PROPAGATION OF CHAOS FOR THE CUCKER-SMALE SYSTEMS UNDER HEAVY TAIL COMMUNICATION

VINH NGUYEN AND ROMAN SHVYDKOY

Abstract. In this work we study propagation of chaos for solutions of the Liouville equation derived from the classical discrete Cucker-Smale system. Assuming that the communication kernel satisfies the heavy tail condition – known to be necessary to induce exponential alignment – we obtain a linear in time convergence rate of the k-th marginals $f^{(k)}$ to the product of k solutions of the corresponding Vlasov-Alignment equation, $f^{\otimes k}$. Specifically, the following estimate holds in terms of Wasserstein-2 metric

$$\mathcal{W}_2(f_t^{(k)}, f_t^{\otimes k}) \leqslant C\sqrt{k}\min\left\{1, \frac{t}{\sqrt{N}}\right\}. \tag{1}$$

For systems with the Rayleigh-type friction and self-propulsion force, we obtain a similar result for sectorial solutions. Such solutions are known to align exponentially fast via the method of Grassmannian reduction, [10]. We recast the method in the kinetic setting and show that the bound (1) persists but with the quadratic dependence on time.

In both the forceless and forced cases, the result represents an improvement over the exponential bounds established earlier in the work of Natalini and Paul, [12], although those bounds hold for general kernels. The main message of our work is that flocking dynamics improves the rate considerably.

1. Background and main results

One of the fundamental questions of the mathematical theory of large systems of particles is a derivation and formal justification of the corresponding kinetic models. Among the many systems describing collective phenomena this question has been successfully settled for the Cucker-Smale model describing the basic mechanism of alignment [3, 4]:

$$\begin{cases} \dot{x}_i = v_i, \qquad x_i(0) = x_i^0 \in \mathbb{R}^n, \\ \dot{v}_i = \dfrac{1}{N}\displaystyle\sum_{j=1}^{N} \phi(x_i - x_j)(v_j - v_i), \quad v_i(0) = v_i^0 \in \mathbb{R}^n. \end{cases} \tag{2}$$

Here ϕ is a non-negative non-increasing smooth communication kernel. The corresponding Vlasov-Alignment equation is given by

$$\partial_t f + v \cdot \nabla_x f + \nabla_v \cdot (f F(f)) = 0, \quad f(0) = f_0 : \mathbb{R}^{2n} \to \mathbb{R}_+, \tag{3}$$

where

$$F(f)(x,v) = \int_{\mathbb{R}^{2n}} \phi(x-y)(w-v) f(y,w,t)\, \mathrm{d}y\, \mathrm{d}w.$$

A formal derivation of (3) via the BBJKY hierarchy was performed in Ha and Tadmor [9], and rigorously via the mean-field limit in Ha and Liu [8].

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The hierarchy approach is based upon the classical idea of propagation of chaos, which postulates that the particles $(x_1, v_1, \dots, x_N, v_N)$ whose joint probability distribution f^N is given by the solution to the Liouville transport equation

$$\partial_t f^N + \sum_{i=1}^N v_i \cdot \nabla_{x_i} f^N + \sum_{i=1}^N \nabla_{v_i} \cdot (f^N F_i^N) = 0, \tag{4}$$

would gradually decorrelate as $N \to \infty$ if initially so

$$f^N(0) = f_0^{\otimes N}, \qquad f_0 : \mathbb{R}^{2n} \to \mathbb{R}_+, \tag{5}$$

and their individual distributions would evolve according to (3). In other words,

$$\langle f^N, \varphi_1 \otimes \ldots \otimes \varphi_k \otimes 1 \otimes \cdots \otimes 1 \rangle \to \prod_{j=1}^k \langle f, \varphi_j \rangle, \qquad \varphi \in C_b(\mathbb{R}^{2nk}). \tag{6}$$

The mean-field limit on the other hand, is based on the weak convergence of a sequence of empirical measures built from solutions to (2),

$$\mu^N = \frac{1}{N} \sum_{j=1}^N \delta_{x_i(t)} \otimes \delta_{v_i(t)} \to f.$$

In fact, a more detailed analysis done in [7, 13] establishes Lipschitz continuity of measure-valued solutions to (3) with respect to the Wasserstein metric,

$$\mathcal{W}_p(\mu_t', \mu_t'') \leqslant C(t) \mathcal{W}_p(\mu_0', \mu_0'').$$

It is well-known, however, that propagation of chaos and the mean-field limit (in a somewhat more specific sense) are equivalent, see Sznitman [14]. In fact, (6) holds if and only if for any $\varphi \in \mathrm{Lip}(\mathbb{R}^{2n})$ one has

$$E_\varphi(t) = \int_{\mathbb{R}^{2nN}} \left| \frac{1}{N} \sum_{j=1}^N \varphi(x_i(t), v_i(t)) - \langle f_t, \varphi \rangle \right|^2 f_0^{\otimes N} \, \mathrm{d}X_0 \, \mathrm{d}V_0 \to 0, \tag{7}$$

where X_0, V_0 are the initial conditions for the characteristic flow $\{x_i(t), v_i(t)\}_{i=1}^N$. Note that initially $E_\varphi(0) \to 0$ by a direct verification. Technically, since not every initial ensemble X_0, V_0 in the support of $f_0^{\otimes N}$ forms an empirical measure weakly close to f_0, the limit (7) does not directly follow from [7, 8, 13]. However, one can restore it using similar estimates on the deformation of the flow-map of (2) and coupling with the characteristics of (3).

In any case, Snitzman's general principle seems to provide little quantitative information on the rate of propagation in (6) as it avoids using any specificity of the system at hand. For stochastically forced systems, the work of Bolley, Cañizo and Carrillo [1] establishes such a quantitative estimate on the Wasserstein-2 distance:

$$\mathcal{W}_2(f_t^{(k)}, f_t^{\otimes k}) \leqslant C(T) \sqrt{\frac{k}{N^{e^{-Ct}}}}, \quad \forall t \leqslant T. \tag{8}$$

Recently, Natalini and Paul addressed the deterministic case in [12] and with additional chemotaxis forces in [11]. For the forceless system, the estimate carries exponential dependence in time,

$$\mathcal{W}_2(f_t^{(k)}, f_t^{\otimes k}) \leqslant C e^{\delta t} \sqrt{\frac{k}{N}}. \tag{9}$$

The estimates (8), (9) are finite-time bounds in spirit, in the sense that they do not take into account any flocking long-time behavior of the system. In this present work we raise the question: can one improve upon the time dependence in the deterministic case (9) when the system is known to flock exponentially fast? It is the result that goes back to Cucker and Smale [3] and improved and extended in [2, 8, 9] that the system (2) with a heavy tail radial communication,

$$\int_0^\infty \phi(r)\,\mathrm{d}r = \infty \tag{10}$$

aligns with an exponential rate. Let us give a quantitative summary of this result for future reference, see also [13] for details.

Proposition 1.1. *Suppose ϕ satisfies (10). For any solution to (2) with initial data in (X_0, V_0) in a compact domain $\Omega \subset \mathbb{R}^{2nN}$ the following flocking estimates hold:*

$$\sup_{t>0} \max_{i,j=1,\dots,N} |x_i - x_j| = D < \infty, \qquad \max_{i,j=1,\dots,N} |v_i - v_j| \leqslant A_0 e^{-t\phi(D)}, \tag{11}$$

where A_0 is the initial velocity fluctuation and D depends only on the initial diameter of the flock and ϕ.

Similarly, for any solution f to (3) with initial compact support one has

$$\sup_{t>0} \operatorname{diam} \operatorname{supp} f_t = D < \infty, \qquad \max_{(x',v'),(x'',v'')\in \operatorname{supp} f_t} |v' - v''| \leqslant A_0 e^{-t\phi(D)}. \tag{12}$$

With the use of this additional flocking information we will improve the estimate (9) to being linear in time.

Theorem 1.2. *Suppose ϕ satisfies (10), and let $f_0 \in C_0^1(\mathbb{R}^{2n})$ be an initial distribution with a compact support. Let f^N be the solution to (4)-(5), while f be the solution to (3). Then there exists a constant C which depends only on $\operatorname{diam}(\operatorname{supp} f_0)$ and ϕ such that for all $N \in \mathbb{N}$, $k \leqslant N$, and $t \geqslant 0$ one has*

$$\mathcal{W}_2(f_t^{(k)}, f_t^{\otimes k}) \leqslant C\sqrt{k} \min\left\{1, \frac{t}{\sqrt{N}}\right\}. \tag{13}$$

Our general methodology relies on the same classical coupling method, which compares characteristic flow of the original system (2) to N copies of the flow-map of the kinetic transport (3), but it differs from [12] in two aspects. First, we run the entire argument from the Lagrangian point of view, which gives a direct access to characteristics and the flocking estimates. This is closer in spirit to the original mean-field approach of [8] or [1] in stochastic settings. Second, we rely on the flocking information of Proposition 1.1 to extract a crucial stabilizing exponential factor in the estimation of kinetic energy, see (24). The linear time dependence here comes primarily from the growth of the potential energy, and it seems not to be removable within the given framework.

Next, we consider the same problem in the context of systems forced with self-propulsion and Rayleigh-type friction force with variable characteristic parameters θ:

$$\begin{cases} \dot{x}_i = v_i, \\ \dot{v}_i = \dfrac{1}{N}\displaystyle\sum_{j=1}^N \phi(x_i - x_j)(v_j - v_i) + \sigma v_i(\theta_i - |v_i|^p), \\ \dot{\theta}_i = \dfrac{\kappa}{N}\displaystyle\sum_{j=1}^N \phi(x_i - x_j)(\theta_j - \theta_i), \end{cases} \qquad (x_i, v_i, \theta_i) \in \mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}_+, \tag{14}$$

where $\kappa > 0$ is a coupling coefficient and $p > 0$. This model is relevant in the study of systems of agents with a tendency to adhere to their preferred characteristic speeds θ_i, see [6, 10]. The recent study [10] introduced a general method of Grassmannian reduction that allows to prove flocking for solutions with velocities confined to a sector Σ of opening $< \pi$, so-called sectorial solutions, see Proposition 3.1 below. We give an extension of this method to the corresponding kinetic Vlasov equation in Proposition 3.3 and use it to prove propagation of chaos for the forced system (14). Specifically, we prove the following theorem:

Theorem 1.3. *Suppose the kernel ϕ satisfies (26). Let $f_0 \in C_0^1(\Omega)$ be a sectorial initial distribution, and f^N, f be the sectorial solutions to the system (61) and (29), respectively. Then there exists a constant C which depends only on* $\operatorname{diam}(\operatorname{supp} f_0)$ *and ϕ such that for all $N \in \mathbb{N}$, $k < N$, and $t \geqslant 0$ one has*

$$W_2(f_t^{(k)}, f_t^{\otimes k}) \leqslant C\sqrt{k} \min\left\{1, \frac{t^2}{\sqrt{N}}\right\}. \tag{15}$$

To achieve this bound we employ monotonicity of the force to control the adverse self-propulsion component. The ultimate effect of its presence, however, is reflected in the quadratic dependence on time in (15).

In the case $\kappa = 0$ our analysis gives no additional improvement over (9). The derived kinetic equation, however, can present an interesting model of opinion dynamics for a large population which takes into account fixed conviction values θ. See Remark 3.12 for more discussion.

2. Propagation of chaos for the forceless system

In this section we focus on establishing propagation of chaos for the pure Cucker-Smale system (2). So, to fix the notation let us consider a solution f^N to the full Liouville equation (4) with the product initial condition (5) on the configuration space $(X, V) \in \mathbb{R}^{2nN}$. We can assume without loss of generality that f_0 is a probability distribution. The forces F_i^N's are given by the Cucker-Smale system

$$F_i^N(X, V) = \frac{1}{N}\sum_{j=1}^N \phi(x_i - x_j)(v_j - v_i).$$

Due to the symmetries of the forces, the solution will remain symmetric with respect to permutations of pairs (x_i, v_i) for all time.

We define the k-th marginal as usual by

$$f_t^{(k)}(x_1, v_1, \dots, x_k, v_k) = \int_{\mathbb{R}^{2n(N-k)}} f_t^N(x_1, v_1, \dots, x_N, v_N)\, \mathrm{d}x_{k+1} \dots \mathrm{d}v_N. \tag{16}$$

Let us introduce various characteristic maps that will be used in the proof. We denote by

$$\Phi_t^N = (x_1(t), v_1(t), \dots, x_N(t), v_N(t)) : \mathbb{R}^{2nN} \to \mathbb{R}^{2nN}$$

the flow-map of the Liouville equation (4), in other words these are solutions to the agent-based system

$$\begin{cases} \dot{x}_i = v_i, \\ \dot{v}_i = \dfrac{1}{N}\displaystyle\sum_{j=1}^N \phi(x_i - x_j)(v_j - v_i). \end{cases} \tag{17}$$

Then, f_t^N at any time $t > 0$ is a push-forward of the initial distribution by Φ_t^N,

$$f_t^N = \Phi_t^N \sharp f_0^{\otimes N}. \tag{18}$$

Now, denote by

$$\bar{\Phi}_t = (\bar{x}(t), \bar{v}(t)) : \mathbb{R}^{2n} \to \mathbb{R}^{2n}$$

the flow-map of the Vlasov equation (3), i.e.

$$\begin{cases} \dot{\bar{x}} = \bar{v}, \\ \dot{\bar{v}} = \displaystyle\int_{\mathbb{R}^{2n}} \phi(\bar{x} - y)(w - \bar{v}) f(y, w, t)\, \mathrm{d}y\, \mathrm{d}w, \end{cases} \tag{19}$$

and by

$$\bar{\Phi}_t^{\otimes N} = (\bar{x}_1(t), \bar{v}_1(t), \dots, \bar{x}_N(t), \bar{v}_N(t)) : \mathbb{R}^{2nN} \to \mathbb{R}^{2nN}$$

the direct product of N copies of $\bar{\Phi}_t$'s. Thus,

$$f_t = \bar{\Phi}_t \sharp f_0, \qquad f_t^{\otimes N} = \bar{\Phi}_t^{\otimes N} \sharp f_0^{\otimes N}. \tag{20}$$

The proof of Theorem 1.2 can be reduced to establishing the following estimate

$$\int_{\mathbb{R}^{2nN}} |\Phi_t^N(X_0,V_0) - \bar{\Phi}_t^{\otimes N}(X_0,V_0)|^2 \, f_0^{\otimes N}(X_0,V_0) \,\mathrm{d}X_0 \,\mathrm{d}V_0 \leqslant C \min\{N, t^2\}. \tag{21}$$

Indeed, let us recall that the Wasserstein-2 distance between two probability measures $\mu, \bar{\mu}$ on $\mathbb{R}^{2nk}$ can be defined in probabilistic sense as

$$\mathcal{W}_2^2(\mu,\bar{\mu}) = \inf \mathbb{E}[|Z - \bar{Z}|^2],$$

where the infimum is taken over $\mathbb{R}^{2nk}$-valued random variables $Z, \bar{Z}$ defined on any probability space with distributions given by μ and $\bar{\mu}$, respectively. To measure the distance between $f_t^{(k)}$ and $f_t^{\otimes k}$ we can pick the probability space $\mathbb{R}^{2nN}$ with measure $f_0^{\otimes N}(X_0,V_0)\,\mathrm{d}X_0\,\mathrm{d}V_0$, and random variables given by any selection of k coordinates of Φ_t^N and $\bar{\Phi}_t^{\otimes N}$, respectively, because their probability distributions relative to the chosen base space are exactly $f_t^{(k)}$ and $f_t^{\otimes k}$ according to (18) and (20).

So, let us denote by Σ_N^k is the set of all ordered subsets of $[1,\dots,N]$ of size k. Clearly, its cardinality is $\binom{N}{k}$. Then, for any $\sigma \in \Sigma_N^k$,

$$\mathcal{W}_2^2(f_t^{(k)}, f_t^{\otimes k}) \leqslant \int_{\mathbb{R}^{2nN}} \sum_{i=1}^k |(x_{\sigma(i)}, v_{\sigma(i)}) - (\bar{x}_{\sigma(i)}, \bar{v}_{\sigma(i)})|^2 \, f_0^{\otimes N}(X_0,V_0)\,\mathrm{d}X_0\,\mathrm{d}V_0.$$

Summing up over all $\sigma \in \Sigma_N^k$, we obtain

$$\binom{N}{k} \mathcal{W}_2^2(f_t^{(k)}, f_t^{\otimes k}) \leqslant \int_{\mathbb{R}^{2nN}} \sum_{\sigma\in\Sigma_N^k} \sum_{i=1}^k |(x_{\sigma(i)}, v_{\sigma(i)}) - (\bar{x}_{\sigma(i)}, \bar{v}_{\sigma(i)})|^2 \, f_0^{\otimes N}(X_0,V_0)\,\mathrm{d}X_0\,\mathrm{d}V_0.$$

Observe that in the double sum inside the integral each coordinate will be repeated $\binom{N-1}{k-1}$ times. So,

$$\binom{N}{k} \mathcal{W}_2^2(f_t^{(k)}, f_t^{\otimes k}) \leqslant \binom{N-1}{k-1} \int_{\mathbb{R}^{2nN}} \sum_{i=1}^N |(x_i, v_i) - (\bar{x}_i, \bar{v}_i)|^2 \, f_0^{\otimes N}(X_0,V_0)\,\mathrm{d}X_0\,\mathrm{d}V_0.$$

Simplifying and using (21), we obtain

$$\mathcal{W}_2^2(f_t^{(k)}, f_t^{\otimes k}) \leqslant Ck \min\left\{1, \frac{t^2}{N}\right\},$$

as desired. Let us note that an alternative argument, relating a distance between k-th marginals to a particular realization (21) appeared in [5], where the authors use the original joint-distribution definition of $\mathcal{W}_2$.

To establish (21) let us break the expression under the integral into potential and kinetic part,

$$\mathcal{P} = \frac{1}{2}\int_{\mathbb{R}^{2nN}} |X_t - \bar{X}_t|^2 \, f_0^{\otimes N}\,\mathrm{d}X_0\,\mathrm{d}V_0, \qquad \mathcal{K} = \frac{1}{2}\int_{\mathbb{R}^{2nN}} |V_t - \bar{V}_t|^2 \, f_0^{\otimes N}\,\mathrm{d}X_0\,\mathrm{d}V_0. \tag{22}$$

Here, X_t, V_t and $\bar{X}_t, \bar{V}_t$ denote the corresponding components of Φ_t^N and $\bar{\Phi}_t^{\otimes N}$, respectively. By the Hölder inequality, we have

$$\frac{\mathrm{d}}{\mathrm{d}t}\mathcal{P} \leqslant 2\mathcal{P}^{1/2}\mathcal{K}^{1/2}. \tag{23}$$

Let us now write out the equation for the kinetic part,

$$
\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}t}\mathcal{K} = \int_{\mathbb{R}^{2nN}} \sum_{i=1}^N (v_i - \bar{v}_i) \cdot \left(\frac{1}{N} \sum_{j=1}^N \phi(x_i - x_j)(v_j - v_i) - \int_{\mathbb{R}^{2n}} \phi(\bar{x}_i - y)(w - \bar{v}_i) f(y,w,t) \,\mathrm{d}y \,\mathrm{d}w \right) \\
\times f_0^{\otimes N} \,\mathrm{d}X_0 \,\mathrm{d}V_0 \\
= A + B + C,
\end{aligned}
$$

where

$$
\begin{aligned}
A &= \int_{\mathbb{R}^{2nN}} \sum_{i=1}^N (v_i - \bar{v}_i) \cdot \frac{1}{N} \sum_{j=1}^N [\phi(x_i - x_j) - \phi(\bar{x}_i - \bar{x}_j)](v_k - v_i) \, f_0^{\otimes N} \,\mathrm{d}X_0 \,\mathrm{d}V_0, \\
B &= \int_{\mathbb{R}^{2nN}} \sum_{i=1}^N (v_i - \bar{v}_i) \cdot \frac{1}{N} \sum_{j=1}^N \phi(\bar{x}_i - \bar{x}_j)[(v_j - \bar{v}_j) - (v_i - \bar{v}_i)] \, f_0^{\otimes N} \,\mathrm{d}X_0 \,\mathrm{d}V_0, \\
C &= \int_{\mathbb{R}^{2nN}} \sum_{i=1}^N (v_i - \bar{v}_i) \cdot \left(\frac{1}{N} \sum_{j=1}^N \phi(\bar{x}_i - \bar{x}_j)(\bar{v}_j - \bar{v}_i) - \int_{\mathbb{R}^{2n}} \phi(\bar{x}_i - y)(w - \bar{v}_i) f(y,w,t) \,\mathrm{d}y \,\mathrm{d}w \right) \\
&\qquad \times f_0^{\otimes N} \,\mathrm{d}X_0 \,\mathrm{d}V_0.
\end{aligned}
$$

Let us start with C. Apply the Hölder inequality first

$$
\begin{aligned}
C^2 &\leqslant \left(\int_{\mathbb{R}^{2nN}} \sum_{i=1}^N |v_i - \bar{v}_i|^2 f_0^{\otimes N} \,\mathrm{d}X_0 \,\mathrm{d}V_0 \right) \\
&\quad \times \left(\int_{\mathbb{R}^{2nN}} \sum_{i=1}^N \left| \frac{1}{N} \sum_{j=1}^N \phi(\bar{x}_i - \bar{x}_j)(\bar{v}_j - \bar{v}_i) - \int_{\mathbb{R}^{2n}} \phi(\bar{x}_i - y)(w - \bar{v}_i) f(y,w,t) \,\mathrm{d}y \,\mathrm{d}w \right|^2 f_0^{\otimes N} \,\mathrm{d}X_0 \,\mathrm{d}V_0 \right) \\
&= 2\mathcal{K} \int_{\mathbb{R}^{2nN}} \sum_{i=1}^N \left| \frac{1}{N} \sum_{j=1}^N \phi(\bar{x}_i - \bar{x}_j)(\bar{v}_j - \bar{v}_i) - \int_{\mathbb{R}^{2n}} \phi(\bar{x}_i - y)(w - \bar{v}_i) f(y,w,t) \,\mathrm{d}y \,\mathrm{d}w \right|^2 f_0^{\otimes N} \,\mathrm{d}X_0 \,\mathrm{d}V_0.
\end{aligned}
$$

Switching back to the Eulerian coordinates, whereby $\bar{x}_i, \bar{v}_i$ become dummy variables, we get

$$
C^2 \leqslant 2\mathcal{K} \int_{\mathbb{R}^{2nN}} \sum_{i=1}^N \left| \frac{1}{N} \sum_{j=1}^N \phi(\bar{x}_i - \bar{x}_j)(\bar{v}_j - \bar{v}_i) - \int_{\mathbb{R}^{2n}} \phi(\bar{x}_i - y)(w - \bar{v}_i) f(y,w,t) \,\mathrm{d}y \,\mathrm{d}w \right|^2 f_t^{\otimes N} \,\mathrm{d}\bar{X} \,\mathrm{d}\bar{V}.
$$

All these terms, due to symmetry are independent of i. According to [12, Lemma 3.3], and our flocking estimate (12), each can be estimated by

$$
\frac{4}{N} \sup_{(\bar{x}', \bar{v}'), (\bar{x}'', \bar{v}'') \in \operatorname{supp} f_t} |\phi(\bar{x}' - \bar{x}'')(\bar{v}' - \bar{v}'')|^2 \leqslant \frac{c}{N} e^{-\delta t}.
$$

Thus,

$$
C \leqslant c e^{-\delta t} \mathcal{K}^{1/2}.
$$

Turning back to A, we use the smoothness of the kernel and exponential flocking estimates (11),

$$
\begin{aligned}
|A| &\leqslant ce^{-\delta t}\sqrt{\mathcal{K}}\left(\int_{\mathbb{R}^{2nN}}\sum_{i=1}^N\left[\frac{1}{N}\sum_{j=1}^N(|x_i-\bar{x}_i|+|x_j-\bar{x}_j|)\right]^2 f_0^{\otimes N}\,\mathrm{d}X_0\,\mathrm{d}V_0\right)^{1/2}\\
&\leqslant ce^{-\delta t}\sqrt{\mathcal{K}}\left(\int_{\mathbb{R}^{2nN}}\sum_{i=1}^N\left[|x_i-\bar{x}_i|^2+\frac{1}{N}\sum_{j=1}^N|x_j-\bar{x}_j|^2\right] f_0^{\otimes N}\,\mathrm{d}X_0\,\mathrm{d}V_0\right)^{1/2}\\
&\leqslant ce^{-\delta t}\sqrt{\mathcal{K}}\left(2\int_{\mathbb{R}^{2nN}}\left[\sum_{i=1}^N|x_i-\bar{x}_i|^2\right] f_0^{\otimes N}\,\mathrm{d}X_0\,\mathrm{d}V_0\right)^{1/2}\\
&= ce^{-\delta t}\sqrt{\mathcal{K}}\sqrt{\mathcal{P}}.
\end{aligned}
$$

Finally, one can see that B contributes a negative term,

$$
\sum_{i=1}^N(v_i-\bar{v}_i)\cdot\frac{1}{N}\sum_{j=1}^N\phi(\bar{x}_i-\bar{x}_j)[(v_j-\bar{v}_j)-(v_i-\bar{v}_i)] = \frac{1}{N}\sum_{i,j=1}^N\phi(\bar{x}_i-\bar{x}_j)((v_i-\bar{v}_i)\cdot(v_j-\bar{v}_j)-|v_i-\bar{v}_i|^2)
$$

and symmetrizing,

$$
\begin{aligned}
&= \frac{1}{2}\frac{1}{N}\sum_{i,j=1}^N\phi(\bar{x}_i-\bar{x}_j)(-|v_j-\bar{v}_j|^2+2(v_i-\bar{v}_i)\cdot(v_j-\bar{v}_j)-|v_i-\bar{v}_i|^2)\\
&= -\frac{1}{2}\frac{1}{N}\sum_{i,j=1}^N\phi(\bar{x}_i-\bar{x}_j)|(v_j-\bar{v}_j)-(v_i-\bar{v}_i)|^2\leqslant 0.
\end{aligned}
$$

Collecting all of the above we obtain

$$
\frac{\mathrm{d}}{\mathrm{d}t}\mathcal{K}\leqslant ce^{-\delta t}(\mathcal{K}^{1/2}+\mathcal{K}^{1/2}\mathcal{P}^{1/2}). \tag{24}
$$

Denoting $p=1+\mathcal{P}^{1/2}$, $k=\mathcal{K}^{1/2}$ we obtain the system

$$
\dot{p}\leqslant k,\ p_0=1;\qquad \dot{k}\leqslant ce^{-\delta t}p,\ k_0=0. \tag{25}
$$

Claim 2.1. Any non-negative solution to (25) obeys an estimate $p\leqslant 1+Ct$, $k\leqslant C\min\{1,t\}$, where $C=C(c,\delta)$.

To see that let us fix an $\varepsilon>0$ to be determined later and compute

$$
\frac{\mathrm{d}}{\mathrm{d}t}(\varepsilon p^2+k^2)\leqslant 2pk(\varepsilon+ce^{-\delta t})\leqslant\sqrt{\varepsilon}(\varepsilon p^2+k^2)+\frac{c}{\sqrt{\varepsilon}}e^{-\delta t}(\varepsilon p^2+k^2).
$$

Thus,

$$
\varepsilon p^2+k^2\leqslant\varepsilon\exp\left\{\sqrt{\varepsilon}t+\frac{1}{\sqrt{\varepsilon}\delta}\right\}.
$$

Setting $\varepsilon=\delta^2$, we can see that the growth rate of p does not exceed $\delta/2$, $p\lesssim e^{\delta t/2}$. Plugging this into k-equation we obtain $\dot{k}\lesssim e^{-\delta t/2}$. This proves the bound on k, and then solving for p, $p\leqslant 1+Ct$.

Going back to the energies, we obtain

$$
\mathcal{K}\leqslant C\min\{1,t^2\},\qquad \mathcal{P}\leqslant Ct^2.
$$

Due to the global bound on the support of the flock (11), (12), we also have $\mathcal{P}\leqslant CN$. Thus,

$$
\mathcal{P}\leqslant C\min\{N,t^2\}.
$$

Consequently, we obtain the required

$$\mathcal{K} + \mathcal{P} \leqslant C \min\{N, t^2\}.$$

3. Propagation of chaos for forced system

Using the basic energy estimates obtained in the previous section, we will now extend the result to the system with friction forces (14) and $\kappa > 0$. It is well-known that the flocking behavior of solutions to (14), even with constant $\theta_i = 1$ does not always hold even for global kernels $\phi \geqslant c_0 > 0$. The example exhibited in [6] shows misalignment dynamics when the initial configuration is symmetric $x_1 = -x_2$ and velocities are aimed in the opposite directions $v_1 = -v_2$. The work [10] proved that this is, in a sense, the only situation when no flocking occurs. As long as the initial condition is *sectorial*, meaning that all $v_i(0) \in \Sigma$, where Σ is an open conical sector of opening less than π, then the solutions align exponentially fast.

Proposition 3.1 ([10]). *Suppose that*

$$\phi(r) \geqslant \frac{\lambda}{(1+r^2)^{\beta/2}}, \quad \lambda > 0, \quad \beta \leqslant 1. \tag{26}$$

For any sectorial solution to (14) *there exists* $v_\infty \in \mathbb{R}^n$ *and* $\theta_\infty > 0$ *with* $|v_\infty|^p = \theta_\infty$, *such that one has*

$$\max_{i=1,\dots,N} (|v_i - v_\infty| + |\theta_i - \theta_\infty|) \leqslant C e^{-\delta t}, \tag{27}$$

$$\sup_{t>0} \max_{i,j=1,\dots,N} |x_i - x_j| = D < \infty. \tag{28}$$

It is within the context of sectorial solutions that we will cast the propagation of chaos result. But first we establish a similar flocking estimates for solutions of the corresponding kinetic model.

3.1. **Grassmannian reduction for Vlasov-alignment equation.** Let us denote $\Omega = \mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}_+$. The Vlasov equation corresponding to (14) is given by

$$\partial_t f + v \cdot \nabla_x f + \nabla_v \cdot (f F(f)) + \nabla_v \cdot (f R) + \nabla_\theta \cdot (f \Theta(f)) = 0, \quad (x, v, \theta) \in \Omega, \quad t > 0, \tag{29}$$

subject to the initial condition

$$f(x, v, \theta, 0) = f_0(x, v, \theta), \tag{30}$$

where

$$F(f)(x, v, \theta) = \int_\Omega \phi(x-y)(w-v) f(y, w, \eta, t) \,\mathrm{d}z \,\mathrm{d}w \,\mathrm{d}\eta,$$
$$R(x, v, \theta) = \sigma v(\theta - |v|^p), \quad \sigma > 0, \quad p > 0,$$
$$\Theta(f)(x, v, \theta) = \kappa \int_\Omega \phi(x-y)(\eta - \theta) f(y, w, \eta, t) \,\mathrm{d}z \,\mathrm{d}w \,\mathrm{d}\eta.$$

In this section, we will prove a similar flocking result for the sectorial solutions of (29). Let us define what they are in the kinetic context.

Definition 3.2. A solution f to (29) is called *sectorial* if there exists a conical region Σ lying on one side of a hyperplane, i.e. with conical opening less than π such that $v \in \Sigma$ for any v in the velocity support of f, $(x, v, \theta) \in \operatorname{supp} f$ for some x, θ.

Since the equation (29) is rotationally invariant, it will be convenient to assume that our solution belong the upper-half space: there exists $\varepsilon > 0$ such that

$$v_n \geqslant \varepsilon |v|, \quad \forall (x, v, \theta) \in \operatorname{supp} f, \tag{31}$$

By the weak maximum principle discussed below in Remark 3.6, it follows that if f is sectorial initially, then it will remain so for all time and the velocity support will lie in the same sector Σ.

Let us state our main result now.

Proposition 3.3. *Suppose the kernel satisfies* (26)*. For any sectorial solution f to* (29) *with initial compact support one has*

$$\sup_{t>0} \operatorname{diam} \operatorname{supp} f_t < \infty, \tag{32}$$

and there exist $v_\infty \in \mathbb{R}^n$, $\theta_\infty \in \mathbb{R}_+$, with $|v_\infty|^p = \theta_\infty$ such that

$$\max_{(x,v,\theta)\in \operatorname{supp} f_t} (|\theta - \theta_\infty| + |v - v_\infty|) \leqslant c e^{-\delta t}. \tag{33}$$

As in the discrete case the proof is based on examination of kinetic characteristics of the equation given by

$$\begin{cases} \dot{x} = v, \qquad x(0) = x_0, \\ \dot{v} = \displaystyle\int_\Omega \phi(x-y)(w-v) f(y,w,\eta,t)\, \mathrm{d}y\, \mathrm{d}w\, \mathrm{d}\eta + \sigma v(\theta - |v|^p), \quad v(0) = v_0, \\ \dot{\theta} = \kappa \displaystyle\int_\Omega \phi(x-y)(\eta - \theta) f(y,w,\eta,t)\, \mathrm{d}y\, \mathrm{d}w\, \mathrm{d}\eta, \quad \theta(0) = \theta_0. \end{cases} \tag{34}$$

Let us denote

$$\begin{aligned} \mathcal{D}(t) &= \max_{(x,v,\theta),(x',v',\theta')\in \operatorname{supp} f_t} |x - x'|, \\ \mathcal{A}(t) &= \max_{(x,v,\theta),(x',v',\theta')\in \operatorname{supp} f_t} |v - v'|, \\ \mathcal{Q}(t) &= \max_{(x,v,\theta),(x',v',\theta')\in \operatorname{supp} f_t} |\theta - \theta'|, \end{aligned}$$

$$\begin{aligned} M &= \int_\Omega f(x,v,\theta,t)\, \mathrm{d}x\, \mathrm{d}v\, \mathrm{d}\theta, \quad \theta_\infty = \frac{1}{M}\int_\Omega \theta f(x,v,\theta,t)\, \mathrm{d}x\, \mathrm{d}v\, \mathrm{d}\theta, \\ \theta_+(t) &= \max_{(x,v,\theta)\in \operatorname{supp} f_t} \theta, \qquad \theta_-(t) = \min_{(x,v,\theta)\in \operatorname{supp} f_t} \theta. \end{aligned}$$

Then we have

$$\frac{\mathrm{d}}{\mathrm{d}t}\mathcal{D} \leqslant \mathcal{A}. \tag{35}$$

Indeed, at time t, let $\ell \in (\mathbb{R}^d)^*, |\ell| = 1, (x,v,\theta), (x',v',\theta') \in \operatorname{supp} f_t$ such that $\mathcal{D}(t) = \ell(x - x')$. By Rademacher's lemma and the first equation in the system (34) we have

$$\frac{\mathrm{d}}{\mathrm{d}t}\mathcal{D} = \ell(\dot{x} - \dot{x}') = \ell(v - v') \leqslant \mathcal{A}.$$

For $\mathcal{Q}$, we have

$$\frac{\mathrm{d}}{\mathrm{d}t}\mathcal{Q} \leqslant -\kappa\phi(\mathcal{D})\mathcal{Q}. \tag{36}$$

To prove that, at time t we choose $\ell \in \mathbb{R}^*, |\ell| = 1, (x,v,\theta), (x',v',\theta') \in \operatorname{supp} f_t$ which satisfy $\mathcal{Q}(t) = \ell(\theta - \theta')$. By Rademacher's lemma and the third equation in the system (34) we get

$$\begin{aligned} \frac{\mathrm{d}}{\mathrm{d}t}\mathcal{Q} &= \kappa \int_\Omega \phi(x-y)\ell(\eta - \theta) f(y,w,\eta,t)\, \mathrm{d}y\, \mathrm{d}w\, \mathrm{d}\eta - \kappa \int_\Omega \phi(x'-y)\ell(\eta - \theta') f(y,w,\eta,t)\, \mathrm{d}y\, \mathrm{d}w\, \mathrm{d}\eta \\ &= \kappa \int_\Omega \phi(x-y)[\ell(\eta - \theta') - \ell(\theta - \theta')] f(y,w,\eta,t)\, \mathrm{d}y\, \mathrm{d}w\, \mathrm{d}\eta \\ &\quad + \kappa \int_\Omega \phi(x'-y)[\ell(\theta - \eta) - \ell(\theta - \theta')] f(y,w,\eta,t)\, \mathrm{d}y\, \mathrm{d}w\, \mathrm{d}\eta. \end{aligned}$$

Since $\ell(\eta-\theta')-\ell(\theta-\theta')\leqslant 0$ and $\ell(\theta-\eta)-\ell(\theta-\theta')\leqslant 0$, the right hand side of the above equality is nonpositive. Note that $\phi(x-y)\geqslant\phi(\mathcal{D})$ for all $x,y\in\operatorname{supp} f_t$. Therefore,

$$\frac{\mathrm{d}}{\mathrm{d}t}\mathcal{Q}\leqslant-\kappa\phi(\mathcal{D})\int_\Omega \ell(\theta-\theta')f(y,w,\eta,t)\,\mathrm{d}y\,\mathrm{d}w\,\mathrm{d}\eta\leqslant-\kappa\phi(\mathcal{D})\mathcal{Q}.$$

Similarly, using the third equation in (34) and Rademacker's lemma, it is not hard to see that θ_+ is decreasing and θ_- is increasing. Thus,

$$\theta_+(t)\leqslant\theta^*,\quad \theta_-(t)\geqslant\theta_*\quad\forall t\geqslant 0, \tag{37}$$

where $\theta^*=\theta_+(0)$ and $\theta_*=\theta_-(0)$.

Before we proceed further let us discuss the boundedness of the velocity support of f and the weak maximum principle.

Lemma 3.4 (boundedness)**.** *There exists a constant C which depends on the initial data such that for any $(x,v,\theta)\in\operatorname{supp} f_t$, one has*

$$|v(t)|\leqslant C,\qquad\forall t>0. \tag{38}$$

Proof. Let

$$|v_+|(t)=\max_{(x,v,\theta)\in\operatorname{supp} f_t}|v|.$$

At time t, let $\ell\in(\mathbb{R}^d)^*, |\ell|=1$, $(x,\,v,\theta)\in\operatorname{supp} f_t$ such that $|v_+|=\ell(v)$. Then, by Rademacher's Lemma,

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}|v_+| &= \int_\Omega \phi(x-z)\,\ell(w-v)f(z,w,\eta,t)\ \mathrm{d}z\,\mathrm{d}w\,\mathrm{d}\eta+\sigma\ell(v)(\theta-|v|^p)\\ &\leqslant \sigma|v_+|(\theta^*-|v_+|^p).\end{aligned}$$

Hence, if $\theta^*\leqslant|v_+|^p$ then

$$|v_+|(t)\leqslant|v_+|(0)\quad\forall t>0.$$

Otherwise, we have

$$\frac{\mathrm{d}}{\mathrm{d}t}|v_+|^p\leqslant\sigma p|v_+|^p(\theta^*-|v_+|^p).$$

Solving the above ODI gives

$$|v_+|(t)\leqslant\frac{\sqrt[p]{\theta^*}e^{\sigma\theta^* t}}{(c+e^{\sigma p\theta^* t})^{1/p}}=\sqrt[p]{\theta^*}+\mathcal{O}(e^{-\sigma\theta^* t}), \tag{39}$$

where c is a positive constant depending on initial data. Thus, $|v_+|(t)$ is bounded for all $t>0$. □

Lemma 3.5 (weak maximum principle)**.** *If for a given functional $\ell\in(\mathbb{R}^n)^*$, all velocity vectors v_0 that lie in the support of the initial flock, $(x_0,v_0,\theta_0)\in\operatorname{supp} f_0$, satisfy*

$$\ell(v_0)\geqslant 0,$$

then at any positive time

$$\ell(v)\geqslant 0,\qquad\forall t>0,\ (x,v,\theta)\in\operatorname{supp} f_t.$$

Proof. At time t, let

$$\ell(v)=\min_{(z,w,\eta)\in\operatorname{supp} f}\ell(w).$$

By Rademacher's Lemma,

$$\frac{\mathrm{d}}{\mathrm{d}t}\ell(v)=\int_\Omega\phi(x-z)\,\ell(w-v)f(z,w,\eta,t)\ \mathrm{d}z\,\mathrm{d}w\,\mathrm{d}\eta+\sigma\ell(v)(\theta-|v|^p)\geqslant\sigma\ell(v)(\theta_*-|v|^p).$$

Then by Lemma 3.4 we get

$$\frac{\mathrm{d}}{\mathrm{d}t}\ell(v)\geqslant c\,\ell(v),$$

where c is constant. Solving this ODI we obtain the desired conclusion,

$$\ell(v) \geqslant \ell(v_0)e^{ct} \geqslant 0, \quad \forall t > 0.$$

□

Remark 3.6. By the weak maximum principle we note that if the support of f_0 in v lies in the convex sector defined by

$$\Sigma_{\mathcal{F}} = \bigcap_{\ell \in \mathcal{F}} \{v \in \mathbb{R}^n : \ell(v) \geqslant 0\},$$

where $\mathcal{F}$ is an arbitrary set of linear functionals on $\mathbb{R}^n$, then the velocity support of f_t will be confined to that sector for all time. Since the system (34) is invariant under rotations, without loss of generality we can assume that the support of f_0 in v lies above the hyperplane $\Pi_n = \{v_n = 0\}$, where v_n is the n-th coordinate of vector v.

Lemma 3.7. *For any sectorial solution f to (29) there exists a positive constant c_0 depending on the initial data such that*

$$|v| \geqslant c_0, \quad \forall (x, v, \theta) \in \operatorname{supp} f_t. \tag{40}$$

Proof. At time t, let (x, v, θ) be a minimizer for $\min\limits_{(x,v,\theta)\in \operatorname{supp} f_t} v_n$. Then

$$\frac{\mathrm{d}}{\mathrm{d}t} v_n = \int_\Omega \phi(x-z)(w_n - v_n) f(z, w, \eta, t)\, \mathrm{d}z\, \mathrm{d}w\, \mathrm{d}\eta + \sigma v_n(\theta - |v|^p) \geqslant \sigma v_n(\theta_* - \varepsilon^{-p} v_n^p). \tag{41}$$

If $\theta_* \leqslant \varepsilon^{-p} v_n^p$ then

$$|v| \geqslant \varepsilon \sqrt[p]{\theta_*}.$$

Otherwise, solving (41) we get

$$v_n \geqslant \frac{\varepsilon \sqrt[p]{\theta_*} e^{\sigma \theta_* t}}{(c + e^{p\sigma\theta_* t})^{1/p}},$$

where c is a positive constant which depends on the initial data. Then the lemma follows. □

Remark 3.8. Lemma 3.7 tells us that for a sectorial solution f, $\operatorname{supp} f(x, \cdot, \theta)$ stays away from the origin. Then, by Lemma 3.4, it implies that $\operatorname{supp} f(x, \cdot, \theta)$ is contained in a sector. Lemma 3.7 also implies that for any sectorial solution f one has

$$|v_-|(t) \geqslant c_0, \quad \forall t > 0, \tag{42}$$

where $|v_-|(t) = \min\limits_{(x,v,\theta)\in \operatorname{supp} f} |v(t)|$.

Proof of Proposition 3.3. From now on we consider a sectorial solution f to the system (29). Denoting $\tilde{r} = \frac{r}{|r|}$ for any vector $r \in \mathbb{R}^n$. One has

$$\frac{\mathrm{d}}{\mathrm{d}t}\tilde{v} = \frac{1}{|v|}\left(\mathrm{Id} - \frac{v}{|v|} \otimes \frac{v}{|v|}\right)\dot{v} = \int_\Omega \frac{|w|}{|v|}\phi(x-z)(\mathrm{Id} - \tilde{v} \otimes \tilde{v})\tilde{w}\, f(z, w, \eta, t)\, \mathrm{d}z\, \mathrm{d}w\, \mathrm{d}\eta. \tag{43}$$

Here, we used $(\mathrm{Id} - \tilde{v} \otimes \tilde{v})v = 0$.

Denoting by $\widehat{(v, u)}$ the angle between two vectors v and u, then $\cos \widehat{(v, u)} = \tilde{v} \cdot \tilde{u}$. Thus, if $(x, v, \theta), (y, u, \zeta)$ are the solutions to (34) with respect to the initial conditions $(x_0, v_0, \theta_0), (y_0, u_0, \zeta_0)$, respectively, then

$$\begin{aligned} \frac{\mathrm{d}}{\mathrm{d}t} \cos \widehat{(v, u)} &= \int_\Omega \frac{|w|}{|v|}\phi(x-z)[\cos \widehat{(u, w)} - \cos \widehat{(v, u)} \cos \widehat{(v, w)}] f(z, w, \eta, t)\, \mathrm{d}z\, \mathrm{d}w\, \mathrm{d}\eta \\ &+ \int_\Omega \frac{|w|}{|u|}\phi(y-z)[\cos \widehat{(v, w)} - \cos \widehat{(v, u)} \cos \widehat{(u, w)}] f(z, w, \eta, t)\, \mathrm{d}z\, \mathrm{d}w\, \mathrm{d}\eta. \end{aligned} \tag{44}$$

Note that if v, u, and w are three vectors lying in the same two dimenstional plane and

$$\widehat{(v,u)} = \widehat{(v,w)} + \widehat{(w,u)} < \pi - \delta \quad \text{for some } \delta > 0, \tag{45}$$

then the followings hold:

$$\begin{aligned}
\cos\widehat{(u,w)} - \cos\widehat{(v,u)}\cos\widehat{(v,w)} &= \cos\left(\widehat{(v,u)} - \widehat{(v,w)}\right) - \cos\widehat{(v,u)}\cos\widehat{(v,w)} \\
&= \sin\widehat{(v,u)}\sin\widehat{(v,w)} \geqslant 0, \\
\cos\widehat{(v,w)} - \cos\widehat{(v,u)}\cos\widehat{(u,w)} &\geqslant 0,
\end{aligned}$$

$$\cos\widehat{(u,w)} + \cos\widehat{(v,w)} = \cos\frac{\widehat{(v,u)}}{2}\cos\frac{\widehat{(u,w)} - \widehat{(v,w)}}{2} \geqslant \left(\cos\frac{\pi-\delta}{2}\right)^2.$$

Therefore, if the support of f in v is on a two dimensional plane and (45) is satisfied, then by Lemma 3.4 , Lemma 3.7 and (44), one has

$$\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}t}\cos\widehat{(v,u)} &\geqslant c\phi(\mathcal{D})\int_\Omega \left(\cos\widehat{(u,w)} + \cos\widehat{(v,w)}\right)\left(1 - \cos\widehat{(v,u)}\right) f(z,w,\eta,t)\,\mathrm{d}z\,\mathrm{d}w\,\mathrm{d}\eta \\
&\geqslant c\phi(\mathcal{D})\left(1 - \cos\widehat{(v,u)}\right).
\end{aligned}$$

Equivalently,

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(1 - \cos\widehat{(v,u)}\right) \leqslant -c\phi(\mathcal{D})\left(1 - \cos\widehat{(v,u)}\right). \tag{46}$$

Now let $\varPi$ be a fixed two dimensional plane which contains the v_n-axis. Denoting by $v^\varPi$ the projection of any $v \in \operatorname{supp} f$ onto $\varPi$. Projecting the second equation in (34) onto $\varPi$ we have the following equation:

$$\dot{v}^\varPi = \int_\Omega \phi(x-z)(w^\varPi - v^\varPi) f(z,w,\eta,t)\,\mathrm{d}z\,\mathrm{d}w\,\mathrm{d}\eta + \sigma v^\varPi(\theta - |v|^p) \tag{47}$$

Therefore, we can write the equation for $\cos\widehat{(v^\varPi, u^\varPi)}$ as follows:

(48)

$$\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}t}\cos\widehat{(v^\varPi, u^\varPi)} &= \int_\Omega \frac{|w^\varPi|}{|v^\varPi|}\phi(x-z)[\cos\widehat{(u^\varPi, w^\varPi)} - \cos\widehat{(v^\varPi, u^\varPi)}\cos\widehat{(v^\varPi, w^\varPi)}] f(z,w,\eta,t)\,\mathrm{d}z\,\mathrm{d}w\,\mathrm{d}\eta \\
&+ \int_\Omega \frac{|w^\varPi|}{|u^\varPi|}\phi(y-z)[\cos\widehat{(v^\varPi, w^\varPi)} - \cos\widehat{(v^\varPi, u^\varPi)}\cos\widehat{(u^\varPi, w^\varPi)}] f(z,w,\eta,t)\,\mathrm{d}z\,\mathrm{d}w\,\mathrm{d}\eta.
\end{aligned}$$

Let us denote $\mathcal{G}(1, n-1)$ the space of all two dimensional subspaces of $\mathbb{R}^n$ which contain v_n-axis. Since $\mathcal{G}(1, n-1)$ can be identified with 1-Grassmannian manifold of $\mathbb{R}^{n-1}$ which is compact, we can define

$$\gamma^{2D} = \max_{\substack{\varPi \in \mathcal{G}(1,n-1) \\ (x,v,\theta),(y,u,\zeta) \in \operatorname{supp} f}} \widehat{(v^\varPi, u^\varPi)}. \tag{49}$$

We note that

$$\gamma^{2D} \leqslant \pi - \delta \quad \text{for some } \delta > 0.$$

Since the n-th coordinate of any $v \in \operatorname{supp} f$ does not change when it is projected onto $\varPi$, $|v^\varPi|$ is still bounded above and below by positive constants. Therefore, choosing a maximizing triple $\varPi, u, v$ for $\widehat{(v^\varPi, u^\varPi)}$, from (48) we deduce that

$$\frac{\mathrm{d}}{\mathrm{d}t}(1 - \cos\gamma^{2D}) \leqslant -c\phi(\mathcal{D})((1 - \cos\gamma^{2D}). \tag{50}$$

Denoting

$$\gamma = \max_{(x,v,\theta),(y,u,\zeta)\in \operatorname{supp} f} \widehat{(u,v)}.$$

Claim 3.9. We have $\gamma \leqslant \gamma^{2D}$.

Proof of Claim 3.9. For any $(x,v,\theta),(y,u,\zeta) \in \operatorname{supp} f$, consider the two dimensional subspace $\mathit{\Pi} = \operatorname{span}\{e_n, \tilde{u}-\tilde{v}\}$ where $e_n = (0,\dots,0,1)$. We have $\mathit{\Pi} \in \mathcal{G}(1,n-1)$ and $\tilde{u}-\tilde{v} = \tilde{u}^{\mathit{\Pi}} - \tilde{v}^{\mathit{\Pi}}$. By the law of cosines, we get

$$\begin{aligned} 2(1-\cos\widehat{(u,v)}) = |\tilde{u}-\tilde{v}|^2 = |\tilde{u}^{\mathit{\Pi}} - \tilde{v}^{\mathit{\Pi}}|^2 &= 2|\tilde{u}^{\mathit{\Pi}}|^2(1-\cos\widehat{(u^{\mathit{\Pi}},v^{\mathit{\Pi}})}) \\ &\leqslant 2(1-\cos\widehat{(u^{\mathit{\Pi}},v^{\mathit{\Pi}})}). \end{aligned}$$

It implies that for any $(x,v,\theta),(y,u,\zeta) \in \operatorname{supp} f$ there exists $\mathit{\Pi} \in \mathcal{G}(1,n-1)$ such that $\widehat{(u,v)} \leqslant \widehat{(u^{\mathit{\Pi}},v^{\mathit{\Pi}})}$. Therefore, the claim is followed. □

Remark 3.10. Claim 3.9 and the inequality (50) imply that if $\mathcal{D}(t) \leqslant D < \infty$ then

$$1-\cos\gamma \leqslant 1-\cos\gamma^{2D} \lesssim e^{-c\phi(D)t}.$$

Now we set

$$\mathcal{R} = \max_{(x,v,\theta),(y,u,\zeta)\in\operatorname{supp} f} \frac{|v|^2}{|u|^2}.$$

Suppose that $(x,v,\theta),(y,u,\zeta)$ maximize $\mathcal{R}$ at time t, we have

$$\begin{aligned} \frac{\mathrm{d}}{\mathrm{d}t}\mathcal{R} &= \frac{2}{|u|^2}\left[\int_\Omega \phi(x-z)(v\cdot w - |v|^2) f(z,w,\eta,t)\,\mathrm{d}z\,\mathrm{d}w\,\mathrm{d}\eta + \sigma|v|^2(\theta-|v|^p)\right] \\ &\quad - \frac{2|v|^2}{|u|^4}\left[\int_\Omega \phi(y-z)(u\cdot w - |u|^2) f(z,w,\eta,t)\,\mathrm{d}z\,\mathrm{d}w\,\mathrm{d}\eta + \sigma|u|^2(\zeta-|u|^p)\right] \\ &= \frac{2}{|u|^2}\int_\Omega \phi(x-z)(v\cdot w - |v|^2) f(z,w,\eta,t)\,\mathrm{d}z\,\mathrm{d}w\,\mathrm{d}\eta \\ &\quad + \frac{2|v|^2}{|u|^4}\int_{\mathbb{R}^{2d}} \phi(y-z)(|u|^2 - u\cdot w) f(z,w,\eta,t)\,\mathrm{d}z\,\mathrm{d}w\,\mathrm{d}\eta + 2\sigma\mathcal{R}(\theta-\zeta+|u|^p-|v|^p). \end{aligned} \tag{51}$$

Since u,v maximize $\mathcal{R}$, we have $v\cdot w - |v|^2 \leqslant |v|(|w|-|v|) \leqslant 0$ for all $w \in \operatorname{supp} f$. Hence, the first term on the right hand side of (51) is nonpositve. For the second term, we have

$$|u|^2 - u\cdot w = |u|^2 - |u||w|\cos\widehat{(u,w)} \lesssim 1-\cos\gamma.$$

Note that $\mathcal{R}$ is bounded from above and below, hence,

$$2\sigma\mathcal{R}(\theta-\zeta+|u|^p-|v|^p) = 2\sigma\mathcal{R}(\theta-\zeta) + \frac{2\sigma\mathcal{R}}{|u|^p}(1-\mathcal{R}^{p/2}) \lesssim \mathcal{Q} + (1-\mathcal{R}).$$

Therefore, there exist positive constants c_1, c_2, c_3 such that

$$\frac{\mathrm{d}}{\mathrm{d}t}(\mathcal{R}-1) \leqslant -c_1(\mathcal{R}-1) + c_2(1-\cos\gamma) + c_3\mathcal{Q}. \tag{52}$$

Firstly, we see that the flock diameter grows at most linearly in time,

$$\mathcal{D}(t) \lesssim t \tag{53}$$

since

$$\frac{\mathrm{d}}{\mathrm{d}t}\mathcal{D}(t) \leqslant \mathcal{A}(t) \tag{54}$$

and $|v|$ is bounded for all $(x,v,\theta) \in \operatorname{supp} f$. It is not hard to see the relation

$$\mathcal{A}^2 \lesssim (\mathcal{R}-1) + (1-\cos\gamma). \tag{55}$$

Thus, to prove an exponential alignment it suffices to show that both $(\mathcal{R}-1)$ and $(1-\cos\gamma)$ decay exponentially fast.

We now consider two cases for β:

Case I: $\beta < 1$. Our assumption on the kernel and (53) imply that

$$\phi(\mathcal{D}) \gtrsim \frac{1}{(1+t^2)^{\beta/2}}. \tag{56}$$

Plugging it into (50) and applying the Grönwall's Lemma we get

$$1-\cos\gamma \leqslant 1-\cos\gamma^{2D} \lesssim e^{-c\langle t\rangle^{1-\beta}}. \tag{57}$$

Plugging (56) into (36) and solving for $\mathcal{Q}$ we also have

$$\mathcal{Q} \lesssim e^{-c\langle t\rangle^{1-\beta}}. \tag{58}$$

Combining these inequalities with (52) and solving for $\mathcal{R}-1$ we obtain

$$\mathcal{R}-1 \lesssim e^{-c\langle t\rangle^{1-\beta}}. \tag{59}$$

From (54), (55), (57) and (59), we have

$$\frac{\mathrm{d}}{\mathrm{d}t}\mathcal{D} \lesssim e^{-c\langle t\rangle^{(1-\beta)/2}}.$$

Solving this ODI gives

$$\mathcal{D}(t) \leqslant D < \infty. \tag{60}$$

Thus, (36) implies that

$$\mathcal{Q}(t) \leqslant \mathcal{Q}(0)e^{-t\phi(D)}.$$

Hence, $\theta(t)$ aligns to θ_∞ exponentially fast for all $(x,v,\theta) \in \operatorname{supp} f$. Due to finite flock diameter (60) and Remark 3.10, we have

$$1-\cos\gamma \lesssim e^{-c\phi(D)t}.$$

Putting the estimates for $\mathcal{Q}$ and $(1-\cos\gamma)$ into (52) and solving for $\mathcal{R}-1$ where we use the Grönwall's Lemma, we obtain the exponential decay for $\mathcal{R}-1$ as well. Therefore, we arrive at an alignment with an exponential rate.

Denoting by E any quantity which decays exponentially fast. So far we have $|\theta-\theta_\infty| = E(t)$, $|v-u| = E(t)$ for any $\theta, v, u \in \operatorname{supp} f$. By (42) and Lemma 3.4, $|v_\pm|(t)$ are bounded, hence, the following equations hold for $|v_\pm|^p(t)-\theta_\infty$:

$$\frac{\mathrm{d}}{\mathrm{d}t}(|v_\pm|^p-\theta_\infty) = (\sigma p|v_\pm|^p(\theta_\infty-|v_\pm|^p)+E) \quad \sim \quad (-(|v_\pm|^p-\theta_\infty)+E).$$

It follows that $|v_\pm|^p(t)$ converges to θ_∞ exponentially fast. Therefore, from the characteristic equation for $v \in \operatorname{supp} f$ in (34) we deduce that

$$\frac{\mathrm{d}}{\mathrm{d}t}v = E, \quad \forall v_0 \in \operatorname{supp} f_0.$$

The existence of v_∞ is followed then.

Case II: $\beta = 1$. In this case, we have $\phi(\mathcal{D}) \gtrsim \dfrac{1}{\sqrt{1+t^2}}$, hence,

$$1-\cos\gamma \leqslant 1-\cos\gamma^{2D} \lesssim \langle t\rangle^{-\alpha}, \quad \text{and}$$

$$\mathcal{Q} \lesssim \langle t\rangle^{-\alpha}, \text{ for some } \alpha > 0.$$

Therefore,

$$\frac{\mathrm{d}}{\mathrm{d}t}(\mathcal{R}-1) \lesssim -(\mathcal{R}-1)+\langle t\rangle^{-\alpha}.$$

Solving this ODI we yield

$$\mathcal{R} - 1 \lesssim \langle t \rangle^{-\alpha}.$$

Here we used the fact that $e^{-ct} * \langle t \rangle^{-\alpha} \sim \langle t \rangle^{-\alpha}$. It implies that

$$\mathcal{A} \lesssim \langle t \rangle^{-\alpha/2},$$

and hence,

$$\mathcal{D} \lesssim \langle t \rangle^{1-\alpha/2}.$$

Thus,

$$\phi(\mathcal{D}) \gtrsim \phi(\langle t \rangle^{1-\alpha/2}) \gtrsim \frac{1}{(1+t^2)^{\tilde{\beta}/2}} \text{ for some } \tilde{\beta} < 1.$$

Now we can argue exactly as in the case $\beta < 1$ replacing β by $\tilde{\beta}$ to reach the conclusions of the theorem. □

3.2. **Propagation of Chaos.** Using Proposition 3.3 as a key ingredient we now prove our main result for the Rayleigh-forced system, Theorem 1.3. So, let us we consider the full Liouville equation for a probability density f^N on Ω^N:

$$\partial_t f^N + \sum_{i=1}^N v_i \cdot \nabla_{x_i} f^N + \sum_{i=1}^N \nabla_{v_i} \cdot (f^N F_i^N) + \sum_{i=1}^N \nabla_{v_i} \cdot (f^N R_i^N) + \sum_{i=1}^N \nabla_{\theta_i} \cdot (f^N \Theta_i^N) = 0, \tag{61}$$

subject to the initial condition

$$f^N(0) = f_0^{\otimes N}, \tag{62}$$

where $f_0 : \Omega \to \mathbb{R}_+$ and for $(X, V, \Theta) = (x_1, \ldots, x_N, v_1, \ldots, v_N, \theta_1, \ldots, \theta_N)$,

$$\begin{aligned} F_i^N(X, V, \Theta) &= \frac{1}{N} \sum_{k=1}^N \phi(x_i - x_k)(v_k - v_i), \\ \Theta_i^N(X, V, \Theta) &= \frac{1}{N} \sum_{k=1}^N \phi(x_i - x_k)(\theta_k - \theta_i), \\ R_i^N(X, V, \Theta) &= \sigma v_i(\theta_i - |v_i|^p). \end{aligned}$$

We introduce a similar notation for the flow-maps. Denote by

$$\Phi_t^N = (x_1(t), v_1(t), \theta_1(t) \ldots, x_N(t), v_N(t), \theta_N(t)) : \Omega^N \to \Omega^N$$

the flow-map of the discrete system (14) which is also the characteristic flow of (61). Then, as before, f^N is the push forward of $f_0^{\otimes N}$ under Φ_t^N,

$$f^N = \Phi_t^N \sharp f_0^{\otimes N}.$$

Let also

$$\bar{\Phi}_t = (\bar{x}(t), \bar{v}(t), \bar{\theta}(t)) : \Omega \to \Omega$$

be the characteristic map of (29), which consists of solutions to (34). The direct product of N copies will be denoted $\bar{\Phi}_t^{\otimes N}$. Then we have

$$f = \bar{\Phi}_t \sharp f_0, \quad f^{\otimes N} = \bar{\Phi}_t^{\otimes N} \sharp f_0^{\otimes N}. \tag{63}$$

By the same logic as before the theorem reduces to establishing the bound

$$\int_{R^{2nN}} |\Phi_t^N(X_0, V_0) - \bar{\Phi}_t^{\otimes N}(X_0, V_0)|^2 \, f_0^{\otimes N}(X_0, V_0) \, \mathrm{d}X_0 \, \mathrm{d}V_0 \leqslant C \min\{N, t^4\}. \tag{64}$$

We split the integrand into three components:

$$
\begin{aligned}
\mathcal{P} &= \frac{1}{2}\int_{\Omega^N} |X_t(X_0,V_0,\Theta_0) - \bar{X}_t(X_0,V_0,\Theta_0)|^2 \, f_0^{\otimes N}(X_0,V_0,\Theta_0)\,\mathrm{d}X_0\,\mathrm{d}V_0\,\mathrm{d}\Theta_0,\\
\mathcal{K} &= \frac{1}{2}\int_{\Omega^N} |V_t(X_0,V_0,\Theta_0) - \bar{V}_t(X_0,V_0,\Theta_0)|^2 \, f_0^{\otimes N}(X_0,V_0,\Theta_0)\,\mathrm{d}X_0\,\mathrm{d}V_0\,\mathrm{d}\Theta_0,\\
\mathcal{C} &= \frac{1}{2}\int_{\Omega^N} |\Theta_t(X_0,V_0,\Theta_0) - \bar{\Theta}_t(X_0,V_0,\Theta_0)|^2 \, f_0^{\otimes N}(X_0,V_0,\Theta_0)\,\mathrm{d}X_0\,\mathrm{d}V_0\,\mathrm{d}\Theta_0.
\end{aligned} \tag{65}
$$

For the potential energy we will use the same inequality as before, (23). For $\mathcal{K}$, we obtain

$$
\frac{\mathrm{d}}{\mathrm{d}t}\mathcal{K} = \mathcal{S}_1 + \mathcal{S}_2,
$$

where $\mathcal{S}_1$ is the exact same alignment term that we handled before, but now with the use of Proposition 3.1 and Proposition 3.3,

$$
\mathcal{S}_1 \leqslant c e^{-\delta t}\mathcal{K}^{1/2}(1+\mathcal{P}^{1/2}). \tag{66}
$$

And $\mathcal{S}_2$ is given by

$$
\mathcal{S}_2 = \int_{\Omega^N}\sum_{i=1}^N (v_i - \bar{v}_i)\cdot\big(\sigma v_i(\theta_i - |v_i|^p) - \sigma\bar{v}_i(\bar{\theta}_i - |\bar{v}_i|^p)\big)\, f_0^{\otimes N}(X_0,V_0,\Theta_0)\,\mathrm{d}X_0\,\mathrm{d}V_0\,\mathrm{d}\Theta_0.
$$

Let us write $\mathcal{S}_2$ as follows

$$
\begin{aligned}
\mathcal{S}_2 &= \sigma\int_{\Omega^N}\sum_{i=1}^N (v_i - \bar{v}_i)\cdot\big(\theta_i v_i - \bar{\theta}_i\bar{v}_i\big)\, f_0^{\otimes N}(X_0,V_0,\Theta_0)\,\mathrm{d}X_0\,\mathrm{d}V_0\,\mathrm{d}\Theta_0\\
&\quad - \sigma\int_{\Omega^N}\sum_{i=1}^N (v_i - \bar{v}_i)\cdot(v_i|v_i|^p) - \bar{v}_i|\bar{v}_i|^p)\, f_0^{\otimes N}(X_0,V_0,\Theta_0)\,\mathrm{d}X_0\,\mathrm{d}V_0\,\mathrm{d}\Theta_0\\
&:= J_1 - J_2.
\end{aligned}
$$

Since

$$
(v_i - \bar{v}_i)\cdot\big(\theta_i v_i - \bar{\theta}_i\bar{v}_i\big) = \frac{1}{2}(\theta_i + \bar{\theta}_i)|v_i - \bar{v}_i|^2 + \frac{1}{2}(v_i - \bar{v}_i)\cdot[(\theta_i - \bar{\theta}_i)(v_i + \bar{v}_i)],
$$

one has

$$
J_1 = \frac{\sigma}{2}\int_{\Omega^N}\sum_{i=1}^N \big((\theta_i + \bar{\theta}_i)|v_i - \bar{v}_i|^2 + (v_i - \bar{v}_i)\cdot[(\theta_i - \bar{\theta}_i)(v_i + \bar{v}_i)]\big)\, f_0^{\otimes N}(X_0,V_0,\Theta_0)\,\mathrm{d}X_0\,\mathrm{d}V_0\,\mathrm{d}\Theta_0.
$$

For J_2, since

$$
(v_i - \bar{v}_i)\cdot(v_i|v_i|^p - \bar{v}_i|\bar{v}_i|^p) = \frac{1}{2}(|v_i|^p + |\bar{v}_i|^p)|v_i - \bar{v}_i|^2 + \frac{1}{2}(|v_i|^2 - |\bar{v}_i|^2)(|v_i|^p - |\bar{v}_i|^p),
$$

and

$$
\frac{1}{2}(|v_i|^2 - |\bar{v}_i|^2)(|v_i|^p - |\bar{v}_i|^p) \geqslant 0,
$$

we get

$$
-J_2 \leqslant -\frac{\sigma}{2}\int_{\Omega^N}\sum_{i=1}^N (|\bar{v}_i|^p + |v_i|^p)|v_i - \bar{v}_i|^2 f_0^{\otimes N}(X_0,V_0,\Theta_0)\,\mathrm{d}X_0\,\mathrm{d}V_0\,\mathrm{d}\Theta_0.
$$

Therefore,

$$\mathcal{S}_2 = J_1 - J_2 \leqslant \frac{\sigma}{2}\int_{\Omega^N}\sum_{i=1}^N(\theta_i - |v_i|^p + \bar\theta_i - |\bar v_i|^p)|v_i - \bar v_i|^2 f_0^{\otimes N}(X_0,V_0,\Theta_0)\,\mathrm{d}X_0\,\mathrm{d}V_0\,\mathrm{d}\Theta_0$$
$$+\frac{\sigma}{2}\int_{\Omega^N}\sum_{i=1}^N(v_i-\bar v_i)\cdot(\theta_i-\bar\theta_i)(\bar v_i + v_i) f_0^{\otimes N}(X_0,V_0,\Theta_0)\,\mathrm{d}X_0\,\mathrm{d}V_0\,\mathrm{d}\Theta_0. \tag{67}$$

Because $|\theta_i - |v_i|^p| \leqslant ce^{-\delta t}$ and $|\bar\theta_i - |\bar v_i|^p| \leqslant ce^{-\delta t}$, the first integral on the right hand side of (67) is less than or equal to $ce^{-\delta t}\mathcal{K}$. Then, we apply the Hölder inequality and the boundedness of $|\bar v_i|$ and $|v_i|$ to the second integral to obtain

$$\mathcal{S}_2 \leqslant c(e^{-\delta t}\mathcal{K} + \mathcal{K}^{1/2}\mathcal{H}^{1/2}). \tag{68}$$

Combining (66) and (68) we get

$$\frac{\mathrm{d}}{\mathrm{d}t}\mathcal{K} \leqslant ce^{-\delta t}\mathcal{K}^{1/2}(\mathcal{K}^{1/2} + 1 + \mathcal{P}^{1/2}) + \mathcal{K}^{1/2}\mathcal{H}^{1/2}. \tag{69}$$

Let us now turn to the characteristic parameters term $\mathcal{C}$:

$$\frac{\mathrm{d}}{\mathrm{d}t}\mathcal{C} = \int_{\Omega^N}\sum_{i=1}^N(\theta_i-\bar\theta_i)\cdot\left(\frac{1}{N}\sum_{k=1}^N\phi(x_i-x_k)(\theta_k-\theta_i) - \int_\Omega \phi(\bar x_i - y)(\eta-\bar\theta_i)f(y,w,\eta,t)\,\mathrm{d}y\,\mathrm{d}w\,\mathrm{d}\eta\right)$$
$$\times f_0^{\otimes N}\,\mathrm{d}X_0\,\mathrm{d}V_0\,\mathrm{d}\Theta_0$$
$$:= I_1 + I_2 + I_3,$$

where

$$I_1 = \int_{\Omega^N}\sum_{i=1}^N(\theta_i-\bar\theta_i)\cdot\frac{1}{N}\sum_{k=1}^N[\phi(x_i-x_k)-\phi(\bar x_i-\bar x_k)](\theta_k-\theta_i)\,f_0^{\otimes N}\,\mathrm{d}X_0\,\mathrm{d}V_0\,\mathrm{d}\Theta_0,$$
$$I_2 = \int_{\Omega^N}\sum_{i=1}^N(\theta_i-\bar\theta_i)\cdot\frac{1}{N}\sum_{k=1}^N\phi(\bar x_i-\bar x_k)[(\theta_k-\bar\theta_k)-(\theta_i-\bar\theta_i)]\,f_0^{\otimes N}\,\mathrm{d}X_0\,\mathrm{d}V_0\,\mathrm{d}\Theta_0,$$
$$I_3 = \int_{\Omega^N}\sum_{i=1}^N(\theta_i-\bar\theta_i)\cdot\left(\frac{1}{N}\sum_{k=1}^N\phi(\bar x_i-\bar x_k)(\bar\theta_k-\bar\theta_i) - \int_\Omega\phi(\bar x_i-y)(\eta-\bar\theta_i)f(y,w,t)\,\mathrm{d}y\,\mathrm{d}w\,\mathrm{d}\eta\right)$$
$$\times f_0^{\otimes N}\,\mathrm{d}X_0\,\mathrm{d}V_0\,\mathrm{d}\Theta_0.$$

We have $I_2 \leqslant 0$ because

$$I_2 = \int_{\Omega^N}\frac{1}{N}\sum_{i,k=1}^N\phi(\bar x_i-\bar x_k)[(\theta_i-\bar\theta_i)\cdot(\theta_k-\bar\theta_k) - |\theta_i-\bar\theta_i|^2]\,f_0^{\otimes N}\,\mathrm{d}X_0\,\mathrm{d}V_0\,\mathrm{d}\Theta_0$$
$$= -\int_{\Omega^N}\frac{1}{2N}\sum_{i,k=1}^N\phi(\bar x_i-\bar x_k)|(\theta_i-\bar\theta_i)-(\theta_k-\bar\theta_k)|^2\,f_0^{\otimes N}\,\mathrm{d}X_0\,\mathrm{d}V_0\,\mathrm{d}\Theta_0.$$

For I_1, we obtain, using Proposition 3.1,

$$
\begin{aligned}
|I_1|^2 &\leqslant 2\mathcal{C}\int_{\Omega^N}\sum_{i=1}^N\left|\frac{1}{N}\sum_{k=1}^N[\phi(x_i-x_k)-\phi(\bar{x}_i-\bar{x}_k)](\theta_k-\theta_i)\right|^2 f_0^{\otimes N}\,\mathrm{d}X_0\,\mathrm{d}V_0\,\mathrm{d}\Theta_0\\
&\leqslant 2|\nabla\phi|_\infty^2\mathcal{C}\int_{\Omega^N}\sum_{i=1}^N\left(\frac{1}{N}\sum_{k=1}^N|(x_i-x_k)-(\bar{x}_i-\bar{x}_k)||\theta_k-\theta_i|\right)^2 f_0^{\otimes N}\,\mathrm{d}X_0\,\mathrm{d}V_0\,\mathrm{d}\Theta_0\\
&\leqslant ce^{-2\delta t}\mathcal{C}\int_{\Omega^N}\sum_{i=1}^N\left(\frac{1}{N}\sum_{k=1}^N(|x_i-\bar{x}_i|+|x_k-\bar{x}_k|)\right)^2 f_0^{\otimes N}\,\mathrm{d}X_0\,\mathrm{d}V_0\,\mathrm{d}\Theta_0\\
&\leqslant ce^{-2\delta t}\mathcal{C}\int_{\Omega^N}\sum_{i=1}^N\left(|x_i-\bar{x}_i|^2+\frac{1}{N}\sum_{k=1}^N|x_k-\bar{x}_k|^2\right) f_0^{\otimes N}\,\mathrm{d}X_0\,\mathrm{d}V_0\,\mathrm{d}\Theta_0\\
&= ce^{-2\delta t}\mathcal{C}\mathcal{P}.
\end{aligned}
$$

Thus,

$$
|I_1|\leqslant ce^{-\delta t}\mathcal{C}^{1/2}\mathcal{P}^{1/2}. \tag{70}
$$

For I_3, we have

$$
\begin{aligned}
|I_3|^2 &\leqslant 2\mathcal{C}\int_{\Omega^N}\sum_{i=1}^N\left|\frac{1}{N}\sum_{k=1}^N\phi(\bar{x}_i-\bar{x}_k)(\bar{\theta}_k-\bar{\theta}_i)-\int_\Omega\phi(\bar{x}_i-y)(\eta-\bar{\theta}_i)f(y,w,\eta,t)\,\mathrm{d}y\,\mathrm{d}w\,\mathrm{d}\eta\right|^2\\
&\qquad\times f_0^{\otimes N}(X_0,V_0,\Theta_0)\,\mathrm{d}X_0\,\mathrm{d}V_0\,\mathrm{d}\Theta_0\\
&= 2\mathcal{C}\int_{\Omega^N}\sum_{i=1}^N\left|\frac{1}{N}\sum_{k=1}^N\phi(\bar{x}_i-\bar{x}_k)(\bar{\theta}_k-\bar{\theta}_i)-\int_\Omega\phi(\bar{x}_i-y)(\eta-\bar{\theta}_i)f(y,w,\eta,t)\,\mathrm{d}y\,\mathrm{d}w\,\mathrm{d}\eta\right|^2\\
&\qquad\times f^{\otimes N}(\bar{X},\bar{V},\bar{\Theta},t)\,\mathrm{d}\bar{X}\,\mathrm{d}\bar{V}\,\mathrm{d}\bar{\Theta}\\
&= 2\mathcal{C}N\int_{\Omega^N}\left|\frac{1}{N}\sum_{k=1}^N\phi(\bar{x}_1-\bar{x}_k)(\bar{\theta}_k-\bar{\theta}_1)-\int_\Omega\phi(\bar{x}_1-y)(\eta-\bar{\theta}_1)f(y,w,\eta,t)\,\mathrm{d}y\,\mathrm{d}w\,\mathrm{d}\eta\right|^2\\
&\qquad\times f^{\otimes N}(\bar{X},\bar{V},\bar{\Theta},t)\,\mathrm{d}\bar{X}\,\mathrm{d}\bar{V}\,\mathrm{d}\bar{\Theta}\\
&\leqslant 2\mathcal{C}N\frac{4}{N}\sup_{(\bar{x},\bar{v},\bar{\theta}),(\bar{x}',\bar{v}',\bar{\theta}')\in\operatorname{supp}f_t}|\phi(\bar{x}-\bar{x}')(\bar{\theta}-\bar{\theta}')|^2\leqslant c\mathcal{C}e^{-2\delta t}.
\end{aligned}
$$

Here in the penultimate step we used again [12, Lemma 3.3]. Therefore,

$$
|I_3|\leqslant ce^{-\delta t}\mathcal{C}^{1/2}. \tag{71}
$$

Combining the three estimates for I_1, I_2, I_3, we obtain

$$
\frac{\mathrm{d}}{\mathrm{d}t}\mathcal{C}\leqslant ce^{-\delta t}(1+\mathcal{P}^{1/2})\mathcal{C}^{1/2}. \tag{72}
$$

Setting $p=1+\mathcal{P}^{1/2}$, $k=\mathcal{K}^{1/2}, q=\mathcal{C}^{1/2}$. By (23), (69) and (72) we obtain the system of ODIs:

$$
\begin{cases}
\dot{p}\leqslant k, & p_0=1,\\
\dot{k}\leqslant ce^{-\delta t}(p+k)+cq, & k_0=0,\\
\dot{q}\leqslant ce^{-\delta t}p, & q_0=0.
\end{cases} \tag{73}
$$

Claim 3.11. For any nonnegative solution (p,k,q) to (73), there exists a constant C depending on c,δ such that

$$
p\leqslant 1+Ct^2,\quad k\leqslant Ct,\quad z\leqslant C\min\{1,t\}. \tag{74}
$$

Proof of the Claim 3.11. Fix $\varepsilon, \tau > 0$ to be chosen later. We have

$$\begin{cases} \frac{\mathrm{d}}{\mathrm{d}t}(\varepsilon p^2) \leqslant 2\varepsilon pky \leqslant \sqrt{\varepsilon}(\varepsilon p^2 + k^2), \\ \frac{\mathrm{d}}{\mathrm{d}t} k^2 \leqslant c e^{-\delta t}(2pk + 2k^2) + 2ckq \leqslant c e^{-\delta t}\left[\frac{1}{\sqrt{\varepsilon}}(\varepsilon p^2 + k^2) + 2k^2\right] + \frac{c}{\sqrt{\tau}}(p^2 + \tau q^2), \\ \frac{\mathrm{d}}{\mathrm{d}t}(\tau q^2) \leqslant 2\tau c e^{-\delta t} pq \leqslant \frac{c\sqrt{\tau} e^{-\delta t}}{\sqrt{\varepsilon}}(\varepsilon p^2 + \tau q^2). \end{cases}$$

It implies that

$$\frac{\mathrm{d}}{\mathrm{d}t}(\varepsilon p^2 + k^2 + \tau q^2) \leqslant c(\tau, \varepsilon) e^{-\delta t}(\varepsilon p^2 + k^2 + \tau q^2) + \left(\sqrt{\varepsilon} + \frac{c}{\sqrt{\tau}}\right)(\varepsilon p^2 + k^2 + \tau q^2).$$

Applying Grönwall's lemma we get

$$\varepsilon p^2 + k^2 + \tau q^2 \leqslant \varepsilon \exp\left(\left(\sqrt{\varepsilon} + \frac{c}{\sqrt{\tau}}\right)t + \frac{c(\varepsilon, \tau)}{\delta}(1 - e^{-\delta t})\right) \leqslant \varepsilon \exp\left(\left(\sqrt{\varepsilon} + \frac{c}{\sqrt{\tau}}\right)t + \frac{c(\varepsilon, \tau)}{\delta}\right).$$

Now choosing $\varepsilon = \delta^2/4, \tau = 4c^2/\delta^2$, we obtain

$$p \lesssim e^{\delta t/2}.$$

Plugging it into the third equation in (73) and solving for q we have

$$q \leqslant c\int_0^t e^{-\delta s/2}\,\mathrm{d}s \leqslant C\min\{1, t\}.$$

Substituting p, q into the second equation in (73) we have

$$\frac{\mathrm{d}}{\mathrm{d}t} k \leqslant c e^{-\delta t} k + c e^{-\delta t/2} + C\min(1, t).$$

It implies that

$$k \leqslant Ct.$$

Hence, by the first equation in (73) we get

$$p \leqslant 1 + Ct^2.$$

The Claim 3.11 follows that

$$\mathcal{P} \leqslant Ct^4, \quad \mathcal{K} \leqslant Ct^2, \quad \mathcal{C} \leqslant C\min\{1, t^2\}.$$

On the other hand, in view of the global estimates on the support of the flock, $\mathcal{P} \leqslant CN$. Due to the alignment we also have $\mathcal{K} \leqslant CN$. Therefore,

$$\mathcal{P} + \mathcal{K} + \mathcal{C} \leqslant C\min\{N, t^4\},$$

as desired. □

Remark 3.12. Our final remark concerns the case $\kappa = 0$. This represents the system with "frozen" characteristic parameters θ. In opinion dynamics such system can be interpreted as a non-cooperative game where "players" come with their fixed convictions θ's but may change their opinions v's to achieve a consensus. In the discrete case this situation was examined in detail in [10] where the consensus was identified as a Nash equilibrium. The equilibrium is unique, stable, and is also a global attractor for the system. While the kinetic version of such result would be highly desirable to achieve – this could be interpreted as a dynamics of infinitely many players – we leave this question to a future research. At this point we note that in the case $\kappa = 0$ no alignment dynamics is possible, however the maximum principle obtained in Lemma 3.4 and Lemma 3.5 still holds. The Grassmannian reduction still works also to show that the support of any sectorial solution f_t narrows down to a kinetic ray $\mathbb{R}_+ v_\infty$ for some $v_\infty \in \mathbb{S}^{n-1}$. It is therefore an essentially unidirectional flow.

Since no global flocking information is available in this case applying our analysis gives the same exponential rate as in Natalini and Paul's estimate (9).

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Department of Mathematics, Statistics and Computer Science, University of Illinois at Chicago, 60607

Email address: vnguye66@uic.edu

Email address: shvydkoy@uic.edu