HODGE AND TEICHMÜLLER

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Abstract. We consider the derivative $D\pi$ of the projection π from a stratum of Abelian or quadratic differentials to Teichmüller space. A closed one-form η determines a relative cohomology class $[\eta]_\Sigma$, which is a tangent vector to the stratum. We give an integral formula for the pairing of $D\pi([\eta]_\Sigma)$ with a cotangent vector to Teichmüller space (a quadratic differential).

We derive from this a comparison between Hodge and Teichmüller norms, which has been used in the work of Arana-Herrera on effective dynamics of mapping class groups, and which may clarify the relationship between dynamical and geometric hyperbolicity results in Teichmüller theory.

1. Introduction

The derivative of the projection. Each stratum $\mathcal{H}(\kappa)$ of genus g Abelian differentials comes equipped with a projection map $\pi\colon \mathcal{H}(\kappa) \to \mathcal{M}_g$ to the moduli space of Riemann surfaces, defined by $\pi(X,\omega) = X$.

Given $(X,\omega) \in \mathcal{H}(\kappa)$, we let $\Sigma = \{z_1, \ldots, z_s\}$ denote the set of zeros of ω. Any complex valued closed differential one-form η on X determines a relative cohomology class $[\eta]_\Sigma \in H^1(X,\Sigma,\mathbb{C})$. Since $H^1(X,\Sigma,\mathbb{C})$ is the tangent space to $\mathcal{H}(\kappa)$, we think of $[\eta]_\Sigma$ as a tangent vector to the stratum.

Using a point-wise decomposition, any such η can be written uniquely as $\eta = \eta^{1,0} + \eta^{0,1}$, where $\eta^{1,0}$ and $\overline{\eta^{0,1}}$ are of type $(1,0)$ and need not be closed. The cotangent space to X is the space $Q(X)$ of quadratic differentials on X, so a tangent vector to $\mathcal{M}_g$ can be thought of as a linear functional on $Q(X)$.

Theorem 1.1. *Let $\eta = \eta^{1,0} + \eta^{0,1}$ be closed as above. Let $\{\gamma_1, \ldots, \gamma_s\}$ denote small disjoint positively oriented loops around the zeros of ω, and let X' be the complement in X of the discs that they bound. Then the pairing of $(D\pi)[\eta]_\Sigma$ with a quadratic differential q is equal to*

$$\int_{X'} q\frac{\eta^{0,1}}{\omega} + \frac{1}{2i}\sum_j \int_{\gamma_j} \frac{F_j}{\omega} q, \tag{1.0.1}$$

where $F_j(z) = \int_{z_j}^{z} \eta$ *is defined by integrating along paths in the disc containing* z_j*.*

See the beginning of Section 2 for how the tensors in this formula should be interpreted. The fact that equation 1.0.1 does not depend on the choice of loops γ_j follows from Stokes' Theorem, as in the proof of Lemma 2.1.

With the help of Lemma 2.1 below (which was suggested to us by McMullen), Theorem 1.1 follows readily from [McM13, Corollary 3.2] by Poincare duality. In Section 2 we give a short, self contained proof of Theorem 1.1 that emphasizes the role of Beltrami coefficients.

In Corollary 2.3, we show that if η is harmonic, one can let the size of the loops go to zero to obtain a formula in terms of residues. In Remark 2.4, we note that Theorem 1.1 holds also for the projection to $\mathcal{M}_{g,s}$ obtained by marking all the zeros of ω. In Section 3.1, we explain how to apply Theorem 1.1 to strata of quadratic differentials.

The principal stratum of quadratic differentials. Any quadratic differential (X, q) admits a canonical double cover $\rho_q \colon \hat{X} \to X$ on which the pullback of q becomes the square of an Abelian differential ω. The Deck group of this cover is an involution τ, and $\tau^*(\omega) = -\omega$. The cover is sometimes called the holonomy double cover, and ω is sometimes called the square-root of q.

Let Σ continue to denote the set of zeros of ω. Denote by $H^1_{-1}(\hat{X}, \Sigma, \mathbb{C})$ and $H^1_{-1}(\hat{X}, \mathbb{C})$ the -1 eigenspace for the action of τ on $H^1(\hat{X}, \Sigma, \mathbb{C})$ and $H^1(\hat{X}, \mathbb{C})$ respectively.

The tangent space to the stratum of (X, q) at the point (X, q) is $H^1_{-1}(\hat{X}, \Sigma, \mathbb{C})$. If q has no even order zeros, then the natural map $H^1_{-1}(\hat{X}, \Sigma, \mathbb{C}) \to H^1_{-1}(\hat{X}, \mathbb{C})$ is an isomorphism, and so $H^1_{-1}(\hat{X}, \mathbb{C})$ can also be viewed as the tangent space. Every element of absolute cohomology can be represented uniquely by a harmonic one-form η, so an arbitrary element of the tangent space uniquely corresponds to an anti-invariant form η on $\hat{X}$ with $\eta^{1,0} \in H^{1,0}(\hat{X})$ and $\eta^{0,1} \in H^{0,1}(\hat{X})$. Here $H^{1,0}(\hat{X})$ denotes the space of Abelian differentials on $\hat{X}$ and $H^{0,1}(\hat{X})$ denotes its complex conjugate.

Since ω and $\eta^{0,1}$ are both in the -1 eigenspace of τ, the Beltrami differential $\eta^{0,1}/\omega$ is τ invariant and hence is the pull back of a Beltrami differential on X, which we will continue to denote $\eta^{0,1}/\omega$.

When (X, q) is in the principal stratum, Corollary 2.3 further simplifies to give the following, where π denotes the projection from the principal stratum of quadratic differentials to the moduli space of Riemann surfaces.

Corollary 1.2. *If (X, q) is in the principal stratum, and η is a harmonic anti-invariant form on $\hat{X}$, then the pairing of $(D\pi)[\eta]_\Sigma$ with a quadratic differential $q' \in Q(X)$ is equal to*

$$\frac{1}{2}\int_X q' \frac{\eta^{0,1}}{\omega}.$$

We warn that the Beltrami differential $\frac{\eta^{0,1}}{\omega}$ is not bounded and hence does not define a tangent vector to $\mathcal{M}_g$ in the usual way. The integrand $q'\frac{\eta^{0,1}}{\omega}$ is integrable (in fact its pullback to $\hat{X}$ is continuous), so $\frac{\eta^{0,1}}{\omega}$ defines a functional on the cotangent space $Q(X)$ via integration, and hence defines a tangent vector indirectly in this way.

Remark 1.3. Corollary 1.2 in particular witnesses that the tangent space to the fiber of π is $H^{1,0}_{-1}(\hat{X})$. In fact, if $q' \in Q(X)$, it is not hard to see that $\rho_q^*(q')/(2\omega)$ is contained in $H^{1,0}_{-1}(\hat{X})$ and moreover is the derivative of the path $(X, q + tq')$ at $t = 0$ [DH75].

Keeping in mind that the kernel of $D\pi$ is $H^{1,0}_{-1}(\hat{X})$, we consider $\eta \in H^{0,1}_{-1}(\hat{X})$, and compare the Hodge norm of η and the Teichmüller norm of $D\pi(\eta)$.

Theorem 1.4. *Assuming ω has area 1, we have*

$$\|\eta\|_{\text{Hodge}} \leq \|D\pi(\eta)\|_{\text{Teich}} \leq \frac{4}{r}\|\eta\|_{\text{Hodge}},$$

for any $\eta \in H^{0,1}_{-1}(\hat{X})$, where $2r$ is the length of the shortest saddle connection on $(\hat{X}, \omega)$.

This is somewhat reminiscent of comparisons between the Teichmüller and Weil-Petersson norms in [BMW12, Lemma 5.4]. The lower bound in Theorem 1.4 is sharp when $\eta = \overline{\omega}$. In Section 3.3, we show that the upper bound in Theorem 1.4 is sharp up to constants.

Other perspectives and previous results. One can of course obtain formulas for $D\pi$ by, for example, triangulating the surface and considering Beltrami differentials of maps that are affine on each triangle; or picking an open cover and using Cech cohomology, as in [HM79]. McMullen gave a formula in terms of complex twists [McM13]. Derivatives of some especially important deformations have been given in [Wol18].

Motivation and significance. We had two specific motivations for writing this paper, both having to do with Theorem 1.4.

(1) Arana-Herrera has used Theorem 1.4 in his proof of an effective version of the lattice point counting problem of Athreya-Bufetov-Eskin-Mirzakhani [AH20, ABEM12]. This in turn is used in his subsequent work on the effective dynamics of the mapping class group [AH21a, AH21b].
(2) Since the work of Forni, the hyperbolicity of the Teichmüller geodesic flow has been studied using the Hodge norm [For02], see also [FM14] for a survey, [EMR19, Fra20] for more recent developments, and [Kon97] for the introduction of the Hodge norm to Teichmüller dynamics by Kontsevich. On the other hand, geometric hyperbolicity results are expressed in terms of Teichmüller distance and are proven using very different techniques [Raf14]. Theorem 1.4 opens the door to links between these dynamical and geometric results, including the possibility of (re)proving geometric hyperbolicity results using dynamical hyperbolicity.

It is also conceivable that Theorem 1.1 could be useful in the study of $GL(2,\mathbb{R})$ orbit closures of translation surfaces, using the restrictions on the period mapping exploited in [MW18].

Corollary 1.2 can be used to re-derive the fact that the canonical symplectic form on the principal stratum, obtained as for the cotangent bundle to any manifold, corresponds to the usual symplectic form on $H^1_{-1}(\hat{X},\mathbb{C})$ [BKN17]. This is notable in part because Teichmüller geodesic flow is easily seen to be Hamiltonian using the later symplectic form [Mas95].

A tangential remark. The following requires only Remark 1.3, but illustrates another potential connection between this paper and other recent work.

Proposition 1.5. *Let $\mathcal{M}$ be a $GL(2,\mathbb{R})$ orbit closure in the principal stratum of quadratic differentials over $\mathcal{M}_{g,n}$, and let π denote the projection to $\mathcal{M}_{g,n}$. Then $\overline{\pi(\mathcal{M})}$ is a totally geodesic subvariety of $\mathcal{M}_{g,n}$.*

Proof. By [EM18, EMM15] each orbit closure is a properly immersed smooth suborbifold, and by [Fil16a, Fil16b] it is moreover an algebraic variety.

When $\mathcal{M}$ is in the principal stratum, its tangent space at $(X,q)\in\mathcal{M}$ is naturally identified with a subspace of $H^1_{-1}(\hat{X},\mathbb{C})$ defined over $\mathbb{R}$. By [Fil16a], this subspace is the direct sum of its intersections with $H^{1,0}_{-1}(\hat{X})$ and $H^{0,1}_{-1}(\hat{X})$. Since these two intersections are complex conjugate, they have the same dimension, which witnesses the fact that $\mathcal{M}$ must have even complex dimension. Since $H^{1,0}_{-1}(\hat{X})$ is the kernel of $D\pi$, we see that the kernel of $D\pi$ restricted to $\mathcal{M}$ has dimension half that of $\mathcal{M}$ at every point.

We conclude that the variety $\pi(\mathcal{M})$ has dimension half that of $\mathcal{M}$. It follows from previous observations that its closure is totally geodesic, as is implicit in [MMW17, EMMW20, Wri20] and explicit in [Gou21, Proposition 1.3]. □

The only known non-trivial primitive totally geodesic subvarieties of dimension greater than 1 have dimension 2 and arise from orbit closures in the principal strata of $\mathcal{M}_{1,3}$, $\mathcal{M}_{1,4}$ and $\mathcal{M}_{2,1}$ [MMW17, EMMW20]. See also [Wri20] for related results, and [Gou21] for a survey.

Before [MMW17], it was not expected that any orbit closures would give rise to totally geodesic subvarieties, and [MMW17, EMMW20] used a detailed understanding to conclude that three orbit closures have sufficiently small projections to give totally geodesic surfaces. So it is perhaps surprising that Proposition 1.5 indicates that, at least for orbit closures in the principal stratum, there is automatically an associated totally geodesic subvariety.

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The title of this paper is inspired by [CJY94].

2. The derivative formula

Before we give the proofs of Theorem 1.1 and Corollary 2.3, we must clarify our conventions. In the integral over X' in (1.0.1), the integrand, as a product of a $(2,0)$, a $(-1,0)$, and a $(0,1)$ form, is a $(1,1)$ form, and hence can be written locally on $U \subset \mathbb{C}$ as $f\,dz\,\overline{dz}$. We then define, as a matter of convention,

$$\int_U f\,dz\,\overline{dz} = \int_U f\,dx\,dy; \tag{2.0.1}$$

we observe that the latter integral will be independent of the choice of coordinate and that (2.0.1) can be used to define a global integral in the usual way. This is the convention used in defining the Teichmüller pairing of a quadratic differential q and a Beltrami differential μ as

$\int_X q\mu$. Since $\overline{dz} \wedge dz = 2i\, dx\, dy$,

$$\int_U f\, dz\, \overline{dz} = \frac{1}{2i} \int_U f\, \overline{dz} \wedge dz. \tag{2.0.2}$$

We continue to use the notation of Theorem 1.1: $X' \subset X$ is the complement of the discs bounded by loops γ_j about the zeros z_j, and $F_j(z) = \int_{z_j}^z \eta$. We start the proof of Theorem 1.1 with the following observation.

Lemma 2.1. *If $[\eta]_\Sigma = 0$ and q is a holomorphic quadratic differential on $X \setminus \Sigma$, then*

$$\int_{X'} q \frac{\eta^{0,1}}{\omega} + \frac{1}{2i} \sum \int_{\gamma_j} \frac{F_j}{\omega} q = 0.$$

Thus, the derivative formula (1.0.1) only depends on the cohomology class of η.

Proof. $[\eta]_\Sigma = 0$ implies that $\eta = df$ for a function f which is zero on Σ. We note

$$d\left(\frac{f}{\omega} q\right) = \overline{\partial}\left(\frac{f}{\omega} q\right) = \eta^{0,1} \wedge \frac{q}{\omega},$$

and hence, by (2.0.2) and Stokes' Theorem,

$$2i \int_{X'} q \frac{\eta^{0,1}}{\omega} = \int_{X'} \eta^{0,1} \wedge \frac{q}{\omega} = -\sum_j \int_{\gamma_j} \frac{f}{\omega} q.$$

Since both f and F_j are zero at z_j and have exterior derivative η, we see that F_j is the restriction of f, so this gives the result. $\square$

Next, consider the Beltrami differential

$$\mu_t = \frac{t\eta^{0,1}}{\omega + t\eta^{1,0}}$$

on X. Assuming η is compactly supported on $X - \Sigma$ and t is sufficiently small, then $\|\mu_t\|_\infty < 1$. In this case, let X_t denote X with the complex structure for which the identity map $X \to X_t$ has Beltrami differential μ_t.

Lemma 2.2. *If η is compactly supported and t is sufficiently small, then the closed one-form $\omega + t\eta$ is holomorphic on X_t.*

Proof. Write $\eta^{1,0} = \eta^{1,0}(z)\, dz$ etc. Since $\omega + t\eta$ is proportional to

$$dz + \frac{t\eta^{0,1}(z)}{\omega(z) + t\eta^{1,0}(z)} \overline{dz} = dz + \mu_t\, \overline{dz},$$

we see that $\omega + t\eta$ is $(1,0)$ form on X_t. (In terms of a conformal chart for X, the derivative at p of a conformal chart for X_t is of the form $z \mapsto A(z+\mu_t(p)\overline{z})$, meaning that the form dz pulls back to $A(dz+\mu_t(p)\overline{dz})$.) Since closed $(1,0)$ forms are holomorphic, this gives the result. $\square$

Proof of Theorem 1.1. By Lemma 2.1, we can assume that η is compactly supported on X'. Lemma 2.2 gives that $(X_t, \omega + t\eta)$ is a path in the stratum whose image in period coordinates is $[\omega + t\eta]_\Sigma$. Since $\frac{d\mu_t}{dt}\big|_{t=0} = \eta^{0,1}/\omega$, the derivative of the family X_t is the Beltrami differential $\eta^{0,1}/\omega$. Since η is is supported in X', the pairing of this Beltrami differential with q coincides with the formula given in Theorem 1.1. $\square$

We now observe the the formula also simplifies if η is harmonic.

Corollary 2.3. *Let η be harmonic, i.e. $\eta^{1,0} \in H^{1,0}(X)$ and $\eta^{0,1} \in H^{0,1}(X)$. Then the pairing of q and $[\eta]_\Sigma$ is given by*

$$\int_X q\frac{\eta^{0,1}}{\omega} + \pi \sum \mathrm{res}_{z_j}\left(\frac{\int_{z_j}^z \eta^{1,0}}{\omega} q\right),$$

where the first term is understood as a Cauchy principal value using flat coordinates provided by ω.

The Cauchy principal value is the limit as $\varepsilon \to 0$ of the integral over the subset of X whose ω-distance to a zero of ω is at least ε.

Proof. It suffices to check that the limit of

$$\int_{\gamma_j} \frac{\int_{z_j}^z \eta^{0,1}}{\omega} q \tag{2.0.3}$$

as the size of the loop γ_j goes to zero is 0. We can pick local coordinates in which $\omega = z^k dz$, and pick γ_j of the form $re^{i\theta}$. We expand $q = (\sum c_\ell z^\ell)dz^2$, and we expand $\int_{z_j}^z \eta^{0,1} = \sum_{m>0} d_m \overline{z}^m$, noting that (because of the integral) $m = 0$ is not included in the index of this last sum. For each value of ℓ and m we obtain a term proportional to $\int_{\gamma_j} z^{-k+\ell}\overline{z}^m dz$, which, using the expression $z = re^{i\theta}$ for γ_j, is equal to

$$\int_0^{2\pi} (re^{i\theta})^{-k+\ell}\overline{(re^{i\theta})}^m d(re^{i\theta}) = ir^{-k+\ell+1+m}\int_0^{2\pi} e^{i\theta(-k+\ell+1-m)}d\theta.$$

This is non-zero only when $-k+\ell+1-m = 0$, in which case $r^{-k+\ell+1+m} = r^{2m}$. Since $m > 0$, this integral goes to zero as $r \to 0$. As the power series expansion for the integrand of (2.0.3) is uniformly and absolutely convergent on the closed disk containing γ_j, it follows that (2.0.3) goes to 0. $\square$

Remark 2.4. All the results of this section also apply to the projection to $\mathcal{M}_{g,s}$, with $s = |\Sigma|$, obtained by marking all the zeros of ω. We can also allow Σ to be a finite set which properly contains the zeros of ω, in which case we think of a point of Σ at which ω does not vanish as a zero of order zero.

3. The principal stratum

3.1. **Arbitrary strata of quadratic differentials.** Let $\mathcal{Q}(\kappa)$ denote a stratum of the bundle of quadratic differentials over $\mathcal{M}_{g,n}$. Given $(X, q) \in \mathcal{Q}(\kappa)$, we continue to let $\rho_q \colon \hat{X} \to X$ denote the double cover on which the pullback of q becomes the square of an Abelian differential ω. We continue to specify a tangent vector to the strata by giving a closed anti-invariant one-form η on $\hat{X}$, and we consider also a cotangent vector $q' \in QD(X)$ to $\mathcal{M}_{g,n}$.

As we now explain, if $\pi \colon \mathcal{Q}(\kappa) \to \mathcal{M}_{g,n}$ continues to denote the projection, then the pairing of the cotangent vector $D\pi([\eta]_\Sigma)$ with the cotangent vector q' is

$$\frac{1}{2}\int_{\hat{X}'} \rho_q^*(q')\frac{\eta^{0,1}}{\omega} + \frac{1}{2}\sum_i \int_{\gamma_i} \frac{F_i}{\omega}\rho_q^*(q'),$$

where $F_i(z) = \int_{z_i}^z \eta$ is defined as before. This will follow from Theorem 1.1 once we clarify the bookkeeping of double covers.

Let U be a small open subset of $H^1_{-1}(\hat{X}, \Sigma, \mathbb{C})$ which contains $[\omega]_\Sigma$ and provides a local coordinate for $\mathcal{Q}(\kappa)$ at (X, q). We restrict π to give a map from U to $\mathcal{M}_{g,n}$.

Let $\hat{g}$ denote the genus of $\hat{X}$. Assuming that U is simply connected, we can lift π to a map $\hat{\pi} \colon U \to \mathcal{T}_{\hat{g}}$. The mapping class τ of the holonomy involution is constant on U, and the image of $\hat{\pi}$ is contained in the subset $\mathcal{G} \subset \mathcal{T}_{\hat{g}}$ on which the mapping class τ contains a holomorphic involution. The quotient by this involution gives a map $\rho \colon \mathcal{G} \to \mathcal{M}_{g,n}$. We consider $\hat{\pi}$ to be a map to $\mathcal{G}$, and we have $\pi = \rho \circ \hat{\pi}$.

Given any differentiable map $\pi = \rho \circ \hat{\pi}$, and any cotangent vector q' in the codomain and tangent vector η in the domain, we have

$$\langle D\pi(\eta), q'\rangle = \langle D\hat{\pi}(\eta), \rho^*(q')\rangle,$$

where $\langle \cdot, \cdot \rangle$ is the pairing between tangent and cotangent vectors and the coderivative ρ^* maps between cotangent spaces.

In our situation, the cotangent space to $\mathcal{G}$ is the space of quadratic differentials which are invariant by the involution, and if q' is a quadratic differential on X then

$$\rho^* q' = \frac{1}{2} \rho_q^* q'.$$

Let $\iota \colon \mathcal{G} \to \mathcal{T}_{\hat{g}}$ denote the inclusion. Any invariant quadratic differential can be considered both as a cotangent vector to $\mathcal{G}$ and to $\mathcal{T}_{\hat{g}}$, and we have $\langle D\iota(v), q'' \rangle = \langle v, q'' \rangle$ for any tangent vector v to $\mathcal{G}$ and any invariant quadratic differential q'' on $\hat{X}$. Thus we get

$$\langle D\pi(\eta), q' \rangle = \langle D(\iota \circ \hat{\pi})(\eta), \rho^*(q') \rangle.$$

This gives the desired formula, since $\iota \circ \hat{\pi}$ is the projection from the stratum $\mathcal{H}$ of $(\hat{X}, \omega)$ to $\mathcal{T}_{\hat{g}}$, whose derivative is given by Theorem 1.1.

Applying this to the principal stratum, we get the following.

Proof of Corollary 1.2. First note that, since q has simple zeros, ω has double zeros. Since each zero of ω is a ramification point of ρ_q, every pull back $\rho_q^*(q')$ has at least a double zero at every zero of ω, so we see that $\rho_q^*(q')/\omega$ is holomorphic at the zeros of ω. This shows that all the residue terms in Corollary 2.3 are zero, giving the result. (The fact that $\rho_q^*(q')/\omega$ is holomorphic gives that pull back of the integrand is continuous on $\hat{X}$, so a Cauchy principal value is not required.) □

3.2. **Norm comparisons.** Consider a tangent vector to a principal stratum of the form $\eta = \overline{\beta}$, where $\beta \in H^{1,0}_{-1}(\hat{X})$. Keeping in mind Corollary 1.2, the Beltrami differential $\mu = \overline{\beta}/\omega$ can be viewed as the tangent vector $D\pi(\eta)$ via pairing with cotangent vectors. We now turn to comparisons between the Hodge norm

$$\|\beta\|_{\text{Hodge}} = \sqrt{\int_{\hat{X}} |\beta|^2}$$

and the Teichmuller norm

$$\|[\mu]\|_{\text{Teich}} = \sup_{\|q'\|=1} \int_X q'\mu = \sup_{\|q'\|=1} \int_{\hat{X}} \frac{1}{2}\rho_q^*(q') \frac{\overline{\beta}}{\omega}.$$

Theorem 3.1. *We have*

$$\|[\mu]\|_{\text{Teich}} \geq \frac{\|\beta\|_{\text{Hodge}}}{\|\omega\|_{\text{Hodge}}} \tag{3.2.1}$$

The reader should keep in mind that the normalization $\|q\| = 1$ corresponds to $\|\omega\|_{\text{Hodge}} = \sqrt{2}$. From now on we will omit the subscript "Hodge", since the only norm we will consider for Abelian differentials is the Hodge norm. On the other hand, $\|q\|$ for a quadratic differential is the L^1 norm.

Proof. Let $q' = (\rho_q)_*(\omega\beta)$, where $(\rho_q)_*$ is defined by summing over fibers, so $\omega\beta = \frac{1}{2}\rho_q^* q'$. Then $\|q'\| = \|\omega\beta\| \leq \|\omega\| \, \|\beta\|$ by the Cauchy-Schwartz inequality. On the other hand,

$$\int_X q'\mu = \int_{\hat{X}} \omega\beta\frac{\overline{\beta}}{\omega} = \int_{\hat{X}} \beta\overline{\beta} = \|\beta\|^2 .$$

Therefore

$$\|[\mu]\|_{\mathrm{Teich}} \geq \frac{\|\beta\|^2}{\|q'\|} \geq \frac{\|\beta\|^2}{\|\beta\| \, \|\omega\|} = \frac{\|\beta\|}{\|\omega\|}. \tag{3.2.2}$$ □

For the other direction it will be helpful to observe the following, for any holomorphic function $f\colon D_r \to \mathbb{C}$, where D_r denotes the disk of radius r:

$$|f(0)| \leq \frac{\int_{D_r} |f(z)|}{\pi r^2} \tag{3.2.3}$$

$$\leq \sqrt{\frac{\int_{D_r} |f(z)|^2}{\pi r^2}} \tag{3.2.4}$$

$$= \frac{\sqrt{\int_{D_r} |f(z)|^2}}{\sqrt{\pi} r}, \tag{3.2.5}$$

where the second inequality follows from Cauchy-Schwarz. In particular, for any $z \in \hat{X}$ that isn't a root of ω, we have

$$\left|\frac{\beta}{\omega}(z)\right| \leq \frac{\|\beta\|}{\sqrt{\pi} r_\omega(z)} \leq \frac{\|\beta\|}{r_\omega(z)}. \tag{3.2.6}$$

where $r_\omega(z)$ is the radius of the largest (open) embedded Euclidean ω-disk around z. We can also control β near a root of ω:

Lemma 3.2. *Suppose that z_0 is an order n root of ω, and the ω-disk of radius $2r$ around z_0 is embedded, without any other singularities of ω. Then for z in the ω-disk of radius r,*

$$\left|\int_{z_0}^{z} \beta\right| \leq \|\beta\| \ln 2(n+1), \tag{3.2.7}$$

and when $n = 2$,

$$\left|\int_{z_0}^{z} \beta\right| \leq \|\beta\| . \tag{3.2.8}$$

Proof. We can find a local coordinate z where z_0 maps to 0, and $\omega = (n+1)z^n dz$ in these coordinates. In the z coordinate, the ω-disk of radius $2r$ will becomes the disc D_a, with $a = (2r)^{1/(n+1)}$. Similarly the ω-disk of radius r becomes the disc $D_{a'}$, with $a' = r^{1/(n+1)} = 2^{-1/(n+1)}a$.

For any $z \in D_{a'}$, we have

$$|\beta(z)| \leq \frac{\|\beta\|}{a - |z|}$$

by (3.2.5). Therefore, for any $z \in D_{a'}$,

$$\begin{aligned}
\left| \int_0^z \beta \right| &\leq \|\beta\| \int_0^{|z|} \frac{1}{a-t} dt \\
&\leq \|\beta\| \int_0^{a'} \frac{1}{a-t} dt \\
&= \|\beta\| \ln \frac{a}{a-a'} \\
&= \|\beta\| \left(-\ln(1 - 2^{-1/(n+1)})\right) \\
&\leq \|\beta\| \ln 2(n+1).
\end{aligned}$$

In the case where $n = 2$, we can use the $\sqrt{\pi}$ from (3.2.5) and a direct numerical estimate of $-\ln(1 - 2^{-1/(n+1)})$ to obtain (3.2.8). $\square$

Theorem 3.3. *Suppose that the ω-distance between any two zeroes of ω is at least $2r$. Then*

$$\left\| \frac{\overline{\beta}}{\omega} \right\|_{\text{Teich}} \leq \frac{4\,\|\beta\|}{r}. \tag{3.2.9}$$

Proof. Let $\epsilon > 0$ be arbitrary. For each zero z_i of ω, we can define a C^1 function $p_i \colon \hat{X} \to [0,1]$ with

$$\|\partial p_i/\omega\|_\infty < (2+\epsilon)/r. \tag{3.2.10}$$

which is 1 on the disk of radius $r/2$ around z_i and supported in the open disk of radius r. We then define a vector field $v = f/\omega$, where $f = \sum_i p_i \int_{z_i} \overline{\beta}$; we have

$$\overline{\partial} v = \frac{1}{\omega} \sum_i \overline{p_i \beta + \partial p_i \int_{z_i} \beta}.$$

If q' is a norm 1 quadratic differential on X then $q'' \equiv \frac{1}{2}\rho_q^* q'$ has double roots at the z_i, so q''/ω is holomorphic. Therefore integration by parts gives

$$\int_{\hat{X}} \overline{\partial} v\, q'' = \int_{\hat{X}} \overline{\partial} f\, \frac{q''}{\omega} = 0,$$

and hence

$$\begin{aligned}\left\|\frac{\overline{\beta}}{\omega}\right\|_{\text{Teich}} &= \sup_{\|q'\|=1} \frac{1}{2}\int_{\hat{X}} q''\frac{\overline{\beta}}{\omega} \\ &= \sup_{\|q'\|=1} \frac{1}{2}\int_{\hat{X}} q''\left(\frac{\overline{\beta}}{\omega} - \overline{\partial} v\right) \\ &\leq \left\|\frac{\overline{\beta}}{\omega} - \overline{\partial} v\right\|_{\infty}.\end{aligned}$$

We can bound the latter norm as follows. In the disk of radius $r/2$ around z_i, it is zero. By (3.2.6) and (3.2.8), in the annulus around z_i with radius between $r/2$ and r,

$$\begin{aligned}\left|\frac{\overline{\beta}}{\omega} - \overline{\partial} v\right| &= \left|\frac{\overline{\beta - p_i\beta - (\partial p_i)\int_{z_i}\beta}}{\omega}\right| \\ &\leq (1-p_i)\left|\frac{\overline{\beta}}{\omega}\right| + \left|\frac{\partial p_i}{\omega}\right|\left|\int_{z_i}\beta\right| \\ &\leq \frac{2\|\beta\|}{r} + \frac{(2+\epsilon)\|\beta\|}{r} = \frac{(4+\epsilon)\|\beta\|}{r}.\end{aligned}$$

Outside of the disks of radius r around the z_i, $\overline{\partial} v$ is zero, and $\left|\overline{\beta}/\omega\right| < \|\beta\|/r$. □

3.3. **Sharpness of the upper bound.** We will now produce a series of examples where

$$\left\|\frac{\overline{\beta}}{\omega}\right\|_{\text{Teich}} \geq \frac{C\|\beta\|}{r},$$

and r is arbitrarily small, which shows that (3.2.9) is sharp up to the value of the constant $C = 4$.

For this we start in the stratum $\mathcal{H}(2)$ of Abelian differentials with 1 double zero. Note that every Abelian differential in $\mathcal{H}(2)$ is the holonomy double cover of a unique surface in $\mathcal{Q}(1,-1^5)$, so this will relate directly to principal strata of quadratic differentials.

We define a family $(X_\epsilon, \omega_\epsilon) \in \mathcal{H}(2)$ by taking a 1 by 1 torus, making a length ϵ horizontal slit, and gluing in an ϵ by ϵ cylinder. This construction is called bubbling a handle in [KZ03].

Let γ_ϵ be the core curve of the small cylinder on $(X_\epsilon, \omega_\epsilon)$. We can write its Poincare dual as sum of its holomorphic and anti-holomorphic part, $\gamma_\epsilon^* = \beta_\epsilon + \overline{\beta}_\epsilon$, which in this case are complex conjugates.

Consider the path $(X_\epsilon, \omega_\epsilon) + t\epsilon\gamma_\epsilon^*$ for $t \in [0,1]$, along which the small cylinder is sheared. This in fact defines a closed loop in $\mathcal{H}(2)$, since when $t = 1$ a full Dehn twist has been accomplished. Since the curve being Dehn twisted about has hyperbolic length bounded above and below, the total Teichmüller distance travelled is bounded above and below. Hence, for some t, the Teichmüller norm of $D\pi_{(X_\epsilon,\omega_\epsilon)+t\epsilon\gamma_\epsilon^*}([\epsilon\gamma_\epsilon^*])$ must be bounded below independently of ϵ. This can also be verified with standard techniques for all t, so we'll assume this is true at $t=0$. In summary, we get

$$\left\|\frac{\overline{\beta}_\epsilon}{\omega}\right\|_{\text{Teich}} = \|D\pi([\gamma_\epsilon^*])\|_{\text{Teich}} \geq \frac{C_0}{\epsilon}$$

for some constant $C_0 > 0$.

We now note that the Hodge norm of β_ϵ is bounded above and below independently of ϵ. This can be obtained as a special case of the estimates in [ABEM12, Section 3], or explained more directly as follows. The surfaces $(X_\epsilon, \omega_\epsilon)$ are converging in the Deligne-Mumford compactification to a nodal surface X_0 obtained by gluing two tori at a point. The collapse maps thus induce isomorphisms on cohomology. The cohomology of X_0 has a well defined Hodge norm, obtained for example via a direct sum, and one can see that the Hodge norm of *any* fixed cohomology class converges as $\epsilon \to 0$ to the Hodge norm of the corresponding class on X_0.

Since the length of the shortest saddle connection on $(X_\epsilon, \omega_\epsilon)$ is ϵ, this example shows that (3.2.9) is sharp up to the value of the constant $C = 4$ for the stratum $\mathcal{Q}(1, -1^5)$.

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