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3 On the Reconstruction of Geodesic Subspaces of $\mathbb{R}^N$

4 Brittany Terese Fasy

5 *School of Computing & Dept. Mathematical Sciences, Montana State University*
6 *Bozeman, MT 59717 USA*
7 *brittany.fasy@montana.edu*

8 Rafal Komendarczyk

9 *Mathematics, Tulane University*
10 *New Orleans, LA 70118 USA*
11 *rako@tulane.edu*

12 Sushovan Majhi*

13 *School of Information, University of California*
14 *Berkeley, CA 94720 USA*
15 *smajhi@berkeley.edu*

16 Carola Wenk

17 *Computer Science, Tulane University*
18 *New Orleans, LA 70118 USA*
19 *cwenk@tulane.edu*

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23 ABSTRACT

24 We consider the topological and geometric reconstruction of a geodesic subspace of $\mathbb{R}^N$ both from
25 the Čech and Vietoris-Rips filtrations on a finite, Hausdorff-close, Euclidean sample. Our recon-
26 struction technique leverages the intrinsic length metric induced by the geodesics on the subspace.
27 We consider the distortion and convexity radius as our sampling parameters for the reconstruc-
28 tion problem. For a geodesic subspace with finite distortion and positive convexity radius, we
29 guarantee a correct computation of its homotopy and homology groups from the sample. This
30 technique provides alternative sampling conditions to the existing and commonly used conditions
31 based on weak feature size and μ -reach, and performs better under certain types of perturbations
32 of the geodesic subspace. For geodesic subspaces of $\mathbb{R}^2$, we also devise an algorithm to output a
33 homotopy equivalent geometric complex that has a very small Hausdorff distance to the unknown
34 underlying space.

35 *Keywords:* Vietoris-Rips complex, Geodesic spaces, Shape reconstruction, Map construction

*Corresponding author

1. Introduction

With the advent of modern sampling technologies, such as GPS, sensors, medical imaging, etc., Euclidean point-clouds are becoming widely available for analysis. In the last decade, the problem of reconstructing an (unknown) Euclidean shape, from a (noisy) sample around it, has received a far and wide attention both in theoretical and applied literature; see [1, 2, 3, 4, 5, 6]. The nature of such a reconstruction attempt can commonly be classified as being *topological* or *geometric*. A topological reconstruction is usually attributed to inferring significant topological features—such as homology and homotopy groups—of the hidden shape of interest. To be more specific, one may also say *homological* reconstruction or *homotopy type* reconstruction. A much stronger paradigm is the *geometric* reconstruction, where one is interested in producing, from the sample, a Euclidean subset that is homotopy equivalent and geometrically “close” (e.g., in Hausdorff distance) to the underlying shape.

The nature of the problem and the techniques of the solution change depending on the type of the shape X and the sample S considered, as well as how their “closeness” is measured. The most natural distance measure between two abstract metric spaces is the Gromov-Hausdorff distance, which measures how “metrically close” two metric spaces are. The reconstruction of a geodesic metric space X from another metric space S that is Gromov-Hausdorff close to X is considered in [7, 8]. For a Euclidean shape X and a Euclidean sample S , however, the sample density is usually quantified by their Hausdorff distance. For the Hausdorff-type reconstruction of Euclidean shapes, see [3, 5, 6, 4].

In many applications, a point cloud approximates a geodesic subspace (see Definition 2.1) of Euclidean space. Examples include GPS trajectories sampled around a road-network (modeled as sampling paths in a graph in $\mathbb{R}^2$), earthquake data sampled around the filamentary trajectory of the shock, or 3D medical imaging. The intrinsic geodesics of these underlying shapes enjoy a rich geometric structure. Capturing that structure from the sampled data is the challenge. The length metric d_L (see (1)) turns them into geodesic subspaces of $\mathbb{R}^N$. In this work, we consider both topological and geometric reconstruction of a geodesic subspace X of $\mathbb{R}^N$ from a finite Hausdorff-close Euclidean sample.

In shape reconstruction, the use of various simplicial complexes built on the point-clouds is becoming increasingly popular; see for example [9, 10, 11, 5, 12]. The most common of them are Vietoris-Rips and Čech complexes. In this work, we use filtrations of both of them, and we recognize the distortion $\delta = \delta(X)$ and convexity radius $\rho = \rho(X)$ of X to be natural sampling parameters when the geodesic subspaces of $\mathbb{R}^N$ are considered; see Section 2 for their formal definitions.

Our homological reconstruction approach is similar to [5], which is based on the weak feature size (**wfs**) of the underlying space. However, the use of partition of unity, for example, in the proof of Theorem 3.10 makes our techniques substantially different. The novelty of this paper is discerned by the introduction of distortion and convexity radius as sampling parameters, which is not related to the known sampling parameters such as the reach, μ -reach or **wfs** [6, 4, 10]. These works are based on an analysis of the gradient flow of the Euclidean distance function to X in $\mathbb{R}^N$ and its critical points. Our techniques are substantially different from that and our results apply to a large class of spaces including smooth

submanifolds of $\mathbb{R}^N$, finite embedded graphs and higher dimensional simplicial complexes. As an application of our reconstruction technique, we develop in Section 4 a new topological approach for the reconstruction of embedded graphs.

1.1. Review of Related Works

This subsection surveys relevant and pivotal results in shape reconstruction from point clouds using topological methods, and compares them to the results of this paper. Table 1 presents a list of some of the most related results alongside the contribution presented in this work. For necessary definitions and background we refer the reader to Section 2.

Reach. The most well-behaved spaces are smooth Euclidean submanifolds, more generally spaces with a positive *reach* $r(X)$. In [3], the authors apply geometric and topological tools to reconstruct a smooth submanifold by the union of Euclidean balls of sufficiently small radius around a dense subset. The work uses the reach of the embedded submanifold as the sampling parameter. In a more recent work ([15]), the authors improve some of the previously known bounds and develop homotopy-type reconstruction of a Euclidean (compact) subset with positive reach (and μ -reach) using Čech and Vietoris-Rips complexes on a sample.

The above results do not apply when considering shapes beyond the class of Euclidean submanifolds or spaces that do not have a positive reach, although such shapes are frequently encountered in practical applications. A common reason for a space to have a vanishing reach is the presence of sharp corners and branchings. Such spaces include graphs, embedded simplicial complexes, manifolds with corners—also the type of shapes we consider in this work for reconstruction. For manifold reconstruction by Vietoris-Rips complexes in a slightly different but related context, see [11, 16].

Weak Feature Size, μ -Reach. In developing a sampling theory for general compact sets in $\mathbb{R}^N$, the notion of *weak feature size* (**wfs**) was introduced in [4] as the infimum of the positive critical values of the distance function to the compact set. Using the **wfs** as a sampling condition, the authors developed a *persistence-based* approach to reconstruct the homology groups and the fundamental group of a hidden shape from the Euclidean thickenings of the sample around it.

The results have been further extended in [5] to facilitate reconstruction of homology groups from Čech, Vietoris-Rips, and witness complexes built on the sample. In comparison with the manifold reconstruction result in [3], the techniques of [4, 5] apply to much less regular subspaces of $\mathbb{R}^N$, such as compact Euclidean neighborhood retracts [17, 18]—as long as they have a positive **wfs**.

The notion of the **wfs** of a Euclidean compact set was generalized in [6] by introducing the concept of μ -reach, denoted $r_\mu(X)$. A homotopy-type reconstruction of spaces with positive μ -reach has been developed in [6, 10]. Although these works consider for reconstruction spaces beyond the class of positive **wfs**, the difficulty lies in applying the results to shapes as simple as an embedded tree. Also, choosing a suitable μ so that the μ -reach is positive is not always clear.

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Table 1: Reconstruction results. Parameters (params.) are: weak feature size ($\mathbf{wfs}$), μ -reach (R), shorted edge length (b), global reach (ξ), smallest turning angle (α), distortion (δ), and convexity radius (ρ).

Authors	Space X	Param.	Condition on S	Result
Niyogi et al. [3]	manifolds	ξ	$\varepsilon < \sqrt{\frac{3}{5}}\xi$ and $S \subset X$ is $\frac{\varepsilon}{2}$ -dense	S^ε deformation retracts to X
Chazal, Lieutier [4]	compact sets	$\mathbf{wfs}$	$d_H(X, S) < \varepsilon < \frac{\mathbf{wfs}(X)}{4}$	$Im(i_*) \simeq H_*(X^\alpha)$, where $i : S^\varepsilon \rightarrow S^{3\varepsilon}$ and α is sufficiently small
Chazal, Oudot [5]	compact sets	$\mathbf{wfs}$	$d_H(X, S) < \varepsilon < \frac{1}{9}\mathbf{wfs}(X)$, S is finite	$Im(i_*) \simeq H_*(X^\alpha)$, where $i : \mathcal{R}_\varepsilon(S) \rightarrow \mathcal{R}_{4\varepsilon}(S)$, α is sufficiently small
Attali et al. [10]	compact sets	μ -reach R	$d_H(X, S) \leq \varepsilon < \lambda^{\text{cech}}(\mu)R$	$\mathcal{C}_\alpha(S)$ is homotopy equivalent to X^η for $\eta \in (0, R)$
Anjaneya et al. [13]	abstract metric graphs	b, r	S is an (ε, R) -approximation, $\frac{15\varepsilon}{2} < b < \min\left\{\frac{R}{4}, \frac{3b-6\varepsilon}{5}\right\}$	homeomorphic graph
Wasserman et al. [14]	embedded metric graphs	μ of each edge, ξ, α, b, τ	S is $\frac{\delta}{2}$ -dense in X^α , $0 < r + \delta < \xi - 2\sigma$, and $0 < \delta < f(b, \alpha, \tau, \xi, \sigma)$	isomorphic pseudo-graph
Theorem 3.5	geodesic spaces	δ, ρ	$d_H(X, S) < \frac{\varepsilon}{4} < \frac{\rho}{2\delta(3\delta+2)}$	$Im(i_*) \simeq H_*(X)$, where $i : \mathcal{R}_\varepsilon(S) \rightarrow \mathcal{R}_{\frac{1}{2}(3\delta+1)\varepsilon}(S)$
Theorem 4.7	planar subspaces	δ, ρ	$d_H(X, S) < \frac{\varepsilon}{3} < \frac{\rho}{\delta(15\delta+2)}$	Hausdorff-close, homotopy equivalent subset

Our topological reconstruction results (Theorem 3.5 and Theorem 3.10), are very similar in style to the results presented in [5]. However, the use of partition of unity for Čech complexes and homotopy equivalence result of Hausmann ([19]) for Vietoris-Rips complexes make our proofs very different. The $\mathbf{wfs}$ -based technique employed in [4, 5] restricts their results to work for homology with coefficients only in a field. Moreover, it's not apparently

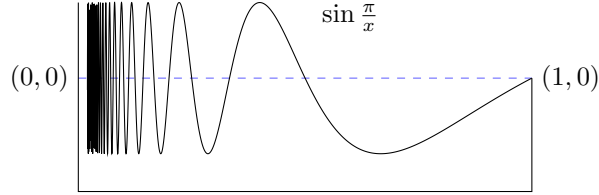


Fig. 1: The compact set X (*Warsaw circle*) has a positive $\mathbf{wfs}$, but X and X^λ do not have the same homotopy type for any $\lambda > 0$. In fact, X has the weak homotopy type of a point, whereas X^λ has the homotopy type of S^1 .

clear whether the results can easily be extended to higher homotopy groups. Our reconstruction results, however, do not suffer such restrictions; see Remark 3.11.

Apart from the fact that we employ $\delta(X)$ and $\rho(X)$ for our sampling condition, all $\mathbf{wfs}$ (and μ -reach) based results guarantee a reconstruction of a thickening X^λ of X and not X directly. There are known pathological examples of spaces where the thickening (however small) is not homotopy equivalent to the underlying space, such as the *Warsaw circle* shown in Figure 1. Although the homological reconstruction results in our work concern the homological reconstruction of the subspace X itself, not the thickening of X , they are not strong enough to apply in the case of the Warsaw circle because of $\delta(X) = +\infty$ in this case.

Another notable difference in the previously discussed approaches appears in the cases where X is “slightly perturbed”, e.g., a submanifold with corners. Such a perturbation is illustrated in Figure 2 for a circle X topologically embedded^a in $\mathbb{R}^2$. The top part of the space X is the graph of a rectifiable curve $\gamma : [0, 1] \rightarrow \mathbb{R}^2$ such that, when restricted to the segment $\left[\frac{1}{n+1}, \frac{1}{n}\right]$, it is a half-circle of diameter $\frac{1}{n(n+1)}$ for n odd and a line-segment for n even. For this space, the set of critical points of the distance function is an infinite set with an accumulation point at $(0, 0)$. Consequently, $\mathbf{wfs}(X) = 0$. However, X has a finite distortion $\delta = \frac{\pi}{2}$ and a positive intrinsic convexity radius: $\rho(X) > 0$. Thus X fails to satisfy the conditions of the reconstruction results of [4, 5], however our results apply to this case. Another important point, suggested by the example of Figure 2, is that any embedded submanifold X in $\mathbb{R}^N$ can be perturbed to a submanifold X' , just by adding a small “spherical cap” at any of its points. Such a small perturbation does not change the distortion and the convexity radius too much, however can produce very small $\mathbf{wfs}$, because we introduce a critical point of the distance function at the center of the cap. Small values of $\mathbf{wfs}$ result in large sample sizes needed for the reconstruction.

Metric Graph Reconstruction. We finish this introduction with a quick summary of some of the existing works on reconstruction of embedded metric graphs ([13, 20, 8]). In [13], the authors consider an abstract metric graph and a sample that is close to it in Gromov-Hausdorff metric, and reconstruct the structure of the metric graph along with the metric

^aa topological embedding is simply a C^0 -embedding.

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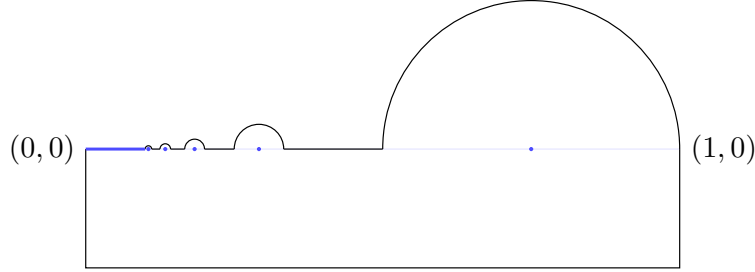


Fig. 2: The space X is a compact Euclidean subspace with $\mathbf{wfs}(X) = 0$ and $r_\mu(X) = 0$. The critical points of the distance function are shown in blue; they accumulate at $(0,0)$. However, X has a finite distortion and a positive convexity radius.

on it. In a more recent work [20], the authors show a statistical treatment of metric graph reconstruction. They consider an embedded metric graph and a Euclidean sample around it. The Gromov-Hausdorff proximity used in [13] is replaced by the density assumption. The algorithm presented in [13] only reconstructs the connectivity of the vertices of the underlying metric graph and outputs an isomorphic pseudo-graph. And lastly, we mention that the first Betti number of an abstract metric graph is computed by considering the persistent cycles in the Vietoris-Rips complexes of a sample that is very close to it, with respect to the Gromov-Hausdorff distance; see [8, Lemma 6.1]. In Gromov-Hausdorff type reconstruction schemes, a small Gromov-Hausdorff distance between the graph and the sample guarantees a successful reconstruction. These methods are not a good choice when embedded graphs in $\mathbb{R}^N$ are considered. For an embedded graph with the induced length metric and a Euclidean sample around it, the Gromov-Hausdorff distance is not guaranteed to be made infinitely small, even if a dense enough sample is taken. Also, most of the above mentioned works may be insufficient to give a geometrically close embedding for the reconstruction. Whereas our technique, presented in Section 4, can successfully be used to reconstruct embedded graphs; see Corollary 4.8.

1.2. Our Contribution

One of the major contributions of this work is to reconstruct geodesic subspaces of $\mathbb{R}^N$, both topologically and geometrically. In our pursuit, we recognize distortion and convexity radius as new sampling parameters. These sampling parameters are very natural properties of geodesic spaces.

In Section 2, along with the other important notions of metric geometry and algebraic topology that we use throughout this paper, we define convexity radius and distortion of a geodesic space.

In Section 3, our main topological reconstruction results for a geodesic subspace X of $\mathbb{R}^N$ are presented. When the distortion is finite and the convexity radius is positive, the Vietoris-Rips and Čech filtrations of the sample are shown to successfully compute the homology and homotopy groups of X (Theorem 3.5 and Theorem 3.10).

In Section 4, we consider geometric reconstruction of geodesic subspaces. We construct a complex on the sample as our geometric reconstruction of the space of interest. Theorem 4.3 establishes the isomorphism of their fundamental groups. As an interesting application in Section 4.2, we consider the geometric reconstruction of planar subspaces and embedded planar graphs (Definition 4.4) in particular. In Theorem 4.7, we compute a homotopy equivalent geometric complex in the same ambient space that is also Hausdorff-close to X . Since the sample S can be taken to be finite, our result gives rise to an efficient algorithm (Algorithm 1) for the geometric reconstruction of planar embedded graphs.

2. Notation and Background

In this section, we provide a brief overview of useful notation and classical results from metric geometry and algebraic topology. For more detailed and complete treatment, we refer the reader to textbooks on metric geometry [21, 22] and algebraic topology [23, 24, 25].

2.1. Geodesic Subspaces, Distortion, Convexity Radius

We first present relevant definitions from metric geometry.

Geodesic Subspaces (of $\mathbb{R}^N$) We start with the unit interval $I := [0, 1] \subset \mathbb{R}$. A continuous function $\gamma: I \rightarrow \mathbb{R}^N$ is called a path. We call $T = \{t_i\}_{i=0}^k$ a discretization of I if $0 = t_0 < t_1 < t_2 < \dots < t_k = 1$. We create a piecewise linear path by using straight line segments to connect $\gamma(t_i)$ with $\gamma(t_{i+1})$ for each $i \in \{0, 1, \dots, k-1\}$. We often equip $\mathbb{R}^N$ with the Euclidean, or L_2 distance, $d_2: \mathbb{R}^N \times \mathbb{R}^N \rightarrow \mathbb{R}$ defined by $d_2(x, y) := \|x - y\|_2$. Let $\gamma: I \rightarrow \mathbb{R}^N$ be a (continuous) path. The *length* of γ is defined as:

$$L(\gamma) := \sup_T \sum_{i \in \{1, 2, \dots, |T|\}} d_2(\gamma(t_i), \gamma(t_{i+1})),$$

where the supremum is taken over all finite discretizations of I . Furthermore, the curve γ is called *rectifiable* if $L(\gamma)$ is finite. For a path-connected subset $X \subseteq \mathbb{R}^N$, we call the restriction of d_2 to X the *restricted metric* on X . We define the *induced length metric* or *geodesic metric*, $d_L: X \times X \rightarrow \mathbb{R}$, by

$$d_L(x, y) = \inf_{\gamma: [0, 1] \rightarrow X} L(\gamma), \quad (1)$$

where the infimum is taken over all paths $\gamma: I \rightarrow X$ such that $\gamma(0) = x$ and $\gamma(1) = y$.

Definition 2.1 (Geodesic Subspace). We call $X \subseteq \mathbb{R}^N$ a *geodesic subspace* if between any pair of points $x, y \in X$, there exists a rectifiable path on X starting at x and ending at y whose length is $d_L(x, y)$.

One example of a geodesic subspace is a connected and compact subset of $\mathbb{R}^N$. The “niceness” of an geodesic subspace is quantified by its distortion, a concept first introduced by M. Gromov in the context of knots on Riemannian manifolds [26, 27, 22]. For a geodesic subspace $X \subseteq \mathbb{R}^N$, we consider the map $f: (X, d_2) \rightarrow (X, d_L)$ induced by the identity map

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205 on X . The distortion of X is the best Lipschitz constant for f . More formally, we have the
 206 following definition.

207 **Definition 2.2 (Distortion).** The *distortion* of the induced length metric d_L with respect
 208 to Euclidean distance over a set $X \subseteq \mathbb{R}^N$ is defined as:

$$209 \quad \delta := \delta(X) = \sup_{x \neq y \in X} \frac{d_L(x, y)}{\|x - y\|_2}.$$

210 For simplicity of exposition, we refer to δ as *the distortion of X* .

211 Since d_L is the induced length metric, δ is bounded below by one and above by $+\infty$. If X
 212 is a straight line segment, then $\delta = 1$. On the other extreme, if X is the subspace $\{(x, y) \in$
 213 $\mathbb{R}^2 \mid x^2 = y^3\}$, then $\delta = +\infty$. To see this, consider the limit as ε approaches zero of the two
 214 points $(-\varepsilon^{3/2}, \varepsilon) \in X$ and $(\varepsilon^{3/2}, \varepsilon) \in X$, getting arbitrarily close to the cusp point $(0, 0)$.
 215 Thus, both the lower and upper bounds on δ are tight. For more on distortion, see [28].

216 **Remark 2.3 (Equivalence of Topologies).** Given a metric space (X, d) , we can topol-
 217 ogize X with metric balls; that is, the topology is generated by sets of the form $B_d(x, r) :=$
 218 $\{y \in X \mid d(x, y) < r\}$, where $x \in X$ and $r \in \mathbb{R}$. If we assume that d_L has finite distortion
 219 with respect to d_2 , then (X, d_L) and (X, d_2) have equivalent topologies. The equivalence of
 220 the two topologies is a direct consequence of the following inequalities for $x, y \in X$:

$$221 \quad \|x - y\|_2 \leq d_L(x, y) \leq \delta \|x - y\|_2. \quad (2)$$

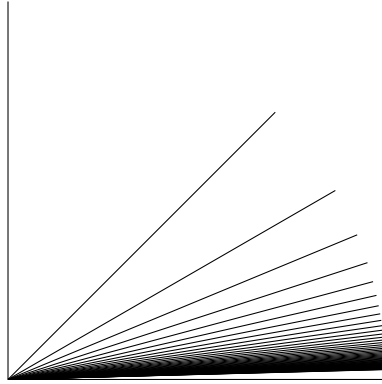


Fig. 3: The set X , the closure of the union of the falling segments in the figure, is known as the *infinite broom*. The topology of (X, d_2) is strictly finer than the length metric topology of (X, d_L) . The latter topology is locally path-connected; whereas, the former topology is not.

222 Equivalence of the topologies does not generally hold if the distortion of X is
 223 not finite. For an example, let $X \subset \mathbb{R}^2$ be the closure of the union of line seg-
 224 ments $\left\{ \left[(0, 0), \left(\cos \frac{\pi}{2i}, \sin \frac{\pi}{2i} \right) \right] \right\}_{i \in \mathbb{N}}$, as shown in Figure 3. Such a space is also known as

the *infinite broom*. We see that the distortion of the space is infinite by considering the sequence $a_i = (\cos \frac{\pi}{2i}, \sin \frac{\pi}{2i})$ of points on the right end of the spokes of the broom:

$$\lim_{i \rightarrow \infty} \frac{d_L((0, 1), a_i)}{\|(0, 1) - a_i\|_2} = \infty.$$

The Euclidean metric topology, in this case, is strictly finer than the length metric topology, as (X, d_L) is locally path-connected, but (X, d_2) is not.

Convexity Radius Convexity radius of the underlying geodesic subspace is one of the parameters of X used in all our reconstruction results. We start with its formal definition from [19]. Although the concept is defined for general length spaces, we restrict ourselves to only geodesic subspaces.

Definition 2.4 (Convexity Radius). We define the *convexity radius*, denoted ρ , of a geodesic subspace $X \subseteq \mathbb{R}^N$ to be the supremum of all $r > 0$ such that:

- (1) For all $x, y \in X$ with $d_L(x, y) < 2r$, there exists a unique (length-minimizing) geodesic path joining x and y .
- (2) If $x, y, z, u \in X$ such that $d_L(x, y) < r$, $d_L(y, z) < r$, $d_L(z, x) < r$, and u is a point on the (length-minimizing) geodesic path joining x and y , then $d_L(u, z) \leq \max \{d_L(x, z), d_L(y, z)\}$.
- (3) If γ and γ' are arc-length parametrized (length-minimizing) geodesics on X such that $\gamma(0) = \gamma'(0)$, then $d_L(\gamma(ts), \gamma'(ts')) \leq d_L(\gamma(s), \gamma'(s'))$ for $0 \leq s, s' < r$ and $0 \leq t \leq 1$.

Consider a circle in $\mathbb{R}^2$ with perimeter R ; its convexity radius is $\frac{R}{4}$. Also, the convexity radius of an embedded graph is $\frac{b}{4}$, where b is the length of its smallest simple cycle. It is well-known that the convexity radius of a compact Riemannian manifold is positive. The convexity radius of a geodesic space is an intrinsic property.

2.2. Simplicial Complexes, Nerve Lemma

We finally conclude this section by outlining a few important notions from algebraic topology. Readers are referred to [23, 24, 25] for more details.

Abstract Simplicial Complex The combinatorial analogue of a topological space, often used in algebraic and combinatorial topology, is an abstract simplicial complex. An *abstract simplicial complex* $\mathcal{K}$ is a collection of finite sets such that if $\sigma \in \mathcal{K}$, then so are all its non-empty subsets.

In general, elements of $\mathcal{K}$ are called *simplices* of $\mathcal{K}$. The singleton sets in $\mathcal{K}$ are often called the *vertices* of $\mathcal{K}$. If a simplex $\sigma \in \mathcal{K}$ has cardinality $(q + 1)$, then it is called a q -simplex (or the *dimension* of σ is q or $\dim(\sigma) = q$). If $\sigma' \subseteq \sigma$, then σ' is called a *face* of σ .

Simplicial Maps and Contiguity Let $\mathcal{K}_1$ and $\mathcal{K}_2$ be abstract simplicial complexes with vertex sets $\mathcal{V}_1$ and $\mathcal{V}_2$, respectively. A *vertex map* is a map between the vertex sets. Let $\phi: \mathcal{V}_1 \rightarrow \mathcal{V}_2$ be a vertex map. If, for all $\sigma \in \mathcal{K}_1$, we have $\phi(\sigma) := \cup_{v \in \sigma} \{\phi(v)\}$ is,

in fact, an element of $\mathcal{K}_2$, then we say that ϕ induces a *simplicial map* $\phi : \mathcal{K}_1 \rightarrow \mathcal{K}_2$. Two simplicial maps $\phi_1, \phi_2 : \mathcal{K}_1 \rightarrow \mathcal{K}_2$ are called *contiguous* if for every simplex $\sigma_1 \in \mathcal{K}_1$, there exists $\sigma_2 \in \mathcal{K}_2$ such that $\phi_1(\sigma_1) \cup \phi_2(\sigma_1) \subseteq \sigma_2$. A simplicial map between abstract simplicial complexes is the combinatorial analogue of a continuous map between topological spaces; likewise, contiguous simplicial maps play the role of homotopic maps in the combinatorial world.

Geometric Complex Although, abstract simplicial complexes have enough combinatorial structure to define simplicial homology and homotopy, they are not topological spaces. For an abstract simplicial complex $\mathcal{K}$ with vertex set $\mathcal{V}$, its *underlying topological space* or *geometric complex*, denoted as $|\mathcal{K}|$, is defined as the space of all functions $\alpha : \mathcal{V} \rightarrow [0, 1]$, also called *barycentric coordinates*, satisfying the following two properties:

- (1) $\text{supp}(\alpha) := \{v \in \mathcal{V} \mid \alpha(v) \neq 0\} \in \mathcal{K}$
- (2) $\sum_{v \in \mathcal{V}} \alpha(v) = 1$.

The details on the topologies on $|\mathcal{K}|$ and their relations can be found in [24, 25]. In this work, we use the standard *metric topology* on $|\mathcal{K}|$, as defined in [25]. Naturally, a simplicial map $\phi : \mathcal{K}_1 \rightarrow \mathcal{K}_2$ induces a continuous map $|\phi| : |\mathcal{K}_1| \rightarrow |\mathcal{K}_2|$ defined by

$$|\phi|(\alpha)(v') = \sum_{\phi(v)=v'} \alpha(v), \text{ for } v' \in \mathcal{K}_2.$$

As one expects, the contiguous simplicial maps induce homotopic continuous maps between their respective underlying topological spaces; see [25] for a proof.

Nerve Lemma A critical ingredient for our Čech reconstruction results is the Nerve Lemma or a modification thereof; therefore, we discuss the concept here. An open cover $\mathcal{U} = \{U_i\}_{i \in \Lambda}$ of a topological space X is called a *good cover* if all finite intersections of its elements are contractible. The *nerve* of $\mathcal{U}$, denoted $\mathcal{N}(\mathcal{U})$, is defined to be the simplicial complex having Λ as its vertex set, and for each non-empty k -way intersection $U_{i_1} \cap U_{i_2} \cap \dots \cap U_{i_k}$, the subset $\{i_1, i_2, \dots, i_k\}$ is a simplex of $\mathcal{N}(\mathcal{U})$. Under the right assumptions, the nerve preserves the homotopy type of the union X , as stated by the following fundamental result.

Lemma 2.5 (Nerve Lemma [29]). *Let $\mathcal{U} = \{U_i\}_{i \in \Lambda}$ be a good open cover of a topological space X . Then, the underlying topological space $|\mathcal{N}(\mathcal{U})|$ is homotopy equivalent to X .*

Remark 2.6. If the open cover $\mathcal{U}$ is locally finite, then the homotopy equivalence in the Nerve Lemma is usually constructed with the help of a *partition of unity* for the cover [23]. Specifically, let $h : X \rightarrow |\mathcal{N}(\mathcal{U})|$ be a homotopy equivalence. Then, a partition of unity is a collection of continuous functions $\{\varphi_i : X \rightarrow [0, 1]\}_{i \in \Lambda}$ such that for all $x \in X$,

$$h(x) = \sum_{i \in \Lambda} \varphi_i(x) v_i, \tag{3}$$

where v_i denotes the vertex of $\mathcal{N}(\mathcal{U})$ corresponding to the cover element U_i . In addition, each φ_i must satisfy the following two requirements: (i) for all $i \in \Lambda$, the support of φ_i ,

denoted $\text{supp}(\varphi_i)$, is a compact proper subset of U_i , and (ii) for all $x \in X$, $\sum_{i \in \Lambda} \varphi_i(x) = 1$.

Čech and Vietoris-Rips Complexes Consider a subspace A of a metric space (M, d) and a positive scale α . The nerve of the collection of open metric balls of radius α centered at the points of A is known as the *Čech complex* of A at scale (radius) α . We are interested in Čech complexes in two metric spaces: Euclidean and the length metric space. Let $X \subseteq \mathbb{R}^N$. Then, the Čech complex under the standard Euclidean metric is: $\mathcal{C}_\alpha(X) := \mathcal{N}(\{\mathbb{B}(x, r)\}_{x \in X})$, where $\mathbb{B}(x, r)$ is the Euclidean ball of radius r centered at x . The Čech complex under the length metric (X, d_L) is $\mathcal{C}_\alpha^L(A) := \mathcal{N}(\{\mathbb{B}^L(x, r)\}_{x \in X})$, where $\mathbb{B}^L(x, r)$ denotes the metric ball of radius r centered at X in (X, d_L) . Note that these complexes may be infinite.

The *Vietoris-Rips Complex* is an abstract simplicial complex having a k -simplex for every set of $(k+1)$ points in A of diameter at most α . Explicit knowledge about the entire metric space (M, d) is not needed to compute the complex. Unlike the Čech complex, the Vietoris-Rips complex is completely determined by the restriction of the metric to the subset A . For $X \subseteq \mathbb{R}^N$ under the standard Euclidean metric, we denote it simply by $\mathcal{R}_\alpha(X)$. In the case when $A \subseteq X$ equipped with length metric (X, d_L) , we denote the Vietoris-Rips complex by $\mathcal{R}_\alpha^L(A)$.

Together, the definition of convexity radius and Nerve Lemma immediately imply the following fact:

Lemma 2.7 (Čech Equivalence). *Let $X \subseteq \mathbb{R}^N$ be a geodesic subspace with a positive convexity radius ρ , and let $0 < \varepsilon < \rho$. Let A be an ε -dense subset of X with respect to the d_L metric. Then, the complex $\mathcal{C}_\varepsilon^L(A)$ is homotopy equivalent to X .*

Proof. Since A is an ε -dense subset of X , we know that $\mathcal{U} := \cup_{a \in A} \mathbb{B}^L(a, \varepsilon)$ is an open cover of (X, d_L) . Since $\varepsilon < \rho$ and by the definition of convexity radius (Definition 2.4), we know that for each $x \in X$ and $y \in \mathbb{B}^L(x, \varepsilon)$, there exists a unique length-minimizing geodesic path between x and y . Using these paths to define a deformation retract from $\mathbb{B}^L(x, \varepsilon)$ to x , we conclude that the metric balls in $\mathcal{U}$ are contractible. Since any finite intersection of metric balls in $\mathcal{U}$ has diameter less than 2ε , by the similar argument it is also contractible. Hence, $\mathcal{U}$ is a good cover of X . By the Nerve Lemma (Lemma 2.5), we conclude that the complex $\mathcal{C}_\varepsilon^L(A)$ is homotopy equivalent to X . $\square$

3. Topological Reconstruction

In this section, we consider the problem of topological reconstruction of a geodesic subspace X of $\mathbb{R}^N$ from a noisy sample S . From now on, unless otherwise stated, we assume that the underlying shape X has a positive convexity radius and a finite distortion, also that the sample S is a finite subset of $\mathbb{R}^N$. We show that both Čech and Vietoris-Rips filtrations of S can be used to compute the homology and homotopy groups of X . Before we treat each type of complex separately, we show how the Čech and Vietoris-Rips complexes behave under Hausdorff perturbation.

Lemma 3.1 (Hausdorff Distance and Complexes). *Let $A, B \subseteq \mathbb{R}^N$ be finite, and ε be*

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a positive number such that $d_H(A, B) < \varepsilon$. Then for any $\alpha > 0$, there exist simplicial maps

$$\mathcal{C}_\alpha(A) \longrightarrow \mathcal{C}_{\alpha+\varepsilon}(B)$$

and

$$\mathcal{R}_\alpha(A) \longrightarrow \mathcal{R}_{\alpha+2\varepsilon}(B)$$

induced by a vertex map $\xi : A \rightarrow B$ such that for every vertex $a \in A$, we have $\|a - \xi(a)\|_2 < \varepsilon$. Moreover, such simplicial maps are unique, up to contiguity.

Proof. We first note the definition

$$d_H(A, B) = \inf \{ \varepsilon > 0 \mid A \subseteq B^\varepsilon, B \subseteq A^\varepsilon \},$$

where A^ε denotes the Euclidean thickening of A .

The definition of Hausdorff distance implies that if $d_H(A, B) < \varepsilon$, there exists a (possibly non-unique, non-continuous) map $\xi : A \rightarrow B$ such that $\|a - \xi(a)\|_2 < \varepsilon$. We show that this vertex map extends to a simplicial map between both Čech and Vietoris-Rips complexes.

Let $\sigma = \{a_0, a_1, \dots, a_k\}$ be a k -simplex of $\mathcal{C}_\alpha(A)$. By definition, there exists a point z in $\mathbb{R}^N$ such that $\|a_i - z\|_2 < \alpha$ for all $i \in \{0, 1, \dots, k\}$. By the triangle inequality, we then have

$$\|\xi(a_i) - z\|_2 \leq \|\xi(a_i) - a_i\|_2 + \|a_i - z\|_2 < \varepsilon + \alpha.$$

So, $\{\xi(a_0), \dots, \xi(a_k)\}$ is a simplex of $\mathcal{C}_{\alpha+\varepsilon}(B)$. Hence, ξ extends to a simplicial map between the Čech complexes. To argue for the uniqueness of the simplicial map, let us assume that η is another simplicial map with the property that for every vertex $a \in A$, we have $\|a - \eta(a)\|_2 < \varepsilon$. Again from the triangle inequality, we have $\|\eta(a_i) - z\|_2 < \varepsilon + \alpha$. So, $\xi(\sigma) \cup \eta(\sigma)$ is a simplex of $\mathcal{C}_{\alpha+\varepsilon}(B)$. Hence, ξ and η are contiguous.

For the Vietoris-Rips complex part, we follow a similar argument. Let $\sigma = \{a_0, a_1, \dots, a_k\}$ be a k -simplex of $\mathcal{R}_\alpha(A)$. By definition, the diameter of σ is not greater than α . From the triangle inequality, we have

$$\|\xi(a_i) - \xi(a_j)\|_2 \leq \|\xi(a_i) - a_i\|_2 + \|a_i - a_j\|_2 + \|\xi(a_j) - a_j\|_2 < 2\varepsilon + \alpha.$$

So, $\{\xi(a_0), \dots, \xi(a_k)\}$ is a simplex of $\mathcal{R}_{\alpha+2\varepsilon}(A)$. Hence, ξ extends to a simplicial map also between Vietoris-Rips complexes. $\square$

3.1. Homology Groups via Vietoris-Rips Complex

We use the following fundamental result from [19] to compute the homology groups of X from a filtration of Vietoris-Rips complexes on a finite sample.

Theorem 3.2 (Hausmann's Theorem [19]). *Let X be a geodesic subspace with a positive convexity radius ρ . For $0 < \varepsilon < \rho$, there exists a homotopy equivalence $T : |\mathcal{R}_\varepsilon^L(X)| \longrightarrow X$.*

Note that $\mathcal{R}_\varepsilon^L(X)$ is usually an infinite Vietoris-Rips complex on the entire space X . A quick corollary of this result is:

Corollary 3.3. *Let X be a geodesic subspace with a positive convexity radius ρ . For $0 < \varepsilon' \leq \varepsilon < \rho$, the inclusion $i : \mathcal{R}_{\varepsilon'}^L(X) \hookrightarrow \mathcal{R}_{\varepsilon}^L(X)$ induces isomorphisms on homology and homotopy groups.*

In order to achieve our result, we use certain simplicial maps to compare $\mathcal{R}_*^L(X)$, $\mathcal{R}_*(X)$, and $\mathcal{R}_*(S)$.

Lemma 3.4 (Euclidean and Intrinsic Rips Complexes). *Let X a geodesic subspace of $\mathbb{R}^N$ with a finite distortion δ . Then for $A \subseteq X$ and any positive number α , we have the following simplicial inclusions*

$$\mathcal{R}_{\alpha}^L(A) \hookrightarrow \mathcal{R}_{\alpha}(A) \hookrightarrow \mathcal{R}_{\delta\alpha}^L(A).$$

Proof. The fact that $\|x - y\|_2 \leq d_L(x, y)$ implies the first inclusion $\mathcal{R}_{\alpha}^L(A) \hookrightarrow \mathcal{R}_{\alpha}(A)$. Similarly, $d_L(x, y) \leq \delta \|x - y\|_2$ implies the second inclusion. $\square$

Theorem 3.5 (Reconstruction via Rips Complex). *Let X be a geodesic subspace of $\mathbb{R}^N$ with a positive convexity radius ρ and finite distortion δ . Let S be a finite subset of $\mathbb{R}^N$, and let ε be a positive number such that*

$$d_H(X, S) < \frac{\varepsilon}{4} < \frac{\rho}{2\delta(3\delta + 2)}.$$

Then, for any non-negative integer k we have the following isomorphism

$$H_k(X) \cong \text{im}(j_* : H_k(\mathcal{R}_{\varepsilon}(S)) \hookrightarrow H_k(\mathcal{R}_{\frac{1}{2}(3\delta+1)\varepsilon}(S)))$$

where j_ is induced by the simplicial inclusion $j : \mathcal{R}_{\varepsilon}(S) \hookrightarrow \mathcal{R}_{\frac{1}{2}(3\delta+1)\varepsilon}(S)$.*

Proof. We derive the following chain of simplicial maps:

$$\mathcal{R}_{\frac{\varepsilon}{2}}^L(X) \xrightarrow{\phi_1} \mathcal{R}_{\varepsilon}(S) \xrightarrow{\phi_2} \mathcal{R}_{\frac{3\varepsilon}{2}\delta}^L(X) \xrightarrow{\phi_3} \mathcal{R}_{(3\delta+1)\frac{\varepsilon}{2}}(S) \xrightarrow{\phi_4} \mathcal{R}_{\frac{1}{2}(3\delta+2)\delta\varepsilon}^L(X). \quad (4)$$

The first map ϕ_1 is the composition of the simplicial inclusion $\mathcal{R}_{\frac{\varepsilon}{2}}^L(X) \hookrightarrow \mathcal{R}_{\frac{\varepsilon}{2}}(X)$ from Lemma 3.4 and the simplicial map $\mathcal{R}_{\frac{\varepsilon}{2}}(X) \hookrightarrow \mathcal{R}_{\varepsilon}(S)$ from Lemma 3.1, thanks to the assumption $d_H(S, X) < \frac{\varepsilon}{4}$.

Now, starting with $\mathcal{R}_{\varepsilon}(S)$ and composing maps from Lemma 3.1 and Lemma 3.4, respectively, we get the second simplicial map ϕ_2 . Similarly, we get the maps ϕ_3 and ϕ_4 .

From Lemma 3.1, we first note that the composition $\phi_3 \circ \phi_2$ is contiguous to the inclusion:

$$j : \mathcal{R}_{\varepsilon}(S) \hookrightarrow \mathcal{R}_{(3\delta+1)\frac{\varepsilon}{2}}(S).$$

Therefore, they induce homotopic maps on the respective underlying topological spaces. Consequently, we have $(\phi_3 \circ \phi_2)_* = j_*$. We first argue that ϕ_{2*} is surjective and ϕ_{3*} is injective.

By the choice of the simplicial maps in Lemma 3.4 and Lemma 3.1, we observe that $\phi_2 \circ \phi_1$ is contiguous to the inclusion

$$\mathcal{R}_{\frac{\varepsilon}{2}}^L(X) \hookrightarrow \mathcal{R}_{\frac{3\varepsilon}{2}\delta}^L(X).$$

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390 By Corollary 3.3, the inclusion induces isomorphism on homology, hence so does $\phi_2 \circ \phi_1$. In
 391 particular, $(\phi_2 \circ \phi_1)_*$ is surjective. Hence, we have ϕ_{2*} is surjective, and ϕ_{1*} is injective.

392 Also, $\phi_4 \circ \phi_3$ is contiguous to the inclusion

$$393 \quad \mathcal{R}_{\frac{3\varepsilon}{2}\delta}^L(X) \hookrightarrow \mathcal{R}_{\frac{1}{2}(3\delta+2)\delta\varepsilon}^L(X),$$

394 which induces an isomorphism on homologies. Therefore, ϕ_{3*} induces an injective homo-
 395 morphism.

396 Since we have $j_* = \phi_{3*} \circ \phi_{2*}$ and ϕ_{2*} is surjective, the image of j_* is the image of ϕ_{3*} .
 397 On the other hand, we know that $\text{Im}(\phi_{3*})$ is isomorphic to $H_*\left(\mathcal{R}_{\frac{3\varepsilon}{2}\delta}^L(X)\right)/\text{Ker}(\phi_{3*})$. As
 398 we have already shown that ϕ_{3*} is injective, its kernel is trivial. Therefore, the image of j_*
 399 is isomorphic to $\mathcal{R}_{\frac{3\varepsilon}{2}\delta}^L(X)$. Since $\frac{3\varepsilon}{2}\delta < \rho$, Theorem 3.2 implies that $\mathcal{R}_{\frac{3\varepsilon}{2}\delta}^L(X)$ is, in fact,
 400 homotopy equivalent to X . This completes the proof. $\square$

401 The Vietoris-Rips reconstruction result works also for an infinite sample S . In applica-
 402 tions, however, we are computationally constrained to use only finite samples.

403 3.2. Homology Groups via Čech Complex

404 The reconstruction of homology groups via the Vietoris-Rips filtration (see Theorem 3.5
 405 in Section 3.1) was due to the homotopy equivalence theorem (Theorem 3.2). In this sub-
 406 section, we use Čech filtration to obtain similar reconstruction results. The Nerve Lemma
 407 (Lemma 2.5) is resorted to as the Čech alternative to Theorem 3.2. Like the Vietoris-Rips
 408 case, we still use different simplicial maps to compare $\mathcal{C}_*^L(X)$, $\mathcal{C}_*(X)$, and $\mathcal{C}_*(S)$. The ap-
 409 proach involves a (controlled) variant of the partition of unity; see Lemma 3.8.

Lemma 3.6 (Euclidean and Intrinsic Čech Complexes). *Let X a geodesic subspace
 of $\mathbb{R}^N$ with a finite distortion δ . Then for $A \subseteq X$ and any positive number α , we have the
 following simplicial inclusions*

$$\mathcal{C}_\varepsilon^L(A) \hookrightarrow \mathcal{C}_\alpha(A) \hookrightarrow \mathcal{C}_{2\delta\alpha}^L(A).$$

410 **Proof.** From $\|x - y\|_2 \leq d_L(x, y)$, we have the first inclusion.

411 On the other hand, for any $x, y \in X$ we have $d_L(x, y) \leq \delta \|x - y\|_2$. Let $\sigma = \{x_0, \dots, x_k\}$
 412 be a simplex of $\mathcal{C}_\alpha(A)$. Then $\|x_i - x_j\|_2 < 2\alpha$, consequently $d_L(x_i, x_j) < 2\delta\alpha$ for all $1 \leq$
 413 $i, j \leq k$. This implies

$$414 \quad \{x_0, x_1, \dots, x_k\} \subset \bigcap_{i=0}^k \mathbb{B}^L(x_i, 2\delta\alpha),$$

415 where $\mathbb{B}^L(x_i, r)$ denotes the ball of radius r centered at x_i in the metric space (X, d_L) .
 416 Therefore $\sigma \in \mathcal{C}_{2\delta\alpha}^L(A)$, and this verifies the second inclusion. $\square$

417 We begin with a lemma that is analogous to Corollary 3.3 in the Čech regime:

Lemma 3.7 (Inclusion of Covers). *Let $\mathcal{U} = \{U_i\}_{i \in \Lambda}$ and $\mathcal{U}' = \{U'_i\}_{i \in \Lambda}$ be locally-finite, good open covers of a para-compact topological space X such that $U_i \subseteq U'_i$ for each i . Then, the inclusion*

$$i : \mathcal{N}(\mathcal{U}) \hookrightarrow \mathcal{N}(\mathcal{U}')$$

induces isomorphisms on the homology and homotopy groups of the respective geometric complexes.

Proof. Consider the following commutative diagram:

$$\begin{array}{ccc} |\mathcal{N}(\mathcal{U})| & \xhookrightarrow{i} & |\mathcal{N}(\mathcal{U}')| \\ & \nwarrow h & \nearrow i \circ h \\ & X & \end{array}$$

where the map $h = \sum \varphi_i u_i$ is obtained from an arbitrary partition of unity $\{\varphi_i\}$ for $\mathcal{U}$. By the Nerve Lemma (Lemma 2.5), h is a homotopy equivalence ([23]). Since $U_i \subseteq U'_i$, $\{\varphi_i\}$ is a partition of unity for $\mathcal{U}'$. So, $i \circ h$ is also a homotopy equivalence. Since the maps h is a homotopy equivalence, we conclude that i induces an isomorphism on homology and homotopy groups. $\square$

We now state the following extension of the partition of unity. Follow [30] for a proof.

Lemma 3.8 (Controlled Partition of Unity). *Let $\{U_i\}$ and $\{V_i\}$ be open covers of a paracompact, Hausdorff space X such that $\overline{V_i} \subseteq U_i$ for each i . Then, there exists a partition of unity $\{\varphi_i\}$ subordinate to $\{U_i\}$ such that $V_i \subseteq \text{supp } \varphi_i \subseteq U_i$ for all i .*

We now use the controlled partition of unity to prove the following important lemma.

Lemma 3.9 (Commuting Diagram). *Let X, Y be paracompact, Hausdorff spaces with a continuous map $f : X \rightarrow Y$. Let $\mathcal{U} = \{U_i\}$ and $\mathcal{V} = \{V_i\}$ be good, locally finite, open covers of X and Y respectively, such that*

- (1) $\bigcap_i V_i \neq \emptyset$ implies $\bigcap_i U_i \neq \emptyset$, i.e., we have the simplicial inclusion $j : \mathcal{N}(\mathcal{V}) \rightarrow \mathcal{N}(\mathcal{U})$ that sends the vertex corresponding to V_i to the vertex corresponding to U_i ,
- (2) $\overline{f^{-1}(V_i)} \subseteq U_i$ for all i .

Then, the following diagram commutes, up to homotopy:

$$\begin{array}{ccc} |\mathcal{N}(\mathcal{V})| & \xhookrightarrow{j} & |\mathcal{N}(\mathcal{U})| \\ \uparrow h_Y & & \uparrow h_X \\ Y & \xleftarrow{f} & X \end{array}$$

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440 where h_X, h_Y are homotopy equivalences from (3).

441 **Proof.** We make use of the controlled partition of unity lemma to prove our result. Let
 442 us choose a partition of unity $\{\phi_i\}$ subordinate to $\{V_i\}$. One can choose h_Y so that for
 443 each $y \in Y$,

$$444 \quad h_Y(y) = \sum_i \phi_i(y) v_i,$$

445 where v_i is the vertex of $\mathcal{N}(\mathcal{V})$ corresponding to V_i .

446 Since $\{f^{-1}(V_i)\}$ is an open cover of X with $\overline{f^{-1}(V_i)} \subseteq U_i$ for each i , by Lemma 3.8 we
 447 can choose a partition of unity $\{\psi_i\}$ subordinate to $\{U_i\}$ such that for each i

$$448 \quad f^{-1}(V_i) \subseteq \text{supp } \psi_i.$$

449 Also, choose h_x such that for each $x \in X$

$$450 \quad h_X(x) = \sum_i \psi_i(x) u_i,$$

451 where u_i is the vertex of $\mathcal{N}(\mathcal{U})$ corresponding to U_i .

452 To see that the diagram commutes, up to homotopy, it suffices to show that $(j \circ h_Y \circ f)$
 453 is homotopic to h_X . We start with a point $x_0 \in X$

$$454 \quad (j \circ h_Y \circ f)(x_0) = j\left(\sum_i \phi_i(f(x_0)) v_i\right) = \sum_i \phi_i(f(x_0)) j(v_i) = \sum_i \phi_i(f(x_0)) u_i.$$

455 On the other hand, $h_X(x_0) = \sum_i \psi_i(x_0) u_i$. Now if $\phi_i(f(x_0))$ is non-zero for some i ,
 456 then $f(x_0) \in V_i$, and consequently $x_0 \in f^{-1}(V_i) \subseteq U_i$. From our choice of the support
 457 of ψ_i and $\psi_i(x_0)$ has to be non-zero. This shows that both $(j \circ h_Y \circ f)(x_0)$ and $h_X(x_0)$ lie
 458 in an (open) simplex of $\mathcal{N}(\mathcal{V})$. Due to convexity of simplices, the following (straight-line)
 459 homotopy is well-defined:

$$460 \quad F(x, t) = \sum_i [t\psi_i(x) + (1-t)\phi_i(x)] u_i.$$

461 This shows that $(j \circ h_Y \circ f)$ is homotopic to h_X . □

462 Now we are in a position to prove our reconstruction result for Čech complexes.

463 **Theorem 3.10 (Reconstruction via Čech complex).** *Let X be a geodesic subspace*
 464 *of $\mathbb{R}^N$ with a positive convexity radius ρ and finite distortion δ . Let S be a finite subset*
 465 *of $\mathbb{R}^N$, and let ε be a positive number such that*

$$466 \quad d_H(X, S) < \varepsilon < \frac{\rho}{2\delta(4\delta+1)}.$$

467 Then, any non-negative integer k we have the following isomorphism

$$468 \quad H_k(X) \cong \text{im}(j_* : H_k(\mathcal{C}_\varepsilon(S)) \hookrightarrow H_k(\mathcal{C}_{(4\delta+1)\varepsilon}(S))) \quad (5)$$

469 where j_* is induced by the simplicial inclusion $j : \mathcal{C}_\varepsilon(S) \rightarrow \mathcal{C}_{(4\delta+1)\varepsilon}(S)$.

Proof. We first note from $d_H(X, S) < \varepsilon$ and Lemma 3.1 that there is a map $\xi : S \rightarrow X$ such that for each $s \in S$,

$$\|s - \xi(s)\|_2 < \varepsilon. \quad (6)$$

Let $X' = \xi(S)$. Then, (6) implies $d_H(S, X') < \varepsilon$, hence $d_H(X, X') < 2\varepsilon$ by the triangle inequality.

We now derive the following chain of simplicial maps:

$$\mathcal{C}_\varepsilon(S) \xrightarrow{\phi_1} \mathcal{C}_{4\varepsilon\delta}^L(X') \xrightarrow{\phi_2} \mathcal{C}_{(4\delta+1)\varepsilon}(S) \xrightarrow{\phi_3} \mathcal{C}_{2\delta(4\delta+1)\varepsilon}^L(X').$$

The first map ϕ_1 is the composition of the simplicial map $\mathcal{C}_\varepsilon(S) \hookrightarrow \mathcal{C}_{2\varepsilon}(X')$ from Lemma 3.1 (due to $d_H(S, X') < \varepsilon$) and the simplicial inclusion $\mathcal{C}_{2\varepsilon}(X') \hookrightarrow \mathcal{C}_{4\varepsilon\delta}^L(X')$ from Lemma 3.6.

Similarly, starting with $\mathcal{C}_{4\varepsilon\delta}^L(X')$ and composing maps from Lemma 3.6 and Lemma 3.1, respectively, we get the second simplicial map ϕ_2 . The other map ϕ_3 is also obtained repeating the exact same argument for a different scale as for ϕ_1 .

We first observe that the choice of simplicial maps in Lemma 3.6 and Lemma 3.1 makes $\phi_2 \circ \phi_1$ contiguous to the given natural inclusion j of $\mathcal{C}_\varepsilon(S)$ into $\mathcal{C}_{2\delta(4\delta+1)\varepsilon}(S)$. We now consider the following diagram:

$$\begin{array}{ccccccc} |\mathcal{C}_\varepsilon(S)| & \xrightarrow{\phi_1} & |\mathcal{C}_{4\varepsilon\delta}^L(X')| & \xrightarrow{\phi_2} & |\mathcal{C}_{(4\delta+1)\varepsilon}(S)| & \xrightarrow{\phi_3} & |\mathcal{C}_{2\delta(4\delta+1)\varepsilon}^L(X')| \\ \uparrow h_1 & & \uparrow h_2 & & & & \uparrow h_3 \\ S^\varepsilon & \xleftarrow{i} & X & \xleftarrow{Id} & X & & X \end{array} \quad (7)$$

To show that the diagram commutes up to homotopy, we first explain the horizontal maps in the bottom row of (7). Since $d_H(X, S) < \varepsilon$, we get the first inclusion $X \subseteq S^\varepsilon$. The three vertical maps are homotopy equivalences that come from the Nerve Lemma (Lemma 2.5) for various good open covers as constructed in Lemma 3.9. The first vertical map h_1 is obtained for the open cover $\mathcal{U}_1 = \{\mathbb{B}(x, \varepsilon)\}_{x \in S}$ of S^ε by Euclidean balls. The other two vertical maps, h_2 and h_3 , are corresponding to the (intrinsic) covers $\mathcal{U}_2$ and $\mathcal{U}_3$ of (X, d_L) by the intrinsic balls of radii $2\delta\varepsilon$ and $4\delta(2\delta+1)\varepsilon$, respectively. The assumption $4\delta(2\delta+1)\varepsilon < \rho$ implies that they are indeed good (intrinsic) covers of X . Therefore, by Lemma 2.7 we get the homotopy equivalences h_2 and h_3 .

Apply Lemma 3.9 to each of the rectangles in (7) to show that the diagram is homotopy commutative, and therefore it commutes on the homology level. The commutativity then implies that ϕ_1 induces a surjective homomorphism and ϕ_2 induces an injective homomorphism on the homology groups. As a consequence, $\text{Im}(\phi_{2*} \circ \phi_{1*}) = \text{Im}(\phi_{2*}) = H_k(X)$ on the k -th homology group. Also, we note that $\phi_2 \circ \phi_1$ is homotopic to the given simplicial inclusion j .

To see that the first rectangle commutes, we consider the covers $\mathcal{U}_1$ and $\mathcal{U}_2$ of S^ε and (X, d_L) . Note that for any $x \in S$, the choice of $\xi(x)$ implies that $i^{-1}(\mathbb{B}(x, \varepsilon)) = \mathbb{B}(x, \varepsilon) \cap$

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502 $X \subseteq \mathbb{B}^L(\xi(x), 2\delta\varepsilon)$. Consequently, $\overline{\mathbb{B}(x, \varepsilon) \cap X} \subseteq \mathbb{B}^L(\xi(x), 4\delta\varepsilon)$. A similar argument also
 503 applies to other rectangle. Therefore by Lemma 3.9, the diagram (7) commutes. $\square$

504 **Remark 3.11.** We remark that Theorem 3.5 and Theorem 3.10 of this section can be
 505 formulated in terms of any natural functor from the category of topological spaces (with
 506 continuous maps as morphisms) to the category of groups (with group homomorphisms). In
 507 particular, the results extend immediately to homology groups $H_*(\cdot; G)$ with coefficients in
 508 any abelian group G , or homotopy groups $\pi_*(\cdot)$.

509 4. Geometric Reconstruction

510 In the previous section, we used filtrations of both the Čech and the Vietoris-Rips complexes
 511 to compute the homology and homotopy groups of our hidden geodesic subspace X from
 512 a noisy sample S around it. The results, however, do not provide us with a topological
 513 space that faithfully carries the topology of X . To remedy this, we consider the problem of
 514 geometric reconstruction of geodesic subspaces.

515 In Section 4.1, we introduce a new metric d_ε on S . As our first step towards capturing
 516 the homotopy type, we show in Theorem 4.3 that the Vietoris-Rips complex of (S, d_ε) and
 517 X have isomorphic fundamental groups. Finally in Section 4.2, we further use this complex
 518 for the geometric reconstruction of embedded graphs.

519 4.1. Recovery of the Fundamental Group

520 For any fixed $\varepsilon > 0$, we first consider the Euclidean Vietoris-Rips complex $\mathcal{R}_\varepsilon(S)$ on the
 521 sample S . Regardless of how dense the sample S is, $\mathcal{R}_\varepsilon(S)$ is not guaranteed to be homo-
 522 topy equivalent to X in general; as shown in Figure 5. This is not surprising, because the
 523 Euclidean metric on S , used to compute the complex, can be very different from the length
 524 metric d_L on X . Our goal is to approximate d_L by the shortest path metric, denoted d_ε , on
 525 the one-skeleton of $\mathcal{R}_\varepsilon(S)$. Let us denote the one-skeleton of $\mathcal{R}_\varepsilon(S)$ by G_ε . Since $\mathcal{R}_\varepsilon(S)$ is
 526 an abstract simplicial complex, G_ε inherits the structure of an abstract graph. However, we
 527 turn its geometric complex $|G_\varepsilon|$ into a metric graph by defining the metric d_ε on it in the
 528 following way: the metric, when restricted to an edge (s, t) , is isometric to a real interval of
 529 length $\|s - t\|_2$.

530 We show in Lemma 4.1 that d_ε nicely approximates the metric d_L , which the Euclidean
 531 sample is oblivious to. For any positive scale α , we denote the Vietoris-Rips complex of S in
 532 the d_ε metric by $\mathcal{R}_\alpha^\varepsilon(S)$. The metric d_ε can be computed in $O(k^3)$ -time from a sample (S, d_2)
 533 of size k . In the following lemma, we compare the metric d_ε with the standard Euclidean
 534 metric d_2 and the length metric d_L .

Lemma 4.1 (Minimal Covering of Paths). *Let X be a geodesic subspace of $\mathbb{R}^N$. Let S be a subset of $\mathbb{R}^N$ and $\varepsilon > 0$ such that $d_H(X, S) < \frac{\varepsilon}{3}$. For any path γ joining any two points $x, y \in X$, we can find a sequence $\{a_i\}_{i=0}^k \subseteq S$ with $\|a_{i+1} - a_i\|_2 < \varepsilon$ such that*

$$\sum_{i=0}^{k-1} \|a_{i+1} - a_i\|_2 < 3l(\gamma).$$

Moreover, a_0 and a_k can be chosen to be any points with $\|x - a_0\|_2 < \frac{\varepsilon}{3}$ and $\|y - a_k\|_2 < \frac{\varepsilon}{3}$.

Proof. Since $d_H(X, S) < \frac{\varepsilon}{3}$, there exists $a_0 \in S$ such that $\|x - a_0\|_2 < \frac{\varepsilon}{3}$. We now iteratively define the sequence $\{a_i\} \subseteq S$, along with a sequence $\{t_i\}_0^k \subset [0, 1]$ that defines a partition of $[0, 1]$. We set $t_0 = 0$. Assuming both a_i and t_i are defined, we define $t_{i+1} \in [0, 1]$ in the following way: if $\gamma([t_i, 1]) \cap \partial\mathbb{B}(a_i, \frac{2\varepsilon}{3}) \neq \emptyset$, we set

$$t_{i+1} = \min\{t \in [t_i, 1] \mid \gamma(t) \in \partial\mathbb{B}(a_i, \frac{2\varepsilon}{3})\}.$$

Otherwise if $\gamma([t_i, 1]) \cap \partial\mathbb{B}(a_i, \frac{2\varepsilon}{3}) = \emptyset$, set $t_{i+1} = 1$. Since $d_H(S, X) < \frac{\varepsilon}{3}$, we set $a_{i+1} \in S$ to be a point in S such that $\|\gamma(t_{i+1}) - a_{i+1}\|_2 < \frac{\varepsilon}{3}$. This procedure forces t_{i+1} to be strictly greater than t_i , hence $\{t_i\}$ defines a partition of $[0, 1]$. Therefore,

$$l(\gamma) = \sum_{i=0}^k l(\gamma|_{[t_i, t_{i+1}]}) \geq \sum_{i=0}^k \|\gamma(t_i) - \gamma(t_{i+1})\|_2 \geq \sum_{i=0}^k \frac{\varepsilon}{3} \geq \frac{1}{3} \sum_{i=0}^k \|a_{i+1} - a_i\|_2.$$

We also note that

$$0 < \|a_{i+1} - a_i\|_2 \leq \|a_{i+1} - \gamma(t_{i+1})\|_2 + \|\gamma(t_{i+1}) - a_i\|_2 < \frac{\varepsilon}{3} + \frac{2\varepsilon}{3} = \varepsilon. \quad \square$$

Analogous to Lemma 3.1, we get the following useful simplicial maps.

Lemma 4.2 (Vietoris-Rips Inclusion by d_ε). Let X a geodesic subspace $X \subseteq \mathbb{R}^N$. Let $S \subseteq \mathbb{R}^N$ and $\varepsilon > 0$ be such that $d_H(X, S) < \frac{\varepsilon}{3}$. For any $\alpha > 0$,

(1) there exists a natural simplicial inclusion

$$\mathcal{R}_\alpha^\varepsilon(S) \hookrightarrow \mathcal{R}_\alpha(S).$$

(2) there exists a simplicial map

$$\xi : \mathcal{R}_\alpha^L(X) \longrightarrow \mathcal{R}_{3\alpha}^\varepsilon(S)$$

induced by the vertex map ξ that sends a vertex $x \in X$ to $s \in S$ such that $\|x - s\|_2 < \frac{\varepsilon}{3}$.

Proof.

(1) Follows immediately from the definition of the metric d_ε .

(2) As observed before in Lemma 3.1, the assumption $d_H(X, S) < \frac{\varepsilon}{3}$ ensures that there is a vertex map $\xi : X \rightarrow S$ such that for each $x \in X$ we have $\|x - \xi(x)\|_2 < \frac{\varepsilon}{3}$.

We show that the map extends to a simplicial map. Let $\sigma = \{x_0, x_1, \dots, x_k\}$ be a k -simplex of $\mathcal{R}_\alpha^L(X)$. Then, $d_L(x_i, x_j) \leq \alpha \forall i, j$. Now by Lemma 4.1, there exists a path joining $\xi(x_i)$ and $\xi(x_j)$ in G_ε , moreover $d_\varepsilon(\xi(x_i), \xi(x_j)) < 3\alpha$. So, $\xi(\sigma)$ is a simplex of $\mathcal{R}_{3\alpha}^\varepsilon(S)$. Hence, the vertex map extends to a simplicial map. $\square$

We now show that the fundamental group of the Vietoris-Rips complex on S under the metric d_ε is isomorphic to that of X . We tolerate the sloppiness from ignoring the basepoint.

Theorem 4.3 (Fundamental Group). *Let X be a connected geodesic subspace of $\mathbb{R}^N$ with a positive convexity radius ρ and a finite distortion δ . Let $S \subseteq \mathbb{R}^N$ and $\varepsilon > 0$ be such that*

$$d_H(X, S) < \frac{\varepsilon}{3} < \frac{\rho}{\delta(15\delta + 2)}.$$

Then, the fundamental groups of $\mathcal{R}_{5\varepsilon\delta}^\varepsilon(S)$ and X are isomorphic.

Proof. We derive the following chain of simplicial maps:

$$\mathcal{R}_\varepsilon(S) \xrightarrow{\phi_1} \mathcal{R}_{\frac{5\varepsilon\delta}{3}}^L(X) \xrightarrow{\phi_2} \mathcal{R}_{5\delta\varepsilon}^\varepsilon(S) \xleftarrow{\phi_3} \mathcal{R}_{5\delta\varepsilon}(S) \xrightarrow{\phi_4} \mathcal{R}_{\delta(15\delta+2)\varepsilon/3}^L(X).$$

The map ϕ_1 is the composition of the simplicial map $\mathcal{R}_\varepsilon(S) \longrightarrow \mathcal{R}_{\frac{5\varepsilon}{3}}(X)$ from Lemma 3.1 and the simplicial inclusion $\mathcal{R}_{\frac{5\varepsilon}{3}}(X) \hookrightarrow \mathcal{R}_{\frac{5\varepsilon\delta}{3}}^L(S)$ from Lemma 3.4, thanks to the assumption $d_H(S, X) < \frac{\varepsilon}{3}$. By a similar composition but at different scales, we get ϕ_4 . We also obtain ϕ_2 from Lemma 4.2 and ϕ_3 from Lemma 4.2.

We argue that ϕ_2 induces the desired isomorphism on the fundamental groups. By Theorem 3.5 and since $\varepsilon < \frac{\rho}{\delta(15\delta+2)}$, the simplicial map $\phi_4 \circ \phi_3 \circ \phi_2$ induces an isomorphism on all homotopy groups. Therefore, ϕ_2 induces an injective homomorphism on the homotopy groups, particularly the fundamental group of X .

We now show that the induced homomorphism is also surjective on the fundamental groups by showing that $\phi_2 \circ \phi_1$ induces a surjection. As observed Theorem 3.5, it suffices to show the surjection for the the natural inclusion $i : \mathcal{R}_\varepsilon(S) \hookrightarrow \mathcal{R}_{5\delta\varepsilon}^\varepsilon(S)$, because i is contiguous to $\phi_2 \circ \phi_1$.

We start with a loop η in $\mathcal{R}_{5\delta\varepsilon}^\varepsilon(S)$. We can assume that η is made up of edges (one-simplices) of $\mathcal{R}_{5\delta\varepsilon}^\varepsilon$. Let us consider an edge $\sigma = \{a, b\}$ in η , then we have $d_\varepsilon(a, b) \leq 5\delta\varepsilon$. By the definition of d_ε , there must be a sequence of points $a = x_0, x_1, \dots, x_k = b$ such that for each i , the segment $[x_i, x_{i+1}]$ is an edge of $\mathcal{R}_\varepsilon(S)$. Moreover, we observe for later that the diameter of the whole set $\{x_0, \dots, x_k\}$ in the d_ε metric is not greater than $5\varepsilon\delta$.

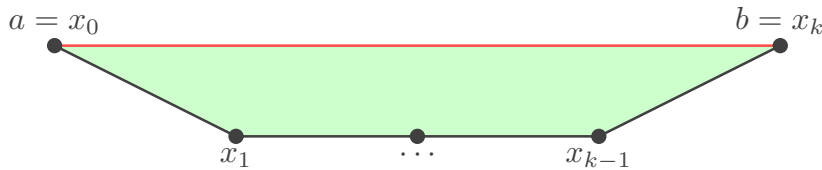


Fig. 4: The red one-simplex $[a, b]$ of $\mathcal{R}_{5\delta\varepsilon}^\varepsilon(S)$ is shown to be pushed off to a path $a = x_0, x_1, \dots, x_k = b$ in $\mathcal{R}_\varepsilon(S)$. All the nodes form a simplex (shown in green) in $\mathcal{R}_{5\delta\varepsilon}^\varepsilon(S)$.

Now, we define a loop η' in $\mathcal{R}_\varepsilon(S)$ by replacing each constituent edge $[a, b]$ of η by the path joining the points in the sequence $a = x_0, x_1, \dots, x_k = b$ consecutively, as shown in Figure 4. We note that η' is indeed a loop in $\mathcal{R}_\varepsilon(S)$. We now show that $(\phi_2 \circ \phi_1)(\eta')$ is homotopic to the loop η in $\mathcal{R}_{5\delta\varepsilon}^\varepsilon(S)$. As observed before, $\{a = x_0, \dots, x_k = b\}$ is a simplex

of $\mathcal{R}_{5\delta\varepsilon}^\varepsilon(S)$. We can then use a (piece-wise) straight line homotopy that maps each edge $[a, b]$ of η to the segment $[a = x_0, x_1] \cup \dots \cup [x_{k-1}, x_k = b]$ of η' . Hence, $[\eta']$ is, in fact, a preimage of $[\eta]$. This shows, in turn, that ϕ_2 induces a surjective homomorphism on π_1 . This completes the proof. $\square$

4.2. Reconstruction of Embedded Graphs

We finally turn our attention to the geometric reconstruction of embedded graphs. We start with the formal definition of an embedded graph.

Definition 4.4 (Embedded Metric Graph). An embedded metric graph G is a subset of $\mathbb{R}^N$ that is homeomorphic to a one-dimensional simplicial complex, where the induced length metric d_L is the shortest path distance on G . For simplicity of exposition, we call such G *embedded graphs*.

We note that if G has finitely many vertices and b is the length of its shortest simple cycle, then the convexity radius ρ is $\frac{b}{4}$. In this paper, we always assume that G has finitely many vertices. We now consider the *shadow* of the Vietoris-Rips complex $\mathcal{R}_\bullet^\varepsilon(S)$, which is defined in Section 4.1.

Definition 4.5 (Shadow of a Complex). Let A be a subset of $\mathbb{R}^N$, and let $\mathcal{K}$ be an abstract simplicial complex whose vertex set is A . For each simplex $\sigma = \{x_1, x_2, \dots, x_k\}$ in $\mathcal{K}$, we define its shadow, denoted $\mathbf{Sh}(\sigma)$, as the convex-hull of the Euclidean point set $\{x_1, x_2, \dots, x_k\}$. The shadow of $\mathcal{K}$ in $\mathbb{R}^N$, denoted by $\mathbf{Sh}(\mathcal{K})$, is the union of the shadows of all its simplices, i.e., $\mathbf{Sh}(\mathcal{K}) := \bigcup_{\sigma \in \mathcal{K}} \mathbf{Sh}(\sigma)$.

We, therefore, have the following natural projection map $p : |\mathcal{K}| \rightarrow \mathbf{Sh}(\mathcal{K})$. In general, $\mathbf{Sh}(\mathcal{K})$ may not have the same homotopy type as $|\mathcal{K}|$. However, as proved in [31], the fundamental group of the Vietoris-Rips complex of a planar point set is isomorphic to the fundamental group of its shadow. In [16], the authors further the understanding of shadows of Euclidean Rips complexes. In the case of planar subsets and $\mathcal{K} = \mathcal{R}_\bullet^\varepsilon(S)$, we prove a similar result now.

Lemma 4.6 (Shadow). Let X be a connected planar subspace with a positive convexity radius ρ and a finite distortion δ . Given $S \subseteq \mathbb{R}^2$ finite and $\varepsilon > 0$ such that

$$d_H(X, S) < \frac{\varepsilon}{3} < \frac{\rho}{\delta(15\delta + 2)}.$$

Then, the shadow projection $p : |\mathcal{R}_{5\varepsilon\delta}^\varepsilon(S)| \longrightarrow \mathbf{Sh}(\mathcal{R}_{5\varepsilon\delta}^\varepsilon(S))$ induces isomorphism on the fundamental groups.

Proof. From Theorem 4.3, we have the following chain of simplicial maps:

$$\mathcal{R}_\varepsilon(S) \xrightarrow{\phi_1} \mathcal{R}_{5\varepsilon\delta/3}^L(X) \xrightarrow{\phi_2} \mathcal{R}_{5\delta\varepsilon}^\varepsilon(S) \xleftarrow{\phi_3} \mathcal{R}_{5\delta\varepsilon}(S) \xrightarrow{\phi_4} \mathcal{R}_{\delta(15\delta+2)\varepsilon/3}^L(X).$$

We have shown that ϕ_2 induces an isomorphism on π_1 . As we have also noted that $(\phi_4 \circ \phi_3 \circ \phi_2)$ induces an isomorphism on all homotopy groups. So, we conclude first that ϕ_3 induces an injective homomorphism on π_1 .

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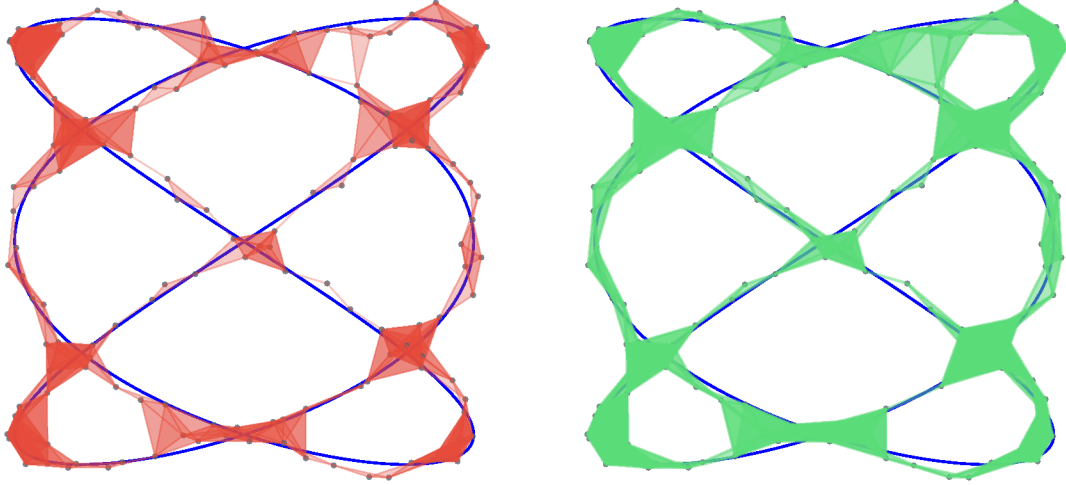


Fig. 5: We implement Algorithm 1 on a Lissajous G with $\beta_1(G) = 8$. On the left, the Euclidean Vietoris-Rips complex $\mathcal{R}_\varepsilon(S)$ (in red) on an ε -dense sample S of 150 points fails to capture the homotopy type, as its $\beta_1 = 9$. On the right, the shadow $\tilde{G}$ (green) of $\mathcal{R}_{5\delta\varepsilon}^\varepsilon(S)$ is shown to correctly reconstruct G . The pictures were generated using the shape reconstruction library available on www.smaajhi.com/shape-reconstruction.

625 Now, we consider the following commutative diagram:

$$\begin{array}{ccccc}
 \mathcal{R}_\varepsilon(S) & \xleftarrow{i} & \mathcal{R}_{5\delta\varepsilon}^\varepsilon(S) & \xleftarrow{\phi_3} & \mathcal{R}_{5\delta\varepsilon}(S) \\
 & & \downarrow p & & \downarrow \tilde{p} \\
 & & \text{Sh}(\mathcal{R}_{5\delta\varepsilon}^\varepsilon(S)) & \xleftarrow{j_2} & \text{Sh}(\mathcal{R}_{5\delta\varepsilon}(S))
 \end{array} \tag{8}$$

627 where i is contiguous to the composition $(\phi_2 \circ \phi_1)$, and $p, \tilde{p}$ are the natural (shadow) pro-
 628 jections.

629 We show that the induced map p_* is an isomorphism on the fundamental groups. From
 630 the commutativity of the diagram (8), we note that p_* is an injection on π_1 , since ϕ_{3*}
 631 is injective and $\tilde{p}_*$ is also injective on π_1 by [31]. For surjectivity, we follow the same lifting
 632 argument presented in [31]. $\square$

633 As a consequence of Lemma 4.6, we finally present our main geometric reconstruction
 634 result.

635 **Theorem 4.7 (Geometric Reconstruction of Planar Subspaces).** *Let X be a con-*
 636 *connected geodesic subspace of $\mathbb{R}^2$ with a positive convexity radius ρ and a finite distortion δ ,*
 637 *which has the homotopy type of a finite simplicial complex. Let $S \subseteq \mathbb{R}^2$ be finite, and $\varepsilon > 0$*

638 be such that

$$639 \quad d_H(X, S) < \frac{\varepsilon}{3} < \frac{\rho}{\delta(15\delta + 2)}. \quad (9)$$

640 Then, the shadow complex $\tilde{X} = \mathbf{Sh}(\mathcal{R}_{5\varepsilon\delta}^\varepsilon(S))$ of $\mathcal{R}_{5\varepsilon\delta}^\varepsilon(S)$ has the homotopy type of X .
641 Moreover,

$$642 \quad d_H(X, \tilde{X}) < \left(5\delta + \frac{1}{3}\right)\varepsilon. \quad (10)$$

643 **Proof.** By Lemma 4.6, the shadow $\tilde{X} = \mathbf{Sh}(\mathcal{R}_{5\varepsilon\delta}^\varepsilon(S))$ and X have isomorphic fundamental
644 groups, via the map p of diagram (8). Note that, by assumption, both $\mathbf{Sh}(\mathcal{R}_{5\varepsilon\delta}^\varepsilon(S))$ and X
645 have a homotopy type of a finite wedge of circles and therefore trivial higher homotopy
646 groups. By the Whitehead's theorem [32], applied to the map p , we conclude that p is a
647 homotopy equivalence.

648 For statement (10), we note that for any finite vertex set $\sigma \subseteq S$ with $\text{diam}(\sigma) < 5\delta\varepsilon$ we
649 have $\sigma \subseteq \mathbf{Sh}(\sigma)$ and $d_H(\sigma, \mathbf{Sh}(\sigma)) \leq \text{diam}(\sigma)$. As a consequence, $d_H(\tilde{X}, S) \leq 5\delta\varepsilon$. By the
650 triangle inequality, we conclude the result. $\square$

651 **Corollary 4.8 (Geometric Reconstruction of Embedded Graphs).** Let G be a finite,
652 connected embedded graph in $\mathbb{R}^2$. Let b be the length of the shortest simple cycle of G , and
653 let δ be its distortion. Let $S \subseteq \mathbb{R}^2$ be finite and $\varepsilon > 0$ be such that

$$654 \quad d_H(G, S) < \frac{\varepsilon}{3} < \frac{b}{4\delta(15\delta + 2)}.$$

655 Then, the shadow of $\tilde{G} = \mathbf{Sh}(\mathcal{R}_{5\varepsilon\delta}^\varepsilon(S))$ has the same homotopy type as G and (10) holds
656 for $X = G$ and $\tilde{X} = \tilde{G}$.

657 **Proof.** It suffices to note that the convexity radius of G is $\frac{b}{4}$ and apply Theorem 4.7.
658 $\square$

659 Based on Corollary 4.8, we devise Algorithm 1 for the geometric reconstruction of (pla-
660 nar) embedded graphs. For a demonstration, see Figure 5.

661 5. Discussion

662 In this paper, we successfully reconstruct homology/homotopy groups of general geodesic
663 spaces. We also reconstruct the geometry of embedded graphs. Currently, the output of
664 such geometric reconstruction is a thick region around the hidden graph; see Figure 5. One
665 can consider a post-processing step to prune the output shadow $\tilde{G}$ in order to output an
666 embedded graph that is isomorphic to the hidden graph G . A natural extension of our work is
667 to consider the geometric reconstruction of higher-dimensional simplicial complexes. Unlike
668 the graphs, such a space may have non-trivial higher homotopy groups. The reconstruction
669 result remains, therefore, an object of future work.

Algorithm 1 Graph Reconstruction Algorithm**Require:** Finite sample $S \subseteq \mathbb{R}^2$, $\varepsilon > 0$, δ , and b **Ensure:** $d_H(\tilde{G}, S) < \frac{\varepsilon}{3} < \frac{b}{4\delta(15\delta+2)}$

- 1: Initialize $\tilde{G} \leftarrow \emptyset$
- 2: Compute the one-skeleton of $\mathcal{R}_\varepsilon(S)$
- 3: Compute (S, d_ε)
- 4: **for all** $\{a, b, c\} \in S$ **do**
- 5: **if** $d_\varepsilon(a, b) \leq 5\delta\varepsilon$ **and** $d_\varepsilon(b, c) \leq 5\delta\varepsilon$ **and** $d_\varepsilon(c, a) \leq 5\delta\varepsilon$ **then**
- 6: $\tilde{G} \leftarrow \tilde{G} \cup \text{CONVEX-HULL}(\{a, b, c\})$
- 7: **end if**
- 8: **end for**
- 9: **return** $\tilde{G}$

On the other hand, we also note that both approaches are not performing well when we deform X , e.g., by “pinching” a pair of points in X , i.e., deforming X to bring these points ϵ -close in the extrinsic Euclidean distance but with bounded intrinsic distance. Creating such an ϵ -pinch generally results in a small **wfs** as well as large distortion of the resulting submanifold.

Based on these considerations, we conjecture that there should be a stability result within an appropriate class of geodesics subspaces of $\mathbb{R}^N$, saying that a fixed sample S satisfying assumptions of Theorem 3.5 and Theorem 3.10, statements (3.5) and (5) should be valid not only for a given X but also for any ϵ -close perturbation within the class. We will address this claim in the forthcoming work.

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