LOCAL WELL-POSEDNESS OF THE TOPOLOGICAL EULER ALIGNMENT MODELS OF COLLECTIVE BEHAVIOR

DAVID N. REYNOLDS AND ROMAN SHVYDKOY

Abstract. In this paper we address the problem of well-posedness of multi-dimensional topological Euler-alignment models introduced in [21]. The main result demonstrates local existence and uniqueness of classical solutions in class $(\rho, u) \in H^{m+\alpha} \times H^{m+1}$ on the periodic domain $\mathbb{T}^n$, where $0 < \alpha < 2$ is the order of singularity of the topological communication kernel $\phi(x, y)$, and $m = m(n, \alpha)$ is large. Our approach is based on new sharp coercivity estimates for the topological alignment operator

$$\mathcal{L}_\phi f(x) = \int_{\mathbb{T}^n} \phi(x, y)(f(y) - f(x))\, \mathrm{d}y,$$

which render proper a priori estimates and help stabilize viscous approximation of the system.

Contents

1. Introduction

Several recent field studies on animal and human behavior revealed that in some cases communication between agents is dominated by so-called *topological* interactions. Topological, as opposed to the classical metric, interactions are based on the principle that a given agent (bird, fish, human, etc) is only capable to sense a limited number of other agents in its immediate proximity to adjust its direction of motion, see [1, 2, 7, 18] and references therein. Kinetic models interpreting such topological interactions as the K-nearest neighbor rule were studied at length by Blanchet and Degond in [3, 4]. In the context of Cucker-Smale type alignment model introduced in [8, 9] Haskovec [15] proposed a topological interaction $\phi(d)$, which depends on asymmetric “distance” $d(x, y)$ between agents x and y defined by counting all agents crowded in the ball of radius $|x - y|$ centered at x. Thus, in crowded directions, propagation of information is hindred by the higher density. Under a global in time graph connectivity assumption – one that is guaranteed to hold, for instance, in the classical metric case such as

$$\phi(x, y) = \frac{H}{(1 + |x - y|^2)^{\beta/2}}, \quad \beta \leq 1, \tag{1}$$

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it is shown that the system flocks.

Recently, newly designed topological protocols resurfaced in the context of a different, although not unrelated, problem of flock self-organization under purely local interactions. To recall let us consider the hydrodynamic Euler-alignment system (macroscopic counterpart of the agent-based Cucker-Smale system):

$$\begin{cases} \rho_t + \nabla \cdot (\rho u) = 0, \\ u_t + u \cdot \nabla u = \displaystyle\int_{\mathcal{D}} \phi(x,y)(u(t,y) - u(t,x))\rho(t,y)\,\mathrm{d}y, \end{cases} \tag{2}$$

where $\mathcal{D}$ is an environment, typically assumed to be $\mathbb{R}^n$ or $\mathbb{T}^n$, see [13, 14, 12] for derivation. For kernels with non-integrable at infinity tails, like those of Cucker-Smale type (1), the system is shown to exhibit unconditional exponential alignment

$$\|u(t) - \bar{u}\|_\infty \lesssim e^{-\delta t},$$

and flocking $\operatorname{diam}\operatorname{supp}\rho(t) \leq D < \infty$, see Tadmor and Tan [26]. While long range communication is indeed relevant in some technological applications, such as its remarkable adaptation to the Darwin mission, [17], in relation to biological systems where communication almost always has a finite reach,

$$\operatorname{supp}\phi \subset \{|x-y| < r_0\},$$

the fundamental issue of self-organization remained open. To be mathematically consistent the problem ought to be considered in the "bulk" of the flock modeled by periodic domain $\mathbb{T}^n$, as it is easy to produce a counterexample on $\mathbb{R}^n$ by placing two distant and disconnected flocks with opposite momenta in the same system.

To address this problem a new topological model was introduced in [21]. The communication protocol involves a singular, local, symmetric kernel $\phi(x,y)$ with adaptive diffusion. The adaptive diffution is a mechanism of active recalibration of communication strength based on the density of the crowd in an intermediate region between a given pair of agents. Specifically, it is postulated that the strength of interactions between (x,y) is inversely proportional to the mass of a *symmetric* region $\Omega(x,y) = \Omega(y,x)$ (a key difference from Haskovec's model) at time t which is encoded into the topological quasi-distance function

$$d(x,y) = \left(\int_{\Omega(x,y)} \rho(t,\xi)\,\mathrm{d}\xi \right)^{1/n}.$$

We define $\phi(x,y)$ as a non-convolution type singular kernel of degree $0 < \alpha < 2$ by

$$\phi(x,y) = \frac{h(|x-y|)}{|x-y|^{n+\alpha-\tau} d^\tau(x,y)}, \tag{3}$$

where $h = h(r)$ is a radial smooth bump function supported on a ball of radius r_0 – a communication cutoff scale, and $\tau \geq 0$ is a parameter that gauges presence of topological effects in the system. The new protocol (3) reflects the core principle of topological interaction – information spreads faster in thinner regions and slower is dense regions.

Theoretical restrictions on what the domain $\Omega(x,y)$ might be are rather loose and can be calibrate according to a specific application. For example, [21] considers the American football body of revolution. Specifically, we require the family $\{\Omega(x,y)\}_{x,y\in\mathcal{D}}$ to be self-similar, i.e. obtained by rescaling and rotating of a basic domain $\Omega_0 = \Omega(-\mathbf{e}_1, \mathbf{e}_1)$ such that

(D1) $\partial\Omega_0$ is smooth except at $\pm\mathbf{e}_1$ where it is Lipschitz of conical opening of degree $< \pi$,
(D2) $\Omega_0 = -\Omega_0$,
(D3) $\Omega_0 \subset B_1(0)$.

Figure 1 shows a prototype family of such domains. The global self-organization of topological systems was established in the following theorem.

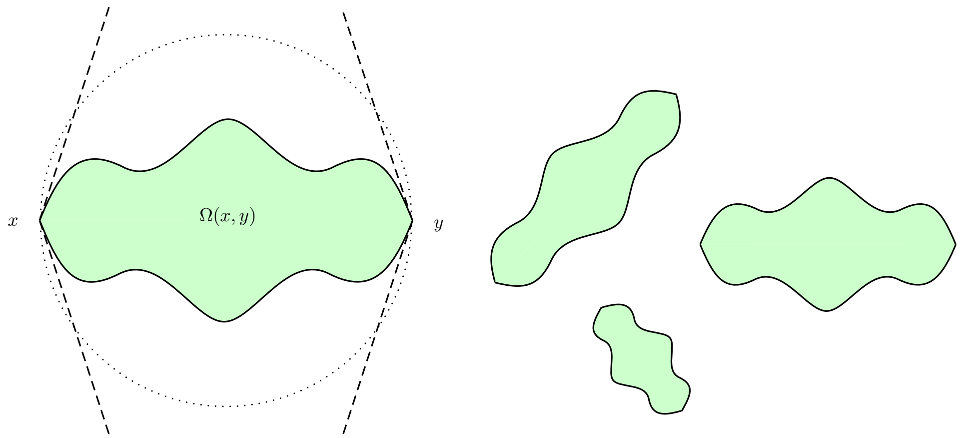

FIGURE 1. Communication domain satisfying assumptions (D1)–(D3)

Theorem 1.1 ([21]). *Suppose $\tau \geq n$. Then any classical solution (u,ρ) to (2) on the torus $\mathbb{T}^n$ satisfying the hydrodynamic connectivity condition*

$$\rho(t,x) \gtrsim \frac{1}{1+t} \tag{4}$$

aligns to its conserved momentum $\bar{u}$ at a logarithmic rate

$$\|u(t)-\bar{u}\|_\infty \lesssim \frac{1}{\sqrt{\ln t}}.$$

Condition (4) presents a degree of hydrodynamic connectivity required for the result to hold – a very common and often necessary assumption in the literature on collective behaviour. Remarkably, it holds true automatically in the one dimensional case, $n=1$, see the discussion in [23, 21].

Regularity theory of metric models (2), i.e. where $\phi(x,y) = \phi(|x-y|)$ is studied in a body of literature [5, 6, 10, 16, 11, 22, 23, 24, 20, 27, 26], and is most completely understood only in one dimensional settings due to an extra conserved quantity

$$e = u_x + \phi * \rho, \quad e_t + (ue)_x = 0, \tag{5}$$

which allows to directly control u_x. For the smooth kernel case this leads to Burgers' type threshold condition $e_0 \geq 0$ to guarantee global existence, see Carrillo et al [5]. For singular communication, $\phi(r) = \frac{1}{r^{1+\alpha}}$, additional parabolic structure leads to regularization and global existence for any smooth non-vacuous data on $\mathbb{T}$, [11, 22, 23, 24]. In multi-D, small initial data results were proved in [10, 20, 16].

Topological models presented a new set of challenges from the perspective of regularity theory as they do not fit directly under any studied class of fractional drift diffusion equations, see [19, 25]. The one dimensional case has been treated in the same article [21] where global wellposedness was established in class $u \in H^{m+1}$, $\rho \in H^{m+\alpha}$ for $\tau \leq \alpha$.

The primary goal of this paper is to lay a technical foundation for the study of topological models by establishing local well-posedness for solutions in higher Sobolev classes in arbitrary spacial dimension $n \geq 1$.

Theorem 1.2. *Let $0 < \alpha < 2$ and $\tau \geq 0$. For any initial data $u_0 \in H^{m+1}(\mathbb{T}^n)$, $\rho_0 \in H^{m+\alpha}(\mathbb{T}^n)$, $m \geq m(\alpha, n)$, with no vacuum $\rho_0(x) > 0$ there exists a unique non-vacuous solution to the system (2)-(3)-$(D2)$ on a time interval $[0,T_0)$ where T_0 depends on the initial conditions, in the class*

$$\begin{aligned} &u \in C_w([0,T_0), H^{m+1}) \cap L^2([0,T_0), H^{m+1+\frac{\alpha}{2}}) \\ &\rho \in C_w([0,T_0), H^{m+\alpha}). \end{aligned} \tag{6}$$

Here, C_w stands for weakly continuous functions.

The main technical challenge in proving Theorem 1.2 is the presence of derivative overload in the continuity equation. In direct energy estimates, it simply takes more derivatives than it can handle a priori. The natural way to handle this difficulty, as was done in previous works on 1D case [11, 22, 23, 24, 21], is to consider a multi-dimensional version of the e-quantity given by

$$e = \nabla \cdot u + \mathcal{L}_\phi \rho,$$

where $\mathcal{L}_\phi$ is the singular alignment operator associated with the topological kernel ϕ:

$$\mathcal{L}_\phi f = \int_{\mathbb{T}^n} \phi(x,y)(f(y) - f(x))\,\mathrm{d}y. \tag{7}$$

One proceeds to replace the continuity equation with an equation for e. However, unlike in 1D, it no longer satisfies the pure conservation law (5) due to first, the appearence of extra stresses as in the metric case, see for example [16], and second, residual terms coming from active dependency of the kernel on ρ:

$$e_t + \nabla \cdot (ue) = (\nabla \cdot u)^2 - \mathrm{Tr}(\nabla u)^2 + \mathcal{L}_{\phi_t}(\rho) + \mathcal{L}_{\nabla \phi \cdot}(\rho u). \tag{8}$$

By replacing the (u,ρ)-system with the new (u,e)-system, we essentially move the overload problem from transport to those last two residual terms in (8). The advntage is that the latter have more geometric structure which we handle by developing an "easing" technique, which allows to transfer analysis from the bulk of communication region to its boundary or other more regular terms in the equation.

The second technical ingredient of the proof is to ensure that the membership of the new pair (u,e) in class $H^{m+1} \times H^m$ is in fact equivalent to the original setting of (u,ρ) being in class $H^{m+1} \times H^{m+\alpha}$. This is done by establishing sharp coercivity estimates

$$\|\mathcal{L}_\phi f\|_{\dot{H}^m} \sim \|f\|_{\dot{H}^{m+\alpha}} + \text{lower order terms}, \tag{9}$$

which reflect a seemingly natural fact that L_ϕ appears to be of order α. This fact, however, is far from obvious given that the active dependency on the density in ϕ may influence the order of the operator due to the density being limited it its own class of regularity. In Proposition 3.1 we detail exactly how the density regularity enters into the equivalence relation (9).

Lastly, we note that the membership of u in $L^2([0,T_0), H^{m+1+\frac{\alpha}{2}})$ is orginated of course from the dissipation. A similar dissipative structure can be seen in the continuity equation if written in the form

$$\rho_t + u \cdot \nabla \rho + e\rho = \rho \mathcal{L}_\phi \rho. \tag{10}$$

However, presence of e injects a rough forcing term $e\rho$ that drives the density out of the expected smoother class $H^{m+\alpha+\frac{\alpha}{2}}$. We therefore cannot ensure a similar membership of ρ in this smoother class.

Taking all the precausions mensioned above into consideration our general strategy in establishing Theorem 1.2 will be to trace the grand quantity

$$Y_m = \|u\|^2_{\dot{H}^{m+1}} + \|e\|^2_{\dot{H}^m} + \|\rho\|^2_{\dot{H}^m} + \overline{\rho} + \underline{\rho}^{-1}, \tag{11}$$

where $\underline{\rho} = \min \rho$, $\overline{\rho} = \max \rho$, and to obtain a short term control over Y_m by proving a priori Riccati type equation

$$\frac{\mathrm{d}}{\mathrm{d}t} Y_m \leq C Y_m^N, \tag{12}$$

where $N \in \mathbb{N}$ may be large. Coercivity estimates (9) demonstrate that Y_m is equivalent to controlling u in H^{m+1} and ρ in $H^{m+\alpha}$. The actual realization of such a priori estimate presents itself in a viscousity-regularized system which produces a unique solution via valishing viscosity limit as we detail in Section 6.

The proof is split between several sections. In Section 2 we set the notation and make elementary a priori estimates on lower order terms in Y_m. Section 3 is entirely devoted to coercivity bounds on the alignment operator via commutator estimates. Sections 4 and 5 detail a priori estimates on the u and e equations, respectively. In Section 6 we conclude by finding local solutions via viscous regularization scheme and establish stability of our a priori estimates under such approximation.

2. Preliminaries

In this section we go through a few quick computations that establish a priori estimates on the lower order terms in the grand quantity Y_m (11), namely, $\|\rho\|^2_{\dot H^m} + \overline{\rho} + \underline{\rho}^{-1}$.

The bound on $\|\rho\|^2_{\dot H^m}$ follows by a simple classical commutator estimate. Indeed, we have

$$\rho_t + u\cdot\nabla\rho + (\nabla\cdot u)\rho = 0.$$

So, testing with $\partial^{2m}\rho$ we obtain

$$\frac{\mathrm{d}}{\mathrm{d}t}\|\rho\|^2_{\dot H^m} = \int(\nabla\cdot u)|\partial^m\rho|^2\,\mathrm{d}x - \int(\partial^m(u\cdot\nabla\rho) - u\cdot\nabla\partial^m\rho)\partial^m\rho\,\mathrm{d}x - \int\partial^m((\nabla\cdot u)\rho)\partial^m\rho\,\mathrm{d}x.$$

Recalling the classical commutator estimate

$$\|\partial^m(fg) - f\partial^m g\|_2 \le |\nabla f|_\infty\|g\|_{\dot H^{m-1}} + \|f\|_{\dot H^m}|g|_\infty, \tag{13}$$

we obtain, for $m > \frac{n}{2}$,

$$\frac{\mathrm{d}}{\mathrm{d}t}\|\rho\|^2_{\dot H^m} \le |\nabla u|_\infty\|\rho\|^2_{\dot H^m} + \|u\|_{\dot H^m}\|\rho\|_{\dot H^m}|\nabla\rho|_\infty + \|u\|_{\dot H^{m+1}}\|\rho\|_{\dot H^m}|\rho|_\infty \le CY_m^{3/2}.$$

Next, differentiating the maximum we obtain

$$\frac{\mathrm{d}}{\mathrm{d}t}\overline{\rho} \le |\nabla u|_\infty\overline{\rho},$$

and similarly,

$$\frac{\mathrm{d}}{\mathrm{d}t}\underline{\rho}^{-1} \le |\nabla u|_\infty\underline{\rho}^{-1}.$$

Thus,

$$\frac{\mathrm{d}}{\mathrm{d}t}(\|\rho\|^2_{\dot H^m} + \overline{\rho} + \underline{\rho}^{-1}) \lesssim Y_m^3.$$

Having these simple bounds out of the way, the main focus now will be on obtaining similar bounds on the first two components of Y_m and ensuring that Y_m is comparable with the spaces in which we are proving local well-posedness.

3. Coercivity bounds on $\mathcal{L}_\phi$

Letting $y = x+z$ and defining the increment operator $\delta_z f(x) = f(x+z) - f(x)$ we can rewrite the operator as

$$\mathcal{L}_\phi f = \int_{\mathbb{T}^n}\phi(x,x+z)\delta_z f(x)\,\mathrm{d}z. \tag{14}$$

Proposition 3.1. *For any sufficiently large $m\in\mathbb{N}$ and $0<\alpha<2$, $\tau\ge 0$, there exists a polynomial p_N of degree $N = N(m,n,\alpha,\tau)\in\mathbb{N}$ such that the following inequalities hold*

$$\begin{aligned}
\|\mathcal{L}_\phi f\|^2_{\dot H^m} &\lesssim \underline{\rho}^{-2\tau/n}(\|f\|^2_{\dot H^{m+\alpha}} + \|\rho\|^2_{\dot H^{m+\alpha}}) + p_N(\overline{\rho}, \underline{\rho}^{-1}, \|\rho\|_{\dot H^{m-1+\alpha}}, \|f\|_{\dot H^{2+\frac{n}{2}}}),\\
\|\mathcal{L}_\phi f\|^2_{\dot H^m} &\gtrsim \overline{\rho}^{-2\tau/n}(\|f\|^2_{\dot H^{m+\alpha}} + \|\rho\|^2_{\dot H^{m+\alpha}}) - p_N(\overline{\rho}, \underline{\rho}^{-1}, \|\rho\|_{\dot H^{m-1+\alpha}}, \|f\|_{\dot H^{2+\frac{n}{2}}}).
\end{aligned} \tag{15}$$

As a consequence of this proposition we obtain control on the key norm $\|\rho\|_{\dot H^{m+\alpha}}$, that will appear in the main estimates on Y_m:

$$\|\rho\|^2_{\dot H^{m+\alpha}} \lesssim Y_m^N, \tag{16}$$

for some large $N \in \mathbb{N}$. Indeed, setting $f = \rho$ in the above, we find (N may change from line to line)

$$\begin{aligned}\|\rho\|^2_{\dot H^{m+\alpha}} &\lesssim \overline{\rho}^{2\tau/n}\|\mathcal{L}_\phi \rho\|^2_{\dot H^m} + p_N(\overline{\rho}, \underline{\rho}^{-1}, \|\rho\|_{\dot H^{m-1+\alpha}})\\ &\le \overline{\rho}^{2\tau/n}\|u\|^2_{\dot H^{m+1}} + \overline{\rho}^{2\tau/n}\|e\|^2_{\dot H^m} + p_N(\overline{\rho}, \underline{\rho}^{-1}, \|\rho\|_{\dot H^{m-1+\alpha}}) \le Y_m^4 + p_N(\overline{\rho}, \underline{\rho}^{-1}, \|\rho\|_{\dot H^{m-1+\alpha}}).\end{aligned}$$

Now by the same estimate applied to $\|\rho\|_{\dot H^{m-1+\alpha}}$ we have

$$\|\rho\|^2_{\dot H^{m-1+\alpha}} \le Y^4_{m-1} + p_N(\overline{\rho}, \underline{\rho}^{-1}, \|\rho\|_{\dot H^{m-2+\alpha}}).$$

However, trivially $Y_{m-1} \lesssim Y_m$, for $m > 1$, and $\|\rho\|_{\dot H^{m-2+\alpha}} \le \|\rho\|_{\dot H^m}$ for all $0 < \alpha < 2$ with the latter being included into the definition of Y_m. Hence,

$$\|\rho\|^2_{\dot H^{m+\alpha}} \lesssim Y_m^4 + p_N(\overline{\rho}, \underline{\rho}^{-1}, Y_m) \le Y_m^N,$$

and (16) follows.

Conversely, it is clear that $\|\rho\|_{\dot H^{m+\alpha}}$ controls $\|\mathcal{L}_\phi \rho\|_{\dot H^m}$ by first in (15). So, along with $\|u\|^2_{\dot H^{m+1}}$ it controls e. We obtain

$$Y_m \sim \|u\|^2_{\dot H^{m+1}} + \|\rho\|^2_{\dot H^{m+\alpha}} + \overline{\rho} + \underline{\rho}^{-1}.$$

Remark 3.2. Although, as we have just seen, estimate (15) is sufficient to establish control over $\|\rho\|_{\dot H^{m+\alpha}}$, what one can actually prove following our argument below is a somewhat sharper version of (15) where the dependence on the density ρ is of order below $m+\alpha$. Namely, for every $\varepsilon > 0$ there exists a $c_\varepsilon > 0$ such that

$$\begin{aligned}\|\mathcal{L}_\phi f\|^2_{\dot H^m} &\lesssim \|f\|^2_{\dot H^{m+\alpha}} + \|\rho\|^N_{\dot H^{m-1+\alpha}}\|f\|^2_{\dot H^{m-1+\alpha}} + \|\rho\|^2_{\dot H^m}\|f\|^2_{\dot H^{2+\frac n2}} + c_\varepsilon\|\rho\|^2_{\dot H^{m-1+\alpha+\varepsilon}}\|f\|^2_{\dot H^{1+\frac n2}},\\ \|\mathcal{L}_\phi f\|^2_{\dot H^m} &\gtrsim \|f\|^2_{\dot H^{m+\alpha}} - \|\rho\|^N_{\dot H^{m-1+\alpha}}\|f\|^2_{\dot H^{m-1+\alpha}} - \|\rho\|^2_{\dot H^m}\|f\|^2_{\dot H^{2+\frac n2}} - c_\varepsilon\|\rho\|^2_{\dot H^{m-1+\alpha+\varepsilon}}\|f\|^2_{\dot H^{1+\frac n2}}.\end{aligned} \tag{17}$$

Here inequality signs $\lesssim, \gtrsim$ mean up to multiples of $\underline{\rho}$ and $\overline{\rho}$.

As a first step in proving Proposition 3.1 we show a basic coercivity estimate.

Lemma 3.3 (Basic coercivity). *For any $0 < \alpha < 2$ the following bounds hold*

$$\begin{aligned}\|\mathcal{L}_\phi f\|_2^2 &\lesssim \underline{\rho}^{-2\tau/n}\|f\|^2_{\dot H^\alpha} + \overline{\rho}^{2\tau/n}\underline{\rho}^{-2-4\tau/n}|\nabla\rho|^2_\infty\|f\|^2_{\dot H^{\alpha/2}},\\ \|\mathcal{L}_\phi f\|_2^2 &\gtrsim \overline{\rho}^{-2\tau/n}\|f\|^2_{\dot H^\alpha} - \overline{\rho}^{2\tau/n}\underline{\rho}^{-2-4\tau/n}|\nabla\rho|^2_\infty\|f\|^2_{\dot H^{\alpha/2}}.\end{aligned} \tag{18}$$

Proof. Let us denote

$$\fint_{\Omega(0,z)} \rho(x+\xi)\,\mathrm{d}\xi = \frac{1}{|\Omega(0,z)|}\int_{\Omega(0,z)} \rho(x+\xi)\,\mathrm{d}\xi.$$

Note that $|\Omega(0,z)| \sim |z|^n$. In order to remove the x-dependence from the kernel we "freeze" the coefficient, meaning replace d with the average value and then replace it with $\rho(x)$:

$$\mathcal{L}_\phi f(x) = \rho(x)^{-\tau/n}\int_{\mathbb{T}^n}\frac{h(|z|)}{|z|^{n+\alpha}}\delta_z f(x)dz + \int_{\mathbb{T}^n}\frac{h(|z|)}{|z|^{n+\alpha}}\left(\frac{1}{\left[\fint_{\Omega(0,z)}\rho(x+\xi)\,\mathrm{d}\xi\right]^{\tau/n}} - \frac{1}{\rho^{\tau/n}(x)}\right)\delta_z f(x)dz.$$

The first integral represents the truncated fractional Laplacian Λ_α, and hence is bounded above and below by $\underline{\rho}^{-\tau/n}\|f\|_{\dot H^\alpha}$ and $\overline{\rho}^{-\tau/n}\|f\|_{\dot H^\alpha}$, respectively. In the residual term we estimate

$$\frac{1}{\left[\fint_{\Omega(0,z)}\rho(x+\xi)\,\mathrm{d}\xi\right]^{\tau/n}} - \frac{1}{\rho^{\tau/n}(x)} = \frac{\rho^{\tau/n}(x) - \left[\fint_{\Omega(0,z)}\rho(x+\xi)\,\mathrm{d}\xi\right]^{\tau/n}}{\left[\fint_{\Omega(0,z)}\rho(x+\xi)\,\mathrm{d}\xi\right]^{\tau/n}\rho^{\tau/n}(x)}$$

and by Taylor expansion,

$$\left|\rho^{\tau/n}(x) - \left[\fint_{\Omega(0,z)} \rho(x+\xi)\,\mathrm{d}\xi\right]^{\tau/n}\right| \leq \overline{\rho}^{\tau/n}\underline{\rho}^{-1}\left|\rho(x) - \fint_{\Omega(0,z)} \rho(x+\xi)\,\mathrm{d}\xi\right| \leq \overline{\rho}^{\tau/n}\underline{\rho}^{-1}|\nabla\rho|_\infty|z|.$$

So, the residual term is bounded by

$$\overline{\rho}^{\tau/n}\underline{\rho}^{-1-2\tau/n}|\nabla\rho|_\infty \int_{\mathbb{T}^n} \frac{h(|z|)}{|z|^{n+\alpha-1}}|\delta_z f(x)|dz.$$

Estimating the L^2-norm of the remaining integral for $\alpha < 1$ we get a bound by $\|f\|_2$ by the Minkowskii inequality, and for $\alpha \geq 1$,

$$\left|\int_{\mathbb{T}^n} \frac{h(|z|)}{|z|^{n+\alpha-1}}|\delta_z f(x)|dz\right|^2 = \left|\int_{\mathbb{T}^n} \frac{h(|z|)}{|z|^{\frac{n}{2}-\varepsilon}} \frac{|\delta_z f(x)|}{|z|^{\frac{n}{2}+\alpha-1+\varepsilon}}dz\right|^2 \leq C_\varepsilon \int_{\mathbb{T}^n} \frac{|\delta_z f(x)|^2}{|z|^{n+2(\alpha-1+\varepsilon)}}dz.$$

Integrating in x we obtain $\leq \|f\|^2_{\dot{H}^{\alpha-1+\varepsilon}}$. In either case, we can increase regularity to $\|f\|_{\dot{H}^{\alpha/2}}$. □

We now want to lift the base regularity into higher order Sobolev spaces H^m. The natural way to obtain such estimates is through a commutator

$$\partial_i^m \mathcal{L}_\phi f = \mathcal{L}_\phi \partial_i^m f + [\mathcal{L}_\phi, \partial_i^m] f. \tag{19}$$

The commutator can be expanded by the Leibniz rule,

$$[\mathcal{L}_\phi, \partial_i^m] f = \sum_{l=0}^{m-1} \binom{m}{l} \mathcal{L}_{\partial_i^{(m-l)}\phi} \partial_i^l f.$$

The main term in (19), upon summation over i enjoys the estimates from Lemma 3.3:

$$\begin{aligned} \sum_{i=1}^n \|\mathcal{L}_\phi \partial_i^m f\|_2^2 &\lesssim \underline{\rho}^{-2\tau/n}\|f\|^2_{\dot{H}^{m+\alpha}} + \overline{\rho}^{2\tau/n}\underline{\rho}^{-2-4\tau/n}|\nabla\rho|^2_\infty \|f\|^2_{\dot{H}^{m+\frac{\alpha}{2}}}, \\ \sum_{i=1}^n \|\mathcal{L}_\phi \partial_i^m f\|_2^2 &\gtrsim \overline{\rho}^{-2\tau/n}\|f\|^2_{\dot{H}^{m+\alpha}} - \overline{\rho}^{2\tau/n}\underline{\rho}^{-2-4\tau/n}|\nabla\rho|^2_\infty \|f\|^2_{\dot{H}^{m+\frac{\alpha}{2}}}. \end{aligned} \tag{20}$$

By interpolation and the generalized Young inequality, we further obtain

$$\begin{aligned} \overline{\rho}^{2\tau/n}\underline{\rho}^{-2-4\tau/n}|\nabla\rho|^2_\infty \|f\|^2_{\dot{H}^{m+\frac{\alpha}{2}}} &\leq \overline{\rho}^{2\tau/n}\underline{\rho}^{-2-4\tau/n}|\nabla\rho|^2_\infty \|f\|^{2\theta_{m,n,\alpha}}_{\dot{H}^{2+\frac{n}{2}}} \|f\|^{2-2\theta_{m,n,\alpha}}_{\dot{H}^{m+\alpha}} \\ &\leq c_\varepsilon p_N(\overline{\rho}, \underline{\rho}^{-1}, \|\rho\|_{\dot{H}^{m-1+\alpha}}, \|f\|_{\dot{H}^{2+\frac{n}{2}}}) + \varepsilon\overline{\rho}^{-2\tau/n}\|f\|^2_{\dot{H}^{m+\alpha}}. \end{aligned} \tag{21}$$

The highest term $\varepsilon\overline{\rho}^{-2\tau/n}\|f\|^2_{\dot{H}^{m+\alpha}}$ for small ε can be absorbed into the leading terms in (20). Thus, we obtain required bounds (15) from the highest term. The rest follows from the following estimate on the commutator.

Lemma 3.4 (Main commutator estimate)**.** *We have the following inequality*

$$\|[\mathcal{L}_\phi, \partial_i^m] f\|_2^2 \lesssim \|\rho\|^N_{\dot{H}^{m-1+\alpha}}(\|f\|^2_{\dot{H}^{m-\frac{1}{2}+\alpha}} + \|f\|^2_{\dot{H}^{m+\frac{\alpha}{2}}}) + (\|\rho\|^2_{\dot{H}^m} + \|\rho\|^2_{\dot{H}^{m-\frac{1}{2}+\alpha}})\|f\|^2_{\dot{H}^{2+\frac{n}{2}}}. \tag{22}$$

for some $N = N(m,n,\alpha,\tau) \in \mathbb{N}$. Here, $\lesssim$ means up to a factor of $\overline{\rho}^a\underline{\rho}^{-b}$.

All the terms on the right hand side of (22) can be treated by interpolation between $H^{m+\alpha}$ and a lower order metric. A computation similar to (21), thus, readily implies (15).

Proof. In the course of this proof all inequalities are understood up to a factor of $\overline{\rho}^a \underline{\rho}^{-b}$, where $a, b > 0$ may change from line to line. We omit those factors for the sake of brevity.

Let us denote by $R(\rho, f)$ the right hand side of (22).

We denote for short $\partial_i = \partial$. To show the commutator is of lower order in f we need obtain bounds on $\|\mathcal{L}_{\partial^{m-l}\phi}\partial^l f\|_2^2$, for $l \in \{0, ..., m-1\}$ but first we expand $\partial^{m-l}\phi$ using Faa di Bruno's Formula.

Writing $\phi(x, y)$ as $\phi(x, x+z)$, we see that the derivatives fall only on the topological part of the kernel. Thus we have

$$\partial^{m-l}\phi(x, x+z) = |z|^{-(n+\alpha-\tau)} h(|z|) \partial^{m-l} d^{-\tau}(x, x+z), \tag{23}$$

$$\partial^{m-l} d^{-\tau}(x, x+z) = \partial^{m-l}\left[\int_{\Omega(x,x+z)} \rho(\xi) d\xi\right]^{-\tau/n} = \partial^{m-l}[d^n(x, x+z)]^{-\tau/n}. \tag{24}$$

Denoting $g = d^n$ and $h(g) = g^{-\tau/n}$, then using Faa di Bruno's Formula gives,

$$\partial^{m-l} d^{-\tau}(x, x+z) = \sum \frac{(m-l)!}{j_1!1!^{j_1} j_2!2!^{j_2} ... j_{m-l}!(m-l)!^{j_{m-l}}} h^{(j_1+...+j_{m-l})}(g) \prod_{k=1}^{m-l} \left(\partial^k g\right)^{j_k} \tag{25}$$

where the sum is over all $(m-l)$-tuples of integers $\mathbf{j} = (j_1, ..., j_{m-l})$ satisfying

$$1j_1 + 2j_2 + ... + (m-l)j_{m-l} = m - l \tag{26}$$

Any term in the commutator takes the form,

$$\mathcal{L}_{\partial^{m-l}\phi}\partial^l f(x) = \int_{\mathbb{T}^n} \frac{h(|z|)}{|z|^{n+\alpha-\tau}} \partial^{m-l}[d^{-\tau}(x, x+z)] \delta_z \partial^l f(x) dz. \tag{27}$$

Then any term in the derivative will take the form

$$I_{\mathbf{j}}[\partial^l f](x) := \int_{\mathbb{T}^n} \frac{h(|z|)}{|z|^{n+\alpha-\tau}} \frac{\prod_{k=1}^{m-l}\left(\int_{\Omega(x,x+z)} \partial^k \rho(\xi) d\xi\right)^{j_k}}{d^{\tau+|\mathbf{j}|n}(x, x+z)} \delta_z \partial^l f(x)\, \mathrm{d}z, \tag{28}$$

where $|\mathbf{j}| = \sum_{k=1}^{m-l} j_k$.

Case $0 < \alpha < 1$. First, we will look at $\int_{\Omega(x,x+z)} \partial^k \rho(\xi)\, \mathrm{d}\xi$. We estimate it with the use of the Hardy-Littlewood maximal function:

$$\left|\int_{\Omega(x,x+z)} \partial^k \rho(\xi)\, \mathrm{d}\xi\right| \leq |z|^n \frac{1}{|z|^n} \int_{\Omega(x,x+z)} |\partial^k \rho(\xi)|\, \mathrm{d}\xi \leq |z|^n M[\partial^k \rho](x),$$

where

$$M[g](x) = \sup_{r>0} \frac{1}{r^n} \int_{B_r(x)} |g(\xi)|\, \mathrm{d}\xi.$$

So,

$$|I_{\mathbf{j}}[\partial^l f](x)| \leq \prod_{k=1}^{m-l} (M[\partial^k \rho](x))^{j_k} \int_{\mathbb{T}^n} h(|z|) |\delta_z \partial^l f(x)| \frac{\mathrm{d}z}{|z|^{n+\alpha}}.$$

To estimate the L^2-norm of $I_{\mathbf{j}}[\partial^l f]$ we pick a set of conjugate exponents p_k, q such that

$$\sum_{k=1}^{m-l} \frac{2j_k}{p_k} + \frac{2}{q} = 1$$

and apply Hölder inequality

$$\|I_{\mathbf{j}}[\partial^l f]\|_2^2 \le \prod_{k=1}^{m-l} \|M[\partial^k \rho]\|_{p_k}^{2j_k} \left(\int_{\mathbb{T}^n} \left(\int_{\mathbb{T}^n} h(|z|) |\delta_z \partial^l f(x)| \frac{\mathrm{d}z}{|z|^{n+\alpha}} \right)^q \mathrm{d}x \right)^{\frac{2}{q}}$$

by the classical Hardy-Littlewood inequality,

$$\lesssim \prod_{k=1}^{m-l} \|\partial^k \rho\|_{p_k}^{2j_k} \left(\int_{\mathbb{T}^n} \left(\int_{\mathbb{T}^n} h(|z|) |\delta_z \partial^l f(x)| \frac{\mathrm{d}z}{|z|^{n+\alpha}} \right)^q \mathrm{d}x \right)^{\frac{2}{q}}$$

$$\lesssim \prod_{k=1}^{m-l} \|\partial^k \rho\|_{p_k}^{2j_k} \|\partial^l f\|_{W^{\alpha+\varepsilon,q}}^2$$

by the Sobolev embeddings,

$$\le \prod_{k=1}^{m-l} \|\rho\|_{\dot{H}^{k+n(\frac{1}{2}-\frac{1}{p_k})}}^{2j_k} \|f\|_{\dot{H}^{l+\alpha+\varepsilon+n(\frac{1}{2}-\frac{1}{q})}}^2.$$

Let us make the following choice of exponents: $p_k = \frac{2m}{k}$, $q = \frac{2m}{l}$. Then

$$\le \prod_{k=1}^{m-l} \|\rho\|_{\dot{H}^{k+\frac{n}{2}(1-\frac{k}{m})}}^{2j_k} \|f\|_{\dot{H}^{l+\alpha+\varepsilon+\frac{n}{2}(1-\frac{l}{m})}}^2.$$

Examining the regularity of the density norms obtained on the last line, we observe that for all $k = 1, \dots, m-1$ we have

$$k + \frac{n}{2}(1 - \frac{k}{m}) \le m - 1 + \alpha,$$

provided m is large enough. So, the whole density product becomes bounded by a lower order term for all $l = 1, \dots, m-1$:

$$\prod_{k=1}^{m-l} \|\rho\|_{\dot{H}^{k+\frac{n}{2}(1-\frac{k}{m})}}^{2j_k} \le \|\rho\|_{\dot{H}^{m-1+\alpha}}^N,$$

for some possibly large N (we take the liberty of changing N from line to line in the sequel). When $l = 0$, the product above still satisfies the same estimate for all multi-indeces $\mathbf{j}$ except one where $k = m$, which can only happen if $\mathbf{j} = (0, \dots, 0, 1)$ due to the restriction given by (26). In this case the density term reaches higher order norm $\|\rho\|_{\dot{H}^m}^2$.

As to the f-term, we have for $l \le m-2$

$$l + \alpha + \varepsilon + \frac{n}{2}(1 - \frac{l}{m}) < m - 1 + \alpha,$$

which contributes the lower order term. So, in this case, given the density estimates above, we have

$$\|I_{\mathbf{j}}[\partial^l f]\|_2^2 \le \|\rho\|_{\dot{H}^{m-1+\alpha}}^N \|f\|_{\dot{H}^{m-1+\alpha}}^2 + \|\rho\|_{\dot{H}^m}^2 \|f\|_{\dot{H}^{2+\frac{n}{2}}}^2 \le R(\rho, f), \quad l = 0, \dots, m-2.$$

For future reference let us record the estimate for the particular subcase when $l = 0$, $j_m = 0$:

$$\|I_{(j_1,\dots,j_{m-1},0)}[f]\|_2^2 \le \|\rho\|_{\dot{H}^{m-1+\alpha}}^N \|f\|_{\dot{H}^{2+\frac{n}{2}}}^2. \tag{29}$$

For the remaining case of $l = m-1$ we have $k = 1$, $j_1 = 1$. So, as far as regularity of f,

$$m - 1 + \alpha + \varepsilon + \frac{n}{2m} < m + \alpha - \frac{1}{2},$$

and hence,

$$\|I_{(1)}[\partial^{m-1} f]\|_2^2 \le \|\rho\|_{\dot{H}^{1+\frac{n}{2}}}^2 \|f\|_{\dot{H}^{m+\alpha-\frac{1}{2}}}^2 \le R(\rho, f).$$

The obtained estimates cover all the cases, so in summary we have obtained

$$\|\mathcal{L}_{\partial^{m-l}\phi}\partial^l f\|_2^2 \leq R(\rho, f). \tag{30}$$

which proves (22).

Case $1 \leq \alpha < 2$. This is a more involved case since for the application of the Gagliardo-Sobolevskii norm one has to include the next term in the Taylor finite difference of f: $\delta_z \partial^l f(x) - z \cdot \nabla \partial^l f(x)$. We therefore add and subtract that term in the formula for $I_{\mathbf{j}}[\partial^l f](x)$:

$$\begin{aligned} I_{\mathbf{j}}[\partial^l f](x) = &\int_{\mathbb{T}^n} \frac{h(|z|)}{|z|^{n+\alpha-\tau}} \frac{\prod_{k=1}^{m-l} \left(\int_{\Omega(x,x+z)} \partial^k \rho(\xi) d\xi\right)^{j_k}}{d^{\tau+|\mathbf{j}|n}(x,x+z)} [\delta_z \partial^l f(x) - z \cdot \nabla \partial^l f(x)] \,\mathrm{d}z \\ &+ \int_{\mathbb{T}^n} \frac{h(|z|)}{|z|^{n+\alpha-\tau}} \frac{\prod_{k=1}^{m-l} \left(\int_{\Omega(x,x+z)} \partial^k \rho(\xi) d\xi\right)^{j_k}}{d^{\tau+|\mathbf{j}|n}(x,x+z)} z \cdot \nabla \partial^l f(x) \,\mathrm{d}z \\ &:= I_{\mathbf{j},1}[\partial^l f](x) + I_{\mathbf{j},2}[\partial^l f](x). \end{aligned}$$

The estimate on $I_{\mathbf{j},1}[\partial^l f]$ goes in exact same way as in the previous case noting that the Gagliardo-Sobolevskii definition applies to smoothness exponents away from the interger values, $2 > \alpha + \varepsilon > 1$. In $I_{\mathbf{j},2}[\partial^l f]$ we symmetrize first

$$\begin{aligned} I_{\mathbf{j},2}[\partial^l f](x) &= \nabla \partial^l f(x) \cdot \int_{\mathbb{T}^n} \frac{h(|z|)}{|z|^{n+\alpha-\tau}} \left[\frac{\prod_{k=1}^{m-l} \left(\int_{\Omega(x,x+z)} \partial^k \rho(\xi) d\xi\right)^{j_k}}{d^{\tau+|\mathbf{j}|n}(x,x+z)} - \frac{\prod_{k=1}^{m-l} \left(\int_{\Omega(x,x-z)} \partial^k \rho(\xi) d\xi\right)^{j_k}}{d^{\tau+|\mathbf{j}|n}(x,x-z)} \right] z \,\mathrm{d}z \\ &= \nabla \partial^l f(x) \cdot \int_{\mathbb{T}^n} \frac{h(|z|)}{|z|^{n+\alpha-\tau}} \prod_{k=1}^{m-l} \left(\int_{\Omega(x,x+z)} \partial^k \rho(\xi) d\xi \right)^{j_k} \left[\frac{d^{\tau+|\mathbf{j}|n}(x,x-z) - d^{\tau+|\mathbf{j}|n}(x,x+z)}{d^{\tau+|\mathbf{j}|n}(x,x+z) d^{\tau+|\mathbf{j}|n}(x,x-z)} \right] z \,\mathrm{d}z \\ &+ \nabla \partial^l f(x) \cdot \int_{\mathbb{T}^n} \frac{h(|z|)}{|z|^{n+\alpha-\tau} d^{\tau+|\mathbf{j}|n}(x,x-z)} \left[\prod_{k=1}^{m-l} \left(\int_{\Omega(x,x+z)} \partial^k \rho(\xi) d\xi \right)^{j_k} - \right. \\ &\qquad \left. - \prod_{k=1}^{m-l} \left(\int_{\Omega(x,x-z)} \partial^k \rho(\xi) d\xi \right)^{j_k} \right] z \,\mathrm{d}z \\ &= I_{\mathbf{j},2,1}[\partial^l f](x) + I_{\mathbf{j},2,2}[\partial^l f](x). \end{aligned}$$

By a straightforward computation,

$$|d^{\tau+|\mathbf{j}|n}(x,x-z) - d^{\tau+|\mathbf{j}|n}(x,x+z)| \leq |\nabla \rho|_\infty |z|^{\tau+|\mathbf{j}|n+1}.$$

With this at hand we proceed to estimate $I_{\mathbf{j},2,1}[\partial^l f](x)$:

$$|I_{\mathbf{j},2,1}[\partial^l f](x)| \leq |\nabla \partial^l f(x)| \prod_{k=1}^{m-l} (M[\partial^k \rho](x))^{j_k} \int_{\mathbb{T}^n} h(|z|) \frac{\mathrm{d}z}{|z|^{n+\alpha-2}}.$$

Since $\alpha < 2$, the integral converges. Thus,

$$\|I_{\mathbf{j},2,1}[\partial^l f]\|_2^2 \leq \prod_{k=1}^{m-l} \|\partial^k \rho\|_{p_k}^{2j_k} \|\partial^{l+1} f\|_q^2 \leq \prod_{k=1}^{m-l} \|\rho\|^{2j_k}_{\dot{H}^{k+n(\frac{1}{2}-\frac{1}{p_k})}} \|f\|^2_{\dot{H}^{l+1+n(\frac{1}{2}-\frac{1}{q})}}.$$

Since $l+1 < l+\alpha+\varepsilon$ by further increasing the smoothness of f the estimate blends with the previous case.

It remains to estimate $I_{\mathbf{j},2,2}[\partial^l f](x)$. To do this we must estimate

$$\prod_{k=1}^{m-l}\left(\int_{\Omega(x,x+z)}\partial^k\rho(\xi)\,\mathrm{d}\xi\right)^{j_k}-\prod_{k=1}^{m-l}\left(\int_{\Omega(x,x-z)}\partial^k\rho(\xi)\,\mathrm{d}\xi\right)^{j_k}.$$

We can rewrite such a difference as

$$\prod_{k=1}^{m-l}a_k^{j_k}-\prod_{k=1}^{m-l}b_k^{j_k}=\sum_{k=1}^{m-l}a_1^{j_1}\cdots a_{k-1}^{j_{k-1}}(a_k^{j_k}-b_k^{j_k})b_{k+1}^{j_{k+1}}\cdots b_{m-l}^{j_{m-l}},$$

and furthermore,

$$a_k^{j_k}-b_k^{j_k}=(a_k-b_k)(a_k^{j_k-1}+a_k^{j_k-2}b_k+\cdots+a_kb_k^{j_k-2}+b_k^{j_k-1}).$$

We will focus on the main difference a_k-b_k, while estimating all other terms with the maximal function like before. We write, letting $s=\alpha-1+\varepsilon<1$,

$$\begin{aligned}&\int_{\Omega(x,x+z)}\partial^k\rho(\xi)\,\mathrm{d}\xi-\int_{\Omega(x,x-z)}\partial^k\rho(\xi)d\xi=\int_{\Omega(0,z)}\partial^k\rho(x+\xi)-\partial^k\rho(x-\xi)\,\mathrm{d}\xi\\&=\int_{\Omega(0,z)}\frac{\partial^k\rho(x+\xi)-\partial^k\rho(x-\xi)}{|\xi|^{\frac{n}{p_k}+s}}|\xi|^{\frac{n}{p_k}+s}d\xi\lesssim\left(\int_{\Omega(0,z)}\frac{|\partial^k\rho(x+\xi)-\partial^k\rho(x-\xi)|^{p_k}}{|\xi|^{n+sp_k}}d\xi\right)^{1/p_k}|z|^{n+s}\\&:=(D_{s,p_k}\partial^k\rho(x))^{1/p_k}|z|^{n+s},\end{aligned}$$

where $\int D_{s,p}g(x)dx=\|g\|_{W^{s,p}}^p$. Then we can estimate the difference in the products by

$$\begin{aligned}&\prod_{k=1}^{m-l}\left(\int_{\Omega(x,x+z)}\partial^k\rho(\xi)\,\mathrm{d}\xi\right)^{j_k}-\prod_{k=1}^{m-l}\left(\int_{\Omega(x,x-z)}\partial^k\rho(\xi)\,\mathrm{d}\xi\right)^{j_k}\\&\quad\lesssim\sum_{k=1}^{m-l}\prod_{\substack{i=1\\i\neq k}}^{m-l}(M[\partial^i\rho](x))^{j_i}(M[\partial^k\rho](x))^{j_k-1}(D_{s,p_k}\partial^k\rho(x))^{1/p_k}|z|^{|\mathbf{j}|n+s}.\end{aligned}\tag{31}$$

Therefore, returning to $I_{\mathbf{j},2,2}[\partial^l f]$, we estimate in L^2, using the same Holder conjugates as before,

$$\begin{aligned}&\|I_{\mathbf{j},2,2}[\partial^l f]\|_2^2\\&\lesssim\int_{\mathbb{T}^n}|\nabla\partial^l f(x)|^2\left(\int_{\mathbb{T}^n}\frac{h(|z|)}{|z|^{n+\alpha-1-s}}\,\mathrm{d}z\right)^2\times\\&\qquad\times\left(\sum_{k=1}^{m-l}\prod_{\substack{i=1\\i\neq k}}^{m-l}(M[\partial^i\rho](x))^{j_i}(M[\partial^k\rho](x))^{j_k-1}(D_{s,p_k}\partial^k\rho(x))^{1/p_k}\right)^2\mathrm{d}x\\&\lesssim\|\partial^{l+1}f\|_q^2\left(\prod_{\substack{i=1\\i\neq k}}^{m-l}\|\partial^i\rho\|_{p_i}^{2j_i}\right)\|\partial^k\rho\|_{p_k}^{2(j_k-1)}\|\partial^k\rho\|_{W^{s,p_k}}^2\\&\le\|f\|^2_{\dot H^{l+1+n(\frac12-\frac1q)}}\prod_{\substack{i=1\\i\neq k}}^{m-l}\|\rho\|^{2j_i}_{\dot H^{i+n(\frac12-\frac1{p_k})}}\|\rho\|^{2(j_k-1)}_{\dot H^{k+n(\frac12-\frac1{p_k})}}\|\rho\|^2_{\dot H^{k+s+n(\frac12-\frac1{p_k})}}\\&=\|f\|^2_{\dot H^{l+1+\frac n2(1-\frac lm)}}\prod_{\substack{i=1\\i\neq k}}^{m-l}\|\rho\|^{2j_i}_{\dot H^{i+\frac n2(1-\frac im)}}\|\rho\|^{2(j_k-1)}_{\dot H^{k+\frac n2(1-\frac km)}}\|\rho\|^2_{\dot H^{k+s+\frac n2(1-\frac km)}}.\end{aligned}\tag{32}$$

As before let us examine regularity of the density first. In any case when the top j-index vanishes, $j_m = 0$, so that $i, k \in \{1, ..., m-1\}$ we have

$$i + \frac{n}{2}(1 - \frac{i}{m}) \leq m - 1 + \alpha$$
$$k + s + \frac{n}{2}(1 - \frac{k}{m}) \leq m - 1 + \alpha,$$

if m is large enough. So, in this case the entire product of densities is controlled by the lower order norm:

$$\prod_{\substack{i=1 \\ i \neq k}}^{m-l} \|\rho\|^{2j_i}_{\dot{H}^{i+\frac{n}{2}(1-\frac{i}{m})}} \|\rho\|^{2(j_k-1)}_{\dot{H}^{k+\frac{n}{2}(1-\frac{k}{m})}} \|\rho\|^2_{\dot{H}^{k+s+\frac{n}{2}(1-\frac{k}{m})}} \leq \|\rho\|^N_{\dot{H}^{m-1+\alpha}}.$$

This applies in particular for all $l = 1, \ldots, m-1$ and even in the case $l = 0$ with $\mathbf{j} = (j_1, ..., j_{m-1}, 0)$. Note that this also extends (29) to the entire range of α's, $0 < \alpha < 2$.

When $k = m$ which is only attainable at $l = 0$, $j_m = 1$ case, we are off by ε: the product collapses to only one norm $\|\rho\|^2_{\dot{H}^{m-1+\alpha+\varepsilon}}$ while the f-term is of low order:

$$\|I_{\mathbf{j},2,2}[f]\|_2^2 \leq \|\rho\|^2_{\dot{H}^{m-1+\alpha+\varepsilon}} \|f\|^2_{\dot{H}^{1+\frac{n}{2}}} \leq \|\rho\|^2_{\dot{H}^{m-\frac{1}{2}+\alpha}} \|f\|^2_{\dot{H}^{1+\frac{n}{2}}} \leq R(\rho, f).$$

Combined with the other $\mathbf{j}$-indeces, the case $l = 0$ altogether gives the estimate above.

Next, for $l = 1, \ldots, m-2$,

$$l + 1 + \frac{n}{2}(1 - \frac{l}{m}) \leq m - 1 + \alpha.$$

So,

$$\|I_{\mathbf{j},2,2}[f]\|_2^2 \leq \|f\|^2_{\dot{H}^{m-1+\alpha}} \|\rho\|^2_{\dot{H}^{m-1+\alpha}} \leq R(\rho, f).$$

For the only remaining case $l = m-1$, the regularity exponent for f is

$$m + \frac{n}{2m} \leq m + \frac{\alpha}{2},$$

while the density product is of course of lower than $m - 1 + \alpha$ order as elucidated above. So, we arrive at

$$\|I_{(1),2,2}[\partial^{m-1} f]\|_2^2 \lesssim \|\rho\|^N_{\dot{H}^{m-1+\alpha}} \|f\|^2_{\dot{H}^{m+\frac{\alpha}{2}}} \leq R(\rho, f).$$

□

4. A priori estimates on the velocity equation

The goal of this section is to establish a priori bound

$$\partial_t \|u\|^2_{\dot{H}^{m+1}} \leq C Y_m^N. \tag{33}$$

Let us rewrite the velocity equation as

$$u_t + u \cdot \nabla u = \mathcal{C}_\phi(u, \rho),$$
$$\mathcal{C}_\phi(u, \rho)(x) = \int_{\mathbb{T}^n} \phi(x, x+z) \delta_z u(x) \rho(x+z) \, \mathrm{d}z = \mathcal{L}_\phi(u\rho) - u \mathcal{L}_\phi \rho.$$

Let us apply ∂^{m+1} and test with $\partial^{m+1} u$. We have (dropping integrals signs)

$$\partial_t \|u\|^2_{\dot{H}^{m+1}} = -\partial^{m+1}(u \cdot \nabla u) \cdot \partial^{m+1} u + \partial^{m+1} \mathcal{C}_\phi(u, \rho) \cdot \partial^{m+1} u.$$

The transport term is estimated using the classical commutator estimate

$$\partial^{m+1}(u \cdot \nabla u) \cdot \partial^{m+1} u = u \cdot \nabla(\partial^{m+1} u) \cdot \partial^{m+1} u + [\partial^{m+1}, u] \nabla u \cdot \partial^{m+1} u$$

Then

$$u \cdot \nabla(\partial^{m+1} u) \cdot \partial^{m+1} u = -\frac{1}{2}(\nabla \cdot u)|\partial^{m+1} u|^2 \leq |\nabla u|_\infty \|u\|^2_{\dot{H}^{m+1}},$$

and using (13) for $f = u$, $g = \nabla u$, we obtain

$$|[\partial^{m+1}, u]\nabla u \cdot \partial^{m+1} u| \leq |\nabla u|_\infty \|u\|^2_{\dot{H}^{m+1}}.$$

Thus,

$$\partial_t \|u\|^2_{\dot{H}^{m+1}} \leq \|u\|^3_{\dot{H}^{m+1}} + \partial^{m+1}\mathcal{C}_\phi(u,\rho)\cdot\partial^{m+1}u.$$

In the rest of the argument we focus on estimating the commutator term. So, we expand by the product rule

$$\partial^{m+1}\mathcal{C}_\phi(u,\rho) = \sum_{k=k_1+k_2=0}^{m+1} \frac{(m+1)!}{k_1!k_2!(m+1-k)!}\mathcal{C}_{\partial^{m+1-k}\phi}(\partial^{k_1}u, \partial^{k_2}\rho). \tag{34}$$

Various term in this expansion will be estimated differently. There is however one end-point term which provides necessary dissipation :

$$\mathcal{C}_\phi(\partial^{m+1}u,\rho)\cdot\partial^{m+1}u \leq -\frac{\rho}{|\rho|_\infty^{\tau/n}}\|u\|^2_{\dot{H}^{m+1+\frac{\alpha}{2}}}. \tag{35}$$

Note that this particular term eventually guarantees inclusion of the velocity in class $L^2([0,T_0);H^{m+1+\frac{\alpha}{2}})$ as claimed in the statement of the main result.

Case $k = 1,\dots,m$. The bulk of the terms can be estimated simultaneously. Those correspond to the range $k = 1,\dots,m$. We start by the standard symmetrization:

$$\begin{aligned}
\int_{\mathbb{T}^n} \mathcal{C}_{\partial^{m+1-k}\phi}(\partial^{k_1}u,\partial^{k_2}\rho)\cdot\partial^{m+1}u\,\mathrm{d}x &= \int_{\mathbb{T}^{2n}} \delta_z\partial^{k_1}u(x)\partial^{k_2}\rho(x+z)\partial^{m+1}u(x)\partial^{m+1-k}\phi(x,x+z)\,\mathrm{d}z\,\mathrm{d}x\\
&= \frac{1}{2}\int_{\mathbb{T}^{2n}} \delta_z\partial^{k_1}u(x)\delta_z\partial^{k_2}\rho(x)\partial^{m+1}u(x)\partial^{m+1-k}\phi(x,x+z)\,\mathrm{d}z\,\mathrm{d}x\\
&+ \frac{1}{2}\int_{\mathbb{T}^{2n}} \delta_z\partial^{k_1}u(x)\partial^{k_2}\rho(x)\delta_z\partial^{m+1}u(x)\partial^{m+1-k}\phi(x,x+z)\,\mathrm{d}z\,\mathrm{d}x\\
&= J_1 + J_2.
\end{aligned}$$

In the Faa di Bruno expansion of the kernel $\partial^{m+1-k}\phi(x,x+z)$ we obtain a set of terms, again, labeled by $\mathbf{j} = (j_1,\dots,j_{m+1-k})$ with

$$1j_1 + \dots + (m+1-k)j_{m+1-k} = m+1-k.$$

With the use of the Hardy-Littlewood maximal function as before we obtain

$$J_1 \leq \sum_{\mathbf{j}} \int_{\mathbb{T}^{2n}} |\delta_z\partial^{k_1}u(x)\delta_z\partial^{k_2}\rho(x)\partial^{m+1}u(x)| \prod_{l=1}^{m+1-k}(M[\partial^l\rho](x))^{j_l}\frac{\mathrm{d}z}{|z|^{n+\alpha}}\,\mathrm{d}x$$

We pick a set of exponents $q_i = \frac{2(m+1)}{k_i}$, $p_l = \frac{2(m+1)}{l}$:

$$\frac{1}{2} + \frac{1}{q_1} + \frac{1}{q_2} + \sum_{l=1}^{m+1-k}\frac{j_l}{p_l} = 1.$$

We have

$$\begin{aligned}
J_1 &\leq \sum_{\mathbf{j}} \int_{\mathbb{T}^{2n}} \frac{|\delta_z\partial^{k_1}u(x)|}{|z|^{\frac{n}{q_1}+\frac{\alpha k}{m}+\varepsilon}} \frac{|\delta_z\partial^{k_2}\rho(x)|}{|z|^{\frac{n}{q_2}+\frac{\alpha(m-k)}{m}}} \frac{|\partial^{m+1}u(x)|}{|z|^{\frac{n}{2}-\varepsilon}} \prod_{l=1}^{m+1-k}\frac{(M[\partial^l\rho](x))^{j_l}}{|z|^{\frac{nj_l}{p_l}}}\,\mathrm{d}z\,\mathrm{d}x\\
&\leq \|u\|_{\dot{H}^{k_1+\frac{\alpha k}{m}+\frac{n}{2}\frac{m+1-k_1}{m+1}+\varepsilon}} \|\rho\|_{\dot{H}^{k_2+\frac{\alpha(m-k)}{m}+\frac{n}{2}\frac{m+1-k_2}{m+1}}} \|u\|_{\dot{H}^{m+1}} \prod_{l=1}^{m+1-k}\|\rho\|^{j_l}_{\dot{H}^{l+\frac{n}{2}\frac{m+1-l}{m+1}}}.
\end{aligned}$$

Provided m is large enough and ε is small enough we have

$$\begin{aligned} u:&\quad k_1+\frac{\alpha k}{m}+\frac{n}{2}\frac{m+1-k_1}{m+1}+\varepsilon<m+1+\frac{\alpha}{2},\\ \rho:&\quad k_2+\frac{\alpha(m-k)}{m}+\frac{n}{2}\frac{m+1-k_2}{m+1}<m+\alpha\\ \rho:&\quad l+\frac{n}{2}\frac{m+1-l}{m+1}<m+\alpha, \end{aligned}$$

for all $k_1+k_2=k$, $l=1,...,m+1-k$, $k=1,...,m$. Thus,

$$J_1\leq Y_m^N+\varepsilon\|u\|^2_{\dot H^{m+1+\frac{\alpha}{2}}}$$

(N will change from line to line). Note that the last term can be hidden into dissipation (35). Moving on to J_2,

$$\begin{aligned} J_2&\leq\sum_{\mathbf{j}}\int_{\mathbb{T}^{2n}}\frac{|\delta_z\partial^{k_1}u(x)|}{|z|^{\frac{n}{q_1}+\frac{\alpha}{2}+\varepsilon}}\frac{|\partial^{k_2}\rho(x)|}{|z|^{\frac{n}{q_2}-\varepsilon}}\frac{|\delta_z\partial^{m+1}u(x)|}{|z|^{\frac{n+\alpha}{2}}}\prod_{l=1}^{m+1-k}\frac{(M[\partial^l\rho](x))^{j_l}}{|z|^{\frac{nj_l}{p_l}}}\,\mathrm{d}z\,\mathrm{d}x\\ &\leq\|u\|_{\dot H^{k_1+\frac{\alpha k}{m}+\frac{n}{2}\frac{m+1-k_1}{m+1}+\varepsilon}}\|\rho\|_{\dot H^{k_2+\frac{\alpha(m-k)}{m}+\frac{n}{2}\frac{m+1-k_2}{m+1}}}\|u\|_{\dot H^{m+1}}\prod_{l=1}^{m+1-k}\|\rho\|^{j_l}_{\dot H^{l+\frac{n}{2}\frac{m+1-l}{m+1}}}. \end{aligned}$$

We now examine the remaining end-point cases.

Case $k=0$. Here we deal with only one term

$$\mathcal{C}_{\partial^{m+1}\phi}(u,\rho)=\mathcal{L}_{\partial^{(m+1)}\phi}[u\rho]-u\mathcal{L}_{\partial^{(m+1)}\phi}\rho.$$

In the Faa di Bruno expansion of the kernel, we single out again the case $\mathbf{j}=(0,...,0,1)$ from the rest, because in the rest of the cases $\mathbf{j}=(j_1,...,j_m,0)$ we do not have to use the commutator structure at all. Instead we have by (29), (noting that $m\to m+1$) and the control bound (16),

$$\begin{aligned} \int I_{\mathbf{j}}[u\rho]\cdot\partial^{m+1}u\,\mathrm{d}x&\leq\|I_{\mathbf{j}}[u\rho]\|_2^2\|u\|_{\dot H^{m+1}}\leq p_N(\|\rho\|_{\dot H^{m+\alpha}})\|u\rho\|^2_{\dot H^{2+\frac{n}{2}}}\|u\|_{\dot H^{m+1}}\\ &\lesssim 1+\|\rho\|^N_{\dot H^{m+\alpha}}+\|u\|^8_{\dot H^{2+\frac{n}{2}}}+\|u\|^2_{\dot H^{m+1}}\leq Y_m^N. \end{aligned}$$

And similarly,

$$\int uI_{\mathbf{j}}[\rho]\cdot\partial^{m+1}u\,\mathrm{d}x\leq|u|_\infty\|I_{\mathbf{j}}[\rho]\|_2^2\|u\|_{\dot H^{m+1}}\leq Y_m^N.$$

Let us consider now the more involved term corresponding to $\mathbf{j}=(0,...,0,1)$. In this case

$$\int(I_{\mathbf{j}}[u\rho]-uI_{\mathbf{j}}[\rho])\cdot\partial^{m+1}u\,\mathrm{d}x=\int_{\mathbb{T}^{2n}}\frac{h(|z|)\int_{\Omega(x,x+z)}\partial^{m+1}\rho(\xi)\,\mathrm{d}\xi}{|z|^{n+\alpha-\tau}d^{\tau+n}(x,x+z)}\delta_zu(x)\rho(x+z)\partial^{m+1}u(x)\,\mathrm{d}z\,\mathrm{d}x$$

after symmetrization,

$$\begin{aligned} &=\frac12\int_{\mathbb{T}^{2n}}\frac{h(|z|)\int_{\Omega(x,x+z)}\partial^{m+1}\rho(\xi)\,\mathrm{d}\xi}{|z|^{n+\alpha-\tau}d^{\tau+n}(x,x+z)}\delta_zu(x)\delta_z\rho(x)\partial^{m+1}u(x)\,\mathrm{d}z\,\mathrm{d}x\\ &+\frac12\int_{\mathbb{T}^{2n}}\frac{h(|z|)\int_{\Omega(x,x+z)}\partial^{m+1}\rho(\xi)\,\mathrm{d}\xi}{|z|^{n+\alpha-\tau}d^{\tau+n}(x,x+z)}\delta_zu(x)\rho(x)\delta_z\partial^{m+1}u(x)\,\mathrm{d}z\,\mathrm{d}x. \end{aligned}$$

The highest density term suffers a derivative overload and needs to be reduced:

$$\int_{\Omega(x,x+z)} \partial^{m+1}\rho(\xi)\,\mathrm{d}\xi = \int_{\partial\Omega(x,x+z)} \partial^m\rho(\xi)\nu_\xi\,\mathrm{d}\xi = \int_{\partial\Omega(0,z)} \partial^m\rho(x+\xi)\nu_\xi\,\mathrm{d}\xi$$
$$= |z|^{n-1}\int_{\partial\Omega(0,\mathbf{e}_1)} \partial^m\rho\left(x+|z|U_z\theta\right)\nu_\theta\,\mathrm{d}\theta$$

where U_z is the orthogonal transformation mapping $\mathbf{e}_1$ to $\hat{z}$,

$$= |z|^{n-1}\int_{\partial\Omega(0,\mathbf{e}_1)} \left[\partial^m\rho\left(x+|z|U_z\theta\right) - \partial^m\rho(x)\right]\nu_\theta\,\mathrm{d}\theta.$$

We recover one power of z by $|\delta_z u| \leq |z|\|\nabla u\|_\infty$ and in the first intergal $|\delta_z\rho| \leq |z|\|\nabla\rho\|_\infty$. Putting together we estimate the integrals by

$$\leq \|\nabla u\|_\infty\|\nabla\rho\|_\infty\int_{\partial\Omega(0,\mathbf{e}_1)}\iint_{\mathbb{T}^{2n}} \frac{h(|z|)\left|\partial^m\rho\left(x+|z|U_z\theta\right)-\partial^m\rho(x)\right|}{|z|^{\frac{n}{2}+\alpha-\frac{1}{2}}}\frac{|\partial^{m+1}u(x)|}{|z|^{\frac{n}{2}-\frac{1}{2}}}\,\mathrm{d}z\,\mathrm{d}x\,\mathrm{d}\theta$$
$$+\|\nabla u\|_\infty\|\rho\|_\infty\int_{\partial\Omega(0,\mathbf{e}_1)}\iint_{\mathbb{T}^{2n}} \frac{h(|z|)\left|\partial^m\rho\left(x+|z|U_z\theta\right)-\partial^m\rho(x)\right|}{|z|^{\frac{n}{2}+\frac{\alpha}{2}}}\frac{|\delta_z\partial^{m+1}u(x)|}{|z|^{\frac{n}{2}+\frac{\alpha}{2}}}\,\mathrm{d}z\,\mathrm{d}x\,\mathrm{d}\theta$$
$$\leq \|\nabla u\|_\infty\|\nabla\rho\|_\infty\|u\|_{\dot{H}^{m+1}} + \|\nabla u\|_\infty\|\rho\|_\infty\|u\|_{\dot{H}^{m+1+\frac{\alpha}{2}}}\left(D_{2\alpha-1}(\partial^m\rho)+D_\alpha(\partial^m\rho)\right),$$

where

$$D_s(g) = \int_{\mathbb{T}^{2n}} \frac{h(z)\left|g\left(x+|z|U_z\theta\right)-g(x)\right|^2}{|z|^{n+s}}\,\mathrm{d}z\,\mathrm{d}x.$$

By Lemma 7.1 this expression is bounded by the $H^{\frac{s}{2}}$ norm. Thus,

$$\int (I_{\mathbf{j}}[u\rho] - uI_{\mathbf{j}}[\rho])\cdot\partial^{m+1}u\,\mathrm{d}x \leq \varepsilon\|u\|^2_{\dot{H}^{m+1+\frac{\alpha}{2}}} + Y^N_m. \tag{36}$$

Case $k = m+1$. In this case the kernel gets no derivatives, however, we deal with a total of m terms $\mathcal{C}_\phi(\partial^l u, \partial^{m+1-l}\rho)$ for $l = 0,\dots,m$ (note that the case $l = m+1$ yields the dissipative term which has been considered already). Let us consider first the end-point case of $l = 0$. In this case the density suffers a derivative overload. We apply the following "easing" technique:

$$\int_{\mathbb{T}^n} \mathcal{C}_\phi(u,\partial^{m+1}\rho)\cdot\partial^{m+1}u\,\mathrm{d}x = \iint_{\mathbb{T}^{2n}} \phi(x,x+z)\delta_z u(x)\partial^{m+1}\rho(x+z)\partial^{m+1}u(x)\,\mathrm{d}z\,\mathrm{d}x.$$

We observe that

$$\partial^{m+1}\rho(x+z) = \partial_z\partial_x^m\rho(x+z) = \partial_z(\partial_x^m\rho(x+z)-\partial_x^m\rho(x)) = \partial_z\delta_z\partial^m\rho(x).$$

Now we integrate by parts in z:

$$\int_{\mathbb{T}^n} \mathcal{C}_\phi(u,\partial^{m+1}\rho)\cdot\partial^{m+1}u\,\mathrm{d}x = \iint_{\mathbb{T}^{2n}} \partial_z\phi(x,x+z)\delta_z u(x)\delta_z\partial^m\rho(x)\partial^{m+1}u(x)\,\mathrm{d}z\,\mathrm{d}x +$$
$$+\iint_{\mathbb{T}^{2n}} \phi(x,x+z)\partial u(x+z)\delta_z\partial^m\rho(x)\partial^{m+1}u(x)\,\mathrm{d}z\,\mathrm{d}x := J_1 + J_2.$$

Let us examine J_2 first. By symmetrization,

$$J_2 = \iint_{\mathbb{T}^{2n}} \delta_z\partial u(x)\delta_z\partial^m\rho(x)\partial^{m+1}u(x)\phi\,\mathrm{d}z\,\mathrm{d}x - \iint_{\mathbb{T}^{2n}} \partial u(x)\delta_z\partial^m\rho(x)\delta_z\partial^{m+1}u(x)\phi\,\mathrm{d}z\,\mathrm{d}x := J_{2,1}+J_{2,2}$$

$$J_{2,1} \leq |\nabla^2 u|_\infty \iint_{\mathbb{T}^{2n}} |\delta_z\partial^m\rho(x)|\frac{\mathrm{d}z}{|z|^{n+\alpha-1}}|\partial^{m+1}u(x)|\,\mathrm{d}x \leq |\nabla^2 u|_\infty\|\rho\|_{\dot{H}^{m-1+\alpha+\varepsilon}}\|u\|_{\dot{H}^{m+1}} \leq Y^N_m,$$

$$J_{2,2} \leq |\nabla u|_\infty\|\rho\|_{\dot{H}^{m+\frac{\alpha}{2}}}\|u\|_{\dot{H}^{m+1+\frac{\alpha}{2}}} \leq \varepsilon\|u\|^2_{\dot{H}^{m+1+\frac{\alpha}{2}}} + Y^N_m.$$

As to J_1, let is first observe that $\partial_z \phi(x,x+z) = \psi(x,x+z)$ is antisymmetric, $\psi(x,y) = -\psi(y,x)$. Then, by symmetrization we have

$$J_1 = \frac{1}{2}\iint_{\mathbb{T}^{2n}} \partial_z \phi(x,x+z)\delta_z u(x)\delta_z \partial^m \rho(x)\delta_z \partial^{m+1} u(x)\,\mathrm{d}z\,\mathrm{d}x.$$

Since

$$\partial_z \phi(x,x+z) = -(n+\alpha-\tau)h(z)\frac{z_i}{|z|^{n+\alpha+2-\tau}d^\tau} + h(z)\frac{\partial_z \int_{\Omega(x,x+z)} \rho(\xi)\,\mathrm{d}\xi}{|z|^{n+\alpha-\tau}d^{\tau+n}} + \frac{\partial_z h(z)}{|z|^{n+\alpha-\tau}d^\tau}$$

and noticing that

$$\left|\partial_z \int_{\Omega(x,x+z)} \rho(\xi)\,\mathrm{d}\xi\right| \leq |\rho|_\infty |z|^{n-1},$$

we can see that this kernel is of order $|z|^{-n-\alpha-1}$ up to the usual quantities bounded by Y_m^N. The one derivative loss is compensated by $|\delta_z u(x)| \leq |z||\nabla u|_\infty$. With this at hand we estimate J_1:

$$J_1 \leq Y_m^N |\nabla u|_\infty \|u\|_{\dot{H}^{m+1+\frac{\alpha}{2}}} \|\rho\|_{\dot{H}^{m+\frac{\alpha}{2}}} \leq \varepsilon \|u\|^2_{\dot{H}^{m+1+\frac{\alpha}{2}}} + Y_m^N.$$

Let us now examine the rest of the commutators $\mathcal{C}_\phi(\partial^l u, \partial^{m+1-l}\rho)$ for $l = 1,\dots,m$. After symmetrization we obtain

$$\begin{aligned}\int_{\mathbb{T}^n} \mathcal{C}_\phi(\partial^l u, \partial^{m+1-l}\rho)\cdot \partial^{m+1} u\,\mathrm{d}x = \frac{1}{2}\int_{\mathbb{T}^{2n}} \delta_z \partial^l u(x)\delta_z \partial^{m+1-l}\rho(x)\partial^{m+1}u(x)\phi\,\mathrm{d}z\,\mathrm{d}x + \\ + \int_{\mathbb{T}^{2n}} \delta_z \partial^l u(x)\partial^{m+1-l}\rho(x)\delta_z\partial^{m+1}u(x)\phi\,\mathrm{d}z\,\mathrm{d}x := J_1 + J_2.\end{aligned}$$

For J_1 we distribute the singularity of the kernel among the three terms

$$J_1 \leq \int_{\mathbb{T}^{2n}} \frac{|\delta_z \partial^l u(x)|}{|z|^{\frac{n}{p}+\frac{2\alpha}{q}+\varepsilon}} \frac{|\delta_z \partial^{m+1-l}\rho(x)|}{|z|^{\frac{n}{q}+\frac{2\alpha}{p}}} \frac{|\partial^{m+1}u(x)|}{|z|^{\frac{n}{2}-\varepsilon}}\,\mathrm{d}z\,\mathrm{d}x,$$

using a Hölder triple

$$\frac{1}{p}+\frac{1}{q}+\frac{1}{2} = 1.$$

We have

$$J_1 \leq \|u\|_{\dot{W}^{l+\varepsilon+\frac{2\alpha}{q},p}} \|\rho\|_{\dot{W}^{m+1-l+\frac{2\alpha}{p},q}} \|u\|_{\dot{H}^{m+1}} \leq \|u\|_{\dot{H}^{l+\varepsilon+\frac{2\alpha}{q}+n(\frac{1}{2}-\frac{1}{p})}} \|\rho\|_{\dot{H}^{m+1-l+\frac{2\alpha}{p}+n(\frac{1}{2}-\frac{1}{q})}} \|u\|_{\dot{H}^{m+1}}.$$

Choosing $p = 2\frac{m+1}{l}$ and $q = 2\frac{m+1}{m+1-l}$ we verify for all $l = 1,\dots,m$

$$\begin{aligned} u: &\quad l+\varepsilon+\frac{(m+1-l)\alpha}{m+1} + \frac{n(m+1-l)}{2(m+1)} < m+1+\frac{\alpha}{2}, \\ \rho: &\quad m+1-l+\frac{l\alpha}{m+1}+\frac{ln}{2(m+1)} < m+\alpha. \end{aligned}$$

We conclude as before

$$J_1 \leq \varepsilon\|u\|^2_{\dot{H}^{m+1+\frac{\alpha}{2}}} + Y_m^N.$$

For J_2 the computation is similar:

$$\begin{aligned} J_2 \leq \int_{\mathbb{T}^{2n}} \frac{|\delta_z \partial^l u(x)|}{|z|^{\frac{n}{p}+\varepsilon+\frac{\alpha}{2}}} \frac{|\partial^{m+1-l}\rho(x)|}{|z|^{\frac{n}{q}-\varepsilon}} \frac{|\delta_z \partial^{m+1}u(x)|}{|z|^{\frac{n}{2}+\frac{\alpha}{2}}}\,\mathrm{d}z\,\mathrm{d}x \leq \|u\|_{\dot{W}^{l+\varepsilon+\frac{\alpha}{2},p}} \|\rho\|_{\dot{W}^{m+1-l,q}} \\ \leq \|u\|_{\dot{H}^{l+\varepsilon+\frac{\alpha}{2}+n(\frac{1}{2}-\frac{1}{p})}} \|\rho\|_{\dot{H}^{m+1-l+n(\frac{1}{2}-\frac{1}{q})}} \|u\|_{\dot{H}^{m+1+\frac{\alpha}{2}}} \leq \varepsilon\|u\|^2_{\dot{H}^{m+1+\frac{\alpha}{2}}} + Y_m^N, \end{aligned}$$

where the last line follows by the same choice of p, q and noting that

$$u: \quad l + \varepsilon + \frac{\alpha}{2} + \frac{n(m+1-l)}{2(m+1)} \leq m+1$$
$$\rho: \quad m+1-l + \frac{ln}{2(m+1)} \leq m+\alpha,$$

for all $l = 1, \ldots, m$.

5. A priori estimates on the e-equation

Consider the quantity

$$e = \nabla \cdot u + \mathcal{L}_\phi \rho.$$

The goal of this section is to show

$$\frac{\mathrm{d}}{\mathrm{d}t}\|e\|^2_{\dot{H}^m} \leq C Y_m^N.$$

We have,

$$\rho_t + \nabla \cdot (\rho u) = 0$$

Due to the topological part of the model, the interaction kernel depends on the density ρ. Therefore the operator $\mathcal{L}_\phi$ does not commute with derivatives. Taking the divergence of the momentum equation and using the density equation and the e-quantity we get the identity

$$e_t + \nabla \cdot (ue) = (\nabla \cdot u)^2 - \mathrm{Tr}(\nabla u)^2 + \partial_t(\mathcal{L}_\phi(\rho)) + \nabla \cdot \mathcal{L}_\phi(\rho u).$$

Let us take a closer look the last two terms and work out a more explicit formula. For the time derivative,

$$\partial_t(\mathcal{L}_\phi(\rho)) = \mathcal{L}_\phi(\rho_t) + \mathcal{L}_{\phi_t}(\rho)$$

where,

$$\begin{aligned}
\mathcal{L}_{\phi_t}(\rho) &:= -\frac{\tau}{n}\int_{\mathbb{T}^n} \frac{h(|z|)}{|z|^{n+\alpha-\tau}} \frac{\int_{\Omega(x,x+z)} \rho_t(\xi)\,\mathrm{d}\xi}{d^{\tau+n}(x,x+z)} \delta_z\rho(x)\,\mathrm{d}z \\
&= \frac{\tau}{n}\int_{\mathbb{T}^n} \frac{h(|z|)}{|z|^{n+\alpha-\tau}} \frac{\int_{\Omega(x,x+z)} \nabla\cdot(\rho u)(\xi)\,\mathrm{d}\xi}{d^{\tau+n}(x,x+z)} \delta_z\rho(x)\,\mathrm{d}z.
\end{aligned}$$

Then looking at the divergence we have,

$$\nabla \cdot \mathcal{L}_\phi(\rho u) = \mathcal{L}_\phi(\nabla \cdot (\rho u)) + \mathcal{L}_{\nabla\phi\cdot}(\rho u)$$

where,

$$\begin{aligned}
\mathcal{L}_{\nabla\phi\cdot}(\rho u) &= \int_{\mathbb{T}^n} \nabla\phi(x,x+z)\cdot\delta_z(\rho u)(x)\,\mathrm{d}z \\
&= -\frac{\tau}{n}\int_{\mathbb{T}^n} \frac{h(|z|)}{|z|^{n+\alpha-\tau}} \frac{\int_{\Omega(x,x+z)} \nabla\rho(\xi)\,\mathrm{d}\xi}{d^{\tau+n}(x,x+z)} \cdot \delta_z(\rho u)(x)\,\mathrm{d}z.
\end{aligned}$$

Now using the density equation we see that the first terms in $\partial_t(\mathcal{L}_\phi(\rho))$ and $\nabla \cdot \mathcal{L}_\phi(\rho u)$ cancel, and the equation becomes,

$$e_t + \nabla \cdot (ue) = (\nabla \cdot u)^2 - \mathrm{Tr}(\nabla u)^2 + \mathcal{L}_{\phi_t}(\rho) + \mathcal{L}_{\nabla\phi\cdot}(\rho u). \tag{37}$$

In order to achieve our estimate we apply ∂^m to (37) and test with $\partial^m e$. Estimating the last two terms will be the main technical component of this section. So, let us make a few quick comments as to the remaining terms. Dropping integral signs we have for the transport term

$$\partial^m(e\nabla\cdot u)\partial^m e + (u\cdot\nabla\partial^m e)\partial^m e + [\partial^m(u\cdot\nabla e) - u\cdot\nabla\partial^m e]\partial^m e.$$

So, it can treated exactly like the similar term in the momentum in the beginning of Section 4. For $\partial^m[(\nabla\cdot u)^2 - \mathrm{Tr}(\nabla u)^2]\partial^m e$ we have quandratic in ∇u expression whose L^2-norm breaks into the product estmate of $\|u\|_{H^{m+1}}|\nabla u|_\infty$. We thus can see that all these terms are bounded by Y_m^3.

We now focus solely on the residual alignment term and start with the "worst" in a sense end point cases.

END-CASE 1. Here we estimate the worst term when all m derivatives fall on the density to form a derivative of order $m+1$:

$$I = \int_{\mathbb{T}^n}\left[\int_{\Omega(0,z)}\partial^m\nabla\rho(x+\xi)\,\mathrm{d}\xi\delta_z(\rho u)(x) - \int_{\Omega(0,z)}\nabla(u\partial^m\rho)(x+\xi)\,\mathrm{d}\xi\delta_z\rho(x)\right]\times$$
$$\times\frac{h(|z|)}{|z|^{n+\alpha-\tau}d^{\tau+n}(x,x+z)}\,\mathrm{d}z.$$

Integrating by parts inside the integrals we obtain the expression

$$\int_{\partial\Omega(0,z)}[\partial^m\rho(x+\xi)\delta_z(\rho u)(x) - (u\partial^m\rho)(x+\xi)\delta_z\rho(x)]\cdot\nu_\xi\,\mathrm{d}\xi.$$

Using that $\delta_z(\rho u)(x) = \delta_z\rho(x)u(x) + \rho(x+z)\delta_z u(x)$, we write the integrand as

$$\partial^m\rho(x+\xi)\delta_z\rho(x)(u(x)-u(x+\xi)) + \partial^m\rho(x+\xi)\delta_z\rho(x)\delta_z u(x) + \partial^m\rho(x+\xi)\rho(x)\delta_z u(x).$$

We focus on the last term which is most difficult. We write

$$\partial^m\rho(x+\xi)\rho(x)\delta_z u(x) = \partial^m\rho(x+\xi)\rho(x)[\delta_z u(x) - \nabla u(x)z] + \partial^m\rho(x+\xi)\rho(x)\nabla u(x)z.$$

We focus on the last term. Let us write the integral to be estimated

$$J = \int_{\mathbb{T}^n}\int_{\partial\Omega(0,z)}[\partial^m\rho(x+\xi) - \partial^m\rho(x)]\rho(x)\nabla u(x)z\cdot\nu_\xi\,\mathrm{d}\xi\frac{h(|z|)}{|z|^{n+\alpha-\tau}d^{\tau+n}(x,x+z)}\,\mathrm{d}z.$$

Changing the variable to $\theta\in\partial\Omega(0,\mathbf{e}_1)$ we obtain

$$J = \int_{\partial\Omega(0,\mathbf{e}_1)}\int_{\mathbb{T}^n}[\partial^m\rho(x+|z|U_z\theta) - \partial^m\rho(x)]\rho(x)\nabla u(x)z\cdot U_z\nu_\theta\frac{h(|z|)}{|z|^{\alpha-\tau+1}d^{\tau+n}(x,x+z)}\,\mathrm{d}z\,\mathrm{d}\theta.$$

Let us freeze the coefficients in the kernel:

$$J = J_1 + J_2,$$

where

$$J_1 = \int_{\partial\Omega(0,\mathbf{e}_1)}\rho^{-\tau/n}(x)\int_{\mathbb{T}^n}[\partial^m\rho(x+|z|U_z\theta) - \partial^m\rho(x)]\nabla u(x)z\cdot U_z\nu_\theta\frac{h(|z|)}{|z|^{n+\alpha+1}}\,\mathrm{d}z\,\mathrm{d}\theta$$

$$J_2 = \int_{\partial\Omega(0,\mathbf{e}_1)}\int_{\mathbb{T}^n}[\partial^m\rho(x+|z|U_z\theta) - \partial^m\rho(x)]\rho(x)\nabla u(x)z\cdot U_z\nu_\theta\frac{h(|z|)}{|z|^{n+\alpha+1}}\times$$
$$\times\left(\frac{1}{\left[⨍_{\Omega(0,z)}\rho(x+\xi)\,\mathrm{d}\xi\right]^{\tau/n+1}} - \frac{1}{\rho^{\tau/n+1}(x)}\right)\mathrm{d}z\,\mathrm{d}\theta.$$

To estimate J_1 we further symmetrize in z noting that $U_{-z} = -U_z$, and so the kernel is even:

$$J_1 = \int_{\partial\Omega(0,\mathbf{e}_1)} \rho^{-\tau/n}(x) \int_{\mathbb{T}^n} [\partial^m \rho(x + |z|U_z\theta) + \partial^m \rho(x - |z|U_z\theta) - 2\partial^m \rho(x)] \times$$
$$\times \nabla u(x) z \cdot U_z \nu_\theta \frac{h(|z|)}{|z|^{n+\alpha+1}} \,\mathrm{d}z \,\mathrm{d}\theta,$$

and we estimate

$$\|J_1\|_2 \le \underline{\rho}^{-\tau/n} |\nabla u|_\infty \sum_{i,j,k} \left\| \int_{\mathbb{T}^n} [\partial^m \rho(\cdot + |z|U_z\theta) + \partial^m \rho(\cdot - |z|U_z\theta) - 2\partial^m \rho(\cdot)] \frac{h(|z|) z_i U_z^{jk}}{|z|^{n+\alpha+1}} dz \right\|_2$$
$$\le \underline{\rho}^{-\tau/n} |\nabla u|_\infty \|\rho\|_{\dot{H}^{m+\alpha}},$$

where the ultimate bound follows from Lemma 7.2.

To estimate J_2 we note that a similar estimate from before gives

$$\left| \frac{1}{\left[\fint_{\Omega(0,z)} \rho(x+\xi) \,\mathrm{d}\xi \right]^{\tau/n+1}} - \frac{1}{\rho^{\tau/n+1}(x)} \right| \le \overline{\rho}^{\frac{\tau}{n}+1} \underline{\rho}^{-3-\frac{2\tau}{n}} |\nabla \rho|_\infty |z|.$$

Therefore by Lemma 7.1

$$|J_2| \le \overline{\rho}^{\frac{\tau}{n}+2} \underline{\rho}^{-3-\frac{2\tau}{n}} |\nabla \rho|_\infty |\nabla u|_\infty \int_{\partial\Omega(0,\mathbf{e}_1)} \int_{\mathbb{T}^n} \frac{|\partial^m \rho(x + |z|U_z\theta) - \partial^m \rho(x)|}{|z|^{\frac{n}{2}+\alpha-\frac{1}{2}}} \frac{h(|z|)}{|z|^{\frac{n}{2}-\frac{1}{2}}} \,\mathrm{d}z \,\mathrm{d}\theta$$
$$\|J_2\|_2 \le \overline{\rho}^{\frac{\tau}{n}+2} \underline{\rho}^{-3-\frac{2\tau}{n}} |\nabla \rho|_\infty |\nabla u|_\infty \|\rho\|_{\dot{H}^{m+\alpha-\frac{1}{2}}}.$$

Now to estimate the first term. The integral we need to estimate is

$$I = \rho(x) \int_{\mathbb{T}^n} \int_{\partial\Omega(0,z)} \partial^m \rho(x+\xi) \cdot \nu_\xi \,\mathrm{d}\xi \frac{[\delta_z u(x) - \nabla u(x) z] h(|z|)}{|z|^{n+\alpha-\tau} d^{\tau+n}(x, x+z)} \,\mathrm{d}z$$
$$= \rho(x) \int_{\mathbb{T}^n} \int_{\partial\Omega(0,\mathbf{e}_1)} |\partial^m \rho(x + |z|U_z\theta) - \partial^m \rho(x)| \cdot U_z \nu_\theta \,\mathrm{d}\theta \frac{[\delta_z u(x) - \nabla u(x) z] h(|z|)}{|z|^{1+\alpha-\tau} d^{\tau+n}(x, x+z)} \,\mathrm{d}z$$
$$|I| \le \overline{\rho} |\nabla^2 u|_\infty \int_{\mathbb{T}^n} \int_{\partial\Omega(0,\mathbf{e}_1)} \frac{h(|z|) |\partial^m \rho(x + |z|U_z\theta) - \partial^m \rho(x)|}{|z|^{n+\alpha-1}} \,\mathrm{d}\theta \,\mathrm{d}z.$$

So estimating in L^2 and applying Lemma 7.1 again, we get,

$$\|I\|_2 \le \overline{\rho} |\nabla^2 u|_\infty \|\rho\|_{\dot{H}^{m+\alpha-\frac{1}{2}}}.$$

Now returning to the first integral in this section, we still need to estimate the first two terms,

$$I_1 = \int_{\mathbb{T}^n} \int_{\partial\Omega(0,z)} \partial^m \rho(x+\xi) \delta_z \rho(x) (u(x) - u(x+\xi)) \cdot \nu_\xi \,\mathrm{d}\xi \frac{h(|z|)}{|z|^{n+\alpha-\tau} d^{\tau+n}(x, x+z)} \,\mathrm{d}z.$$

To estimate this we add and subtract $\partial^m \rho(x) u(x)$ in the integrand to get,

$$I_{11} = u(x) \int_{\mathbb{T}^n} \int_{\partial\Omega(0,z)} (\partial^m \rho(x+\xi) - \partial^m \rho(x)) \cdot \nu_\xi \,\mathrm{d}\xi \frac{h(|z|) \delta_z \rho(x)}{|z|^{n+\alpha-\tau} d^{\tau+n}(x, x+z)} \,\mathrm{d}z,$$
$$I_{12} = -\int_{\mathbb{T}^n} \int_{\partial\Omega(0,z)} (\partial^m \rho(x+\xi) u(x+\xi) - \partial^m \rho(x) u(x)) \cdot \nu_\xi \,\mathrm{d}\xi \frac{h(|z|) \delta_z \rho(x)}{|z|^{n+\alpha-\tau} d^{\tau+n}(x, x+z)} \,\mathrm{d}z.$$

Looking at I_{11} we include the next term in the Taylor finite difference.

$$I_{111} = u(x)\int_{\mathbb{T}^n}\int_{\partial\Omega(0,z)} (\partial^m\rho(x+\xi) - \partial^m\rho(x))\cdot\nu_\xi\,\mathrm{d}\xi\frac{h(|z|)[\delta_z\rho(x) - z\nabla\rho(x)]}{|z|^{n+\alpha-\tau}d^{\tau+n}(x,x+z)}\,\mathrm{d}z,$$

$$I_{112} = u(x)\int_{\mathbb{T}^n}\int_{\partial\Omega(0,z)} (\partial^m\rho(x+\xi) - \partial^m\rho(x))\cdot\nu_\xi\,\mathrm{d}\xi\frac{h(|z|)z\nabla\rho(x)}{|z|^{n+\alpha-\tau}d^{\tau+n}(x,x+z)}\,\mathrm{d}z.$$

Notice that shifting to $\partial\Omega(0,\mathbf{e}_1)$ and symmetrizing makes I_{111} and I_{112} take the same form as J_1 above, so Lemma 7.2 gives

$$\|I_{11}\|_2 \leq \underline{\rho}^{\tau/n}|\nabla\rho|_\infty|u|_\infty\|\rho\|_{\dot{H}^{m+\alpha}}$$

Proceeding the same way for I_{12} we get

$$I_{121} = -\int_{\mathbb{T}^n}\int_{\partial\Omega(0,z)} (\partial^m\rho(x+\xi)u(x+\xi) - \partial^m\rho(x)u(x))\cdot\nu_\xi\,\mathrm{d}\xi\frac{h(|z|)[\delta_z\rho(x) - z\nabla\rho(x)]}{|z|^{n+\alpha-\tau}d^{\tau+n}(x,x+z)}\,\mathrm{d}z,$$

$$I_{122} = -\int_{\mathbb{T}^n}\int_{\partial\Omega(0,z)} (\partial^m\rho(x+\xi)u(x+\xi) - \partial^m\rho(x)u(x))\cdot\nu_\xi\,\mathrm{d}\xi\frac{h(|z|)z\nabla\rho(x)}{|z|^{n+\alpha-\tau}d^{\tau+n}(x,x+z)}\,\mathrm{d}z.$$

Shifting to $\partial\Omega(0,\mathbf{e}_1)$, symmetrizing and using Lemma 7.2 with $g = u\partial^m\rho$ also gives

$$\|I_{12}\|_2 \leq \underline{\rho}^{\tau/n}|\nabla\rho|_\infty\|u\partial^m\rho\|_{\dot{H}^\alpha} \leq \underline{\rho}^{\tau/n}|\nabla\rho|_\infty|u|_\infty\|\rho\|_{\dot{H}^{m+\alpha}}.$$

The second term in the first integral to estimate is

$$I_2 = \int_{\mathbb{T}^n}\int_{\partial\Omega(0,z)} \partial^m\rho(x+\xi)\cdot\nu_\xi\,\mathrm{d}\xi\frac{h(|z|)\delta_z\rho(x)\delta_z u(x)}{|z|^{n+\alpha-\tau}d^{\tau+n}(x,x+z)}\,\mathrm{d}z.$$

We pick up two powers of z from $\delta_z\rho(x)$ and $\delta_z u(x)$ to get

$$|I_2| \leq |\nabla\rho|_\infty|\nabla u|_\infty\int_{\Omega(0,e_1)}\int_{\mathbb{T}^n}\frac{h(|z|)|\partial^m\rho(x+|z|U_z\theta) - \partial^m\rho(x)|}{|z|^{n+\alpha-1}}\,\mathrm{d}z\,\mathrm{d}\theta.$$

Applying Holder's inequality and using Lemma 7.1 we get

$$\|I_2\| \leq |\nabla\rho|_\infty|\nabla u|_\infty\|\rho\|_{\dot{H}^{m+\alpha-\frac{1}{2}}}.$$

Now let us look at the other endpoint where all m derivatives fall inside the increment $\delta_z f$ in the residual terms.

END-CASE 2. Here we need to combine terms from $\mathcal{L}_{\nabla\phi\cdot}(\rho u)$ and $\mathcal{L}_{\phi_t}(\rho)$ again.

$$I = \int_{\mathbb{T}^n}\left[\int_{\Omega(0,z)} \nabla\cdot(\rho u)(x+\xi) - \nabla\rho(x+\xi)\cdot u(x)\,\mathrm{d}\xi\right]\frac{h(|z|)\delta_z\partial^m\rho(x)}{|z|^{n+\alpha-\tau}d^{\tau+n}(x,x+z)}\,\mathrm{d}z. \tag{38}$$

Expanding $\nabla\cdot(\rho u) = \nabla\rho\cdot u + \rho(\nabla\cdot u)$ we get two terms to be estimated. We focus on the last first.

$$J = \int_{\mathbb{T}^n}\left[\int_{\Omega(0,z)} \rho(x+\xi)(\nabla\cdot u)(x+\xi)\,\mathrm{d}\xi\right]\frac{h(|z|)\delta_z\partial^m\rho(x)}{|z|^{n+\alpha-\tau}d^{\tau+n}(x,x+z)}\,\mathrm{d}z.$$

As before we will freeze the coefficients, splitting this into $J = J_1 + J_2$ with,

$$J_1 = \frac{\rho(x)(\nabla\cdot u)(x)}{\rho^{\frac{\tau}{n}+1}(x)}\int_{\mathbb{T}^n}\frac{h(|z|)}{|z|^{n+\alpha}}\delta_z\partial^m\rho(x)\,\mathrm{d}z$$

$$J_2 = \int_{\mathbb{T}^n}\frac{h(|z|)}{|z|^{n+\alpha}}\left(\frac{\fint_{\Omega(0,z)}\rho(x+\xi)(\nabla\cdot u)(x+\xi)\,\mathrm{d}\xi}{\left(\fint_{\Omega(0,z)}\rho(x+\xi)\,\mathrm{d}\xi\right)^{\frac{\tau}{n}+1}} - \frac{\rho(x)(\nabla\cdot u)(x)}{\rho^{\frac{\tau}{n}+1}(x)}\right)\delta_z\partial^m\rho(x)\,\mathrm{d}z.$$

The integral in J_1 is the truncated fractional Laplacian, so is bounded by $\underline{\rho}^{-\frac{\tau}{n}+1}\overline{\rho}|\nabla u|_\infty \|\rho\|_{\dot{H}^{m+\alpha}}$. Then for J_2 we need to control the difference, by adding and subtracting appropriately.

$$J_{2,1} = \int_{\mathbb{T}^n} \frac{h(|z|)}{|z|^{n+\alpha}} \fint_{\Omega(0,z)} \rho(x+\xi)(\nabla\cdot u)(x+\xi)\,\mathrm{d}\xi \left(\frac{\rho^{\frac{\tau}{n}+1}(x) - \left(\fint_{\Omega(0,z)} \rho(x+\xi)\,\mathrm{d}\xi\right)^{\frac{\tau}{n}+1}}{\left(\fint_{\Omega(0,z)} \rho(x+\xi)\,\mathrm{d}\xi\right)^{\frac{\tau}{n}+1} \rho^{\frac{\tau}{n}+1}(x)} \right) \delta_z \partial^m \rho(x)\,\mathrm{d}z$$
$$\leq \overline{\rho}^{\frac{\tau}{n}+2} \underline{\rho}^{-3-\frac{2\tau}{n}} |\nabla u|_\infty |\nabla\rho|_\infty \int_{\mathbb{T}^n} \frac{h(|z|)}{|z|^{n+\alpha-1}} |\delta_z \partial^m \rho(x)|\,\mathrm{d}z,$$

for $\alpha < 1$ estimating in L^2 we get a bound by $\|\rho\|_{\dot{H}^m}$ by the Minkowskii inequality, and for $\alpha \geq 1$ we get a bound by $\|\rho\|_{\dot{H}^{m+\alpha-1+\varepsilon}}$. Then looking at $J_{2,2}$ we get,

$$J_{2,2} = \int_{\mathbb{T}^n} \frac{h(|z|)}{|z|^{n+\alpha}} \left(\frac{\fint_{\Omega(0,z)} \rho(x+\xi)(\nabla\cdot u)(x+\xi)\,\mathrm{d}\xi - \rho(x)(\nabla\cdot u)(x)}{\rho^{\frac{\tau}{n}+1}(x)} \right) \delta_z \partial^m \rho(x)\,\mathrm{d}z$$
$$\leq \underline{\rho}^{-1-\frac{\tau}{n}} (|\nabla^2 u|_\infty \overline{\rho} + |\nabla u|_\infty |\nabla \rho|_\infty) \int_{\mathbb{T}^n} \frac{h(|z|)}{|z|^{n+\alpha-1}} |\delta_z \partial^m \rho(x)|\,\mathrm{d}z,$$

where we can estimate the integral in the same way as for $J_{2,1}$. Now we still need to estimate the first term from expanding $\nabla\cdot(\rho u)$. The term we need to estimate is

$$J = \int_{\mathbb{T}^n} \left[\int_{\Omega(0,z)} \nabla\rho(x+\xi)\cdot(u(x+\xi)-u(x))\,\mathrm{d}\xi \right] \frac{h(|z|)\delta_z\partial^m\rho(x)}{|z|^{n+\alpha-\tau} d^{\tau+n}(x,x+z)}\,\mathrm{d}z \tag{39}$$
$$|J| \leq |\nabla\rho|_\infty |\nabla u|_\infty \int_{\mathbb{T}^n} \frac{h(|z|)|\delta_z\partial^m\rho(x)|}{|z|^{n+\alpha-1}}\,\mathrm{d}z$$

which again is bounded by $\|\rho\|_{\dot{H}^m}$ for $\alpha < 1$ and $\|\rho\|_{\dot{H}^{m+\alpha-1+\varepsilon}}$ for $\alpha \geq 1$.

We no longer need to combine terms from the two residual terms so we will now proceed to estimate the remainder of the terms from $\mathcal{L}_{\nabla\phi\cdot}(\rho u)$ and $\mathcal{L}_{\phi_t}(\rho)$ individually. First looking at $\mathcal{L}_{\nabla\phi\cdot}(\rho u)$ we will estimate some of the higher order terms where all m derivatives hit the density, and then combine the rest of the intermediary terms in one estimate.

End-Case 3. In the previous case we used $u(x)\delta_z\partial^m\rho(x)$ from $\delta_z(u\partial^m\rho(x)) = \delta_z u(x)\delta_z\partial^m\rho(x) + u(x)\delta_z\partial^m\rho(x) + \partial^m\rho(x)\delta_z u(x)$, we still need to estimate the other two terms.

$$I_1 = \int_{\mathbb{T}^n} \frac{h(|z|)\delta_z u(x)\delta_z\partial^m\rho(x)}{|z|^{n+\alpha-\tau} d^{\tau+n}(x,x+z)} \int_{\Omega(x,x+z)} \nabla\rho(x)\,\mathrm{d}\xi\,\mathrm{d}z \tag{40}$$
$$|I_1| \leq |\nabla u|_\infty |\nabla\rho|_\infty \int_{\mathbb{T}^n} \frac{h(|z|)|\delta_z\partial^m\rho(x)|}{|z|^{n+\alpha-1}}\,\mathrm{d}z.$$

Then estimating in L^2 the integral is bounded by $\|\rho\|_{\dot{H}^m}$ for $\alpha < 1$ and $\|\rho\|_{\dot{H}^{m+\alpha-1+\varepsilon}}$ for $\alpha \geq 1$.

For the second term we need to look at separately for $\alpha < 1$ and for $\alpha \geq 1$. First $\alpha < 1$,

$$I_2 = \int_{\mathbb{T}^n} \frac{h(|z|)(\delta_z u(x))\partial^m\rho(x)}{|z|^{n+\alpha-\tau} d^{\tau+n}(x,x+z)} \int_{\Omega(x,x+z)} \nabla\rho(x)\,\mathrm{d}\xi\,\mathrm{d}z \tag{41}$$
$$|I_2| \leq |\nabla u|_\infty |\nabla\rho|_\infty |\partial^m\rho(x)| \int_{\mathbb{T}^n} \frac{h(|z|)}{|z|^{n+\alpha-1}}\,\mathrm{d}z,$$

which in L^2 is bounded by $\|\rho\|_{\dot{H}^m}$. For $\alpha \geq 1$ we add and subtract the next Taylor term to get $I_2 = I_{21} + I_{22}$

$$I_{21} = \int_{\mathbb{T}^n} \frac{h(|z|)\partial^m \rho(x)}{|z|^{n+\alpha-\tau} d^{\tau+n}(x, x+z)} \int_{\Omega(x,x+z)} \nabla\rho(x)\, \mathrm{d}\xi\, [\delta_z u(x) - z\nabla u(x)]\, \mathrm{d}z,$$

$$I_{22} = \nabla u(x)\partial^m \rho(x) \int_{\mathbb{T}^n} \frac{h(|z|)}{|z|^{n+\alpha-\tau} d^{\tau+n}(x, x+z)} \int_{\Omega(x,x+z)} \nabla\rho(x)\, \mathrm{d}\xi z\, \mathrm{d}z.$$

For I_{21} we use $|\delta_z u(x) - z\nabla u(x)| \leq |\nabla^2 u|_\infty |z|^2$ to get

$$|I_{21}| \leq |\nabla^2 u|_\infty |\nabla\rho|_\infty |\partial^m \rho(x)| \int_{\mathbb{T}^n} \frac{h(|z|)}{|z|^{n+\alpha-2}}\, \mathrm{d}z,$$

which in L^2 is bounded by $\|\rho\|_{\dot{H}^m}$ again. To estimate I_{22} we symmetrize first and split into two parts,

$$\begin{aligned}
I_{22} &= \nabla u(x)\partial^m \rho(x) \int_{\mathbb{T}^n} \frac{h(|z|)}{|z|^{n+\alpha-\tau}} \left(\frac{\int_{\Omega(x,x+z)} \nabla\rho(x)\, \mathrm{d}\xi}{d^{\tau+n}(x, x+z)} - \frac{\int_{\Omega(x,x-z)} \nabla\rho(x)\, \mathrm{d}\xi}{d^{\tau+n}(x, x-z)} \right) z\, \mathrm{d}z \\
&= \nabla u(x)\partial^m \rho(x) \int_{\mathbb{T}^n} \frac{h(|z|)}{|z|^{n+\alpha-\tau}} d^{-\tau-n}(x, x+z) \left(\int_{\Omega(x,x+z)} \nabla\rho(x)\, \mathrm{d}\xi - \int_{\Omega(x,x-z)} \nabla\rho(x)\, \mathrm{d}\xi \right) z\, \mathrm{d}z \\
&\quad + \nabla u(x)\partial^m \rho(x) \int_{\mathbb{T}^n} \frac{h(|z|)}{|z|^{n+\alpha-\tau}} \int_{\Omega(0,z)} \nabla\rho(\xi)\, \mathrm{d}\xi \left(\frac{d^{\tau+n}(x, x+z) - d^{\tau+n}(x, x-z)}{d^{\tau+n}(x, x+z) d^{\tau+n}(x, x-z)} \right) z\, \mathrm{d}z \\
&= I_{221} + I_{222}.
\end{aligned}$$

Now for I_{221} we notice that a similar computation as before gives,

$$\int_{\Omega(x,x+z)} \nabla\rho(\xi)\, \mathrm{d}\xi - \int_{\Omega(x,x-z)} \nabla\rho(\xi)\, \mathrm{d}\xi \lesssim (D_{s,p}\partial\rho(x))^{1/p} |z|^{n+s},$$

where $s = \alpha - 1 + \varepsilon < 1$, and so $n + \alpha - 1 - s < n$,

$$|I_{221}| \leq |\nabla u|_\infty |\partial^m \rho(x)| (D_{s,p}\partial\rho(x))^{1/p} \int_{\mathbb{T}^n} \frac{h(|z|)}{|z|^{n+\alpha-1-s}}\, \mathrm{d}z.$$

Then using Holder's inequality in L^2 with $\frac{2}{p} + \frac{2}{q} = 1$ we get,

$$\|I_{221}\|_2^2 \leq |\nabla u|_\infty^2 \|\rho\|^2_{\dot{H}^{m+n(\frac{1}{2}-\frac{1}{q})}} \|\rho\|^2_{\dot{H}^{1+s+n(\frac{1}{2}-\frac{1}{p})}}.$$

Then choosing $q = \frac{2m}{m-1}$ and $p = 2m$ gives

$$\begin{aligned}
\rho: &\quad m + n\left(\frac{1}{2} - \frac{1}{q}\right) = m + \frac{n}{2m} < m + \alpha, \\
\rho: &\quad 1 + s + n\left(\frac{1}{2} - \frac{1}{p}\right) = 1 + s + \frac{n}{2}\frac{m-1}{m} < m + \alpha.
\end{aligned}$$

For I_{222} we have already shown how to estimate the difference $d^{\tau+n}(x, x+z) - d^{\tau+n}(x, x-z)$ so we get,

$$\begin{aligned}
|I_{222}| &\leq |\nabla u|_\infty |\partial^m \rho(x)| |\nabla\rho|_\infty^2 \int_{\mathbb{T}^n} \frac{h(|z|)}{|z|^{n+\alpha-2}}\, \mathrm{d}z \\
\|I_{222}\|_2^2 &\leq |\nabla u|_\infty^2 |\nabla\rho|_\infty^4 \|\rho\|^2_{\dot{H}^m}.
\end{aligned}$$

END-CASE 4. Since $\partial^m(\rho u) = \partial^{m-1}(\rho\partial u) + u\partial^m\rho$, we still need to estimate the term

$$I_{0,0}[\partial^{m-1}(\rho\partial u)](x) = \int_{\mathbb{T}^n} \frac{h(|z|)}{|z|^{n+\alpha-\tau}} \frac{\int_{\Omega(x,x+z)} \nabla\rho(\xi)\,\mathrm{d}\xi}{d^{\tau+n}(x,x+z)} \delta_z(\partial^{m-1}(\rho\partial u))(x)\,\mathrm{d}z. \tag{42}$$

For $\alpha < 1$ and ε so that $\alpha + \varepsilon < 1$, we get

$$\begin{aligned}
|I_{0,0}[\partial^{m-1}(\rho\partial u)](x)| &\le |\nabla\rho|_\infty \int_{\mathbb{T}^n} \frac{h(|z|)}{|z|^{n+\alpha}} |\delta_z(\partial^{m-1}(\rho\partial u))(x)|\,\mathrm{d}z \\
\|I_{0,0}[\partial^{m-1}(\rho\partial u)]\|_2^2 &\le |\nabla\rho|_\infty^2 \|\rho\partial u\|^2_{\dot H^{m-1+\alpha+\varepsilon}} \\
&\le |\nabla\rho|_\infty^2 \|\rho\|^2_{\dot H^{m-1+\alpha+\varepsilon}} \|u\|^2_{\dot H^{m+1}}.
\end{aligned}$$

For $\alpha \ge 1$, we again add and subtract the next Taylor term, focus on the second one, and symmetrize

$$I_{0,0,2}[\partial^{m-1}(\rho\partial u)](x) = \nabla(\partial^{m-1}(\rho\partial u))(x) \int_{\mathbb{T}^n} \frac{h(|z|)}{|z|^{n+\alpha-\tau}} \left(\frac{\int_{\Omega(x,x+z)} \nabla\rho(\xi)\,\mathrm{d}\xi}{d^{\tau+n}(x,x+z)} - \frac{\int_{\Omega(x,x-z)} \nabla\rho(\xi)\,\mathrm{d}\xi}{d^{\tau+n}(x,x-z)} \right) z\,\mathrm{d}z$$

splitting this into two parts

$$\begin{aligned}
I_{0,0,2,1}[\partial^{m-1}(\rho\partial u)](x) &= \nabla(\partial^{m-1}(\rho\partial u))(x) \times \\
&\times \int_{\mathbb{T}^n} \frac{h(|z|)}{|z|^{n+\alpha-\tau} d^{\tau+n}(x,x+z)} \left(\int_{\Omega(x,x+z)} \nabla\rho(\xi)\,\mathrm{d}\xi - \int_{\Omega(x,x-z)} \nabla\rho(\xi)\,\mathrm{d}\xi \right) z\,\mathrm{d}z \\
I_{0,0,2,2}[\partial^{m-1}(\rho\partial u)](x) &= \nabla(\partial^{m-1}(\rho\partial u))(x) \times \\
&\times \int_{\mathbb{T}^n} \frac{h(|z|) \int_{\Omega(x,x-z)} \nabla\rho(\xi)\,\mathrm{d}\xi}{|z|^{n+\alpha-\tau}} \left(\frac{1}{d^{\tau+n}(x,x+z)} - \frac{1}{d^{\tau+n}(x,x-z)} \right) z\,\mathrm{d}z
\end{aligned}$$

and estimating these as before we get,

$$\begin{aligned}
|I_{0,0,2,1}[\partial^{m-1}(\rho\partial u)](x)| &\le |\nabla\partial^{m-1}(\rho\partial u)(x)| (D_{s,p}\partial\rho(x))^{1/p} \int_{\mathbb{T}^n} \frac{h(|z|)}{|z|^{n+\alpha-1-s}}\,\mathrm{d}z \\
\|I_{0,0,2,1}[\partial^{m-1}(\rho\partial u)]\|_2^2 &\le \|\nabla\partial^{m-1}(\rho\partial u)\|_q^2 \|\partial\rho\|_p^2 \\
&\le \|\rho\|^2_{\dot H^{m+n(\frac{1}{2}-\frac{1}{p})}} \|u\|^2_{\dot H^{m+1+n(\frac{1}{2}-\frac{1}{p})}} \|\rho\|^2_{\dot H^{1+s+n(\frac{1}{2}-\frac{1}{q})}}.
\end{aligned}$$

Choosing $q = \frac{2m}{m-1}$ and $p = 2m$ we get

$$\begin{aligned}
\|J_{2,1}\|_2^2 &\le \|\rho\|^4_{\dot H^{m+\alpha}} \|u\|_{\dot H^{m+1+\frac{n}{2m}}} \\
&\le Y_m^N \|u\|_{\dot H^{m+1}} + \varepsilon \|u\|^2_{\dot H^{m+1+\frac{\alpha}{2}}}
\end{aligned}$$

and for $I_{0,0,2,2}[\partial^{m-1}(\rho\partial u)](x)$ we get,

$$\begin{aligned}
|I_{0,0,2,2}[\partial^{m-1}(\rho\partial u)](x)| &\le |\nabla\partial^{m-1}(\rho\partial u)(x)| |\nabla\rho|_\infty \int_{\mathbb{T}^n} \frac{h(|z|)}{|z|^{n+\alpha-2}}\,\mathrm{d}z \\
\|I_{0,0,2,2}[\partial^{m-1}(\rho\partial u)]\|_2^2 &\le |\nabla\rho|_\infty^2 \|\rho\|^2_{\dot H^m} \|u\|^2_{\dot H^{m+1}}.
\end{aligned}$$

END-CASE 5. All m derivatives on $\nabla\phi$, and $l = 0, ..., m-1$. Have to estimate

$$I_{\mathbf{j},l}[\rho u](x) = \int_{\mathbb{T}^n} \frac{h(|z|)}{|z|^{n+\alpha-\tau}} \int_{\Omega(x,x+z)} \partial^l \nabla\rho(\xi)\,\mathrm{d}\xi \frac{\prod_{k=1}^{m-l} \left(\int_{\Omega(x,x+z)} \partial^k \rho(\xi) d\xi \right)^{j_k}}{d^{\tau+(|\mathbf{j}|+1)n}} \cdot \delta_z(\rho u)(x)\,\mathrm{d}z$$

Using the maximal functions we get,

$$|I_{\mathbf{j},l}[\rho u](x)| \le \prod_{k=1}^{m-l} (M[\partial^k\rho](x))^{j_k} M[\partial^{l+1}\rho](x) \int_{\mathbb{T}^n} \frac{h(|z|)}{|z|^{n+\alpha}} |\delta_z \rho u(x)|\,\mathrm{d}z.$$

Then using Holder's inequality with

$$\sum_{k=1}^{m-l} \frac{2j_k}{p_k} + \frac{2}{q_1} + \frac{2}{q_2} = 1,$$

and the Hardy-Littlewood inequality, we get for $0 < \alpha < 1$,

$$\begin{aligned} \|I_{\mathbf{j},l}[\rho u]\|_2^2 &\lesssim \prod_{k=1}^{m-l} \|\partial^k \rho\|_{p_k}^{2j_k} \|\partial^{l+1}\rho\|_{q_1}^2 \|\rho u\|_{W^{\alpha+\varepsilon,q_2}}^2 \\ &\leq \prod_{k=1}^{m-l} \|\rho\|_{\dot{H}^{k+n(\frac{1}{2}-\frac{1}{p_k})}}^{2j_k} \|\rho\|_{\dot{H}^{l+1+n(\frac{1}{2}-\frac{1}{q_1})}}^2 \|\rho u\|_{\dot{H}^{\alpha+\varepsilon+n(\frac{1}{2}-\frac{1}{q_2})}}^2. \end{aligned}$$

Now we choose, for $l \neq 0$, $p_k = \frac{2m}{k}$, $q_1 = \frac{2m-1}{l}$, and $q_2 = \frac{2m(2m-1)}{l}$. Then we get

$$\rho: \qquad k + n\left(\frac{1}{2} - \frac{1}{p_k}\right) = k + \frac{n}{2}\left(\frac{m-k}{m}\right) \leq m + \alpha$$

for m large enough, for all $k = 1, ..., m$. Then for $l = 1, ..., m-1$,

$$\begin{aligned} \rho: &\qquad l + 1 + n\left(\frac{1}{2} - \frac{1}{q_1}\right) = l + 1 + n\left(\frac{2m-1-2l}{2(2m-1)}\right) \leq m + \alpha \\ \rho u: &\qquad \alpha + \varepsilon + n\left(\frac{1}{2} - \frac{1}{q_2}\right) = \alpha + \varepsilon + \frac{n}{2}\left(\frac{2m^2-m-l}{2m^2-m}\right) < 2 + \frac{n}{2} \end{aligned}$$

and for $l = 0$, instead of using the maximal function on $\nabla\rho$ we simply estimate with $|\nabla\rho|_\infty$ and use $|\delta_z(\rho u)| \leq |z|(|\nabla\rho|_\infty |u|_\infty + \overline{\rho}|u|_\infty)$ to get

$$\begin{aligned} |I_{\mathbf{j},0}[\rho u](x)| &\leq \prod_{k=1}^{m} (M[\partial^k \rho](x))^{j_k} |\nabla\rho|_\infty (|\nabla\rho|_\infty |u|_\infty + \overline{\rho}|\nabla u|_\infty) \\ \|I_{\mathbf{j},0}[\rho u]\|_2^2 &\leq \prod_{k=1}^{m} \|\rho\|_{\dot{H}^k}^{2j_k} |\nabla\rho|_\infty^2 (|\nabla\rho|_\infty |u|_\infty + \overline{\rho}|\nabla u|_\infty)^2. \end{aligned}$$

For $\alpha \geq 1$ we add and subtract the next Taylor term to get $I_{\mathbf{j},l}[\rho u] = I_{\mathbf{j},l,1}[\rho u] + I_{\mathbf{j},l,2}[\rho u]$

$$\begin{aligned} I_{\mathbf{j},l,1}[\rho u](x) &= \int_{\mathbb{T}^n} \frac{h(|z|)}{|z|^{n+\alpha-\tau}} \frac{\prod_{k=1}^{m-l}\left(\int_{\Omega(x,x+z)} \partial^k\rho(\xi)\,\mathrm{d}\xi\right)^{j_k}}{d^{\tau+(|\mathbf{j}|+1)n}(x,x+z)} \times \\ &\qquad \times \left(\int_{\Omega(x,x+z)} \partial^l \nabla\rho(\xi)\,\mathrm{d}\xi\right) [\delta_z(\rho u)(x) - z\nabla(\rho u)(x)]\,\mathrm{d}z \\ I_{\mathbf{j},l,2}[\rho u](x) &= \nabla(\rho u)(x) \int_{\mathbb{T}^n} \frac{h(|z|)}{|z|^{n+\alpha-\tau}} \int_{\Omega(x,x+z)} \partial^l \nabla\rho(\xi)\,\mathrm{d}\xi \frac{\prod_{k=1}^{m-l}\left(\int_{\Omega(x,x+z)} \partial^k\rho(\xi)\,\mathrm{d}\xi\right)^{j_k}}{d^{\tau+(|\mathbf{j}|+1)n}(x,x+z)} z\,\mathrm{d}z. \end{aligned}$$

The argument for $I_{\mathbf{j},l,1}[\rho u]$ goes just as above, noting again that the Gagliardo-Sobolevskii definition applies to smoothness exponents away from the integer values, $2 > \alpha + \varepsilon > 1$. Looking at $I_{\mathbf{j},l,2}[\rho u]$

we symmetrize and split further into three parts getting,

$$\begin{aligned}
I_{\mathbf{j},l,2}[\rho u](x) &= \nabla(\rho u)(x)\int_{\mathbb{T}^n}\frac{h(|z|)}{|z|^{n+\alpha-\tau}}\Big[\int_{\Omega(x,x+z)}\partial^l\nabla\rho(\xi)\,\mathrm{d}\xi\frac{\prod_{k=1}^{m-l}\left(\int_{\Omega(x,x+z)}\partial^k\rho(\xi)\,\mathrm{d}\xi\right)^{j_k}}{d^{\tau+(|\mathbf{j}|+1)n}(x,x+z)} \\
&\qquad -\int_{\Omega(x,x-z)}\partial^l\nabla\rho(\xi)\,\mathrm{d}\xi\frac{\prod_{k=1}^{m-l}\left(\int_{\Omega(x,x-z)}\partial^k\rho(\xi)\,\mathrm{d}\xi\right)^{j_k}}{d^{\tau+(|\mathbf{j}|+1)n}(x,x-z)}\Big]z\,\mathrm{d}z \\
&= \nabla(\rho u)(x)\int_{\mathbb{T}^n}\frac{h(|z|)}{|z|^{n+\alpha-\tau}}\int_{\Omega(x,x+z)}\partial^l\nabla\rho(\xi)\,\mathrm{d}\xi\prod_{k=1}^{m-l}\left(\int_{\Omega(x,x+z)}\partial^k\rho(\xi)\,\mathrm{d}\xi\right)^{j_k}\times \\
&\qquad\times\left(d^{-\tau-(|\mathbf{j}|+1)n}(x,x+z)-d^{-\tau-(|\mathbf{j}|+1)n}(x,x-z)\right)z\,\mathrm{d}z \\
&\quad+\nabla(\rho u)(x)\int_{\mathbb{T}^n}\frac{h(|z|)}{|z|^{n+\alpha-\tau}}\int_{\Omega(x,x+z)}\partial^l\nabla\rho(\xi)\,\mathrm{d}\xi d^{-\tau-(|\mathbf{j}|+1)n}(x,x-z)\times \\
&\qquad\times\left(\prod_{k=1}^{m-l}\left(\int_{\Omega(x,x+z)}\partial^k\rho(\xi)\,\mathrm{d}\xi\right)^{j_k}-\prod_{k=1}^{m-l}\left(\int_{\Omega(x,x-z)}\partial^k\rho(\xi)\,\mathrm{d}\xi\right)^{j_k}\right)z\,\mathrm{d}z \\
&\quad+\nabla(\rho u)(x)\int_{\mathbb{T}^n}\frac{h(|z|)}{|z|^{n+\alpha-\tau}}\prod_{k=1}^{m-l}\left(\int_{\Omega(x,x-z)}\partial^k\rho(\xi)\,\mathrm{d}\xi\right)^{j_k}d^{-\tau-(|\mathbf{j}|+1)n}(x,x-z)\times \\
&\qquad\times\left(\int_{\Omega(x,x+z)}\partial^l\nabla\rho(\xi)\,\mathrm{d}\xi-\int_{\Omega(x,x-z)}\partial^l\nabla\rho(\xi)\,\mathrm{d}\xi\right)z\,\mathrm{d}z \\
&= I_{\mathbf{j},l,2,1}[\rho u](x)+I_{\mathbf{j},l,2,2}[\rho u](x)+I_{\mathbf{j},l,2,3}[\rho u](x).
\end{aligned}$$

For $I_{\mathbf{j},l,2,1}[\rho u](x)$ and $I_{\mathbf{j},l,2,2}[\rho u](x)$ we make the same estimates as before, using

$$|d^{\tau+(|\mathbf{j}|+1)n}(x,x+z)-d^{\tau+(|\mathbf{j}|+1)n}(x,x-z)|\leq|\nabla\rho|_\infty|z|^{\tau+(\mathbf{j}+1)n+1}$$

and also applying the Maximal function to $\int_{\Omega(x,x+z)}\nabla\partial^l\rho(\xi)\,\mathrm{d}\xi\leq|z|^nM[\partial^{l+1}(\rho)](x)$ to get,

$$\begin{aligned}
|I_{\mathbf{j},l,2,1}[\rho u](x)| &\leq|\nabla(\rho u)|_\infty|\nabla\rho|_\infty M[\partial^{l+1}\rho](x)\prod_{k=1}^{m-l}(M[\partial^k\rho](x))^{j_k}\times \\
&\qquad\times\left(\int_{\mathbb{T}^n}\frac{h(|z|)}{|z|^{n+\alpha-2}}\,\mathrm{d}z\right) \\
\|I_{\mathbf{j},l,2,1}[\rho u]\|_2^2 &\leq|\nabla(\rho u)|_\infty^2|\nabla\rho|_\infty^2\|\partial^{l+1}\rho\|_q^2\prod_{k=1}^{m-l}\|\partial^k\rho\|_{p_k}^{2j_k} \\
&\leq|\nabla(\rho u)|_\infty^2|\nabla\rho|_\infty^2\|\rho\|^2_{\dot{H}^{l+1+n(\frac{1}{2}-\frac{1}{q})}}\prod_{k=1}^{m-l}\|\rho\|^{2j_k}_{\dot{H}^{k+n(\frac{1}{2}-\frac{1}{p_k})}}
\end{aligned}$$

where we used Holder's inequality with

$$\sum_{k=1}^{m-l}\frac{2j_k}{p_k}+\frac{2}{q}=1.$$

Picking $q = \frac{2m}{l}$ and $p_k = \frac{2m}{k}$ gives

$$\begin{aligned} \rho: &\quad l+1+\frac{n(m-l)}{2m} \leq m+\alpha, \\ \rho: &\quad k+\frac{n(m-k)}{2m} \leq m+\alpha. \end{aligned}$$

Then for $I_{\mathbf{j},l,2,2}[\rho u](x)$ we get

$$\begin{aligned} |I_{\mathbf{j},l,2,2}[\rho u](x)| &\leq |\nabla(\rho u)|_\infty M[\partial^{l+1}\rho](x) \sum_{k=1}^{m-l} \prod_{\substack{i=1\\ i\neq k}}^{m-l} (M[\partial^i \rho](x))^{j_i} M([\partial^k \rho](x))^{j_k-1} (D_{s,p_k}\partial^k \rho(x))^{1/p_k} \times \\ &\qquad \times \left(\int_{\mathbb{T}^n} \frac{h(|z|)}{|z|^{n+\alpha-1-s}} \,\mathrm{d}z \right) \\ \|I_{\mathbf{j},l,2,2}[\rho u]\|_2^2 &\leq |\nabla(\rho u)|_\infty^2 \|\partial^{l+1}\rho\|_q^2 \prod_{\substack{i=1\\ i\neq k}}^{m-l} \|\partial^i \rho\|_{p_i}^{2j_i} \|\partial^k \rho\|_{p_k}^{2(j_k-1)} \|\partial^k \rho\|_{W^{s,p_k}}^2 \\ &\leq |\nabla(\rho u)|_\infty^2 \|\partial^{l+1}\rho\|^2_{\dot{H}^{l+1+n(\frac{1}{2}-\frac{1}{q})}} \prod_{\substack{i=1\\ i\neq k}}^{m-l} \|\rho\|^{2j_i}_{\dot{H}^{i+n(\frac{1}{2}-\frac{1}{p_i})}} \|\rho\|^{2(j_k-1)}_{\dot{H}^{k+n(\frac{1}{2}-\frac{1}{p_k})}} \|\rho\|^2_{\dot{H}^{k+s+n(\frac{1}{2}-\frac{1}{p_k})}}. \end{aligned}$$

Picking the same Holder conjugates gives

$$\begin{aligned} \rho: &\quad l+1+\frac{n(m-l)}{2m} \leq m+\alpha \\ \rho: &\quad k+\frac{n(m-k)}{2m} \leq m+\alpha \\ \rho: &\quad k+s+\frac{n(m-k)}{2m} \leq m+\alpha \end{aligned}$$

To estimate $I_{\mathbf{j},l,2,3}[\rho u](x)$ we note that a similar computation as before gives

$$\int_{\Omega(x,x+z)} \partial^l \nabla \rho(\xi)\,\mathrm{d}\xi - \int_{\Omega(x,x-z)} \partial^l \nabla \rho(\xi)\,\mathrm{d}\xi \leq |z|^{n+s} (D_{s,q}\partial^{l+1}\rho(x))^{1/q}.$$

Therefore, again using the maximal function we get,

$$\begin{aligned} |I_{\mathbf{j},l,2,3}[\rho u](x)| &\leq |\nabla(\rho u)|_\infty (D_{s,q}\partial^{l+1}\rho(x))^{1/q} \prod_{k=1}^{m-l} (M[\partial^k \rho](x))^{j_k} \int_{\mathbb{T}^n} \frac{h(|z|)}{|z|^{n+\alpha-1-s}} \,\mathrm{d}z \\ \|I_{\mathbf{j},l,2,3}[\rho u]\|_2^2 &\leq |\nabla(\rho u)|_\infty^2 \|\rho\|^2_{\dot{H}^{l+1+s+n(\frac{1}{2}-\frac{1}{q})}} \prod_{k=1}^{m-l} \|\rho\|^{2j_k}_{\dot{H}^{k+n(\frac{1}{2}-\frac{1}{p_k})}}. \end{aligned}$$

Choosing the same Holder conjugates gives,

$$\begin{aligned} \rho: &\quad l+1+s+\frac{n(m-l)}{2m} \leq m+\alpha, \\ \rho: &\quad k+\frac{n(m-k)}{2m} \leq m+\alpha. \end{aligned}$$

Intermediary Cases. For all $l = 1, ..., m-1$, $i = 0, ..., m-l$, and $k = 1, ..., m-l-i$, we have to estimate

$$I_{\mathbf{j},i}[\partial^l(\rho u)](x) = \int_{\mathbb{T}^n} \frac{h(|z|)}{|z|^{n+\alpha-\tau}} \int_{\Omega(x,x+z)} \partial^i \nabla\rho(\xi)\,\mathrm{d}\xi \frac{\prod_{k=1}^{m-l-i}\left(\int_{\Omega(x,x+z)} \partial^k\rho(\xi)\,\mathrm{d}\xi\right)^{j_k}}{d^{\tau+(|j|+1)n}(x,x+z)} \cdot \delta_z\partial^l(\rho u)(x)\,\mathrm{d}z.$$

First, for $0 < \alpha < 1$, we employ the Maximal functions again to get,

$$|I_{\mathbf{j},i}[\partial^l(\rho u)](x)| \lesssim \prod_{k=1}^{m-l-i}\Big(M[\partial^k\rho](x)\Big)^{j_k} M[\partial^{i+1}\rho](x)\int_{\mathbb{T}^n}\frac{h(|z|)}{|z|^{n+\alpha}}|\delta_z\partial^l(\rho u)(x)|\,\mathrm{d}z.$$

Then estimating in L^2-norm, applying Holder's inequality with

$$\sum_{k=1}^{m-l-i}\frac{2j_k}{p_k}+\frac{2}{q_1}+\frac{2}{q_2}=1,$$

and using the Hardy-Littlewood inequality, we get

$$\|I_{\mathbf{j},i}[\partial^l(\rho u)]\|_2^2 \lesssim \prod_{k=1}^{m-l-i}\|\rho\|^{2j_k}_{\dot{H}^{k+n(\frac{1}{2}-\frac{1}{p_k})}}\|\rho\|^2_{\dot{H}^{i+1+n(\frac{1}{2}-\frac{1}{q_1})}}\|\rho u\|^2_{\dot{H}^{l+\alpha+\varepsilon+n(\frac{1}{2}-\frac{1}{q_2})}}.$$

Now we choose $p_k=\frac{2m}{k}$, $q_1=\frac{2m}{i}$, and $q_2=\frac{2m}{l}$. Provided m is large enough and ε is small enough,

$$\begin{aligned}
\rho:&\quad k+\frac{n}{2}\left(\frac{m-k}{m}\right)\le m-1+\alpha\\
\rho:&\quad i+1+\frac{n}{2}\left(\frac{m-i}{m}\right)\le m+\alpha\\
\rho u:&\quad l+\alpha+\varepsilon+\frac{n}{2}\left(\frac{m-l}{m}\right)\le m+\alpha,
\end{aligned}$$

for all $l=1,...,m-1, i=0,...,m-l$, and $k=1,...,m-l-j$.

As before, to extend the argument to include $\alpha\ge1$, we must include the next term in the Taylor finite difference

$$\begin{aligned}
I_{\mathbf{j},i}[\partial^l(\rho u)](x) = &\int_{\mathbb{T}^n}\frac{h(|z|)}{|z|^{n+\alpha-\tau}}\frac{\prod_{k=1}^{m-l-i}\left(\int_{\Omega(x,x+z)}\partial^k\rho(\xi)\,\mathrm{d}\xi\right)^{j_k}}{d^{\tau+(|\mathbf{j}|+1)n}}\times\\
&\times\left(\int_{\Omega(x,x+z)}\partial^i\nabla\rho(\xi)\,\mathrm{d}\xi\right)\cdot[\delta_z\partial^l(\rho u)(x)-z\nabla\partial^l\rho u(x)]\,\mathrm{d}z\\
&+\int_{\mathbb{T}^n}\frac{h(|z|)}{|z|^{n+\alpha-\tau}}\int_{\Omega(x,x+z)}\partial^i\nabla\rho(\xi)\,\mathrm{d}\xi\frac{\prod_{k=1}^{m-l-i}\left(\int_{\Omega(x,x+z)}\partial^k\rho(\xi)\,\mathrm{d}\xi\right)^{j_k}}{d^{\tau+(|\mathbf{j}|+1)n}}\cdot z\nabla\partial^l(\rho u)(x)\,\mathrm{d}z\\
&:= I_{\mathbf{j},i,1}[\partial^l(\rho u)](x)+I_{\mathbf{j},i,2}[\partial^l(\rho u)](x).
\end{aligned}$$

Again, the estimate on $I_{\mathbf{j},i,1}[\partial^l(\rho u)]$ goes as before, and for $I_{\mathbf{j},i,2}[\partial^l(\rho u)](x)$ we symmetrize

$$
\begin{aligned}
I_{\mathbf{j},i,2}[\partial^l(\rho u)](x) = \int_{\mathbb{T}^n} \frac{h(|z|)}{|z|^{n+\alpha-\tau}} \Big[\int_{\Omega(x,x+z)} \partial^i \nabla\rho(\xi)\,\mathrm{d}\xi \frac{\prod_{k=1}^{m-l-i} \left(\int_{\Omega(x,x+z)} \partial^k \rho(\xi)\,\mathrm{d}\xi \right)^{j_k}}{d^{\tau+(|\mathbf{j}|+1)n}(x,x+z)} \\
- \int_{\Omega(x,x-z)} \partial^i \nabla\rho(\xi)\,\mathrm{d}\xi \frac{\prod_{k=1}^{m-l-i} \left(\int_{\Omega(x,x-z)} \partial^k \rho(\xi)\,\mathrm{d}\xi \right)^{j_k}}{d^{\tau+(|\mathbf{j}|+1)n}(x,x-z)} \Big] \cdot z \nabla \partial^l(\rho u(x))\,\mathrm{d}z \\
= \nabla\partial^l(\rho u)(x) \int_{\mathbb{T}^n} \frac{h(|z|)}{|z|^{n+\alpha-\tau}} \int_{\Omega(x,x+z)} \partial^i \nabla\rho(\xi)\,\mathrm{d}\xi \prod_{k=1}^{m-l-i} \left(\int_{\Omega(x,x+z)} \partial^k \rho(\xi)\,\mathrm{d}\xi \right)^{j_k} \times \\
\times \left(d^{-\tau-(|\mathbf{j}|+1)n}(x,x+z) - d^{-\tau-(|\mathbf{j}|+1)n}(x,x-z) \right) z\,\mathrm{d}z \\
+ \nabla\partial^l(\rho u)(x) \int_{\mathbb{T}^n} \frac{h(|z|)}{|z|^{n+\alpha-\tau}} \int_{\Omega(x,x+z)} \partial^i \nabla\rho(\xi)\,\mathrm{d}\xi d^{-\tau-(|\mathbf{j}|+1)n}(x,x+z) \times \\
\times \left(\prod_{k=1}^{m-l-i} \left(\int_{\Omega(x,x+z)} \partial^k \rho(\xi)\,\mathrm{d}\xi \right)^{j_k} - \prod_{k=1}^{m-l-i} \left(\int_{\Omega(x,x-z)} \partial^k \rho(\xi)\,\mathrm{d}\xi \right)^{j_k} \right) z\,\mathrm{d}z \\
+ \nabla\partial^l(\rho u)(x) \int_{\mathbb{T}^n} \frac{h(|z|)}{|z|^{n+\alpha-\tau}} \prod_{k=1}^{m-l-i} \left(\int_{\Omega(x,x+z)} \partial^k \rho(\xi)\,\mathrm{d}\xi \right)^{j_k} d^{-\tau-(|\mathbf{j}|+1)n}(x,x+z) \times \\
\times \left(\int_{\Omega(x,x+z)} \partial^i \nabla\rho(\xi)\,\mathrm{d}\xi - \int_{\Omega(x,x-z)} \partial^i \nabla\rho(\xi)\,\mathrm{d}\xi \right) z\,\mathrm{d}z \\
= I_{\mathbf{j},i,2,1}[\partial^l(\rho u)](x) + I_{\mathbf{j},i,2,3}[\partial^l(\rho u)](x) + I_{\mathbf{j},i,2,3}[\partial^l(\rho u)](x).
\end{aligned}
$$

For $I_{\mathbf{j},i,2,1}[\partial^l(\rho u)](x)$ and $I_{\mathbf{j},i,2,2}[\partial^l(\rho u)](x)$ we apply the same estimates as above to get

$$
|I_{\mathbf{j},i,2,1}[\partial^l(\rho u)](x)| \leq |\nabla\partial^l(\rho u)(x)| \prod_{k=1}^{m-l-i} (M[\partial^k \rho](x))^{j_k} M[\partial^{i+1}\rho](x) \int_{\mathbb{T}^n} h(|z|) \frac{\mathrm{d}z}{|z|^{n+\alpha-2}}.
$$

Since $\alpha < 2$, the integral converges, and

$$
\begin{aligned}
\|I_{\mathbf{j},i,2,1}[\partial^l(\rho u)]\|_2^2 \lesssim \|\partial^{l+1}(\rho u)\|_{q_2}^2 \|\partial^{i+1}\rho\|_{q_1}^2 \prod_{k=1}^{m-l-i} \|\partial^k \rho\|_{p_k}^{2j_k} \\
\leq \|\rho u\|^2_{\dot{H}^{l+1+n(\frac{1}{2}-\frac{1}{q_2})}} \|\rho\|^2_{\dot{H}^{i+1+n(\frac{1}{2}-\frac{1}{q_1})}} \prod_{k=1}^{m-l-i} \|\rho\|^{2j_k}_{\dot{H}^{k+n(\frac{1}{2}-\frac{1}{p_k})}}.
\end{aligned}
$$

Choosing the Holder conjugates as before blends this into the previous case. For $I_{\mathbf{j},i,2,2}[\partial^l(\rho u)](x)$ we have,

$$
\begin{aligned}
|I_{\mathbf{j},i,2,2}[\partial^l(\rho u)](x)| \leq |\nabla\partial^l(\rho u)(x)| M[\partial^{i+1}\rho](x) \times \\
\times \sum_{k=1}^{m-l-i} \prod_{\substack{\lambda=1 \\ i\neq k}}^{m-l-i} (M[\partial^\lambda \rho](x))^{j_\lambda} (M[\partial^k \rho](x))^{j_k-1} (D_{s,p_k} \partial^k \rho(x))^{1/p_k} \int_{\mathbb{T}^n} \frac{h(|z|)}{|z|^{n+\alpha-1-s}}\,\mathrm{d}z.
\end{aligned}
$$

Since $\alpha < 1 + s < 2$, the integral converges, so for any $k = 1, ..., m-l-i$,

$$\|I_{\mathbf{j},i,2,2}[\partial^l(\rho u)]\|_2^2 \leq \|\partial^{l+1}(\rho u)\|_{q_2}^2 \|\partial^{i+1}\rho\|_{q_1}^2 \prod_{\substack{\lambda=1 \\ i\neq k}}^{m-l-i} \|\partial^\lambda \rho\|_{p_\lambda}^{2j_\lambda} \|\partial^k \rho\|_{p_k}^{2(j_k-1)} \|\partial^k\rho\|_{W^{s,p_k}}^2$$

$$\leq \|(\rho u)\|^2_{\dot H^{l+1+n(\frac12-\frac{1}{q_2})}} \|\rho\|^2_{\dot H^{i+1+n(\frac12-\frac{1}{q_1})}} \prod_{\substack{\lambda=1 \\ i\neq k}}^{m-l-i} \|\rho\|^{2j_\lambda}_{\dot H^{\lambda+n(\frac12-\frac{1}{p_\lambda})}} \|\rho\|^{2(j_k-1)}_{\dot H^{k+n(\frac12-\frac{1}{p_k})}} \|\rho\|^2_{\dot H^{k+s+n(\frac12-\frac{1}{p_k})}}.$$

Again choosing the same Holder conjugates as before gives the necessary bound. Now for $I_{\mathbf{j},i,2,3}[\partial^l(\rho u)]$ we get,

$$\|I_{\mathbf{j},i,2,3}[\partial^l(\rho u)]\|_2^2 \lesssim \|(\rho u)\|^2_{\dot H^{l+1+n(\frac12-\frac{1}{q_2})}} \|\rho\|^2_{\dot H^{i+1+s+n(\frac12-\frac{1}{q_1})}} \prod_{k=1}^{m-l-i} \|\rho\|^{2j_k}_{\dot H^{k+n(\frac12-\frac{1}{p_k})}}.$$

Choosing the same Holder conjugates again gives the desired bound. Therefore we have the necessary bounds for every term in $\partial^m \mathcal{L}_{\nabla\phi\cdot}(\rho u)$.

Now let us examine $\mathcal{L}_{\phi_t}(\rho)$. Notice that any term in $\partial^m\mathcal{L}_{\phi_t}(\rho)$ takes the form

$$I = \int_{\mathbb{T}^n} \frac{h(|z|)}{|z|^{n+\alpha-\tau}} \int_{\Omega(x,x+z)} \partial^i \nabla\cdot(\rho u)(\xi)\,\mathrm{d}\xi \frac{\prod_{k=1}^{m-l-i}\left(\int_{\Omega(x,x+z)} \partial^k\rho(\xi)\,\mathrm{d}\xi\right)^{j_k}}{d^{\tau+(|\mathbf{j}|+1)n}(x,x+z)} \delta_z\partial^l\rho(x)\,\mathrm{d}z.$$

The cases where $l = 1, ..., m-1$ are estimated exactly the same as the Intermediary case for $\mathcal{L}_{\nabla\phi\cdot}(\rho u)$ above by switching the roles of ρu and ρ in the increment δ_z and in the first integral that contains the gradient.

Similarly the case where $l = 0$ and $i = 0, ..., m-1$ is taken care of by End Case 5. Further, we have already used the case where $l = m$ during the estimates in End Case 2, and part of the term $l = 0, i = m$ in End Case 1. Since $\nabla\partial^m(\rho u) = \nabla(u\partial^m\rho) + \nabla\partial^{m-1}(\rho\partial u)$ we still have to estimate the term

$$J = \int_{\mathbb{T}^n} \frac{h(|z|)}{|z|^{n+\alpha-\tau}} \frac{\int_{\Omega(x,x+z)} \nabla\partial^{m-1}(\rho\partial u)\,\mathrm{d}\xi}{d^{\tau+n}(x,x+z)} \delta_z\rho(x)\,\mathrm{d}z.$$

For $\alpha < 1$ we use $|\delta_z\rho(x)| \leq |\nabla\rho|_\infty|z|$ and the maximal function to get

$$|J| \leq |\nabla\rho|_\infty M[\partial^m(\rho\partial u)](x) \int_{\mathbb{T}^n} \frac{h(|z|)}{|z|^{n+\alpha-1}}\,\mathrm{d}z$$
$$\|J\|_2^2 \leq |\nabla\rho|_\infty^2 \|\rho\|^2_{\dot H^m} \|u\|^2_{\dot H^{m+1}}.$$

For $1 \leq \alpha < 2$ we utilize the next Taylor term again and estimate the second of these by symmetrizing and splitting into two parts to get,

$$J_2 = \nabla\rho(x) \int_{\mathbb{T}^n} \frac{h(|z|)}{|z|^{n+\alpha-\tau}} \frac{\int_{\Omega(x,x+z)} \nabla\partial^{m-1}(\rho\partial u)\,\mathrm{d}\xi}{d^{\tau+n}(x,x+z)} z\,\mathrm{d}z$$
$$J_{21} = \nabla\rho(x) \int_{\mathbb{T}^n} \frac{h(|z|)}{|z|^{n+\alpha-\tau}} \frac{\int_{\Omega(x,x+z)} \nabla\partial^{m-1}(\rho\partial u)\,\mathrm{d}\xi - \int_{\Omega(x,x-z)} \nabla\partial^{m-1}(\rho\partial u)\,\mathrm{d}\xi}{d^{\tau+n}(x,x+z)} z\,\mathrm{d}z$$
$$J_{22} = \nabla\rho(x) \int_{\mathbb{T}^n} \frac{h(|z|)}{|z|^{n+\alpha-\tau}} \int_{\Omega(x,x-z)} \nabla\partial^{m-1}(\rho\partial u)\,\mathrm{d}\xi \left(d^{-\tau-n}(x,x+z) - d^{-\tau-n}(x,x-z)\right) z\,\mathrm{d}z.$$

Estimating J_{21} gives

$$
\begin{aligned}
|J_{21}| &\leq |\nabla \rho|_\infty (D_{s,2}(\partial^m(\rho \partial u)))^{1/2} \int_{\mathbb{T}^n} \frac{h(|z|)}{|z|^{n+\alpha-1-s}} \, \mathrm{d}z \\
\|J_{21}\|_2^2 &\leq |\nabla \rho|_\infty^2 \|\rho\|_{\dot{H}^{m+\alpha}}^2 \|u\|_{\dot{H}^{m+1+s}}^2 \\
&\leq Y_m^N \|u\|_{\dot{H}^{m+1}}^2 + \varepsilon \|u\|_{\dot{H}^{m+1+\frac{\alpha}{2}}},
\end{aligned}
$$

where we used Interpolation and Young's inequality to get the last inequality. Since $1 \leq \alpha < 2$ it is possible to find an s such that $s \leq \alpha/2 < 1$ for interpolation and $1 + s > \alpha$ to make the above integral finite.

For J_{22} we use the differences in $d^{-\tau-n}$ to get

$$
\begin{aligned}
|J_{22}| &\leq |\nabla \rho|_\infty^2 M[\partial^m(\rho \partial u)] \int_{\mathbb{T}^n} \frac{h(|z|)}{|z|^{n+\alpha-2}} \, \mathrm{d}z \\
\|J_{22}\|_2^2 &\leq |\nabla \rho|_\infty^2 \|\rho\|_{\dot{H}^m}^2 \|u\|_{\dot{H}^{m+1}}^2.
\end{aligned}
$$

This covers all the terms in $\partial^m \mathcal{L}_{\phi_t}(\rho)$. Recalling that the goal is to bound everything by the grand quantity Y_m^N, we have shown that

$$
\|\partial^m(\mathcal{L}_{\phi_t}(\rho) + \mathcal{L}_{\nabla\phi\cdot}(\rho u))\partial^m e\|_2 \leq Y_m^N.
$$

Combined with the transport terms we have estimated in the beginning we therefore have proved the desired a priori bound

$$
\frac{\mathrm{d}}{\mathrm{d}t} \|e\|_{\dot{H}^m}^2 \leq C Y_m^N.
$$

6. Viscous regularization, local existence and uniqueness

To actually produce local solutions we consider viscous regularization of the system

$$
\begin{aligned}
\rho_t + \nabla \cdot (u\rho) &= \varepsilon \Delta \rho \\
u_t + u \cdot \nabla u &= \mathcal{C}_\phi(u, \rho) + \varepsilon \Delta u,
\end{aligned}
\tag{43}
$$

First, we show that this regularization is sufficient to obtain local solutions via the standard fixed point argument. Second, we show that such regularization does not interfere with the a priori estimates we have obtained in the previous sections.

To prove local estimates of smooth solutions to (43) we consider the mild formulation

$$
\begin{aligned}
\rho(t) &= e^{\varepsilon t \Delta} \rho_0 - \int_0^t e^{\varepsilon(t-s)\Delta} \nabla \cdot (u\rho)(s) \, \mathrm{d}s \\
u(t) &= e^{\varepsilon t \Delta} u_0 - \int_0^t e^{\varepsilon(t-s)\Delta} u \cdot \nabla u(s) \, \mathrm{d}s + \int_0^t e^{\varepsilon(t-s)\Delta} \mathcal{C}_\phi(u, \rho)(s) \, \mathrm{d}s.
\end{aligned}
\tag{44}
$$

Let us denote by $Z = (\rho, u)$ the state variable of our system and by $T[Z](t)$ the right hand side of the mild formulation. In order to apply the stadard fixed point argument we have to show that T leaves the set $C([0, T_{\delta,\varepsilon}); B_\delta(Z_0))$ invariant, where $B_\delta(Z_0)$ is the ball of radius ε around initial condition Z_0, and that it is a contraction. We limit ourselves to showing details for invariance as the estimates involved there are identical to those required to also prove Lipschitzness.

First we assume that ρ has no vacuum: $\rho_0(x) \geq c_0 > 0$. The metric we are using the same as before $\rho \in \dot{H}^{m+\alpha} \cap L^1$, $u \in H^{m+1}$. Note that if $\delta > 0$ is small enough then for any $\|\rho - \rho_0\|_{\dot{H}^{m+\alpha}} < \delta$ which has the same mass $\int \rho = \int \rho_0$, one obtains

$$
\rho(x) > \frac{1}{2} c_0. \tag{45}
$$

So, let us assume that $Z \in C([0,T_\delta); B_\delta(Z_0))$. It is clear that $\|e^{\varepsilon t\Delta} Z_0 - Z_0\| < \frac{\delta}{2}$ provided time t is short enough. The Z has some bound $\|Z\| \leq C$. Using that let us estimate the norms under the integrals. First, recall that $\|\Lambda_\alpha e^{\varepsilon t\Delta}\|_{L^2 \to L^2} \lesssim \frac{1}{t^{\alpha/2}}$. In the case $\alpha \geq 1$, we have

$$\begin{aligned}\left\| \partial^m \Lambda_\alpha \int_0^t e^{\varepsilon(t-s)\Delta} \nabla \cdot (u\rho)(s)\,\mathrm{d}s \right\|_2 &\leq \int_0^t \frac{1}{(t-s)^{\alpha/2}} \|\partial^{m+1}(u\rho)(s)\|_2 \,\mathrm{d}s \\ &\leq \int_0^t \frac{1}{(t-s)^{\alpha/2}} \|u\|_{\dot H^{m+1}} \|\rho\|_{\dot H^{m+\alpha}} \,\mathrm{d}s \leq C^2 t^{1-\alpha/2} < \frac{\delta}{2},\end{aligned}$$

provided T_δ is small enough. In the case $\alpha < 1$, we combine instead one full derivatives with the heat semigroup, and the rest $\partial^{m+\alpha}$ gets applied to $u\rho$, which produces a similar bound.

Moving on to the u-equation, we have

$$\begin{aligned}\left\| \partial^{m+1} \int_0^t e^{\varepsilon(t-s)\Delta} u \cdot \nabla u(s)\,\mathrm{d}s \right\|_2 &\leq \int_0^t \frac{1}{(t-s)^{1/2}} \|\partial^m (u\cdot\nabla u)(s)\|_2 \,\mathrm{d}s \\ &\leq \int_0^t \frac{1}{(t-s)^{\alpha/2}} \|u\|_{\dot H^{m+1}} \|u\|_{\dot H^m} \,\mathrm{d}s \leq C^2 t^{1/2} < \frac{\delta}{2}.\end{aligned}$$

As to the commutator form, for $\alpha \leq 1$ the computation is very similar: we combine one derivative with the heat semigroup and for the rest we use (15):

$$\|\partial^m \mathcal{C}_\phi(u,\rho)\|_2 \leq \|u\|_{m+\alpha}^N \|\rho\|_{m+\alpha}^N < C^{2N},$$

and the rest follows as before. When $\alpha > 1$ we need to use the refined estimate (17). Namely, it follows from the first in (17) by keeping the highest norms only,

$$\begin{aligned}\|\mathcal{L}_\phi f\|_{\dot H^m} &\lesssim c_\varepsilon \|\rho\|_{\dot H^{m-1+\alpha+\varepsilon}}^N \|f\|_{\dot H^{m+\alpha}} \\ \|\mathcal{L}_\phi f\|_{\dot H^{m-1}} &\lesssim c_\varepsilon \|\rho\|_{\dot H^{m-2+\alpha+\varepsilon}}^N \|f\|_{\dot H^{m-1+\alpha}}\end{aligned}$$

Therefore, by interpolation, we have an estimate in the fractional space $\dot H^{m-1+s}$ for $0 < s < 1$:

$$\|\mathcal{L}_\phi f\|_{\dot H^{m-1+s}} \lesssim \|\rho\|_{\dot H^{m-1+\alpha+\varepsilon}}^N \|f\|_{\dot H^{m-1+\alpha+s}} \tag{46}$$

Taking $s = 2-\alpha$ yields

$$\|\mathcal{L}_\phi f\|_{\dot H^{m+1-\alpha}} \lesssim \|\rho\|_{\dot H^{m-1+\alpha+\varepsilon}}^N \|f\|_{\dot H^{m+1}}. \tag{47}$$

Combining α derivatives with the heat, and using the inequality above with $\varepsilon = 1$, we obtain

$$\begin{aligned}\left\| \int_0^t \Lambda^\alpha e^{\varepsilon(t-s)\Delta} \Lambda^{m+1-\alpha} \mathcal{C}_\phi(u,\rho)u(s)\,\mathrm{d}s \right\|_2 &\leq \int_0^t \frac{1}{(t-s)^{\alpha/2}} [\|\mathcal{L}_\phi(u\rho)\|_{\dot H^{m+1-\alpha}} + \|u\mathcal{L}_\phi(\rho)\|_{\dot H^{m+1-\alpha}}]\,\mathrm{d}s \\ &\leq \int_0^t \frac{1}{(t-s)^{\alpha/2}} \|\rho\|_{\dot H^{m+\alpha}}^N \|u\|_{\dot H^{m+1}} \,\mathrm{d}s \leq C^2 t^{1-\alpha/2} < \frac{\delta}{2}.\end{aligned}$$

We have proved that $\|T[Z](t) - Z_0\| < \delta$, and the proof is complete.

The obtained interval of existence of course depends on ε as it enters into all the estimates of the integrals. In order to conclude the local existence argument we still have to show that our a priori bound

$$\frac{\mathrm{d}}{\mathrm{d}t} Y_m \lesssim Y_m^N \tag{48}$$

is independent of ε. This would allow us to extend $T_{\varepsilon,\delta}$ to a time dependent on the initial condition only. Then the classical compactness argument would apply to pass to the limit as $\varepsilon \to 0$ in the same state space $(u,\rho) \in C_w([0,T); (\dot H^{m+\alpha} \cap L^1) \times H^{m+1})$.

It is clear that the u-equation will not see the effect of viscous regularization because the term produced by the energy method is $-\varepsilon\|\partial^{m+2}u\|_2^2$. The e-equation, however, will produce several extra terms:

$$e_t + \nabla\cdot(ue) = (\nabla\cdot u)^2 - \mathrm{Tr}(\nabla u)^2 + \mathcal{L}_{\phi_t}(\rho) + \mathcal{L}_{\nabla\phi\cdot}(\rho u) - 2\varepsilon\mathcal{L}_{\nabla\phi}\nabla\rho - \varepsilon\mathcal{L}_{\Delta\phi}\rho + \varepsilon\Delta e. \tag{49}$$

After the test, the extra terms become

$$\begin{aligned} & -\varepsilon\|e\|^2_{\dot H^{m+1}} - 2\varepsilon\langle\partial^{m-1}\mathcal{L}_{\nabla\phi}\nabla\rho, \partial^{m+1}e\rangle - \varepsilon\langle\partial^{m-1}\mathcal{L}_{\Delta\phi}\rho, \partial^{m+1}e\rangle \\ & \qquad\qquad \le -\frac{1}{2}\varepsilon\|e\|^2_{\dot H^{m+1}} + 8\varepsilon\|\partial^{m-1}\mathcal{L}_{\nabla\phi}\nabla\rho\|_2^2 + 4\varepsilon\|\partial^{m-1}\mathcal{L}_{\Delta\phi}\rho\|_2^2. \end{aligned} \tag{50}$$

Let us observe that the residual two terms present special parts of the expansion of the commutator we have estimated in Lemma 3.4 for $m \to m+1$. So, from (22) we obtain

$$\begin{aligned} \|\partial^{m-1}\mathcal{L}_{\nabla\phi}\nabla\rho\|_2^2 + \|\partial^{m-1}\mathcal{L}_{\Delta\phi}\rho\|_2^2 \lesssim \|\rho\|^N_{\dot H^{m+\alpha}}(\|\rho\|^2_{\dot H^{m+\frac12+\alpha}} + \|\rho\|^2_{\dot H^{m+1+\frac{\alpha}{2}}}) + \\ + (\|\rho\|^2_{\dot H^{m+1}} + \|\rho\|^2_{\dot H^{m+\frac12+\alpha}})\|\rho\|^2_{\dot H^{2+\frac{n}{2}}}. \end{aligned}$$

Let us recall that we have another ε-gain term from viscous regularization:

$$-\varepsilon\|\partial^{m+2}u\|^2_{\dot H^{m+1}} - \frac12\varepsilon\|e\|^2_{\dot H^{m+1}} \lesssim -\varepsilon\underline{\rho}^{-2\tau/n}\|\rho\|^2_{\dot H^{m+1+\alpha}} + \varepsilon Y_m^N.$$

By a computation similar to (21) the residual term can be estimated by

$$\varepsilon\|\partial^{m-1}\mathcal{L}_{\nabla\phi}\nabla\rho\|_2^2 + \varepsilon\|\partial^{m-1}\mathcal{L}_{\Delta\phi}\rho\|_2^2 \lesssim \frac12\varepsilon\underline{\rho}^{-2\tau/n}\|\rho\|^2_{\dot H^{m+1+\alpha}} + \varepsilon Y_m^N.$$

So, the total influence of the viscous term on a priori estimates will be an additional εY_m^N added to (48) which has no effect.

Having obtained uniformly bounded solutions $(u^\varepsilon, \rho^\varepsilon) \in C([0,T); H^{m+1}\times H^{m+\alpha})$ on a common time interval we pass to the w^*-limit in the top space and strong limit in any lower regularity space $H^{m+1-\delta}\times H^{m+\alpha-\delta}$, which guarantees that the limit will actually be weakly continuous in the top space. This concludes the proof of local existence.

Let us briefly address the uniqueness as it is essentially a straightforward consequence of the estimates we obtained so far. One assumes that there is a pair of solutions (u', ρ'), (u'', ρ'') in the same local class (6) sharing the same initial data. Let us note that the kernels being active will differ as well, denote them ϕ', ϕ'', respectively. Denote $\rho = \rho' - \rho''$, $u = u' - u''$, $\phi = \phi' - \phi''$, $e = e' - e''$. We write the system for the tripple (u, ρ, e):

(51)

$$\begin{aligned} &\partial_t\rho + \nabla\cdot(u\rho') + \nabla\cdot(u''\rho) = 0 \\ &\quad u_t + u\cdot\nabla u' + u''\cdot\nabla u = \mathcal{C}_\phi(u', \rho') + \mathcal{C}_{\phi''}(u, \rho') + \mathcal{C}_{\phi''}(u'', \rho), \\ &\quad e_t + \nabla\cdot(ue') + \nabla\cdot(u''e) = \nabla\cdot u\nabla\cdot u' + \nabla\cdot u''\nabla\cdot u - \mathrm{Tr}(\nabla u\nabla u') - \mathrm{Tr}(\nabla u''\nabla u) \\ &\qquad\qquad + \partial_t(\mathcal{L}_\phi(\rho')) + \partial_t(\mathcal{L}_{\phi''}(\rho)) + \nabla\cdot\mathcal{L}_\phi(\rho' u') + \nabla\cdot\mathcal{L}_{\phi''}(\rho u') + \nabla\cdot\mathcal{L}_{\phi''}(\rho'' u). \end{aligned}$$

The analysis of this system resembles very closely the analysis we undertook to reach the Riccati estimate for the grand quantity Y_m. We simply note that the u-equation shares the same dissipative structure with $\mathcal{C}_{\phi''}(u, \rho')$ being the princial diffusion term, albeit with kernel being dependent on ρ'' rather than ρ'. This discrepancy, however, does not alter the estimate (35), as we assume no-vacuum for both densities to hold. With the dissipation at hand, the rest of the estimates repeat those presented above, and it would be impractical to reproduce the entire argument for the remaining parts of the system (51).

7. Appendix: variants of intrinsic definitions of a Sobolev space.

In the proof we encountered the following quantity

$$D_\alpha(g) = \int_{\mathbb{T}^{2n}} \frac{h(z)\,|g(x+|z|U_z\theta) - g(x)|^2}{|z|^{n+\alpha}}\,\mathrm{d}z\,\mathrm{d}x,$$

where $\theta \in \partial\Omega(0,\mathbf{e}_1)$. It is not essential where exactly θ is localted as long as it is uniformly bounded.

Lemma 7.1. *Let $0<\alpha<2$. Then there exists a constant $C_\alpha>0$ such that for all $g \in H^{\alpha/2}$ one has*

$$D_\alpha(g) \le C_s \|g\|_{\dot{H}^{\alpha/2}}.$$

Proof.

$$\int_{\mathbb{T}^{2n}} \frac{h(z)\,|g(x+|z|U_z\theta) - g(x)|^2}{|z|^{n+\alpha}}\,\mathrm{d}z\,\mathrm{d}x = \sum_{\mathbf{k}\in\mathbb{Z}^n} |\widehat{g}(\mathbf{k})|^2 \int_{\mathbb{T}^n} \frac{h(z)\,\left|e^{i\mathbf{k}\cdot|z|U_z\theta} - 1\right|^2}{|z|^{n+\alpha}}\,\mathrm{d}z.$$

Since

$$\left|e^{i\mathbf{k}\cdot|z|U_z\theta} - 1\right| \le \min\{2, |\mathbf{k}||z|\}$$

the splitting of the integral into small scale $|z|<1/|\mathbf{k}|$ and large scale $|z|>1/|\mathbf{k}|$ as in the classical case, shows that the integral is bounded by $|\mathbf{k}|^\alpha$ which implies the claim. □

Similarly goes the proof of the next lemma.

Lemma 7.2. *Let $0<\alpha<2$. Then there exists a constant $C_\alpha>0$ such that for all $g\in H^\alpha$ one has*

$$\left\| \int_{\mathbb{T}^n} [g(\cdot + |z|U_z\theta) + g(\cdot - |z|U_z\theta) - 2g(\cdot)] \frac{h(|z|)z_i U_z^{jk}}{|z|^{n+\alpha+1}} dz \right\|_2 \le C_\alpha \|g\|_{\dot{H}^\alpha}.$$

Proof.

$$\begin{aligned}
&\left\| \int_{\mathbb{T}^n} [g(\cdot + |z|U_z\theta) + g(\cdot - |z|U_z\theta) - 2g(\cdot)] \frac{h(|z|)z_i U_z^{jk}}{|z|^{n+\alpha+1}} dz \right\|_2^2 = \\
&= \sum_{\mathbf{k}\in\mathbb{Z}^n} |\widehat{g}(\mathbf{k})|^2 \left| \int_{\mathbb{T}^n} (e^{i\mathbf{k}\cdot|z|U_z\theta} + e^{-i\mathbf{k}\cdot|z|U_z\theta} - 2) \frac{h(|z|)z_i U_z^{jk}}{|z|^{n+\alpha+1}} dz \right|^2 \\
&\le \sum_{\mathbf{k}\in\mathbb{Z}^n} |\widehat{g}(\mathbf{k})|^2 \left| \int_{\mathbb{T}^n} |e^{i\mathbf{k}\cdot|z|U_z\theta} + e^{-i\mathbf{k}\cdot|z|U_z\theta} - 2| \frac{h(|z|)}{|z|^{n+\alpha}} dz \right|^2.
\end{aligned}$$

The integral is estimated with the use of

$$|e^{i\mathbf{k}\cdot|z|U_z\theta} + e^{-i\mathbf{k}\cdot|z|U_z\theta} - 2| \le \min\{3, |z|^2|\mathbf{k}|^2\}$$

and splitting as before into $|z|<1/|\mathbf{k}|$ and $|z|>1/|\mathbf{k}|$. The result is $|\mathbf{k}|^\alpha$ and the formula follows. □

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Department of Mathematics, Statistics and Computer Science, University of Illinois at Chicago, 60607
Email address: dreyno8@uic.edu
Email address: shvydkoy@uic.edu