^-)^{-1} \\ &= \tilde{\mathbf{M}}(x, t, q_o^2/z_n) \mathbf{M}_\infty^- \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -\check{D}_n b_{11}(q_o^2/z_n) \\ 0 & 0 & 0 \end{pmatrix} (\mathbf{M}_\infty^-)^{-1}, \end{aligned} \quad (4.13e)$$

$$\begin{aligned} \operatorname{Res}_{z=q_o^2/z_n^*} \tilde{\mathbf{M}}(x, t, z) &= \hat{D}_n(\mathbf{0}, \mu_{+,3}(q_o^2/z_n^*), \mathbf{0}) (\mathbf{M}_\infty^+)^{-1} \\ &= \tilde{\mathbf{M}}(x, t, q_o^2/z_n^*) \mathbf{M}_\infty^+ \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & \hat{D}_n \gamma(q_o^2/z_n^*) & 0 \end{pmatrix} (\mathbf{M}_\infty^+)^{-1}, \end{aligned} \quad (4.13f)$$

where $C_n, \overline{C}_n, D_n, \overline{D}_n, \check{D}_n$ and $\hat{D}_n$ are defined in Lemma 4.3 with $n = 1, \dots, N_1$ for equations involving ζ_n and $n = 1, \dots, N_2$ for equation involving z_n .

The symmetries of those norming constants were given in Lemma 4.4. We are now ready to derive the formal solution of the RHP and the reconstruction formula. To solve (4.10), we subtract from both sides of (4.10) the asymptotic behavior of $\tilde{\mathbf{M}}$ as $z \rightarrow \infty$ and $z \rightarrow 0$ as well as the residue contributions from the poles inside and on the circle of radius q_o . Namely, recalling (4.12) and (4.13), we subtract the following quantity from both sides of (4.10):

$$\begin{aligned} \mathbf{I} + \frac{i}{z} \tilde{\mathbf{M}}_0^+ &+ \sum_{j=1}^{N_1} \left(\frac{\operatorname{Res}_{z=\zeta_j} \tilde{\mathbf{M}}(z)}{z - \zeta_j} + \frac{\operatorname{Res}_{z=\zeta_j^*} \tilde{\mathbf{M}}(z)}{z - \zeta_j^*} \right) + \sum_{j=1}^{N_2} \left(\frac{\operatorname{Res}_{z=z_j^*} \tilde{\mathbf{M}}(z)}{z - z_j^*} + \frac{\operatorname{Res}_{z=z_j} \tilde{\mathbf{M}}(z)}{z - z_j} \right) \\ &+ \sum_{j=1}^{N_2} \left(\frac{\operatorname{Res}_{z=q_o^2/z_j} \tilde{\mathbf{M}}(z)}{z - q_o^2/z_j} + \frac{\operatorname{Res}_{z=q_o^2/z_j^*} \tilde{\mathbf{M}}(z)}{z - q_o^2/z_j^*} \right). \end{aligned}$$

The left-hand side of the resulting, regularized RHP is analytic in the upper half z -plane and is $\mathcal{O}(1/z)$ as $z \rightarrow \infty$ there. Also, the right-hand side is analytic in the lower half z -plane and is $\mathcal{O}(1/z)$ as $z \rightarrow \infty$ there. Applying again $P_\pm$ to the regularized RHP, we then obtain

Theorem 4.3. *If the RHP defined by Lemmas 4.5-4.7 admits a unique solution, it is given by*

$$\tilde{\mathbf{M}}(x, t, z) = \mathbf{I} + \frac{i}{z} \tilde{\mathbf{M}}_0(z) + \frac{1}{2\pi i} \int_{\mathbb{R}} \left(\frac{i}{\zeta} \Delta \tilde{\mathbf{M}}_0 + \tilde{\mathbf{M}}^-(x, t, \zeta) e^{i\Delta(\zeta)} \tilde{\mathbf{K}}(\zeta) e^{-i\Delta(\zeta)} \right) \frac{d\zeta}{\zeta - z}$$

$$\begin{aligned}
& + \sum_{j=1}^{N_1} \left(\frac{\text{Res}_{z=\zeta_j} \tilde{\mathbf{M}}(z)}{z - \zeta_j} + \frac{\text{Res}_{z=\zeta_j^*} \tilde{\mathbf{M}}(z)}{z - \zeta_j^*} \right) + \sum_{j=1}^{N_2} \left(\frac{\text{Res}_{z=z_j^*} \tilde{\mathbf{M}}(z)}{z - z_j^*} + \frac{\text{Res}_{z=z_j} \tilde{\mathbf{M}}(z)}{z - z_j} \right) \\
& + \sum_{j=1}^{N_2} \left(\frac{\text{Res}_{z=q_o^2/z_j} \tilde{\mathbf{M}}(z)}{z - q_o^2/z_j} + \frac{\text{Res}_{z=q_o^2/z_j^*} \tilde{\mathbf{M}}(z)}{z - q_o^2/z_j^*} \right) \quad (4.14)
\end{aligned}$$

for all $z \in \mathbb{C} \setminus \mathbb{R}$, where $\tilde{\mathbf{K}} = \tilde{\mathbf{L}} - \mathbf{I}$, $\tilde{\mathbf{M}}_0(z) = \tilde{\mathbf{M}}_0^\pm$ for $z \in \mathbb{C}^\pm$ with $\Delta \tilde{\mathbf{M}}_0 = \tilde{\mathbf{M}}_0^- - \tilde{\mathbf{M}}_0^+$ and the left-hand side is $\tilde{\mathbf{M}}(x, t, z) = \tilde{\mathbf{M}}^\pm(x, t, z)$ for $\Im z \gtrless 0$, respectively. Moreover, the eigenfunctions in the residue conditions in the Lemma 4.7 are given by

$$\begin{aligned}
& \tilde{\mathbf{M}}(x, t, w) \mathbf{M}_{\infty,1}^- = \begin{pmatrix} 1 \\ i\mathbf{q}_+/w \end{pmatrix} \\
& + \frac{1}{2\pi i} \int_{\mathbb{R}} \left(\frac{i}{\zeta} \Delta \tilde{\mathbf{M}}_{0,1} + (\tilde{\mathbf{M}}^-(x, t, \zeta) e^{i\Delta(\zeta)} \tilde{\mathbf{K}}(\zeta) e^{-i\Delta(\zeta)})_1 \right) \frac{d\zeta}{\zeta - w} \\
& + \sum_{j=1}^{N_1} \left[\frac{C_j}{w - \zeta_j} \gamma(\zeta_j) \tilde{\mathbf{M}}(\zeta_j) \mathbf{M}_{\infty,3}^+ \right] + \sum_{j=1}^{N_2} \left[\frac{D_j}{w - z_j} b_{33}(z_j) \tilde{\mathbf{M}}(z_j) \mathbf{M}_{\infty,2}^+ \right], \quad w = \zeta_n^*, z_n^*, \\
& \tilde{\mathbf{M}}(x, t, w) \mathbf{M}_{\infty,3}^+ = \begin{pmatrix} -iq_o/w \\ \mathbf{q}_+/q_o \end{pmatrix} \\
& + \frac{1}{2\pi i} \int_{\mathbb{R}} \left(\frac{i}{\zeta} \Delta \tilde{\mathbf{M}}_0 + \tilde{\mathbf{M}}^-(x, t, \zeta) e^{i\Delta(\zeta)} \tilde{\mathbf{K}}(\zeta) e^{-i\Delta(\zeta)} \right) \mathbf{M}_{\infty,3}^+ \frac{d\zeta}{\zeta - w} \\
& + \sum_{j=1}^{N_1} \left[\frac{\bar{C}_j}{w - \zeta_j^*} \gamma(\zeta_j^*) \tilde{\mathbf{M}}(\zeta_j^*) \mathbf{M}_{\infty,1}^- \right] - \sum_{j=1}^{N_2} \left[\frac{\check{D}_j}{w - q_o^2/z_j} b_{11}(q_o^2/z_j) \tilde{\mathbf{M}}(q_o^2/z_j) \mathbf{M}_{\infty,2}^- \right] \\
& - \sum_{j=1}^{N_2} \left[\frac{(q_\perp^\dagger)^\dagger q_+}{q_+^\dagger q_-} \frac{\bar{D}_j}{w - z_j^*} \gamma(z_j^*) \tilde{\mathbf{M}}(z_j^*) \mathbf{M}_{\infty,1}^- \right], \quad w = \zeta_n, q_o^2/z_n^*, \\
& \tilde{\mathbf{M}}(x, t, q_o^2/z_\ell) \mathbf{M}_{\infty,2}^- = \begin{pmatrix} 0 \\ \mathbf{q}_\perp^\dagger/q_o \end{pmatrix} \\
& + \frac{1}{2\pi i} \int_{\mathbb{R}} \left(\frac{i}{\zeta} \Delta \tilde{\mathbf{M}}_0 + \tilde{\mathbf{M}}^-(x, t, \zeta) e^{i\Delta(\zeta)} \tilde{\mathbf{K}}(\zeta) e^{-i\Delta(\zeta)} \right) \mathbf{M}_{\infty,2}^- \frac{d\zeta}{\zeta - q_o^2/z_\ell} \\
& - \sum_{j=1}^{N_2} \left[\frac{\bar{D}_j}{q_o^2/z_\ell - z_j^*} \gamma(z_j^*) \tilde{\mathbf{M}}(z_j^*) \mathbf{M}_{\infty,1}^- \right] + \sum_{j=1}^{N_2} \left[\frac{\hat{D}_j}{q_o^2/z_\ell - q_o^2/z_j^*} \gamma(q_o^2/z_j^*) \tilde{\mathbf{M}}(q_o^2/z_j^*) \mathbf{M}_{\infty,3}^+ \right], \\
& \tilde{\mathbf{M}}(x, t, z_\ell) \mathbf{M}_{\infty,2}^+ = \begin{pmatrix} 0 \\ (q_o/q_-^\dagger q_+) \mathbf{q}_\perp^\dagger \end{pmatrix} \\
& + \frac{1}{2\pi i} \int_{\mathbb{R}} \left(\frac{i}{\zeta} \Delta \tilde{\mathbf{M}}_0 + \tilde{\mathbf{M}}^-(x, t, \zeta) e^{i\Delta(\zeta)} \tilde{\mathbf{K}}(\zeta) e^{-i\Delta(\zeta)} \right) \mathbf{M}_{\infty,2}^+ \frac{d\zeta}{\zeta - z_\ell} \\
& + \sum_{j=1}^{N_1} \left[\frac{q_+^\dagger q_-^\dagger}{q_-^\dagger q_+} \frac{\bar{C}_j}{z_\ell - \zeta_j^*} \gamma(\zeta_j^*) \tilde{\mathbf{M}}(\zeta_j^*) \mathbf{M}_{\infty,1}^- \right] - \sum_{j=1}^{N_2} \left[\frac{q_o^4}{|q_+^\dagger q_-|^2} \frac{\bar{D}_j}{z_\ell - z_j^*} \gamma(z_j^*) \tilde{\mathbf{M}}(z_j^*) \mathbf{M}_{\infty,1}^- \right]
\end{aligned}$$

$$\begin{aligned}
& + \sum_{j=1}^{N_2} \left[\frac{\hat{D}_j}{z_\ell - q_o^2/z_j^*} \gamma(q_o^2/z_j^*) \tilde{\mathbf{M}}(q_o^2/z_j^*) \mathbf{M}_{\infty,3}^+ \right] \\
& - \sum_{j=1}^{N_2} \left[\frac{q_+^\dagger q_-^\perp}{q_-^\dagger q_+} \frac{\check{D}_j}{z_\ell - q_o^2/z_j} b_{11}(q_o^2/z_j) \tilde{\mathbf{M}}(q_o^2/z_j) \mathbf{M}_{\infty,2}^- \right].
\end{aligned}$$

For simplicity, we suppressed x and t dependence when doing so introduces no confusion.

Lemma 4.8. *If the RHP defined by Lemmas 4.5-4.7 admits a unique solution, the matrix $\tilde{\mathbf{M}}(x, t, z)$ satisfies the modified Lax pair*

$$\begin{aligned}
& \tilde{\mathbf{M}}_x(x, t, z) + ik[J, \tilde{\mathbf{M}}(x, t, z)] \\
& = \mathbf{Q}(x, t) \tilde{\mathbf{M}}(x, t, z) - i \tilde{\mathbf{M}}(x, t, z) \mathbf{M}_\infty^\pm (kJ + \Lambda(z)) (\mathbf{M}_\infty^\pm)^{-1}, \quad z \in \mathbb{C}^\pm, \\
& \tilde{\mathbf{M}}_t(x, t, z) - 2ik^2[J, \tilde{\mathbf{M}}(x, t, z)] \\
& = (iJ(\mathbf{Q}^2(x, t) - \mathbf{Q}_x(x, t) - q_o^2) - 2k\mathbf{Q}(x, t)) \tilde{\mathbf{M}}(x, t, z) \\
& \quad + i \tilde{\mathbf{M}}(x, t, z) \mathbf{M}_\infty^\pm (2k^2J + \Omega(z)) (\mathbf{M}_\infty^\pm)^{-1}, \quad z \in \mathbb{C}^\pm
\end{aligned}$$

with $\Lambda(z)$ and $\Omega(z)$ as in (2.8), and

$$\mathbf{Q}(x, t) = -2i \lim_{z \rightarrow \infty} z [J, \tilde{\mathbf{M}}(x, t, z)]. \quad (4.15)$$

The modified Lax pair (4.15) coincides with the one satisfied by $\mathbf{M}(x, t, z)$ except for the last term in the right hand side of each equation. (The reason for the presence of these extra terms is that the transformation from $\mathbf{M}(x, t, z)$ to $\tilde{\mathbf{M}}(x, t, z)$ “mixes” the columns, each of which corresponds to an eigenfunction that is multiplied by a different exponential.) Nonetheless, one can verify by direct computation that the compatibility condition of (4.15) (which is satisfied automatically since the ODEs are derived by assuming $\tilde{\mathbf{M}}(x, t, z)$ is the unique solution of the RHP) is still equivalent to the requirement that $\mathbf{q}(x, t)$ as obtained from (4.15) satisfies the Manakov system. In other words, we have the following:

Corollary 4.1. *Let $\tilde{\mathbf{M}}(x, t, z)$ satisfy Eqs. (4.15). The function $\mathbf{q}(x, t)$ defined by (4.15) satisfies the Manakov system (1.3).*

Taking $\tilde{\mathbf{M}}(x, t, z)$ when $z \in \mathbb{C}^-$ in (4.14) and comparing its 2,1 and 3,1 elements in the limit as $z \rightarrow \infty$ with the corresponding elements in (2.28), as before, we obtain the following:

Theorem 4.4 (Reconstruction formula). *Let $\tilde{\mathbf{M}}(x, t, z)$ be the solution of the Riemann-Hilbert problem in Theorem 4.3. The corresponding solution $\mathbf{q}(x, t) = (q_1(x, t), q_2(x, t))^T$ of the defocusing Manakov system with non-orthogonal boundary conditions (1.2) is reconstructed as*

$$q_\ell(x, t) = \mathbf{q}_{+, \ell} + \frac{1}{2\pi} \int_{\mathbb{R}} \left(\frac{i}{\zeta} \mathbf{q}_{+, \ell} + (\tilde{\mathbf{M}}^-(x, t, \zeta) e^{i\Delta(x, t, \zeta)} \tilde{\mathbf{K}}(x, t, \zeta) e^{-i\Delta(x, t, \zeta)})_{(\ell+1)1} \right) d\zeta$$

$$\begin{aligned}
& -i \sum_{j=1}^{N_1} C_j(x, t) \gamma(\zeta_j) (\tilde{\mathbf{M}}(x, t, \zeta_j) \mathbf{M}_\infty^+)_{(\ell+1)3} \\
& -i \sum_{j=1}^{N_2} D_j(x, t) b_{33}(z_j) (\tilde{\mathbf{M}}(x, t, z_j) \mathbf{M}_\infty^+)_{(\ell+1)2}, \quad \ell = 1, 2.
\end{aligned}$$

It should be noted that, even though the function $\mathbf{q}(x, t)$ defined by the reconstruction formula solves the Manakov system, verifying that it also satisfies the initial and boundary conditions remains a nontrivial task. This is true even in the case of the scalar NLS equation.

4.1.3. Trace formula and the relation between $\mathbf{q}_+$ and $\mathbf{q}_-$

The last task of the inverse problem is the reconstruction of the analytic scattering coefficients in terms of the scattering data — i.e., discrete eigenvalues, reflection coefficients, and, possibly, norming constants. As in [12] we prove the following:

Lemma 4.9 (Trace formula). *For all $z \in \mathbb{C}^+$, the scattering coefficient $a_{11}(z)$ defined in (2.13) is given by*

$$\begin{aligned}
a_{11}(z) &= \prod_{n=1}^{N_1} \frac{z - \zeta_n}{z - \zeta_n^*} \prod_{n=1}^{N_2} \frac{z - z_n}{z - z_n^*} \\
&\quad \times \exp \left\{ -\frac{1}{2\pi i} \int_{\mathbb{R}} \log \left(1 - |\rho_1(\zeta)|^2 - \frac{\zeta^2 - q_o^2}{\zeta^2} |\rho_2(\zeta)|^2 \right) \frac{d\zeta}{\zeta - z} \right\}. \quad (4.16)
\end{aligned}$$

The expressions of the other analytic coefficients follow immediately from the symmetries (2.19) and (2.24). Explicitly, $b_{11}(z) = b_{33}(q_o^2/z) = a_{11}^*(z^*) = a_{33}^*(q_o^2/z^*)$. The integral in (4.16) is classically convergent for all z in the upper-half plane. (For example, the logarithmic singularity of the integrand at $\zeta = \pm q_o$ resulting from the behavior of the scattering coefficients is integrable. Moreover, using (2.32) one can show that $|\rho_2(\zeta)|^2/\zeta^2$ is finite when $\zeta \rightarrow 0$ on the real axis.) On the other hand, when evaluating the trace formula for $z \in \mathbb{R}$, the integrand has an additional singularity at $\zeta = z$. The same kind of singularity is also present in the trace formulae for the scalar NLS equations, even in the case of zero boundary conditions. As in those cases, when $z \in \mathbb{R}$ the integral should be interpreted in the principal value sense. On the other hand, as $z \rightarrow \pm q_o$, this logarithmic singularity combines with the term $1/(\zeta - z)$ to produce the pole of $a_{11}(z)$ at $z = \pm q_o$. This issue is discussed in detail in [25], and we refer the reader to the treatment of this issue there.

The trace formula (4.16) is essentially the same as in the case of parallel polarization vectors $\mathbf{p}_+$ and $\mathbf{p}_-$ [12, 40]. On the other hand, the derivation of the trace formula and the formula itself break down in the orthogonal case, because in that case $a_{11}(z)$ goes to zero as $z \rightarrow 0$, and therefore its logarithm becomes singular. Even in the non-orthogonal case, the relation between $\mathbf{p}_+$ and $\mathbf{p}_-$ is more complicated than in the parallel case, since, in the parallel case, the proportionality factor between the two is simply a phase (given by the so called “theta condition” [25]), but that is not true anymore in our case. Nonetheless,

one can obtain a relation between $\mathbf{q}_+$ and $\mathbf{q}_-$ by observing that $\mathbf{q}_-$ and $\mathbf{q}_-^\perp$ form a basis for the space of two-component vectors, and comparing (4.16) with the asymptotic behavior of $a_{11}(z)$ and $\rho_2(z)$ as $z \rightarrow 0$ in Corollary 2.7, yields

Corollary 4.2. *The limiting values of the potential as $x \rightarrow \pm\infty$ are related by the following expression:*

$$\mathbf{q}_+ = c_1 \mathbf{q}_- + c_2 \mathbf{q}_-^\perp, \quad (4.17)$$

where

$$\begin{aligned} c_1 &= \frac{\mathbf{q}_-^\dagger \mathbf{q}_+}{q_o^2} = \prod_{n=1}^{N_1} \frac{\zeta_n^*}{\zeta_n} \prod_{n=1}^{N_2} \frac{z_n^*}{z_n} \\ &\quad \times \exp \left\{ \frac{1}{2\pi i} \int_{\mathbb{R}} \log \left(1 - |\rho_1(\zeta)|^2 - \frac{\zeta^2 - q_o^2}{\zeta^2} |\rho_2(\zeta)|^2 \right) \frac{d\zeta}{\zeta} \right\}, \\ c_2 &= \frac{(\mathbf{q}_-^\perp)^\dagger \mathbf{q}_+}{q_o^2} = -i \frac{\mathbf{q}_+^\dagger \mathbf{q}_-}{q_o} \lim_{z \rightarrow 0} \frac{\rho_2(z)}{z}. \end{aligned} \quad (4.18)$$

Again, the integral in (4.18) must be interpreted in the principal value sense. In the special case in which $\mathbf{q}_+$ and $\mathbf{q}_-$ are parallel, one has $c_2 = 0$, and (4.17) reduces to the theta condition in [12, 40]. More in general, note that, as with the theta condition in the case of parallel boundary conditions, the relation (4.17) is not a constraint on the scattering data. Instead, the relation is consistent with the fact that the scattering data completely determine the potential for all $x \in \mathbb{R}$ up to a single arbitrary constant polarization vector $\mathbf{q}_+$ or $\mathbf{q}_-$. Therefore, once one of them is assigned, the other is completely determined.

4.2. Riemann-Hilbert problem: orthogonal boundary conditions

As we discuss next, the case of orthogonal polarization vectors presents additional difficulties. The main problem is that, since $\mathbf{q}_+^\dagger \mathbf{q}_- = 0$, one has $a_{33}(z) = \mathcal{O}(1/z)$ and $a_{11}(z) = \mathcal{O}(z)$ as $z \rightarrow \infty$ and as $z \rightarrow 0$ respectively [cf. Corollary 2.6]. This asymptotic behavior significantly complicates the formulation of the inverse problem in the IST. On one hand, it introduces a pole at $z = 0$ for both $\mathbf{M}^\pm$ in Lemma 4.1. Most importantly, it has the consequence that some of the terms in the leading order asymptotic behavior of $\mathbf{M}^\pm$ (namely, $\mathbf{M}_\infty^\pm$ and $\mathbf{M}_0^-$) diverge. To overcome this difficulty, using Lemma 4.1 we can define new modified meromorphic matrices $\tilde{\mathbf{M}}^\pm(x, t, z)$ as in the following lemma:

Lemma 4.10. *The modified meromorphic matrices $\tilde{\mathbf{M}}^\pm(x, t, z)$, defined as*

$$\tilde{\mathbf{M}}(x, t, z) = \begin{cases} \mathbf{M}^+(x, t, z) \operatorname{diag}(1, 1/z, 1) = \begin{pmatrix} \frac{\mu_{-,1}}{a_{11}}, \frac{m}{z b_{33}}, \frac{\mu_{+,3}}{\gamma} \end{pmatrix}, & \Im z > 0, \\ \mathbf{M}^-(x, t, z) \operatorname{diag}(1, 1, 1/z) = \begin{pmatrix} \frac{\mu_{+,1}}{\gamma}, -\frac{\bar{m}}{b_{11}}, \frac{\mu_{-,3}}{z a_{33}} \end{pmatrix}, & \Im z < 0, \end{cases} \quad (4.19)$$

satisfy the new jump condition

$$\tilde{\mathbf{M}}^+(x, t, z) = \tilde{\mathbf{M}}^-(x, t, z) e^{i\Theta(x, t, z)} \tilde{\mathbf{L}}(z) e^{-i\Theta(x, t, z)}, \quad z \in \mathbb{R} \setminus \{0, \pm q_o\}, \quad (4.20)$$

where $\tilde{\mathbf{M}}^\pm(x, t, z)$ denote the projection of $\tilde{\mathbf{M}}(x, t, z)$ to the real z -axis from above/below, and

$$\tilde{\mathbf{L}}(z) = \text{diag}(1, 1, z)\mathbf{L}(z)\text{diag}(1, 1/z, 1).$$

Lemma 4.11. *The matrices $\tilde{\mathbf{M}}^\pm(x, t, z)$ defined in (4.19) have the following asymptotic behavior:*

$$\begin{aligned}\tilde{\mathbf{M}}^\pm(x, t, z) &= \tilde{\mathbf{M}}_\infty^\pm + \mathcal{O}\left(\frac{1}{z}\right) \quad \text{as } z \rightarrow \infty, \\ \tilde{\mathbf{M}}^\pm(x, t, z) &= \frac{1}{z}\tilde{\mathbf{M}}_0^\pm + \mathcal{O}(1) \quad \text{as } z \rightarrow 0,\end{aligned}\tag{4.21}$$

where

$$\begin{aligned}\tilde{\mathbf{M}}_\infty^+ &= \begin{pmatrix} 1 & 0 & 0 \\ 0 & (iq_o/\alpha^*)\mathbf{q}_\perp^\perp & (1/q_o)\mathbf{q}_+ \end{pmatrix}, \quad \tilde{\mathbf{M}}_\infty^- = \begin{pmatrix} 1 & 0 & 0 \\ 0 & (1/q_o)\mathbf{q}_+^\perp & -(iq_o/\alpha)\mathbf{q}_- \end{pmatrix}, \\ \tilde{\mathbf{M}}_0^+ &= \begin{pmatrix} 0 & 0 & -iq_o \\ 0 & \mathbf{q}_+^\perp/q_o & 0 \end{pmatrix}, \quad \tilde{\mathbf{M}}_0^- = \begin{pmatrix} 0 & 0 & 0 \\ i\mathbf{q}_+ & (iq_o^3/\alpha^*)\mathbf{q}_-^\perp & 0 \end{pmatrix}.\end{aligned}$$

The residue conditions in the case of orthogonal NZBC are obtained using exactly the same method as in the non-orthogonal NZBC. However, the resulting conditions are slightly different due to the different normalization adopted for the sectionally analytic matrix $\tilde{\mathbf{M}}$. Specifically, recalling that symmetries of those norming constants were given in Lemma 4.4, we have:

Lemma 4.12. *The modified meromorphic matrices defined in Lemma 4.10 satisfy the following residue conditions:*

$$\text{Res}_{z=\zeta_n} \tilde{\mathbf{M}}(x, t, z) = C_n(\mu_{+,3}(\zeta_n), \mathbf{0}, \mathbf{0}) = \tilde{\mathbf{M}}(x, t, \zeta_n) \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ \gamma(\zeta_n)C_n & 0 & 0 \end{pmatrix}, \tag{4.22a}$$

$$\text{Res}_{z=\zeta_n^*} \tilde{\mathbf{M}}(x, t, z) = \frac{\overline{C}_n}{\zeta_n^*}(\mathbf{0}, \mathbf{0}, \mu_{+,1}(\zeta_n^*)) = \tilde{\mathbf{M}}(x, t, \zeta_n^*) \begin{pmatrix} 0 & 0 & \overline{C}_n/2\lambda(\zeta_n^*) \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \tag{4.22b}$$

$$\text{Res}_{z=z_n} \tilde{\mathbf{M}}(x, t, z) = D_n(m(z_n), \mathbf{0}, \mathbf{0}) = \tilde{\mathbf{M}}(x, t, z_n) \begin{pmatrix} 0 & 0 & 0 \\ z_n D_n & b_{33}(z_n) & 0 \\ 0 & 0 & 0 \end{pmatrix}, \tag{4.22c}$$

$$\text{Res}_{z=z_n^*} \tilde{\mathbf{M}}(x, t, z) = -\overline{D}_n(\mathbf{0}, \mu_{+,1}(z_n^*), \mathbf{0}) = \tilde{\mathbf{M}}(x, t, z_n^*) \begin{pmatrix} 0 & -\gamma(z_n^*)\overline{D}_n & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \tag{4.22d}$$

$$\begin{aligned}\text{Res}_{z=q_o^2/z_n} \tilde{\mathbf{M}}(x, t, z) &= \frac{z_n}{q_o^2} \check{D}_n(\mathbf{0}, \mathbf{0}, \overline{m}(q_o^2/z_n)) \\ &= \tilde{\mathbf{M}}(x, t, q_o^2/z_n) \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -z_n/q_o^2 \check{D}_n b_{11}(q_o^2/z_n) \\ 0 & 0 & 0 \end{pmatrix},\end{aligned}\tag{4.22e}$$

$$\begin{aligned}
\operatorname{Res}_{z=q_o^2/z_n^*} \tilde{\mathbf{M}}(x, t, z) &= \frac{z_n^*}{q_o^2} \hat{D}_n(\mathbf{0}, \mu_{+,3}(q_o^2/z_n^*), \mathbf{0}) \\
&= \tilde{\mathbf{M}}(x, t, q_o^2/z_n^*) \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & -\hat{D}_n/2\lambda(z_n^*) & 0 \end{pmatrix}, \tag{4.22f}
\end{aligned}$$

where $C_n, \bar{C}_n, D_n, \bar{D}_n, \check{D}_n$ and $\hat{D}_n$ are defined in Lemma 4.3 with $n = 1, \dots, N_1$ for the equations involving ζ_n , and $n = 1, \dots, N_2$ for the equations involving z_n .

To solve (4.20), we subtract from both sides the asymptotic behavior of $\tilde{\mathbf{M}}$ as $z \rightarrow \infty$ and $z \rightarrow 0$ as well as the residue contributions from the poles inside and on the circle of radius q_o . Namely, recalling (4.21), (4.22), we subtract the following quantity from both sides of (4.20):

$$\begin{aligned}
&\tilde{\mathbf{M}}_\infty^+ + \frac{1}{z} \tilde{\mathbf{M}}_0^+ + \sum_{j=1}^{N_1} \left(\frac{\operatorname{Res}_{z=\zeta_j} \tilde{\mathbf{M}}(z)}{z - \zeta_j} + \frac{\operatorname{Res}_{z=\zeta_j^*} \tilde{\mathbf{M}}(z)}{z - \zeta_j^*} \right) + \sum_{j=1}^{N_2} \left(\frac{\operatorname{Res}_{z=z_j^*} \tilde{\mathbf{M}}(z)}{z - z_j^*} + \frac{\operatorname{Res}_{z=z_j} \tilde{\mathbf{M}}(z)}{z - z_j} \right) \\
&+ \sum_{j=1}^{N_2} \left(\frac{\operatorname{Res}_{z=q_o^2/z_j} \tilde{\mathbf{M}}(z)}{z - q_o^2/z_j} + \frac{\operatorname{Res}_{z=q_o^2/z_j^*} \tilde{\mathbf{M}}(z)}{z - q_o^2/z_j^*} \right).
\end{aligned}$$

The left-hand side of the resulting, regularized RHP is analytic in the upper half z -plane. Also, the right-hand side is analytic in the lower half z -plane. Now, proceeding as in the first method described in Section 4.1, we obtain:

Theorem 4.5. *If the RHP defined by Lemmas 4.10-4.12 admits a unique solution, it is given by*

$$\begin{aligned}
\tilde{\mathbf{M}}(x, t, z) &= \tilde{\mathbf{M}}_\infty(z) + \frac{1}{z} \tilde{\mathbf{M}}_0(z) \\
&+ \frac{1}{2\pi i} \int_{\mathbb{R}} \left(\Delta \tilde{\mathbf{M}}_\infty + \frac{i}{\zeta} \Delta \tilde{\mathbf{M}}_0 + \tilde{\mathbf{M}}^-(x, t, \zeta) e^{i\Theta(\zeta)} \tilde{\mathbf{K}}(\zeta) e^{-i\Theta(\zeta)} \right) \frac{d\zeta}{\zeta - z} \\
&+ \sum_{j=1}^{N_1} \left(\frac{\operatorname{Res}_{z=\zeta_j} \tilde{\mathbf{M}}(z)}{z - \zeta_j} + \frac{\operatorname{Res}_{z=\zeta_j^*} \tilde{\mathbf{M}}(z)}{z - \zeta_j^*} \right) + \sum_{j=1}^{N_2} \left(\frac{\operatorname{Res}_{z=z_j^*} \tilde{\mathbf{M}}(z)}{z - z_j^*} + \frac{\operatorname{Res}_{z=z_j} \tilde{\mathbf{M}}(z)}{z - z_j} \right) \\
&+ \sum_{j=1}^{N_2} \left(\frac{\operatorname{Res}_{z=q_o^2/z_j} \tilde{\mathbf{M}}(z)}{z - q_o^2/z_j} + \frac{\operatorname{Res}_{z=q_o^2/z_j^*} \tilde{\mathbf{M}}(z)}{z - q_o^2/z_j^*} \right) \tag{4.23}
\end{aligned}$$

for all $z \in \mathbb{C} \setminus \mathbb{R}$, where $\tilde{\mathbf{K}} = \tilde{\mathbf{L}} - \mathbf{I}$, $\tilde{\mathbf{M}}_\infty(z) = \tilde{\mathbf{M}}_\infty^\pm$ and $\tilde{\mathbf{M}}_0(z) = \tilde{\mathbf{M}}_0^\pm$ when $z \in \mathbb{C}^\pm$ with $\Delta \tilde{\mathbf{M}}_\infty = \tilde{\mathbf{M}}_\infty^- - \tilde{\mathbf{M}}_\infty^+$ and $\Delta \tilde{\mathbf{M}}_0 = \tilde{\mathbf{M}}_0^- - \tilde{\mathbf{M}}_0^+$. Moreover, the eigenfunctions in the residue conditions in Lemma 4.12 are given by

$$\mathbf{M}_1(w) = \begin{pmatrix} 1 \\ i\mathbf{q}_+/w \end{pmatrix} + \frac{1}{2\pi i} \int_{\mathbb{R}} \left(\Delta \tilde{\mathbf{M}}_{\infty,1} + \frac{i}{\zeta} \Delta \tilde{\mathbf{M}}_{0,1} + (\tilde{\mathbf{M}}^-(\zeta) e^{i\Theta(\zeta)} \tilde{\mathbf{K}}(\zeta) e^{-i\Theta(\zeta)})_1 \right) \frac{d\zeta}{\zeta - w}$$

$$\begin{aligned}
& + \sum_{j=1}^{N_1} \left[\frac{\gamma(\zeta_j) C_j}{w - \zeta_j} \mathbf{M}_3(\zeta_j) \right] + \sum_{j=1}^{N_2} \left[\frac{\gamma(z_j) D_j}{w - z_j} 2\lambda(z_j) b_{33}(z_j) \mathbf{M}_2(z_j) \right], \quad w = \zeta_n^*, z_n^*, \\
\mathbf{M}_3(w) & = \begin{pmatrix} -iq_o/w \\ \mathbf{q}_+/q_o \end{pmatrix} + \frac{1}{2\pi i} \int_{\mathbb{R}} \left(\Delta \tilde{\mathbf{M}}_{\infty,3} + \frac{i}{\zeta} \Delta \tilde{\mathbf{M}}_{0,3} + (\tilde{\mathbf{M}}^-(\zeta) e^{i\Theta(\zeta)} \tilde{\mathbf{K}}(\zeta) e^{-i\Theta(\zeta)})_3 \right) \frac{d\zeta}{\zeta - w} \\
& + \sum_{j=1}^{N_1} \left[\frac{\bar{C}_j}{2\lambda(\zeta_j^*)(w - \zeta_j^*)} \mathbf{M}_1(\zeta_j^*) \right] \\
& - \sum_{j=1}^{N_2} \left[\frac{\check{D}_j}{2\lambda(q_o^2/z_j)(w - q_o^2/z_j)} \frac{b_{11}(q_o^2/z_j)}{\gamma(q_o^2/z_j)} \mathbf{M}_2(q_o^2/z_j) \right], \quad w = \zeta_n, q_o^2/z_n^*, \\
\mathbf{M}_2(q_o^2/z_\ell) & = \begin{pmatrix} 0 \\ \mathbf{q}_+^\perp/q_o + iq_o z_\ell \mathbf{q}_+^\perp/\alpha^* \end{pmatrix} \\
& + \frac{1}{2\pi i} \int_{\mathbb{R}} \left(\Delta \tilde{\mathbf{M}}_{\infty,2} + \frac{i}{\zeta} \Delta \tilde{\mathbf{M}}_{0,2} + (\tilde{\mathbf{M}}^-(\zeta) e^{i\Theta(\zeta)} \tilde{\mathbf{K}}(\zeta) e^{-i\Theta(\zeta)})_2 \right) \frac{d\zeta}{\zeta - q_o^2/z_\ell} \\
& - \sum_{j=1}^{N_2} \left[\frac{\gamma(z_j^*) \bar{D}_j}{q_o^2/z_\ell - z_j^*} \mathbf{M}_1(z_j^*) \right] + \sum_{j=1}^{N_2} \left[\frac{\hat{D}_j}{2\lambda(q_o^2/z_j^*)(q_o^2/z_\ell - q_o^2/z_j^*)} \mathbf{M}_3(q_o^2/z_j^*) \right], \\
\mathbf{M}_2(z_\ell) & = \begin{pmatrix} 0 \\ iq_o \mathbf{q}_+^\perp/\alpha^* + \mathbf{q}_+^\perp/(q_o z_\ell) \end{pmatrix} \\
& + \frac{1}{2\pi i} \int_{\mathbb{R}} \left(\Delta \tilde{\mathbf{M}}_{\infty,2} + \frac{i}{\zeta} \Delta \tilde{\mathbf{M}}_{0,2} + (\tilde{\mathbf{M}}^-(\zeta) e^{i\Theta(\zeta)} \tilde{\mathbf{K}}(\zeta) e^{-i\Theta(\zeta)})_2 \right) \frac{d\zeta}{\zeta - z_\ell} \\
& - \sum_{j=1}^{N_2} \left[\frac{\gamma(z_j^*) \bar{D}_j}{z_\ell - z_j^*} \mathbf{M}_1(z_j^*) \right] + \sum_{j=1}^{N_2} \left[\frac{\hat{D}_j}{2\lambda(q_o^2/z_j^*)(z_\ell - q_o^2/z_j^*)} \mathbf{M}_3(q_o^2/z_j^*) \right].
\end{aligned}$$

Again, for simplicity, we suppressed x and t dependence when doing so introduces no confusion. And again, the asymptotic behavior of (2.28) implies that (4.8) still holds. As before, we take $\tilde{\mathbf{M}}(x, t, z)$ when $z \in \mathbb{C}^-$ in (4.23) and compare its 2, 1 and 3, 1 elements in the limit as $z \rightarrow \infty$ with the corresponding elements found in (2.28) to obtain the following:

Theorem 4.6 (Reconstruction formula). *Let $\tilde{\mathbf{M}}(x, t, z)$ be the solution of the Riemann-Hilbert problem in Theorem 4.3. The corresponding solution $\mathbf{q}(x, t) = (q_1(x, t), q_2(x, t))^T$ of the defocusing Manakov system with orthogonal boundary conditions (1.2) is reconstructed as*

$$\begin{aligned}
q_\ell(x, t) & = \mathbf{q}_{+, \ell} + \frac{1}{2\pi} \int_{\mathbb{R}} \left(\frac{i}{\zeta} \mathbf{q}_{+, \ell} + (\tilde{\mathbf{M}}^-(x, t, \zeta) e^{i\Theta(\zeta)} \tilde{\mathbf{K}}(\zeta) e^{-i\Theta(\zeta)})_{(\ell+1)1} \right) d\zeta \\
& - i \sum_{j=1}^{N_1} \gamma(\zeta_j) C_j(x, t) \mathbf{M}_{(\ell+1)3}(x, t, \zeta_j)
\end{aligned}$$

$$-i \sum_{j=1}^{N_2} \gamma(\zeta_i) D_j(x, t) 2\lambda(z_j) b_{33}(z_j) \mathbf{M}_{(\ell+1)2}(x, t, z_j), \quad \ell = 1, 2.$$

Again we used the fact that the first columns of $\tilde{\mathbf{M}}_{\infty}^{-}$ and $\tilde{\mathbf{M}}_{\infty}^{+}$ coincide.

5. Final Remarks

In summary, we have presented the formulation of the IST to solve the IVP for the defocusing Manakov system with non-parallel boundary conditions. We made essential use of the uniformization variable $z = k + \lambda(k)$, as in [12, 13, 15, 25, 40]. It should be noted that one could also formulate the whole IST without making use of the uniformization variable, in a similar way to what was done in the scalar defocusing case in [11] and the scalar focusing case in [14]. On the other hand, we believe that, when the problem admits a uniformization variable, its use is advantageous for a number of reasons, including the fact that it avoids having to deal with branch cut discontinuities and as a consequence it results in a RHP that admits solutions in closed form for reflectionless potentials (when they exist).

Apart from the technical complications in the formulation of the IST, one of the most important differences between the parallel and non-parallel cases is that, in the latter, the reflection coefficients can never vanish identically. In this respect, it should be noted that the non-existence of reflectionless potentials is simply a consequence of the direct problem, and is therefore independent of whatever method one uses to solve the inverse problem. The existence of a non-zero continuous spectrum necessarily implies that there is always a dispersive component to the solution. (Recall that, in the context of the IST, the contribution to the solution of the IVP coming from the reflection coefficient is commonly referred to as the “dispersive component of the solution”. This is because this component is the direct nonlinearization of the Fourier transform solution of the linear limit of the PDE.) Thus, even when the discrete spectrum is non-empty and the solution contains solitonic components, the solitons are always accompanied by radiation. In other words, the problem does not admit pure soliton solutions, independently of whether one looks for them with direct methods or as an algebraic-only reduction of the Riemann-Hilbert problem. (Of course direct methods could also be used in principle to include the radiative components of the solutions, in addition to the solitons.)

From an applied point of view, we expect the results of this paper to be useful in characterizing recent experiments in nonlinear optics [18, 26, 46] and Bose-Einstein condensation [29, 48]. Conversely, from a theoretical point of view, the results in this work open up a number of interesting avenues for further study, as we discuss next.

A first obvious problem for further study is the issue of existence and uniqueness of solutions of the RHP defined by Lemmas 4.5-4.7 and of the RHP defined by Lemmas 4.10-4.12. It is possible that this question could be studied using similar techniques as in [7, 8, 12, 47]. A powerful approach to prove existence and uniqueness of solutions of Riemann-Hilbert problems was pioneered by Zhou [51] and was recently used in [10]. However, it is not

clear whether such an approach is viable here, because in the present case the jump matrix is not self-adjoint (unlike what happens for the scalar NLS equation, say). An alternative approach to prove existence and uniqueness of solutions of the inverse problem consists in formulating the inverse problem via Gelfand-Levitan-Marchenko equations. However, a Gelfand-Levitan-Marchenko formulation of the inverse problem for the Manakov system with NZBC is still missing to the best of our knowledge, even in the parallel case. The main difficulty lies in the lack of analyticity of the middle columns of the Jost eigenfunctions. This defect of analyticity is circumvented by the introduction of the auxiliary eigenfunctions. However, these are not scattering eigenfunctions — i.e., their behavior as $x \rightarrow \pm\infty$ is not simple — and they do not admit triangular representation. This is precisely the main reason for formulating the inverse problem as a RHP. It should be clear that resolving these kinds of issues remains a nontrivial task, which is outside the scope of the present work. A related issue left for future work is the characterization of the required properties on the minimal set of scattering data that guarantee the unique solvability of the inverse problem.

Nonetheless, we point out that the formulation of the IST and the conversion of the RHP to a set of linear algebraic-integral equations are valuable even in the absence of rigorous results on existence and uniqueness of solutions, even for problems (such as the present one) when no exact closed-form solutions are possible. There are three reasons why this is the case. The first is that the availability of an explicit RHP formulation is itself the starting point for an existence and uniqueness proof. The second is that the reduction of the RHP to a set of linear algebraic-integral equations makes it possible to efficiently compute its solutions numerically — e.g. following similar methods as in [47], which bypass the need for (and in many cases are more efficient than) direct numerical simulations to study the behavior of solutions. The third and final reason is that one could use the formulation of the RHP to study the long-time behavior of solutions even in the absence of rigorous existence and uniqueness results (which is exactly what was done in [14] for a related RHP whose solutions are not unique [10]).

Another interesting theoretical issue is the possible existence of spectral singularities, namely zeros of the analytic scattering coefficients inside the continuous spectrum. It is well known that no such zeros can exist in the scalar defocusing case, whereas in the focusing case several examples are known of potentials that do give rise to such zeros. (Indeed, generically, new eigenvalues bifurcate from the continuous spectrum in the focusing case.) However, it is an open question whether such spectral singularities can exist in the defocusing Manakov system. This is an open question even in the case of parallel NZBC. Finally, the derivation of a trace formula in the case of orthogonal polarization vectors remains an open problem.

Yet another open problem is the investigation of the long-time asymptotics of solutions — e.g. using the Deift-Zhou nonlinear steepest descent method for oscillatory RHPs [4, 20, 21]. This is another problem that is also open in the case of parallel NZBC. The problem is especially interesting in the case of non-parallel NZBC, however, because of the absence of reflectionless potentials, which suggests that these kinds of situations could give rise to the generation of dispersive shocks — e.g. as in [1, 2, 24].

Finally, the results of the present work suggest several natural possible extensions and

generalizations. A first natural extension will be the case of one-sided NZBC, namely, $\mathbf{q}_- = \mathbf{0}$ (or equivalently $\mathbf{q}_+ = \mathbf{0}$ thanks to the symmetry of the Manakov system with respect to spatial reflections — i.e. $x \mapsto -x$). The case of one-sided NZBC was studied in [11] for the scalar defocusing NLS equation and in [42] for the scalar focusing NLS equation. However, the problem of one-sided NZBC is open both for the focusing and the defocusing Manakov systems. In this regard, note that, even though here we allowed $\mathbf{q}_-$ and $\mathbf{q}_+$ to be non-parallel, the formulation of the IST presented in this work still requires $\|\mathbf{q}_+\| = \|\mathbf{q}_-\|$. (For example, if $\mathbf{q}_- = \mathbf{0}$, the regions of analyticity of the Jost eigenfunctions $\phi_-(x, t, z)$ normalized as $x \rightarrow -\infty$ will be different.) Hence, one cannot obtain one-sided NZBC as a special case of the non-parallel NZBC studied here. Nonetheless, we believe that one will be able to formulate the IST to solve the one-sided NZBC for the two-component defocusing Manakov system by combining the approaches used to study one-sided NZBC in the scalar defocusing case [11], scalar focusing case [42] and focusing Manakov with parallel NZBC [35].

For the same reason, the formulation of the IST will also need to be modified to study the case of NZBC with asymmetric amplitudes, namely, the case $\|\mathbf{q}_-\| \neq \|\mathbf{q}_+\|$. The additional complication in this case is that four different branch points are present, and therefore one cannot use a uniformization variable. One will therefore need to combine the approach of the present work and that used in [11] for the scalar defocusing NLS equation with asymmetric NZBC.

Another interesting generalization will be to the case of non-parallel NZBC for the three-component coupled NLS equations. Note that the three-component case with parallel NZBC, which was studied in [13, 41], already presents enormous complications compared to the two-component case, because two non-analytic eigenfunctions are present, and the symmetries of the scattering data are much more complicated. (This complexity reflects the fact that the corresponding behavior of solutions in the three-component case is much richer.) One can therefore expect that the IST for the three-component case with non-parallel NZBC will require a combination of the approach of this work with those of [13, 41].

Finally, all of these scenarios can also be considered for the focusing variants of the two- and three-component coupled NLS equations. (Note that only the IST for the two-component focusing Manakov system with parallel NZBC has been considered so far [35], and the three-component focusing case with NZBC is still completely open.) We hope that the present discussion will stimulate future work on some of the above problems.

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