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Well-posedness and Ill-posedness for Linear Fifth-Order Dispersive Equations in the Presence of Backwards Diffusion

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Abstract

Fifth-order dispersive equations arise in the context of higher-order models for phenomena such as water waves. For fifth-order variable-coefficient linear dispersive equations, we provide conditions under which the intitial value problem is either well-posed or ill-posed. For well-posedness, a balance must be struck between the leading-order dispersion and possible backwards diffusion from the fourth-derivative term. This generalizes work by the first author and Wright for third-order equations. In addition to inherent interest in fifth-order dispersive equations, this work is also motivated by a question from numerical analysis: finite difference schemes for third-order numerical equations can yield approximate solutions which effectively satisfy fifth-order equations. We find that such a fifth-order equation is well-posed if and only if the underlying third-order equation is ill-posed.

Keywords Dispersion · Anti-diffusion · Well-posedness · Fifth-order dispersive equations

1 Introduction

We study fifth-order linear constant-coefficient equations of the form

$$\begin{aligned} u_t &= a(x,t)u_{xxxxx} + b(x,t)u_{xxxx} + c(x,t)u_{xxx} \\ &\quad + d(x,t)u_{xx} + e(x,t)u_x + f(x,t)u + h(x,t), \end{aligned} \tag{1}$$

for given functions a, b, c, d, e, f, and h, subject to the initial condition

$$u(x,0) = u_0(x). \tag{2}$$

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We take the spatial variable x to be in the interval $X = [0, M]$ for some $M > 0$, and we impose periodic boundary conditions at the ends of this interval. We assume that the coefficient functions satisfy periodic boundary conditions on X and that they are defined for all $t \in [0, T]$, for a given value $T > 0$. If the coefficient $b(x, t)$ is ever positive, then we say that anti-diffusion or backwards diffusion is present. It is of course well known that backwards diffusion can cause catastrophic growth of solutions, leading to instability and ill-posedness. However, in some cases for third-order equations, dispersion has been shown to ameliorate the growth from backwards diffusion [1,2,4,7], suggesting that the initial value problem (1), (2) could be well-posed even in the presence of backwards diffusion. In this paper, we will demonstrate conditions under which the initial value problem is well-posed, as well as conditions under which the initial value problem is ill-posed.

A number of authors have studied fifth-order dispersive equations, as they have been found to be useful as higher-order models in the theory of water waves [11], and the stronger dispersion (that is, stronger than the third-order dispersion in the Korteweg-de Vries equation) has been found to have stabilizing effects on solitary waves [12]. There are a number of works on well-posedness of particular equations or particular families of equations with fifth-order dispersion, such as [5,10], and also studies of well-posedness of families of more general equations which include fifth-order equations [13]. These papers, though, do not explore the central question of the present work, which is how the presence of anti-diffusion affects well-posedness of fifth-order dispersive equations.

In addition to this interest in fifth-order equations in their own right, another motivation for studying the well-posedness of the initial value problem for (1) comes from numerical analysis. When finite difference schemes are used to compute solutions, higher-order equations which the numerical approximations effectively satisfy may be derived [9]. Thus, for a third-order linear constant coefficient equation, such as those studied in [4], the effective equations for numerical schemes would be in the class (1). By studying well-posedness of these effective equations, guidance can be provided for design and implementation of numerical methods. For some numerical schemes, nonlinear effective equations have been found to be satisfied by the approximate solutions, such as by Goodman and Lax and by Zumbrun [8,14]. An analysis of nonlinear equations related to (1) can be made in these cases, such as was carried out in [3,4], with ill-posedness of the effective equations explaining in part behavior seen in simulations.

The paper [4] considered well-posedness and ill-posedness of the initial value problem for the family of third-order equations

$$u_t = \tilde{a}(x, t)u_{xxx} + \tilde{b}(x, t)u_{xx} + \tilde{c}(x, t)u_x + \tilde{d}(x, t)u + \tilde{e}(x, t).$$

The initial value problem was found to be well-posed under three conditions, two of which are routine (regularity of the coefficients and nonvanishing of the leading coefficient). The most interesting of the three conditions expresses the necessary balance between dispersion and anti-diffusion, and is

$$\frac{1}{M}\int_0^M \frac{\tilde{b}(y, t)}{\tilde{a}(y, t)}\, dy \geq 0. \tag{3}$$

In Sect. 3 we develop the criteria on coefficients of (1) for well-posedness and find that the condition on the highest order terms coefficient is opposite from what was found in [4] for the third-order variant, namely

$$\frac{1}{M}\int_0^M \frac{b(y, t)}{a(y, t)}\, dy < 0. \tag{4}$$

This difference in the required sign of the integrals on the left-hand sides of (3) and (4) is what leads to the interesting interpretation from numerical analysis presented below in Sect. 2. Considering the case that the sign of the integral on the left-hand side of (4) is instead positive, we give an ill-posedness result in Sect. 4. This ill-posedness result requires a further assumption, that the coefficients are independent of t.

We note that in order for (4) to be satisfied, there must of course be some forward fourth-order diffusion present in the problem. In the absence of leading-order dispersion (i.e., if $a = 0$), the presence of any pointwise value where $b > 0$ would make the initial value problem ill-posed. So, we are demonstrating that a combination of some forward diffusion and leading-order dispersion can lead to a well-posed initial value problem even in the presence of backwards diffusion at places in the domain. Our ill-posedness result demonstrates that this well-posedness is not guaranteed, and it is still possible for the backwards diffusion to overwhelm the effects of forward diffusion and dispersion.

The plan of the paper is as follows. In Sect. 2, we provide the calculations related to numerical analysis of finite difference schemes we have mentioned in the introduction. In Sect. 3 we prove that under certain condition, the initial value problem (1), (2) is well-posed in Sobolev spaces H^n for $n \geq 6$. In Sect. 4, we prove that in the case of time-independent coefficients, the initial value problem is ill-posed if our condition on the balance of dispersion to anti-diffusion is violated.

2 Motivation from Numerical Analysis

Now we will relate this result to a third-order variant with respect to numerical analysis. We note that a related calculation appears in [6] as motivation for study of some related third-order dispersive equations. We consider the initial value problem

$$\begin{aligned} &v_t = a(x,t)v_{xxx} + b(x,t)v_{xx} + c(x,t)v_x + d(x,t)v + e(x,t), \\ &v(\cdot,0) = v_0 \in H^n. \end{aligned} \tag{5}$$

It was established in [4] that this initial value problem is well-posed if the coefficients are sufficiently well-behaved (i.e. they must be sufficiently regular and the leading coefficient, $a(x,t)$, must be bounded away from zero) and also, taking $a > 0$ without loss of generality, if

$$\frac{1}{M}\int_0^M \frac{b(y,t)}{a(y,t)}\,dy \geq 0.$$

This condition balances dispersion and backwards diffusion.

For a fifth-order variant of (5),

$$\begin{aligned} w_t = &a(x,t)w_{xxxxx} + b(x,t)w_{xxxx} + c(x,t)w_{xxx} + d(x,t)w_{xx} \\ &+e(x,t)w_x + f(x,t)w + h(x,t), \end{aligned} \tag{6}$$

in addition to regularity of the coefficients and that the leading coefficient be bounded away from zero, the additional condition is instead

$$\frac{1}{M}\int_0^M \frac{b(y,t)}{a(y,t)}\,dy < 0. \tag{7}$$

Again, we are taking $a > 0$ without loss of generality. (That (7) is the condition for well-posedness is one of the main theorems of the present work.)

We consider a numerical approximation of (5) around some x_i. Making Taylor expansions of v about the base point x_i, we have the following formulas:

$$\begin{aligned}
v(x_{i-2}) &\approx \sum_{n=0}^{k} \frac{(-2\Delta x)^n \partial_x^n v(x_i)}{n!} = v(x_i) + (-2\Delta x)\partial_x v(x_i) + \frac{(-2\Delta x)^2 \partial_x^2 v(x_i)}{2} \\
&\quad + \frac{(-2\Delta x)^3 \partial_x^3 v(x_i)}{6} + \frac{(-2\Delta x)^4 \partial_x^4 v(x_i)}{24} + O\left((2\Delta x)^4\right), \\
v(x_{i-1}) &\approx \sum_{n=0}^{k} \frac{(-\Delta x)^n \partial_x^n v(x_i)}{n!} = v(x_i) + (-\Delta x)\partial_x v(x_i) + \frac{(-\Delta x)^2 \partial_x^2 v(x_i)}{2} \\
&\quad + \frac{(-\Delta x)^3 \partial_x^3 v(x_i)}{6} + \frac{(-\Delta x)^4 \partial_x^4 v(x_i)}{24} + O\left((\Delta x)^4\right), \\
v(x_{i+1}) &\approx \sum_{n=0}^{k} \frac{(\Delta x)^n \partial_x^n v(x_i)}{n!} = v(x_i) + (\Delta x)\partial_x v(x_i) + \frac{(\Delta x)^2 \partial_x^2 v(x_i)}{2} \\
&\quad + \frac{(\Delta x)^3 \partial_x^3 v(x_i)}{6} + \frac{(\Delta x)^4 \partial_x^4 v(x_i)}{24} + O\left((\Delta x)^4\right), \\
v(x_{i+2}) &\approx \sum_{n=0}^{k} \frac{(2\Delta x)^n \partial_x^n v(x_i)}{n!} = v(x_i) + (2\Delta x)\partial_x v(x_i) + \frac{(2\Delta x)^2 \partial_x^2 v(x_i)}{2} \\
&\quad + \frac{(2\Delta x)^3 \partial_x^3 v(x_i)}{6} + \frac{(2\Delta x)^4 \partial_x^4 v(x_i)}{24} + O\left((2\Delta x)^4\right).
\end{aligned}$$

We also have the following finite-difference approximations for spatial derivatives of v:

$$\begin{aligned}
v(x_i)_x &\approx \frac{v(x_{i+1}) - v(x_{i-1})}{2\Delta x}, \\
v(x_i)_{xx} &\approx \frac{v(x_{i+1}) - 2v(x_i) + v(x_i - 1)}{\Delta x^2}, \\
v(x_i)_{xxx} &\approx \frac{v(x_{i+2}) - 2v(x_{i+1}) + 2v(x_{i-1}) - v(x_{i-2})}{2\Delta x^3}.
\end{aligned}$$

Substituting, we find

$$v(x_i)_x \approx \partial_x v(x_i) + \frac{(2\Delta x)^2 \partial_x^3 v(x_i)}{6} + O((\Delta x)^4), \tag{8}$$

$$v(x_i)_{xx} \approx \partial_x^2 v(x_i) + \frac{(\Delta x)^2 \partial_x^4 v(x_i)}{12} + O((\Delta x)^4), \tag{9}$$

and

$$v(x_i)_{xxx} \approx \partial_x^3 v(x_i) + \frac{(\Delta x)^2 \partial_x^5 v(x_i)}{4} + O((\Delta x)^4). \tag{10}$$

Of course, in each of (8), (9), and (10), the quantity on the left-hand side is the finite-difference approximation to the given derivative at the point x_i while the first term on the right-hand side is the true value of the given derivative at the point x_i.

Using these formulas, we have a finite-difference approximation for (5):

$$v_t \approx a(x,t)\left(\partial_x^3 v(x_i) + \frac{(\Delta x)^2 \partial_x^5 v(x_i)}{4} + O((\Delta x)^4)\right)$$

$$+b(x,t)\left(\partial_x^2 v(x_i) + \frac{(\Delta x)^2 \partial_x^4 v(x_i)}{12} + O((\Delta x)^4)\right)$$
$$+c(x,t)\left(\partial_x v(x_i) + \frac{(2\Delta x)^2 \partial_x^3 v(x_i)}{6} + O((\Delta x)^4)\right) + d(x,t)v + e(x,t).$$

We rewrite this to order the terms on the right-hand side by the number of derivatives on v :

$$\begin{aligned} v_t(x_i) \approx & \left(\frac{(\Delta x)^2 a(x,t)}{4}\right)\partial_x^5 v(x_i) + \left(\frac{(\Delta x)^2 b(x,t)}{12}\right)\partial_x^4 v(x_i) \\ & + \left(a(x,t) + c(x,t)\frac{(2\Delta x)^2}{6}\right)\partial_x^3 v(x_i) + b(x,t)\partial_x^2 v(x_i) + c(x,t)\partial_x v(x_i) \\ & + d(x,t)v + e(x,t) + (a(x,t)+b(x,t)+c(x,t))O((\Delta x)^4). \end{aligned} \tag{11}$$

We see that the finite-difference approximation of (5) yields an effective equation in the form of our fifth-order variant, (6). We apply the condition (7) to our effective equation (11), finding that for well-posedness of the initial value problem for (11), the quantity of interest is

$$\frac{1}{M}\int_0^M \frac{\frac{(\Delta x)^2 b(y,t)}{12})}{\frac{(\Delta x)^2 a(y,t)}{4}}\,dy = \frac{1}{3M}\int_0^M \frac{b(y,t)}{a(y,t)}\,dy. \tag{12}$$

This leads to an interesting result; assume without loss of generality that the leading coefficient a is positive. If the quantity in (12) is positive then the initial value problem for the original equation (5) is well-posed while the initial value problem for the effective equation (11) (neglecting the smallest terms) is ill-posed. Alternatively, if the quantity in (12) is negative, then the effective equation for the finite difference scheme (11) (again, neglecting the smallest terms) is well-posed while the original initial value problem for (5) is ill-posed.

This kind of problem from numerical analysis has been observed before; Goodman and Lax noted that a finite difference scheme for the Korteweg–de Vries equation had an effective equation with degenerate dispersion, and they noted that the scheme worked well as long as the solution remained away from zero [8]. This is explained theoretically by the third-order results in [4] showing that equations with degenerate dispersion have well-posed initial value problems when the data is away from zero, and by the ill-posedness result of [3], showing that equations with degenerate dispersion can have ill-posed initial value problems when solutions cross zero. A related equation arose in a similar discussion of finite difference schemes in [14].

3 Well-posedness Theorem

In this section we prove the first of our two main theorems; this theorem states that under three kinds of assumptions, the initial value problem (1), (2) is well-posed in sufficiently regular Sobolev spaces.

Theorem 1 *Let $T > 0$ be given. Let $n \in \mathbb{N}$ satisfying $n \geq 6$ be given. Let $u_0 \in H^n(X)$ be given. Assume the following conditions hold:*

(A1) *The coefficients have the following regularity:*

- $a(x,t) \in C([0,T]; C^{n+5})$,
- $a_t(x,t) \in C([0,T]; C^n)$,

- $b(x,t) \in C([0,T]; C^{n+4})$,
- $b_t(x,t) \in C([0,T]; C^n)$,
- $c(x,t) \in C([0,T]; C^n)$,
- $d(x,t) \in C([0,T]; C^n)$,
- $e(x,t) \in C([0,T]; C^n)$,
- $f(x,t) \in C([0,T]; C^n)$,
- $h(x,t) \in L^\infty([0,T]; H^n)$.

(A2) *The leading coefficient $a(x,t)$ is bounded away from* 0*, i.e. there exists $\mathfrak{a} > 0$ such that for all x and t, we have $|a(x,t)| \geq \mathfrak{a} > 0$.*

(A3) *For all t, the coefficients a and b satisfy*

$$\frac{1}{M}\int_0^M \frac{b(y,t)}{|a(y,t)|}\,dy < 0.$$

Then there exists a unique solution $u \in C([0,T]; H^n(X))$ to the initial value problem (1)*,* (2)*. Moreover, the solution gains regularity in the sense that $u \in L^2([0,T]; H^{n+2}(X))$ as well.*

Remark 2 We stress that the sign of a is not important—it may either be positive or negative—but assumption (A2) expresses that we must have a bounded away from zero. There are multiple different ways to see that the sign is unimportant; most directly, by changing the spatial variable x to $-x$, one sees that the coefficient of the fifth-order term changes sign while the coefficient of the fourth-order term does not. Thus any well-posedness theory (which must balance the fifth-order and fourth-order terms against each other) may not rely on this leading sign. Another way is to understand the role that dispersion plays; the fifth-order term causes dispersion in the equation, and the sign of the leading coefficient determines whether wave packets move to the left or the right. The fourth-order term, depending on its sign, has the potential to cause backward parabolic growth. If a wavepacket is in a "growth region" (i.e., a region in which $b(x,t) > 0$), the amount of growth depends on the amount of time the wavepacket is in this region. This amount of time is related to the coefficient $a(x,t)$, which keeps the wave packet in motion. To avoid unbounded growth, it does not matter if the wave packet moves through this "growth region" to the left or to the right, as long as it keeps moving.

Since the sign of a is immaterial, to simplify notations we will henceforth and without loss of generality take $a(x,t) > 0$. If instead $a(x,t) < 0$, then only trivial changes to the proofs are necessary.

The proof of Theorem 1 will be by the gauged energy method. In order to make our energy estimates, then, we need to introduce the gauge. We do not choose the exact gauge we will use right now, but we do specify the properties we wish to be satisfied by our gauge. We want the function g_n to have the following properties:

(C1) $g_n \in C([0,T]; C^{n+5})$ and $\partial_t g_n \in C([0,T]; C^n)$,
(C2) There exist constants $A_1 > 0$, $A_2 > 0$ such that $A_1 < g_n < A_2$, and
(C3) g_n and its first $n-1$ spatial derivatives satisfy periodic boundary conditions.

Given such a g_n, we define v through the equation

$$u = v g_n(x,t).$$

We then have the following lemma on the equivalence of regularity of u and v :

Lemma 3 *Assume g_n satisfies the assumptions* (C1)–(C3). *Then $u \in C([0,T]; H^n)$ if and only if $v \in C([0,T]; H^n)$. Moreover, there is a constant $C_g \geq 1$, depending only on norms of g_n, such that*

$$C_g^{-1} \sup_{t\in[0,T]} \|u\|_{H^n} \leq \sup_{t\in[0,T]} \|v\|_{H^n} \leq C_g \sup_{t\in[0,T]} \|u\|_{H^n}.$$

The proof of this lemma is straightforward and is omitted.

Of course the point is that a favorable choice of g_n will allow energy estimates to be made for v; we look now at the evolution equation for v:

$$\begin{aligned} v_t &= a(x,t)v_{xxxxx} + \bar{b}(x,t)v_{xxxx} + \bar{c}(x,t)v_{xxx} \\ &\quad + \bar{d}(x,t)v_{xx} + \bar{e}(x,t)v_x + \bar{f}(x,t)v + \bar{h}(x,t), \end{aligned} \tag{13}$$

where the new coefficients are given by

$$\begin{aligned} \bar{b} &= 5a\frac{\partial_x g_n}{g_n} + b, \\ \bar{c} &= 10a\frac{\partial_x^2 g_n}{g_n} + 4b\frac{\partial_x g_n}{g_n} + c, \\ \bar{d} &= 10a\frac{\partial_x^3 g_n}{g_n} + 6b\frac{\partial_x^2 g_n}{g_n} + 3c\frac{\partial_x g_n}{g_n} + d, \\ \bar{e} &= 5a\frac{\partial_x^4 g_n}{g_n} + 4b\frac{\partial_x^3 g_n}{g_n} + 3c\frac{\partial_x^2 g_n}{g_n} + 2d\frac{\partial_x g_n}{g_n} + e, \\ \bar{f} &= \frac{-\partial_t g}{g_n} + a\frac{\partial_x^5 g_n}{g_n} + b\frac{\partial_x^4 g_n}{g_n} + c\frac{\partial_x^3 g_n}{g_n} + d\frac{\partial_x^2 g_n}{g_n} + e\frac{\partial_x g_n}{g_n} + f, \end{aligned} \tag{14}$$

and, finally,

$$\bar{h} = \frac{h}{g_n}.$$

Remark 4 We now remark on the strategy of proof for Theorem 1. For a particular choice of gauge, g_n, we prove the existence and uniqueness of solutions for the initial value problem for (13). This immediately implies existence and uniqueness of solutions for the initial value problem (1), (2). The proof of existence for (13) starts with introduction of an approximate system, and then Lemma 5 below gives existence of solutions for the approximate system on a very short time interval. A uniform bound for the approximate solutions is proved in Proposition 6 below, and this yields a uniform time of existence for the approximate solutions. The proof of existence is then able to be carried out. Uniqueness is then proved in Proposition 8 below.

Before we perform our energy estimates we must introduce an approximate problem. To this end we define the operator $\mathcal{X}_m$ to be a truncation operator acting on Fourier series. If we have the Fourier series for a function f,

$$f(x) = \sum_{k=-\infty}^{\infty} f_k e^{ikx},$$

then we define $\mathcal{X}_m$ by

$$(\mathcal{X}_m f)(x) = \sum_{k=-m}^{m} f_k e^{ikx}.$$

The following properties of $\mathcal{X}_m$ will be useful for our energy estimates:

- $\mathcal{X}_m(\partial_x f) = \partial_x(\mathcal{X}_m f)$,
- for all $s \in \mathbb{N}$, $\|\mathcal{X}_m(\partial_x f)\|_{H^{s+1}} \leq m\|\mathcal{X}_m f\|_{H^s}$,
- $\int_X g\mathcal{X}_m f \, dx = \int_X (\mathcal{X}_m g) f \, dx$,
- $\mathcal{X}_m f = \mathcal{X}_m^2 f$,
- for all $s \in \mathbb{N}$, $\|\mathcal{X}_m f\|_{H^s} \leq \|f\|_{H^s}$.

We note these properties without proof.

We now incorporate $\mathcal{X}_m$ into our equation for v to make a sequence of approximations. Our approximate initial value problem is given by

$$v_t^m = \mathcal{X}_m\Bigg(a\mathcal{X}_m(v_{xxxxx}^m) + \bar{b}\mathcal{X}_m(v_{xxxx}^m) + \bar{c}\mathcal{X}_m(v_{xxx}^m) + \bar{d}\mathcal{X}_m(v_{xx}^m) + \bar{e}\mathcal{X}_m(v_x^m) + \bar{f}\mathcal{X}_m(v^m) + \bar{h}\Bigg), \tag{15}$$

$$v^m(x, 0) = v_0(x). \tag{16}$$

We have existence of solutions for this sequence of approximations, and this is the content of our next lemma.

Lemma 5 *For all $m \in \mathbb{N}$ there exists a $T_m > 0$ such that there exists $v^m \in C^1([0, T_m]; H^n)$ which solves the initial value problem (15), (16).*

We omit the proof of Lemma 5; the proof uses the Picard Theorem. That the right-hand side of (15) is Lipschitz follows from the properties of our operator $\mathcal{X}_m$ and the regularity of the coefficients.

The time of existence guaranteed by Lemma 5 for our approximations v_m unfortunately depends badly on the approximation parameter m. To be able to take the limit as m goes to infinity, we must prove that these approximations exist on the common time interval $[0, T]$. This requires proving an energy estimate which is uniform with respect to m.

Proposition 6 *For the given $T > 0$, there exists $k > 0$ such that for all $m \in \mathbb{N}$, the approximate solutions v^m are in $C([0, T]; H^n)$, with the bound*

$$\|v^m\|_{H^n}^2 \leq k(\|v_0\|_{H^n}^2 + 1). \tag{17}$$

Proof We define an energy functional,

$$E = \frac{1}{2}\int (v^m)^2 + (\partial_x^n v^m)^2 \, dx.$$

This functional is equivalent to the square of the H^n-norm of v_m.

We take the time derivative of the energy:

$$\frac{dE}{dt} = \int v^m v_t^m + (\partial_x^n v^m)(\partial_x^n v_t^m) \, dx.$$

The first term on the right-hand side can be easily bounded since we know that $n \geq 6$ and that the highest derivative in $\partial_t v^m$ is of fifth order:

$$\int v^n v_t^m = \int \Bigg[\mathcal{X}_m(v^n)\Bigg(a(x)\mathcal{X}_m(v_{xxxxx}^m) + \bar{b}(x)\mathcal{X}_m(v_{xxxx}^m) + \bar{c}(x)\mathcal{X}_m(v_{xxx}^m)$$

$$+\bar{d}(x)\mathcal{X}_m(v^m_{xx})+\bar{e}(x)\mathcal{X}_m(v^m_x)+\bar{f}(x)\mathcal{X}_m(v^m)+\bar{h}(x)\Big)\Big]\,dx\le C(E+1).$$

This immediately implies

$$\frac{dE}{dt}\le C(E+1)+\int(\partial_x^n v^m)(\partial_x^n v^m_t)\,dx. \tag{18}$$

We must still bound the integral on the right-hand side of (18), which we expand using (15):

$$\begin{aligned}\int(\partial_x^n v^m)(\partial_x^n v^m_t)\,dx &=\int\Big[\mathcal{X}_m(\partial_x^n v^m)\Big(\partial_x^n\Big(a(x)\mathcal{X}_m(v^m_{xxxxx})+\bar{b}(x)\mathcal{X}_m(v^m_{xxxx})\\ &\quad+\bar{c}(x)\mathcal{X}_m(v^m_{xxx})+\bar{d}(x)\mathcal{X}_m(v^m_{xx})+\bar{e}(x)\mathcal{X}_m(v^m_x)\\ &\quad+\bar{f}(x)\mathcal{X}_m(v^m)+\bar{h}(x)\Big)\Big)\Big]\,dx\\ &=I+II+III+IV+V+VI+VII.\end{aligned} \tag{19}$$

Here, the term I corresponds to the contribution of the coefficient a, the term II corresponds to the contribution of the coefficient $\bar{b}$, and so on, through the term VII which corresponds to the contribution of the coefficient $\bar{h}$.

We expand the term I, by using the product rule:

$$\begin{aligned}I&=\int\mathcal{X}_m(\partial_x^n v^m)\partial_x^n(a\mathcal{X}_m(\partial_x^5 v^m))\,dx\\ &=\int a(\partial_x^{n+5}\mathcal{X}_m(v^m))\partial_x^n\mathcal{X}_m(v^m)\,dx+n\int(\partial_x a)\left(\partial_x^{n+4}\mathcal{X}_m(v^m)\right)\partial_x^n\mathcal{X}_m(v^m)\,dx\\ &\quad+\binom{n}{2}\int(\partial_x^2 a)\left(\partial_x^{n+3}\mathcal{X}_m(v^m)\right)\partial_x^n\mathcal{X}_m(v^m)\,dx\\ &\quad+\binom{n}{3}\int(\partial_x^3 a)\left(\partial_x^{n+2}\mathcal{X}_m(v^m)\right)\partial_x^n\mathcal{X}_m(v^m)\,dx\\ &\quad+\binom{n}{4}\int(\partial_x^4 a)\left(\partial_x^{n+1}\mathcal{X}_m(v^m)\right)\partial_x^n\mathcal{X}_m(v^m)\,dx\\ &\quad+\sum_{j=0}^{n-5}\binom{n}{j}\int(\partial_x^n\mathcal{X}_m(v^m))(\partial_x^{n-j}a)(\partial_x^{j+5}\mathcal{X}_m(v^m))\,dx.\end{aligned} \tag{20}$$

We next similarly expand the term II :

$$\begin{aligned}II&=\int(\partial_x^n\mathcal{X}_m(v^m))\partial_x^n(\bar{b}\partial_x^4\mathcal{X}_m(v^m))\,dx\\ &\quad+\int\bar{b}\left(\partial_x^{n+4}\mathcal{X}_m(v^m)\right)\partial_x^n\mathcal{X}_m(v^m)\,dx+n\int(\partial_x\bar{b})\left(\partial_x^{n+3}\mathcal{X}_m(v^m)\right)\partial_x^n\mathcal{X}_m(v^m)\,dx\\ &\quad+\binom{n}{2}\int(\partial_x^2\bar{b})\left(\partial_x^{n+2}\mathcal{X}_m(v^m)\right)\partial_x^n\mathcal{X}_m(v^m)\,dx\\ &\quad+\binom{n}{3}\int(\partial_x^3\bar{b})\left(\partial_x^{n+1}\mathcal{X}_m(v^m)\right)\partial_x^n\mathcal{X}_m(v^m)\,dx\\ &\quad+\sum_{j=0}^{n-4}\binom{n}{j}\int(\partial_x^n\mathcal{X}_m(v^m))(\partial_x^{n-j}\bar{b})(\partial_x^{j+4}\mathcal{X}_m(v^m))\,dx.\end{aligned} \tag{21}$$

We can expand the other terms in the same fashion; a number of the resulting terms are bounded in terms of the energy. All the terms in the final summations on the right-hand sides of (20) and (21) can be bounded by CE since at most n derivatives of $\mathcal{X}_m(v^m)$ appear there. We can also integrate by parts in terms in the form $\gamma(x)\left(\partial_x^{n+1}\mathcal{X}_m(v^m)\right)\partial_x^n\mathcal{X}_m(v^m)$, and then bound the result by CE. The term VII can be immediately bounded by $CE^{1/2}$, which can then be bounded by $C(E+1)$.

After making these further expansions and bounding those terms which we have just described, we are left with the following:

$$\begin{aligned}\frac{dE}{dt} &\leq C(E+1) + \int a\left(\partial_x^{n+5}\mathcal{X}_m(v^m)\right)\partial_x^n\mathcal{X}_m(v^m)\,dx \\ &\quad + \int\left[n\partial_x a(x) + \bar{b}(x)\right]\left(\partial_x^{n+4}\mathcal{X}_m(v^m)\right)\partial_x^n\mathcal{X}_m(v^m)\,dx \\ &\quad + \int\left[\binom{n}{2}\partial_x^2 a(x) + n\partial_x\bar{b}(x) + \bar{c}(x)\right]\left(\partial_x^{n+3}\mathcal{X}_m(v^m)\right)\partial_x^n\mathcal{X}_m(v^m)\,dx \\ &\quad + \int\left[\binom{n}{3}\partial_x^3 a(x) + \binom{n}{2}\partial_x^2\bar{b}(x) + n\partial_x\bar{c}(x) + \bar{d}(x)\right]\left(\partial_x^{n+2}\mathcal{X}_m(v^m)\right)\partial_x^n\mathcal{X}_m(v^m)\,dx.\end{aligned} \tag{22}$$

To estimate the remaining terms, we will first rewrite them using some identities based on the product rule, and then we will choose our gauge. The identities we use are the following (we state this for a general function V, but will use it above for $V = \partial_x^n\mathcal{X}_m v$).

$$\begin{aligned}VV_{xxxxx} &= \frac{1}{2}\partial_x^5(V^2) - \frac{5}{2}\partial_x^3(V_x^2) + \frac{5}{2}\partial_x(V_{xx}^2), \\ VV_{xxxx} &= \frac{1}{2}\partial_x^4(V^2) - 2\partial_x^2(V_x^2) + V_{xx}^2, \\ VV_{xxx} &= \frac{1}{2}\partial_x^3(V^2) - \frac{3}{2}\partial_x(V_x^2), \\ VV_{xx} &= \frac{1}{2}\partial_x^2(V^2) - V_x^2.\end{aligned}$$

Using these, (22) becomes

$$\begin{aligned}\frac{dE}{dt} &\leq C(E+1) \\ &\quad + \int a(x)\left(\frac{1}{2}\partial_x^5[(\partial_x^n\mathcal{X}_m v^m)^2] - \frac{5}{2}\partial_x^3[(\partial_x^{n+1}\mathcal{X}_m v^m)^2] + \frac{5}{2}\partial_x[(\partial_x^{n+2}\mathcal{X}_m v^m)^2]\right)dx \\ &\quad + \int[n\partial_x a(x)+\bar{b}(x)]\left(\frac{1}{2}\partial_x^4[(\partial_x^n\mathcal{X}_m v^m)^2] - 2\partial_x^2[(\partial_x^{n+1}\mathcal{X}_m v^m)^2] + (\partial_x^{n+2}\mathcal{X}_m v^m)^2\right)dx \\ &\quad + \int\left[\binom{n}{2}\partial_x^2 a(x) + n\partial_x\bar{b}(x) + \bar{c}(x)\right]\left(\frac{1}{2}\partial_x^3[(\partial_x^n\mathcal{X}_m v^m)^2] - \frac{3}{2}\partial_x[(\partial_x^{n+1}\mathcal{X}_m v^m)]^2\right)dx \\ &\quad + \int\left[\binom{n}{3}\partial_x^3 a(x) + \binom{n}{2}\partial_x^2\bar{b}(x) + n\partial_x\bar{c}(x) + \bar{d}(x)\right] \\ &\quad \left(\frac{1}{2}\partial_x^2[(\partial_x^n\mathcal{X}_m v^m)^2] - [(\partial_x^{n+1}\mathcal{X}_m v^m)]^2\right)dx.\end{aligned} \tag{23}$$

The first term in each of the integrals on the right-hand side of (23) can be bounded in terms of the energy, after performing the appropriate number of integrations by parts. For the remaining terms in each integral we perform some further integrations by parts, leading us to the following:

$$\frac{dE}{dt} \leq C(E+1) + \int \left(\left(n - \frac{5}{2}\right) a_x + \bar{b}\right) [\partial_x^{n+2} \mathcal{X}_m v^m]^2 \, dx + \int \beta [\partial_x^{n+1} \mathcal{X}_m v^m]^2 \, dx,$$

where we have made the definition

$$\beta = \left(\frac{5}{2} + \frac{3}{2}\binom{n}{2} - 2n - \binom{n}{3}\right) \partial_x^3 a + \left(\frac{3n}{2} - 2 - \binom{n}{2}\right) \partial_x^2 \bar{b} + \left(\frac{3}{2} - n\right) \partial_x \bar{c} - \bar{d}.$$

We now add and subtract, introducing a function $a\delta$ (with δ to be defined) with the intention of having it control the term with the $(n+1)$-st derivatives:

$$\begin{aligned} \frac{dE}{dt} &\leq C(E+1) + \int \left(\left(n - \frac{5}{2}\right) a_x + \bar{b} - a\delta\right) [\partial_x^{n+2} \mathcal{X}_m v^m]^2 \, dx \\ &\quad + \int a\delta [\partial_x^{n+2} \mathcal{X}_m v^m]^2 \, dx + \int \beta [\partial_x^{n+1} \mathcal{X}_m v^m]^2 \, dx. \end{aligned} \tag{24}$$

For the final integral on the right-hand side of (24), we integrate by parts as follows:

$$\int \beta (\partial_x^{n+1} \mathcal{X}_m v^m)^2 \, dx = -\int (\beta \partial_x^{n+1} \mathcal{X}_m v^m)_x (\partial_x^n \mathcal{X}_m v^m) \, dx.$$

We apply the derivative and integrate by parts again to find the following:

$$\begin{aligned} \int \beta (\partial_x^{n+1} \mathcal{X}_m v^m)^2 \, dx &= \int \frac{\beta_x}{2} (\partial_x^n \mathcal{X}_m v^m)^2 \, dx \\ &\quad - \int \beta (\partial_x^{n+2} \mathcal{X}_m v^m)(\partial_x^n \mathcal{X}_m v^m) \, dx. \end{aligned}$$

The first integral on the right-hand side is bounded in terms of the energy, and for the second integral on the right-hand side, we use Young's inequality with parameter $\epsilon > 0$. We combine this with the second integral on the right-hand side of (24), finding the bound

$$\begin{aligned} \int a\delta [\partial_x^{n+2} \mathcal{X}_m v^m]^2 + \beta [\partial_x^{n+1} \mathcal{X}_m v^m]^2 \, dx &\leq C(E+1) \\ + \int a\delta [\partial_x^{n+2} \mathcal{X}_m v^m]^2 + \epsilon [\partial_x^{n+2} \mathcal{X}_m v^m]^2 &+ \frac{1}{\epsilon} (\beta \partial_x^n \mathcal{X}_m v^m)^2 \, dx. \end{aligned}$$

Since the energy controls up to n spatial derivatives, we may bound the final term on the right-hand side by the energy:

$$\begin{aligned} &\int a\delta(x) [\partial_x^{n+2} \mathcal{X}_m v^m]^2 + \beta(x) [\partial_x^{n+1} \mathcal{X}_m v^m]^2 \\ &\leq C(E+1) + \int a\delta(x) [\partial_x^{n+2} \mathcal{X}_m v^m]^2 + \epsilon [\partial_x^{n+2} \mathcal{X}_m v^m]^2 \, dx. \end{aligned}$$

We recall that we have taken $\inf_x a(x) > 0$, and we then note that as long as $\delta(x) \leq \frac{-\epsilon}{\inf_x a(x)} < 0$ the integral in the right-hand side will be non-positive. This leads to the following bound:

$$\int a\delta(x) [\partial_x^{n+2} \mathcal{X}_m v^m]^2 + \beta(x) [\partial_x^{n+1} \mathcal{X}_m v^m]^2 \, dx \leq C(E+1).$$

Using this with (24), we now have

$$\frac{\partial E}{\partial t} \leq C(E+1) + \int \left(\left(n - \frac{5}{2} \right) a_x(x) + \bar{b}(x) - a\delta(x) \right) [\partial_x^{n+2} \mathcal{X}_m v^m]^2 \, dx.$$

If we had $\left(n - \frac{5}{2}\right) a_x + \bar{b} - a\delta = 0$ then our energy estimate would be complete. Substituting for $\bar{b}$ from (14), we see that we seek δ such that

$$\left(n - \frac{5}{2} \right) \partial_x a(x) + 5a \frac{\partial_x g_n}{g_n} + b - a\delta = 0.$$

This equation may be integrated. We do so, and we solve for the gauge, g_n :

$$g_n = a^{\left(\frac{1}{2} - \frac{n}{5}\right)} \exp \left\{ -\frac{1}{5} \int_0^x \left(\frac{b}{a} - \delta \right) \, dx' \right\}. \tag{25}$$

We then make a favorable choice of $\delta(t)$, so as to maintain periodic boundary conditions for g_n:

$$\delta(t) = \frac{1}{M} \int_0^M \frac{b}{a} \, dx. \tag{26}$$

We still must choose the value of ϵ above. Now that we have defined δ, we have the condition

$$\frac{1}{M} \int_0^M \frac{b}{a} \, dx \leq \frac{-\epsilon}{\inf_x a(x)} < 0.$$

So, we take $\epsilon > 0$ such that

$$\left(\sup_{t \in [0,T]} \frac{1}{M} \int_0^M \frac{b(x,t)}{a(x,t)} \, dx \right) \left(\inf_{x \in [0,M]} a(x) \right) < -\epsilon.$$

(Recall here that the first factor on the left-hand side is strictly negative, and the second factor on the left-hand side is strictly positive.)

We have therefore demonstrated the bound

$$\frac{dE}{dt} \leq C(E+1).$$

This immediately implies that the energy remains bounded over the time interval $[0, T]$. (Note that we have fundamentally used the fact that $\frac{1}{M} \int \frac{b}{a} < 0$; without this, we could not control the energy.) □

Remark 7 The proof also establishes that $\mathcal{X}_m v^m$ is uniformly bounded in the space $L^2([0, T]; H^{n+2})$. This fact will be used later.

Having established that the approximate solutions exist on a common time interval, we now work to take the limit as m goes to infinity.

Using the uniform bound (17) and the approximate evolution equations (15), we have the uniform bound for the time derivatives,

$$\sup_{t \in [0,T]} \|v_t^m(t)\|_{L^\infty} \leq k(\|v_0\|_{H^n} + 1).$$

The uniform bound (17) also implies that v_x^m is uniformly bounded. Thus $\{v^m\}$ is a uniformly bounded and equicontinuous family of functions on the domain $[0, M] \times [0, T]$. By the

Arzeala-Ascoli theorem there exist a function $v^* \in C([0, M] \times [0, T])$ such that (taking a subsequence, which we do not relabel)

$$\lim_{m\to\infty} \|v^m - v^*\|_{C([0,M]\times[0,T])} = 0.$$

Looking at $n' \in [0, n)$ we can use a typical Sobolev interpolation inequality,

$$\|v^m - v^{m'}\|_{H^{n'}} \leq \|v^m - v^{m'}\|_{L^2}^{1-\frac{n'}{n}} \|v^m - v^{m;}\|_{H^n}^{\frac{n'}{n}},$$

The convergence in $C([0, M] \times [0, T])$ implies convergence in $L^2([0, M])$ at each time, and the bound (17) applies to v^m as well as $v^{m'}$. From this we can see that $\{v^m\}$ is a Cauchy sequence in $C([0, T]; H^{n'})$, and is therefore convergent of course, and the limit must be v^*. So, for all $n \in [0, n')$,

$$v^* \in C([0, T]; H^{n'}).$$

Furthermore, we know $\{v^m\}$ is bounded in H^n at each time, so there exists a weak limit in H^n. We call this weak limit $v^{**} \in L^\infty([0, T]; H^n)$. By the uniqueness of limits we know $v^{**} = v^*$, thus

$$v^* \in L^\infty([0, T]; H^n).$$

We next may conclude that v^* solves the initial value problem. We denote the right-hand side of (15) as $\mathcal{B}_m$; of course, $\mathcal{B}_m$ involves at most five derivatives of v^m. We similarly denote the right-hand side of (13) as $\mathcal{B}[v]$. Integrating (15) in time, and using (16), we have

$$v^m(\cdot, t) = v_0 + \int_0^t \mathcal{B}_m(\cdot, \tau)\, d\tau.$$

Since $\mathcal{B}_m$ involves at most five derivatives and we have uniform convergence with more regularity than this, we may pass to the limit in the integral. We know of course that the limit of v^m is v^*, and thus we see that

$$v^*(\cdot, t) = v_0 + \int_0^t \mathcal{B}[v^*](\cdot, \tau)\, d\tau.$$

Thus $v^*(0) = v_0$ and v^* satisfies (13).

We now are able to conclude also that $v^* \in L^2([0, T]; H^{n+2})$. We noted in Remark 7 that the proof of Proposition 6 implies that the family $\mathcal{X}_m v^m$ is bounded with respect to m in $L^2([0, T]; H^{n+2})$. Since this is a Hilbert space, we conclude that $\mathcal{X}_m v_m$ has a weak limit in this space. We know, however, that in less regular spaces, $\mathcal{X}_m v^m$ converges to v^*. By uniqueness of limits, we conclude that $v^* \in L^2([0, T]; H^{n+2})$.

All that remains is to show $v^* \in C([0, T]; H^n)$. Since $v^* \in L^2([0, T]; H^{n+2})$, we see that for almost every $t \in [0, T]$, we have $v^*(\cdot, t) \in H^{n+2}$. Let $\tau > 0$ be such a time. Then we can re-run the existence argument in its entirety using $v^*(\cdot, \tau) \in H^{n+2}$ as the initial condition. Call this new solution v^{**}. We have $v^{**} \in C([\tau, T]; H^{n'+2})$ for all $n' \in [0, n)$. By uniqueness of solutions (to be proven in Proposition 8 below), it must be the case that $v^{**} = v^*$. Since τ can be taken arbitrarily close to zero, this proves that v^* is continuous in H^n for any $t \in (0, T]$. Showing right-continuity of the solution in H^n at time zero is all that remains.

That v^* is bounded in H^n at all times implies that v^* is weakly continuous in time, and thus

$$\|v^*(\cdot, 0)\|_{H^n} \leq \liminf_{t\to 0^+} \|v^*(\cdot, t)\|_{H^n}.$$

Our energy estimates imply

$$\limsup_{t\to 0^+} \|v^*(\cdot,t)\|_{H^n} \le \|v^*(\cdot,0)\|_{H^n}.$$

Combining these, and the weak continuity in time, we conclude that the solution is indeed continuous in time in H^n. This completes the proof of existence of solutions in Theorem 1.

It remains to establish the uniqueness result of Theorem 1; this will follow immediately from the following proposition.

Proposition 8 *Let $u_1 \in C([0,T]; H^n(X))$ and $u_2 \in C([0,T]; H^n(X))$ be two solutions of (1), each with initial data in H^n. Then we have the estimate*

$$\sup_{t\in[0,T]} \|u_1 - u_2\|_{L^2} \le c\|u_1(\cdot,0) - u_2(\cdot,0)\|_{L^2}. \tag{27}$$

Proof We consider the difference of the two solutions, $w = u_1 - u_2$. Since the Eq. (1) is linear, w satisfies almost the same equation, just with h replaced with zero. So we have

$$w_t = a(x,t)w_{xxxxx} + b(x,t)w_{xxxx} + c(x,t)w_{xxx} + d(x,t)w_{xx} + e(x,t)w_x + f(x,t)w.$$

We may follow the arguments of Proposition 6 to define a gauge. The choice of gauge depends on the regularity we choose to estimate, and we are only making an estimate in L^2 at present. Thus the gauge is given by (25) but with n replaced with zero. Specifically, we have

$$g_0 = a^{1/2}\exp\left\{-\frac{1}{5}\int_0^x \left(\frac{b}{a} - \delta\right)\, dx'\right\}.$$

Give the gauge, we define v as before, i.e. $w = vg_0$. We define the energy to be

$$E_d = \int_X v^2\, dx,$$

and (again, following the arguments of Proposition 6) we find

$$\frac{dE_d}{dt} \le cE_d. \tag{28}$$

Note that in Proposition 6, we had $\frac{dE}{dt} \le c(E+1)$. We do not have a constant on the right-hand side of the present bound (28) because in the present case the term h has been replaced by zero. Gronwall's inequality applied to (28) now implies the desired bound. □

4 Ill-posedness Result

As we have mentioned in the introduction, we also have an ill-posedness result when the fourth-order backwards diffusion is stronger. The following is our second main theorem. (We remark again that without loss of generality we are taking $a > 0$.) We state the theorem under the assumption that the coefficients are C^∞ (as was done in [4]) for simplicity. We make Remark 10 afterwards regarding relaxing this assumption.

Theorem 9 *Let Lu be given by the right-hand side of (1), and let δ be defined as in (26) above. Assume that the coefficients a, b, c, d, e, f, and h are in C^∞. If $\delta > 0$ and if the coefficients a, b, c, d, e, f, and h do not depend on t, then the operator L has a sequence of eigenvalues whose real parts go to infinity. Thus the initial value problem (1), (2) is ill-posed.*

Proof To demonstrate ill-posedness we will start by showing it for a special case with constant dispersion,

$$u_t = u_{xxxxx} + \delta u_{xxxx} + c_0(x)u_{xxx} + d_0(x)u_{xx} + e_0(x)u_x + f_0(x)u + h_0(x), \quad (29)$$

taken with data

$$u(x,0) = u_0(x) \in H^{n+5}.$$

After treating this case, the general case will follow by a change of variables.

First we will show that L, for the linear operator L associated with (29), has a sequence of eigenvalues λ_j for which

$$\lim_{j\to\infty} |\lambda_j| = \infty.$$

This is straightforward since L is a relatively compact perturbation of the operator $\partial_x^5 + I$, as long as the coefficients c_0, d_0, e_0, f_0, and h_0 are in L^2. Thus we know $\sigma_{\text{ess}}(L) = \sigma_{\text{ess}}(\partial_x^5 + I) = \emptyset$. This can be shown by choosing $\theta \in \mathbb{C}$ which is not a eigenvalue of ∂_x^5; then, the inverse operator $(\partial_x^5 - \theta)^{-1}$ is a bounded operator defined for all $f \in H^0$.

In fact, the operator $(\partial_x^5 - \theta)^{-1}$ is a bounded map from $H^0 \to H^5$. This means the the resolvent set of L is nonempty. We take Q in the resolvent and let the operator $\bar{L}$ be given by $\bar{L} = (L - Q)^{-1}$. We know that $\bar{L}$ is a bounded linear map from $L^2 \to H^5$. We know H^5 is compactly embedded in L^2 so we know $\bar{L}$ is a compact map from L^2 to itself. This implies that $\bar{L}$ has a non-zero sequence of eigenvalues $\{\mu_j\}$ with a corresponding sequence of eigenfunctions $\{u_j\}$ such that $\lim_{j\to\infty} \mu_j = 0$. So we have the following relations:

$$\begin{aligned}
&\bar{L}u_j = \mu_j u_j,\\
&(L - Q)^{-1}u_j = \mu_j u_j,\\
&L(u_j) = \left(Q + \frac{1}{\mu_j}\right) u_j.
\end{aligned}$$

It is easy to see that $\lim_{j\to\infty} |Q + \frac{1}{\mu_j}| = \infty$ since $\lim_{j\to\infty} \mu_j = 0$. This is our sequence of eigenvalues of L so we call $Q + \frac{1}{\mu_j} = \lambda_j$.

For such an eigenvalue λ of L, we let u be the associated eigenfunction such that $\|u\|_{L^2} = 1$. Then we have $\bar{u}Lu = \lambda u\bar{u}$ and thus, after integrating,

$$\int_X \bar{u}(u_{xxxxx} + \delta u_{xxxx} + c_0(x)u_{xxx} + d_0(x)u_{xx} + e_0(x)u_x + f_0(x)u)\,dx = \lambda.$$

We see by integration by parts that $\int_X \bar{u}u_{xxxxx}\,dx = -\int_X \bar{u}_{xxxxx}u\,dx$, thus it must be that $\Re\int_X \bar{u}u_{xxxxx}\,dx = 0$. (Note that here $\Re z$ indicates the real part of the complex number z.) Also by integration by parts, we see that $\int_X \bar{u}\delta u_{xxxx}\,dx = \delta\|u_{xx}\|_{L^2}^2$. Looking at the real part of λ, then, we see

$$\Re(\lambda) = \delta\|u_{xx}\|_{L^2}^2 + C_R,$$

where C_R is given by

$$C_R = \Re\left[\int_X \bar{u}(c_0(x)u_{xxx} + d_0(x)u_{xx} + e_0(x)u_x + f_0(x)u)dx\right].$$

We can then easily see that

$$|C_R| \le k_1(\|u_x\|_{L^2}^2 + \|u\|_{L^2}^2) = k_1\|u_x\|_{L^2}^2 + k_1.$$

Thus we have for $\delta > 0$ that

$$\Re(\lambda) \geq \delta \|u_{xx}\|_{L^2}^2 - k_1 \|u_x\|_{L^2}^2 - k_1 \geq -\widetilde{k}_1, \tag{30}$$

for an appropriate $\widetilde{k}_1 > 0$. Continuing, we see that

$$\Re(\lambda) + k_1 \|u_x\|_{L^2}^2 + k_1 \geq \delta \|u_{xx}\|_{L^2}^2,$$

and thus we may conclude

$$\delta^{-1}(\Re(\lambda) + k_1 \|u_x\|_{L^2}^2 + k_1) \geq \|u_{xx}\|_{L^2}^2. \tag{31}$$

Now we bound the imaginary part of λ. To begin, we introduce the decomposition

$$\Im(\lambda) = \int_X \bar{u} u_{xxxxx} dx + C_I, \tag{32}$$

where

$$C_I = \Im \left[\int_X \bar{u}(c_0(x)u_{xxx} + d_0(x)u_{xx} + e_0(x)u_x + f_0(x)u)\, dx \right]. \tag{33}$$

Integrating by parts and using the Cauchy-Schwarz inequality on the first term of the right-hand side of (32), we have

$$\int_X \bar{u} u_{xxxxx}\, dx = \int_X \bar{u}_{xx} u_{xxx}\, dx \leq \|u_{xxx}\|_{L^2} \|u_{xx}\|_{L^2}.$$

With an eye toward C_I, we make the following calculation:

$$\begin{aligned}
&\int_X c_0(x)\bar{u}u_{xxx} + d_0(x)\bar{u}u_{xx} + e_0(x)\bar{u}u_x + f_0(x)\bar{u}u\, dx \\
&= \int_X -c_0(x)\bar{u}_x u_{xx} + (d_0(x) - c_{0x}(x))\bar{u}u_{xx} + e_0(x)\bar{u}u_x + f_0(x)\bar{u}u\, dx \\
&= \int_X -c_0(x)\bar{u}_x u_{xx} - (d_0(x) - c_{0x}(x))\bar{u}_x u_x \\
&\quad + (e_0(x) - d_{0x}(x) + c_{0xx}(x))\bar{u}u_x + f_0(x)\bar{u}u\, dx.
\end{aligned}$$

We take the imaginary part of this, finding

$$C_I = \Im \left[\int_X -c_0(x)\bar{u}_x u_{xx} + (e_0(x) - d_{0x}(x) + c_{0xx}(x))\bar{u}u_x\, dx \right].$$

Using the Cauchy-Schwarz inequality we conclude

$$C_I \leq k_2 \|u_x\|_{L^2} \|u_{xx}\|_{L^2} + k_3 \|u_x\|_{L^2}.$$

Putting this together, we have

$$|\Im(\lambda)| \leq \|u_{xxx}\|_{L^2} \|u_{xx}\|_{L^2} + k_2 \|u_x\|_{L^2} \|u_{xx}\|_{L^2} + k_3 \|u_x\|_{L^2}. \tag{34}$$

Next we take the equation $Lu = \lambda u$, and we take the inner product with u_x :

$$\lambda \int_X \bar{u}_x u\, dx = \int_X \bar{u}_x (u_{xxxxx} + \delta u_{xxxx} + c_0(x)u_{xxx} + d_0(x)u_{xx} + e_0(x)u_x + f_0(x)u)\, dx.$$

Integrating by parts shows that the first term on the right-hand side is equal to $\|u_{xxx}\|_{L^2}^2$; thus, we have

$$\|u_{xxx}\|_{L^2}^2 = \lambda \int_X \bar{u}_x u\, dx - \int_X \bar{u}_x (\delta u_{xxxx} + c_0(x)u_{xxx}$$

$$+d_0(x)u_{xx} + e_0(x)u_x + f_0(x)u)\,dx. \tag{35}$$

Integrating by parts demonstrates that $\lambda \int_X \bar{u}_x u\,dx = -\lambda \int_X \bar{u} u_x\,dx$, so $\Re\lambda \int_X \bar{u}_x u\,dx = 0$, and similarly $\delta \int_X \bar{u}_x u_{xxxx}\,dx = -\delta \int_X \bar{u}_{xxxx} u_x\,dx$, so $\Re\delta \int_X \bar{u}_x u_{xxxx} = 0$. Combining this with (35) yields

$$\|u_{xxx}\|_{L^2}^2 = -\Re\left\{\int_X \bar{u}_x (c_0(x)u_{xxx} + d_0(x)u_{xx} + e_0(x)u_x + f_0(x)u)\,dx\right\}.$$

We manipulate the integral on the right-hand side by integrating by parts and using the Cauchy-Schwarz inequality, finding

$$\begin{aligned}
&\int_X \bar{u}_x (c_0(x)u_{xxx} + d_0(x)u_{xx} + e_0(x)u_x + f_0(x)u)\,dx \\
&= \int_X -c_0(x)\bar{u}_{xx}u_{xx} + (d_0(x) - c_{0x}(x))u_{xx}\bar{u}_x + e_0(x)u_x\bar{u}_x + f_0(x)u\bar{u}_x\,dx \\
&\leq k_1\|u_{xx}\|_{L^2} + k_2\|u_x\|_{L^2}\|u_{xx}\|_{L^2} + k_3\|u_x\|_{L^2}^2 + k_4\|u_x\|_{L^2}.
\end{aligned}$$

Letting $q = \max(k_1, k_2, k_3, k_4)$, we rewrite this as

$$\|u_{xxx}\|_{L^2}^2 \leq q(\|u_{xx}\|_{L^2} + \|u_x\|_{L^2}\|u_{xx}\|_{L^2} + \|u_x\|_{L^2}^2 + \|u_x\|_{L^2}). \tag{36}$$

We next find a bound for $\|u_x\|_{L^2}$. We begin by introducing the notation

$$u_{-n} = \sum_{k\in\mathbf{Z}\setminus\{0\}} \frac{e^{ikx}\hat{u}(k)}{(ik)^n}.$$

Using this notation, we integrate the equation $\lambda u\bar{u}_{-2} = Lu\bar{u}_{-2}$:

$$\begin{aligned}
\lambda \int_X \bar{u}_{-2}u\,dx = \int_X \bar{u}_{-2}(&u_{xxxxx} + \delta u_{xxxx} + c_0(x)u_{xxx} \\
&+d_0(x)u_{xx} + e_0(x)u_x + f_0(x)u)\,dx.
\end{aligned}$$

Considering the real part, we see that $\Re(\int_X \bar{u}_{-2}u_{xxxxx}\,dx) = 0$ and $\Re(\int_X \delta u_{xxxx}\bar{u}_{-2}\,dx) = -\delta\|u_x\|_{L^2}^2$ by integration by parts. We therefore have the following:

$$\delta\|u_x\|_{L^2}^2 = \int_X \bar{u}_{-2}(c_0(x)u_{xxx} + d_0(x)u_{xx} + e_0(x)u_x + (f_0(x) + \Re(\lambda))u)\,dx.$$

We continue to integrate by parts, yielding

$$\begin{aligned}
\delta\|u_x\|_{L^2}^2 = \Re\Bigg\{&\int_X c_0(x)\bar{u}_x u + (3c_{0x} + d_0(x))\bar{u}_0 u + (e_0(x) + 2d_{0x} + c_{0xx})\bar{u}_{-1}u_x\,dx \\
&+ \int_X (f_0(x) + e_{0x} + d_{0xx} + c_{0xxx} + \Re(\lambda))\bar{u}_{-2}u\,dx\Bigg\}.
\end{aligned}$$

Considering the first term on the right-hand side, we may estimate it as follows:

$$\begin{aligned}
\Re\left\{\int_X c_0(x)\bar{u}_x u\right\}\,dx &= \frac{1}{2}\int_X c_0(x)\bar{u}_x u + c_0(x)\bar{u}u_x\,dx \\
&= \frac{1}{2}\int_X \partial_x(c_0(x)\bar{u}u) - c_0'(x)\bar{u}u\,dx \\
&= -\frac{1}{2}\int_X c_0'(x)\bar{u}u\,dx \leq p_1\|u\|_{L^2}^2.
\end{aligned}$$

Since $\|u\|_{L^2}^2 = 1$, we use this to conclude

$$\delta\|u_x\|_{L^2}^2 \leq p_1 + \Re\left\{\int_X (3c_{0x} + d_0(x))\bar{u}_0 u + (e_0(x) + 2d_{0x} + c_{0xx})\bar{u}_{-1}u_x\, dx + \int_X (f_0(x) + e_{0x} + d_{0xx} + c_{0xxx} + \Re(\lambda))\bar{u}_{-2}u\, dx\right\}.$$

From Parseval's identity and the definition of u_{-n}, it is clear that $\|u_{-n}\|_{L^2} \leq \|u\|_{L^2}$. Using this together with the Cauchy-Schwarz inequality, we have the existence of constants p_i such that

$$\begin{aligned}\Re\Bigg\{&\int_X (3c_{0x} + d_0(x))\bar{u}_0 u + (e_0(x) + 2d_{0x} + c_{0xx})\bar{u}_{-1}u_x\, dx \\ &+ \int_X (f_0(x) + e_{0x} + d_{0xx} + c_{0xxx} + \Re(\lambda))\bar{u}_{-2}u\, dx\Bigg\} \\ \leq{}& (p_2 + p_3 + p_4 + \Re[\lambda])\|u\|_{L^2}^2.\end{aligned}$$

Setting $P = \sum_{n=1}^4 p_n$, we conclude

$$\delta\|u_x\|_{L^2}^2 \leq P + \Re[\lambda]. \tag{37}$$

We now combine the bounds (31), (36), and (37) on derivatives of u with the bound on the imaginary part of the eigenvalue (34). This implies that the imaginary part of the eigenvalue λ is bounded in terms of its real part. Recall that we also know that the real part cannot go to negative infinity by (30). We also know that there is a sequence of eigenvalues of L for which $|\lambda_n| \to \infty$. For this to be true we must have that $\Re[\lambda_n] \to \infty$. Thus we have proven our result for $a = 1$ and $b = \delta$.

Now that we have proven ill-posedness for a special form of L (i.e. when $a = 1$ and $b = \delta$), we will show that through a change of variables we can treat the general case. We start by defining

$$\Phi(x) = \int_0^x a^{-\frac{1}{5}}(y)\, dy.$$

In making this definition, we recall that a is bounded away from zero; this implies that Φ is invertible. Note that $d\Phi/dx = a^{-\frac{1}{5}}(x)$. We next define $v(z)$ where $u(x) = v(\Phi(x))$, i.e. $v(z) = u(\Phi^{-1}(z))$. Using again the definition of the operator L applied to u, namely

$$Lu = au_{xxxxx} + bu_{xxxx} + cu_{xxx} + du_{xx} + eu_x + fu,$$

and applying the change of variable, we see that

$$Lu = a(v_{zzzzz}\Phi'^5) + v_{zzzz}(10a\Phi'^3\Phi'' + b\Phi'^4) + c_1 v_{zzz} + d_1 v_{zz} + e_1 v_z + f_1 u. \tag{38}$$

(The explicit equations for the trailing coefficients are omitted as their specific form is not entirely relevant for our purposes, although they are specified in Remark 10 below.)

Using the definition of Φ, we rewrite (38) as

$$Lu = v_{zzzzz} + v_{zzzz}(10a\Phi'^3\Phi'' + b\Phi'^4) + c_1 v_{zzz} + d_1 v_{zz} + e_1 v_z + f_1 v.$$

We calculate the coefficient of the fourth-order term, using the definition of Φ :

$$10a\Phi'^3\Phi'' + b\Phi'^4 = a^{-4/5}(x)(b(x) - 2a'(x)).$$

We introduce the variable $z = \Phi(x)$, and introduce the function $\Gamma(z)$ defined through the equation

$$\Gamma(z) = a^{-4/5}(x)(b(x) - 2a'(x)).$$

Defining M' through the equation $\Phi(M') = M$, we have

$$\int_0^{M'} \Gamma(z)\, dz = \int_0^{\Phi(M')} \Gamma(\Phi(x))\Phi'(x)\, dx.$$

This implies that

$$\int_0^{M'} \Gamma(z)\, dz = \int_0^{M} a^{-1}(x)(b(x) - 2a'(x))\, dx = M\delta(x).$$

We introduce the notation $\bar{\Gamma} = \frac{M}{M'}\delta$, and we define the operator $\bar{L}$ through

$$\bar{L}\bar{v} = \bar{v}_{zzzzz} + \bar{v}_{zzzz}\Gamma(z) + c_1\bar{v}_{zzz} + d_1\bar{v}_{zz} + e_1\bar{v}_z + f_1\bar{v}.$$

Since $\bar{L}$ is simply a change of variables away from L, we know they have the same eigenvalues. We next use our gauge, i.e. we change again to $\bar{v} = vg$, with g as described in our energy estimates:

$$g = \exp\left\{\frac{-1}{5}\int_0^z \Gamma(s) - \bar{\Gamma}\, ds\right\}.$$

We may then calculate that if $\bar{L}\bar{v} = \lambda\bar{v}$, then

$$v_{zzzzz} + v_{zzzz}\left(5\frac{g_z}{g} + \Gamma(z)\right) + c_2v_{zzz} + d_2v_{zz} + e_2v_z + f_2v = \lambda v.$$

(We note again that formulas for the coefficients c_2, d_2, e_2, and f_2 can be found below in Remark 10.) We can simplify the coefficient of the fourth derivative,

$$5\frac{g_z}{g} + \Gamma(z) = (-(\Gamma(z) - \bar{\Gamma}) + \Gamma(z) = \bar{\Gamma}.$$

Then our eigenvalue problem becomes

$$v_{zzzzz} + \bar{\Gamma}v_{zzzz} + c_2v_{zzz} + d_2v_{zz} + e_2v_z + f_2v = \lambda v.$$

This is the same form of the operator that was studied at the beginning of this proof. This completes the proof of ill-posedness for the general case. □

Remark 10 As we said before beginning the proof of Theorem 9, we will comment now on the regularity needed for the coefficients in the proof of our ill-posedness theorem. The argument at the beginning of the proof, relating to a relatively compact perturbation, required coefficients to be in L^2. Specifically, the coefficients c_2, d_2, e_2, and f_2 must be in L^2. The formulas for these coefficients are

$$\begin{aligned}
c_2 &= \frac{10\partial_z^2 g}{g} + \frac{4\Gamma g_z}{g} + c_1,\\
d_2 &= \frac{10\partial_z^3 g}{g} + \frac{6\Gamma\partial_z^2 g}{g} + \frac{3c_1 g_z}{g} + d_1,\\
e_2 &= \frac{5\partial_z^4 g}{g} + \frac{4\Gamma\partial_z^3 g}{g} + \frac{3c_1\partial_z^2 g}{g} + \frac{2d_1 g_z}{g} + e_1,
\end{aligned}$$

$$f_2 = \frac{\partial_z^5 g}{g} + \frac{\Gamma \partial_z^4 g}{g} + \frac{c_1 \partial_z^3 g}{g} + \frac{d_1 \partial_z^2 g}{g} + \frac{e_1 g_z}{g} + f_1.$$

We then must give the formulas for the coefficients c_1, d_1, e_1, and f_1; these are

$$\begin{aligned}
c_1 &= 10a\Phi'^2\Phi''' + 15a\Phi'\Phi''^2 + 6b\Phi^2\Phi'' + c\Phi'^3, \\
d_1 &= 10a\Phi''\Phi''' + 5a\Phi'\Phi'''' + b\Phi''^2 + 4b\Phi'\Phi''' + 3c\Phi'\Phi'' + d\Phi'^2, \\
e_1 &= a\Phi''''' + b\Phi'''' + c\Phi''' + d\Phi'' + e\Phi', \\
f_1 &= f.
\end{aligned}$$

The formulas for g, Φ, and Γ are given in the proof of Theorem 9.

Our assumption that all the coefficients be C^∞ is clearly sufficient, but of course lower regularity assumptions are possible as long as one may conclude that c_2, d_2, e_2, and f_2 are in L^2.

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Conflict of interest The authors declare that they have no conflict of interest.

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