Hochster *Res Math Sci* (2022) 9:56
https://doi.org/10.1007/s40687-022-00353-z

Research in
the Mathematical Sciences

RESEARCH

Tight closure and strongly F-regular rings

Melvin Hochster

*Correspondence:
hochster@umich.edu
Department of Mathematics,
University of Michigan, East Hall,
530 Church St., Ann Arbor, MI
48109–1043, USA

This paper is dedicated to Jürgen Herzog on the occasion of his 80th birthday, in celebration of his fundamental contributions to commutative algebra.
The author was partially supported by National Science Foundation Grant DMS—1902116.
Data sharing is not applicable to this article, as no datasets were generated or analyzed in connection with this study.

Abstract

We describe several aspects of the theory of strongly F-regular rings, including how they should be defined without the hypothesis of F-finiteness, and its relationship to tight closure theory, to F-signature, and to cluster algebras. As a necessary prerequisite, we give a quick introduction to tight closure theory, without proofs, but with discussion of underlying ideas. This treatment includes characterizations, important applications, and material concerning the existence of various kinds of test elements, since test elements play a considerable role in the theory of strongly F-regular rings. We introduce both weakly F-regular and strongly F-regular rings. We give a number of characterizations of strong F-regularity. We discuss techniques for proving strong F-regularity, including Glassbrenner's criterion and several methods that have been used in the literature. Many open questions are raised.

Keywords: Big test element, Cluster algebra, Completely stable test element, Excellent ring, F-rational ring, F-regular ring, F-pure regular, Frobenius endomorphism, Frobenius functor, F-signature, F-split ring, Glassbrenner criterion, Hilbert–Kunz multiplicity, Peskine–Szpiro functor, Plus closure, Purity, Regular ring, Solid closure, Splinter, Strongly F-regular ring, Tight closure, Very strongly F-regular ring, Weakly F-regular ring

Mathematics Subject Classification: Primary 13A35; Secondary 13D45, 13C14, 13F40, 13H05, 13H10

1 Introduction

Unless otherwise specified, all given rings are commutative, associative and have a multiplicative identity. By a *local ring*, we always mean a Noetherian quasilocal ring. Let $\mathbb{N}$, $\mathbb{N}_+$, $\mathbb{Z}$, and $\mathbb{Q}$ denote, respectively, the nonnegative integers, the positive integers, the integers, and the field of rational numbers.

Our focus will be on Noetherian rings R of positive prime characteristic p. In fact, we are primarily concerned with what is true in the excellent case, but we will not make a blanket assumption of excellence for a while. General references for material not made explicit here are [13,73].

The theory of strongly F-regular rings grew out of the positive characteristic theory of tight closure. After giving some general background and an introduction to tight closure theory, we discuss the notions of weak F-regularity, F-regularity, and strong F-regularity. It is annoying to have three notions when, at least for locally excellent rings, it is an open question whether all three are equivalent. Weak and strong F-regularity are known to be equivalent for finitely generated $\mathbb{N}$-graded algebras over a field and in the Gorenstein case. Note that, in the locally excellent case, all of these conditions imply that the ring is Cohen–Macaulay and normal.

We shall give many characterizations of strongly F-regular rings. The notion was originally defined only for *F-finite rings*, i.e, for rings R such that the Frobenius endomorphism F_R (or, simply, F) of R, where $F(r) = r^p$ for all $r \in R$, makes R into a finite module over itself. We will discuss issues that arise when one generalizes the notion to the case where the ring is not necessarily F-finite. We present large classes of rings that are strongly F-regular. We also discuss a great many open questions.

This paper is intended to be accessible to researchers who are, relatively, newcomers to the study of characteristic p phenomena in commutative algebra.

Before embarking on a detailed study, we want to mention some down-to-earth examples of strongly F-regular rings. Let $1 \leq r \leq s$ and $a_1, \ldots, a_n$ be positive integers. Examples include the following rings. Note that, examples (2)–(7) are finitely generated $\mathbb{N}$-graded rings (in (6), the grading is obtained by weighting the variables) over a field.

(1) Regular rings
(2) Generic determinantal rings: quotients of a polynomial ring by the ideal generated by size t minors of an $r \times s$ matrix formed from the indeterminates ([50, Sect. 7])
(3) Toric rings: integrally closed rings generated by monomials in a polynomial ring (including Veronese subrings of and iterated Segre products of polynomial rings)
(4) Homogeneous coordinates rings of Grassmann varieties: these are generated by the $r \times r$ minors of an $r \times s$ matrix of indeterminates (and so are subrings of polynomial rings) ([50, Sect. 7])
(5) Generic Pfaffian rings: quotients of a polynomial ring by the ideal generated by the Pfaffians of a given size of an alternating matrix of indeterminates (if $t = 2h$ is even, the symmetrically placed size t minors have determinants that are perfect squares: these minors are the squares of the Pfaffians) ([5])
(6) For all sufficiently large primes p, the hypersurface defined by $x_1^{a_1} + \cdots + x_n^{a_n}$ over a field K of characteristic p when $\sum_i \frac{1}{a_i} > 1$ (the condition asserts that in a certain precise sense, the a_i are small compared to n)
(7) Rings of invariants of certain actions of linear algebraic groups on polynomial rings (cf. Theorem 10.4)
(8) Locally acyclic cluster algebras (cf. Theorem 10.5)
(9) Direct summands (or pure subrings) of any of the above.

This is just a brief sampling of results we discuss in more detail later. But we will mention here that if $R \hookrightarrow S$ is a homomorphism of Noetherian rings of prime characteristic $p > 0$ such that R is a direct summand of S as an R-module (or a pure R-submodule of S: cf. Theorem 9.7) and S is strongly F-regular, then so is R, which explains part (9) above.

The next section provides a very brief exposition of background material on integral closure of ideals,Cohen–Macaulay rings, and excellent rings.

The author would like to thank Tigran Ananyan and the anonymous referees for their comments on this paper.

2 Background

2.1 Integral closure of ideals

A reference for material in this subsection is [88]. We define an element $f \in R$ to be in the *integral closure* $\overline{I}$ of I if, equivalently

(IC1) With t an indeterminate over R, the element $ft \in R[t]$ is in the integral closure of the Rees ring $R[It] \subseteq R[t]$ (which is generated over R by the set $It := \{gt : g \in I\}$; it suffices to let g run through a set of generators of I as an ideal of R to get algebra generators for $R[It]$ over R).

(IC2) There exist $n \in \mathbb{N}_+$ and a monic polynomial $H(z) = z^n + \sum_{i=1}^{n} r_i z^{n-i}$ in the polynomial ring $R[z]$ such that for $1 \le i \le n$, $r_i \in I^i$.

(IC3) For every homomorphism $R \to V$, where V is a valuation domain, $f \in IV$.

It turns out that $\overline{I}$ is an ideal of R. Given a ring homomorphism $R \to S$, $\overline{I}S \subseteq \overline{IS}$. Moreover, $f \in \overline{I}$ if and only if the image of f in R/P is in the integral closure of $I(R/P)$ for all minimal primes P of R.

For simplicity, we assume now that R is a Noetherian domain, which is the main case. For $f \in R$ and an ideal $I \subseteq R$, we also have that the following are equivalent to the statement that $f \in \overline{I}$:

(IC4) For every discrete valuation ring V between R and its fraction field, $f \in IV$.

(IC5) There is a nonzero element $c \in R$ such that $cf^n \in I^n$ for infinitely many $n \in \mathbb{N}_+$.

(IC6) There is a nonzero element $c \in R$ such that $cf^n \in I$ for all $n \gg 1$.

(IC7) There is a nonzero element $c \in R$ such that $cf^n \in I$ for all $n \geq 1$.

Later, we study the relationship between integral closure and tight closure and we shall use tight closure to generalize the Briançon-Skoda theorem. See Theorem 4.6.

2.2 Cohen–Macaulay rings and modules

We simultaneously treat Cohen–Macaulay rings and small Cohen–Macaulay modules, which are also called *maximal Cohen–Macaulay modules*. We shall not deal with finitely generated Cohen–Macaulay modules that are not maximal, although we do define them.

A Noetherian local ring $(R, \mathfrak{m}, K)$ is Cohen–Macaulay (respectively, a finitely generated module M over R is small Cohen–Macaulay or maximal Cohen–Macaulay) if the following three equivalent conditions hold:

(CM1) Some system of parameters for R is a regular sequence on R (respectively, M)

(CM2) Every system of parameters for R is a regular sequence on R (respectively, M) .

(CM3) The depth of R (respectively, M) on $\mathfrak{m}$ is equal to the Krull dimension of R.

A maximal Cohen–Macaulay module M automatically has dimension equal to that of R. We remark that a finitely generated module M over R is called Cohen–Macaulay if it is maximal Cohen–Macaulay over $R/\mathrm{Ann}_R M$. This is equivalent to requiring that the depth of M on $\mathfrak{m}$ is equal to the Krull dimension of M.

More generally, if R is Noetherian but not necessarily local, the following conditions are equivalent and define the notion of Cohen–Macaulay for rings that are not necessarily local:

(CM4) Every local ring of R at a maximal ideal is Cohen–Macaulay.

(CM5) Every local ring of R at a prime ideal is Cohen–Macaulay.

(CM6) For every proper ideal I of R, the height of I is equal to the depth of R on I (which is the length of any maximal regular sequence in I).

We also note the following properties:

(CM7) Regular rings are Cohen–Macaulay.

(CM8) If R is module-finite over a subring A that is regular, then R is Cohen–Macaulay if and only if R is projective as an A-module. If A is local or a polynomial ring over a field, R is Cohen–Macaulay if and only if it is free as an A-module.

(CM9) If R is Cohen–Macaulay, so is every polynomial ring or formal power series ring in finitely many variables over R.

(CM10) If R Cohen–Macaulay, then R is universally catenary (i.e., in any algebra S essentially of finite type over R, if $P \subseteq Q$ are primes of S, all saturated chains of primes joining P to Q have the same length).

(CM11) If R is Cohen–Macaulay and either local or finitely generated and $\mathbb{N}$-graded over a field, for every minimal prime of P of R, the Krull dimension of R/P is the same as the Krull dimension of R.

(CM12) If R is Cohen–Macaulay and $f_1, \ldots, f_h$ are elements of R generating an ideal I of height h, then every associated prime of I is minimal and has height h. Moreover, R/I is again Cohen–Macaulay. In particular, these statements hold when R is regular.

Cohen–Macaulay rings are useful for many purposes. Various duality theories in commutative algebra and algebraic geometry are simpler in the Cohen–Macaulay case, and Serre intersection multiplicities become lengths of tensor products (instead of alternating sums of lengths of Tor modules) in the Cohen–Macaulay case. See [13] and the review of that book in [36].

The *type* of a Cohen–Macaulay local ring $(R, \mathfrak{m}, K)$ of Krull dimension d is the K-vector space dimension of the socle in $R/(x_1, \ldots, x_d)R$ for some (equivalently, every) system of parameters $x_1, \ldots, x_d$ for R and may also be characterized the K-vector space dimension of $\operatorname{Ext}^d_R(K, R)$. A local ring R is called *Gorenstein* if, equivalently:

(Gor1) R has type 1.

(Gor2) R has finite injective dimension as a module over itself, in which case its injective dimension is d.

If R of dimension d is Cohen–Macaulay local and a homomorphic image of a Gorenstein local ring S of dimension $d + h$, the module $\omega_R := \operatorname{Ext}^h_S(R, S)$ is, up to isomorphism, independent of the choice of S and is called a *canonical module* for R. The type of R is also the minimum number of generators of ω_R in this case.

Discussion 2.1 **Big Cohen–Macaulay modules and algebras.** A module M over a local ring $(R, \mathfrak{m}, K)$ that is not necessarily finitely generated is called a balanced *big Cohen–Macaulay* module for R if every system of parameters is a regular sequence on M (which implies $\mathfrak{m}M \neq M$). It is not necessarily the case if one system of parameters is a regular sequence on M, then every system of parameters is a regular sequence on M, but this is true for the $\mathfrak{m}$-adic completion of M. Thus, if R has a big Cohen–Macaulay module, it has a balanced big Cohen–Macaulay module. In the rest of this manuscript, we typically omit the word "balanced." An R-algebra that is a big Cohen–Macaulay module is called a big Cohen–Macaulay algebra over R.

2.3 Colon ideals

We shall have many occasions to talk about the colon ideals $I :_R J$ where $I, J, J' \subseteq R$ are ideals. This ideal is simply $\{r \in R : rJ \subseteq I\}$. When $J = uR$ we often write $I :_R u$ for $I :_R J$. We note the following:

(Col1) $R/I \supseteq (I :_R J)/I \cong \mathrm{Ann}_{R/I} J$. Hence, $\mathrm{Ann}_R Rx = (0) :_R x$.
(Col2) $x_1, \ldots, x_n$ is a regular sequence in R if and only if $(x_1, \ldots, x_n)$ is a proper ideal and $(x_1, \ldots, x_i) :_R x_{i+1} = (x_1, \ldots, x_i)$ for $1 \leq i \leq n-1$. (If $i = 0$, $(x_1, \ldots, x_i) = (0)$.)
(Col3) $I :_R (J + J') = (I :_R J) \cap (I :_R J')$.
(Col4) $I :_R (JJ') = (I :_R J) :_R J'$.

2.4 Excellent rings

We recall that a Noetherian ring R is excellent if and only if it satisfies the following conditions:

(E1) R is universally catenary,
(E2) the singular locus in any finitely generated R-algebra is closed, and
(E3) for every local ring S of R, the fibers of the map $S \to \widehat{S}$ are geometrically regular.

Conditions that are, on the face of it, much weaker, suffice to characterize excellence. See [73] for a very readable treatment.

All fields as well as $\mathbb{Z}$ are regular rings. An R-algebra essentially of finite type over an excellent ring R is excellent, and complete local rings are excellent. Convergent power series rings over $\mathbb{R}$ and $\mathbb{C}$ are also excellent. In an excellent ring R, the set of primes P such that R_P is Cohen–Macaulay (respectively, Gorenstein) is Zariski open.

The normalization of an excellent domain R is module-finite over R, and completion for excellent local rings preserves the properties of being reduced and of being normal.

A Noetherian ring is called *locally excellent* if all of its localizations at maximal (equivalently, prime) ideals are excellent.

2.5 Purity

A map of R-modules $A \to B$ is called *pure* if it is injective and for every R-module M, the map $A \otimes M \to B \otimes M$ is injective. This holds when $A \to B$ splits, i.e., when $B \cong A \oplus A'$ and the map is $A \to B$ is the associated injection of A into B. When B/A is finitely presented, $A \to B$ is pure if and only if the map splits. See [56,57] for treatments of purity. We note that a direct limit of pure maps of R-modules is pure, and that if R is Noetherian, $A \subseteq B$ is pure if and only if for every submodule B_0 of B containing A such that B_0/A is finitely generated, A is a direct summand of B_0.

3 Some positive characteristic phenomena

3.1 Functors related to the Frobenius endomorphism

If $\theta : R \to S$ is a ring homomorphism, we have a base change functor $S \otimes_R \_$ from R-modules to S-modules, and a restriction of scalars functor θ_* from S-modules to R-modules. In base change, if elements m_i generate M, the elements $1 \otimes m_i$ generate $S \otimes M$.

The base change functor sends R to S, commutes with direct sums and direct limits, preserves finite generation, and preserves the module properties of being free, or projective, or flat. Base change is right exact: It commutes with taking cokernels. Matrices act on the left. Thus, if an R-module M is the cokernel of the $m \times n$ matrix $\big(r_{ij}\big)$ thought of as a map from $R^n \to R^m$, then $S \otimes_R M$ is the cokernel of the matrix $\big(\theta(r_{ij})\big)$.

On the other hand, restriction of scalars is an exact functor: It does not change the underlying abelian groups nor their maps. It does not ordinarily preserve finite generation. If M is an S-module and $m \in M$, the R-module structure is given by $r \cdot m = \theta(r)m$.

We shall use e to represent a varying integer in $\mathbb{N}$, the set of nonnegative integers. We shall write F (or, if more precision is needed, F_R) for the Frobenius endomorphism of R, so that $F(r) = r^p$ for all $r \in R$. We write F^e for the e-fold iterate of F with itself under composition, so that $F^e(r) = r^{p^e}$. We shall be working with the base change and restriction of scalars functors when θ is F^e. This is a powerful tool, but the situation can be confusing.

When $\theta = F^e$, we shall denote the corresponding base change functor as $\mathcal{F}^e$ (or $\mathcal{F}^e_R$)). We refer to these functors as *Frobenius functors* or *Peskine–Szpiro functors*. C. Peskine and L. Szpiro obtained beautiful results using these functors systematically in [76,77]. Thus,

$$\mathcal{F}^e\Big(\mathrm{Coker}\big((r_{ij})\big)\Big) = \mathrm{Coker}\big((r_{ij}^{p^e})\big).$$

When M is the cyclic module R/I, $\mathcal{F}^e(M)$ may be identified with $R/I^{[p^e]}$, where $I^{[p^e]}$ is the ideal of R *generated* by all r^{p^e} for $r \in I$. The ideal $I^{[p^e]}$ is the same as the extension of $I \subseteq R$ to R where the map $R \to R$ being used is F^e. If one has generators f_j of I, the elements $f_j^{p^e}$ generate $I^{[p^e]}$. The ideals $I^{[p^e]}$ are called the *bracket* powers of I.

Note that, the ideal $I^{[p^e]}$ contains *all* R-linear combinations of the elements f^{p^e} for $f \in I$, not just the elements f^{p^e}. E.g., in the polynomial ring $R = K[w, x, y, z]$, if $I = (x, y)R$, $I^{[p^e]}$ contains $wx^{pe} + zy^{pe}$. Notice also that, typically, $I^{[p^e]}$ is much smaller than the ordinary power I^{p^e}. For example, in characteristic 3, $(x, y)^{[9]}$ is (x^9, y^9) while $(x, y)^9$ is generated by the ten monomials $x^i y^{9-i}$, $0 \le i \le 9$.

When $\theta = F^e$, we denote the restriction of scalars functor by using the exponent ^{e} on the left, so that $F^e_*(M) = {}^eM$. We note that when R is reduced, the algebra map $R \to {}^eR$ may be identified either with the inclusion map $F^e(R) \subseteq R$, where $F^e(R) = \{r^{p^e} : r \in R\}$ or with the inclusion map $R \subseteq R^{1/p^e}$, where $R^{1/p^e} = \{r^{1/q} : r \in R\}$. We write ${}^er$ for the element of ${}^eR$ corresponding to r. If we identify ${}^eR$ with R^{1/p^e}, then ${}^er$ is r^{1/p^e}.

3.2 F-finite rings

We have already defined the notion of F-finiteness in Sect. 1. We note here that a field K is F-finite if and only if $[K : K^p]$ is finite, where $K^p = \{c^p : c \in K\}$. The class of F-finite rings is closed under taking quotients and localizations, as well as under adjoining finitely many polynomial or power series indeterminates. A complete local ring is F-finite if and only if its residue class field is. A very useful fact due to Kunz [66] is that every F-finite ring is excellent.

4 Tight closure in positive characteristic

4.1 Tight closure for ideals

Let R be a Noetherian ring of prime characteristic $p > 0$. Suppose $f \in R$ and $I \subseteq R$ is an ideal. Let R° denote the multiplicative system of elements of R not in any minimal prime.

If R is reduced, then R° is the set of nonzerodivisors, while if R is a domain , R° is the set of nonzero elements of R. We use e to denote a varying element of $\mathbb{N}$. We define $f \in R$ to be in the *tight closure* of I in R if there exists c in R° such that $cf^{p^e} \in I^{[p^e]}$ for all $e \gg 0$. The tight closure of I turns out to be an ideal, and we use I^* to denote this ideal. We shall make a considerable effort to make this definition, which seems rather technical at first sight, more transparent. The notion was first defined by Craig Huneke and the author after a conference held at the University of Illinois at Urbana-Champaign in late October, 1986. Making this definition led quickly to surprisingly simple proofs of a number of theorems and led to substantial generalizations of them. We discuss a number of these below. See Proposition 4.1, and Theorems 4.2, 4.4, 4.5, and 4.6 .

The word "tight" was chosen because this closure is much smaller than other closures, such as integral closure, that had been considered earlier. It turns out to be a "tight fit" for the ideal. Many basic properties of tight closure are given in [44]. Other introductory references for the theory include [41–43,48–50], and [62], as well as [52] for the equal characteristic zero theory. In this manuscript, we focus in positive characteristic.

We begin with some very basic facts. Let R be a Noetherian ring of prime characteristic $p > 0$, with $f \in R$ and $I, J \subseteq R$ ideals.

(TC1) If $I \subseteq J$, then $I^* \subseteq J^*$.
(TC2) $f \in I^*$ if and only if for every minimal prime P of R, the image of f is in the tight closure of $I(R/P)$ in the domain R/P.
(TC3) $(I^*)^* = I^*$.

From (TC2), one easily sees

(TC4) For all $I \subseteq R$, I^* is the inverse image in R of $(IR_{\text{red}})^*$ calculated in R_{red}.

That is, in a certain sense, nilpotent elements are "irrelevant" when working with tight closure.

Note that, if R is a domain, then

(TC5) $f \in I^*$ if and only if there exists a nonzero $c \in R$ such that $cf^{p^e} \in I^{[p^e]}$ for all $e \gg 0$.
(TC6) $f \in I^*$ if and only if there exists a nonzero $c \in R$ such that $cf^{p^e} \in I^{[p^e]}$ for all $e \geq 0$.

As mentioned earlier, we want to give immediately some more substantial results that help to explain why tight closure is so useful. We give very few proofs here. However, the following statement is immediate from our discussion of integral closure of ideals, from (TC2), (IC4), (TC4), and from the fact that $I^{[p^e]} \subseteq I^{p^e}$ for all $e \in \mathbb{N}$:

Proposition 4.1 *Let R be a Noetherian ring of prime characteristic $p > 0$. For every ideal I of R, we have that $I^* \subseteq \overline{I}$. Hence, for any integrally closed ideal I, we have that $I^* = I$. In particular, this holds for radical (and, hence, prime) ideals.*

The next result is of very great importance:

Theorem 4.2 *Let R be a Noetherian ring of prime characteristic $p > 0$. If R is regular, then every ideal of R is tightly closed.*

Although tight closure is a trivial operation for regular rings, it gives an extremely useful criterion for membership in ideal that, on the face of it, is a bit weaker than membership. The proof of this result depends heavily on the following result of Ernst Kunz [65]

Theorem 4.3 *Let R be a Noetherian ring of prime characteristic $p > 0$. Then, R is regular if and only if the Frobenius endomorphism is flat.*

A critical consequence of the flatness of the Frobenius endomorphism for regular rings is that for all ideals $I, J \subseteq R$, $I^{[p^e]} :_R J^{[p^e]} = (I :_R J)^{[p^e]}$, and this plays a key role in the proof of Theorem 4.2. We also note that there are "flatness" results for Frobenius acting on modules of finite projective dimension [76], and a converse due to Jürgen Herzog in [31].

Rings that are not Cohen–Macaulay are significantly harder to work with than rings that do have this property. It is therefore that a "small" closure like tight closure can be used to control the failure of the Cohen–Macaulay property under mild hypotheses. Here is the key fact:

Theorem 4.4 (Colon-capturing) *Let $(R, \mathfrak{m}, K)$ be an excellent local domain and let $x_1, \ldots, x_{k+1}$ be part of a system of parameters. Let $I_k = (x_1, \ldots, x_k)$. Then, $I_k :_R x_{k+1} \subseteq I_k^*$.*

We have stated this version to avoid technicalities. One does not need that the ring be a domain, and there are also generalizations to the case where the ring is not local. In the excellent local case, the elements must form part a system of parameters modulo every minimal prime. We shall see in Sect. 6 that under mild conditions on R, if every ideal is tightly closed, then R is Cohen–Macaulay: cf. Theorem 6.5.

Tight closure also captures contracted extensions of ideals to integral extension rings:

Theorem 4.5 *Let $R \subseteq S$ be Noetherian rings of prime characteristic $p > 0$ such that S is integral over R. Then, for every ideal I of R, we have that $IS \cap R \subseteq I^*$.*

This result has important implications for understanding rings that are *splinters*: see 7.2.

The Briançon-Skoda theorem was first proved for regular rings of equal characteristic 0 using an analytic criterion for membership in an ideal (expressed in terms of the finiteness of an integral) due to Skoda. See [12,83]. Later, Lipman and Sathaye [67] gave an algebraic proof valid for all regular rings. Tight closure theory [44, Sect. 5] gives a quite simple proof of a stronger result:

Theorem 4.6 (generalized Briançon–Skoda theorem) *Let R be a Noetherian ring of prime characteristic $p > 0$. Let I be an ideal of R with at most n generators. Then, $\overline{I^n} \in I^*$. Hence, if R is regular, $\overline{I^n} \subseteq I$.*

More generally, for every $k \in \mathbb{N}$, $\overline{I^{n+k}} \subseteq (I^{k+1})^*$.

The result is also valid in equal characteristic 0: One can prove this by introducing tight closure theory for rings containing a field of characteristic 0 as in [52].

In [19], the authors obtain a surprising comparison theorem for ordinary and symbolic powers of ideals in regular rings of equal characteristic 0 using the theory of (asymptotic) multiplier ideals and results that rest on analytic techniques. Tight closure [54] yields comparable results in characteristic 0 that are, in some ways, stronger and gives new proofs in equal characteristic 0 by the method of reduction in characteristic p. These results are refined in [55]. The following results using tight closure for the proof in characteristic $p > 0$ and reduction in characteristic p for the proof in equal characteristic 0 are contained in [54, Theorems 2.6, 4.4].

Theorem 4.7 *Let I be ideal of a Noetherian ring containing a field . Let h be the largest height of any associated prime of I (or let h be the largest analytic spread of IR_P for an associated prime P of I). Then, if R is regular, $I^{(hn)} \subseteq I^n$ for all $n \in \mathbb{N}_+$ (and, more generally, $I^{(hn+kn)} \subseteq I^{(k+1)n}$ for all $k \in \mathbb{N}$ and $n \in \mathbb{N}_+$). If R need not be regular but I has finite projective dimension, then $I^{(hn)} \subseteq (I^n)^*$ for all $n \in \mathbb{N}_+$.*

Results like this for regular rings of mixed characteristic are proved in [71].

4.2 Tight closure and localization

It was an open question for many years whether tight closure commutes with localization. In [11], it is proved even for algebras over a field K that it does not when K contains an element that is transcendental over the prime field. We mention only briefly here that there are many instances when it either commutes with localization or has some other form of compatibility with localization. See, for example, [3,53]. We later discuss the notion of plus closure, and theorems about when plus closure and tight closure are the same: see Sect. 7.3. One point of interest is that plus closure *does* commute with localization.

4.3 Tight closure for submodules

Let R be a Noetherian ring of prime characteristic $p > 0$ We next want to define tight closure for submodules N of arbitrary modules M. We do not need to assume that these are finitely generated, but that will be the main case in the sequel. We first give a rather abstract definition using the Frobenius or Peskine–Szpiro functors introduced in Sect. 3. We then explain why this definition is really the same, in the case of $N := I \subseteq R =: M$ as the definition that we have already given.

Discussion 4.8 Let R be a Noetherian ring of prime characteristic $p > 0$. Let $N \subseteq M$ be R-modules, and let $N^{[p^e]}$ or, more precisely, $N_M^{[p^e]}$, denote the image of $\mathcal{F}^e(N) \to \mathcal{F}^e(M)$. In general, for any base change from R to an R-algebra S, there is a map $M \to S \otimes_R M$ such that $u \mapsto 1 \otimes u$. This gives a map $M \to \mathcal{F}^e(M)$ for each R-module M, and we denote the image of u under this map as u^{p^e}. We can now define the tight closure N_M^* of N in M as the set of elements $u \in M$ for which there exists $c \in R^\circ$ such that $cu^{p^e} \in N_M^{[p^e]}$ for all $e \gg 1$.

The tight closure of N in M is a submodule of M containing N.

Proposition 4.9 *Let R be a Noetherian ring of prime characteristic $p > 0$. If $u \in M$, then $u \in N_M^*$ if and only if the image of u in M/N is in $0_{M/N}^*$. Hence, if G is a free module that maps onto M, $H \subseteq G$ is the inverse image of N, and $v \in G$ is a pre-image of u, then $u \in N_M^*$ if and only if $v \in H_G^*$.*

Thus, for many purposes, understanding tight closure comes down to understanding tight closure in free modules. Because $\mathcal{F}^e(R) = R$, if we fix a basis for the free module G, we may identify $F^e(G)$ with G, since F^e commutes with direct sum: for each basis element b, we identify $F^e(Rb)$ with Rb. When we do this, $u^{p^e} \in G$ is simply the element obtained from u by raising all of the coefficients in its representation in terms of the specified free basis to the p^e power. When $G = R$, r^{p^e} as defined in the third sentence of Discussion 4.8 therefore has its usual meaning. Note that, $H^{[p^e]}$ is then identified in G with the submodule generated over R by all the elements h^{p^e} for $h \in H$. We note

Proposition 4.10 *If R is regular, then every submodule of every module is tightly closed.*

In the remainder of this manuscript, we shall often assume that M is finitely generated, although many statements are valid in greater generality. We do want to include several remarks about what happens when the finite generation condition is relaxed, particularly because considering this situation gives a characterization of strongly F-regular rings. Of particular importance is the case where M is the injective hull of the residue class field of a local ring.

5 Test elements and persistence

One difficulty in studying tight closure is that the definition gives no indication of what element $c \in R^\circ$ one should use in testing for tight closure. The theory of test elements and test ideals addresses this particular problem.

Throughout this section, unless otherwise specified, R is a reduced Noetherian ring of prime characteristic $p > 0$. The assumption that R is reduced simplifies the theory and is reasonable because of property (TC4).

The following three conditions on element $c \in R^\circ$ are equivalent:

(TE1) For every ideal I and element $r \in R$, $r \in I^*$ if and only if for all $e \in \mathbb{N}$, $cr^{p^e} \in I^{[p^e]}$.

(TE2) $c \in \bigcap_{I \subseteq R} (I :_R I^*)$.

(TE3) For every pair of finitely generated modules $N \subseteq M$ and element $u \in M$, $u \in N^*_M$ if and only if for all $e \in N$, $cu^{p^e} \in N^{[p^e]}_M$.

We call elements of R° satisfying these equivalent conditions *test elements*. It is highly non-trivial that test elements exist, but this is known to be true in many cases. It is natural to study the ideal $\tau(R) = \bigcap_{I \subseteq R} I :_R I^*$, which is called the *test ideal* of R. Typically, in the cases where test elements are known to exist, $\tau(R)$ is generated by $\tau(R) \cap R^\circ$: the latter is the set of test elements. We refer the reader to [43,44,49,52,64,69] for more information.

We also note that a test element is called *locally stable* (respectively, *completely stable* if it is a test element in every local ring of R (respectively, and in the completion of every local ring of R)). We also define $c \in R^\circ$ to be a *big test element* if condition (TE3) holds for all pairs of modules $N \subseteq M$ without the condition that either module be finitely generated.

Test elements play a considerable role in the theory of strongly F-regular rings, and conversely. We shall discuss this point further once we have introduced the notion of a strongly F-regular ring.

The following result gives several important instances in which test elements are known to exist.

Theorem 5.1 *Let R be a Noetherian ring of prime characteristic $p > 0$, and assume that R is reduced.*

(a) *Suppose that R is module-finite over a regular domain A with fraction field $\mathcal{L}$, and that $\mathcal{L} \otimes_A R$ is a finite product of separable field extensions of $\mathcal{L}$, i.e., is generically* *étale over A. Then, R has a completely stable test element. Hence, every localization of R has a test element.*

(b) *Suppose that R is essentially of finite type over an excellent local ring. Suppose that $c \in R^\circ$ is such that R_c is regular. Then, c has a power that is a completely stable element.*

(c) *Let K be a field with algebraic closure $\overline{K}$. If R is a finitely generated algebra over a field K (respectively, a complete local K-algebra with coefficient field K) such that $\overline{K} \otimes_K R$ (respectively, $\overline{K}\widehat{\otimes}_K R$) is reduced and equidimensional, then the Jacobian ideal $\mathcal{J}_{R/K}$ of R over K (which defines the singular locus) intersected with R° consists of completely stable test elements.*

For the proofs of parts (a), (b), and (c), we refer the reader to [44, Theorem 6.13], [49, Theorem 6.20], [52, Corollary 1.5.5] and [64] (for the complete case) respectively. Note that, part (c) of the result above is related to but does not follow from part (b), which only yields that every element of $\mathcal{J} \cap R^\circ$ has a power that is a completely stable test element. Note that, [64] also obtains results in equal characteristic 0.

5.1 Persistence

Let $R \to S$ be a homomorphism of Noetherian rings of characteristic $p > 0$ and let $N \subseteq M$ be finitely generated R-modules. Suppose that $u \in N^*_M$. One often wants to know that $1 \otimes u$ is in the tight closure of the image of $S \otimes_R N$ in $S \otimes_R M$. There are many cases where this is obvious, e.g., for $R \subseteq S$ and $R^\circ \subseteq S^\circ$. This happens whenever R and S are domains. However, the situation becomes delicate when $R \to S$ has a kernel. The issue may be considered mod the minimal primes of S, and so one needs to focus on the situation where $R \to S$ is replaced by the map $R \to R/P$ where P is the inverse image of a minimal prime of S. The problem is that the element one used in testing tight closure over R may map to 0 in R/P. It turns out that if one has a sufficient supply of test elements in the rings one is working with, there is a way around the problem. One gets from R to R/P by taking successive quotients by height one primes. In the process, it turns out one can replace R by a normal domain. Then, in many cases, the test ideal has height at least two, and so not all test elements are killed when one takes the quotient by a height one prime. There is a detailed study of persistence [49]. We record only one result here, which is part of [49, Theorem 6.24].

We mention two applications of persistence. One is the proof of the vanishing theorem for maps of Tor sketched in Remark 7.5. Another is that one can obtain considerable information about which elements are in or not in I^* for $I \subseteq R$ by mapping R to another ring where tight closure is easier to calculate. For instance, one sees in this way that if $f \in I^*$, then $f \in IS$ for every regular ring R-algebra S.

Theorem 5.2 *Let $R \to S$ be a homomorphism of Noetherian rings of characteristic $p > 0$. Let $N \subseteq M$ be finitely generated R-modules and let $w \in M$ be an element of M in N^*_M Assume that at least one of the following conditions holds:*

(1) *R is locally excellent and S has a locally stable test element (or S is local), or*
(2) *S has a completely stable test element (or S is a complete local ring).*

Then, $1 \otimes w$ is in the tight closure of the image of $S \otimes_R N$ in $S \otimes_R M$.

6 Weakly F-regular rings

Let R be a Noetherian ring of prime characteristic $p > 0$. We say that R is *weakly F-regular* if every ideal is tightly closed. Some properties of weakly F-regular rings are given in the results below, which provide a summary of a substantial number of non-trivial theorems.

Theorem 6.1 *Let R be a Noetherian ring of prime characteristic $p > 0$ The following conditions are equivalent.*

(1) *R is weakly F-regular.*
(2) *Every submodule of every finitely generated module is tightly closed.*
(3) *The localization of R at every maximal ideal is weakly F-regular.*
(4) *Every ideal primary to a maximal ideal is tightly closed.*

We note

Theorem 6.2 *Every regular ring is weakly F-regular.*

Theorem 6.3 *A weakly F-regular ring is normal.*

In fact, one shows very generally that the tight closure of a principal ideal generated by a nonzerodivisor is the same as its integral closure, which turns out to imply the result just above. However, in general, the tight closure of an ideal is much smaller than the integral closure.

Theorem 6.4 *if $R \subseteq S$ is pure as a map of R-modules, e.g., if it splits as a map of R-modules, and S is weakly F-regular, then R is weakly F-regular.*

Theorem 6.5 *If R is weakly F-regular and locally excellent, then R is Cohen–Macaulay.*

It is not known whether the property of being weakly F-regular passes to localizations in the locally excellent case. It does pass to localizations at maximal ideals. Thus, the problem is whether, given a weakly F-regular local ring, its localization at every prime ideal is weakly F-regular. We note a proposition about weak F-regularity and then raise this question formally.

Proposition 6.6 *. Let R be a Noetherian ring of prime characteristic $p > 0$. The following conditions on R are equivalent.*

(a) *The localization of R at every prime ideal is weakly F-regular, i.e., R is F-regular.*
(b) *The localization of R at every multiplicative system is weakly F-regular.*

A weakly F-regular ring satisfying these conditions is called *F-regular*.

Open question 6.7 *Let R be a Noetherian ring of prime characteristic $p > 0$. Suppose that R is locally excellent ring and weakly F-regular. Is R F-regular?*

It would suffice to prove this in the case where R is complete and F-finite, and it would suffice to show that if such a ring is weakly F-regular, then so is its localization at a prime ideal Q such that the Krull dimension of R/Q is 1. The idea of the reduction in this case is to replace the local ring by its completion and then by an F-finite ring using the Γ construction in [49, Sect. 6]. One can localize at any prime by repeatedly localizing at primes whose quotient has dimension one.

6.1 Invariant theory

For simplicity, we consider here only the case of an algebraically closed field K. A Zariski closed subgroup G of $\mathrm{GL}(n, K)$ is called a *linear algebraic group*. In the case of a finite-dimensional vector space V over K we consider only actions of G on V such that the map homomorphism $G \to \mathrm{GL}(n, K)$ is regular. We then refer to V as G-module: K is understood. In the case of infinite-dimensional vector space V, we consider only actions of G such that V is a directed union of finite-dimensional G-stable subspaces that are G-modules. A G-module is *simple* if it V has no proper nonzero G-stable subspace. The group G is called *linearly reductive* if every G-module is a direct sum of simple G-modules (which will be finite-dimensional over K). In characteristic $p > 0$, there are very few linearly reductive groups (finite groups with order not divisible by p and products of copies of the multiplicative group of the field are included).

Over a field of characteristic 0, additionally, all of the semisimple groups are linearly reductive, including the general and special linear groups, the orthogonal and special orthogonal group, and the symplectic group.

When one has a linearly reductive group, there is a covariant functor on G-modules $V \mapsto V^G$, where $V^G := \{v \in V : \text{forall}\, g \in G, g(v) = v\}$. This functor turns out to be invariant. In fact, V splits functorially and uniquely as a direct sum $V^G \oplus V_G$, where V_G contains no invariant elements other than 0. One therefore has a canonical retraction $\rho : V \to V^G$ in the category of G-modules, called the *Reynolds operator*. When G is finite and $|G|$ is invertible in K, then $\rho : v \mapsto (1/|G|) \sum_{g \in G} g(v)$.

When G is linearly reductive and acts by K-algebra automorphisms on a K-algebra R, $R^G \subseteq R$ is a K-subalgebra, and $\rho : R \to R^G$ is R^G-linear. That is, R^G is a direct summand of R as a module over itself. By Theorem 6.4, we know that if R is regular of characteristic p (or weakly F-regular) then so is R^G. Hence, R^G is Cohen–Macaulay under mild conditions. By generalizing tight closure theory to equal characteristic 0 (or by reducing to characteristic p in another way), one can show

Theorem 6.8 *If R is a direct sum of a regular ring containing a field, then R is Cohen–Macaulay. In particular, if G is a linearly reductive algebraic group over K acting on a regular K-algebra R, then R^G is Cohen–Macaulay.*

We refer the reader to [44,52] for more details. Major results in this direction were obtained in [56]. A proof not using tight closure for the equal characteristic 0 case is given in [8], where it is proved that if R is regular and affine over K, R^G has rational singularities (the argument uses resolution of singularities and the Grauert-Riemenschnieder vanishing theorem [23]). It is now known that direct summands of regular rings are Cohen–Macaulay even when the ring does not contain a field, using techniques of almost mathematics and perfectoid geometry. See, for example, [30].

7 Further applications of tight closure

7.1 Phantom acyclicity and the vanishing theorem for maps of Tor

There is a tight closure analog of the Buchsbaum-Eisenbud acyclicity criterion [6].

Discussion and Notation 7.1 In the discussion here, we first assume that the ring is Noetherian but do not impose any restrictions on the characteristic.

Recall that if $J \subseteq R$, the depth of R on J is the length of some (equivalently, every) maximal regular sequence in J when $J \neq R$ and is $+\infty$ if $J = R$.

Consider a left complex $G_\bullet$ of finitely generated free R-modules:

$$0 \to R^{b_n} \to \cdots \to R^{b_i} \xrightarrow{A_i} R^{b_{i-1}} \to \cdots \to R^{b_0} \to 0.$$

Here, A_i is the $b_{i-1} \times b_i$ matrix of the map $R^{b_i} \xrightarrow{A_i} R^{b_{i-1}}$. We consider three functions on ideals J of R: the height of J, the depth of R on J, and the *minheight* of J, which we define to be the smallest height of any of the ideals $J(R/\mathfrak{p})$ when $\mathfrak{p}$ is a minimal prime of R. We make the convention that all three of these functions have the value $+\infty$ when $J = R$ is the unit ideal.

Note that, when R is a domain, minheight coincides with height. For simplicity, the reader may wish to consider the results we are about to state only in the domain case. Let r_i denote the determinantal rank of A_i. We also make the convention that $r_{n+1} = 0$. We shall say the complex $G_\bullet$ *satisfies the standard conditions on rank and minheight (or depth, or height)* if

(1) For all i, $1 \le i \le n$, $b_i = r_{i+1} + r_i$.
(2) For all i, $1 \le i \le n$, the minheight of $I_{r_i}(A_i)$ (or the depth of R on $I_{r_i}(A_i)$, or the height of $I_{r_i}(A_i)$) is at least i.

We say that a complex $\cdots \to M_{i+1} \xrightarrow{d_{i+1}} M_i \xrightarrow{d_i} M_{i-1} \cdots$ has *phantom homology* at M_i if the kernel Z_i of d_i is in the tight closure of the image B of d_{i+1} is M_i. When this condition holds and R is weakly F-regular, the homology at M_i is zero. More generally, when one tensors with a weakly F-regular ring S, the image of the homology over R in the homology of $S \otimes_R M_\bullet$ is 0.

One of the main results of [6] is that $G_\bullet$ is acyclic if and only if it satisfies the standard conditions on rank and depth. An analogous result for tight closure is this

Theorem 7.2 (Phantom acyclicty criterion) *Let R be a Noetherian ring of prime characteristic $p > 0$. Assume that R is reduced, locally equidimensional, and is a homomorphic image of Cohen–Macaulay ring. Let $G_\bullet$ be as in 7.1. Then, $\mathcal{F}^e(G_\bullet)$ is phantom acyclic for all $e \ge 0$ (meaning that all of the homology modules with index at least 1 are phantom) if and only if $G_\bullet$ satisfies the standard conditions on rank and height.*

We refer the reader to [44, Sect. 9] and [48] for a much more detailed discussion. The result above can be extended to many other cases. But if R is not locally equidimensional, one needs to work modulo every minimal prime, and one needs the standard conditions on rank and minheight rather than rank and height. Assuming the existence of sufficiently many completely stable test elements, one can reduce to the case where the base ring is complete local. In particular

Theorem 7.3 (Phantom acyclicty criterion, second version) *Let R be a Noetherian ring of prime characteristic $p > 0$. Assume that R is reduced and is essentially of finite type over an excellent local ring. Let $G_\bullet$ be as in 7.1. Then, $\mathcal{F}^e(G_\bullet)$ is phantom acyclic for all $e \ge 0$ (meaning that all of the homology modules with index at least 1 are phantom) if and only if $G_\bullet$ satisfies the standard conditions on rank and minheight.*

The phantom acyclicity criterion can be used to prove the following result [44]. The argument can be given first in characteristic $p > 0$ and then in equal characteristic zero by reduction in characteristic p.

Theorem 7.4 (Vanishing theorem for maps of Tor) *Let R be module-finite and torsion-free over a regular domain A, and let $R \to S$ be any homomorphism to a regular ring. Let M be any R-module. Then, for all $i \geq 1$, the map $\mathrm{Tor}_i^A(M, R) \to \mathrm{Tor}_i^A(M, S)$ is 0.*

Remark 7.5 This reduces to the case where M is finitely generated, by a direct limit argument. If one has a counterexample, it remains one after one localizes S at maximal ideal and completes. One may also replace A by a suitable localized completion. Take a finite free resolution of M over A. It satisfies the standard conditions on rank and depth. When one tensors with R, the resulting complex $G_\bullet$ will satisfy the standard conditions on rank and height. Hence, it has phantom homology. From this, one deduces that the image of the homology of the complex $R \otimes_A G_\bullet$ in the homology of the complex $S \otimes_A G_\bullet$ is 0, since tight closure is persistent and every module is tightly closed over S.

Remark 7.6 This result is non-trivial even when the map $R \to S$ is simply the map from a local ring $(R, \mathfrak{m}, K)$ to its residue class field! In fact, this special case implies the direct summand theorem (that regular rings are direct summands of their module finite extensions) in characteristic p, and also in mixed characteristic. The direct summan

Remark 7.7 Theorem 7.4 has many generalizations. For example, the argument sketched in Remark 7.5 is valid if we assume that S is weakly F-regular and locally excellent. See also the discussion in [51, Sect. 4] and [39].

7.2 Phantom extensions and splinters

Discussion and Notation 7.8 Let R be a Noetherian ring of prime characteristic $p > 0$. Suppose also that R is reduced. Consider a short exact sequence

$$(\dagger) \quad 0 \to R \xrightarrow{\alpha} M \xrightarrow{\beta} N \to 0$$

of finitely generated R-modules. Choose a projective resolution of N by finitely generated projective R-modules, say

$$\cdots \to G_2 \xrightarrow{d_2} G_1 \xrightarrow{d_1} G_0 \to 0,$$

so that $N = \mathrm{Coker}(d_1)$. Note that, one may use this projective resolution to calculate $\mathrm{Ext}_R^1(N, R)$, which may be calculated as

$$\mathrm{Ker}\big(\mathrm{Hom}_R(G_1, R) \to \mathrm{Hom}_R(G_2, R)\big)/\mathrm{Image}\big(\mathrm{Hom}_R(G_0, R) \to \mathrm{Hom}(G_1, R)\big).$$

Hence, $\mathrm{Ext}_R^1(N, R)$ may be identified with a submodule of

$$\mathrm{Hom}_R(G_1, R)/\mathrm{Image}(\mathrm{Hom}_R(G_0, R) \to \mathrm{Hom}(G_1, R)\big).$$

We use this identification in the statement of (Ph1) below.

The short exact sequence $(\dagger)$ or the map $R \hookrightarrow M$ is said to be a *phantom extension* if the following equivalent conditions hold:

(Ph1) For some (equivalently, every) choice of $G_\bullet$ as above, the element of the module $\mathrm{Ext}^1_R(M/N, R)$ represented by (†) is in the tight closure of of $\mathrm{Image}\big(\mathrm{Hom}_R(G_0, R) \to \mathrm{Hom}(G_1, R)\big)$ in $\mathrm{Hom}_R(G_1, R)$.

(Ph2) There exist $c \in R^\circ$ such that for all $e \gg 0$ there is a map $\gamma_e : \mathcal{F}^e(M) \to \mathcal{F}^e(R)$ such that $\gamma_e \mathcal{F}^e(\alpha) = c\,\mathrm{id}_{\mathcal{F}^e(R)}$.

We refer the reader to [50, Sect. 5] for a detailed treatment that is somewhat more general. The following results should give ample motivation for making this definition.

First, from (Ph1) we obviously have

Proposition 7.9 *Let R be a Noetherian ring of prime characteristic $p > 0$. If R is weakly F-regular and $R \to M$ is a phantom extension, then $R \to M$ splits over R, i.e., R is a direct summand of M.*

Theorem 7.10 *Let R be a Noetherian ring of prime characteristic $p > 0$. If R is reduced and S is any module-finite extension ring, then $R \to S$ is a phantom extension.*

This immediately yields

Theorem 7.11 *Let R be a Noetherian ring of prime characteristic $p > 0$. If R is weakly F-regular, then R is a direct summand of every module-finite extension.*

A domain R that is a direct summand of every module-finite extension S is called a *splinter*. Because one may consider the issue after forming the quotient by a minimal prime of S disjoint from $R \setminus \{0\}$, it is equivalent to make the splitting requirement for module-finite extension domains S.

Every splinter is normal. This notion is more interesting in equal characteristic $p > 0$ (and in mixed characteristic) than in equal characteristic 0: If a ring contains the rationals, it is a direct summand of every module-finite extension if and only if it is normal. We refer the reader to [32,33] for further discussion. We note the following:

Theorem 7.12 *Let R be a normal Noetherian domain R of arbitrary characteristic. The following conditions are equivalent:*

(1) *R is a splinter.*
(2) *Every local ring of R at a maximal ideal is a splinter.*
(3) *Every localization of R at any multiplicative system is a splinter.*
(4) *For every module-finite extension S of R and every ideal I of R, $IS \cap R = I$.*
(5) *For every integral extension S of R and every ideal I of R, $IS \cap R = I$.*

Remark 7.13 In conditions (4) and (5), one also gets an equivalence if one restricts S to be a domain extension of R.

We can restate Theorem 7.11 as follows.

Theorem 7.14 *A weakly F-regular domain is a splinter.*

7.3 Plus closure

The *absolute integral closure* R^+ of a domain in its integral closure in an algebraic closure of its fraction field and is a maximal domain that is an integral extension of R. The *plus* *closure* I^+ of an ideal I of a Noetherian domain R is $IR^+ \cap R$, and it is easy to see that $I^+ = \bigcup_S IS \cap R$ as S runs through all module-finite extension domains of R. Hence, we have already seen in 4.5:

Theorem 7.15 *Let R be a Noetherian ring of prime characteristic $p > 0$. Let $I \subseteq R$ be an ideal. Then, $I^+ \subseteq I^*$.*

A deep result of Karen Smith [84] gives a partial converse. Call an ideal I of a Noetherian domain a *parameter ideal* if $\mathfrak{m}R_\mathfrak{m}$ is generated by part of a system of parameters for every maximal ideal containing I.

Theorem 7.16 (K. E. Smith) *Let R be a Noetherian ring of prime characteristic $p > 0$. Suppose that R is a locally excellent ring, and that I is a parameter ideal. Then, $I^* = I^+$.*

A Noetherian ring of characteristic p in which every parameter ideal is tightly closed is called *F-rational*. This is a characteristic $p > 0$ analog of the notion of rational singularities[1] We restrict further discussion to the case where the ring is locally excellent.

A locally excellent splinter is F-rational by Theorem 7.16. This property passes to all localizations. Because parameter ideals are tightly closed, the colon-capturing property for tight closure implies that F-rational rings are Cohen–Macaulay, and they are also normal. Moreover, in the Gorenstein case, the fact that ideals generated by a system of parameters are tightly closed gives a family of irreducible $\mathfrak{m}$-primary ideals cofinal with the powers of the maximal ideal, and this enables one to show that every ideal is tightly closed. Hence, F-rational is equivalent to F-regular in the Gorenstein case. Putting all this together, we have:

Theorem 7.17 *Let R be a Noetherian ring of prime characteristic $p > 0$. Suppose that R is locally excellent. If R is splinter, then R is F-rational and, in particular, Cohen–Macaulay. If, moreover, R is Gorenstein, then R is F-regular if and only if R is F-rational.*

For a more detailed treatment, we refer the reader to [49,50]

The papers [26,74,85] make a quite explicit connection between the notions of having rational singularity and of being F-rational. Consider a finitely generated algebra R over a field of characteristic K 0. Then, R can be written as $K \otimes_A R_A$, where A is a finitely generated $\mathbb{Z}$-subalgebra of K. The quotients fields A/μ of A by maximal ideals μ are finite fields. One has

Theorem 7.18 (Smith, Hara, Mehta-Srinivas) *With notation as in the paragraph just above, R has rational singularities if and only if for all maximal ideals μ in a dense open subset of* $\mathrm{MaxSpec}(A)$*, the positive characteristic ring $(A/\mu) \otimes_A R_A$ is F-rational.*

The reader may be interested in [72, Sect. 8], where a method for deciding by computer the rationality of some singularities over a field of characteristic 0 is discussed.

[1] An affine algebra over a field of characteristic 0 is said to have *rational singularities* if it is normal and the higher direct images of the structure sheaf of a desingularization are 0. This implies the Cohen–Macaulay property.

7.4 Phantom extensions and big Cohen–Macaulay modules and algebras

The notion of a phantom extension can be used to prove the existence of big Cohen–Macaulay modules and algebras in equal characteristic p. We give here just the barest sketch of the idea, but we also discuss briefly its extension to mixed characteristic.

Big Cohen–Macaulay modules may be constructed by forming a direct limit system of extensions of a ring R. These extensions are the result of a sequence of extensions each of which is called a *modification*. There are different kinds, depending on whether one is constructing a big Cohen–Macaulay module or a big Cohen–Macaulay algebra: There are *module modifications* and there are *partial algebra modifications*. In making one modification, one trivializes a relation on part of system of parameters for R in an essentially universal way. The problem with the construction is to show that when one takes the direct limit B of all these finite sequences of modifications, the element $1 \in R$ is not in $\mathfrak{m}_R B$. One proves this as follows. If it happens, it happens from some specific finite sequence of modifications. To prove that it does not happen with the sequence $R \to M_1 \to M_2 \to \cdots \to M_h$, one proves by induction on h that $R \to M_h$ is a phantom extension. It is then not difficult to prove that $1 \notin \mathfrak{m}_R M_h$. The crucial fact need for the induction is that if $R \to M$ is phantom and one makes a modification $M \to M'$, then $R \to M'$ is phantom. The details for the case of big Cohen–Macaulay modules are given in [35, Sect. 5].

Closure operations that resemble but may be different from tight closure can be used to construct big Cohen–Macaulay modules and algebras. We refer the reader to [16–18,64,80,81] for more detailed discussion of these ideas, especially [18,64,81].

8 Alternative characterizations of tight closure

Under assumptions like the existence of a completely stable test element, tight closure is determined by what happens after base change to the completions of the local rings at a maximal ideal of R, and then to the local rings obtained by taking the quotient by a minimal prime. Under the same hypotheses, it is determined by the tight closures of ideals, or submodules N of a finitely generated module M, such that the quotient M/N is supported at one maximal ideal and has finite length, because $N_M* = \bigcap_I (N + IM)$, where I runs through ideals primary to a maximal ideal. The results in this section therefore give alternative characterizations of tight closure in good cases. For simplicity, we have only stated these results for ideals, and we have assumed the ring is a complete local domain. There are counterparts for modules of all of the stated results. For a detailed discussion, we refer the reader to [44, §8] and [34]. Before stating the characterizations, we give some prerequisites.

8.1 Hilbert–Kunz multiplicity

In order to state part one of our characterizations, we need the notion of the *Hilbert–Kunz multiplicity* of a primary ideal in a local ring $(R, \mathfrak{m}, K)$ of prime characteristic $p > 0$ and Krull dimension d. The notion was introduced by Paul Monsky [24,75].

If I is such a primary ideal, a theorem of Monsky asserts that

$$\ell(R/I^{[q]}) = \gamma q^d + \mathrm{O}(q^{d-1})$$

as $q \to \infty$ for a positive real constant γ. Here, q is running through the values p^e for $e \in \mathbb{N}$. We define the *Hilbert–Kunz* multiplicity $\mathrm{e}_{\mathrm{HK}}(I)$ of I in R to be the constant γ.

Thus, $|\ell(R/I^{[q]}) - \gamma q^d|$ is bounded by positive real constant times q^{d-1} as d varies, and $e_{HK}(I) = \lim_{q\to\infty} \frac{\ell(R/I^{[q]})}{q^d}$.

8.2 Solid algebras

We define a module M over a domain R to be *solid* if $Hom_R(M, R) \neq 0$. If an R-algebra S is called *solid* if it is solid as an R-module. In this case, there is always an R-linear map $\beta : S \to R$ such that $\beta(1) \neq 0$, for if α is a map such that $\alpha(s_0) \neq 0$ we may define $\beta(s) = \alpha(ss_0)$. A module-finite extension of a domain is always solid, but, in general, it may be difficult to decide with a finitely generated extension algebra of R is solid.

For those familiar with local cohomology, we note the following fact from [34]:

Theorem 8.1 *A module M over a complete local domain $(R, \mathfrak{m}, K)$ of Krull dimension d is solid if and only if $H^d_m(M) \neq 0$*

We note that one can deduce easily from this that a big Cohen–Macaulay algebra over a complete local domain is solid.

Although we are generally omitting proofs, the following result has a very short demonstration.

Proposition 8.2 *Let R be a Noetherian domain of prime characteristic $p > 0$, and suppose $I \subseteq R$ and $f \in R$. If S is a solid R-algebra and $f \in IS$, then $f \in I^*$.*

Proof We have $(*)$ $f \cdot 1 = \sum_{t=1}^{h} i_t s_t$ for some h, with $i_t \in I$ and $s_t \in S$. If $\beta : S \to R$ with $\beta(1) = c \neq 0$, raise both sides of $(*)$ to the $q = p^e$ power and apply β to get $(**)$ $f^q\beta(1) = \sum_{t=1}^{h} i_t^q \beta(s_t^q)$. Since the elements $\beta(s_t^q) \in R$, this shows that $cf^q \in I^{[q]}$ for all $q = p^e$, and so $f \in I^*$. □

We can now state the following result characterizing tight closure in important instances.

Theorem 8.3 *Let $(R, \mathfrak{m}, K)$ be a complete local domain of prime characteristic $p > 0$. Let I be an ideal of R and $f \in R$. The following conditions are equivalent:*

(a) $f \in I^*$.
(b) *There exists a big Cohen–Macaulay algebra B over R such that $f \in IB$.*
(c) *There exists a solid R-algebra S such that $f \in IS$.*

Moreover, if I is $\mathfrak{m}$-primary, the following condition is also equivalent:

(d) *The Hilbert–Kunz multiplicities of $I + fR$ and IR are equal.*

We refer the reader [61] and, for example, [9,10] for further characterizations of tight closure. The work of Brenner starts with the local cohomology criterion for solidity and gives beautiful geometric criteria based on it.

9 Strong F-regularity

In this section, we discuss several notions all of which imply weak F-regularity. Since all of the definitions that one might give imply that the ring is normal Noetherian and, hence, a product of finitely many Noetherian domains, we shall, for simplicity, assume throughout that the ring is a domain.

The notion of strong F-regularity was first defined in the F-finite case. The notion has been extended in various ways to the situation where the ring is not necessarily F-finite. For detailed treatments of the F-finite case, we refer the reader to [43,49].

Since the variant definitions of strong F-regularity all imply that R is reduced, and, in fact normal, we may assume that R is a finite product of domains, and then, each variant definition holds if and only if it holds for all factors. Thus, in the sequel, we shall simply assume that R is a domain. When R is F-finite, all the notions agree. When it is not, there are two notions: one we shall call *very strongly F-regularity*, following Hashimoto [28] (the same property is called *F-pure regularity in* [15]), and we shall call the second notion *strong F-regularity*. For a more detailed treatment, see the two papers mentioned and [59, Sect. 2].

Definition 9.1 Let R be a domain of characteristic $p > 0$. If R is F-finite, we define R to be *strongly F-regular* if for every $c \in R^0$ there exists $q = p^e, e \geq 1$ such that the R-linear map $\alpha : R \to R^{1/q}$ with $1 \mapsto c^{1/q}$ splits over R. Let β be the map $R^{1/q} \to R$ sending $c^{1/q} \to 1$. When this happens for a choice $c \in R^{\circ}$, one also gets that R is F-split and that the same condition holds for all $q' = p^{e'}$ with $e' \geq e$.

We note at once:

Theorem 9.2 *If R is strongly F-regular, then every submodule of every module is tightly closed (finite generation is not required).*

Proof In the case of an ideal $I \subseteq R$ $cu^q \in I^{[q]}$ for all $q \gg 0$ implies $c^{1/q}u \in IR^{1/q}$. Applying the splitting β that exists for $q \gg 1$ yields $u \in I$. We may give exactly the same argument when R is replaced by an arbitrary direct sum of copies of R and I by an submodule of that free module. □

Thus, strongly F-regular rings are weakly F-regular, and, hence, normal. The converse has been known in some cases for a considerable time, e.g., when the ring is Gorenstein[2] is a finitely generated graded algebra over a field [68]. Whether the notions are equivalent in the locally excellent case is an open question. We have the following (cf. [28,38,59]):

Theorem 9.3 *Let R be a Noetherian ring of prime characteristic* $p > 0$*. Suppose that R is F-finite. Then, the following conditions are equivalent.*

(1) *R is strongly F-regular.*
(2) $R_{\mathfrak{m}}$ *is strongly F-regular for every maximal ideal* $\mathfrak{m}$*.*
(3) *Every submodule of every module (even if the modules are not finitely generated) is tightly closed.*
(4) *0 is tightly closed in the injective hull of* $R/\mathfrak{m}$ *over R for every maximal ideal* $\mathfrak{m}$*.*

When R is not necessarily F-finite, one cannot expect a splitting for α, as in the definition of strongly F-regular in the F-finite case, but one can hope for a pure map. Therefore, one can extend the notion of strongly F-regular to the general case in two ways: they turn out to be distinct.

We shall say that an arbitrary Noetherian domain of positive characteristic is *strongly F-regular* if every submodule of every module is tightly closed, as in condition (3) of the

[2]More generally, if there are only isolated non-$\mathbb{Q}$-Gorenstein point [70,91].

theorem: We are not assuming finite generation of the modules. This definition follows [38]. We follow here the terminology of [28]: a domain R is *very strongly F-regular* if for every $c \in R^0$ there exists $q = p^e$, $e \geq 1$ such that the R-linear map $\alpha : R \to R^{1/q}$ with $1 \mapsto c^{1/q}$ is pure over R. This condition is called *F-pure regular* in [15]. We refer the reader to [15,28,38,43,49,59] for detailed discussion and proofs of statements given here. With these conditions, we have

Theorem 9.4 *A very strongly F-regular ring is strongly F-regular. Both properties are inherited by arbitrary localizations. If R is local, the notions are equivalent. A Noetherian ring is strongly F-regular if and only if its localization at every maximal (equivalently, every prime) is very strongly F-regular.*

Remark 9.5 The notions of very strongly F-regular and strongly F-regular are different, even if the ring is regular. In fact, every regular ring is strongly F-regular, but not necessarily very strongly F-regular: see, for example, [59, Sect. 6].

Remark 9.6 It is proved in [28] that a strongly F-regular local ring essentially of finite type over an excellent local ring B is very strongly F-regular. In [14, Remark 3.2.2(2)], it is proved that the result holds if B is only assumed to be a local G-ring, while in [59, Theorem 2.23], it is observed that the result holds if B is only assumed to be a semilocal excellent ring.

Yongwei Yao and the author have recently proved that every excellent strongly F-regular ring is very strongly F-regular [60, Sect. 7]

From this point on, we focus on the property of strong F-regularity. Occasionally, for simplicity, we restrict to the F-finite case.

A very useful fact is the following (see, for example, [59, Corollary 2.13].

Theorem 9.7 *Let R, S be Noetherian domains of positive characteristic and $R \to S$ a ring homomorphism that is pure as a map of R-modules. If S is strongly F-regular, then so is R. In particular, this holds when R is a direct summand of S as an R-module (and, hence, if R is an algebra retract of S), and also when $R \to S$ is faithfully flat.*

For affine algebras R over K, this implies that if $L \otimes_K R$ is a strongly F-regular domain for a field extension L of K, then R is strongly F-regular.

9.1 Strong F-regularity and test elements in the F-finite case

There is a strong interaction of the notion of strong F-regularity and the existence of test elements. For simplicity, we assume that all rings in this section are F-finite, although there are also results that hold more general (see, for example, [59]. We give two important theorems. The first is [43, Theorem 3.3].

Theorem 9.8 *Let R be an F-finite Noetherian domain of characteristic $p; > 0$.*

(a) *If $c \in R^\circ$ is such that R_c is strongly F-regular*[3] *then R is strongly F-regular if and only if there exists $q = p^e$ such that the map $R \to R^{1/q}$ with $1 \mapsto c^{1/q}$ splits.*
(b) *The set of primes P in* Spec(R) *such that R_P is strongly F-regular is open.*

[3]Since R is F-finite it is excellent, and one may even choose $c \in R^\circ$ such that R_c is regular. Thus, such choices of c always exist.

Thus, one can check that R is strongly F-regular by establishing the condition in the definition for just one value of c instead of for every nonzero c.

The second is a source of test elements [43, Theorem 3.4] as improved in [38].

Theorem 9.9 *Let R be an F-finite Noetherian domain of characteristic $p; > 0$. Then, every element c of R° such that R_c is strongly F-regular has a power that is a big completely stable test element.*

10 Tools for proving strong F-regularity

10.1 The graded case

A striking result[4] of Gennady Lyubeznik and Karen Smith [68] asserts

Theorem 10.1 *Let R be a finitely generated $\mathbb{N}$-graded algebra over an F-finite field K with $R_0 = K$ of characteristic $p > 0$. Let $\mathfrak{m}$ be the homogeneous maximal ideal of R. Let L be the perfect closure of K. Then, the following conditions are equivalent:*

(1) *$R_{\mathfrak{m}}$ is weakly F-regular.*
(2) *R is weakly F-regular.*
(3) *R is strongly F-regular.*
(4) *$L \otimes_K R$ is strongly F-regular.*

The equivalence of parts (1), (2), and (4) is [68, Corollary 4.4], while it is clear that (3) implies (2), and (4) implies (3) by Theorem 9.7.

Knowing that a standard graded ring R is strongly F-regular has substantial consequences for the geometry of Proj(R), which is termed *globally F-regular* by Karen Smith in [86]. See also [82].

10.2 A criterion for Gorenstein rings

Let $(R, \mathfrak{m})$ be a standard graded K-algebra over a field K. We recall that the $\mathfrak{a}$-invariant $\mathfrak{a}(R)$ is the largest integer a such that $[H^d_{\mathfrak{m}}(R)]_a \neq 0$. We note that the following criterion is essentially [50, Corollary 7.13], combined with Theorem 10.1.

Theorem 10.2 *Let $(R, \mathfrak{m})$ be a standard graded Gorenstein K-algebra over a field K of characteristic $p > 0$. Then, R is strongly F-regular if and only if the following conditions hold:*

(1) *R_P is weakly F-regular for all primes $P \neq \mathfrak{m}$.*
(2) *Some homogeneous system of parameters for R generates an ideal I that is Frobeniusly closed (i.e., if $r \in R$ and $r^{p^e} \in I^{[p^e]}$ then $r \in I$).*
(3) *$\mathfrak{a}(R) < 0$.*

10.3 The Glassbrenner criterion

The following result of Glassbrenner [22] is reminiscent of the Fedder criterion [20] for a quotient of an F-finite regular ring to be F-split.

Theorem 10.3 (Glassbrenner) *Let S be an F-finite regular local ring of prime characteristic p with maximal ideal $\mathfrak{m}$. Let R denote S/I for some proper radical ideal I. Let s be an*

[4] Related results are given in [87].

element of S not in any minimal prime of I such that R_s is regular (or even just strongly F-regular: such elements always exist). The following are equivalent.

(1) *R is strongly F-regular.*
(2) *For each element c of S not in any minimal prime of I, $c(I^{[p^e]} : I) \not\subseteq m^{[p^e]}$ for all sufficiently large positive integers e.*
(3) *I is prime and $I = \bigcap_{e\geq 1}(m^{[p^e]} : (I^{[p^e]} : I))$.*
(4) *There exists a positive integer e such that $s(I^{[p^e]} : I) \not\subseteq m^{[p^e]}$.*

10.4 Results of Hashimoto on rings of invariants

The following contains quite useful results of Hashimoto from [27,29]. The result from [27] can be deduced from the result of [29].

Theorem 10.4 (Hashimoto) *Let G be a reductive linear algebraic group over an algebraically closed field K of characteristic $p > 0$. Let V be a finite-dimensional G-module, let P be a parabolic subgroup of G, and U_P the unipotent radical of P. Let S by the polynomial ring over K which is the symmetric algebra of V. Assume that S is good*[5] *as a G-module. Then, the ring of invariants S^{U_P} is a finitely generated strongly F-regular Gorenstein UFD. Moreover, S^G is a direct summand of S^{U_P} and so is strongly F-regular.*

10.5 Cluster algebras

There has been a recent explosion of interest in cluster algebras. it would take us far afield to survey this area or even to give basic definitions. However, we do want to mention the following result from [7], to which we refer the reader for definitions.

Theorem 10.5 *A locally acyclic cluster algebra over a field of positive characteristic is strongly F-regular.*

11 F-signature of local rings

We give a very brief introduction to the theory of F-signature, a characteristic $p > 0$ invariant that can be used to characterize strongly F-regular rings. The notion was introduced in [63] in the F-finite case, and a definition valid in general was given in [92]. After a series of partial results, it was proved that the defining limit involved exists in general in [89]. Many authors have made contributions, and the reader is referred to [1,2,4,63,78,79,89,90,92] for further background.

Definition 11.1 Let $(R, \mathfrak{m}, K)$ be an F-finite local ring with $d = \dim R$ and M a finitely generated R-module. For each $e \in \mathbb{N}$, write ${}^eM \cong R^{a_e} \oplus M_e$ as R-modules, where M_e has no nonzero free direct summand. We let $\#({}^eM, R) := a_e$ and let $\alpha(R)$ be such that $[K : K^p] = p^{\alpha(R)}$: this degree is finite when R and, hence, K, is F-finite. We then define

$$\mathrm{s}(M) := \lim_{e\to\infty} \frac{\#({}^eM, R)}{q^{\alpha(R)+d}}.$$

The limit always exists by [89]. We define $\mathrm{s}(R)$ to be the *F-signature* of R.

As mentioned above, in [92], a definition of F-signature is given for all Noetherian local rings of prime characteristic p and it is equivalent to Definition 11.1 whenever R is F-

[5] For the somewhat technical explanation of what it means for a representation to be *good*, we refer the reader to [29].

finite. However, we shall not give details here. But we note that Theorem 11.2 is valid for all excellent local rings of positive characteristic.

Thus, in the F-finite case, the F-signature gives a normalized measure of the size of the largest R-free direct summand of ${}^{e}R$ when e is large. Like Hilbert–Kunz multiplicity, it is very hard to calculate. It is at most 1 and is 1 if and only if the local ring R is regular.

Theorem 11.2 ([4,63]) *If R is an excellent local ring, then R is strongly F-regular if and only if* $\mathrm{s}(R) > 0$.

Thus, strongly, F-regular local rings are characterized by the fact that ${}^{e}R$ splits off a relatively large direct summand in the F-finite case.

We also note that [79] proves that

$$\mathrm{s}(R) = \inf\{\mathrm{e_{HK}}(I_1) - \mathrm{e_{HK}}(I_2) \mid I_1 \subset I_2, \sqrt{I_1} = \mathfrak{m}, I_2/I_1 \cong k\}$$

(this was conjectured in [90]) for approximately Gorenstein rings (these include all normal rings [33]), and, in particular, all weakly (or strongly) F-regular rings. This was a starting point for the results of [58].

12 Another splitting characterization of strongly F-regular rings

This section gives another characterization of strongly F-regular rings: [59, Corollary 4.3]. For simplicity, we have limited the statement to the F-finite case and omitted some other equivalences.

Theorem 12.1 (Hochster-Yao) *Let R be an F-finite ring. The following statements are equivalent.*

(1) *R is strongly F-regular.*
(2) *For every finitely generated R-module M supported on all of* $\mathrm{Spec}(R)$, *${}^{e}M$ has R as a direct summand for all $e \gg 0$.*
(3) *For every finitely generated faithful R-module M, there exists $e > 0$ such that ${}^{e}M$ has R as aa direct summand.*
(4) *R is normal, and for every $P \in \mathrm{Spec}(R)$ such that $\dim(R_P) \geq 2$, there exists $e > 0$ such that ${}^{e}P_P$ has R_P as a direct summand.*

Received: 2 February 2022 Accepted: 2 August 2022 Published online: 22 August 2022

References

1. Aberbach, I.M.: The existence of the F-signature for rings with large Q-Gorenstein locus. J. Algebra **319**(7), 2994–3005 (2008)
2. Aberbach, I.M., Enescu, F.: When does the F-signature exist. Ann. Fac. Sci. Toulouse Math. **6**(15), 195–201 (2006)
3. Aberbach, I.M., Hochster, M., Huneke, C.: Localization of tight closure and modules of finite phantom projective dimension. J. Reine Angew. Math. **434**, 67–114 (1993)
4. Aberbach, I.M., Leuschke, G.J.: The F-signature and strong F-regularity. Math. Res. Lett. **10**, 51–56 (2003)
5. Băeţică, C.: F-rationality of algebras defined by Pfaffians: Memorial issue dedicated to Nicolae Radu. Math. Rep. **3**(53), 139–144 (2001)
6. Buchsbaum, D., Eisenbud, D.: What makes a complex exact. J. Algebra **25**, 259–268 (1973)
7. Benito, A., Muller, G., Rajchgot, J., Smith, K.E.: Singularities of locally acyclic cluster algebras. Algebra Number Theory **9**, 913–936 (2015)
8. Boutot, J.-F.: Singularités rationelles et quotients par les groupes réductifs. Invent. Math. **88**, 65–68 (1987)
9. Brenner, H.: Tight closure and projective bundles. J. Algebra **265**, 45–78 (2003)
10. Brenner, H.: Slopes of vector bundles on projective curves and applications to tight closure problems. Trans. Amer. Math. Soc. **356**, 371–392 (2004)
11. Brenner, H., Monsky, P.: Tight closure does not commute with localization. Annals Math. **171**, 571–588 (2010)

12. Briançon, J., Skoda, H.: Sur la clôture intégrale d'un ideal de germes de fonctions holomorphes en un point de C^n. C. R. Acad. Sci. Paris Sér. A **278**, 949–951 (1974)
13. Bruns, W., Herzog, J.: *Cohen-Macaulay Rings*, Cambridge studies in advanced mathematics **39**, 1993
14. Datta, R., Murayama, T.: with an Appendix by K. E. Smith, *Excellence, F-singularities, and solidity*, arXiv:2007.10383 [math.AC], July 20, (2020)
15. Datta, R., Smith, K. E.: *Frobenius and valuation rings*, Algebra & Number Theory **10** (2016), 1057–1090 [Correction to the article: Algebra & Number Theory **11**, 1003–1007 (2017)
16. Dietz, G.D.: Big Cohen-Macaulay algebras and seeds. Trans. Amer. Math. Soc. **359**, 5959–5990 (2007)
17. Dietz, G.D.: A characterization of closure operators that induce big Cohen-Macaulay modules. Proc. Amer. Math. Soc. **138**, 3849–3862 (2010)
18. Dietz, G.D.: Axiomatic closure operations, phantom extensions, and solidity. J. Algebra **502**, 123–145 (2018)
19. Ein, L., Lazarsfeld, R., Smith, K.E.: Uniform bounds and symbolic powers on smooth varieties Invent. Math **144**, 241–252 (2001)
20. Fedder, R.: F-purity and rational singularity. Trans. Amer. Math. Soc. **278**, 461–480 (1983)
21. Fedder, R., Watanabe, K.-i.: A characterization of F-regularity in terms of F-purity, in *Commutative Algebra*, Math. Sci. Research Inst. Publ. **15** Springer-Verlag, New York-Berlin-Heidelberg 227–245 (1989)
22. Glassbrenner, D.J.: Strong F-regularity in images of regular rings. Proc. Amer. Math. Soc. **124**, 345–354 (1996)
23. Grauert, H., Riemenschneider, O.: Verschwindungsätze für analytische kohomologiegruppen auf komplexen Räuman. Invent. Math. **11**, 263–290 (1970)
24. Han, C., Monsky, P.: Some surprising Hilbert-Kunz functions. Math. Z. **214**, 119–135 (1993)
25. Hara, N.: A characterization of rational singularities in terms of injectivity of Frobenius maps, Research into the homological characteristics of graded commutative rings (Japanese) (Kyoto, 1996). Sūrikaisekikenkū Kōkyūroku No. **964**, 138–144 (1996)
26. Hara, N.: A characterization of rational singularities in terms of injectivity of Frobenius maps. Amer. J. Math. **120**, 981–996 (1998)
27. Hashimoto, M.: Good filtrations of symmetric algebras and strong F -regularity of invariant subrings. Math. Z. **236**, 605–623 (2001)
28. Hashimoto, M.: F-pure homomorphisms, strong F-regularity, and F-injectivity. Comm. Algebra **38**, 4569–4596 (2010)
29. Hashimoto, M.: Good filtrations and strong F-regularity of the ring of U_P-invariants. J. Algebra **370**, 198–220 (2012)
30. Heitmann, R., Ma, L.: Big Cohen-Macaulay algebras and the vanishing conjecture for maps of Tor in mixed characteristic. Algebra Number Theory **12**, 1659–1674 (2018)
31. Herzog, Jürgen.: Ringe der Charakteristik p und Frobeniusfunktoren. Math. Z. **140**, 67–78 (1974)
32. Hochster, M.: Contracted ideals from integral extensions of regular rings. Nagoya Math. J. **51**, 25–43 (1973)
33. Hochster, M.: Cyclic purity versus purity in excellent Noetherian rings. Trans. Amer. Math. Soc. **231**, 463–488 (1977)
34. Hochster, M.: *Solid closure*, in *Commutative Algebra: Syzygies, Multiplicities and Birational Algebra*, Contemp. Math. **159**, Amer. Math. Soc., Providence, R. I., 103–172 (1994)
35. Hochster, M.: *Tight closure in equal characteristic, big Cohen-Macaulay algebras, and solid closure* in *Commutative Algebra: Syzygies, Multiplicities and Birational Algebra*, Contemp. Math. **159** Amer Math Soc., Providence, R. I., 173–196 (1994)
36. Hochster, M.: Book Review of rings, by Winfried Bruns and Jürgen Herzog. Bull. M. S. **32**, 265–275 (1995)
37. Hochster, M.: *The notion of tight closure in equal characteristic zero* in *Proc. of the CBMS Conference on Tight Closure and Its Applications* (Fargo, North Dakota, July, 1995), Appendix to the notes on the main lectures by Craig Huneke, C.B.M.S. Regional Conference Series, Amer. Math. Soc., Providence, R.I., (1996)
38. Hochster, M.: *Foundations of Tight Closure Theory*, 276 pp., lecture notes available at http://www.math.lsa.umich.edu/~hochster/711F07/711.html
39. Hochster, M.: *Homological conjectures and lim Cohen-Macaulay sequences*, in Homological and Computational Methods in Commutative Algebra, Springer INdAM Series 20, Springer, 181–197 (2017)
40. Hochster, M., Eagon, J.A.: Cohen-Macaulay rings, invariant theory, and the generic perfection of determinantal loci. Amer. J. Math. **93**, 1020–1058 (1971)
41. Hochster, M., Huneke, C.: Tightly closed ideals. Bull. Amer. Math. Soc. **18**, 45–48 (1988)
42. Hochster, M., Huneke, C.: *Tight closure* in *Commutative Algebra*, Math. Sci. Research Inst. Publ. **15** Springer, New York-Berlin-Heidelberg, 305–324 (1989)
43. Hochster, M., Huneke, C.: Tight closure and strong F-regularity. Mémoires de la Société Mathématique de France, numéro **38**, 119–133 (1989)
44. Hochster, M., Huneke, C.: Tight closure, invariant theory, and the Briançon-Skoda theorem. J. Amer. Math Soc. **3**, 31–116 (1990)
45. Hochster, M., Huneke, C.: Absolute integral closures are big Cohen-Macaulay algebras in characteristic p. Bull. Amer. Math. Soc. (New Series) **24**, 137–143 (1991)
46. Hochster, M., Huneke, C.: Tight closure and elements of small order in integral extensions. J. of Pure and Appl. Algebra **71**, 233–247 (1991)
47. Hochster, M., Huneke, C.: Infinite integral extensions and big Cohen-Macaulay algebras. Annals of Math. **135**, 53–89 (1992)
48. Hochster, M., Huneke, C.: Phantom homology. Memoirs Amer. Math. Soc. **103**(490), 1–91 (1993)
49. Hochster, M., Huneke, C.: F-regularity, test elements, and smooth base change. Trans. Amer. Math. Soc. **346**, 1–62 (1994)
50. Hochster, M., Huneke, C.: Tight closure of parameter ideals and splitting in module-finite extensions. J. Algebr. Geom. **3**, 599–670 (1994)
51. Hochster, M., Huneke, C.: Applications of the existence of big Cohen-Macaulay algebras. Adv. Math. **113**, 45–117 (1995)
52. Hochster, M., Huneke, C.: Tight closure in equal characteristic 0, preprint. Partially revised in 2019 and 2021
53. Hochster, M., Huneke, C.: Test exponents and localization of tight closure. Michigan Math. J. **48**, 305–329 (2000)

54. Hochster, M., Huneke, C.: Comparison of symbolic and ordinary powers of ideals. Invent. Math. **147**, 349–369 (2002)
55. Hochster, M., Huneke, C.: Fine behavior of symbolic powers of ideals llinois. J. Math. **51**, 171–183 (2007)
56. Hochster, M., Roberts, J.L.: Rings of invariants of reductive groups acting on regular rings are Cohen-Macaulay. Adv. Math. **13**, 115–175 (1974)
57. Hochster, M., Roberts, J.L.: The purity of the Frobenius and local cohomology. Adv. Math. **20**, 117–172 (1976)
58. Hochster, M., Yao, Y.: *F-rational signature and drops in the Hilbert-Kunz multiplicity*, Algebra & Number Theory, to appear
59. Hochster, M., Yao, Y.: Splitting of ${}^{e}M$, strong F-regularity, and the existence of small Cohen-Macaulay modules, preprint, (2022)
60. Hochster, M., Yao, Y.: *Generic local duality and purity exponents*, preprint, (2022)
61. Hochster, M., Zhang, W.: Content of local cohomology, parameter ideals, and robust algebras. Trans. Amer. Math. Soc. **370**, 7789–7814 (2018)
62. Huneke, C.: *Tight Closure and Its Applications*, Proc. of the C.B.M.S. Conference held at Fargo, North Dakota, July, 1995, C.B.M.S. Regional Conference Series, Amer. Math. Soc., Providence, R.I., (1996)
63. Huneke, C., Leuschke, G.J.: Two theorems about maximal Cohen-Macaulay modules. Math. Ann. **324**, 391–404 (2002)
64. Jiang, Z.: *Test elements in equal characteristic semianalytic algebras*, preprint, arXiv:2104.12867v1 [math.AC]
65. Kunz, E.: Characterizations of regular local rings of characteristic p. Amer. J. Math. **91**, 772–784 (1969)
66. Kunz, E.: On Noetherian rings of characteristic p. Amer. J. Math. **98**, 999–1013 (1976)
67. Lipman, J., Sathaye, A.: Jacobian ideals and a theorem of Briançdon-Skoda. Michigan Math. J. **28**, 199–222 (1981)
68. Lyubeznick, G., Smith, K.E.: Strong and weak F-regularity are equivalent for graded rings. Amer. J. Math. **121**, 1279–1290 (1999)
69. Lyubeznick, G., Smith, K.E.: On the commutation of the test ideal with localization and completion. Trans. Amer. Math. Soc. **353**, 3149–3180 (2001)
70. MacCrimmon, B.: Strong F-regularity and Boundedness Questions in Tight Closure. University of Michigan, Thesis (1996)
71. Ma, L., Schwede, K.: Perfectoid multiplier/test ideals in regular rings and bounds on symbolic powers Invent. Math. **214**, 913–955 (2018)
72. Ma, L., Schwede, K.: Singularities in mixed characteristic via perfectoid big Cohen-Macaulay algebras. Duke Math. J. **170**, 2815–2890 (2021)
73. Matsumura, H.: *Commutative algebra*, Benjamin, (1970)
74. Mehta, V.B., Srinivas, V.: A characterization of rational singularities. Asian. J. Math. **1**, 249–278 (1997)
75. Monsky, P.: The Hilbert-Kunz function. Math. Ann. **263**, 43–49 (1983)
76. Peskine, C., Szpiro, L.: Dimension projective finie et cohomologie locale. Publ. Math. I.H.E.S. **42**, 323–395 (1973)
77. Peskine, C., Szpiro, L.: Syzygies et multiplicités, C. R. Acad. Sci. Paris Sér. A **278**, 1421–1424 (1974)
78. Polstra, T.: Uniform bounds in F-finite rings and lower semi-continuity of the F-signature. Trans. Amer. Math. Soc. **370**, 3147–3169 (2018)
79. Polstra, T., Tucker, K.: F-signature and Hilbert-Kunz multiplicity: a combined approach and comparison. Algebra Number Theory **12**, 61–97 (2018)
80. Rebecca, R.G.: Closure operations that induce big Cohen-Macaulay modules and classification of singularities. J. Algebra **467**, 237–267 (2016)
81. Rebecca, R.G.: Closure operations that induce big Cohen-Macaulay algebras. J. Pure Appl. Algebra **222**, 1878–1897 (2018)
82. Schwede, K., Smith, K.E.: Globally F-regular and log Fano varieties Adv. Math. **224**, 863–894 (2010)
83. Skoda, H.: Applications des techniques L^2 a la théorie des idéaux d'une algèbre de fonctions holomorphes avec poids. Ann. Sci. École Norm. Sup. **5**, 545–579 (1972)
84. Smith, K.E.: Tight closure of parameter ideals. Inventiones Math. **115**, 41–60 (1994)
85. Smith, K.E.: F-rational rings have rational singularities. Amer. J. Math. **119**, 159–180 (1997)
86. Smith, K.E.: Globally F-regular varieties: applications to vanishing theorems for quotients of Fano varieties. Michigan Math. J. **48**, 553–572 (2000)
87. Stubbs, J.F.: Potent Elements and Tight Closure in Artinian Modules. University of Michigan, Thesis (2008)
88. Swanson, I., Huneke, C.: *Integral closure of ideals, rings and modules*, London Math. Soc. Lecture Note Ser. **336**, Cambridge University Press, Cambridge, England, (2006)
89. Tucker, K.: F-signature exists. Invent. Math. **190**, 743–765 (2012)
90. Watanabe, K.-I., Yoshida, K.: Minimal relative Hilbert-Kunz multiplicity. Illinois J. Math. **48**, 273–294 (2004)
91. Williams, L.: Uniform stability of kernels of Koszul cohomology indexed by the Frobenius endomorphism. J. Algebra **172**, 721–743 (1995)
92. Yao, Y.: Observations on the F-signature of local rings of characteristic p. J. Algebra **299**, 198–218 (2006)

Publisher's Note