



Periodic points on the regular and double n -gon surfaces

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Abstract

Using the transfer principle of Lie theory, we classify the periodic points on the regular n -gon and double n -gon translation surfaces and deduce consequences for the finite blocking problem on rational triangles that unfold to these surfaces.

Keywords Translation surfaces · Periodic points · Dynamics on moduli space · Rational billiards · Veech surfaces

Mathematics Subject Classification 32G15 · 37D40 · 14H15

1 Introduction

The group $\mathrm{GL}^+(2, \mathbb{R})$ acts on the moduli space of translation surfaces, which is stratified by specifying the number of singularities of the flat metric and their cone angles. This action, which is generated by complex scalar multiplication and Teichmüller geodesic flow, preserves these strata. In the sequel, an element of a stratum will be denoted (X, ω) where X is a Riemann surface and ω is a holomorphic 1-form that induces the translation surface structure. Given such a point, its stabilizer $\mathrm{SL}(X, \omega)$ in $\mathrm{SL}(2, \mathbb{R})$ is called the *Veech group*, and (X, ω) is called a *Veech surface* when this group is a lattice.

Definition 1.1 A point p on a Veech surface (X, ω) is called *periodic* if p is not a zero of ω and if its orbit under $\mathrm{Aff}(X, \omega)$ —the affine diffeomorphism group of (X, ω) —is finite.

Remark 1.2 Our definition is equivalent to the one used in Apisa [2]. A version of this definition which includes the zeros of ω first appeared in Gutkin-Hubert-Schmidt [10]. Under this definition, an equivalent notion of a periodic point is a point marked by a holomorphic multisection of the universal curve over a suitable finite cover of the Teichmüller curve associated to (X, ω) . See Möller [22, Lemma 1.2] for details.

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Consider the following translation surfaces. For n even, the *regular n -gon surface* is the regular n -gon with opposite sides identified, and for n odd, the *double n -gon surface* is two copies of a regular n -gon differing by a rotation by π with parallel sides glued together. The regular 10-gon and the double 7-gon surfaces are depicted in Fig. 2. Both the regular and double n -gon surfaces are hyperelliptic with their hyperelliptic involutions being affine diffeomorphisms of derivative $-\text{Id}$. The *Weierstrass points* are the fixed points of this involution. In [24], Veech proved that the regular n -gon and double n -gon surfaces are Veech surfaces for all $n \geq 3$. Our main result is a classification of periodic points on these surfaces.

Theorem 1.3 *When $n \geq 5$ and $n \neq 6$, the periodic points of the regular n -gon and double n -gon surfaces are exactly the Weierstrass points that are not singularities of the flat metric.*

Remark 1.4 For the regular n -gon surface, the Weierstrass points are the center, midpoints of the sides, and, when $4 \mid n$, the vertices of the regular n -gon that comprises the surface. For the double n -gon surface, the Weierstrass points are the midpoints of the sides and vertices of the two regular n -gons that comprise the surface.

Remark 1.5 When $n = 5, 8$, and 10 this result was shown by Möller [22, Theorems 5.1, 5.2]. When $n = 3, 4$ or 6 the surfaces are tori and have infinitely many periodic points coming from torsion points.

The proof can be divided into two steps. First, we use the transfer principle,¹ recalled in the first sentence of the proof of Proposition 3.2, to reduce the problem to classifying periodic points on an explicit set of saddle connections.² Second, we classify the periodic points on these saddle connections by covering them with two collections of non-parallel cylinders and using the “rational height lemma”, recalled as Lemma 3.5 below.

The regular n -gon and double n -gon surfaces belong to a larger infinite family of Veech surfaces called the Veech-Ward-Bouw-Möller surfaces [4]. It would be interesting to know whether our methods could be used to classify the periodic points on these surfaces, which, by Hooper [11] and Wright [26], also admit a presentation as a disjoint union of regular polygons with side identifications.

Theorem 1.3 has consequences for the finite blocking problem.

Definition 1.6 Two points P, Q on a billiard table (resp. translation surface) M are *finitely blocked* if there is a finite set of points $S \subset M - \{P, Q\}$ such that all billiard trajectories (resp. straight line segments that do not contain singularities in their interior) from P to Q pass through a point in S .

The following corollaries will be proven after the proof of the main theorem.

Corollary 1.7 *When $n \geq 5$ and $n \neq 6$, the pairs of finitely blocked points on the regular n -gon and double n -gon surfaces consist precisely of any point that is not a singularity and its image under the hyperelliptic involution.*

Via the unfolding construction of Katok-Zemlyakov [27], the $\left(\frac{\pi}{2}, \frac{\pi}{n}, \frac{(n-2)\pi}{2n}\right)$ triangle unfolds to the regular n -gon or double n -gon surface when n is even or odd respectively. Therefore, a consequence of the previous corollary is the following.

¹ Our use of the phrase “transfer principle” is inspired by and analogous to the usage in Calsamiglia-Deroi-Francaviglia [5]

² A saddle connection is a closed geodesic arc on a translation surface whose endpoints are singular points and which has no singular points in its interior.

Corollary 1.8 When $n \geq 5$ and $n \neq 6$, the $\left(\frac{\pi}{2}, \frac{\pi}{n}, \frac{(n-2)\pi}{2n}\right)$ triangle admits a pair of finitely blocked points if and only if n is even, in which case the only such pair is the vertex of angle $\frac{\pi}{n}$ and itself.

Remark 1.9 Similar statements can be deduced for the $\left(\frac{\pi}{n}, \frac{\pi}{n}, \frac{(n-2)\pi}{n}\right)$ and $\left(\frac{2\pi}{n}, \frac{(n-2)\pi}{2n}, \frac{(n-2)\pi}{2n}\right)$ triangles, which unfold to the regular n -gon surface, the double n -gon surface, or a double cover of one of those surfaces.

The remainder of the paper is divided into three sections. In Sect. 2 we establish some facts about the flat geometry of the regular and double n -gon surfaces. This will require a lemma on the rationality of ratios of sines that will be proven in Sect. 4. Section 3 contains the proof of Theorem 1.3 and its corollaries.

Context. Some of the earliest results on periodic points and the finite blocking problem, especially for Veech surfaces, are due to Gutkin et al. [10], Hubert et al. [14], and Monteil [19, 20]; see especially [19, Theorem 1] for related work on the finite blocking problem in the regular n -gon.

Periodic points in strata of Abelian and quadratic differentials were classified by Apisa [2] and Apisa and Wright [3] respectively. Periodic points for every genus two translation surface were classified by Möller [22] and Apisa [1]. These works use the classification of $\mathrm{GL}(2, \mathbb{R})$ orbit closures in genus two strata of Abelian differentials by McMullen [18]. The only periodic point on a genus two translation surface that is not a Weierstrass point was discovered by Kumar and Mukamel [15] and relates to orbit closures discovered by Eskin et al. [9].

For recent work on the finite blocking and illumination problems that was inspired by work of Eskin et al. [7, 8], see Lelièvre et al. [16], Apisa and Wright [3], and Wolecki [25].

2 Preliminaries

Fix an integer n so that either $n = 5$ or $n \geq 7$. Let R_1 denote a regular n -gon circumscribed in the unit circle in \mathbb{C} centered at the origin and so that one of its vertices lies at the point i . When n is even, the regular n -gon surface is R_1 with opposite sides identified. To form the double n -gon surface when n is odd we take a copy of R_1 rotated by $\frac{\pi}{n}$, which we call R_2 , and identify parallel sides between R_1 and R_2 . By triangulating these polygons and computing Euler characteristic, it is easy to see that the genus of the regular n -gon surface (resp. double n -gon surface) is $\lfloor \frac{n}{4} \rfloor$ (resp. $\frac{n-1}{2}$).

Let Γ_n denote the Veech group of the regular n -gon surface when n is even and the double n -gon surface when n is odd. Make the following definitions,

$$r_n = \begin{pmatrix} \cos \frac{\pi}{n} & -\sin \frac{\pi}{n} \\ \sin \frac{\pi}{n} & \cos \frac{\pi}{n} \end{pmatrix}, \quad s_n = \begin{pmatrix} 1 & 2 \cot \frac{\pi}{n} \\ 0 & 1 \end{pmatrix}.$$

Theorem 2.1 ([24] (Definition 5.7, Theorem 5.8); see also [21] (Theorem 5.4)) When n is even, Γ_n is generated by $\{r_n^2, s_n, r_n s_n r_n^{-1}\}$ and is isomorphic to the $(n/2, \infty, \infty)$ triangle group. In particular, \mathbb{H}/Γ_n has two cusps.

When n is odd, Γ_n is generated by $\{r_n, s_n\}$ and is isomorphic to the $(2, n, \infty)$ triangle group. In particular, \mathbb{H}/Γ_n has one cusp.

Remark 2.2 In fact when n is even, Veech considered a double cover of the regular n -gon surface; however, it is well-known that the Veech group of the two surfaces are identical. Nevertheless, we will only ever use that, when n is even, Γ_n is contained in the Veech group of the regular n -gon surface, which is clear since each generator has that property.

Remark 2.3 It is well known (see for instance Veech [24, Section 3] or Hubert-Schmidt [13, Lemma 4]) that for Veech surfaces, the maximal parabolic subgroups of the Veech group are in one-to-one correspondence with cylinder directions. The correspondence is given by associating to each cylinder direction, its stabilizer in the Veech group. Under this correspondence, the action of the Veech group on cylinder directions corresponds to its action by conjugation on maximal parabolic subgroups. Since conjugacy classes of maximal parabolic subgroups correspond to cusps of the quotient of the upper half plane by the Veech group, we see that each cusp corresponds to the orbit of a cylinder direction under the Veech group. Thus, every cylinder direction can be moved to one of these prescribed directions (described below) by an element of the Veech group.

In light of this observation, by Theorem 2.1, on the double n -gon surface any cylinder direction may be sent to any other by an affine diffeomorphism. Similarly, there are two orbits of cylinders under the action of the affine diffeomorphism group on the regular n -gon surface. We will now describe these two cylinder directions.

The first is the horizontal direction, which is covered by $\left\lceil \frac{n}{4} \right\rceil$ (resp. $\frac{n-1}{2}$) cylinders when n is even (resp. odd), as seen on the left in Fig. 1. Let g'_n denote the number of horizontal cylinders. Notice that g'_n is greater than or equal to the genus of the surface. Since the vertices of R_1 lie at the points $\left\{ i \exp\left(\frac{2j\pi i}{n}\right) \right\}_{j=0}^{n-1}$, it is easy to see that the heights of the horizontal cylinders are

$$h_j := \operatorname{Im} \left(i \exp\left(\frac{2j\pi i}{n}\right) - i \exp\left(\frac{(2j+2)\pi i}{n}\right) \right)$$

for $j \in \{0, \dots, g'_n - 1\}$. We can simplify the expression for the heights as follows,

$$\begin{aligned} h_j &= \operatorname{Im} \left(i \exp\left(\frac{(2j+1)\pi i}{n}\right) \left(\exp\left(\frac{-\pi i}{n}\right) - \exp\left(\frac{\pi i}{n}\right) \right) \right) \\ &= 2 \sin\left(\frac{\pi}{n}\right) \sin\left(\frac{(2j+1)\pi}{n}\right). \end{aligned} \tag{1}$$

When n is odd, notice that since $\sin(x) = \sin(\pi - x)$ for any x , for $j > \left\lfloor \frac{n-3}{4} \right\rfloor$ we can write $\sin\left(\frac{(2j+1)\pi}{n}\right) = \sin\left(\frac{(n-2j-1)\pi}{n}\right)$, and so

$$\{h_j\}_{j=0}^{g'_n-1} = \left\{ 2 \sin\left(\frac{\pi}{n}\right) \sin\left(\frac{k\pi}{n}\right) \right\}_{k=1}^{g'_n}. \tag{2}$$

When n is even, there are two orbits of cylinder directions under the action of the Veech group. Under the equivalences explained above, these two cylinder directions can be chosen to be stabilized by the maximal parabolic subgroup of Γ_n generated by s_n and $r_n s_n r_n^{-1}$. Using this observation (which is explained more fully in [24, Section 5]), a cylinder direction that cannot be sent to the horizontal one by the Veech group can be described as follows. Rotate the regular n -gon surface so that R_1 remains circumscribed in a unit circle but with one of its edges being horizontal. Let R'_1 denote this rotated copy of R_1 . The horizontal direction is now covered by $g_n := \left\lfloor \frac{n}{4} \right\rfloor$ cylinders, as seen on the right in Fig. 1; the notation g_n is

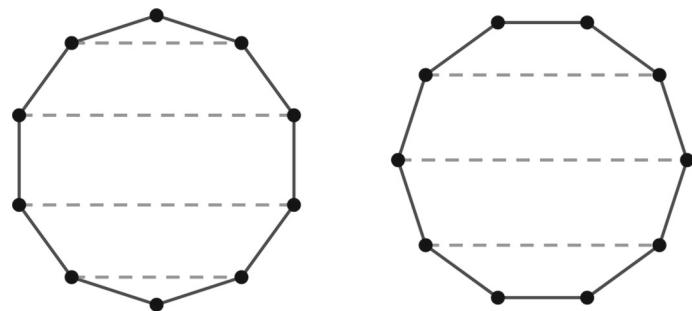


Fig. 1 Two cylinder directions for the regular decagon

chosen since it is equal to the genus of the surface. Since the vertices of R'_1 lie at the points $\left\{i \exp\left(\frac{(2j+1)\pi i}{n}\right)\right\}_{j=0}^{n-1}$, it is easy to see that the heights of the horizontal cylinders are

$$h'_j := \operatorname{Im}\left(i \exp\left(\frac{(2j+1)\pi i}{n}\right) - i \exp\left(\frac{(2j+3)\pi i}{n}\right)\right)$$

for $j \in \{0, \dots, g_n - 1\}$. We can simplify the expression for the heights as follows,

$$\begin{aligned} h'_j &= \operatorname{Im}\left(i \exp\left(\frac{(2j+2)\pi i}{n}\right) \left(\exp\left(\frac{-\pi i}{n}\right) - \exp\left(\frac{\pi i}{n}\right)\right)\right) \\ &= 2 \sin\left(\frac{\pi}{n}\right) \sin\left(\frac{(2j+2)\pi}{n}\right). \end{aligned} \quad (3)$$

The following result was stated in McMullen [17, page 7] where it is indicated that its proof follows from an application of the bounds in the proof of [17, Theorem 2.1]. Since the deduction is not entirely trivial, we offer a proof in Sect. 4. In our deduction, we will not use the full strength of [17, Theorem 2.1], which shows that the number of sine ratios of any fixed degree over \mathbb{Q} is finite and which can be used to find all such ratios.

Lemma 2.4 *For rational numbers $0 < \alpha < \beta \leq \frac{1}{2}$, $\frac{\sin(\pi\alpha)}{\sin(\pi\beta)}$ is rational if and only if $\alpha = \frac{1}{6}$ and $\beta = \frac{1}{2}$.*

Lemma 2.5 *On the regular n -gon and double n -gon surfaces, at least one cylinder direction has the property that the ratio of heights and circumferences of distinct cylinders in this direction have irrational ratio; every cylinder direction has this property whenever n is not congruent to 0 or 6 mod 12. Moreover, for any two parallel cylinders sharing a boundary saddle connection, their heights and circumferences have an irrational ratio.*

Proof By Remark 2.3, any cylinder direction can be sent to one of the two directions specified in Remark 2.3 by an element of the Veech group. In particular, the ratio of heights of two distinct parallel cylinders is either $\frac{h_j}{h_k}$ or $\frac{h'_j}{h'_k}$. By Eqs. (1)–(3), up to inverting the ratio, these ratios always have the form $\frac{\sin(\pi\alpha)}{\sin(\pi\beta)}$ for rational numbers $0 < \alpha < \beta \leq \frac{1}{2}$ where $n\alpha$ and $n\beta$ are integers. By Lemma 2.4 such a ratio is rational if and only if $\alpha = \frac{1}{6}$ and $\beta = \frac{1}{2}$. In particular by Eq. (2) the ratio of heights of distinct parallel cylinders is irrational when n is odd.

Suppose therefore that n is even. By Equation (1), $\frac{h_j}{h_k} = \frac{\sin\left(\frac{(2j+1)\pi}{n}\right)}{\sin\left(\frac{(2k+1)\pi}{n}\right)}$. If $j < k$ then this ratio is rational if and only if $\frac{2j+1}{n} = \frac{1}{6}$ and $\frac{2k+1}{n} = \frac{1}{2}$. This implies that $n = 12j + 6$ and $k = 3j + 1$. Therefore, the ratio of heights of distinct cylinders in this cylinder direction is irrational if and only if n is not congruent to 6 mod 12. Moreover, since j and k correspond to cylinders that share a boundary saddle connection if and only if $|j - k| = 1$ we have that cylinders that share a boundary saddle connection have an irrational ratio of height as long as $j > 0$, which is the case since we have assumed that $n \neq 6$.

By Eq. (3), $\frac{h'_j}{h'_k} = \frac{\sin\left(\frac{(2j+2)\pi}{n}\right)}{\sin\left(\frac{(2k+2)\pi}{n}\right)}$. If $j < k$ then this ratio is rational if and only if $\frac{2j+2}{n} = \frac{1}{6}$ and $\frac{2k+2}{n} = \frac{1}{2}$. This implies that $n = 12j + 12$ and $k = 3j + 2$. Therefore, the ratio of heights of distinct cylinders in this cylinder direction is irrational if and only if n is not congruent to 0 mod 12. As before cylinders that share a boundary saddle connection have an irrational ratio of height.

By Remark 2.3 any cylinder direction can be moved by an element of the Veech group to one of the two cylinder directions analyzed in the preceding paragraphs, so the result follows. The claims for circumferences hold since the ratio of moduli³ of parallel cylinders is rational for any Veech surface. \square

Definition 2.6 Given translation surfaces (X, ω) and (X', ω') a *translation cover* $f : (X, \omega) \rightarrow (X', \omega')$ is a holomorphic map $f : X \rightarrow X'$ such that $f^*\omega' = \omega$. Similarly, if (Y, q) is a quadratic differential that is not the square of a holomorphic one-form, then we say that $f : (X, \omega) \rightarrow (Y, q)$ is a *half-translation cover* if $f : X \rightarrow Y$ is holomorphic and $f^*q = \omega^2$.

Lemma 2.7 Let (X, ω) be the regular n -gon or double n -gon surface. If (X', ω') is a translation surface so that $f : (X, \omega) \rightarrow (X', \omega')$ is a translation cover then $(X, \omega) = (X', \omega')$.

Proof Suppose to a contradiction that there is a translation cover $f : (X, \omega) \rightarrow (X', \omega')$ where the genus g' of X' is less than the genus g of X . For each cylinder C of circumference c on (X, ω) , there is an integer m so that $f(C)$ is a cylinder of circumference c/m . By Lemma 2.5, there is a cylinder direction on (X, ω) in which all distinct pairs of cylinders have an irrational ratio of circumferences and hence map to distinct cylinders on (X', ω') under f . Since every cylinder direction on (X, ω) has at least g cylinders (see the description of cylinder directions in Remark 2.3), it follows that there is a cylinder direction on (X', ω') with g distinct cylinders. Since ω has at most two zeros, the number of zeros s' of ω' is also at most two. Therefore, $g \leq g' + 1$, since the number of cylinders on (X', ω') is bounded above by $g' + s' - 1$. Since we have assumed that $g' < g$ we see that $g' = g - 1$ and $s' = 2$.

By the Riemann-Hurwitz formula, this implies that $g \leq 3$ and hence that $n \in \{5, 7, 8, 10, 12, 14\}$. The condition that ω has two singularities reduces the possibilities to just $n \in \{10, 14\}$. Since 10 and 14 are not congruent to 0 or 6 mod 12 it follows from Lemma 2.5 that these surfaces possess cylinder directions with $g + 1$ cylinders so that the ratio of circumferences of distinct cylinders is irrational. As argued above, this implies that (X', ω') also has such a cylinder direction and hence that $g = g'$, which is a contradiction. \square

Corollary 2.8 For the regular n -gon and double n -gon, the affine diffeomorphism group is isomorphic to the Veech group.

³ The modulus of a cylinder of height h and circumference c on a translation surface is $\frac{h}{c}$.

Proof Let (X, ω) be the regular n -gon or double n -gon surface. Letting $\text{Aut}(X, \omega)$ be the group of affine diffeomorphisms of derivative Id , we have the following short exact sequence (see [24, Equation (2.6)]).

$$0 \longrightarrow \text{Aut}(X, \omega) \longrightarrow \text{Aff}(X, \omega) \longrightarrow \text{SL}(X, \omega) \longrightarrow 0.$$

It suffices to show that $\text{Aut}(X, \omega)$ is trivial. This follows from Lemma 2.7 since the cover $(X, \omega) \rightarrow (X, \omega)/\text{Aut}(X, \omega)$ must be the identity. \square

From now on, we consider the Veech group acting on the regular or double n -gon by the above isomorphism.

Remark 2.9 It is standard that the Weierstrass points that do not coincide with singularities of the flat metric are periodic points for any translation surface in a hyperelliptic locus of a stratum. However, in our case, this is particularly easy to see since the isomorphism between $\text{Aff}(X, \omega)$ and $\text{SL}(X, \omega)$ shows that the hyperelliptic involution is in the center of $\text{Aff}(X, \omega)$ —since it is sent to $-\text{Id}$ —and hence that $\text{Aff}(X, \omega)$ permutes its fixed points.

Remark 2.10 Let n be even and let p provisionally denote the center of the regular n -gon R_1 whose opposite sides are identified to form the regular n -gon surface. We will show that p is fixed by every element of the affine diffeomorphism group. It is obvious that it is fixed by the rotation r_n . The remaining generators of the affine diffeomorphism group are shears in the cylinder directions identified in Remark 2.3. In cylinder direction of slope $\frac{\pi}{n}$, p lies on a boundary saddle connection of a cylinder and is trivially fixed. In the horizontal cylinder direction, p lies on the core curve of a cylinder C of modulus $\tan(\pi/n)$. The corresponding generator of the maximal parabolic subgroup is s_n , which performs two Dehn twists in C and hence fixes p . Since p is fixed by the generators of the affine diffeomorphism group, it is fixed by every element in it.

3 Proof of theorem 1.3 and its corollaries

We now begin our study of periodic points using the transfer principle. Our goal is to reduce the main theorem to identifying the periodic points on finitely many saddle connections. We start with the following definition.

Definition 3.1 When n is even let P_n denote the point in Remark 2.10. When n is odd, let P_n denote the unique cone point of the flat metric of the double n -gon surface. Both points are fixed by every element of Γ_n .

Proposition 3.2 *The Γ_n orbit of any periodic point on the regular n -gon or double n -gon surface contains a point lying on the leaf of the horizontal foliation passing through P_n or, when n is even, a point lying on the leaf of the foliation passing through P_n that makes an angle of $\frac{\pi}{n}$ with the horizontal (see Fig. 2). These leaves are saddle connections or possibly, when n is even, the core curve of a cylinder.*

Proof The transfer principle states that if G and H are topological groups acting continuously, from the left and right respectively, on a topological space \mathcal{X} then the following are in bijective correspondence:

1. Closed (resp. dense) G orbits on \mathcal{X}/H .

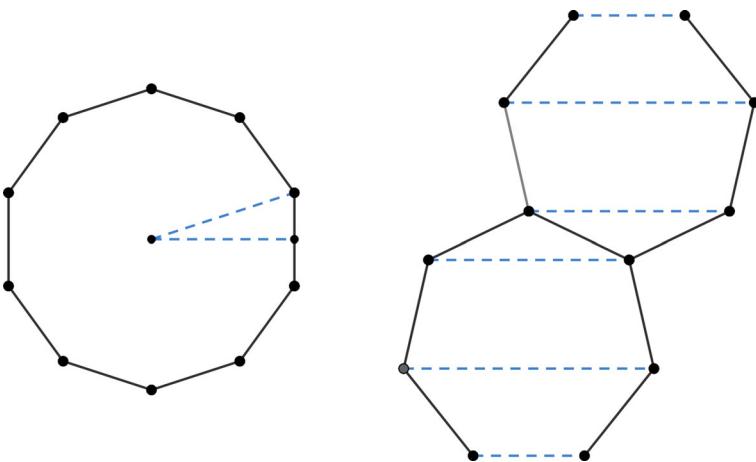


Fig. 2 After applying Proposition 3.2, any periodic point can be assumed to lie on one of the dashed lines or its image under the hyperelliptic involution

2. Closed (resp. dense) $G \times H$ orbits on \mathcal{X} .
3. Closed (resp. dense) H orbits on $G \backslash \mathcal{X}$

We will briefly sketch a proof of this claim for closed orbits (the case of dense orbits is similar). Let $\pi : \mathcal{X} \rightarrow \mathcal{X}/H$ be the quotient map. If $C \subseteq \mathcal{X}/H$ is a closed G -invariant set, then its preimage $\pi^{-1}(C)$ is a closed $G \times H$ -invariant set. Conversely, if $D \subseteq \mathcal{X}$ is a closed $G \times H$ invariant set, then, since it is H -invariant, $\pi(D)$ is the complement of $\pi(\mathcal{X} - D)$. Since π is an open map, $\pi(\mathcal{X} - D)$ is open and hence $\pi(D)$ is a closed G -invariant set. Therefore, taking images and preimages under π establishes a bijection between closed G -invariant subsets of \mathcal{X}/H and closed $G \times H$ invariant subsets of \mathcal{X} . This bijection restricts to the desired bijection between closed orbits.

Under these correspondences, a $G \times H$ orbit of $x \in \mathcal{X}$ will be sent to the G orbit of xH or the H orbit of Gx . In our context, G is the Veech group Γ_n , \mathcal{X} is $\text{SL}(2, \mathbb{R})$, and H is the unipotent subgroup $U := \left\{ \begin{pmatrix} 1 & s \\ 0 & 1 \end{pmatrix} : s \in \mathbb{R} \right\}$. The quotient $\text{SL}(2, \mathbb{R})/U$ can be identified with $\mathbb{R}^2 - \{0\}$ by sending $g \in \text{SL}(2, \mathbb{R})$ to $g \cdot \begin{pmatrix} 1 \\ 0 \end{pmatrix}$. Under this identification, the action of Γ_n on $\text{SL}(2, \mathbb{R})/U$ is given by the standard linear action of $\text{SL}(2, \mathbb{R})$ on $\mathbb{R}^2 - \{0\}$. Define $a_t := \begin{pmatrix} e^t & 0 \\ 0 & e^{-t} \end{pmatrix}$ where t is any real number.

It is a foundational result of Dani [6, Theorem A] that the only U orbits of $\Gamma_n \backslash \text{SL}(2, \mathbb{R})$ are closed or dense, and the closed orbits are horocycles around the cusps. Recall from Theorem 2.1 that cusps of $\Gamma_n \backslash \text{SL}(2, \mathbb{R})$ correspond to conjugacy classes of maximal parabolic subgroups, and these are generated by s_n and, when n is even, $r_n s_n r_n^{-1}$. In particular, the closed horocycles corresponding to the cusps of $\Gamma_n \backslash \text{SL}(2, \mathbb{R})$ are given by $\Gamma_n a_t U$ and also, when n is even, $\Gamma_n r_n a_t U$ where t is any real number. By the transfer principle, the only vectors in $\mathbb{R}^2 - \{0\}$ that do not have dense Γ_n orbit are vectors parallel to a vector in $\Gamma_n \cdot \begin{pmatrix} 1 \\ 0 \end{pmatrix}$ or, when n is even, parallel to a vector in $\Gamma_n \cdot \begin{pmatrix} \cos \frac{\pi}{n} \\ \sin \frac{\pi}{n} \end{pmatrix}$.

Now let p be any periodic point that is distinct from P_n . By definition, the orbit of p under Γ_n is finite. In particular, $\Gamma_n \cdot p$ remains a bounded distance away from P_n . Since the regular n -gon surface is comprised of a convex polygon with opposite sides identified and since the double n -gon surface is comprised of two regular n -gons whose vertices correspond to P_n , it follows that there is a straight line segment γ from P_n to p , the holonomy of which we will denote by v . Since p remains a bounded distance away from P_n we have that $\Gamma_n \cdot v$ is not dense in \mathbb{R}^2 (in particular, the orbit is not dense in a neighborhood of 0) and hence there is a vector in the Γ_n orbit of v that is parallel to a vector in $\Gamma_n \cdot \begin{pmatrix} 1 \\ 0 \end{pmatrix}$ or, when n is even, one in $\Gamma_n \cdot \begin{pmatrix} \cos \frac{\pi}{n} \\ \sin \frac{\pi}{n} \end{pmatrix}$. This shows that $\Gamma_n \cdot p$ contains a point on either the horizontal leaf through P_n or, in the case of n even, the leaf that makes an angle of $\frac{\pi}{n}$ with the horizontal. \square

Definition 3.3 When n is odd, call the saddle connections identified in Proposition 3.2 *candidate line segments*. When n is even, notice that the hyperelliptic involution fixes both line segments identified in Proposition 3.2. Each line segment can be partitioned into two sub-segments, which are exchanged by the hyperelliptic involution and have one endpoint at P_n . For each line segment identified in Proposition 3.2 we choose one of these subsegments and call them *candidate line segments* (see Fig. 2). Since the hyperelliptic involution preserves the collection of periodic points, any periodic point can be moved by an element of the Veech group to a candidate line segment. (Recall that, as observed in Remark 2.9, the hyperelliptic involution can be identified with the element $-\text{Id}$ of the Veech group.)

In the sequel, we will adopt the convention that all cylinders are assumed to be closed; that is, they contain their boundary.

Definition 3.4 A point contained in a cylinder C is said to have *rational height in C* if its distance from the boundary of C is a rational multiple of the height of C .

The following observation is well-known and a version of it appears in [12, Lemma 4] and [2, Lemma 5.4].

Lemma 3.5 *A periodic point on a Veech surface has rational height in any cylinder containing it.*

Proof Let p be the periodic point and suppose that it is contained in a cylinder C . After perhaps rotating the surface we may suppose without loss of generality that C is horizontal. Denote its height and circumference by h and c respectively. Since the surface is Veech, the Veech group contains an element $g := \begin{pmatrix} 1 & s \\ 0 & 1 \end{pmatrix}$ where $s = kc/h$ for some nonzero integer k . Choosing flat coordinates so that the bottom boundary of C lies on the x -axis, we see that g sends a point $(x, y) \in C$ to $(x + ykc/h, y)$ where the x coordinate is taken modulo c . Thus, if (x, y) has finite orbit under the Veech group, then y is a rational multiple of h . \square

In the following lemma, it will be useful to use the notation \overline{PQ} to refer to a straight line segment on a flat surface between points P and Q . In general on a flat surface there are infinitely many straight line segments between any two points, so we emphasize that this notation presupposes a choice of a line segment between P and Q and that the line segment is not uniquely determined by its endpoints.

Lemma 3.6 *Let C_2 and C_3 be two adjacent parallel cylinders whose ratio of heights is irrational. Let C_1 be another cylinder. Suppose that \overline{PQ} is a line segment satisfying the following:*

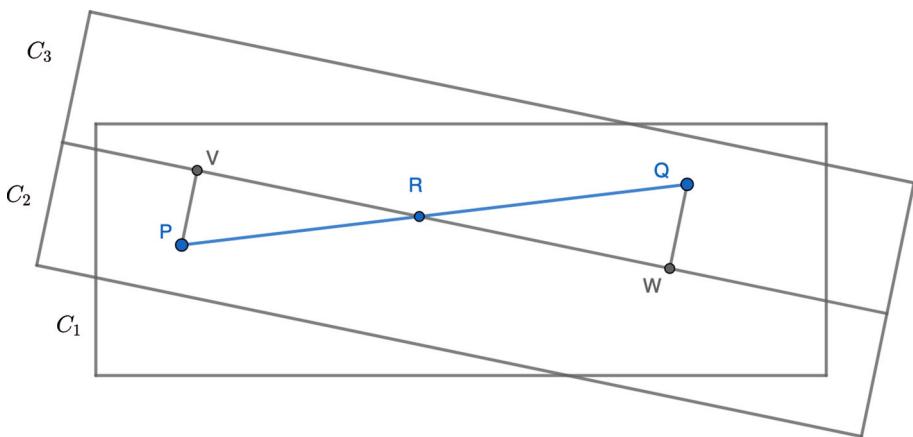


Fig. 3 The three cylinders in Lemma 3.6

1. \overline{PQ} is neither parallel nor perpendicular to the core curves of C_1 , C_2 , and C_3 .
2. \overline{PQ} is contained in C_1 and its interior does not intersect the boundary of C_1 .
3. \overline{PQ} is contained in $C_2 \cup C_3$ and its interior intersects the boundary of C_2 and C_3 in a unique point R . \overline{PR} (resp. \overline{RQ}) is contained in C_2 (resp. C_3) and has nonzero length, see Fig. 3.
4. The orthogonal projection of \overline{PQ} (resp. \overline{PR} , \overline{RQ}) to the core curve of C_1 (resp. C_2 , C_3) is a proper subset of the core curve.
5. P (resp. Q) has rational height in C_1 and C_2 (resp. C_1 and C_3).

Then the only point on \overline{PR} (resp. \overline{RQ}) that has rational height in both C_1 and C_2 (resp. C_1 and C_3) is P (resp. Q).

Proof Assume for the sake of contradiction that there is a point S on \overline{PR} other than P with rational height inside both C_1 and C_2 . Without loss of generality, perhaps after rotating the surface, we may suppose that C_1 is horizontal.

When P is contained in the interior of C_1 , let ℓ denote the leaf of the horizontal foliation passing through P . When P is contained in the boundary of C_1 , let ℓ denote the boundary of C_1 containing P . Since the interior of \overline{PQ} is contained in the interior of C_1 (by (2)), we may think of C_1 as a Euclidean cylinder and orthogonally project PQ onto ℓ . By (1) and (4), this projection is a line segment PU where $U \neq P$. Let QU denote the vertical line segment contained in C_1 from Q to U . The triangle, which we will denote ΔPQU , formed by \overline{PQ} , \overline{PU} , and \overline{QU} is, by (2), a right triangle contained in C_1 as shown in Fig. 4. Let T be the point on \overline{QU} so that ΔSQT is similar to ΔPQU (see Fig. 4).

Since, by (5), P , Q , and S (and hence also T and U) have rational height in C_1 , it follows that the length of \overline{UT} is a rational multiple of the length of \overline{TQ} . Since ΔSQT and ΔPQU are similar, it follows that the length of \overline{PS} is a rational multiple of the length of \overline{SQ} .

The preceding argument (with the role of \overline{PQ} being played by \overline{PR} and that of C_1 by C_2) shows that the length of \overline{PS} is a rational multiple of the length of \overline{SR} (notice that (3) is the analogue of (2) here). It follows that the lengths of \overline{PR} and \overline{RQ} have a rational ratio.

Finally, let V (resp. W) denote the orthogonal projection of P (resp. Q) to the component of the boundary of C_2 (resp. C_3) containing R (see Fig. 3). We note that the triangles ΔPVR and ΔQWR , which are formed in the same way we formed ΔPQU , are similar. Since the

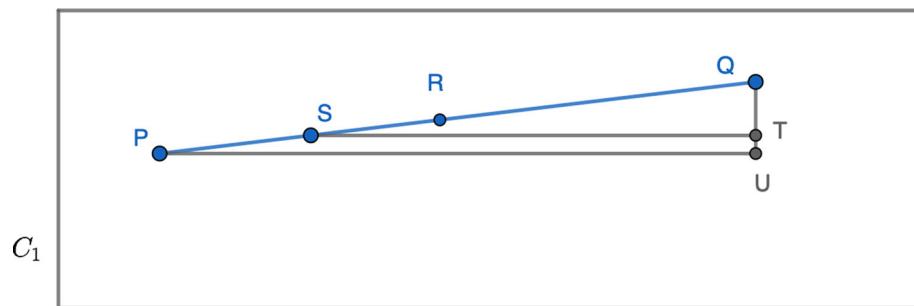
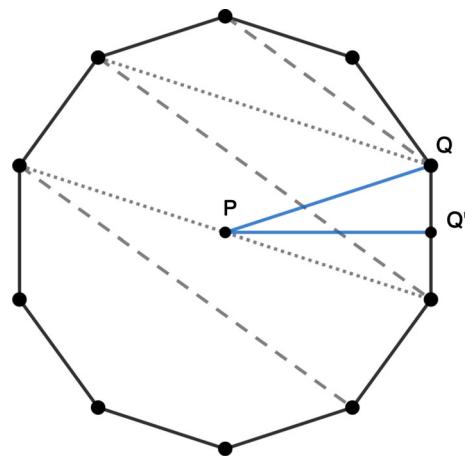


Fig. 4 The triangles SQT and PQU are similar

Fig. 5 The cylinders dividing candidate line segments in the even case. Both Q and Q' are endpoints of candidate line segments



lengths of \overline{PR} and \overline{RQ} have a rational ratio, so do the lengths of \overline{PV} and \overline{RW} . However, by (5), the lengths of \overline{PV} and \overline{RW} are rational multiples of the height of C_2 and C_3 respectively. Therefore, we have shown that C_2 and C_3 have a rational ratio of heights, which is a contradiction. \square

Proof of theorem 1.3 Suppose first that n is even. By Proposition 3.2, any periodic point can be moved by an element of the Veech group to one of the two candidate line segments (see Definition 3.3 and Fig. 5). The endpoints of the candidate line segments are singularities of the flat metric or Weierstrass points. It suffices to show that these endpoints are the only periodic points on a candidate line segment. Choosing a candidate line segment, let P denote P_n , which is one endpoint, and let Q denote the other endpoint. We will let \overline{PQ} denote the candidate line segment. Notice that \overline{PQ} is contained in a single cylinder C_1 that makes an angle of $-\frac{\pi}{n}$ with the horizontal and to which \overline{PQ} is not parallel. This is the dotted cylinder in Fig. 5.

The line segment \overline{PQ} is also contained in the union of two parallel cylinders C_2 and C_3 in the cylinder direction that makes an angle of $-\frac{2\pi}{n}$ with the horizontal. The cylinders C_2 and C_3 share a boundary saddle connection and so by Lemma 2.5 they have an irrational ratio of heights. These cylinders are the dashed cylinders in Fig. 5.

Since the endpoints of the candidate line segments are either singularities of the flat metric or periodic points they have rational height in any cylinder containing them by Lemma 3.5.

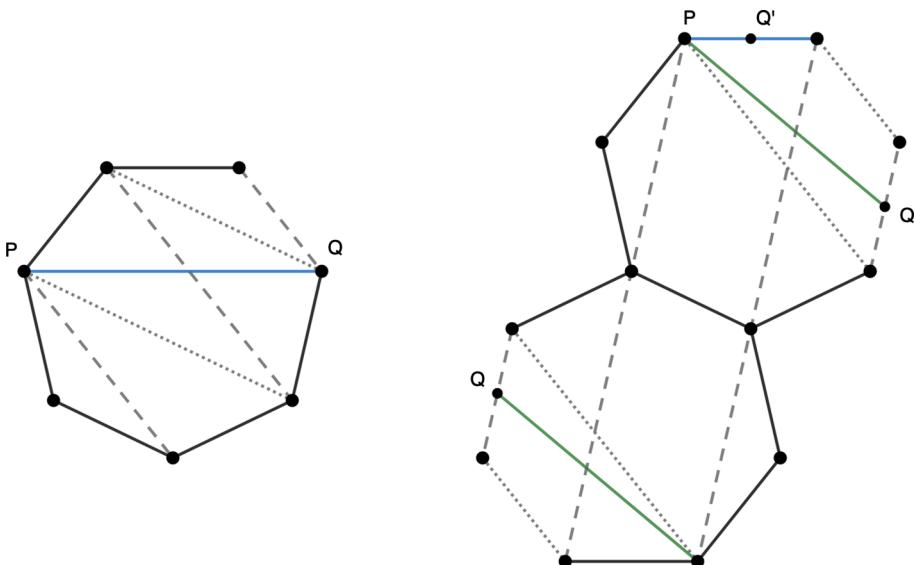


Fig. 6 The cylinders dividing candidate line segments in the odd case. The boundary of C_1 (resp. C_2 and C_3) are the dotted (resp. dashed) lines

By Lemma 3.6 it follows that any point lying in the interior of \overline{PQ} has irrational height in at least one of C_1 , C_2 , and C_3 . Therefore, any point lying in the interior of a candidate line segment cannot be periodic by Lemma 3.5 as desired.

Now suppose that n is odd. Recall that the double n -gon surface is comprised of two regular n -gons, which we denote by R_1 and R_2 , that differ from each other by a rotation of π/n and so that parallel sides are identified. Recall too that the hyperelliptic involution exchanges R_1 and R_2 . Since every candidate line segment is contained in either R_1 or R_2 it suffices to classify the periodic points on the candidate line segments in just R_2 . Recall that we have supposed that R_2 is circumscribed in a circle centered at the origin in \mathbb{C} and has a vertex lying at the point $-i$.

We will begin by showing that if a candidate line segment passes through the interior of R_2 then its interior contains no periodic points. Notice that such a candidate line segment is contained in the interior of a cylinder C_1 that makes an angle of $-\frac{\pi}{n}$ with the horizontal and in the union of two cylinders C_2 and C_3 that share a boundary saddle connection and make an angle of $-\frac{2\pi}{n}$ with the horizontal (see the left subfigure in Fig. 6). The claim that this candidate line segment contains no periodic points in its interior is now identical to the argument in the case of n even.

It remains to consider the candidate line segment ℓ that is an edge of R_2 (see the right subfigure in Fig. 6). Let Q' denote the midpoint of this candidate line segment, which is a Weierstrass point. We will show that Q' is the only periodic point on the interior of ℓ . As before, let P denote P_n and let $\overline{PQ'}$ denote the line segment contained in ℓ that begins at P , travels in the positive horizontal direction and ends at Q' .

Notice that ℓ is entirely contained in a cylinder C_1 that makes an angle of $-\frac{\pi}{n}$ with the horizontal. Apply the element $r_n^{-1}s_nr_n$ of the Veech group, which shears the cylinders parallel to C_1 . When n is odd, any two parallel cylinders have equal moduli (Veech showed this for one specific cylinder direction in [24, Equation 5.2] and so it holds for every cylinder

direction by Theorem 2.1, see Remark 2.3). Therefore, $r_n^{-1}s_n r_n$ acts on C_1 by performing a Dehn twist. In particular, the image $\overline{PQ'}$ under $r_n^{-1}s_n r_n$ remains in C_1 and becomes a line segment from P to a new point Q (see the right subfigure of Fig. 6). Let \overline{PQ} denote this line segment.

It is easy to see that the line segment \overline{PQ} is contained in two cylinders C_2 and C_3 that share a boundary saddle connection and make an angle of $-\frac{\pi}{n}$ with the vertical. Proceeding as in the case of n even yields that \overline{PQ} contains no periodic points in its interior and so the same must hold for $\overline{PQ'}$. Since ℓ is the union of $\overline{PQ'}$ and its image under the hyperelliptic involution, we have that Q' is the only periodic point contained in the interior of ℓ as desired. \square

Proof of corollary 1.7 By [22, Theorem 2.6] whenever a translation surface (X, ω) is not a translation cover of a torus there is a translation cover $\pi_{X_{min}} : (X, \omega) \rightarrow (X_{min}, \omega_{min})$ so that any translation cover with domain (X, ω) is a factor of $\pi_{X_{min}}$. Similarly, by [3, Lemma 3.3], there is a quadratic differential (Q_{min}, q_{min}) and a degree one or two (half)-translation cover $\pi : (X_{min}, \omega_{min}) \rightarrow (Q_{min}, q_{min})$ so that any half-translation cover is a factor of $\pi_{Q_{min}} := \pi \circ \pi_{X_{min}}$. By [3, Theorem 3.6 and Lemma 3.8], if (p, q) are finitely blocked on (X, ω) then one of the following occurs:

1. p and q are periodic points or zeros of ω and the blocking set may be taken to be the collection of all other distinct periodic points.
2. Neither p nor q are periodic points or zeros of ω , but $\pi_{Q_{min}}(p) = \pi_{Q_{min}}(q)$ and the blocking set may be taken to be the union of the periodic points with $\pi_{Q_{min}}^{-1}(\pi_{Q_{min}}(p))$.

Let (X, ω) now denote the regular n -gon or double n -gon surface. By Lemma 2.7, (X, ω) is not a translation cover of a torus and $\pi_{X_{min}}$ is the identity. Since $(X, \omega) = (X_{min}, \omega_{min})$, the discussion above shows that $\pi_{Q_{min}}$ must be degree one or two. By uniqueness of $\pi_{Q_{min}}$, if (X, ω) admits any degree two map to a quadratic differential this map must be $\pi_{Q_{min}}$. Since (X, ω) is hyperelliptic, the quotient by the hyperelliptic involution is such a map and hence must be $\pi_{Q_{min}}$.

By [3, Lemma 3.1], all pairs (p, q) where p is not a zero of ω and q is its image under the hyperelliptic involution are finitely blocked. (Note that the statement of [3, Lemma 3.1] does not include the case when $p = q$ is a Weierstrass point that is not a zero of ω . However, the proof is identical.) By Remark 1.4, it is clear that a zero of ω is never blocked from itself by the collection of Weierstrass points and hence is never finitely blocked from itself.

It remains to show that if p and q are distinct points that are Weierstrass points or zeros of ω , then they are not finitely blocked from each other. Since the blocking set would have to consist of the other Weierstrass points, convexity of the n -gons comprising (X, ω) and the explicit description of the Weierstrass points in the preceding paragraph shows that this is never possible. Thus, the only finitely blocked points are the ones listed in the statement of the corollary. \square

Proof of corollary 1.8 Let T be a billiard table that unfolds to a translation surface (X, ω) . Two points p and q on T are finitely blocked if and only if every preimage of p is finitely blocked from every preimage of q on (X, ω) . When (X, ω) is the regular n -gon or double n -gon surface, each point is finitely blocked from at most one other by Corollary 1.7. When T is the $\left(\frac{\pi}{2}, \frac{\pi}{n}, \frac{(n-2)\pi}{2n}\right)$ triangle the only points on T that have two or fewer preimages on (X, ω) are the vertices of angle $\frac{(n-2)\pi}{2n}$ and $\frac{\pi}{n}$, which, in the first case, unfolds to the zeros of ω and, in the second, to either the Weierstrass point P_n when n is even or to two points exchanged by the hyperelliptic involution when n is odd. Since the zeros of ω are not finitely

blocked from any other point, we see that the only possible pair of finitely blocked points on T is the vertex of angle $\frac{\pi}{n}$ from itself. When n is even, the preimage of this vertex on (X, ω) is P_n , which is finitely blocked from itself. When n is odd, the preimage of this vertex consists of two points swapped by the hyperelliptic involution, and each point is not finitely blocked from itself. \square

4 Proof of lemma 2.4

In this section we will show that for rational numbers $0 < \alpha < \beta \leq \frac{1}{2}$, $\frac{\sin(\pi\alpha)}{\sin(\pi\beta)}$ is rational if and only if $\alpha = \frac{1}{6}$ and $\beta = \frac{1}{2}$. McMullen stated this result in [17, page 7] and indicated that its proof follows from an application of the bounds in the proof of [17, Theorem 2.1]. Since we were unable to find an explicit proof in the literature, we provide one here.

For any positive integer m , let $\zeta_m := \exp\left(\frac{2\pi i}{m}\right)$ and

$$g(m) := \begin{cases} m & m \equiv 2 \pmod{4} \\ 2m & 4 \mid m \\ 4m & m \text{ odd.} \end{cases}$$

We begin with the following simple lemma, which is well known. Throughout this section, if p and q are positive integers we will let (p, q) denote their greatest common divisor.

Lemma 4.1 *For positive integers k and n with $(k, n) = 1$, $\mathbb{Q}(\sin(\frac{\pi k}{n}))$ is the maximal real subfield of $\mathbb{Q}(\zeta_{g(n)})$.*

Proof It is well known that for positive integers ℓ and m with $(\ell, m) = 1$, $\cos\left(\frac{2\pi\ell}{m}\right)$ generates the maximal real subfield of $\mathbb{Q}(\zeta_m)$. Notice that

$$\sin\left(\frac{\pi k}{n}\right) = \cos\left(\frac{\pi}{2} - \frac{\pi k}{n}\right) = \cos\left(\frac{2\pi(n-2k)}{4n}\right).$$

Since $(k, n) = 1$, the only prime that might divide both $n - 2k$ and $4n$ is 2. When n is odd, we see that the numerator is odd, so $(n - 2k, 4n) = 1$ and the claim holds. Similarly, when $4 \mid n$, k is odd and so $\frac{n-2k}{4n} = \frac{\frac{n}{2}-k}{2n}$, where $(\frac{n}{2}-k, 2n) = 1$ since the numerator is odd. Finally, suppose that $n \equiv 2 \pmod{4}$, which in particular implies that $\mathbb{Q}(\zeta_n) = \mathbb{Q}(\zeta_{n/2})$. We see that $4 \mid n - 2k$ and that 8 is the largest power of 2 that divides $4n$. Therefore, when $\frac{n-2k}{4n}$ is put into lowest terms the denominator is either n or $\frac{n}{2}$ and the result follows. \square

Now let $\alpha = \frac{k_1}{n_1}$ and $\beta = \frac{k_2}{n_2}$ be rational numbers expressed in lowest terms where $0 < \alpha < \beta \leq \frac{1}{2}$ and so that $\frac{\sin(\pi\alpha)}{\sin(\pi\beta)}$ is rational. If $n_2 = 2$, then it is well-known that $\alpha = \frac{1}{6}$ and $\beta = \frac{1}{2}$, see Niven [23, Cor. 3.12]. So suppose in order to deduce a contradiction that n_2 and hence also n_1 are greater than 2. Let N be the least common multiple of n_1 and n_2 .

Lemma 4.2 *If n_1 and n_2 are as in the preceding paragraph, then $n_1 = n_2$*

Proof Suppose in order to deduce a contradiction that $n_1 \neq n_2$. Since g is an injection onto the even integers, $\mathbb{Q}(\zeta_{g(n_1)}) \neq \mathbb{Q}(\zeta_{g(n_2)})$. The compositum of these two fields is $\mathbb{Q}(\zeta_M)$ where M is the least common multiple of $g(n_1)$ and $g(n_2)$. Since, by Lemma 4.1, $\mathbb{Q}(\sin(\frac{\pi k_i}{n_i}))$ is the maximal real subfield of $\mathbb{Q}(\zeta_{g(n_i)})$ for $i \in \{1, 2\}$ and since $\frac{\sin(\pi\alpha)}{\sin(\pi\beta)}$ is rational, we see that the

maximal real subfields of $\mathbb{Q}(\zeta_{g(n_1)})$ and $\mathbb{Q}(\zeta_{g(n_2)})$ coincide. In general, if E and F are subfields of \mathbb{C} so that, letting $K := E \cap F$, we have that E/K and F/K are Galois, then the compositum EF/K is Galois and $\text{Gal}(EF/K) \cong \text{Gal}(E/K) \times \text{Gal}(F/K)$. In our case, since we assumed that $n_i > 2$ for $i \in \{1, 2\}$, $\mathbb{Q}(\zeta_{g(n_i)})$ is a degree two extension of its maximal real subfield K . This shows that $\mathbb{Q}(\zeta_M)$ is a degree four extension of K and so $\phi(M) = 2\phi(g(n_1)) = 2\phi(g(n_2))$ where ϕ denotes the Euler phi function. Using that g is injective and that $n_1 \neq n_2$, this implies that $M = 12k$ where $3 \nmid k$ for a positive integer k and where, up to exchanging n_1 with n_2 , $g(n_1) = 6k$ and $g(n_2) = 4k$. Since $\text{Gal}(\mathbb{Q}(\zeta_M)/\mathbb{Q})$ is isomorphic to $(\mathbb{Z}/M\mathbb{Z})^\times$, we see that under the Galois correspondence $\mathbb{Q}(\zeta_{g(n_1)}) = \mathbb{Q}(\zeta_M^2)$ corresponds to the subgroup generated by $6k + 1$ and the maximal real subfield of $\mathbb{Q}(\zeta_M)$ corresponds to the subgroup generated by -1 . Again by the Galois correspondence, $\mathbb{Q}(\zeta_{g(n_2)}) = \mathbb{Q}(\zeta_M^3)$ must correspond to the subgroup generated by $-(6k + 1)$. This implies that $-3(6k + 1) \equiv 3 \pmod{12k}$, equivalently $6k \equiv 6 \pmod{12k}$, which implies that $k = 1$. However, in this case $g(n_2) = 4$ which implies that $n_2 = 1$, a contradiction to the assumption that $n_2 > 2$. \square

Following McMullen [17, Proof of Theorem 2.1], there is a constant C_1 such that $\frac{N}{n_1} \leq C_1$. By Lemma 4.2 we may set $C_1 = 1$. By [17, Proof of Theorem 2.1], $\frac{1}{2} \leq \frac{5\log(2N)}{N}$, which implies that $N < 45$. By Lemma 4.2 we have the following

$$\frac{\sin(\pi\alpha)}{\sin(\pi\beta)} = \frac{\zeta_{2N}^{k_2} - \zeta_{2N}^{-k_2}}{\zeta_{2N}^{k_1} - \zeta_{2N}^{-k_1}} = q$$

for some positive rational number $q \in \mathbb{Q}$. Since $k_2 > k_1$, this implies that ζ_{2N} is a root of the polynomial

$$F(x) := x^{2k_2} - qx^{k_1+k_2} + qx^{k_2-k_1} - 1.$$

The minimal polynomial for ζ_{2N} , which is the $(2N)$ th cyclotomic polynomial, has degree $\phi(2N)$ and divides F .

Lemma 4.3 *F is the $(2N)$ th cyclotomic polynomial.*

Proof If not, then because $\phi(2N) \mid 2k_2$, it would follow that $2\phi(2N) \leq 2k_2$. Since $\frac{k_2}{N} = \beta < \frac{1}{2}$ we have, $\frac{\phi(2N)}{2N} < \frac{1}{4}$. Let Π be the set of primes that divide $2N$, then we have

$$\frac{\phi(2N)}{2N} = \prod_{p \in \Pi} \left(\frac{p-1}{p} \right) < \frac{1}{4}$$

For this inequality to hold, N would need to have at least three prime factors aside from 2. The smallest such number is 105, which is greater than 45. \square

Since k_2 is coprime to N and since $2k_2 = \phi(2N)$, we have that $(N, \phi(N)) = 1$. This implies that N is squarefree and since $N > 2$, that N is odd. Since $N < 45$, N is prime or $N \in \{15, 21, 33, 35, 39\}$. We can discard the cases of $N \in \{21, 39\}$ since $(N, \phi(N)) \neq 1$. When N is prime, the $(2N)$ th cyclotomic polynomial is $\sum_{k=0}^{N-1} (-1)^k x^k$, which is never the same as F . The $(2N)$ th cyclotomic polynomials for $N \in \{15, 33, 35\}$ all have more than four nonzero coefficients of monomial terms, so again they cannot be F and we have a contradiction.

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Declarations

Conflicts of interest The authors have no relevant financial or non-financial interests to disclose.

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