Bi-Lipschitz geometry of quasiconformal trees

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Abstract A quasiconformal tree is a doubling metric tree in which the diameter of each arc is bounded above by a fixed multiple of the distance between its endpoints. We study the geometry of these trees in two directions. First, we construct a catalog of metric trees in a purely combinatorial way, and show that every quasiconformal tree is bi-Lipschitz equivalent to one of the trees in our catalog. This is inspired by results of Herron and Meyer and of Rohde for quasi-arcs. Second, we show that a quasiconformal tree bi-Lipschitz embeds in a Euclidean space if and only if its set of leaves admits such an embedding. In particular, all quasi-arcs bi-Lipschitz embed into some Euclidean space.

Contents

1. Introduction

In this paper, a *(metric) tree* is a compact, connected, locally connected metric space with the property that each pair of distinct points forms the endpoints of a unique arc. In some sense, trees make up the simplest class of one-dimensional continua and are ubiquitous in analysis and geometry.

Within the class of all trees, an important role has been played by the class of *quasiconformal trees* studied in [2, 3, 16]. By definition, these are trees T that satisfy two simple geometric properties:

- T is *doubling*: there is a constant N such that each ball in T can be covered by N balls of half the radius.
- T is *bounded turning*: there is a constant C such that each pair of points $x, y \in T$ can be joined by a continuum whose diameter is at most $Cd(x, y)$.

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These conditions are both invariant under quasisymmetric mappings, making the class of quasiconformal trees a natural quasisymmetrically invariant class. We do not recall the definition of quasisymmetric mappings here (see [2] or [13]) but merely note that they are an important generalization of conformal mappings to arbitrary metric spaces.

Quasiconformal trees appear in several fields of analysis. For instance, Julia sets of semihyperbolic polynomials (e.g., $z^2 + i$) are quasiconformal trees (see [5, p. 95] and [6]), and quasiconformal trees T in $\mathbb{R}^2$ (often called *Gehring trees*) were recently characterized by Lin and Rhode [19] in terms of the laminations of the conformal map $f : \mathbb{C} \setminus \mathbb{D} \to \mathbb{C} \setminus T$.

Quasiconformal trees generalize two more well-known types of spaces. For one, quasiconformal trees that are simply topological arcs (i.e., have no branching) are called *quasi-arcs* and have been studied in complex analysis and analysis on metric spaces for decades [9]. For example, the famous von Koch snowflake is a quasi-arc. A well-known result of Tukia and Väisälä [26] shows that quasi-arcs are exactly those spaces that are quasisymmetrically equivalent to the unit interval $[0, 1]$.

Quasiconformal trees also generalize (doubling) *geodesic trees*. Geodesic trees are trees in which, for each pair of points x, y, the unique arc joining them has (finite) length equal to $d(x, y)$. Thus, in geodesic trees all paths are “straight” (isometric to intervals in the real line), whereas paths in quasiconformal trees may be fractal, like the von Koch snowflake. Geodesic trees are of course standard objects of study in many parts of mathematics and computer science. Recently, Bonk and Meyer [2] generalized the result of Tukia and Väisälä mentioned above by showing that each quasiconformal tree is quasisymmetric to a geodesic tree.

Rather than studying the quasisymmetric geometry of quasiconformal trees, this paper is concerned with the finer notion of bi-Lipschitz geometry. Recall that a mapping f between two metric spaces is called *bi-Lipschitz* (or L-bi-Lipschitz to emphasize the constant) if there is a constant $L \geq 1$ such that

$$L^{-1} d(x, y) \leq d\big(f(x), f(y)\big) \leq L d(x, y), \qquad \text{for all } x, y \in X.$$

Thus, bi-Lipschitz mappings preserve distances up to constant factors. All bi-Lipschitz mappings are quasisymmetric, but the converse is false. For example, one may parametrize the von Koch snowflake K by a quasisymmetric map $[0, 1] \to K$ but not by a bi-Lipschitz map.

Given a metric space X, natural questions in the bi-Lipschitz world are as follows:

- *Uniformization*: Which metric spaces are bi-Lipschitz equivalent to X (i.e., admit a surjective bi-Lipschitz mapping onto X)?
- *Embeddability*: Does X admit a bi-Lipschitz embedding into some Euclidean space $\mathbb{R}^n$ (i.e., a bi-Lipschitz mapping from X into $\mathbb{R}^n$)?

The first of these questions is about recognizing or providing models for spaces up to bi-Lipschitz equivalence—that is, up to bounded distortion of their metrics. The second is about understanding which spaces can be viewed as subsets of Euclidean space up to bounded distortion, and in complete generality is a major problem in analysis, geometry, and computer science [12, 21].

We study both of these questions for quasiconformal trees. Concerning the first, we give a "combinatorial model" for generating quasiconformal trees based on a purely discrete construction and then show that every quasiconformal tree is bi-Lipschitz equivalent to one of our combinatorial constructions. This is in the vein of the combinatorial models for quasi-arcs up to bi-Lipschitz equivalence given by Herron and Meyer [14] and by Rohde [22], although the construction for trees is more elaborate. Our main theorem on this topic is Theorem 1.4.

Concerning the second question, we build on ideas from [24] to show that every **quasi-arc** admits a bi-Lipschitz embedding into some Euclidean space and use this to show that the bi-Lipschitz embedding properties of quasiconformal trees are completely controlled by their sets of *leaves* (Theorem 1.8). We leave open the main question of whether all quasiconformal trees admit bi-Lipschitz embeddings into Euclidean space; see below for additional background and discussion.

We now discuss these ideas in more detail.

1.1. Combinatorial models for quasiconformal trees up to bi-Lipschitz equivalence

We first give a way to define metric spaces using certain sequences of combinatorial graphs—that is, $G = (V, E)$, where V is the vertex set and E is the edge set. This is inspired by the ideas of [14] and [22] concerning quasi-arcs, with a number of new wrinkles in the case of trees. To simplify the presentation as much as possible, a number of definitions are postponed until Section 2.

Let A be an "alphabet": a set of the form $\{1, \dots, n\}$, or $A = \mathbb{N}$. Denote by ε the empty word and by $|w|$ the length of a word—that is, the number of letters. Let $A^0 = \{\varepsilon\}$, and for each $k \in \mathbb{N}$ denote by A^k the set of all words made from the alphabet A of length exactly k. Define the set of finite words

$$A^* = \bigcup_{k=0}^{\infty} A^k.$$

Denote also by $A^{\mathbb{N}}$ the set of infinite words formed by the alphabet A, and $A^{\mathbb{N}}_u \subseteq A^{\mathbb{N}}$ the set of all infinite words that begin with a given finite word $u \in A^*$.

DEFINITION 1.1

We consider the following **combinatorial data** $\mathcal{C} = (A, (G_k)_{k \in \mathbb{N}})$, where

(1) A is a finite or infinite alphabet: $A = \{1, \dots, M\}$ for some integer $M \geq 2$, or $A = \mathbb{N}$;
(2) for each $k \in \mathbb{N}$, $G_k = (A^k, E_k)$ is a connected combinatorial graph on the vertex set A^k with the following properties:
   (a) For each $w \in A^k$, the subgraph of G_{k+1} induced by the vertex set $\{wi : i \in A\}$ is connected.
   (b) If $\{w, u\} \in E_k$, then there is a pair $(i, j) \in A \times A$ such that $\{wi, uj\} \in E_{k+1}$.

We next define a way to "move between" different infinite word sets $A^{\mathbb{N}}_u$ using the structure of the combinatorial data. Moves between $A^{\mathbb{N}}_u$ and $A^{\mathbb{N}}_v$ are always permitted if

u and v are adjacent words of equal length, but in general we take into account the full scope of the combinatorial data.

Thus, given combinatorial data $\mathcal{C} = (A, (G_k)_{k\in\mathbb{N}})$, we say that two infinite word sets $A^{\mathbb{N}}_{u_1}$ and $A^{\mathbb{N}}_{u_2}$ *combinatorially intersect*, and write $A^{\mathbb{N}}_{u_1} \wedge_{\mathcal{C}} A^{\mathbb{N}}_{u_2} \neq \emptyset$ if the following holds:

$$\text{For each } n > \max\{|u_1|, |u_2|\}, \text{ there exist words } w_1, w_2 \in A^n, \text{ beginning with } u_1 \text{ and } u_2, \text{ respectively, that are adjacent in } G_n. \tag{1.1}$$

In other words, two word sets $A^{\mathbb{N}}_{u_1}$ and $A^{\mathbb{N}}_{u_2}$ combinatorially intersect if their restrictions to every sufficiently large finite level are adjacent. Below, in Definition 3.2, we will give a precise definition of the set $A^{\mathbb{N}}_{u_1} \wedge_{\mathcal{C}} A^{\mathbb{N}}_{u_2}$ and show that its non-emptiness is equivalent to (1.1).

Given this notion of combinatorial intersection, we can describe how to move between two infinite words, as follows.

DEFINITION 1.2
Given two words $w, w' \in A^{\mathbb{N}}$, we say that $\{A^{\mathbb{N}}_{w_1}, \dots, A^{\mathbb{N}}_{w_n}\}$ is a *chain joining w with w'* if $w \in A^{\mathbb{N}}_{w_1}$, $w' \in A^{\mathbb{N}}_{w_n}$ and for every $i = 1, \dots, n-1$, we have $A^{\mathbb{N}}_{w_i} \wedge_{\mathcal{C}} A^{\mathbb{N}}_{w_{i+1}} \neq \emptyset$.

Now that we have a way to move between two infinite words, we can define a distance on $A^{\mathbb{N}}$ by assigning costs to each chain with a "diameter function" as follows.

DEFINITION 1.3
Given an alphabet A, a *diameter function* is a function $\Delta : A^* \to [0,1]$ such that

(1) $\Delta(\varepsilon) = 1$;
(2) for each $w \in A^k$ and $i \in A$, $\Delta(wi) = 0$ for all but finitely many $i \in A$; and
(3) $\lim_{n\to\infty} \max\{\Delta(w) : w \in A^n\} = 0$.

The class of all diameter functions on A is defined by $\mathcal{D}(A)$. Given $0 < \delta_1 \le \delta_2 \le 1$ and finite A, we denote by $\mathcal{D}(A, \delta_1, \delta_2)$ the collection of all diameter functions on the alphabet A such that

$$\text{for each } w \in A^* \text{ and } i, j \in A, \qquad \Delta(wi) = \Delta(wj) \qquad \text{and} \qquad \frac{\Delta(wi)}{\Delta(w)} \in \{\delta_1, \delta_2\}.$$

Note that, in Definition 1.3, (2) is automatic if A is finite, and (3) is automatic if $\Delta \in \mathcal{D}(A, \delta_1, \delta_2)$ and $\delta_2 < 1$. In (3), Condition (2) implies that the maximum is actually achieved, even if A is infinite.

Given combinatorial data $\mathcal{C} = (A, (G_k)_{k\in\mathbb{N}})$ and $\Delta \in \mathcal{D}(A)$, we define a pseudo-metric $D_{\mathcal{C},\Delta}$ on $A^{\mathbb{N}}$ by

$$D_{\mathcal{C},\Delta}(w, u) = \inf \sum_{i=0}^{N} \Delta(v_i), \tag{1.2}$$

where the infimum is taken over all chains $\{A^{\mathbb{N}}_{v_i}\}$ joining w with u.

We prove in Lemma 3.8 that $D_{\mathcal{C},\Delta}$ is indeed always a pseudometric on $A^{\mathbb{N}}$. Taking the quotient space $\mathcal{A} := A^{\mathbb{N}}/\sim$ under the equivalence relation $w \sim w'$ whenever $D_{\mathcal{C},\Delta}(w,w') = 0$, we obtain a metric space

$$(\mathcal{A}, d_{\mathcal{C},\Delta}),$$

where $d_{\mathcal{C},\Delta}([w],[v]) = D_{\mathcal{C},\Delta}(w,v)$.

To help digest the definition, we provide a number of examples illustrating this combinatorial construction in Section 6 below.

Our main theorem on these combinatorial models is as follows.

THEOREM 1.4

(1) *If $\mathcal{C}$ defines combinatorial data and $\Delta \in \mathcal{D}(A)$, then the space $(\mathcal{A}, d_{\mathcal{C},\Delta})$ is compact, connected, and bounded turning with constant $C = 1$.*
(2) *If, in addition, each graph G_k in the combinatorial data is a combinatorial tree, then the space $(\mathcal{A}, d_{\mathcal{C},\Delta})$ is a metric tree.*
(3) *Conversely, if X is an arbitrary quasiconformal tree, then there exist combinatorial data $\mathcal{C} = (A, (G_k)_{k\in\mathbb{N}})$ and a diameter function $\Delta \in \mathcal{D}(A, K_1, K_2)$ such that each G_k is a combinatorial tree and X is bi-Lipschitz equivalent to the space $(\mathcal{A}, d_{\mathcal{C},\Delta})$. The choice of alphabet, the constants K_1 and K_2, and the bi-Lipschitz constant depend only on the doubling and bounded turning constants of X and on* $\operatorname{diam}(X)$.

Parts (1) and (2) of Theorem 1.4 are proven in Proposition 3.10, and Part (3) is proven (with a more detailed statement) in Theorem 5.1.

We emphasize that an important feature of Theorem 1.4 is that all quasiconformal trees are built (up to bi-Lipschitz equivalence) not only from combinatorial objects but from the simple *homogeneous* word sets $A^{\mathbb{N}}$ and the additional data provided by $\{G_k\}$ and the diameter function. In some sense, one can view the construction in [14], which combinatorially builds bi-Lipschitz models of all quasi-*arcs*, as being a special case of the above construction in the case where A has two elements and the graphs G_k are combinatorial arcs, and so we show that the above re-interpretation and expansion of their construction yields all quasiconformal *trees* up to bi-Lipschitz equivalence. Later, in Section 6, we provide some concrete examples and pictures of the combinatorial construction described above, including describing in more detail how quasi-arcs fit into our picture.

REMARK 1.5

The metric space $(\mathcal{A}, d_{\mathcal{C},\Delta})$ constructed from given combinatorial data and diameter function need not be doubling in general, even if the alphabet A is finite, the graphs G_k are all combinatorial trees, and the diameter function Δ lies in $\mathcal{D}(A, \delta_1, \delta_2)$ for $0 < \delta_1 < \delta_2 < 1$.

However, in Proposition 4.1 we give some sufficient conditions that imply that the space $(\mathcal{A}, d_{\mathcal{C},\Delta})$ is doubling. In Theorem 1.4(3), the space $(\mathcal{A}, d_{\mathcal{C},\Delta})$ that we construct always satisfies these conditions. This is stated explicitly in Theorem 5.1.

1.2. Combinatorial descriptions of metric spaces with good tilings

Some techniques in the proof of Theorem 1.4(3) can be used for a more general class of metric spaces that can be tiled in a uniform fashion. Roughly speaking, we say that a metric space has a "good tiling" if there exists an alphabet A, a constant $r \in (0,1)$, and a tiling decomposition $\{X_w : w \in A^*\}$ of X such that each tile X_w has diameter comparable to $r^{|w|}$ and any two nonintersecting tiles X_w, X_u have distance at least a fixed multiple of $\max\{r^{|w|}, r^{|u|}\}$. See Section 7 for a precise definition.

In Proposition 7.1, we show that any such space is bi-Lipschitz equivalent to a space $(\mathcal{A}, d_{\mathcal{C},\Delta})$ for some combinatorial data $\mathcal{C}$ and $\Delta(w) = r^{|w|}$. Spaces with good tilings include many attractors of iterated function systems such as the square, the cube, the Sierpiński carpet, the Sierpiński gasket, and others; see Example 7.2 for further discussion.

We note that Proposition 7.1 is not a generalization of Theorem 1.4: if X is a quasiconformal tree, the combinatorial data that Proposition 7.1 will provide may not consist of combinatorial trees, as required by Theorem 1.4. The proof also proceeds differently, and in fact we do not know if every quasiconformal tree possesses a good tiling in the sense given in Section 7.

1.3. Bi-Lipschitz embeddings of quasi-arcs and quasiconformal trees

We now turn our attention to the problem of finding bi-Lipschitz embeddings of quasiconformal trees into Euclidean space. The most natural question is as follows.

QUESTION 1.6

Does every quasiconformal tree have a bi-Lipschitz embedding into some Euclidean space $\mathbb{R}^n$?

We do not answer this question here and, indeed, it may be rather difficult to answer. In the special case of doubling, *geodesic* trees, the answer is known to be positive, by a theorem of Gupta–Krauthgammer–Lee [10]; see also [11, Corollary 8]. Lee–Naor–Peres also give an alternative proof of the result for geodesic trees in [18, Theorem 2.12].

By adapting techniques of Romney and the second named author, we make progress on Question 1.6 in the case where the quasiconformal tree has no branching, as follows.

PROPOSITION 1.7

Every quasi-arc admits a bi-Lipschitz embedding into some Euclidean space $\mathbb{R}^n$.

Proposition 1.7 is a simplified version of Proposition 8.2 below, where we identify the sharp dimension n for the embedding. We note that Herron and Meyer proved Proposition 1.7 in the special case of quasi-arcs with Assouad dimension less than 2; see [14, Theorem C].

Using Proposition 8.2, and results of Lang and Plaut [17] and of Seo [25], we end by giving a criterion that can answer Question 1.6 in certain examples. If X is a metric tree, we denote by $\mathcal{L}(X)$ be the set of *leaves* of X; that is,

$$\mathcal{L}(X) := \{x \in X : X \setminus \{x\} \text{ is connected}\}.$$

THEOREM 1.8

A quasiconformal tree X admits a bi-Lipschitz embedding into some Euclidean space if and only if $\mathcal{L}(X)$ admits a bi-Lipschitz embedding into some Euclidean space.

Theorem 1.8 is a simplified version of the quantitative statement of Theorem 8.1.

REMARK 1.9

If X is a quasiconformal tree, the set $\mathcal{L}(X)$ need not be closed and may even be dense in X. Thus, Theorem 1.8 does not necessarily always reduce Question 1.6 to a simpler problem.

In many particular cases, however, it may be significantly easier to check the embeddability of $\mathcal{L}(X)$ rather than X itself. For example, in many concrete settings, the leaf set $\mathcal{L}(X)$ is an *ultrametric space*, and every doubling ultrametric space bi-Lipschitz embeds into some Euclidean space [20].

REMARK 1.10

An equivalent reformulation of Theorem 1.8 is that a subset E of a quasiconformal tree X admits a bi-Lipschitz embedding into some Euclidean space if and only if the minimal subtree of X containing E does.

1.4. Outline of the paper

In Section 2, we review some elementary notions from graph theory concerning combinatorial graphs and trees. In Section 3, we provide more details on our combinatorial models and prove parts (1) and (2) of Theorem 1.4. In Section 4, we work in the case of combinatorial trees and identify conditions on A, $\mathcal{C}$, and Δ that guarantee that the metric tree $(\mathcal{A}, d_{\mathcal{C},\Delta})$ is doubling.

In Section 5, we prove a more detailed version of Part (3) of Theorem 1.4. The basic idea is to construct an n-adic decomposition $(X_w)_{w\in\{1,\dots,n\}^*}$ of a given quasiconformal tree X for some $n \geq 2$ that satisfies the following properties:

(1) Each X_w is the union of its children $X_{w1}, \dots, X_{wn}$, which are themselves trees. Each two of the children intersect in at most one point, which has valency 2 in X.
(2) Each child X_{wi} of X_w has diameter comparable to that of X_w.
(3) Any two points x, y on $X_w \cap \overline{X \setminus X_w}$ have distance comparable to the diameter of X_w.

This is accomplished by performing certain subdivisions and gluings on top of a construction of Bonk and Meyer [2]. Once we have such a decomposition, we can build combinatorial data $\mathcal{C}$ and a diameter function Δ such that $(\mathcal{A}, d_{\mathcal{C},\Delta})$ is bi-Lipschitz equivalent to X.

Section 6 contains some examples and pictures that illustrate how our combinatorial data yields metric spaces in a few concrete cases.

Section 7 considers more general metric spaces, not necessarily trees, that admit a notion of "good tiling." We show that such spaces can also be viewed from our combinatorial data, in a slightly different way than Theorem 1.4. In particular, we describe how some self-similar spaces like the unit square and the Sierpiński gasket can be constructed in our framework.

Finally, in Section 8, we prove a quantitative version of Proposition 1.7 and then apply a bi-Lipschitz welding result of Lang and Plaut [17] and a bi-Lipschitz embedding characterization of Seo [25] to complete the proof of Theorem 1.8.

2. Preliminaries

In this section, we introduce some further preliminary definitions and results related to the combinatorial models defined in Section 1.1.

2.1. *Words*

Recall from Section 1.1 that we start with an alphabet $A = \{1, \dots, M\}$ for some integer $M \geq 2$, or $A = \mathbb{N}$. In addition to the sets A^*, $A^{\mathbb{N}}$, $A_u^{\mathbb{N}}$ defined above, we also set a few other pieces of notation. For $w \in A^*$ and $k \geq |w|$, define

$$A_w^k = \{wu : u \in A^{k-|w|}\}, \qquad A_w^* = \{wu : u \in A^*\}.$$

Given $n \in \mathbb{N}$ and $w \in A^{\mathbb{N}}$, denote by $w(n)$ the unique word $u \in A^n$ such that $w = uw'$ for some $w' \in A^{\mathbb{N}}$. Similarly, if $n \in \mathbb{N}$ and $w \in A^*$, $w(n)$ denotes the initial subword of w with length n, and we set $w(n) = w$ if $n \geq |w|$.

Finally, given $k \in \mathbb{N}$ and $u \in A^k$, denote by $u^{\uparrow}$ the unique element of A^{k-1} such that $u \in A_{u^{\uparrow}}^k$.

2.2. *Combinatorial graphs and trees*

Definition 1.1 above uses some graph theory terminology. A *combinatorial graph* is a pair $G = (V, E)$ of a finite or countable vertex set V and an edge set

$$E \subset \big\{\{v, v'\} : v, v' \in V \text{ and } v \neq v'\big\}.$$

If $\{v, v'\} \in E$, we say that the vertices v and v' are *adjacent* in G.

A combinatorial graph $G' = (V', E')$ is a *subgraph* of $G = (V, E)$ (and we write $G \subset G'$) if $V' \subset V$ and $E' \subset E$. We commonly generate subgraphs of $G = (V, E)$ by starting with a vertex set $V' \subset V$ and considering the *subgraph of G induced by V'*: the graph $G' = (V', E')$, where E' is the set of all edges between two vertices of V'.

A *path* in G is a set $\gamma = \{\{v_1, v_2\}, \dots, \{v_{n-1}, v_n\}\} \subset E$; in this case, we say that γ joins v_1, v_n. The path $\gamma = \{\{v_1, v_2\}, \dots, \{v_{n-1}, v_n\}\}$ is a *combinatorial arc* or *simple path* if for all $i, j \in \{1, \dots, n\}$, $v_i = v_j$ if and only if $i = j$; in this case we say that the endpoints of the arc γ are the points v_1, v_n. A combinatorial graph $G = (V, E)$ is connected if for any distinct $v, v' \in V$ there exists a path γ in G that joins v with v'. A *component* of a combinatorial graph G is a maximal connected subgraph of G.

A graph $T = (V, E)$ is a *combinatorial tree* if for any distinct v, v' there exists unique combinatorial arc γ whose endpoints are v and v'. Given a combinatorial tree $T = (V, E)$ and a point $v \in V$, define the valencies

$$\mathrm{Val}(T, v) := \mathrm{card}\{e \in E : v \in e\} \qquad \text{and} \qquad \mathrm{Val}(T) := \max_{v \in V} \mathrm{Val}(T, v)$$

and the set of leaves $\mathrm{Leaves}(T) := \{v \in V : \mathrm{Val}(T, v) = 1\}$. Here, card denotes the cardinality of a finite or countable set, taking values in $\mathbb{N} \cup \{\infty\}$.

Given a combinatorial graph $G = (V, E)$ and a vertex $v \in V$, we write $G \setminus \{v\}$ to be the subgraph of G induced by $V \setminus \{v\}$. Note that, if T is a tree, then every component of $T \setminus \{v\}$ is a tree.

3. A model for bounded turning metric spaces and trees

3.1. Combinatorial data

Recall the notion of combinatorial data $\mathcal{C} = (A, (G_k)_{k \in \mathbb{N}})$ from Definition 1.1, where A is an alphabet and $G_k = (A^k, E_k)$ are combinatorial graphs on the vertex sets A^k, satisfying certain axioms. **For the remainder of Section 3, we fix combinatorial data $\mathcal{C} = (A, (G_k)_{k \in \mathbb{N}})$.**

Our first lemma gives some basic structural properties of these graphs. In particular, if each G_k is a combinatorial tree, then the pair $(i, j) \in A \times A$ of Definition 1.1(2b) is unique.

LEMMA 3.1

Let $k \geq j$ and $v \neq w \in A^j$.

(1) *If v and w are adjacent in G_j, then there are words v' and w' in A^{k-j} such that vv' and ww' are adjacent in G_k.*
(2) *If G_k is a combinatorial tree and there are words v' and w' in A^{k-j} such that vv' and ww' are adjacent in G_k, then v and w are adjacent in G_j.*
(3) *If G_k is a combinatorial tree and v and w are adjacent in G_j, then there is a unique pair of words (v', w') in $A^{k-j} \times A^{k-j}$ such that vv' and ww' are adjacent in G_k.*

Proof

The first statement is an immediate consequence of (2b) in Definition 1.1, and induction on $k - j$.

For the second, suppose that v and w were not adjacent in G_j, under these assumptions.

Let $v = u_0, u_1, \ldots, u_{n-1}, u_n = w$ be a path from v to w in G_j. Note that $n \geq 2$. Then, by the first statement in the lemma and Part (2) of Definition 1.1, there is a simple path from $vv' \in A_v^k$ to $ww' \in A_w^k$ in G_k of the form

$$\text{elements of } A_{u_0}^k, \text{ elements of } A_{u_1}^k, \ldots, \text{ elements of } A_{u_n}^k.$$

On the other hand, there is also an adjacency between vv' and ww' in G_k. This contradicts the assumption that G_k is a tree.

For the third claim, the existence of v' and w' follows from (1). Suppose that the uniqueness failed. We consider the following two possible cases.

Suppose first that there are two distinct $v', v'' \in A$ and $w' \in A$ such that both vv' and vv'' are adjacent to ww'. Then there exists two combinatorial arcs in G_k that join vv' with vv'': one through the vertices of G_k restricted on A^k_v (by (2a) in Definition 1.1), and another is $\{\{vv', ww'\}, \{ww', vv''\}\}$. This contradicts the fact that G_k is a tree.

The other possibility is that there are two distinct $v', v'' \in A$ and two distinct $w', w'' \in A$ such that vv' is adjacent to ww', and vv'' are adjacent to ww''. Then there exist two combinatorial arcs in G_k that join vv' with vv'': one through the vertices of G_k restricted on A^k_v (by (2a) in Definition 1.1) and another through the vertices of G_k restricted on A^k_w along with edges $\{vv', ww''\}$ and $\{vv'', ww''\}$. This again contradicts the fact that G_k is a tree. □

3.2. Combinatorial intersection and chains

Recall the notion of combinatorial intersection $A^{\mathbb{N}}_u \wedge_{\mathcal{C}} A^{\mathbb{N}}_v$ defined in (1.1) in Section 1.1. There, we defined only what it means for this set to be non-empty, but here we actually give a meaning to the set itself.

DEFINITION 3.2

Given $u_1, u_2 \in A^*$, we define

$$\begin{aligned} A^{\mathbb{N}}_{u_1} \wedge_{\mathcal{C}} A^{\mathbb{N}}_{u_2} := \{ & w \in A^{\mathbb{N}}_{u_1} : \forall n > \max\{|u_1|, |u_2|\} \text{ there exists } u \in A^n_{u_2} \\ & \text{with } \{w(n), u\} \in E_n\} \\ & \cup \{w \in A^{\mathbb{N}}_{u_2} : \forall n > \max\{|u_1|, |u_2|\} \text{ there exists } u \in A^n_{u_1} \\ & \text{with } \{w(n), u\} \in E_n\}. \end{aligned} \tag{3.1}$$

The set $A^{\mathbb{N}}_{u_1} \wedge_{\mathcal{C}} A^{\mathbb{N}}_{u_2}$ is called the *combinatorial intersection* of $A^{\mathbb{N}}_{u_1}$ and $A^{\mathbb{N}}_{u_2}$.

We now show that this definition agrees with that in (1.1) and give an equivalent reformulation in the case of trees.

LEMMA 3.3

Let $u_1, u_2 \in A^*$. The following are equivalent:

(1) *The set $A^{\mathbb{N}}_{u_1} \wedge_{\mathcal{C}} A^{\mathbb{N}}_{u_2}$ is non-empty.*
(2) *For every $k > \max\{|u_1|, |u_2|\}$ there exists $v_1 \in A^k_{u_1}$ and $v_2 \in A^k_{u_2}$ such that $\{v_1, v_2\} \in E_k$.*

If each graph G_k is a combinatorial tree, then (1) and (2) are also equivalent to the following:

(3) *There exists $k > \max\{|u_1|, |u_2|\}$ and $v_1 \in A^k_{u_1}$, $v_2 \in A^k_{u_2}$ such that $\{v_1, v_2\} \in E_k$.*

Proof

We start by showing the equivalence of (1) and (2). That (1) implies (2) follows immediately from the definition of $A^{\mathbb{N}}_{u_1} \wedge_{\mathcal{C}} A^{\mathbb{N}}_{u_2}$.

To show that (2) implies (1), we will inductively construct elements of $A^{\mathbb{N}}_{u_1} \wedge_{\mathcal{C}} A^{\mathbb{N}}_{u_2}$. Let $k_0 = \max\{|u_1|, |u_2|\}$ and choose $u_1 i_1 \in A^{k_0+1}_{u_1}$, $u_2 j_1 \in A^{k_0+1}_{u_2}$ such that $\{u_1 i_1, u_2 j_1\} \in E_{k_0+1}$. By (2b) in Definition 1.1, given that $\{u_1 i_1 \cdots i_{n-k}, u_2 j_1 \cdots j_{n-k}\} \in E_n$ for some $n \geq k+1$, there exist $i_{n-k+1}, j_{n-k+1} \in A$ such that $\{u_1 i_1 \cdots i_{n-k+1}, u_2 j_1 \cdots j_{n-k+1}\} \in E_{n+1}$. Set now

$$w_1 = u_1 i_1 i_2 \cdots \qquad \text{and} \qquad w_2 = u_2 j_1 j_2 \cdots$$

and note that both w_1 and w_2 are in $A^{\mathbb{N}}_{u_1} \wedge_{\mathcal{C}} A^{\mathbb{N}}_{u_2}$.

Assume now that each graph G_k is a combinatorial tree. Clearly, (2) implies (3) so it suffices to show that (3) implies (2). Assume there is an integer $k_0 \geq \max\{|u_1|, |u_2|\}$ and words $w_1 \in A^{k_0}_{u_1}$ and $w_2 \in A^{k_0}_{u_2}$ such that $\{w_1, w_2\} \in E_{k_0}$. If $k \geq k_0$, then by Lemma 3.1(1), there exist $v_1 \in A^k_{w_1}$ and $v_2 \in A^k_{w_2}$ (hence, $v_1 \in A^k_{u_1}$ and $v_2 \in A^k_{u_2}$) such that $\{v_1, v_2\} \in E_k$. If k is an integer with $\max\{|u_1|, |u_2|\} \leq k \leq k_0$, then by Lemma 3.1(2), there exist $v_1 \in A^k_{u_1}$ and $v_2 \in A^k_{u_2}$ such that $w_1 \in A^k_{v_1}$, $w_2 \in A^k_{v_2}$ and $\{v_1, v_2\} \in E_k$. Therefore, (2) holds. □

The next lemma gives a description of the set $A^{\mathbb{N}}_{u_1} \wedge_{\mathcal{C}} A^{\mathbb{N}}_{u_2}$ in the case that each G_k is a combinatorial tree.

LEMMA 3.4

Let $u_1, u_2 \in A^*$ with $|u_1| \leq |u_2|$, let $k_1 = |u_1|$, and let $u_2' = u_2(k_1)$.

(1) *If $u_2' = u_1$ (that is, $u_2 \in A^*_{u_1}$), then $A^{\mathbb{N}}_{u_2} \subset A^{\mathbb{N}}_{u_1} \wedge_{\mathcal{C}} A^{\mathbb{N}}_{u_2}$.*

Suppose additionally that each G_k is a combinatorial tree. Then:

(2) *If $A^{\mathbb{N}}_{u_1} \wedge_{\mathcal{C}} A^{\mathbb{N}}_{u_2} \neq \emptyset$, then either $\{u_1, u_2'\} \in E_{k_1}$ or $u_1 = u_2'$.*

(3) *If $\{u_1, u_2'\} \in E_{k_1}$, then $A^{\mathbb{N}}_{u_1} \wedge_{\mathcal{C}} A^{\mathbb{N}}_{u_2}$ contains exactly two elements: one in $A^{\mathbb{N}}_{u_1}$ and one in $A^{\mathbb{N}}_{u_2}$. The converse is also true.*

Proof

Let $u_1, u_2, v \in A^*$, and $k_1 \in \mathbb{N}$ be as in the statement, and let $k_2 = |u_2|$.

To prove (1), assume that $u_2' = u_1$; that is, $u_2 \in A^{k_2}_{u_1}$. Let $w \in A^{\mathbb{N}}_{u_2}$. By Definition 1.1(2a), the subgraph of G_{k_2+1} induced by $A^{k_2+1}_{u_1}$ is connected. Fix $v \in A^{k_2+1}_{u_1}$ adjacent to $w(k_2+1)$. Applying Definition 1.1(2b), we find a sequence $\{i_1, i_2, \dots\} \subset A$ such that for each $n \in \mathbb{N}$, $v i_1 \cdots i_n$ is adjacent to $w(k_2+n+1)$. Since $v i_1 \cdots i_n \in A^{k_2+n+1}_{u_1}$, by definition, $w \in A^{\mathbb{N}}_{u_1} \wedge_{\mathcal{C}} A^{\mathbb{N}}_{u_2}$.

Assume now for the rest of the proof that each G_k is a combinatorial tree. To prove (2), assume that $A^{\mathbb{N}}_{u_1} \wedge_{\mathcal{C}} A^{\mathbb{N}}_{u_2} \neq \emptyset$. By Lemma 3.3(2), we have that there exists $v_1 \in A^{k_2+1}_{u_1}$ and $v_2 \in A^{k_2+1}_{u_2}$ such that $\{v_1, v_2\} \in E_{k_2+1}$. Applying Lemma 3.1(2), we have that either $u_1 = u_2'$ or $\{u_1, u_2'\} \in E_{k_1}$.

To prove (3), assume that $\{u_1, u_2'\} \in E_{k_1}$, and let v_1 and v_2 be as in the proof of (2)—that is, $v_1 \in A^{k_2+1}_{u_1}$, $v_2 \in A^{k_2+1}_{u_2}$, and $\{v_1, v_2\} \in E_{k_2+1}$. By Definition (2b) of 1.1, there exist $i_1, i_2, \dots \in A$ and $j_1, j_2, \dots \in A$ such that for all $m \in \mathbb{N}$, $\{v_1 i_1 \cdots i_m, v_2 j_1 \cdots j_m\} \in E_{k_2+1+m}$. It follows that the words $w_1 = v_1 i_1 i_2 \cdots \in A^{\mathbb{N}}_{u_1}$ and $w_2 = v_2 j_1 j_2 \cdots \in A^{\mathbb{N}}_{u_2}$ are in $A^{\mathbb{N}}_{u_1} \wedge_{\mathcal{C}} A^{\mathbb{N}}_{u_2}$.

Suppose now that there exist two distinct $w_1', w_1 \in A^{\mathbb{N}}_{u_1}$ such that $w_1', w_1 \in A^{\mathbb{N}}_{u_1} \wedge_{\mathcal{C}} A^{\mathbb{N}}_{u_2}$. Let $l > k_2$ be an integer such that $w_1(l) \neq w_1'(l)$. By Definition 3.2, there exist $v, v' \in A^{\mathbb{N}}_{u_2} \subseteq A^{\mathbb{N}}_{u_2'}$ such that $\{w_1(l), v\}$ and $\{w_1'(l), v'\}$ are in E_l. This contradicts the uniqueness statement of Lemma 3.1(3).

Finally, for the converse of (3), simply note that if $A^{\mathbb{N}}_{u_1} \wedge_{\mathcal{C}} A^{\mathbb{N}}_{u_2}$ contains exactly two elements, then by (2), either $u_1 = u_2'$, or u_1 is adjacent to u_2'. However, the former is false since in that case, by (1), $A^{\mathbb{N}}_{u_1} \wedge_{\mathcal{C}} A^{\mathbb{N}}_{u_2}$ would be an infinite set. □

We now study chains, as defined in Definition 1.2 of Section 1.1. The following lemma shows that, if each G_k in the combinatorial data is a combinatorial tree, chains must respect the “between-ness” relation in each G_k.

LEMMA 3.5

Suppose that each graph G_k is a combinatorial tree. Let $w_1, w_2, w_3 \in A^k$, and let w_2 be on the unique combinatorial arc in G_k that joins w_1 and w_3. If $u_1 \in A^{\mathbb{N}}_{w_1}$ and $u_3 \in A^{\mathbb{N}}_{w_3}$, then for every chain $\{A^{\mathbb{N}}_{v_1}, \dots, A^{\mathbb{N}}_{v_n}\}$ joining u_1 with u_3, there exists $v \in A^*$ and $i \in \{1, \dots, n\}$ such that $A^{\mathbb{N}}_{w_2 v} \subset A^{\mathbb{N}}_{v_i}$.

Proof

We may assume that the three words w_1, w_2, w_3 are distinct; otherwise, the lemma is trivial.

As a start, we note that u_1 has an initial w_1 substring and an initial v_1 substring, so either v_1 is an initial substring of w_1 or vice versa. A similar consideration applies to u_3, v_n, and w_3.

For each $i \in 1, \dots, n$, we define a subset $P_i \subseteq A^k = V(G_k)$ as follows: If $|v_i| < k$, then let $P_i = A^k_{v_i}$. If $|v_i| \geq k$, then let $P_i = \{v_i(k)\}$. In either case, P_i induces a connected subgraph of G_k.

CLAIM 3.6

P_1 contains w_1 and P_n contains w_3.

Proof

If $|v_1| < k$, then v_1 is an initial substring of w_1, and so $P_1 = A^k_{v_1} \ni w_1$. If $|v_1| \geq k$, then $w_1 = v_1(k) \in P_1$.

By the same argument, P_n contains w_3. □

CLAIM 3.7

For each $i \in \{1, \dots, n-1\}$, either $P_i \cap P_{i+1} \neq \emptyset$ or there is an edge $\{a, b\} \in E_k$ with $a \in P_i$ and $b \in P_{i+1}$.

Proof
Assume without loss of generality that $|v_i| \geq |v_{i+1}|$. Since $\{v_i\}$ is a chain, $A^{\mathbb{N}}_{v_i} \wedge_{\mathcal{C}} A^{\mathbb{N}}_{v_{i+1}} \neq \emptyset$.

Case 1: If $|v_{i+1}| \leq |v_i| < k$, then $P_i = A^k_{v_i}$ and $P_{i+1} = A^k_{v_{i+1}}$. These contain adjacent elements by Lemma 3.3(2).

Case 2: If $k \leq |v_{i+1}| \leq |v_i|$, then $P_i = \{v_i(k)\}$ and $P_{i+1} = \{v_{i+1}(k)\}$. By Lemma 3.3(3) and Lemma 3.1(2), the elements $v_i(k)$ and $v_{i+1}(k)$ are either equal or adjacent in G_k.

Case 3: If $|v_{i+1}| < k \leq |v_i|$, then $P_i = \{v_i(k)\}$ and $P_{i+1} = A^k_{v_{i+1}}$. If $v_i(k) \in A^k_{v_{i+1}}$, then clearly $P_i \subseteq P_{i+1}$. Otherwise, since $A^{\mathbb{N}}_{v_1} \wedge_{\mathcal{C}} A^{\mathbb{N}}_{v_{i+1}} \neq \emptyset$, by Lemma 3.1(2), there exist $j, l \in A$ and $v' \in A^{|v_i|-|v_{i+1}|}$ such that $v_i j$ is adjacent to $v_i v' l$, and since $v_i(k) \notin A^k_{v_{i+1}}$, we have by Lemma 3.1(2) that $v_i(k)$ (which is in P_i) is adjacent to $(v_{i+1}v')(k)$ (which is in P_{i+1}). This completes the proof of the claim. □

Thus, the union of the sets $P_1, P_2, \dots, P_n$ induces a connected subgraph of G_k that contains w_1 and w_3. It therefore must contain w_2, so $w_2 \in P_i$ for some i.

If $|v_i| < k$, then this means that $w_2 \in P_i = A^k_{v_i}$. Thus, $A^{\mathbb{N}}_{w_2} \subseteq A^{\mathbb{N}}_{v_i}$, which proves the lemma in this case.

If $|v_i| \geq k$, then $w_2 \in P_i = \{v_i(k)\}$. Thus, $w_2 v = v_i$ for some word v, which proves the lemma in this case. □

3.3. Diameter functions and metrics

Recall the notion of a diameter function Δ on an alphabet A (and the class $\mathcal{D}(A)$ of all diameter functions on A) from Definition 1.3. **For the remainder of Section 3, we fix a diameter function $\Delta \in \mathcal{D}(A)$.**

Given $\mathcal{C}$ and Δ, we defined the distance $D_{\mathcal{C},\Delta}$ on $A^{\mathbb{N}}$ in (1.2) by taking an infimum over chains. We first prove that $D_{\mathcal{C},\Delta}$ is indeed a pseudometric as claimed.

LEMMA 3.8
The function $D_{\mathcal{C},\Delta}$ is a pseudometric on $A^{\mathbb{N}}$.

Proof
First, notice that for any $w \in A^{\mathbb{N}}$ and any $n \in \mathbb{N}$, $\{A^{\mathbb{N}}_{w(n)}\}$ is a chain that joins w with w. Thus,

$$D_{\mathcal{C},\Delta}(w,w) \leq \Delta\big(w(n)\big) \leq \max_{v \in A^n} \Delta(v),$$

which vanishes as $n \to \infty$ by property (3) in Definition 1.3. Hence, $D_{\mathcal{C},\Delta}(w,w) = 0$.

The symmetry of $D_{\mathcal{C},\Delta}$ is trivial, as any chain joining w with u is also a chain joining u with w.

For the triangle inequality, fix $\epsilon > 0$. Let $\{A^{\mathbb{N}}_{w_1}, \dots, A^{\mathbb{N}}_{w_n}\}$ be a chain joining w with u, and let $\{A^{\mathbb{N}}_{u_1}, \dots, A^{\mathbb{N}}_{u_m}\}$ be a chain joining u with v such that

$$\sum_{i=1}^{n} \Delta(w_i) < \frac{\epsilon}{2} + D_{\mathcal{C},\Delta}(w,u) \qquad \text{and} \qquad \sum_{j=1}^{n} \Delta(u_j) < \frac{\epsilon}{2} + D_{\mathcal{C},\Delta}(u,v).$$

By Lemma 3.4(1), we have that $A_{w_n}^{\mathbb{N}} \wedge_{\mathcal{C}} A_{u_1}^{\mathbb{N}} \neq \emptyset$, and so $\{A_{w_1}^{\mathbb{N}}, \dots, A_{w_n}^{\mathbb{N}}, A_{u_1}^{\mathbb{N}}, \dots, A_{u_m}^{\mathbb{N}}\}$ is a chain joining w with v. Thus, $D_{\mathcal{C},\Delta}(w,v) \le D_{\mathcal{C},\Delta}(w,u) + D_{\mathcal{C},\Delta}(u,v) + \epsilon$. As ϵ was chosen arbitrarily, the lemma follows. □

We now describe more precisely the metric space $(\mathcal{A}, d_{\mathcal{C},\Delta})$ associated to a given combinatorial data $\mathcal{C}$ and diameter function Δ on A, introduced briefly in Section 1.1.

To turn $D_{\mathcal{C},\Delta}$ into a metric, we define a relation on $A^{\mathbb{N}}$. In particular, we write $w \sim u$ (for convenience we drop the dependence on $\mathcal{C}$, Δ) if and only if $D_{\mathcal{C},\Delta}(w,u) = 0$. Since $D_{\mathcal{C},\Delta}$ is a pseudometric, it follows that $\sim$ is an equivalence relation. Using this identification, we define

$$\mathcal{A} = A^{\mathbb{N}}/\sim \qquad \text{and} \qquad \mathcal{A}_w = A_w^{\mathbb{N}}/\sim \quad \text{for each } w \in A^*.$$

Based on $D_{\mathcal{C},\Delta}$, we define a function $d_{\mathcal{C},\Delta}$ on $\mathcal{A} \times \mathcal{A}$ in the usual way: if $[w], [u] \in \mathcal{A}$, then set

$$d_{\mathcal{C},\Delta}\big([w],[u]\big) := D_{\mathcal{C},\Delta}(w,u).$$

The function $d_{\mathcal{C},\Delta}$ is well-defined. To see why this is true, let $w, w', u, \in A^{\mathbb{N}}$ such that $[w] = [w']$. By Lemma 3.8, we have that $D_{\mathcal{C},\Delta}(w,u) \le D_{\mathcal{C},\Delta}(w, w') + D_{\mathcal{C},\Delta}(w',u) = D_{\mathcal{C},\Delta}(w',u)$. Similarly, $D_{\mathcal{C},\Delta}(w',u) \le D_{\mathcal{C},\Delta}(w,u)$ and thus, $D_{\mathcal{C},\Delta}(w',u) = D_{\mathcal{C},\Delta}(w,u)$.

LEMMA 3.9
The function $d_{\mathcal{C},\Delta}$ is a metric on $\mathcal{A}$ and for each $w \in A^*$, $\operatorname{diam} \mathcal{A}_w \le \Delta(w)$.

Proof
We first show that $d_{\mathcal{C},\Delta}$ is a metric. It is clear that $d_{\mathcal{C},\Delta}$ is non-negative, symmetric and $d_{\mathcal{C},\Delta}([w],[u]) = 0$ if and only if $[w] = [u]$ in $\mathcal{A}$. The triangle inequality follows from Lemma 3.8.

Let $w \in A^*$ and $[u_1], [u_2] \in \mathcal{A}_w$. We may choose u_1 and u_2 in A_w. The set $\{A_w^{\mathbb{N}}\}$ is then a chain joining u_1 with u_2 and $d_{\mathcal{C},\Delta}([u_1],[u_2]) \le \Delta(w)$. Therefore, $\operatorname{diam} \mathcal{A}_w^{\mathbb{N}} \le \Delta(w)$. □

We use standard metric space terminology when discussing $(\mathcal{A}, d_{\mathcal{C},\Delta})$. In particular, if $[w] \in \mathcal{A}$ and $r > 0$, we write $B([w], r)$ for the open ball centered at $[w]$ of radius r in this space.

3.4. Bounded turning spaces
We now work toward the following proposition, which proves Parts (1) and (2) of Theorem 1.4.

PROPOSITION 3.10
The metric space $(\mathcal{A}, d_{\mathcal{C},\Delta})$ is compact, path-connected, and 1-bounded turning. Moreover, if each combinatorial graph G_k is a combinatorial tree, then the metric space is a tree.

(Here we are using the shorthand "C-bounded turning" for "bounded turning with constant C".)

The separate statements of Proposition 3.10 are proven in Lemmas 3.12, 3.14, 3.15, and 3.17.

LEMMA 3.11
Fix $w \in A^*$. Let

$$I = \{i \in A : \Delta(wi) > 0\}.$$

If $\operatorname{diam}(\mathcal{A}_w) > 0$, *then*

$$\mathcal{A}_w \subseteq \bigcup_{i \in I} \mathcal{A}_{wi}.$$

Proof
The assumption that $\operatorname{diam}(\mathcal{A}_w) > 0$ implies that I is non-empty. Let $k = |w|$.

Consider any $[v] \in \mathcal{A}_w$; without loss of generality, $v(k) = w$. We will show that $[v] = [u]$ for some $u \in \cup_{i \in I} A^{\mathbb{N}}_{wi}$. If $v(k+1) \in \{wi : i \in I\}$, then we are done, so suppose it is not. Then there is a simple path

$$u_1, u_2, \dots, u_n$$

in the combinatorial tree G_{k+1} such that $u_1 = v(k+1)$, $u_n = wi$ f c bor some $i \in I$, and $u_j \notin \{wi : i \in I\}$ for $1 \leq j \leq n-1$.

By Lemma 3.1, there is $u \in A^{\mathbb{N}}_{u_n}$ such that for each m, either $u(k+m) \in A^{k+m}_{u_{n-1}}$ or $u(k+m)$ is adjacent to some element of $A^{k+m}_{u_{n-1}}$. In either case, for each $m \geq 1$, we have that $A^{\mathbb{N}}_{u_{n-1}} \wedge_{\mathcal{C}} A^{\mathbb{N}}_{u(k+m)} \neq \emptyset$. Therefore, the set

$$\{A^{\mathbb{N}}_{u_1}, \dots, A^{\mathbb{N}}_{u_{n-1}}, A^{\mathbb{N}}_{u(k+m)}\}$$

is a chain that joins v to $u \in \mathcal{A}_{wi}$. Note that $\Delta(u_j) = 0$ for $1 \leq j \leq n-1$. Therefore,

$$D_{\mathcal{C},\Delta}(v,u) \leq \Delta\big(u(k+m)\big) \leq \max\{\Delta(r) : r \in A^{k+m}\} \to 0 \quad \text{as } m \to \infty.$$

It follows that $[v] = [u] \in \mathcal{A}_{wi}$. This completes the proof. □

We can now prove a slightly stronger version of the first statement in Proposition 3.10.

LEMMA 3.12
For each $w \in A^*$, the metric space $(\mathcal{A}_w, d_{\mathcal{C},\Delta})$ is compact.

In particular, taking $w = \varepsilon$, we see that $(\mathcal{A}, d_{\mathcal{C},\Delta})$ is compact, as required in Proposition 3.10.

Proof
We show that $(\mathcal{A}_w, d_{\mathcal{C},\Delta})$ is sequentially compact. Let $([w_n])$ be a sequence in $\mathcal{A}_w$. Suppose that this sequence has no convergent subsequence. This implies that $\operatorname{diam}(\mathcal{A}_w) > 0$; otherwise, $([w_n])$ would be constant.

Let

$$I_1 = \{i \in A : \Delta(wi) > 0\}.$$

Note that I_1 is finite by Definition 1.3. Thus, by Lemma 3.11, there exists $i_1 \in I_1$ and a subsequence $([w_n^1])$ of $([w_n])$ in $\mathcal{A}_{wi_1}$.

We proceed by induction to construct sets $I_m \subseteq A$, indices $i_m \in I_m$ and subsequences $([w_n^m])$ of $([w_n])$ contained in $\mathcal{A}_{wi_1 i_2 \dots i_m}$.

Assuming that there is a subsequence $([w_n^m]) \subseteq \mathcal{A}_{wi_1 \cdots i_m}$, let

$$I_{m+1} = \{i \in A : \Delta(wi_1 \cdots i_m i) > 0\},$$

which is finite as above. As above, $\operatorname{diam}(\mathcal{A}_{wi_1 \cdots i_m}) > 0$; otherwise, $([w_n^m])$ would be constant—-hence, convergent. Thus, by Lemma 3.11, there is $i_{m+1} \in I_{m+1} \subseteq A$ and a subsequence $([w_n^{m+1}])$ of $([w_n^m])$ in $\mathcal{A}_{wi_1 \cdots i_{m+1}}$.

Set $u = wi_1 i_2 \cdots \in A^{\mathbb{N}}$ and consider the subsequence $([w_n^n])$ of $([w_n])$. Then $d_{\mathcal{C},\Delta}([w_n^n], [u]) \leq \Delta(u(n)) \to 0$ as $n \to \infty$, contradicting our assumption. Thus, $(\mathcal{A}_w, d_{\mathcal{C},\Delta})$ is compact. □

We now work toward the connectedness properties. The following definition is convenient: An *ϵ-path* in a metric space (X, d) is a finite sequence $(x_1, \dots, x_n)$ such that $d(x_i, x_{i+1}) \leq \epsilon$ for each $i \in \{1, \dots, n-1\}$. We say that the ϵ-path *joins a and b* if $a = x_1$ and $b = x_n$.

LEMMA 3.13
Let $[w_1], [w_2] \in \mathcal{A}$ with $d_{\mathcal{C},\Delta}([w_1], [w_2]) < r$, and let $\epsilon > 0$. Then there is an ϵ-path joining $[w_1]$ and $[w_2]$ of diameter less than r.

Proof
Fix $[w_1]$, $[w_2]$, $r > 0$, and $\epsilon > 0$ as in the statement of the lemma. Let $\{A_{u_1}^{\mathbb{N}}, \dots, A_{u_k}^{\mathbb{N}}\}$ be a chain joining w_1 with w_2 such that

$$\sum_{i=1}^{k} \Delta(u_i) \leq d_{\mathcal{C},\Delta}\big([w_1], [w_2]\big) + \frac{r - d_{\mathcal{C},\Delta}([w_1], [w_2])}{2} < r.$$

Note that for any $i, j \in \{1, \dots, k\}$ and any $w_i \in A_{u_i}^{\mathbb{N}}$ and $w_j \in A_{u_j}^{\mathbb{N}}$, we may use a subset of this same chain to join them and so obtain

$$d_{\mathcal{C},\Delta}\big([w_i], [w_j]\big) < r. \tag{3.2}$$

By Property (3) in Definition 1.3, there exists $m \in \mathbb{N}$ such that $\Delta(u) \leq \epsilon/2$ for all $u \in A^m$. By the properties of G_m and Lemma 3.3, there exists a path

$$\gamma = \big\{\{u_1', u_2'\}, \dots \{u_{k-1}', u_k'\}\big\} \subset \bigcup_{i=1}^{k} A_{u_i}^m$$

such that $w_1 \in A_{u_1'}^{\mathbb{N}}$ and $w_2 \in A_{u_n'}^{\mathbb{N}}$. For each $i \in \{1, \dots, n\}$, let $v_i = u_i' 1^\infty$, and let $v_0 = w_1$ and $v_{n+1} = w_2$. Then for each $i = 1, \dots, n-1$,

$$d_{\mathcal{C},\Delta}\big([v_i], [v_{i+1}]\big) \leq \Delta(u_i') + \Delta(u_{i+1}') \leq \epsilon/2 + \epsilon/2 = \epsilon,$$

and similarly, $d_{\mathcal{C},\Delta}([w_1], [v_1]) \leq \Delta(u_1') \leq \epsilon$ and $d_{\mathcal{C},\Delta}([w_2], [v_n]) \leq \Delta(u_n') \leq \epsilon$.

Thus, $([v_0],[v_1],\dots,[v_{n+1}])$ is an ϵ-path joining $[w_1]$ to $[w_2]$. Its diameter is less than r by (3.2). □

The following lemma completes the proof of the topological properties in Proposition 3.10.

LEMMA 3.14

The metric space $(\mathcal{A}, d_{\mathcal{C},\Delta})$ has the property that $\overline{B([w_0],r)}$ is connected for each $[w_0]\in\mathcal{A}$ and $r>0$.

In particular, the space is connected, locally connected, and path-connected.

Proof

The second sentence follows from the first: connectedness by taking $r = 1 \geq \operatorname{diam}(\mathcal{A})$, local connectedness by, e.g., [27, Chapter I, Line 15.1], and path-connectedness by the Hahn–Mazurkiewicz theorem and Lemma 3.12.

For the first sentence, fix $w_0 \in A^{\mathbb{N}}$ and $r>0$. To show that $\overline{B([w_0],r)}$ is connected, it suffices to show that for any $\epsilon>0$, each $[w]\in\overline{B([w_0],r)}$ can be joined to $[w_0]$ by an ϵ-path contained in $\overline{B([w_0],r)}$.

The point $[w]$ is less than ϵ-distance away from an element $[w']$ of $B([w_0],r)$. There is an ϵ-path joining $[w_0]$ to $[w']$ inside $B([w_0],r)$, by Lemma 3.13. Since $d_{\mathcal{C},\Delta}([w'],[w])<\epsilon$, appending $[w]$ to this path yields an ϵ-path joining $[w_0]$ to $[w]$ inside $\overline{B([w_0],r)}$. □

LEMMA 3.15

The metric space $(\mathcal{A}, d_{\mathcal{C},\Delta})$ is 1-bounded turning.

Proof

Let $[w_1],[w_2]\in\mathcal{A}$, with $r = d_{\mathcal{C},\Delta}([w_1],[w_2])>0$. Let $\epsilon>0$. By Lemma 3.13, there is an ϵ-path $(v_0,v_1,\dots,v_n)$ joining $[w_1]$ to $[w_2]$ with diameter at most $r+\epsilon$.

Define a compact set $K_\epsilon\subseteq\mathcal{A}$ by

$$K_\epsilon = \bigcup_{j=0}^{n} \overline{B\big([v_j],2\epsilon\big)}.$$

Note that each ball in this union is connected, by Lemma 3.14. Since $\overline{B([v_j],2\epsilon)}\cap\overline{B([v_{j+1}],2\epsilon)}\neq\emptyset$ for each $j=0\dots n-1$, it follows that K_ϵ is also connected. Moreover,

$$\operatorname{diam}(K_\epsilon)\leq r+5\epsilon. \tag{3.3}$$

The sets $K_1, K_{1/2}, K_{1/3},\dots$ are each compact, connected, and contain both $[w_1]$ and $[w_2]$. They therefore admit a subsequence that converges in the Hausdorff metric to a compact, connected set that contains $[w_1]$ and $[w_2]$. By (3.3), this set has diameter r. This completes the proof. □

3.5. Metric trees

We now prove the second half of Proposition 3.10—namely, that if each combinatorial graph in our data is in fact a combinatorial tree, then the resulting metric space is a metric tree. **Thus, for the remainder of Section 3, we assume that each combinatorial graph G_k is a metric tree, and we rename the graphs T_k to reflect this.**

LEMMA 3.16

Suppose that $w, w', w_0 \in A^k$ and w_0 is on the unique combinatorial arc in T_k that joins w with w'. If there exist $u \in A^{\mathbb{N}}_w$ and $u' \in A^{\mathbb{N}}_{w'}$ such that $[u] = [u']$, then $[u] \in \mathcal{A}_{w_0}$.

Proof

Let w, w', w_0 be as in the statement of the lemma. We claim that for any $\epsilon > 0$ sufficiently small, there exists $v \in A^{\mathbb{N}}_{w_0}$ such that $D_{\mathcal{C},\Delta}(u, v) < \epsilon$. Assuming this claim, by Lemma 3.12, it follows that there exists $u_0 \in \mathcal{A}_{w_0}$ such that $D_{\mathcal{C},\Delta}(u, u_0) = 0$, and we obtain that $[u] \in \mathcal{A}_{w_0}$.

To prove the claim, fix $\epsilon > 0$. Since $D_{\mathcal{C},\Delta}(u, u') = 0$, there exists a chain $\{A^{\mathbb{N}}_{w_1}, \dots, A^{\mathbb{N}}_{w_m}\}$ that joins u with u' such that $\sum_{l=1}^{m} \Delta(w_l) < \epsilon$. By Lemma 3.5, there exist $l_0 \in \{1, \dots, m\}$ and $v \in A^{\mathbb{N}}_{w_0} \cap A^{\mathbb{N}}_{w_{l_0}}$. In particular, $\{A^{\mathbb{N}}_{w_1}, \dots, A^{\mathbb{N}}_{w_{l_0}}\}$ is a chain joining u with v. It follows that

$$D_{\mathcal{C},\Delta}(u, v) \le \sum_{l=1}^{l_0} \Delta(w_l) \le \sum_{l=1}^{m} \Delta(w_l) < \epsilon.$$

As $\epsilon > 0$ was arbitrary, this proves the initial claim and hence the lemma. □

LEMMA 3.17

The metric space $(\mathcal{A}, d_{\mathcal{C},\Delta})$ is a metric tree.

Proof

First of all, since $(\mathcal{A}, d_{\mathcal{C},\Delta})$ is Hausdorff and path-connected, it is also arcwise connected; see, e.g., [28, Section 31]. Let $[w_1]$, $[w_2]$ be two distinct arbitrary points in $\mathcal{A}$. We will show that there is a point of $\mathcal{A} \setminus \{[w_1], [w_2]\}$ (in fact, a whole continuum) that every path γ from $[w_1]$ to $[w_2]$ must contain. This clearly implies that there can be no simple closed path containing $[w_1]$ and $[w_2]$, and therefore that $\mathcal{A}$ is a metric tree. (See [7, Theorem 1.1] for various characterizations of metric trees, called dendrites there, from which we are using characterization (20).)

For each $n \in \mathbb{N}$, let

$$\{v_{n,1}, \dots, v_{n,m(n)}\} \subseteq A^n$$

be all the vertices of T_n lying on the unique combinatorial arc that joins $w_1(n)$ with $w_2(n)$, ordered so that $v_{n,1} = w_1(n)$, $v_{n,m(n)} = w_2(n)$, and $\{v_{n,i}, v_{n,i+1}\} \in E_n$ for all $i = 1, \dots, m(n) - 1$.

Note that for each $n \in \mathbb{N}$ and $i \in \{1, \dots, m(n+1)\}$, the word $v_{n+1,i}(n)$ lies on the combinatorial arc from $w_1(n)$ to $w_2(n)$—i.e., is equal to $v_{n,j}$ for some

$j \in \{1, \dots, m(n)\}$. Indeed, if not, then the combinatorial arc $\{v_{n,1}, \dots, v_{n,m(n)}\}$ avoids $v_{n+1,i}(n)$, and so by Definition 1.1, Properties (2a) and (2b), we can form an arc from $w_1(n+1)$ to $w_2(n+1)$ that avoids $v_{n+1,i}$, contradicting the uniqueness of this arc in T_{n+1}.

Conversely, if $n \in \mathbb{N}$ and $i \in \{1, \dots, m(n)\}$, then some $v_{n+1,j}$ has $v_{n+1,j}(n) = v_{n,i}$. If not, then using Definition 1.1, Properties (2a) and (2b), we could construct a separate combinatorial arc joining $w_1(n+1)$ and $w_2(n+1)$ that does contain some child of $v_{n,i}$, violating the tree condition.

The upshot of the previous two paragraphs is that each $\mathcal{A}_{v_{n+1,i}}$ is contained in some $\mathcal{A}_{v_{n,j}}$, and each $\mathcal{A}_{v_{n,i}}$ contains some $\mathcal{A}_{v_{n+1,j}}$.

In particular, for each $n \in \mathbb{N}$,

$$\bigcup_{i=1}^{m(n+1)} \mathcal{A}_{v_{n+1,i}} \subseteq \bigcup_{i=1}^{m(n)} \mathcal{A}_{v_{n,i}}.$$

Let

$$K_n := \bigcup_{i=1}^{m(n)} \mathcal{A}_{v_{n,i}} \subseteq \mathcal{A}, \qquad \text{and} \qquad K := \bigcap_{n=1}^{\infty} K_n \subseteq \mathcal{A}.$$

Note that the above sets are all compact by Lemma 3.12.

CLAIM 3.18

We have that $[w_1], [w_2] \in K$.

Proof

We have that $w_1 = v_{n,1}$ for each n, so $w_1 \in A^{\mathbb{N}}_{v_{n,1}}$ for each n. Hence, $[w_1] \in \mathcal{A}_{v_{n,1}} \subseteq K_n$ for each n, and $[w_1]$ is therefore in K. Similarly, $[w_2] \in K$. □

CLAIM 3.19

The set K *contains a continuum that joins* $[w_1]$ *with* $[w_2]$.

Proof

For any $\delta > 0$, there exists $n \in \mathbb{N}$ such that $\sup_{w \in A^n} \Delta(w) < \delta/2$. We first claim that for any $i = 1, \dots, m(n)$, there exists a point $[v_i] \in \mathcal{A}_{v_{n,i}} \cap K$. Indeed, by the discussion at the beginning of the proof of this lemma, there is a sequence

$$\mathcal{A}_{v_{n,i}} \supseteq \mathcal{A}_{v_{n+1,i_1}} \supseteq \mathcal{A}_{v_{n+2,i_2}} \supseteq \cdots$$

By compactness of $\mathcal{A}$ and the definition of K, there is an element of K in the intersection of these.

It is then immediate that $([w_1], [v_1], [v_2], \dots, [v_{m(n)}], [w_2])$ is a δ-path in K joining $[w_1]$ with $[w_2]$. As the choice of $\delta > 0$ was arbitrary, it follows that $[w_1]$ and $[w_2]$ must lie in the same connected component of K (see [27, Chapter I, (9.2), p. 15]), which must also be closed as K is compact. □

CLAIM 3.20

The set K is contained in every path γ from $[w_1]$ to $[w_2]$ in $\mathcal{A}$.

Proof

Fix such a path γ, and let $\epsilon > 0$ and $[v_0] \in K$. Choose $n \in \mathbb{N}$ such that $\sup_{w \in A^n} \Delta(w) < \epsilon$. Let $i \in \{1, \ldots, m(n)\}$ such that $[v_0] \in \mathcal{A}_{v_{n,i}}$. Let $\{T_{n,j} = (V_j, E_j)\}_j$ enumerate the components of $T_n \setminus \{v_{n,i}\}$. For each j, let $X_j = \bigcup_{w \in V_j} \mathcal{A}_w$. These are compact sets: each can be rewritten as $X_j = \bigcup_{w \in V_j, \Delta(w)=0} \mathcal{A}_w$, and this is a finite union of compact sets by Definition 1.3(2) and Lemma 3.12.

Moreover, the union of these sets contains $\mathcal{A} \setminus \mathcal{A}_{v_{n,i}}$. Finally, the sets $\{X_j\}$ also have the property that $X_j \cap X_{j'} \subseteq \mathcal{A}_{v_{n,i}}$ whenever $j \neq j'$. Indeed, if $[v] \in X_j \cap X_{j'}$, then $[v] = [u] = [u']$, where $u(n) \in T_{n,j}$ and $u(n) \in T_{n,j'}$. The unique combinatorial arc from $u(n)$ to $u(n')$ in T_n contains $v_{n,i}$, so by Lemma 3.16 we have that $[v] = [u] \in \mathcal{A}_{v_{n,i}}$.

If neither of $[w_1]$ or $[w_2]$ is contained in $\mathcal{A}_{v_{n,i}}$, then $w_1(n)$ and $w_2(n)$ are contained in different subgraphs $T_{n,j}$ and hence $[w_1]$, $[w_2]$ are contained in different sets X_j. In either case, the path γ must intersect $\mathcal{A}_{v_{n,i}}$. Thus,

$$d_{\mathcal{C},\Delta}\big(\gamma, [v_0]\big) \leq \Delta(w_{n,i}) < \epsilon.$$

Since $\epsilon > 0$ was arbitrary, we have $[v_0] \in \gamma$. □

Thus, every path in $\mathcal{A}$ from $[w_1]$ to $[w_2]$ contains K, which contains a fixed continuum joining $[w_1]$ and $[w_2]$. In particular, any two such paths must intersect somewhere other than their endpoints. This shows that $\mathcal{A}$ is a metric tree. □

REMARK 3.21

Given $w_1, w_2 \in A^{\mathbb{N}}$, let $K \subset \mathcal{A}$ be as in the proof of Lemma 3.17. We showed above that K contains a continuum that joins $[w_1]$ with $[w_2]$ and, conversely, that every path in $\mathcal{A}$ that joins $[w_1]$ with $[w_2]$ contains K. Therefore, K is the unique arc that joins $[w_1]$ with $[w_2]$ in $\mathcal{A}$.

Together, Lemmas 3.14, 3.15, and 3.17 prove Proposition 3.10.

4. Doubling metric trees

Recall that a metric space is C-doubling if there exists a constant $C \geq 1$ such that for any $x \in X$ and $r > 0$, the ball $B(x, r)$ can be covered by at most C balls of radius $r/2$. Our goal here is to give some sufficient conditions for our combinatorial construction to yield a doubling metric tree.

For the remainder of Section 4, we assume that A is an alphabet and $\mathcal{C} = (A, (T_k)_{k \in \mathbb{N}})$ is combinatorial data as in Definition 1.1, with the additional assumption that each graph T_k is a combinatorial tree.

PROPOSITION 4.1

Fix $N, n_0 \in \mathbb{N}$, $c > 1$, and $\delta_1, \delta_2 \in (0,1)$. There exists $C > 1$, depending only on these constants, with the following property. Assume that:

- (P1) card $A \leq N$.
- (P2) *$Val(T_k) \leq n_0$ for all $k \in \mathbb{N}$.*
- (P3) *For all $w \in A^*$ and $i \in A$, $\delta_1 \Delta(w) \leq \Delta(wi) \leq \delta_2 \Delta(w)$.*
- (P4) *Suppose that for some $k \in \mathbb{N}$ and some distinct $u, u_1, u_2 \in A^n$, we have $A_u^{\mathbb{N}} \wedge_{\mathcal{C}} A_{u_i}^{\mathbb{N}} \neq \emptyset$ for $i = 1, 2$. If $w_i \in A_u^{\mathbb{N}} \wedge_{\mathcal{C}} A_{u_i}^{\mathbb{N}}$ for $i = 1, 2$, then $d_{\mathcal{C},\Delta}([w_1],[w_2]) \geq c^{-1}\Delta(u)$.*

Then $(\mathcal{A}, d_{\mathcal{C},\Delta})$ is C-doubling.

REMARK 4.2

Items (P1), (P2), and (P3) of Proposition 4.1 are rather innocuous, while (P4) requires some more thought. Essentially, (P4) prevents the space from "collapsing" too many far away points close together, which may violate doubling. In Lemma 4.8, we provide a more easily checkable condition that implies (P4), and in Example 6.10 we show that (P4) is necessary in Proposition 4.1.

Note also that if $w_i, w_i' \in A_u^{\mathbb{N}} \wedge_{\mathcal{C}} A_{u_i}^{\mathbb{N}}$, then $d_{\mathcal{C},\Delta}([w_i],[w_i']) = 0$. Therefore, in (P4), we may assume that $w_i \in (A_u^{\mathbb{N}} \wedge_{\mathcal{C}} A_{u_i}^{\mathbb{N}}) \cap A_u^{\mathbb{N}}$.

Recall the definition of a parent word $u^{\uparrow}$. For the proof of Proposition 4.1, we make the following definition. Given $r > 0$, define

$$A^*(r) := \{w \in A^* : \Delta(w) < r \text{ and } \Delta(w^{\uparrow}) \geq r\}.$$

REMARK 4.3

The set $A^*(r)$ induces a partition on $A^{\mathbb{N}}$—namely, $A^{\mathbb{N}} = \bigcup_{u \in A^*(r)} A_u^{\mathbb{N}}$, and for distinct $w, u \in A^*(r)$, we have $A_w^{\mathbb{N}} \cap A_u^{\mathbb{N}} = \emptyset$.

LEMMA 4.4

Let A and $\mathcal{C}$ satisfy (P2). Then for each $r > 0$ and for each $w \in A^*(r)$, there exist at most n_0 words $u \in A^*(r) \setminus \{w\}$ such that $A_w^{\mathbb{N}} \wedge_{\mathcal{C}} A_u^{\mathbb{N}} \neq \emptyset$.

Proof

Let $r > 0$ and $w \in A^*(r)$. To prove the claim, let $u_1, \dots, u_n$ be words in $A^*(r) \setminus \{w\}$ such that $A_w^{\mathbb{N}} \wedge_{\mathcal{C}} A_{u_i}^{\mathbb{N}} \neq \emptyset$ for each i.

Let $k_0 = |w|$. If $|u_i| < k_0$, then by Lemmas 3.3 and 3.4, there exists a unique $u_i' \in A_{u_i}^{k_0}$ such that $\{w, u_i'\} \in E_{k_0}$. If $|u_i| \geq k_0$, then let $u_i' = u_i(k_0)$, and by Lemma 3.1, we have that $\{w, u_i'\} \in E_{k_0}$. We claim that if $i \neq j$, then $u_i' \neq u_j'$. Assuming the claim, by (P2) we have that $n \leq n_0$, and so the proof is complete once we establish this claim. To do so, we fix distinct $i, j \in \{1, \dots, n\}$ and consider three possible cases.

Case 1. Suppose that $|u_i| \geq k_0$ and $|u_j| \geq k_0$. For a contradiction, assume that $u_i' = u_j' = u'$. By Remark 4.3, we have that $u' \neq w$. Therefore, by Lemma 3.4, $\{u', w\} \in E_{k_0}$. Let $k = \max\{|u_i|, |u_j|\}$. By Lemma 3.1, there exist unique $w'' \in A^k$

and unique $u'' \in A^k_{u'}$ such that $\{w'', u''\} \in E_k$. By Remark 4.3, either $u'' \notin A^k_{u_i}$ or $u'' \notin A^k_{u_j}$. Assuming the former (without loss of generality), by Lemma 3.3, we have $A^{\mathbb{N}}_w \wedge_{\mathcal{C}} A^{\mathbb{N}}_{u_i} = \emptyset$, which is a contradiction.

Case 2. Suppose that $|u_i| \le k_0$ and $|u_j| \le k_0$. For a contradiction, assume that $u'_i = u'_j = u'$. Then $A^{\mathbb{N}}_{u_i} \cap A^{\mathbb{N}}_{u_j} \neq \emptyset$, which contradicts Remark 4.3.

Case 3. Suppose that $|u_i| \le k_0$ and $|u_j| \ge k_0$. By Remark 4.3, $u'_i \neq w$. Now apply the arguments of Case 1 to the triple u'_i, w, and u_j. □

Proof of Proposition 4.1
Let $[w] \in \mathcal{A}$ and $r > 0$. To prove the proposition, it suffices to prove that the doubling property holds for the ball $B([w], r)$ if $r < c^{-1} \operatorname{diam} \mathcal{A}$. Let u_0 be the unique element of $A^*(c\delta_1^{-1} r)$ such that $w \in A^{\mathbb{N}}_{u_0}$.

CLAIM 4.5
There exist at most n_0 words $u \in A^*(c\delta_1^{-1} r) \setminus \{u_0\}$ such that $A^{\mathbb{N}}_{u_0} \wedge_{\mathcal{C}} A^{\mathbb{N}}_u \neq \emptyset$, and each such word u satisfies

$$c\delta_1^{-1} r > \Delta(u) \ge c r.$$

Proof of Claim 4.5
By Lemma 4.4, there exist at most n_0 such words $u \in A^*(c\delta_1^{-1} r) \setminus \{u_0\}$. Moreover, by (P3), for each $u \in A^*(c\delta_1^{-1} r)$,

$$c\delta_1^{-1} r > \Delta(u) \ge \delta_1 \Delta(u^{\uparrow}) \ge c r.$$ □

CLAIM 4.6
If $u \in A^*(c\delta_1^{-1} r)$ and $A^{\mathbb{N}}_{u_0} \wedge_{\mathcal{C}} A^{\mathbb{N}}_u = \emptyset$, then for any $w' \in A^{\mathbb{N}}_u$, we have $d_{\mathcal{C},\Delta}([w], [w']) \ge r$.

Proof of Claim 4.6
Let $\gamma \subset \mathcal{A}$ be the unique arc with endpoints $[w]$ and $[w']$. For each k, let P_k be the simple path in T_k from $w(k)$ to $w'(k)$.

Let $n = \max\{|u|, |u_0|\}$. Then P_n must contain a vertex $v \in A^n \setminus (A^n_{u_0} \cup A^n_u)$; otherwise, $A^{\mathbb{N}}_{u_0} \wedge_{\mathcal{C}} A^{\mathbb{N}}_u \neq \emptyset$. Consider the following two possible cases.

Case 1. Suppose that $v \in A^*(c\delta_1^{-1} r)$ or v has a descendent in $A^*(c\delta_1^{-1} r)$. Then v is adjacent to two distinct vertices v_1 and v_2 of P_n. For $i = 1, 2$, let $w_i \in A^{\mathbb{N}}_v$ be such that $w_i(k) \in P_k$ and is adjacent to an element of $A^k_{v_i}$ for each $k \ge n$. By Remark 3.21, both $[w_1]$ and $[w_2]$ are in γ. Therefore, by the 1-bounded turning property of $\mathcal{A}$, by (P3), and by (P4),

$$d_{\mathcal{C},\Delta}\big([w], [w']\big) = \operatorname{diam} \gamma \ge d_{\mathcal{C},\Delta}\big([w_1], [w_2]\big) \ge c^{-1} \Delta(v) \ge r.$$

Case 2. Suppose that v is contained in $A^*_{v'}$ for some $v' \in A^*(c\delta_1^{-1} r)$. Let $m = |v'|$. First, note that P_m must contain v'; if not, then out of P_m we could construct a combinatorial arc in T_n that does not contain v, which implies that there are two distinct combinatorial arcs in T_n with the same endpoints. The latter, however, contradicts the fact that T_n is a tree. Second, by Remark 4.3, we have that $A^{\mathbb{N}}_{v'} \cap A^{\mathbb{N}}_{u_0} = \emptyset$. Since $A^{\mathbb{N}}_{u_0} \subset A^{\mathbb{N}}_{u_0(m)} = A^{N}_{w(m)}$, it follows that $A^{\mathbb{N}}_{v'} \cap A^{\mathbb{N}}_{w(m)} = \emptyset$. Similarly, $A^{\mathbb{N}}_{v'} \cap A^{\mathbb{N}}_{w(m)} = \emptyset$. Therefore, v' is adjacent to two distinct vertices of P_m. Now working as in Case 1, we obtain that $d_{\mathcal{C},\Delta}([w],[w']) \geq c^{-1}\Delta(v') \geq r$. □

CLAIM 4.7

Let $u \in A^*(c\delta_1^{-1}r)$, and let k be the smallest positive integer such that

$$k \geq \frac{\log((2c)^{-1}\delta_1)}{\log(\delta_2)}.$$

Then

$$\operatorname{diam}(\mathcal{A}_v) < r/2$$

for each $v \in A_u^{|u|+k}$.

Proof of Claim 4.7

By the upper bound in (P3), we have that for every $v \in A_u^{|u|+k}$,

$$\operatorname{diam}(\mathcal{A}_v) \leq \Delta(v) \leq \delta_2^k \Delta(u) < \delta_2^k \delta_1^{-1} c r \leq r/2. \qquad \square$$

Let $\{u_1, \dots, u_p\}$ be all the words $u \in A^*(c\delta_1^{-1}r) \setminus \{u_0\}$ such that $A^{\mathbb{N}}_{u_0} \wedge_{\mathcal{C}} A^{\mathbb{N}}_u \neq \emptyset$. By Claim 4.6,

$$B\big([w],r\big) \subseteq \bigcup_{i=0}^{p} \mathcal{A}_{u_i}.$$

Claim 4.5 implies that $p \leq n_0$. Claim 4.7 implies that each of the sets $\mathcal{A}_{u_i}$ in this union can be covered by at most N^k sets of diameter $< r/2$; hence, N^k balls of radius $r/2$. This completes the proof. □

We now give some sufficient conditions for (P4) which are easier to verify.

For the next lemma, we use the following notation. Consider combinatorial data $\mathcal{C} = (A, (T_k)_{k\in\mathbb{N}})$ as fixed at the beginning of this section. For each $k \in \mathbb{N}$ and $w \in A^k$, let $\partial_{\mathcal{C}} A^{k+1}_w$ be all words $u \in A^{k+1}_w$ for which there exists $u' \in A^{k+1} \setminus A^{k+1}_w$ with $\{u,u'\} \in E_{k+1}$.

LEMMA 4.8

Let $\mathcal{C} = (A, (T_k)_{k\in\mathbb{N}})$ be combinatorial data as fixed at the beginning of this section, and let $\Delta \in \mathcal{D}(A)$. Assume that the following conditions hold for each $k \geq 0$.

1. *Suppose that $w, u, u' \in A^k$ are distinct with $\{w,u\}, \{w,u'\} \in E_k$. If $wi, wj, ul, u'l' \in A^{k+1}$ with $\{wi, ul\}, \{wj, u'l'\} \in E_{k+1}$, then $i \neq j$.*
2. *For any $w \in A^k$ and any distinct $u, u' \in \partial_{\mathcal{C}} A^{k+1}_w$, the arc $\{\{u,u_1\}, \dots, \{u_n, u'\}\}$ joining u with u' in T_{k+1} satisfies*

$$\Delta(u) + \Delta(u_1) + \cdots + \Delta(u_n) + \Delta(u') \geq \Delta(w).$$

Then (P4) of Proposition 4.1 holds with $c = 1$.

In particular, $\operatorname{diam}(\mathcal{A}_u) = \Delta(u)$ *for each* $u \in A^*$ *with at least two neighbors in* $T_{|u|}$.

For the proof of the lemma, given a chain $\mathcal{C} = \{A^{\mathbb{N}}_{u_1}, \dots, A^{\mathbb{N}}_{u_n}\}$ joining two words in $A^{\mathbb{N}}$, we define the *depth* of $\mathcal{C}$ to be the number $\operatorname{Depth}(\mathcal{C}) := \max\{|u_1|, \dots, |u_n|\}$ and the Δ*-length of* $\mathcal{C}$ to be

$$\ell(\mathcal{C}) := \sum_{i=1}^{n} \Delta(u_i).$$

Proof

Fix $k \in \mathbb{N}$ and let $w, u_1, u_2 \in A^k$ be distinct points such that $\{w, u_1\}$ and $\{w, u_2\}$ are in E_k. Let $w_1, w_2 \in A^{\mathbb{N}}_w$, $w_1' \in A^{\mathbb{N}}_{u_1}$, and $w_2' \in A^{\mathbb{N}}_{u_2}$ such that for any $n \geq k$ and any $i \in \{1,2\}$, $w_i(n)$ is adjacent to $w_i'(n)$. We will show that $d_{\mathcal{C},\Delta}([w_1],[w_2]) = \Delta(w)$.

On the one hand, $\{A^{\mathbb{N}}_w\}$ is a chain joining w_1 with w_2, so $d_{\mathcal{C},\Delta}([w_1],[w_2]) \leq \Delta(w)$. For the opposite inequality, fix $\mathcal{C} = \{A^{\mathbb{N}}_{v_1}, \dots, A^{\mathbb{N}}_{v_n}\}$ to be a chain in $A^{\mathbb{N}}$ joining w_1 with w_2. We start by doing four reductions.

First, if $A^{\mathbb{N}}_w \subset A^{\mathbb{N}}_{v_i}$ for some i, then we can replace $\mathcal{C}$ with $\mathcal{C}' = \{A^{\mathbb{N}}_w\}$, which has smaller Δ-length. Therefore, we may assume that for all i, either $A^{\mathbb{N}}_w \cap A^{\mathbb{N}}_{v_i} = \emptyset$, or $A^{\mathbb{N}}_{v_i} \subset A^{\mathbb{N}}_w$.

Second, dropping some of the sets in the chain, if necessary, we may assume that $A^{\mathbb{N}}_{v_i} \subset A^{\mathbb{N}}_w$ for all i.

Third, if $A^{\mathbb{N}}_{v_i} \subset A^{\mathbb{N}}_{v_j}$, then we can drop $A^{\mathbb{N}}_{v_i}$.

Fourth, let P_l be the combinatorial arc in T_l that joins $w_1'(l)$ with $w_2'(l)$. We first claim that P_l contains $w_1(l)$, $w_2(l)$. By Definition 2a, the subgraph of T_l induced by the vertex set A^l_w is connected so there exists a combinatorial arc P' with vertices in A^l_w that has endpoints $w_1(l)$, $w_2(l)$. Adding the two points $w_1'(l)$, $w_2'(l)$ along with edges $\{w_1(l), w_1'(l)\}$, $\{w_2(l), w_2'(l)\}$, we obtain a combinatorial arc in T_l that has endpoints $w_1'(l)$, $w_2'(l)$ and contains $w_1(l)$, $w_2(l)$. By uniqueness of this arc, it must be P_l, and the proof of the claim is complete. Now, it follows from Lemma 3.5 that for any $l \geq \operatorname{Depth}(\mathcal{C})$,

$$\bigcup_{v \in P_l} A^{\mathbb{N}}_v \subset A^{\mathbb{N}}_{v_1} \cup \dots \cup A^{\mathbb{N}}_{v_n}.$$

CLAIM 4.9

The collection

$$\mathcal{C}' = \left\{A^{\mathbb{N}}_{v_i} : A^{\mathbb{N}}_{v_i} \cap \bigcup_{v \in P_l} A^{\mathbb{N}}_v \neq \emptyset\right\}$$

forms a chain joining w_1 *and* w_2.

Proof

First, there exists $v \in P_l$ such that $w_1 \in A^{\mathbb{N}}_v$, which implies that there exists v_i such that $w_i \in A^{\mathbb{N}}_v \subset A^{\mathbb{N}}_{v_i}$. Therefore, w_1 is contained in some element of $\mathcal{C}'$ and similarly for w_2.

Enumerate the arc $P_l = \{v'_0, \dots, v'_{p+1}\}$ so that $v'_0 = w'_1(l)$, $v'_1 = w_1(l)$, $v'_p = w_2(l)$, $v'_{p+1} = w'_2(l)$, and for any j, v'_j is adjacent to v'_{j+1}. Now there exists a set $\{m_1, \dots, m_s\} \subset \{1, \dots, n\}$ such that

(1) $A^{\mathbb{N}}_{v'_1} \subset A^{\mathbb{N}}_{v_{m_1}}$, and $A^{\mathbb{N}}_{v'_p} \subset A^{\mathbb{N}}_{v_{m_s}}$;
(2) for all v'_i, there exists v_{m_j} such that $A^{\mathbb{N}}_{v'_i} \subset A^{\mathbb{N}}_{v_{m_j}}$; and
(3) if $A^{\mathbb{N}}_{v'_i} \subset A^{\mathbb{N}}_{v_{m_j}}$ and $A^{\mathbb{N}}_{v'_{i+1}} \subset A^{\mathbb{N}}_{v_{m_s}}$, then $m_j \le m_s$.

Now it is easy to see that $A^{\mathbb{N}}_{v_{m_i}} \wedge_{\mathcal{C}} A^{\mathbb{N}}_{v_{m_{i+1}}} \neq \emptyset$, so the set $\mathcal{C}' = \{A^{\mathbb{N}}_{v_{m_i}} : i = 1, \dots, s\}$ forms a chain joining w_1 and w_2. □

The fourth reduction says, in other words, that we may drop all sets $A^{\mathbb{N}}_{v_i}$ from the chain such that $A^{\mathbb{N}}_{v_i} \cap \bigcup_{v \in P_l} A^{\mathbb{N}}_v = \emptyset$.

The four reductions imply that we may assume that for all i,

(i) $A^{\mathbb{N}}_{v_i} \subset A^{\mathbb{N}}_w$;
(ii) if $j \neq i$, then $A^{\mathbb{N}}_{v_i} \cap A^{\mathbb{N}}_{v_j} = \emptyset$;
(iii) for all $l \ge \mathrm{Depth}(\mathcal{C})$, there exists $v \in P_l$ such that $A^{\mathbb{N}}_v \subset A^{\mathbb{N}}_{v_i}$; and
(iv) for all $l \ge \mathrm{Depth}(\mathcal{C})$, $\bigcup_{v \in P_l} A^{\mathbb{N}}_v \subset A^{\mathbb{N}}_{v_1} \cup \dots \cup A^{\mathbb{N}}_{v_n}$.

Let $k_0 = \mathrm{Depth}(\mathcal{C})$ and $i_0 \in \{1, \dots, n\}$ such that $|v_{i_0}| = k_0$. If $k_0 = |w|$, then $\mathcal{C} = \{A^{\mathbb{N}}_w\}$, and the Δ-length of $\mathcal{C}$ is equal to $\Delta(w)$.

Assume now that $k_0 > |w|$. Then $v^{\uparrow}_{i_0}$ is contained in P_{k_0-1}. Moreover, $v^{\uparrow}_{i_0}$ has valency 2 in P_{k_0-1} because the endpoints of P_{k_0-1} are in $A^{k_0-1}_{u_i}$ and not in $A^{k_0-1}_w$.

By (iii) and Assumption (1) of the lemma, $A^{k_0}_{v^{\uparrow}_{i_0}} \cap P_{k_0}$ has at least two elements. By (ii), (iv), and the assumption that $|v_{i_0}| = \mathrm{Depth}(\mathcal{C})$, each element of $A^{k_0}_{v^{\uparrow}_{i_0}} \cap P_{k_0}$ must be in $\{v_1, \dots, v_n\}$. Enumerate them as $\{v_{j_1}, v_{j_2}, \dots, v_{j_p}\}$. Since $v^{\uparrow}_{i_0}$ has valency 2 in P_{k_0-1}, the elements of $\{v_{j_1}, v_{j_2}, \dots, v_{j_p}\}$ contain the vertices of a simple path joining two distinct points of $\partial_{\mathcal{C}} A^{k_0}_{v^{\uparrow}_{i_0}}$.

But then, by Assumption (2) of the lemma,

$$\Delta(v_{j_1}) + \dots + \Delta(v_{j_p}) \ge \Delta(v^{\uparrow}_{i_0}),$$

and we can replace $\mathcal{C}$ with the chain

$$\mathcal{C} \cup \{A^{\mathbb{N}}_{v^{\uparrow}_{i_0}}\} \setminus \{A^{\mathbb{N}}_{v_i} : v_i \in A^{k_0}_{v^{\uparrow}_{i_0}}\},$$

which has at most the Δ-length of $\mathcal{C}$.

Working in similar fashion, we can show that if $\mathrm{Depth}(\mathcal{C}) > |w|$, then there exists a chain $\mathcal{C}'$ joining w_1 with w_2 such that $\mathrm{Depth}(\mathcal{C}') = \mathrm{Depth}(\mathcal{C}) - 1$ and has at most the Δ-length of $\mathcal{C}$. Applying a backward induction on the depth of $\mathcal{C}$, we obtain that

$$\ell(\mathcal{C}) \ge \ell\big(\{A^{\mathbb{N}}_w\}\big) = \Delta(w).$$

Therefore, $d_{\mathcal{C},\Delta}([w_1], [w_2]) \ge \Delta(w)$.

For the final statement in the lemma, any $u \in A^k$ with two distinct neighbors must have at least two distinct words in its combinatorial boundary, and so

$$\mathrm{diam}(\mathcal{A}_u) \geq \Delta(w)$$

by the first part of the lemma. The reverse inequality follows from Lemma 3.9. □

For examples of combinatorial data and diameter functions satisfying the assumptions of Proposition 4.1 and Lemma 4.8, see Section 6.

5. Characterization of quasiconformal trees

We now claim that our combinatorial constructions above describe all quasiconformal trees up to bi-Lipschitz equivalence. The following result proves Part (3) of Theorem 1.4, while providing additional details, and is the goal of this section.

THEOREM 5.1

Let (X,d) be an N-doubling, C-bounded turning tree. Then for any $M \in \mathbb{N}$ sufficiently large, $K_1 > 0$ sufficiently small, and $K_2 \in [\frac{1}{2}, 1)$, there exist

(1) *an alphabet $A = \{1, \dots, M\}$,*
(2) *combinatorial data $\mathcal{C} = (A, (T_k)_{k\in\mathbb{N}})$ with each T_k a combinatorial tree, and*
(3) *a diameter function $\Delta \in \mathcal{D}(A, K_1, K_2)$*

such that $(\mathcal{A}, d_{\mathcal{C},\Delta})$ is bi-Lipschitz equivalent to X.

The sufficient condition on M depends only on N and C. The sufficient condition on K_1 depends only on M, N, and C. The bi-Lipschitz constant depends only on N, C, K_2/K_1, and $\mathrm{diam}(X)$.

Moreover, $(\mathcal{C}, \Delta)$ satisfies the conditions of Proposition 4.1.

We first make some small reductions. If X is a single point, then Theorem 5.1 is easy. For example, one may take $M = 2$, $\Delta \in \mathcal{D}(A, \frac{1}{3}, \frac{1}{3})$, and each T_k a combinatorial arc. Thus, we may assume that $\mathrm{diam}(X) > 0$ and so, by rescaling, that $\mathrm{diam}(X) = 1$. We may also assume that the bounded turning constant C is equal to 1 by replacing the metric d on X with a bi-Lipschitz equivalent 1-bounded turning metric (see [2, Lemma 2.5]). **All these assumptions are in force for the remainder of Section 5. Thus, we fix an N-doubling, 1-bounded turning metric tree X of diameter 1.**

5.1. Subdividing into a uniform number of pieces

To prove Theorem 5.1, we use a construction of Bonk and Meyer [2] to decompose the tree X into suitable pieces. We then modify this construction to decompose X into an equal number of pieces at each scale. We first summarize the results we need from [2, Section 5].

PROPOSITION 5.2 (Bonk–Meyer [2])

Let $\delta > 0$ sufficiently small, depending on N. Then there is a constant $M(N,\delta) \in \mathbb{N}$, *and for each $n \in \mathbb{N}$, there exists a δ^n-separated set $V_n \subseteq X$ satisfying*

$$V_1 \subseteq V_2 \subseteq \dots$$

with the following properties.

Write $\mathcal{T}_n$ for the collection of closures of components of $X \setminus V_n$. Then

(1) *Each $T \in \mathcal{T}_n$ is a connected subset (hence, subtree) of X with $\emptyset \neq T \cap \overline{X \setminus T} \subseteq V_n$.*
(2) *Distinct elements $T, T' \in \mathcal{T}_n$ have at most one point in common, and such a common point is an element of V_n.*
(3) *Each element of V_n is in exactly two elements of $\mathcal{T}_n$.*
(4) *Each element of $\mathcal{T}_{n+1}$ ($n \geq 1$) is in exactly one element of $\mathcal{T}_n$, and each element of $\mathcal{T}_n$ is the union of all elements of $\mathcal{T}_{n+1}$ inside it.*
(5) *We have $\delta^n \leq \operatorname{diam}(T) \leq K\delta^n$ for each $T \in \mathcal{T}_n$, where K is a constant depending only on N.*
(6) *Each element of $\mathcal{T}_n$ contains at least two and at most $M(N,\delta)$ elements of $\mathcal{T}_{n+1}$.*
(7) *Each element of $\mathcal{T}_n$ intersects at most $M(N,\delta)$ other elements of $\mathcal{T}_n$.*

Proof

The first four items appear explicitly in [2, Lemma 5.1]. The fifth appears in [2, Equation (5.3)]. The existence of the upper bound $M(N,\delta)$ in (6) and (7) is an immediate consequence of (1)–(5) and the doubling property, as in [2, Lemma 5.7]. The lower bound of two in (6) follows from (4) and (5) if $\delta < 1/K$. □

Bonk and Meyer refer to the elements of $\mathcal{T}_n$ as "n-tiles," but we will reserve the word "tiles" for the modifications we construct below. Before that, we observe that adjacency graphs induced by these sets form combinatorial trees.

LEMMA 5.3

Let X be a metric tree. Let $\mathcal{S}$ be a finite collection of compact, connected subsets of X such that $\cup_{S\in\mathcal{S}} S = X$ and no point of X is in more than two different sets of $\mathcal{S}$.

Then the graph G such that

$$V(G) = \{S \in \mathcal{S}\}, E(G) = \big\{\{S, S'\} \subseteq V(G) : S \neq S' \text{ and } S \cap S' \neq \emptyset\big\}$$

is a combinatorial tree.

Proof

The connectedness of G follows easily from the facts that X is connected, all $S \in \mathcal{S}$ are compact, and $\cup_{S\in\mathcal{S}} S = X$.

To see that G is a combinatorial tree, we will use the following simple equivalent characterization of combinatorial trees: A connected finite graph is a combinatorial tree if and only if the removal of any edge disconnects it.

Thus, suppose that the removal of an edge $\{S, S'\}$ from G left it connected. Let $S = S_0, S_1, \dots, S_n = S'$ be the ordered vertices along a simple path from S to S' in G

avoiding this edge; note that $n \geq 2$. Let $x \in S \cap S'$, $p \in S \cap S_1$, and $q \in S' \cap S_{n-1}$. The points x, p, and q are distinct, by the assumption that no point is in more than two elements of $\mathcal{S}$. Similarly, x is disjoint from S_i for each $1 \leq i \leq n-1$.

There is an arc from p to q in $S \cup S'$, which must pass through x. Since X is a metric tree, p and q must be in distinct connected components of $X \setminus \{x\}$. On the other hand, $\cup_{i=1}^{n-1} S_i$ is a connected subset of $X \setminus \{x\}$ containing both, and we reach a contradiction. $\square$

We now modify the construction of Proposition 5.2 so that each tile has an equal number of children. This requires us to give up some control on the diameters of the tiles. However, it is crucial to retain the property that the boundary points of a given tile are "well-separated," in the sense that the distance between two distinct boundary points of a tile is always comparable to the diameter of the tile. This is Property (6) of Lemma 5.4 below.

Fix δ sufficiently small, depending on N, so that Proposition 5.2 holds and so that in addition $K\delta < 1/2$, where K is the constant from Proposition 5.2(5). Thus, we have constants $K = K(N)$ and $M(N, \delta)$ from Proposition 5.2, Items (5) and (6).

LEMMA 5.4

Let $M \geq M(N, \delta)$, $K_1 \in (0, K^{-1}\delta^{\log_2(M)+1}]$, and $K_2 \in [\frac{1}{2}, 1)$. Let $A = \{1, \dots, M\}$.

Then there is a collection of closed subsets $X_w \subset X$ for all $w \in A^*$, satisfying the following properties:

(1) *For each $w \in A^*$, X_w is a connected subset (hence, subtree) of X, and $X_\varepsilon = X$.*
(2) *For each $w \in A^*$ and $i \in A$, $X_{wi} \subseteq X_w$. Moreover, $X_w = \bigcup_{i \in A} X_{wi}$.*
(3) *For each $w \in A^*$ and $i \in A$,*

$$K_1 \operatorname{diam} X_w \leq \operatorname{diam} X_{wi} \leq K_2 \operatorname{diam} X_w.$$

(4) *For each $w \in A^* \setminus \{\varepsilon\}$ and every $x \in X_w \cap \overline{X \setminus X_w}$, we have that x is a leaf of X_w and contained in $X_{w'}$ for exactly one $w' \in A^{|w|} \setminus \{w\}$.*
(5) *For every distinct $w, w' \in A^*$ with $|w| = |w'|$ we have that $X_w \cap X_{w'}$ is either a point or empty.*
(6) *There exists $K_3 \in (0, 1)$ such that for all $w \in A^*$ and for all distinct $x, y \in X_w \cap \overline{X \setminus X_w}$, we have*

$$d(x, y) \geq K_3 \operatorname{diam} X_w.$$

Proof

Fix δ, M, K, K_1, K_2 as above, and let $A = \{1, \dots, M\}$. We prove the lemma for $K_3 = \delta/K$.

We start the proof by noting that it suffices to prove the lemma with (5) replaced by the following property:

(5′) For every $w \in A^*$ and distinct $i, j \in A$, we have that $X_{wi} \cap X_{wj}$ is either a point or empty.

Indeed, assume that the lemma holds with (5) replaced by (5′). Given distinct $w, w' \in A^*$ with $|w| = |w'|$, there exists maximal (in word length) $w_0 \in A^*$ such that $w, w' \in A^{|w|}_{w_0}$. There also exist distinct $i, j \in A$ such that $w \in A^{|w|}_{w_0 i}$ and $w \in A^{|w|}_{w_0 j}$. By (2) and (5′), we have that $X_w \cap X_{w'} \subset X_{w_0 j} \cap X_{w_0 j}$, which is either a point or empty.

We relabel the collections $\mathcal{T}_n$ constructed in Proposition 5.2. Set $T_\varepsilon = X$. We write $\mathcal{T}_1 = \{T_1, \dots, T_{m_\varepsilon}\}$. Assume now that for some $n \in \mathbb{N}$ and some $w \in \mathbb{N}^n$, we have defined T_w to be an element of $\mathcal{T}_n$. Then we write $\{T_{w1}, \dots, T_{wm_w}\}$ to be the elements of $\mathcal{T}_{n+1}$ contained in T_w. By Proposition 5.2(6), we have $2 \le m_w \le M$. Therefore, for every T_w defined, we have $w \in A^*$. We set $\mathcal{W}$ to be the set of all words w in A^* for which T_w has been defined. Given integer $n \ge 0$ and $w \in A^*$, we denote $\mathcal{W}^n = \mathcal{W} \cap A^n$, $\mathcal{W}_w = \mathcal{W} \cap A^*_w$, and $\mathcal{W}^n_w = \mathcal{W} \cap A^n_w$.

We now define the family $\{X_w\}_{w \in A^*}$ in an inductive manner.

STEP 0. Set $X_\varepsilon = T_\varepsilon = X$.

INDUCTIVE HYPOTHESIS. Suppose that for some integer $k \ge 0$, we have defined closed sets $\{X_w\}_{w \in A^k}$ such that the properties of the lemma up to level k hold, with $K_3 = \delta/K$; that is, we assume that the following conditions hold:

(1) For each $l \le k$ and $w \in A^l$, X_w is a connected subset of X.
(2) For each $l \le k-1$, $w \in A^l$, and $i \in A$, we have $X_{wi} \subseteq X_w$. Moreover, $X_w = \bigcup_{i \in A} X_{wi}$.
(3) For each $l \le k-1$, $w \in A^l$, and $i \in A$, we have
$$K_1 \operatorname{diam} X_w \le \operatorname{diam} X_{wi} \le K_2 \operatorname{diam} X_w.$$
(4) For each $l \le k$, $w \in A^l \setminus \{\varepsilon\}$ and every $x \in X_w \cap \overline{X \setminus X_w}$, we have that x is a leaf of X_w and contained in $X_{w'}$ for exactly one $w' \in A^l \setminus \{w\}$.
(5) For each $l \le k-1$, $w \in A^l$ and distinct $i, j \in A$ we have that $X_{wi} \cap X_{wj}$ is either a point or empty.
(6) For each $l \le k$, $w \in A^l$ and distinct $x, y \in X_w \cap \overline{X \setminus X_w}$, we have
$$d(x, y) \ge (\delta/K) \operatorname{diam} X_w.$$

In addition, we make the following inductive assumption:

(7) For each $w \in A^k$, there exists $u \in \mathcal{W}$ and distinct $ui_1, \dots, ui_q \in \mathcal{W}^{|u|+1}_u$ such that $X_w = \bigcup_{j=1}^q T_{ui_j}$.

Note that (7) holds when $k = 0$.

INDUCTIVE STEP. We now describe the construction of the sets $\{X_w\}_{w \in A^{k+1}}$. Fix a word $w \in A^k$. By Assumption (7), $X_w = T_{ui_1} \cup \dots \cup T_{ui_q}$. For simplicity, we assume that $i_j = j$ for all j. By Proposition 5.2(6), $q \le M$.

Case 1: $q = M$. In this case, we set $X_{wj} = T_{uj}$ for $j = 1, \dots, M$.

Case 2: $q < M$. Let n be the smallest integer such that

$$\sum_{j=1}^{q} \operatorname{card}(\mathcal{W}^{n+|u|}_{uj}) \ge M. \tag{5.1}$$

By Proposition 5.2(6), $2 \le n \le \log_2 M + 1$.

Case 2.1: The sum in (5.1) is equal to M. In this case, we set

$$\{X_{wi} : i \in A\} := \Big\{T_v : v \in \bigcup_{j=1}^{q} \mathcal{W}_{uj}^{n+|u|}\Big\}.$$

Case 2.2: The sum in (5.1) is strictly greater than M. Enumerate the elements of $\bigcup_{j=1}^{q} \mathcal{W}_{uj}^{n-1+|u|} = \{u_1, \dots, u_r\}$ so that for each $i \in \{1, \dots, r\}$, the set

$$T_{u_i} \cap \overline{X_w \setminus (T_{u_1} \cup \dots \cup T_{u_i})}$$

contains only one point. In other words, the sets $X_w \setminus T_{u_1}$, $(X_w \setminus T_{u_1}) \setminus T_{u_2}$, etc. are connected. That this is possible follows from Lemma 5.3 and the fact that every finite combinatorial tree has a leaf.

By minimality of n, we have that $r < M$. Now let m be the smallest integer in $\{1, \dots, r\}$ such that

$$\sum_{i=1}^{m} \mathrm{card}(\mathcal{W}_{u_i}^{n+|u|}) + (r - m) \geq M. \tag{5.2}$$

Note that if $m = r$, then (5.2) holds by (5.1), so such a minimal m exists.

Case 2.2.1: The sum in (5.2) is equal to M. Then, by the assumption of Case 2.2, we have $m < r$, and we set

$$\{X_{wi} : i \in A\} := \Big\{T_v : v \in \bigcup_{j=1}^{m} \mathcal{W}_{u_i}^{n+|u|} \cup \{u_{m+1}, \dots, u_r\}\Big\}.$$

Case 2.2.2: The sum in (5.2) is strictly greater than M. As before, enumerate the elements of $\mathcal{W}_{u_m}^{n+|u|} = \{u_m i_1, \dots, u_m i_l\}$ so that for each $j \in \{1, \dots, l\}$, the set

$$T_{u_m i_j} \cap \overline{T_{u_m} \setminus (T_{u_m i_1} \cup \dots \cup T_{u_m i_j})}$$

contains only one point.

By the minimality of m (and the fact that $r < M$), we have

$$\sum_{i=1}^{m-1} \mathrm{card}(\mathcal{W}_{u_i}^{n+|u|}) + \big(r - (m-1)\big) \leq M - 1,$$

and so

$$\sum_{i=1}^{m-1} \mathrm{card}(\mathcal{W}_{u_i}^{n+|u|}) + (r - m) \leq M - 2. \tag{5.3}$$

Let

$$p = M - 1 - (r - m) - \sum_{i=1}^{m-1} \mathrm{card}(\mathcal{W}_{u_i}^{n+|u|}).$$

Note that $p \geq 1$ by (5.3). Moreover, $p \leq l - 1 = \mathrm{card}\, \mathcal{W}_{u_m}^{n+|u|} - 1$; otherwise,

$$\sum_{i=1}^{m} \mathrm{card}(\mathcal{W}_{u_i}^{n+|u|}) + (r - m) \leq M - 1,$$

contradicting (5.2).

Define now

$$\mathcal{U} := \bigcup_{i=1}^{m-1} \mathcal{W}_{u_i}^{n+|u|} \cup \{u_m i_1, \ldots, u_m i_p\} \cup \{u_{m+1}, \ldots, u_r\}.$$

Note that $\operatorname{card}(\mathcal{U}) = M - 1$ by choice of p. Set

$$\{X_{wi} : i \in A\} := \{T_v : v \in \mathcal{U}\} \cup \left\{\overline{T_{u_m} \setminus (T_{u_m i_1} \cup \cdots \cup T_{u_m i_p})}\right\}.$$

To complete the inductive step and the proof of Lemma 5.4, it remains to check that the inductive properties (1)–(7) above are satisfied up to level $k+1$.

Property (1) holds: If $w \in A^k$, each X_{wi} is either equal to some T_v constructed in Proposition 5.2, and hence connected by Proposition 5.2(1), or (as is possible in Case 2.2.2) is a connected union of finitely many such T_v.

It also straightforward to check that Property (7) holds. In Cases 1, 2.1, and 2.2.1 of the construction, each X_{wi} for $wi \in A^{k+1}$ is exactly equal to some set T_u as constructed in Proposition 5.2 and therefore is a finite union of sets T_{uj}. In Case 2.2.2, there is also the possibility that X_{wi} is of the form $\overline{T_{u_m} \setminus (T_{u_m i_1} \cup \cdots \cup T_{u_m i_p})}$, where $u_m \in \mathcal{W}$ and $i_k \in A$. In that case, X_{wi} is also equal to a finite union of children of T_{u_m}; namely, $\{T_{u_m k} : k \neq i_1, \ldots, i_p\}$.

To see that Property (2) holds, set $w \in A^k$. In the construction of $\{X_{wi} : i \in A\}$, we write X_w as a finite union $T_{u1} \cup \cdots \cup T_{uq}$, where these sets come from Proposition 5.2. In each case, the sets X_{wi} are constructed to be subsets of these T_{uj} and exhaust each of them.

For Property (4), set $wi \in A^{k+1}$ and $x \in X_{wi} \cap \overline{X \setminus X_{wi}}$. The construction of X_{wi} and Proposition 5.2(2,3) ensures that x is contained in at most one other X_{wj} $(j \neq i)$ and is a leaf of X_{wi} in this case.

If $x \in X_{wi} \cap X_{w'i'}$ for some $w \neq w' \in A^k$, then by induction, x is a leaf of X_w and hence of X_{wi}. Moreover, in this case, x cannot be contained in any other $X_{w''}$ by induction or in any other element X_{wj} $(j \neq i)$ since a leaf of X_w can only be in one of the nontrivial connected subsets X_{wj}.

To see that Property (5) holds, consider $w \in A^k$ and the set $X_{wi} \cap X_{wj}$ (for $i \neq j$). By (1), this intersection is either empty, a point, or a non-trivial continuum. By construction, each of the two sets $X_{wi} \cap X_{wj}$ is a finite union of distinct elements of some $\mathcal{T}_n$ constructed in Proposition 5.2, and so the intersection cannot be a continuum by Proposition 5.2(2).

For Property (3), fix $w \in A^k$ and $i \in A$. By (7), there exists $u \in \mathcal{W}^l$ and $uj \in \mathcal{W}^{l+1}$ such that $T_{uj} \subset X_w \subset T_u$. By the design above, there exists $v \in \mathcal{W}_u^{l+n}$ and $vj' \in \mathcal{W}_u^{l+n+1}$ such that $T_{vj'} \subset X_{wi} \subset T_v$ and $2 \leq n \leq \log_2 M + 1$. Therefore, applying Proposition 5.2(5),

$$K_1 \leq K^{-1} \delta^{\log_2 M + 1} \leq \frac{\operatorname{diam} X_{wi}}{\operatorname{diam} X_w} \leq K\delta \leq K_2.$$

Finally, for Property (6), fix $w \in A^{k+1}$ and distinct $x, y \in X_w \cap \overline{X \setminus X_w}$. By (7), we know that $X_w = T_{ui_1} \cup \cdots \cup T_{ui_n}$ for some $u \in \mathcal{W}^l$ and $ui_1, \ldots, ui_n \in \mathcal{W}^{l+1}$. By Proposition 5.2(1), x, y have distance at least δ^{l+1}, so

$$\operatorname{dist}(x, y) \geq \delta^{l+1} \geq (\delta/K) \operatorname{diam} T_u \geq (\delta/K) \operatorname{diam} X_w. \qquad \square$$

We call the sets X_w constructed in Lemma 5.4 "tiles." We observe that these new tiles also maintain the property that they can touch only a controlled number of tiles of the same scale.

LEMMA 5.5

There is a constant n_0, depending only on the doubling constant of X and the constants from Lemma 5.4, such that if $w \in A^*$, then

$$\operatorname{card}\{v \in A^{|w|} : v \neq w, X_v \cap X_w \neq \emptyset\} \leq n_0.$$

Proof

Let

$$W = \{v \in A^{|w|} : v \neq w, X_v \cap X_w \neq \emptyset\}.$$

For each $v \in W$, Lemma 5.4(5) implies that $X_w \cap X_v$ is a single point, which we call $x_v \in X_w \cap X_v$. Moreover, if $v, v' \in W$ and $v \neq v'$, then $x_v, x_{v'} \in X_w \cap \overline{X \setminus X_w}$. By Property (4) of Lemma 5.4, we have that $x_v \neq x_{v'}$, and by Property (6), we have that

$$d(x_v, x_{v'}) \geq K_3 \operatorname{diam}(X_w).$$

Since all the points $\{x_v : v \in W\}$ are contained in X_w, the doubling property of X completes the proof. □

5.2. Definition of combinatorial data

Fix δ as above in Lemma 5.4, and apply Lemma 5.4 with fixed parameters $M \in \mathbb{N}$ and $K_1, K_2 \in (0,1)$ as in the statement of that lemma. Let $A = \{1, \ldots, M\}$. We define combinatorial data $\mathcal{C} = (A, (T_k)_{k\in\mathbb{N}})$ by setting $T_k = (A^k, E_k)$, where two words v, w of A^k are adjacent if and only if $X_v \cap X_w \neq \emptyset$.

LEMMA 5.6

$\mathcal{C}$ satisfies the conditions of Definition 1.1, and each graph T_k is a combinatorial tree.

Proof

Property (1) of Definition 1.1 is immediate. That T_k is a (connected) combinatorial tree follows from Lemma 5.3.

Property (2a) of Definition 1.1 holds similarly, taking $X = X_w$, which is connected, and again using Lemma 5.3.

For Property (2b), consider $\{w, u\} \in E_k$. Then there is a point $x \in X_w \cap X_u$. By Lemma 5.4(2), there are words wi and uj such that $x \in X_{wi} \cap X_{uj}$, and therefore $\{wi, uj\} \in E_{k+1}$. □

One basic consequence of this construction of combinatorial data is the following.

LEMMA 5.7

If $w, u \in A^*$ and $A_w^{\mathbb{N}} \wedge_{\mathcal{C}} A_u^N \neq \emptyset$, then $X_w \cap X_u \neq \emptyset$.

Proof
Let $w, u \in A^*$ with $A_w^{\mathbb{N}} \wedge_{\mathcal{C}} A_u^N \neq \emptyset$. By Lemma 3.3, there are then $k \in \mathbb{N}$, $w' \in A_w^k$, and $u' \in A_u^k$ with $\{w', u'\} \in E_k$. It follows from the definition of $\mathcal{C}$ that $X_{w'} \cap X_{u'} \neq \emptyset$, $X_{w'} \subseteq X_w$, and $X_{u'} \subseteq X_u$. This proves the lemma. □

5.3. Definition of diameter function
We continue to use the quasiconformal tree X fixed at the start of Section 5, and the constants M, K_1, K_2 and combinatorial data $\mathcal{C} = (A, (T_k)_{k\in\mathbb{N}})$ fixed at the start of Section 5.2.

We now define a diameter function $\Delta \in \mathcal{D}(A, K_1, K_2)$ with the following two rules.

- $\Delta(\varepsilon) = 1$.
- Suppose that for some $w \in A^*$, we have defined $\Delta(w)$:
  (1) If $\Delta(w) \leq \operatorname{diam} X_w$, then we define $\Delta(wi) = K_2\Delta(w)$ for all $i \in A$.
  (2) If $\Delta(w) > \operatorname{diam} X_w$, then we define $\Delta(wi) = K_1\Delta(w)$ for all $i \in A$.

This satisfies Definition 1.3, with Property (3) following from the fact that $K_1 < K_2 < 1$.

We now show that $\Delta(w)$ is always comparable to $\operatorname{diam}(X_w)$. This argument is very similar to the proof of Theorem A in [14, Section 4.1].

LEMMA 5.8
For all $w \in A^*$,

$$(K_2/K_1)^{-1}\Delta(w) \leq \operatorname{diam}(X_w) \leq (K_2/K_1)\Delta(w). \tag{5.4}$$

Proof
By Lemma 5.4(3), we have for all $w \in A^*$:

$$K_1 \operatorname{diam}(X_w) \leq \operatorname{diam}(X_{wi}) \leq K_2 \operatorname{diam}(X_w).$$

Note that (5.4) holds for $w = \varepsilon$ since $\Delta(\varepsilon) = \operatorname{diam}(X_\varepsilon) = 1$. Assume by induction that we have a word w such that (5.4) holds. Consider any $i \in A$. There are two possibilities.

Case 1: $\Delta(w) \leq \operatorname{diam}(X_w)$. In this case, we have

$$\Delta(wi) = K_2\Delta(w) \leq K_2 \operatorname{diam}(X_w) \leq (K_2/K_1) \operatorname{diam}(X_{wi})$$

and

$$\operatorname{diam}(X_{wi}) \leq K_2 \operatorname{diam}(X_w) \leq K_2(K_2/K_1)\Delta(w) = (K_2/K_1)\Delta(wi),$$

which together prove (5.4) for the word wi in case 1.

Case 2: $\Delta(w) > \operatorname{diam}(X_w)$. In this case, we have

$$\Delta(wi) = K_1\Delta(w) \leq K_1(K_2/K_1) \operatorname{diam}(X_w) \leq (K_2/K_1) \operatorname{diam}(X_{wi})$$

and

$$\operatorname{diam}(X_{wi}) \leq K_2 \operatorname{diam}(X_w) < K_2\Delta(w) = (K_2/K_1)\Delta(wi),$$

which together prove (5.4) for the word wi in case 2. □

As in Section 3.3, let $\sim$ be the equivalence relation on $A^{\mathbb{N}}$ induced by the diameter function Δ, and let $\mathcal{A} = A^{\mathbb{N}}/\sim$ and $\mathcal{A}_w = A^{\mathbb{N}}_w/\sim$.

5.4. Proof of Theorem 5.1

A consequence of Lemma 5.4(2) is that for each $x \in X$, there exists an infinite word $w_x \in A^{\mathbb{N}}$ such that $x \in X_{w(n)}$ for all $n \in \mathbb{N}$. We therefore define a map $f\colon X \to \mathcal{A}$ by $f(x) = [w_x]$.

LEMMA 5.9

The map $f : X \to \mathcal{A}$ defined above is well-defined and surjective.

Proof

Suppose that there exist two words $w, u \in A^{\mathbb{N}}$ such that for all $n \in \mathbb{N}$, $x \in X_{w(n)} \cap X_{u(n)}$. Then, by the construction of the combinatorial data $\mathcal{C}$, for each $n \in \mathbb{N}$ we have $\{w(n), u(n)\} \in E_n$. (Recall that E_n is the set of edges of T_n.) Thus, for each $n \in \mathbb{N}$, the set $\{A^{\mathbb{N}}_{w(n)}, A^{\mathbb{N}}_{u(n)}\}$ is a chain that joins w with u, and so $d_{\mathcal{C},\Delta}([w],[u]) \leq \Delta(w(n)) + \Delta(u(n)) \to 0$ as $n \to \infty$. We therefore have that $d_{\mathcal{C},\Delta}([w],[u]) = 0$, which implies that $[w] = [u]$. This shows that f is well-defined.

To show that f is surjective, consider an arbitrary $[u] \in \mathcal{A}$. We have nested compact tiles

$$X_{u(1)} \supseteq X_{u(2)} \supseteq X_{u(3)} \dots$$

in X. Let $x \in \cap_{n\in\mathbb{N}} X_{u(n)}$. If $f(x) = w \in \mathcal{A}$, then by definition of f, we have

$$x \in X_{w(n)} \cap X_{u(n)} \quad \text{for all } n \in \mathbb{N}.$$

As before, $u(n)$ and $w(n)$ are adjacent in T_n for each n, and hence again

$$d_{\mathcal{C},\Delta}\big([u],[w]\big) \leq \Delta\big(u(n)\big) + \Delta\big(w(n)\big) \to 0.$$

Thus, $[u] = [w] = f(x)$, and f is surjective. □

The proof of Theorem 5.1 concludes with the next two results.

PROPOSITION 5.10

The map $f : (X, d) \to (\mathcal{A}, d_{\mathcal{C},\Delta})$ is bi-Lipschitz, with constant depending only on K_1, K_2, and K_3.

Proof

Fix $x, y \in X$.

We first claim that $d_{\mathcal{C},\Delta}(f(x), f(y)) \geq \frac{K_1}{K_2} d(x,y)$. Suppose that $f(x) = [w]$ and $f(y) = [u]$. Let $\{A^{\mathbb{N}}_{w_1}, \dots, A^{\mathbb{N}}_{w_m}\}$ be a chain joining w with u. Since $w \in A^{\mathbb{N}}_{w_1}$, we have $w_1 = w(|w_1|)$, and therefore $x \in X_{w_1}$; similarly, $y \in X_{w_m}$.

We also have $X_{w_i} \cap X_{w_{i+1}} \neq \emptyset$ for each $i \in \{1, \dots, m-1\}$, by Lemma 5.7.

Therefore, using the triangle inequality and (5.4), we have

$$\sum_{i=1}^{m} \Delta(w_i) \geq \frac{K_1}{K_2} \sum_{i=1}^{m} \operatorname{diam} X_{w_i} \geq \frac{K_1}{K_2} d(x,y). \tag{5.5}$$

Taking the infimum over all possible chains, we obtain $d_{\mathcal{C},\Delta}(f(x), f(y)) \geq \frac{K_1}{K_2} d(x,y)$, as desired.

We now claim that

$$d_{\mathcal{C},\Delta}\big(f(x), f(y)\big) \lesssim d(x,y), \tag{5.6}$$

with implied constant depending only on K_1, K_2, K_3.

Let w_0 be a word in $\mathcal{W}$ of maximal length such that $x, y \in X_{w_0}$. Then, there exists distinct $i, j \in A$ such that $w_0 i, w_0 j \in \mathcal{W}$, $x \in X_{w_0 i}$ and $y \in X_{w_0 j}$. Set $k = |w_0|$. We consider the following two possible cases.

Suppose first that $X_{w_0 i} \cap X_{w_0 j} = \emptyset$. Let γ be the unique arc in X with endpoints x, y. Note that $\gamma \subseteq X_{w_0}$ as X_{w_0} is connected. Assuming $X_{w_0 i} \cap X_{w_0 j} = \emptyset$, it follows that $\gamma \setminus (X_{w_0 i} \cup X_{w_0 j})$ is a nonempty relatively open subset of γ. There must therefore exist some $l \in A \setminus \{i, j\}$ such that $\gamma \cap \partial X_{w_0 l}$ contains two distinct points v, v' of $\partial X_{w_0 l}$.

By the 1-bounded turning property of X and Lemma 5.4(6),

$$d(x,y) \geq \operatorname{diam} \gamma \geq d(v, v') \geq K_3 \operatorname{diam}(X_{w_0 l}).$$

On the other hand, $f(x), f(y) \in \mathcal{A}_{w_0}$ and so, by Lemma 3.9 and (5.4), we have

$$d_{\mathcal{C},\Delta}\big(f(x), f(y)\big) \leq \operatorname{diam} \mathcal{A}_{w_0} \leq \Delta(w_0) \leq \frac{K_2}{K_1} \operatorname{diam}(X_{w_0}).$$

Therefore, using Lemma 5.4(3),

$$d(x,y) \geq K_3 \operatorname{diam}(X_{w_0 l}) \geq K_3 K_1 \operatorname{diam}(X_{w_0}) \geq \frac{K_1^2 K_3}{K_2} d_{\mathcal{C},\Delta}\big(f(x), f(y)\big).$$

This completes the proof of (5.6) in the case where $X_{w_0 i} \cap X_{w_0 j} = \emptyset$.

Suppose now that $X_{w_0 i} \cap X_{w_0 j} \neq \emptyset$. Find words $w, u \in A^*$ of maximal lengths such that $w_0 w, w_0 u \in \mathcal{W}^*$, $x \in X_{w_0 w}$, $y \in X_{w_0 u}$ and $X_{w_0 w} \cap X_{w_0 u} \neq \emptyset$. Then there exist $w_0 w i, w_0 u j \in A^*$ such that $X_{w_0 w i} \cap X_{w_0 u} = \emptyset$, $X_{w_0 u j} \cap X_{w_0 w} = \emptyset$, $x \in X_{w_0 w i}$ and $y \in X_{w_0 u j}$.

Let z be the unique point of $X_{w_0 w} \cap X_{w_0 u}$ and again set γ to be the unique arc from x to y in X, which must pass through z. Choose $k \in A$ such that $z \in X_{w_0 w k}$. Note that $k \neq i$ by the maximality of w, and that $z \in \partial X_{w_0 w k}$. The sub-arc of γ from x to z must also contain a point $v \in \partial X_{w_0 w k}$ distinct from z, by Lemma 5.4(4).

Hence, again by 1-bounded turning and Lemma 5.4(6), we have

$$d(x,z) \geq d(v,z) \geq K_3 \operatorname{diam}(X_{w_0 w k}).$$

Similarly,

$$d(y,z) \geq K_3 \operatorname{diam}(X_{w_0 u l}),$$

for some $l \in A$.

By the 1-bounded turning property and Lemma 5.4(3),

$$d(x,y) \geq \frac{1}{2}\big(d(x,z)+d(y,z)\big) \geq \frac{1}{2}K_3\big(\operatorname{diam}(X_{w_0wk})+\operatorname{diam}(X_{w_0ul})\big)$$
$$\geq \frac{1}{2}K_3K_1\big(\operatorname{diam}(X_{w_0w})+\operatorname{diam}(X_{w_0u})\big).$$

On the other hand, $f(x) \in \mathcal{A}_{w_0w}$, $f(y) \in \mathcal{A}_{w_0u}$ and $\{A^{\mathbb{N}}_{w_0w}, A^{\mathbb{N}}_{w_0u}\}$ is a chain joining $f(x)$ and $f(y)$. Therefore, by Lemma 3.9 and by (5.4),

$$d_{\mathcal{C},\Delta}\big(f(x),f(y)\big) \leq \operatorname{diam}\mathcal{A}_{w_0w} + \operatorname{diam}\mathcal{A}_{w_0w} \leq \Delta(w_0w)+\Delta(w_0u)$$
$$\leq \frac{K_2}{K_1}\big(\operatorname{diam}(X_{w_0w})+\operatorname{diam}(X_{w_0u})\big).$$

Therefore,

$$d_{\mathcal{C},\Delta}\big(f(x),f(y)\big) \leq \frac{2K_2}{K_1^2K_3}d(x,y).$$

This completes the proof of (5.6) and hence of the proposition. □

Finally, to prove the "moreover" piece of Theorem 5.1, we now show the following.

LEMMA 5.11

The combinatorial data $\mathcal{C}$ and diameter function Δ defined above satisfy the conditions of Proposition 4.1 for some choice of N, n_0, c, δ_1, δ_2.

Proof

Property (P1) of Proposition 4.1 follows from our choice of a finite alphabet $A = \{1,\dots,M\}$. Property (P2) follows from Lemma 5.5 and the definition of the combinatorial trees T_k in our combinatorial data. Property (P3) is immediate from our construction of Δ, with $\delta_1 = K_1$ and $\delta_2 = K_2$.

It remains to verify Property (P4) of Proposition 4.1. Consider $k \in \mathbb{N}$ and distinct $u, u_1, u_2 \in A^*$ such that $\{u,u_1\}$ and $\{u,u_2\}$ are in E_n. Let also $w_1, w_2 \in A^{\mathbb{N}}_u$, $v_1 \in A^{\mathbb{N}}_{u_1}$, and $v_2 \in A^{\mathbb{N}}_{u_2}$ such that for all $n \geq k$ and $i \in \{1,2\}$, $\{w_i(n), v_i(n)\} \in E_n$.

For each $i \in \{1,2\}$, let $x_i \in X$ denote the unique point such that

$$x_i \in \bigcap_{n=0}^{\infty} X_{w_i(n)}.$$

By definition, we have $f(x_i) = w_i$. Notice that x_1 and x_2 are both in X_u as $w_i \in A^{\mathbb{N}}_u$.

We first claim that for $i \in \{1,2\}$,

$$x_i \in X_{u_i} \cap X_u \subseteq \partial X_u. \tag{5.7}$$

It follows from the definition of $\mathcal{C}$ that

$$\emptyset \neq X_{w_i(n)} \cap X_{v_i(n)} \subseteq X_{w_i(n)} \cap X_{u_i}$$

for all $n > k$. Hence,

$$\operatorname{dist}(x_i, X_{u_i}) \leq \operatorname{diam}(X_{w_i(n)}) \to 0 \quad \text{as } n \to \infty,$$

and so $x_i \in X_u \cap X_{u_i} \subseteq \partial X_u$.

We next claim that $x_1 \neq x_2$. Suppose to the contrary that $x_1 = x_2 = x$, and choose $n > k$ such that $w_1(n) \neq w_2(n)$. Then $X_{w_1(n)}$ and $X_{w_2(n)}$ are distinct subsets of X_u with $x \in X_{w_1(n)} \cap X_{w_2(n)}$. In addition, we showed in (5.7) that $x \in X_{u_1}$. It follows that there is an element $v \in A^n_{u_1}$ with $x \in X_v$. The word v, beginning as it does with $u_1 \neq u$, is distinct from both $w_1(n)$ and $w_2(n)$, and so the three words v, $w_1(n)$, and $w_2(n)$ are distinct and of the same length n. Moreover, $x \in X_{w_1(n)} \cap X_{w_2(n)} \cap X_v$. However, this contradicts Lemma 5.4(4).

Thus, x_1 and x_2 are distinct elements of ∂X_u. By Lemma 5.4(6) and (5.4),

$$d(x_1, x_2) \geq K_3 \operatorname{diam}(X_u) \geq (K_3 K_1/K_2)\Delta(u).$$

By Proposition 5.10, f is bi-Lipschitz with constant depending only on K_1, K_2, K_3. Therefore,

$$d_{\mathcal{C},\Delta}\big([w_1],[w_2]\big) = d_{\mathcal{C},\Delta}\big(f(x_1), f(x_2)\big) \geq c\,\Delta(u),$$

for some c depending only on K_1, K_2, K_3. This completes the proof. □

6. Examples and simple cases of quasiconformal trees

In this section, we discuss some examples and simple special cases of quasiconformal trees based on our construction.

6.1. Quasi-arcs

Here we discuss combinatorial data and diameter functions that give rise to quasi-arcs. We start with a corollary in which the conditions of Proposition 4.1 can be verified, using Lemma 4.8.

LEMMA 6.1

Let $\mathcal{C} = (A, (T_k)_{k\in\mathbb{N}})$ be combinatorial data such that card $A = N \geq 2$ *and each $T_k = (A^k, E_k)$ is a combinatorial arc. Let $\Delta \in \mathcal{D}(A)$ satisfy Property (P3) of Proposition 4.1 and assume that for all $k \geq 0$ and $w \in A^k$,*

$$\sum_{wi\in A_w^{k+1}} \Delta(wi) \geq \Delta(w). \tag{6.1}$$

Then $(\mathcal{A}, d_{\mathcal{C},\Delta})$ is a doubling bounded turning arc.

Proof

First, since card $A = M$, (P1) of Proposition 4.1 is immediately satisfied, and since each T_k is a combinatorial arc, $\mathrm{Val}(T_k) = 2$ and Condition (P2) of Proposition 4.1 is also satisfied. Since card $A \geq 2$, Assumption (1) of Lemma 4.8 is satisfied and by (6.1), Assumption (2) of Lemma 4.8 is satisfied. Hence, by Lemma 4.8 and Proposition 3.10, $(\mathcal{A}, d_{\mathcal{C},\Delta})$ is doubling and bounded turning.

It remains to show that $(\mathcal{A}, d_{\mathcal{C},\Delta})$ is an arc. By design, there exist exactly two words $w_1, w_2 \in A^{\mathbb{N}}$ such that for all $n \in \mathbb{N}$, the valency of $w_i(n)$ in T_n is 1. Recalling the definition of K from the proof of Lemma 3.17, we note that $K = \mathcal{A}$. Therefore, $(\mathcal{A}, d_{\mathcal{C},\Delta})$ is an arc. □

EXAMPLE 6.2
Let $M \in \{2, 3, \dots\}$ and $A = \{1, \dots, M\}$. Let $\mathcal{C}_M = (A, (G_k)_{k\in\mathbb{N}})$, where for each $k \in \mathbb{N}$, the graph G_k is a simple path with the following two rules:

(1) For each $w \in A^*$ and $i \in \{1, \dots, M-1\}$, we have that wi is adjacent to wi', where $i' = i + 1$.
(2) If $wiv, wjv' \in A^*$ with $i < j$, $|v| = |v'|$ and wiv is adjacent to wjv', then $wivM$ is adjacent to $wjv'1$.

In other words, each word in A^k is simply adjacent to the following word in lexicographic order in G_k.

Let $\delta \in (M^{-1}, 1]$ and $\Delta \in \mathcal{D}(A, M^{-1}, \delta)$. We write $\mathcal{A} = A^{\mathbb{N}}/\sim$ and for each $w \in A^*$, $\mathcal{A}_w = A^{\mathbb{N}}_w/\sim$.

The following lemma summarizes some properties of this construction.

LEMMA 6.3
(1) *Suppose $v, v' \in A^k$, with v coming earlier than v' in lexicographic order. Then $\mathcal{A}_v \cap \mathcal{A}_{v'} \neq \emptyset$ if and only if v and v' are adjacent in G_k.*
(2) *In case (1), $[vM^\infty] = [v'1^\infty]$ is the unique element of $\mathcal{A}_v \cap \mathcal{A}_{v'} \neq \emptyset$.*
(3) *For each $v \in A^*$, the set $\mathcal{A}_v$ is a topological arc with $M^{-1}\Delta(v) \leq \operatorname{diam} \mathcal{A}_v \leq \Delta(v)$.*

Proof
We begin with (1). Suppose $v, v' \in A^k$, with v preceding v' in lexicographic order, and $\mathcal{A}_v \cap \mathcal{A}_{v'} \neq \emptyset$. This means that there are infinite words w, w' with $[vw] = [v'w']$. Suppose v and v' were not adjacent; let u be a word on the simple path T_k between them (and hence lexicographically between v and v'). Let $n \in \mathbb{N}$ be such that $\Delta(t) < \frac{1}{2}\Delta(u)$ for all $t \in A^n$.

Because u is lexicographically between v and v', each $t \in A^n_u$ is lexicographically between $(vw)(n)$ and $(v'w')(n)$, and hence is on the unique simple path between $(vw)(n)$ and $(v'w')(n)$ in T_n. By Lemma 3.16, $[vw]$ and $[v'w']$ are both in $\mathcal{A}_t$ for each $t \in A^n_u$. In particular, all $\mathcal{A}_t$ for $t \in A^n_u$ share a common point. Therefore, by Lemma 3.9 and our choice of n above,

$$\operatorname{diam}(\mathcal{A}_u) \leq 2\max\{\operatorname{diam}(\mathcal{A}_t) : t \in A^n_u\} < \Delta(u). \tag{6.2}$$

On the other hand, our combinatorial data $\mathcal{C}_M$ satisfies the assumptions of Lemma 4.8. Indeed, Lemma 4.8(1) holds because the graphs G_k in $\mathcal{C}_M$ consist simply of arcs in lexicographical order, and Lemma 4.8(2) holds because any pair u, u' of distinct vertices in some $\partial_{\mathcal{C}} A^{k+1}_w$ are separated by at least M other vertices, each with diameter function giving weight $\geq M^{-1}\Delta(w)$.

Therefore, by Lemma 4.8, $\operatorname{diam}(\mathcal{A}_u) = \Delta(u)$, which contradicts (6.2).

This proves the "forward direction" of (1). For the other direction, it is immediate from the construction of $\mathcal{C}_M$ that if v and v' are adjacent in T_k, with v lexicographically preceding, then for each $n \in \mathbb{N}$,

$$d_{\mathcal{C},\Delta}(vM^\infty, v'1^\infty) \leq \Delta(vM^n) + \Delta(v'1^n) \to 0 \quad \text{as } n \to \infty,$$

and so $[vM^\infty] = [v'1^\infty] \in \mathcal{A}_v \cap \mathcal{A}_{v'}$.

For (2), suppose there was a point p other than $[vM^\infty] = [v'1^\infty]$ in $\mathcal{A}_v \cap \mathcal{A}_{v'}$. Then there would be an infinite word $w \in A^{\mathbb{N}}$, $w \neq M^\infty$ such that $p = [vw]$. Choose n such that the nth letter of w is not M. Then $vw(n)$ and $v'1^n$ are not adjacent in T_{k+n}, but $[vw] \in \mathcal{A}_{vw(n)} \cap \mathcal{A}_{v'1^n}$. This contradicts (1).

For Fact (3), it is an immediate consequence of Remark 3.21 that each $\mathcal{A}_v$ is a topological arc. The diameter of $\mathcal{A}_v$ is at most $\Delta(v)$ by Lemma 3.9. If v has at least two neighbors in $T_{|v|}$, then $\operatorname{diam}(T_{|v|}) = \Delta(v)$ by Lemma 4.8. Otherwise, vi has at least two neighbors in $T_{|v|+1}$ for some $i \in A$, and so Lemma 4.8 says that $\operatorname{diam}(\mathcal{A}_{vi}) = \Delta(vi)$. Therefore,

$$\operatorname{diam} \mathcal{A}_v \geq \operatorname{diam} \mathcal{A}_{vi} = \Delta(vi) \geq M^{-1}\Delta(v). \qquad \square$$

PROPOSITION 6.4

The space $(\mathcal{A}, d_{\mathcal{C},\Delta})$ is a quasi-arc.

Proof

By Lemma 6.3, we know that $(\mathcal{A}, d_{\mathcal{C},\Delta})$ is a topological arc, and by Theorem 3.10, we know that $(\mathcal{A}, d_{\mathcal{C},\Delta})$ is bounded turning. Moreover, Property (P3) of Proposition 4.1 is satisfied, and for any $w \in A^*$ and $i \in A$,

$$\sum_{i \in A} \Delta(wi) \geq \sum_{i=1}^{M} \frac{1}{M} = 1,$$

and (6.1) holds. Therefore, by Lemma 6.1, $(\mathcal{A}, d_{\mathcal{C},\Delta})$ is doubling. $\square$

We note that a more refined statement holds; see Lemma 8.3. Furthermore, the converse of Proposition 6.4 is also true: every quasi-arc is bi-Lipschitz equivalent to $(\mathcal{A}, d_{\mathcal{C},\Delta})$ for some $\delta \in [M^{-1}, 1)$ and some $\Delta \in \mathcal{D}(A, M^{-1}, \delta)$; see Proposition 8.4.

6.2. The Vicsek tree and variations

Here we discuss a concrete example of a self-similar quasiconformal tree—the Vicsek tree—and how it can be viewed through our construction.

EXAMPLE 6.5

The *Vicsek tree* $\mathbb{V}$ is defined as the attractor of the iterated function system $\{\phi_1, \dots, \phi_5\}$ on $\mathbb{C}$ with

$$\phi_1(z) = \frac{1}{3}(z - 2 + 2i), \qquad \phi_2(z) = \frac{1}{3}(z + 2 + 2i),$$
$$\phi_3(z) = \frac{1}{3}(z + 2 - 2i), \qquad \phi_4(z) = \frac{1}{3}z, \qquad \phi_5(z) = \frac{1}{3}(z - 2 - 2i).$$

Let $A = \{1, \dots, 5\}$. For $k \in \mathbb{N}$, we define trees $T_k = (A^k, E_k)$ as follows. First, $E_1 = \{\{i, 4\} : i = 1, 2, 3, 5\}$. Inductively, assume that for some $k \in \mathbb{N}$, we have defined $T_k = (A^k, E_k)$ such that

- if $w \in A^{k-1}$ and $i \in \{1, 2, 3, 5\}$, then wi and $w4$ are adjacent.

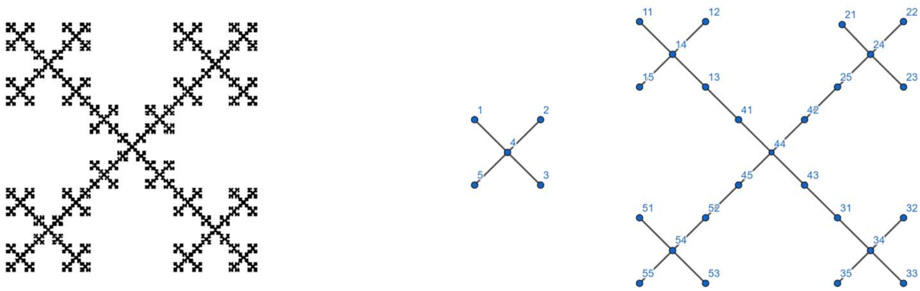

Figure 1. (Color online) On the left, we have $\mathbb{V}$ while on the right, we have the trees T_1 and T_2 of $\mathcal{C}$.

- if $w, u \in A^{k-1}$, $i, j \in A$ with $i \leq j$, and wi is adjacent to uj, then either $(i, j) = (1, 3)$, or $(i, j) = (1, 4)$, or $(i, j) = (2, 4)$, or $(i, j) = (2, 5)$, or $(i, j) = (3, 4)$, or $(i, j) = (4, 5)$.

For the definition of T_{k+1}, fix $w, u \in A^{k-1}$.

(1) If $i \in A$, then wii_1 is adjacent to wii_2 with $i_1, i_2 \in A$ if and only if $i_1 \in \{1, 2, 3, 5\}$ and $i_2 = 4$.
(2) If $w1$ is adjacent to $u3$, then $w11$ is adjacent to $u33$.
(3) If $w1$ is adjacent to $u4$, then $w13$ is adjacent to $u41$.
(4) If $w2$ is adjacent to $u4$, then $w25$ is adjacent to $u42$.
(5) If $w2$ is adjacent to $u5$, then $w22$ is adjacent to $u55$.
(6) If $w3$ is adjacent to $u4$, then $w31$ is adjacent to $u43$.
(7) If $w4$ is adjacent to $u5$, then $w45$ is adjacent to $u52$.

Figure 1 shows an illustration of $\mathbb{V}$ as well as the first two combinatorial trees, T_1 and T_2.[1]

Define a diameter function $\Delta : A^* \to [0, 1]$ by simply setting $\Delta(w) = 3^{-|w|}$. Clearly, $\Delta \in \mathcal{D}(A, 1/3, 1/3)$.

CLAIM 6.6
The space $(\mathcal{A}, d_{\mathcal{C},\Delta})$ is bi-Lipschitz equivalent to $\mathbb{V}$.

Proof
The proof essentially follows that of Theorem 5.1. For each $w = i_1 \cdots i_n \in A^*$, let $X_w = \phi_{i_1} \circ \cdots \circ \phi_{i_n}(\mathbb{V})$. The collection of sets $\{X_w : w \in A^*\}$ satisfies the conclusions of Lemma 5.4. Moreover, given $k \in \mathbb{N}$ and distinct $w, u \in A^k$, we have that $X_w \cap$

1. This picture of $\mathbb{V}$ was generated using the IFS Construction Kit (version April 11, 2019) created by Larry Riddle. It is available at http://larryriddle.agnesscott.org/ifskit/download.htm.

$X_u \neq \emptyset$ if and only if w is adjacent to u in T_k. Define now $F : \mathcal{A} \to \mathbb{V}$ such that if $w = i_1 i_2 \cdots \in A^{\mathbb{N}}$, then

$$F\big([w]\big) := \bigcap_{n=1}^{\infty} X_{i_1 \cdots i_n}, \quad \text{for } w = i_1 i_2 \cdots \in A^{\mathbb{N}}.$$

The rest of the proof is as in Section 5.4, and we leave the details to the reader. □

It follows immediately from Claim 6.6 that $(\mathcal{A}, d_{\mathcal{C},\Delta})$ is doubling since $\mathbb{V}$ is. One could also see this by noting that Conditions (P1), (P2), and (P3) from Proposition 4.1 are clearly satisfied by this combinatorial data. To show that (P4) also holds, we verify Lemma 4.8. Item (1) of Lemma 4.8 is easy to check. For Item (2), take any $w \in A^k$ and any $u, u' \in \partial_{\mathcal{C}} A_w^{k+1}$. The combinatorial arc that joins u with u' in T_{k+1} contains three vertices, $\{u, w4, u'\}$, and so the total Δ-length of this combinatorial arc is

$$\Delta(u) + \Delta(w4) + \Delta(u') = \frac{1}{3}\Delta(w) + \frac{1}{3}\Delta(w) + \frac{1}{3}\Delta(w) = \Delta(w).$$

Therefore, Lemma 4.8 holds in this example, and so does Assumption (P4) of Proposition 4.1. Thus, all the conditions of Proposition 4.1 are satisfied, and $(\mathcal{A}, d_{\mathcal{C},\Delta})$ can be seen to be doubling by this proposition.

One may obtain new self-similar quasiconformal trees by keeping the same combinatorial data as the Vicsek tree but altering the diameter function Δ. We describe two examples.

EXAMPLE 6.7

Keep the same combinatorial data $\mathcal{C} = \{T_k\}$ for $\mathbb{V}$ defined above, but now use the diameter function $\Delta_2(w) = 2^{-|w|}$ rather than $\Delta(w) = 3^{-|w|}$ as before. Then the associated quotient space $(\mathcal{A}', d_{\mathcal{C},\Delta_2})$ is a "snowflake" of the previous example, in the following sense: It is bi-Lipschitz equivalent to the space $(\mathbb{V}, |\cdot|^p)$, where $p = \frac{\log(2)}{\log(3)}$. The proof parallels that of Claim 6.6, with the only difference being that the tiles X_w of $\mathbb{V}$ under the snowflaked Euclidean metric $|\cdot|^p$ have diameters $(3^{-|w|})^p = 2^{-|w|} = \Delta_2(w)$.

EXAMPLE 6.8

We again keep the combinatorial data $\mathcal{C} = \{T_k\}$ of the Vicsek tree but modify the diameter function once more. Define a diameter function Δ_3 by setting $\Delta_3(\varepsilon) = 1$ and inductively setting

$$\Delta_3(wi) = \begin{cases} \frac{1}{2}\Delta_3(w) & \text{if } i \in \{2,4,5\} \\ \frac{1}{4}\Delta_3(w) & \text{if } i \in \{1,3\}. \end{cases}$$

In this case, the space $(\mathcal{A}'', d_{\mathcal{C},\Delta_3})$ is a quasiconformal tree which contains both geodesic segments (e.g., the path from $[1^\infty]$ to $[3^\infty]$) and nongeodesic "snowflake" segments (e.g., the path from $[2^\infty]$ to $[5^\infty]$).

REMARK 6.9

A similar example to the Vicsek tree appears in [3, 4] in the form of the *continuum self-similar tree* (*CSST*). The CSST is a quasiconformal tree, and hence by Theorem 1.4 is bi-Lipschitz to one of our combinatorial models. However, it is not obvious to us that there is a simple concrete or dynamical way to form "tiles" in the CSST that satisfy all the assumptions in Lemma 5.4, as we did for the Vicsek tree.

6.3. A nondoubling tree

Below we give an example which illustrates the importance of Condition (P4) for the conclusions of Proposition 4.1. Thus, we will construct combinatorial data $\mathscr{C}$ in which all graphs G_k are trees, satisfying all the conditions of Proposition 4.1 except (P4), and for which the resulting metric tree is not doubling.

EXAMPLE 6.10

Let $\mathscr{C}$ be the combinatorial data of Example 6.5. For each $n \in \mathbb{N}$, let $w_n = 2\cdots 2 = 2^n \in A^n$, and let $u_{n,1},\dots,u_{n,N_n}$ denote those elements of A^n such that $w_n 1 u_{n,i}$ has valence 1 in T_{2n+1}.

Define $\Delta : A^* \to [0,1]$ with the following rules:

(1) If w is a word of the form $w_n 1 v u$, where $v \in A^n \setminus \{u_{n,1},\dots,u_{n,N_n}\}$ and $u \in A^*$, then let $\Delta(w_n 1 v u) = 4^{-u|}\Delta(w_n 1 v) = 3^{-2n-1}4^{-|u|}$.
(2) For all other words $w \in A^*$, let $\Delta(w) = 3^{-|w|}$.

We see that for each $n \in \mathbb{N}$, the following hold:

- If $v \in \{u_{n,1},\dots,u_{n,N_n}\}$, then $\mathscr{A}_{w_n 1 v}$ is bi-Lipschitz homeomorphic to $\mathbb{V}$ scaled by a factor of 3^{-2n-1}.
- If $v \in A^n \setminus \{u_{n,1},\dots,u_{n,N_n}\}$, then $\operatorname{diam} \mathscr{A}_{w_n 1 v} = 0$. (Indeed, two elements of $\mathscr{A}_{w_n 1 v}$ can be joined by a chain of 3^k steps at level k, each with the Δ value being $3^{-2n-1}4^{-k}$ for arbitrary $k \in \mathbb{N}$, which forces the distance to be zero.)

Therefore, for each $n \in \mathbb{N}$, the point $[w_n 1 3^\infty] \in \mathscr{A}$ has at least N_n branches, each of diameter at least 3^{-2n-1}. Since $N_n \to \infty$ as $n \to \infty$, it follows that $(\mathscr{A}, d_{\mathscr{C},\Delta})$ is not doubling.

Note also that A, $\mathscr{C}$, and Δ satisfy Properties (P1), (P2), (P3), but not (P4). Indeed, the fact that $\operatorname{diam} \mathscr{A}_{w_n 1 v} = 0$ for certain words, as in the second bullet above, already violates (P4).

7. Combinatorial descriptions of more general spaces with "good tilings"

In this section, we axiomatize a notion of a "good tiling" of a compact space and show that every compact space with such a tiling (not necessarily a tree) can be built from our combinatorial data.

Let X be a compact space for which there is a finite alphabet A, constants $r \in (0,1)$, $C > 1$, and a collection of nonempty closed, connected subsets $\{X_w : w \in A^*\}$ with the following properties:

(1) $X_\varepsilon = X$.
(2) For all $w \in A^*$ and all $i \in A$, $X_{wi} \subset X_w$. Moreover, $\bigcup_{i\in A} X_{wi} = X_w$.
(3) For all $w \in A^*$, $C^{-1} r^{|w|} \le \operatorname{diam} X_w \le C r^{|w|}$.

(4) If for $k \in \mathbb{N}$ and $w, u \in A^k$, we have $X_w \cap X_u = \emptyset$, then $d(X_w, X_u) \geq C^{-1} r^k$.

Tilings of metric spaces with very similar properties have certainly been considered by other authors (e.g., [2, 15]). The goal here is simply to write down some simple conditions that can be interpreted in our framework.

For each $k \in \mathbb{N}$, define a graph $G_k = (A^k, E_k)$ with the rule that for words $w, u \in A^k$, w is adjacent to u if and only if $X_w \cap X_u \neq \emptyset$. It is easy to see that the collection $\mathcal{C} = (A, (G_k)_{k \in \mathbb{N}})$ is combinatorial data in the sense of Definition 1.1. Define also $\Delta : A^* \to [0, 1]$, with $\Delta(w) = r^{|w|}$. Clearly, $\Delta \in \mathcal{D}(A, r, r)$.

PROPOSITION 7.1

The space (X, d) is bi-Lipschitz homeomorphic to $(\mathcal{A}, d_{\mathcal{C},\Delta})$.

Before the proof, we re-emphasize two points about Proposition 7.1. First, even if X is a metric tree, Proposition 7.1 does not force the combinatorial data $\mathcal{C}$ to consist of combinatorial trees. The second point is that in general, it is not obvious to us which spaces admit good tilings in the sense of this section. Thus, Proposition 7.1 is not in itself a generalization of Theorem 1.4 and proceeds along different lines. The tiles we constructed for quasiconformal trees in Lemma 5.4 do not satisfy the conditions of this section as they may in principle fail Conditions (3) or (4) of this section.

However, Proposition 7.1 does yield descriptions of some natural examples, as we show following the proof.

Proof of Proposition 7.1

Since $r < 1$, Property (3) in conjunction with the compactness of sets $\{X_w : w \in A^*\}$, gives that for any $w \in A^{\mathbb{N}}$, the set $\bigcap_{n \in \mathbb{N}} X_{w(n)}$ contains exactly one point which we denote by x_w.

Let $w, u \in A^{\mathbb{N}}$ such that $x_w \neq x_u$. Then there exists $n \in \mathbb{N}$ such that $X_{w(n)} \cap X_{u(n)} = \emptyset$. Let $A^{\mathbb{N}}_{w_1}, \ldots, A^{\mathbb{N}}_{w_n}$ be a chain joining w with u. Then $x_w \in X_{w_1}$, $x_u \in X_{w_n}$ and $X_{w_i} \cap X_{w_{i+1}} \neq \emptyset$ for all $i \in \{1, \ldots, n-1\}$. By the triangle inequality,

$$d(x_w, x_u) \leq \sum_{i=1}^{n} \operatorname{diam} X_{w_i} \leq C \sum_{i=1}^{n} \Delta(w_i).$$

Taking the infimum over all such chains, we obtain that $d(x_w, x_u) \leq C d_{\mathcal{C},\Delta}([w], [u])$. Therefore, if $[w] = [u]$, then $x_w = x_u$.

We can now define $F : \mathcal{A} \to X$ with $F([w]) = x_w$. By the preceding paragraph, F is well defined and C-Lipschitz.

To see why F is bi-Lipschitz, fix $w, u \in A^{\mathbb{N}}$.

If $d(x_w, x_u) = 0$ (i.e., $x_w = x_u$), then for all $n \in \mathbb{N}$, $X_{w(n)} \cap X_{u(n)} \neq \emptyset$. Therefore, $w(n)$ is adjacent or equal to $u(n)$ for all n, and it follows that $d_{\mathcal{C},\Delta}([w], [u]) = 0$.

If $x_w \neq x_u$, then there exists $n \in \mathbb{N}$ such that $X_{w(n)} \cap X_{u(n)} \neq \emptyset$ and $X_{w(n+1)} \cap X_{u(n+1)} = \emptyset$. It follows that $w(n)$ is adjacent to $u(n)$ in G_n and $\{A^{\mathbb{N}}_{w(n)}, A^{\mathbb{N}}_{u(n)}\}$ is a

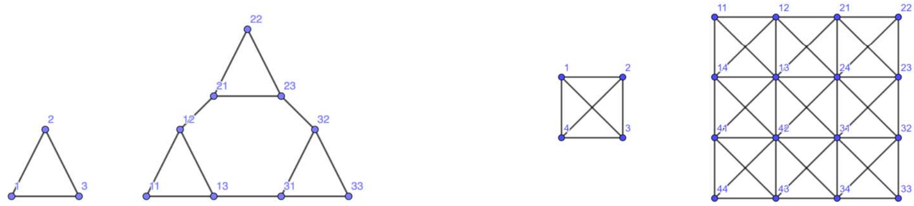

Figure 2. (Color online) On the left we have the graphs G_1, G_2 for the Sierpiński gasket, while on the right we have the graphs G_1, G_2 for the square.

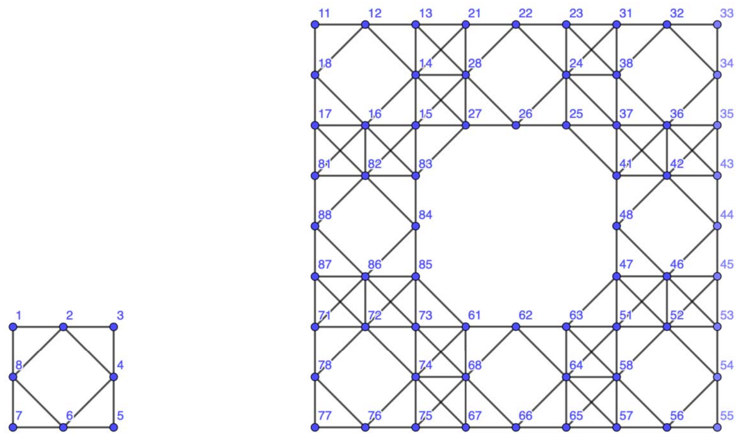

Figure 3. (Color online) Possible graphs G_1 and G_2 for the standard Sierpiński carpet.

chain joining w with u. Therefore,

$$\begin{aligned} d\big(F\big([w]\big), F\big([u]\big)\big) &\geq \operatorname{dist}(X_{w(n+1)}, X_{u(n+1)}) \\ &\geq C^{-1} r^{n+1} = (2C/r)^{-1} 2r^n \\ &= (2C/r)^{-1}\big(\Delta\big(w(n)\big) + \Delta\big(u(n)\big)\big) \\ &\geq (2C/r)^{-1} d_{\mathcal{C},\Delta}\big([w],[u]\big). \end{aligned} \qquad \square$$

EXAMPLE 7.2

Proposition 7.1 applies to many metric spaces which are attractors for certain iterated function systems, like the square, the Sierpiński gasket, and the Sierpiński carpet. See Figures 2 and 3 for possible graphs G_1 and G_2 for the gasket, square, and carpet.

8. Bi-Lipschitz embeddability of quasiconformal trees

This section is devoted to the proof of the following quantitative version of Theorem 1.8.

THEOREM 8.1

Let X be a C-doubling, c-bounded turning tree. Assume that $\mathcal{L}(X)$ admits an L-bi-Lipschitz embedding into some $\mathbb{R}^M$. Then X admits an L'-bi-Lipschitz embedding into some $\mathbb{R}^N$. Here N and L' depend only on C, c, M, and L'.

The proof of Theorem 8.1 consists of two steps. In Section 8.1, we prove the special case of embeddability of quasi-arcs—i.e., quasiconformal trees in which the set of leaves consists of exactly two points. This is done in Proposition 8.2 below, which is a stronger version of Proposition 1.7 from the introduction.

Then, in Section 8.2, we employ a bi-Lipschitz welding theorem of Lang and Plaut [17] and a characterization of metric spaces admitting bi-Lipschitz embedding into Euclidean spaces by Seo [25] to complete the proof of Theorem 8.1.

8.1. Bi-Lipschitz embeddability of quasi-arcs

The main result of this subsection is the following special case of Theorem 1.8, where the leaf set $\mathcal{L}(X)$ consists of only two points. In particular, this gives a detailed, sharp version of Proposition 1.7.

We first introduce a piece of terminology: A metric space X is (C,s)-*homogeneous* for some $C, s \geq 0$, if every subset of diameter d can be covered by at most $C\epsilon^{-s}$ sets of diameter at most ϵd. In particular, every doubling metric space is (C,s)-homogeneous for some C and s, depending on the doubling constant [13, Section 10.13].

PROPOSITION 8.2

Given $s \geq 1$, $C > 0$, and $c \geq 1$, there exists $L = L(c,C,s) > 1$ with the following property: If $\Gamma = ([0,1],d)$ is c-bounded turning and (C,s)-homogeneous, then it is L-bi-Lipschitz embeddable in $\mathbb{R}^{\lfloor s\rfloor+1}$.

Proposition 8.2 generalizes Theorem C in [14], where it was assumed that $s < 2$. We remark that the dimension $\lfloor s\rfloor + 1$ in Proposition 8.2 is sharp when $s > 1$, in the sense that there exists a 1-bounded turning, (C,s)-homogeneous metric d on $[0,1]$ (namely the snowflaked Euclidean metric $|\cdot|^{1/s}$) such that $([0,1],d)$ cannot be bi-Lipschitz embedded in $\mathbb{R}^{\lfloor s\rfloor}$.

For the proof of Proposition 8.2, we may assume that $\operatorname{diam}\Gamma = 1$. The proof uses a construction of Herron and Meyer [14] and a bi-Lipschitz embedding method of Romney and Vellis [24] (see also [1] and [29]).

Let $M \in \{2,3,\dots\}$, $A = \{1,\dots,M\}$ and $\mathcal{C}_M = (A,(G_k)_{k\in\mathbb{N}})$ be as in Example 6.2.

LEMMA 8.3 ([14, Lemma 3.1])

If $d \in (M^{-1},1)$ and $\Delta \in \mathcal{D}(A,M^{-1},\delta)$, then the space $(\mathcal{A}, d_{\mathcal{C}_M,\Delta})$ is s-homogeneous with $s = \log(M)/\log(1/\delta)$.

The following result can be obtained following the arguments of [14, Theorem B] essentially verbatim; we provide a brief reference to the necessary arguments.

PROPOSITION 8.4

Let $s \geq 1$, $c \geq 1$, and Γ a c-bounded turning and s-homogeneous metric arc with $\operatorname{diam}\Gamma = 1$. *Then for any $M \in \{2, 3, \dots\}$ and any $\delta \in (M^{-1/s}, 1)$, there exists $\Delta \in \mathcal{D}(A, M^{-1}, \delta)$ and an L-bi-Lipschitz homeomorphism $f : \Gamma \to (\mathcal{A}, d_{\mathcal{C}_M, \Delta})$. The constant L depends only on c, s, and M.*

Proof
Exactly following the procedure in [14, p. 622], we divide Γ into M sub-arcs of equal diameter then iterate this procedure on each sub-arc. Letting $\mathcal{C}_M = (A = \{1, \dots, M\}, G_k)$ as above yields an assignment to each element $w \in A^*$ of an arc $\gamma_w \subseteq \Gamma$, with nesting and adjacency properties reflecting that of $\mathcal{C}_M$ and $\sup_{w \in A^k} \operatorname{diam}(\gamma_w) \to 0$ as $k \to \infty$.

The argument in [14, pp. 622–623] provides a diameter function $\Delta \in \mathcal{D}(A, M^{-1}, \delta)$ such that

$$\Delta(w) \approx \operatorname{diam}(\gamma_w),$$

with implied constant depending only on c, s, and M.

Defining $F \colon \mathcal{A} \to \Gamma$ by $F([w]) = \cap_{k=1}^{\infty} \gamma_{w(k)}$, we see exactly as in Lemma 5.9 and Proposition 5.10 of the present paper that F is well-defined, surjective, and bi-Lipschitz. Taking $f = F^{-1}$ completes the proof. □

We now fix parameters M and δ that will enable us to use a construction from [24]. Given $s \geq 1$, let

- n be the minimal integer satisfying $n > (\lfloor s \rfloor + 1 - s)^{-1}$,
- $p = \lfloor s \rfloor - 1 + \frac{n-1}{n} = \lfloor s \rfloor - \frac{1}{n} > 0$,
- $M_0 = 9^{n(\lfloor s \rfloor + 1)}$,
- $M = M_0^{1+p}$, and
- $\delta = M_0^{-1}$.

The above parameters all depend on s, but we suppress this in the notation. Observe that $\delta > M^{-1/s} \geq M^{-1}$ in all cases, and in fact δ is an integer multiple of M^{-1}. Only δ and M will play a direct role below.

Given Proposition 8.4, the proof of Proposition 8.2 now reduces to the following lemma.

LEMMA 8.5

Let $s \geq 1$ and choose M and δ as above. Let $\Delta \in \mathcal{D}(A, M^{-1}, \delta)$. Then there is a bi-Lipschitz embedding of $(\mathcal{A}, d_{\mathcal{C}_M, \Delta})$ into $\mathbb{R}^{\lfloor s \rfloor + 1}$ with bi-Lipschitz constant depending only on M, δ and s, and thus only on s.

The construction of the embedding follows ideas and notation from [24]. We fix parameters M and δ as in the statement of Lemma 8.5 and write $\mathcal{C} = \mathcal{C}_M$. We also fix $\Delta \in \mathcal{D}(A, M^{-1}, \delta)$ for the remainder of this subsection.

Let

$$I = [0,1] \times \{0\}^{\lfloor s \rfloor}$$
$$L = \big((\{0\} \times [0,1/2]) \cup ([0,1/2] \times \{1/2\}) \big) \times \{0\}^{\lfloor s \rfloor - 1},$$

with the convention that $E \times \{0\}^0 = E$. An I-segment (resp. L-segment) is the image of I (resp. L) under a similarity mapping of $\mathbb{R}^{\lfloor s \rfloor + 1}$ and is parallel to the coordinate axes.

Given an I- or L- segment τ with length ℓ and endpoints x^*, y^*, we define the *cubic thickening* $Q(\tau)$ of τ to be the union of all closed cubes parallel to coordinate axes, of side length $(1-2\delta)\ell$ and centered on points $z \in \tau$ such that

$$\min\big\{|z - x^*|, |z - y^*|\big\} \geq \ell(1-2\delta)/2.$$

Define also $\mathscr{C}(\tau)$ to be the closed cube which is parallel to coordinate axes, has side length ℓ, and is centered on the midpoint of τ. The intersection $Q(\tau) \cap \partial\mathscr{C}(\tau)$ has exactly two components which we call the *entrances* of $Q(\tau)$.

For each $\tau \in \{I, L\}$, we define two polygonal arcs $\mathscr{J}(\tau)$ and $\mathscr{J}_0(\tau)$ in the following lemma.

LEMMA 8.6

Given $\tau \in \{I, L\}$, there exist two polygonal arcs $\mathscr{J}(\tau)$ and $\mathscr{J}_0(\tau)$, each contained in $Q(\tau)$, whose endpoints are the same as those of τ and that satisfy the following properties:

(J1) *The arcs $\mathscr{J}(\tau)$, $\mathscr{J}_0(\tau)$ consist of M-many I-segments and L-segments σ_i, $i \in \{1, \dots, M\}$, labeled according to their order in $\mathscr{J}(\tau)$ with σ_1 containing the origin. Each σ_i in $\mathscr{J}(\tau)$ has length δ, and each σ_i in $\mathscr{J}_0(\tau)$ has length M^{-1}.*

(J2) *The segments σ_1 and σ_M are I-segments.*

(J3) *For all $i \in \{1, \dots, M-1\}$, $Q(\sigma_i) \cap Q(\sigma_{i+1})$ is an entrance of $Q(\sigma_i)$ and an entrance of $Q(\sigma_{i+1})$. If $i, j \in \{1, \dots, M\}$, with $|i - j| > 1$, then $Q(\sigma_i) \cap Q(\sigma_j) = \emptyset$.*

(J4) *If E_1, E_2 are the entrances of $Q(\tau)$, then an entrance of $Q(\sigma_1)$ is contained in E_1, and an entrance of $Q(\sigma_M)$ is contained in E_2. Moreover, for any $i \in \{2, \dots, M-1\}$, $Q(\sigma_i) \cap \partial Q(\tau) = \emptyset$.*

Proof

The constructions of $\mathscr{J}_0(I)$ and $\mathscr{J}_0(L)$ are quite simple. Write $I = \bigcup_{m=1}^{M} \sigma_m$ with

$$\sigma_m = \left[\frac{m-1}{M}, \frac{m}{M}\right] \times \{0\}^{\lfloor s \rfloor} \subset \mathbb{R}^{\lfloor s \rfloor + 1}$$

and set $\mathscr{J}_0(I) = \bigcup_{m=1}^{M} \sigma_m = I$. Similarly write $L = \bigcup_{m=1}^{M} \sigma_m$, where σ_m is an L-segment if $m = \frac{M+1}{2}$ and an I-segment otherwise and each σ_m has length $1/M$. Set $\mathscr{J}_0(L) = \bigcup_{m=1}^{M} \sigma_m$.

The constructions of $\mathscr{J}(I)$ and $\mathscr{J}(L)$ are more complicated and can be found in [24, Sections 6.1 and 6.2] (where they are denoted as $J_I(N,n)$ and $J_L(N,n)$, respectively).

Without describing the construction, we briefly explain how our parameters match with those of [24]. The parameter N appearing in [24, p. 1181] matches our $\lfloor s \rfloor - 1$. Our parameters p and n match the ones given there. Our parameter M_0 corresponds to M in [24, p. 1182], and our parameter M corresponds to M^{1+p} in [24, p. 1182]. Making allowances for the changes in notation, our desired properties of $\mathcal{J}(I)$ and $\mathcal{J}(L)$ are listed in [24, Section 3.3] as Properties (1)–(3). □

We record a few more simple consequences of Properties (J1)–(J4).

LEMMA 8.7

Consider $\tau \in \{I, L\}$, $J \in \{\mathcal{J}(\tau), \mathcal{J}_0(\tau)\}$. Recall that J is a union of sets $\{\sigma_i\}_{i=1}^M$, each of which is an I-segment or L-segment. Then

(1) *For each $i \in \{1, \dots, M\}$, $Q(\sigma_i) \subset Q(\tau)$.*

(2) *For each $i \in \{2, \dots, M-1\}$,*
$$\operatorname{dist}\big(Q(\sigma_i), \partial Q(\tau)\big) \geq M^{-2}.$$

(3) *If $i, j \in \{1, \dots, M\}$ with $|i - j| > 1$, then*
$$\operatorname{dist}\big(Q(\sigma_i), Q(\sigma_j)\big) \geq M^{-2}.$$

(4) *Let E be the entrance of $Q(\tau)$ that contains an endpoint of σ_1 (resp. endpoint of σ_M), and let P be the $\lfloor s \rfloor$-dimensional plane that contains E. Then for all $i \in \{2, \dots, M\}$ (resp. $i \in \{1, \dots, M-1\}$),*
$$\operatorname{dist}\big(Q(\sigma_i), P\big) \geq M^{-2}.$$

Proof
All four statements are obvious in the case $J = \mathcal{J}_0(\tau)$, so we now assume that $J = \mathcal{J}(\tau)$. Statement (1) is an immediate consequence of the fact that $J \subseteq Q(\tau)$ and of Property (J4) of Lemma 8.6.

For the remaining three properties, it is useful to first observe that since δ is an integer multiple of M^{-1}, the sets $Q(\tau)$ and $Q(\sigma_i)$ are each unions of axis-parallel cubes whose vertices lie on the M^{-2}-scale grid $M^{-2}\mathbb{Z}^{\lfloor s \rfloor + 1}$.

Statements (2) and (4) follow immediately from this observation and (J4). Statement (3) follows immediately from this observation and (J3). □

We now use Lemma 8.6 to construct arcs in $\mathbb{R}^{\lfloor s \rfloor + 1}$ that mimic the metric properties of the combinatorial construction $\mathcal{C}$, Δ fixed below the statement of Lemma 8.5.

LEMMA 8.8

For each $w \in A^*$, there exists an I- or L-segment τ_w with the following properties:

(1) *If $w, u \in A^k$ are adjacent, then τ_w and τ_u intersect at an endpoint, while $Q(\tau_w) \cap Q(\tau_u)$ is contained in an entrance of $Q(\tau_w)$ and an entrance of $Q(\tau_u)$. If $w, u \in A^k$ are distinct but not adjacent, then $Q(\tau_w) \cap Q(\tau_u)$ and $\tau_w \cap \tau_u$ are empty.*

(2) *For any $w \in A^*$, there exists $\tau \in \{I, L\}$ such that τ_w and $Q(\tau_w)$ are scaled copies of τ and $Q(\tau)$, respectively, by a factor of $\Delta(w)$.*

Proof
The construction is done in an inductive manner.

Let $\tau_\varepsilon := I \subset \mathbb{R}^{\lfloor s \rfloor + 1}$. Property (1) of the lemma is vacuous in this base case, while property (2) is immediate.

Assume now that for some integer $k \geq 0$, we have defined I- and L-segments τ_w (for all $j \leq k$ and $w \in A^j$) satisfying the properties of the lemma. Fix $w \in A^k$, and let u be the preceding vertex of A^k in lexicographic order, assuming for the moment that such a vertex exists. Let E be the entrance of $Q(\tau_w)$ that intersects an entrance of $Q(\tau_u)$. Suppose that τ_w is a rescaled copy of $\tau \in \{I, L\}$. Let $\phi_w : \mathbb{R}^{\lfloor s \rfloor + 1} \to \mathbb{R}^{\lfloor s \rfloor + 1}$ be a similarity map such that $Q(\tau)$ is mapped onto $Q(\tau_w)$, the entrance of $Q(\tau)$ that contains the origin is mapped onto the entrance of $Q(\tau_w)$ that contains $Q(\tau_w) \cap Q(\tau_u)$, and the other entrance of $Q(\tau)$ is mapped to the other entrance of $Q(\tau_w)$.

If there is no $u \in A^k$ preceding w in lexicographic order, then $w = 1^k$ for some $k \geq 0$. In that case, if $k = 0$, we set ϕ_w to be the identity, and if $k \geq 1$, we set $u = 1^{k-1}2$ and do the analogous construction of ϕ_w to arrange that the entrance of $Q(\tau)$ that does not contain the origin is mapped onto the entrance of $Q(\tau_w)$ that contains $Q(\tau_w) \cap Q(\tau_u)$.

We now define τ_{wi} for each $i \in A$:

- If $\Delta(w1) = M^{-1}\Delta(w)$, then for each $i \in A$, set $\tau_{wi} = \phi_w(\sigma_i)$, where $\sigma_i \subset \mathcal{J}_0(\tau)$.
- If $\Delta(w1) = \delta\Delta(w)$, then for each $i \in A$, set $\tau_{wi} = \phi_w(\sigma_i)$, where $\sigma_i \subset \mathcal{J}(\tau)$.

This completes the definition of the arcs τ_w for all $w \in A^{k+1}$. We now prove that the family $\{\tau_w : w \in A^{k+1}\}$ satisfies Properties (1) and (2) of the lemma.

For Property (2) of the lemma, by design, and the inductive hypothesis (2), for all $i \in A$,

$$\operatorname{diam} \tau_{wi} = \operatorname{diam} \phi_w(\sigma_i) = \frac{\operatorname{diam} Q(\tau_w)}{\operatorname{diam} Q(\tau)} \operatorname{diam} \sigma_i = \Delta(w) \operatorname{diam} \sigma_i = \Delta(wi) \operatorname{diam} \tau'$$

for some $\tau' \in \{I, L\}$. Therefore, $\operatorname{diam} Q(\tau_{wi}) = \Delta(wi) \operatorname{diam} Q(\tau')$ for some $\tau' \in \{I, L\}$, and Property (2) holds for $k + 1$.

We now turn to the proof of (1). Let $w \in A^k$ and $i \in A$. Let also $u \in A^k$ and $j \in A$. We consider two cases.

Case 1. Assume that $w = u$ and $i \neq j$. If wi is adjacent to wj, then by design of paths $\mathcal{J}(\tau)$ and $\mathcal{J}_0(\tau)$, we have that τ_{wi} and τ_{wj} share an endpoint and by (J3), $Q(\tau_{wi}) \cap Q(\tau_{wj})$ is a common entrance of $Q(\tau_{wi})$ and $Q(\tau_{wj})$. If wi is not adjacent to wj, then again by (J3), $Q(\tau_{wi}) \cap Q(\tau_{wj}) = \emptyset$ which also implies that $\tau_{wi} \cap \tau_{wj} = \emptyset$.

Case 2. Assume that $u \neq w$. The proof splits in two subcases.

Case 2.1. Assume that $i \notin \{1, M\}$. Then wi is not adjacent to uj, and by (J4), $Q(\tau_{wi})$ is contained in the interior of $Q(\tau_w)$ which is disjoint from Q_u by the inductive hypothesis. Therefore, $Q(\tau_{wi}) \cap Q(\tau_{uj})$ and $\tau_{wi} \cap \tau_{uj}$ are both empty.

Case 2.2 Assume that $i \in \{1, M\}$. Without loss of generality, we assume that $i = 1$; the case $i = M$ is similar. By design, $Q(\tau_{wi})$ intersects one entrance of $Q(\tau_w)$ but not the other. Therefore, if u is not adjacent to w or if it is adjacent to w but is preceded by w, then the inductive hypothesis implies that $Q(\tau_{w1}) \cap Q(\tau_{uj})$ and $\tau_{w1} \cap \tau_{uj}$ are both empty. Assume now that u is adjacent to w and precedes w. Then the only $j \in A$ for which $Q(\tau_{uj})$ intersects the entrance of $Q(\tau_u)$ which contains $Q(\tau_w) \cap Q(\tau_u)$ is $j = M$. In this case, $\tau_{uM} \cap \tau_{w1}$ is the common endpoint of τ_w and τ_u. Therefore, $Q(\tau_{wi}) \cap Q(\tau_{uj})$ is nonempty and is contained in an entrance of $Q(\tau_{wi})$ and an entrance of $Q(\tau_{uj})$. □

Lemma 8.8(2) implies that for all $w \in A^*$,

$$2^{-1/2}\Delta(w) \leq \operatorname{diam} \tau_w \leq \Delta(w). \tag{8.1}$$

LEMMA 8.9

Let $w, u \in A^k$ be adjacent words, with w preceding u in lexicographic order. If $i \in A \setminus \{M\}$ or if $j \in A \setminus \{1\}$, then

$$\operatorname{dist}\big(Q(\tau_{wi}), Q(\tau_{uj})\big) \gtrsim_s \max\big\{\Delta(w), \Delta(u)\big\}.$$

Proof

Set $E = Q(\tau_w) \cap Q(\tau_u)$. By Lemma 8.8, E is contained in an entrance of $Q(\tau_w)$ and in an entrance of $Q(\tau_u)$. Let P be the $\lfloor s \rfloor$-dimensional plane in $\mathbb{R}^{\lfloor s \rfloor+1}$ that contains E. Then P separates the interior of $Q(\tau_{wi})$ from the interior $Q(\tau_{uj})$. By Lemma 8.7,

$$\begin{aligned}\operatorname{dist}\big(Q(\tau_{wi}), Q(\tau_{uj})\big) &\geq \max\big\{\operatorname{dist}\big(Q(\tau_{wi}), P\big), \operatorname{dist}\big(Q(\tau_{uj}), P\big)\big\}\\ &\gtrsim_s \max\big\{\Delta(w), \Delta(u)\big\}.\end{aligned}$$ □

For each $w \in A^*$ and $k \geq |w|$, set

$$\mathcal{Q}_w^{(k)} := \bigcup_{u \in A_w^k} Q(\tau_u), \qquad \mathcal{Q}^{(k)} := \bigcup_{u \in A^k} Q(\tau_u), \qquad \mathcal{Q}_w := \bigcap_{n \geq |w|} \mathcal{Q}_w^{(n)}.$$

By (8.1), if $w \in A^{\mathbb{N}}$, then $\lim_{n\to\infty} \operatorname{diam} Q(\tau_{w(n)}) \leq \lim_{n\to\infty}(\lfloor s \rfloor + 1)^{1/2}\delta^n = 0$. For each $w \in A^{\mathbb{N}}$, denote by x_w the unique point

$$\{x_w\} := \bigcap_{n\in\mathbb{N}} Q(\tau_{w(n)}) = \bigcap_{n\in\mathbb{N}} \mathcal{Q}_{w(n)}.$$

Define a map $F : (\mathcal{A}, d_{\mathcal{C},\Delta}) \to \mathcal{Q}_\varepsilon \subset \mathbb{R}^{\lfloor s \rfloor+1}$ by $F([w]) = x_w$.

LEMMA 8.10

F is well-defined, and $F(\mathcal{A}_w) = \mathcal{Q}_w$ for all $w \in A^*$.

Proof

Let $[w] = [v] \in \mathcal{A}$, with $w \neq v$. By Lemma 6.3, there is an $n \in \mathbb{N}$ and u, u' adjacent in A^n such that $w = uM^\infty$ and $v = u'1^\infty$ (or vice versa).

For each $n \in \mathbb{N}$, $Q(\tau_{uM^n})$ intersects with $Q(\tau_{u'1^n})$ on a common entrance. Denote by p the unique point in $\bigcap_{n\in\mathbb{N}}(Q(\tau_{uM^n}) \cap Q(\tau_{u'1^n}))$. Then $Q(\tau_{w(k)})$ and $Q(\tau_{v(k)})$ both contain p for all k, and hence $F([v]) = F([w]) = p$. So F is well-defined.

For the second part, fix $n \in \mathbb{N}$ and $w \in A^n$. For $k \geq n$, note that $\{\mathcal{Q}_w^{(k)}\}$ converges in Hausdorff distance to $\mathcal{Q}_w$. By construction, each point of $F([\mathcal{A}_w])$ is contained in the Hausdorff limit of the sets $\mathcal{Q}_w^{(k)}$, and hence in $\mathcal{Q}_w$. Thus, $F(\mathcal{A}_w) \subseteq \mathcal{Q}_w$.

For the other inclusion, fix $p \in \mathcal{Q}_w$. Let $v_0 = w$. For each $k \geq 1$, we inductively set $v_k \in A_{v_{k-1}}^{|w|+k} \subseteq A_w^{|w|+k}$ to be a word with $p \in \mathcal{Q}_{v_k}$. Let v be the infinite word such that $v(|w|+k) = v_k$ for all $k \geq 0$. Then immediately $p = F([v])$. Therefore, $\mathcal{Q}_w \subseteq F(\mathcal{A}_w)$. □

It remains to show now that F is L-bi-Lipschitz with L depending only on s.

Proof of Lemma 8.5
Fix distinct $[w], [w'] \in \mathcal{A}$. Without loss of generality, assume that w precedes w' in lexicographic order. Let σ be the unique arc in $\mathcal{A}$ whose endpoints are $[w]$ and $[w']$. Let also $w_0 \in A^*$ be the longest word such that $[w], [w'] \in \mathcal{A}_{w_0}$. Let also $i, j \in A$ such that $[w] \in \mathcal{A}_{w_0 i}$ and $[w'] \in \mathcal{A}_{w_0 j}$. By maximality of w_0, we have that $i \neq j$. We consider the following possible two cases.

Case 1. Suppose that $|i - j| > 1$. On one hand, there exists $i' \in A$ such that $\mathcal{A}_{w_0 i'} \subset \sigma$ which implies that

$$M^{-1}\Delta(w_0) \leq \Delta(w_0 i') \leq \operatorname{diam}\sigma = d_{\mathcal{C},\Delta}\big([w],[w']\big) \leq \Delta(w_0).$$

On the other hand, $F([w]) \in Q(\tau_{w_0 i})$, $F([w']) \in Q(\tau_{w_0 j})$, and by Lemma 8.7,

$$M^{-2}\Delta(w_0) \leq \operatorname{dist}\big(Q(\tau_{w_0 i}), Q(\tau_{w_0 j})\big) \leq \big|F\big([w]\big) - F\big([w']\big)\big|$$

$$\leq \operatorname{diam} Q(\tau_{w_0}) \leq \big(\lfloor s \rfloor + 1\big)^{1/2}\Delta(w_0).$$

Therefore, $d_{\mathcal{C},\Delta}([w],[w']) \approx_s \Delta(w_0) \approx_s |F([w]) - F([w'])|$. This completes the proof in Case 1.

Case 2. Suppose that $|i - j| = 1$. Without loss of generality, assume that $j = i+1$. Let k and l be the unique integers such that

$$\mathcal{A}_{w_0 i M^k} \cup \mathcal{A}_{w_0 j 1^l} \subset \sigma \subset \mathcal{A}_{w_0 i M^{k-1}} \cup \mathcal{A}_{w_0 j 1^{l-1}}.$$

Let also $i', j' \in A$ such that $[w] \in \mathcal{A}_{w_0 i M^{k-1} i'}$ and $[w'] \in \mathcal{A}_{w_0 j 1^{l-1} j'}$. Note that $i' \neq M$, while $j' \neq 1$. On one hand, using the 1-bounded turning property of $(\mathcal{A}, d_{\mathcal{C},\Delta})$ and Lemma 6.3, we have

$$\begin{aligned}
\max\big\{\Delta(w_0 i M^k), \Delta(w_0 j 1^l)\big\} &\leq M d_{\mathcal{C},\Delta}\big([w],[w']\big) \\
&\leq M \operatorname{diam}(\mathcal{A}_{w_0 i M^{k-1}} \cup \mathcal{A}_{w_0 j 1^{l-1}}) \\
&\leq 2M \max\big\{\Delta(w_0 i M^{k-1}), \Delta(w_0 j 1^{l-1})\big\} \\
&\leq 2M^2 \max\big\{\Delta(w_0 i M^k), \Delta(w_0 j 1^l)\big\}.
\end{aligned}$$

On the other hand, by Lemma 8.9,

$$\begin{aligned}\big|F\big([w]\big)-F\big([w']\big)\big| &\lesssim \max\{\operatorname{diam} Q(\tau_{w_0 i M^{k-1}}), \operatorname{diam} Q(\tau_{w_0 j 1^{l-1}})\}\\ &\lesssim \max\{\Delta(w_0 i M^{k-1}), \Delta(w_0 j 1^{l-1})\}\\ &\lesssim \operatorname{dist}\big(Q(\tau_{w_0 i M^{k-1} i'}), Q(\tau_{w_0 i M^{k-1} i'})\big)\\ &\le \big|F\big([w]\big)-F\big([w']\big)\big|,\end{aligned}$$

with implied constants depending on the parameter s.

Therefore,

$$\big|F\big([w]\big)-F\big([w']\big)\big| \approx_s \max\{\Delta(w_0 i M^{k-1}), \Delta(w_0 j 1^{l-1})\} \approx_s d_{\mathcal{C},\Delta}\big([w],[w']\big).$$

This completes the proof in Case 2 and the proof of the lemma. □

8.2. Proof of Theorem 8.1

Here we prove Theorem 1.8 using two bi-Lipschitz embedding results of Lang and Plaut [17] and of Seo [25]. The first result says that one can “glue” two bi-Lipschitz embeddings into a single embedding.

THEOREM 8.11 ([17, Theorem 3.2])

Let X be a metric space, and let $X_1, X_2 \subset X$ be closed subsets such that $X = X_1 \cup X_2$. If X_1 L_1-bi-Lipschitz embeds in $\mathbb{R}^{n_1}$ and X_2 L_2-bi-Lipschitz embeds in $\mathbb{R}^{n_2}$, then X L-bi-Lipschitz embeds in $\mathbb{R}^{n_1+n_2+1}$, with L depending on L_1, L_2, n_1, and n_2.

Using Theorem 8.11, we show that balls of X that are appropriately far from $\mathcal{L}(X)$ admit a bi-Lipschitz embedding into some Euclidean space quantitatively.

LEMMA 8.12

Let X be a doubling, bounded turning tree. For every $0 < \beta < 1$, there exist L and N depending only on the doubling constant of X, the bounded turning constant of X, and β such that if $B(x,r)$ is a ball with $x \in X \setminus \mathcal{L}(X)$ and $r < \beta \operatorname{dist}(x, \mathcal{L}(X))$, then $B(x,r)$ admits an L-bi-Lipschitz embedding into $\mathbb{R}^N$.

Proof

Fix $0 < \beta < 1$. Let $B = \overline{B}(x,r)$ be a ball with $x \in X \setminus \mathcal{L}(X)$ and $r < \beta \operatorname{dist}(x, \mathcal{L}(X))$. Let D denote the doubling constant of X and H the bounded turning constant. We will argue that B is contained in a union of at most $K = K(\beta, D, H)$ quasi-arcs. By Proposition 8.2 and Theorem 8.11, the latter implies that B admits an L-bi-Lipschitz embedding into $\mathbb{R}^N$ with N and L, depending only on K and D; hence, only on β, D, and H.

Let Γ be the collection of all arcs in X that join x to a leaf of X. For each $\gamma \in \Gamma$, parametrize it by a continuous $\gamma\colon [0,1] \to X$ such that $\gamma(0) = x$ and $\gamma(1) \in \mathcal{L}(X)$. Let $x_\gamma = \gamma(t_\gamma)$, where

$$t_\gamma = \sup\{t \in [0,1] : \gamma(t) \in B\}.$$

In other words, x_γ is the "last" point on γ contained in B. Similarly, let y_γ denote the last point on γ contained in $\overline{B}(x, r/\beta)$. Note that B and $\overline{B}(x, r/\beta)$ are disjoint from $\mathcal{L}(X)$ by assumption, so the points x_γ and y_γ must exist for each $\gamma \in \Gamma$.

Two properties of these points are clear:

(1) If $x_\gamma \neq x_{\gamma'}$, then $y_\gamma \neq y_{\gamma'}$. In particular,

$$\operatorname{card}\{x_\gamma : \gamma \in \Gamma\} \leq \operatorname{card}\{y_\gamma : \gamma \in \Gamma\}. \tag{8.2}$$

(2) We have $d(x_\gamma, x) = r$ and $d(y_\gamma, x) = r/\beta$ for each $\gamma \in \Gamma$.

Finally, let Γ_0 be the collection of arcs joining x to x_γ, as γ ranges in Γ. We will show that Γ_0 contains a controlled finite number of distinct elements, by showing that the collection $\{x_\gamma : \gamma \in \Gamma\}$ contains a controlled number of distinct elements. Since B is contained in the union of all arcs of Γ_0, this will complete the proof.

Suppose $\gamma, \gamma' \in \Gamma$ have $x_\gamma \neq x_{\gamma'}$. We then claim that

$$d(y_\gamma, y_{\gamma'}) \geq \eta r$$

for some constant η depending only on D and H.

Indeed, the arc $[y_\gamma, y_{\gamma'}]$ must contain x_γ, and hence its diameter is at least

$$d(y_\gamma, x_\gamma) \geq \Big(\frac{1}{\beta} - 1\Big) r,$$

and so

$$d(y_\gamma, y_{\gamma'}) \geq \frac{1}{H} \operatorname{diam}\big([y_\gamma, y_{\gamma'}]\big) \geq \frac{1}{H}\Big(\frac{1}{\beta} - 1\Big) r = \eta r.$$

The total number of different arcs in Γ_0 is controlled by the total number of distinct x_γ, which is controlled by $\operatorname{card}\{y_\gamma : \gamma \in \Gamma\}$ by (8.2). The points y_γ form an ηr-separated set in $\overline{B}(x, r/\beta)$, and so the cardinality of this set is bounded by a constant K depending only on η, β, and the doubling constant D. □

The second bi-Lipschitz embedding result that we need is Seo's general bi-Lipschitz embeddability criterion [25]. In fact, we use a simplified version of Seo's result presented by Romney in [23, Theorem 2.2]. Before stating the result, we recall a generalized notion of Whitney decomposition for metric measure spaces due to Christ [8] and Seo [25].

DEFINITION 8.13 ([8, 23, 25])
Let (X, d, μ) be a metric measure space, and let Ω be an open proper subset of X. A collection $\mathcal{Q}$ of open subsets of Ω is a *Christ–Whitney decomposition* of Ω if there exist constants $\delta \in (0,1)$, $C_1 > c_0 > 0$, and $a \geq 4$ such that the following properties are satisfied:

(1) $\bigcup_{Q \in \mathcal{Q}} Q$ is dense in Ω.
(2) For every $Q, Q' \in \mathcal{Q}$ with $Q \neq Q'$, we have $Q \cap Q' = \emptyset$.

(3) For every $Q \in \mathcal{Q}$, there exists $x \in \Omega$ and $k \in \mathbb{Z}$ such that

$$B(x, c_0\delta^k) \subset Q \subset B(x, C_1\delta^k)$$

and

$$(a-2)C_1\delta^k \leq \operatorname{dist}(Q, X \setminus \Omega) \leq \left(\frac{aC_1}{\delta}\right)\delta^k.$$

LEMMA 8.14 ([8, Theorem 11], [23, Lemma 2.5], [25, Lemma 2.1])
Let X be a doubling metric space and Y be a nonempty closed proper subset of X. Then $X \setminus Y$ has a Christ–Whitney decomposition, with constants δ, c_0, C_1, a absolute.

THEOREM 8.15 ([23, Theorem 2.2] [25, Theorem 1.1])
Let X be a complete metric measure space. Then X admits an L-bi-Lipschitz embedding into some Euclidean space $\mathbb{R}^M$ if and only if the following conditions hold for some constants L_1, L_2, M_1, M_2:

(1) *X is doubling.*
(2) *There is a nonempty closed subset of $Y \subseteq X$ which admits an L_1-bi-Lipschitz embedding into some $\mathbb{R}^{M_1}$.*
(3) *There is a Christ–Whitney decomposition of $X \setminus Y$ such that each cube admits an L_2-bi-Lipschitz embedding into some $\mathbb{R}^{M_2}$.*

The distortion L and target dimension M of the embedding of X depend only on the doubling constant of μ, M_1, M_2, and L_1, L_2.

Proof of Theorem 8.1
It suffices to show that X satisfies the conditions of Theorem 8.15 with $Y = \overline{\mathcal{L}(X)}$. The doubling property (1) in Theorem 8.15 is satisfied by assumption. We assume that $\mathcal{L}(X)$; hence, Y admits a bi-Lipschitz embedding into some $\mathbb{R}^{M_1}$, so (2) is assumed to hold in Theorem 5.1. It remains to prove (3).

By Lemma 8.14, there exists a Christ–Whitney decomposition $\mathcal{Q}$ for some constants $\delta \in (0,1)$, $C_1 > c_0 > 0$, and $a \geq 4$. Let $Q \in \mathcal{Q}$ be an arbitrary cube of this decomposition.

The doubling property of X implies that there exists $N \in \mathbb{N}$, depending only on the doubling constant of X and the constants of the Christ–Whitney decomposition, and there exist at most N balls $B_1, \dots, B_n$ with centers on Q and of radius $\frac{1}{3}\operatorname{dist}(Q,Y)$, such that $Q \subset B_1 \cup \dots \cup B_n$. In particular, the balls B_i each satisfy the assumptions of Lemma 8.12 with $\beta = \frac{1}{2}$.

Thus, by Lemma 8.12, each B_i admits an L'-bi-Lipschitz embedding into $\mathbb{R}^{M'}$, where L' and M' depend only on the doubling and bounded turning constants of X. By Theorem 8.11, $Q \subseteq B_1 \cup \dots \cup B_n$ admits an L_2-bi-Lipschitz embedding into $\mathbb{R}^{M_2}$, where L_2 and M_2 depend only on the doubling and bounded turning constants of X. This verifies Condition (3) of Theorem 8.15 and completes the proof of Theorem 8.1.
□

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