LONG TIME DYNAMICS FOR COMBUSTION IN RANDOM MEDIA

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Abstract. We study long time dynamics of combustive processes in random media, modeled by reaction-diffusion equations with random ignition reactions. One expects that under reasonable hypotheses on the randomness, large space-time scale dynamics of solutions to these equations is almost surely governed by a different effective PDE, which should be a homogeneous Hamilton-Jacobi equation. While this was previously proved in one dimension as well as for isotropic reactions in several dimensions (i.e., with radially symmetric laws), we provide here the first proof of this phenomenon in the general non-isotropic multidimensional setting. Our results hold for reactions that have finite ranges of dependence (i.e., their values are independent at sufficiently distant points in space) as well as for some with infinite ranges of dependence, and are based on proving existence of deterministic front (propagation) speeds in all directions for these reactions.

1. Introduction

The reaction-diffusion equation

$$u_t = \Delta u + f(x, u, \omega), \tag{1.1}$$

with $(t, x) \in (0, \infty) \times \mathbb{R}^d$ and ω an element of some probability space $(\Omega, \mathcal{F}, \mathbb{P})$, models a host of physical phenomena occurring in random media. These phenomena all exhibit diffusion, modeled by the Laplacian, as well as some reactive process, modeled by the nonlinear *reaction function* f. The nature of the latter process determines the behavior of f in the variable u, which models the property under study and will take values between its minimum and maximum, customarily normalized to be 0 and 1.

When $u = 0$ is an unstable equilibrium for the (x, ω)-dependent ODE $\dot{u} = f(x, u, \omega)$ and $u = 1$ a stable one (e.g., when $f > 0$ for $u \in (0, 1)$), the reaction is of the *monostable type*. A special case of this are the *Kolmogorov-Petrovskii-Piskunov (KPP)* or *Fisher-KPP type* reactions [9,13], for which the growth rate $u^{-1} f(x, u, \omega)$ of the reactive process is largest near $u = 0$ for each (x, ω) (e.g., in the case of logistic growth functions $f(x, u, \omega) = g(x, \omega)u(1-u)$). These reactions are used in, for instance, population dynamics models, with u being the normalized population density and $u^{-1} f(x, u, \omega)$ the sum of the birth and death rates. When both $u = 0$ and $u = 1$ are asymptotically stable equilibria for $\dot{u} = f(x, u, \omega)$ (e.g., when $f(x, u, \omega) = g(x, \omega)u(1 - u)(u - h(x, \omega))$ with $h(x, \omega) \in (0, 1)$), the reaction is of the *bistable type*, used in modeling phase transition processes.

In this paper we will consider the third main type of reactions, modeling various combustive processes, including forest fires. Here u is the normalized temperature and f vanishes for all u below some possibly (x, ω)-dependent ignition temperature (so $u = 0$ is typically a stable but not asymptotically stable equilibrium), which is why these reactions are of the *ignition*

type. Our interest is in the long term dynamics of solutions to (1.1). The PDE typically exhibits ballistic propagation of solutions, which means that the state $u \sim 1$ invades the region where initially $u \sim 0$ at a linear-in-time rate. If the medium is sufficiently random, one expects this invasion to acquire a deterministic asymptotic speed as $t \to \infty$, which may depend on the invading direction but not on the position (or ω), due to averaging of the variations in the medium over long distances.

This phenomenon is called *homogenization*, because over large space-time scales, solutions behave as if the medium were possibly non-isotropic but homogeneous (i.e., direction- but not position-dependent). One can study solutions on these scales by rescaling them via the transformation

$$u_\varepsilon(t,x,\omega) := u\left(\varepsilon^{-1}t, \varepsilon^{-1}x, \omega\right), \tag{1.2}$$

with $\varepsilon > 0$ small, which turns (1.1) into

$$(u_\varepsilon)_t = \varepsilon \Delta u_\varepsilon + \varepsilon^{-1} f\left(\varepsilon^{-1}x, u_\varepsilon, \omega\right). \tag{1.3}$$

If we now take $\varepsilon \to 0$, the hope is to recover some (almost surely) ω-independent limit $u_\varepsilon \to \bar{u}$, in an appropriate sense and for appropriate initial data $u_\varepsilon(0,\cdot,\omega)$, that should ideally also satisfy some limiting *effective PDE*.

However, unlike in typical homogenization scenarios, the limiting PDE for reaction-diffusion equations cannot be another reaction-diffusion equation, or even another second order parabolic PDE. The reason for this is that one expects solutions to exhibit uniformly bounded in time width of the regions where transition between values $u \sim 0$ and $u \sim 1$ happens, which means that this width becomes zero in the scaling from (1.2) as $\varepsilon \to 0$, and any limiting function $\bar{u}$ takes only values 0 and 1. For instance, in the homogeneous deterministic reaction case $f(x,u,\omega) = f(u)$, the simplest solutions are *traveling fronts*, which are of the form $u(t,x) = U(x \cdot e - ct)$ for some vector $e \in \mathbb{S}^{d-1}$, where the front profile and speed (U,c) solve the ODE $U'' + cU' + f(U) = 0$ with boundary values $U(-\infty) = 1$ and $U(\infty) = 0$. Clearly, the region where $u(t,\cdot) \in [\eta, 1-\eta]$ for any fixed $\eta > 0$ is a slab of a constant-in-t width that shrinks to zero as we take $\varepsilon \to 0$ in (1.2). But then the limiting solution will be the (discontinuous) characteristic function of the half-space-time $\{x \cdot e < ct\}$, which does not solve a second order parabolic PDE.

This suggests that any effective equation should be of the first order, with any limiting function $\bar{u}$ being its discontinuous solution, taking only values 0 and 1. The expectation of the effective (asymptotic) propagation speeds being direction- but not position-dependent then suggests that the effective PDE should be the *Hamilton-Jacobi equation*

$$\bar{u}_t = c^*\left(-\frac{\nabla \bar{u}}{|\nabla \bar{u}|}\right)|\nabla \bar{u}|, \tag{1.4}$$

with $c^*(e)$ being the (x,ω)-independent effective propagation speed in direction $e \in \mathbb{S}^{d-1}$. Moreover, the traveling front solutions above suggest that in the deterministic homogeneous reaction case, the speed $c^*(e)$ should be precisely the traveling front speed c (which is also direction-independent in that case). One may therefore hope that in the general random case, it is also possible to find some front-like solutions in all directions $e \in \mathbb{S}^{d-1}$, and that each of these has an associated speed $c^*(e)$ in some sense.

Unfortunately, there are some serious obstacles to realizing this hope. The first is that its basic premise, that the width of the transition region where $u(t,\cdot,\omega) \in [\eta, 1-\eta]$ stays uniformly bounded in time (or at least $o(t)$) for any fixed $\eta > 0$, may not be true in some media. The second author in fact showed that this need not happen for bistable reactions, even for periodic ones in one dimension [28], where solutions can develop linearly-in-time growing intervals on which they are close to periodic functions with values strictly away from 0 and 1. As a result, there may be no analog of a traveling front for such reactions, and hence no homogenization as described above.

Recalling pictures of forest fires, which are usually actively burning only along the margins of the already burnt area, one may hope that such issues do not occur for ignition reactions. The second author showed that this is indeed the case in dimensions $d \leq 3$ [29], where the widths of the transition regions (properly defined, as these regions may have complicated geometries in heterogeneous media; see (1.7) below) indeed remain uniformly bounded in time, by constants depending on η above and some bounds on the reaction. However, he also showed in [29] that this need not be the case in dimensions $d \geq 4$, where these widths may grow linearly in time as in the above bistable example. Nevertheless, the relevant examples have a special structure and it is not clear to what extent they indicate possible almost sure behaviors of solutions for various stochastic reactions (in particular, those with finite ranges of spatial dependence).

All this demonstrates the difficulties associated with even the question whether solutions to (1.1) have some basic properties required for one to be able to initiate the study of homogenization for (1.1). This is the reason for relatively little progress in this area, until recently, particularly in the multi-dimensional case $d \geq 2$. In the one-dimensional setting, there are only two directions of propagation of solutions, and homogenization simply refers to showing that solutions starting from large enough compactly supported initial data propagate almost surely with some deterministic asymptotic speeds c_+ (to the right) and c_- (to the left). Moreover, the transition regions (which are intervals) have trivial geometries. This allowed several authors to obtain such "homogenization" results in this setting for all three types of stationary ergodic reactions — KPP [10], ignition [19, 27], and bistable [19, 22, 28] — although with some non-trivial limitations in the latter case, due to the counterexamples from [28] mentioned above. There are also a number of 1D and quasi-1D results concerning related models and/or periodic reactions, which we do not discuss here.

Once we move to higher dimensions, the geometry of the level sets of solutions becomes much more complicated, and relatively little is known. One previous result appears in the paper [17] by Lions and Souganidis, which studies homogenization for viscous Hamilton-Jacobi equations. Their Theorem 9.3 states that homogenization also holds for general stationary ergodic KPP reactions in any dimension. (While it is indicated in [17] that its proof can be obtained via methods from [17, 18] and two other papers, a proof is not provided there.) The reason why Hamilton-Jacobi homogenization techniques should be applicable to KPP reaction-diffusion equations is that the dynamics of solutions for these reactions is determined, to the leading order, by the linearization of (1.1) (i.e., of f) at $u = 0$. This linear

PDE can then be turned into a viscous Hamilton-Jacobi equation with a convex Hamiltonian via the Hopf-Cole transformation.

This linearization approach can only work for KPP reactions, and is not applicable to other types, including other monostable ones. In particular, it cannot be used in ignition-reaction-based models of combustion, where one has to work with the original non-linear PDE. Because of this complication, so far there has only been a single result proving homogenization for (non-KPP) stationary ergodic reactions in several dimensions. This is a conditional result by Lin and the second author [16], who proved homogenization for ignition reactions whose Wulff shapes exist and have no corners (a Wulff shape for (1.1), if it exists, is an open set $\mathcal{S} \subseteq \mathbb{R}^d$ such that solutions starting from any large enough compactly supported initial data converge to $\chi_{\mathcal{S}}$ as $t \to \infty$, after being scaled down by t in space). They also showed that these properties hold for isotropic ignition reactions in dimensions $d \leq 3$, with the dimension limitation being used to show that the Wulff shape exists (recall the above-mentioned examples of solutions with linearly growing widths of transition regions in dimensions $d \geq 4$ from [29]) and isotropy then guaranteeing that the Wulff shape is a (corner-less) ball centered at the origin. We also note that it follows from a result of Caffarelli, Lee, and Mellet [6] that Wulff shapes can have corners, even for periodic ignition reactions in two dimensions.

In fact, even homogenization for periodic reactions in several dimensions has seen fairly limited progress until recently, despite many results concerning existence of pulsating fronts and Wulff shapes for such reactions (see [5, 16, 23, 24] and references therein). While Theorem 9.3 in [17] applies to periodic KPP reactions (and is based in part on methods from [18], applicable to KPP reactions in periodic media), homogenization for periodic non-KPP reactions in several dimensions has only recently been obtained for ignition reactions as a byproduct of the method in [16], as well as for monostable reactions by Alfaro and Giletti [1] (for initial data with smooth convex supports, later extended to general convex supports in [16]).

In this paper we prove for the first time unconditional stochastic homogenization for ignition reactions, without assuming the reaction to be isotropic. Our Theorems 1.3 and 1.4 below are valid for random pure ignition reactions (see Definition 1.2) in dimensions $d \leq 3$ that either have a finite range of dependence (see Definition 1.1) or can be uniformly approximated by such reactions. We also extend these results in Theorems 1.7 and 1.8 to ignition reactions in any dimension, provided some a priori assumptions on the dynamics of certain special solutions to (1.1) are satisfied.

Our proof uses a result from [16], which shows that to prove homogenization, it suffices to show that the above-mentioned propagation speeds $c^*(e)$ (called *front speeds*) exist for all directions $e \in \mathbb{S}^{d-1}$, are almost-surely ω-independent, and also *exclusive* (see Definition 6.1). This is, however, a difficult problem in general, and [16] was only able to show existence of a deterministic front speed in direction e when the reaction has a Wulff shape with outer normal vector e at some point (this is where the absence of corners is needed), because then the expanding Wulff shape can be used at large times to locally approximate a front-like solution propagating in direction e.

To show existence of deterministic front speeds, we apply a method modeled on the one employed by Armstrong and Cardaliaguet [2] in their proof of homogenization for Hamilton-Jacobi equations with α-homogeneous (for $\alpha \geq 1$) non-convex (in ∇u) Hamiltonians with finite ranges of dependence. This was the first proof of stochastic Hamilton-Jacobi homogenization for non-convex Hamiltonians in several dimensions without special structural hypotheses (such as $H(x, \nabla u, \omega) = H(\nabla u) + V(x, \omega)$). While there are many homogenization results for convex and level-set-convex Hamiltonians, including in the paper [3] by Armstrong, Cardaliaguet, and Souganidis where the method from [2] was first used to study Hamilton-Jacobi equations (it is based on ideas from the study of first passage percolation [12, 25]), non-convexity of the Hamiltonian presents serious issues. In fact, similarly to our reaction-diffusion setting, there are examples when homogenization does not happen for Hamilton-Jacobi equations with stationary ergodic non-convex Hamiltonians [8, 26], even in one dimension. The approach in [2] overcomes these problems by leveraging the finite range of dependence hypothesis (which is akin to an i.i.d. medium setting) and the resulting mixing properties of the environment to obtain strong quantitative estimates on the solutions where a soft approach via ergodic theorems does not appear to work. These estimates involve fluctuations of the values of solutions to the so-called *metric problem* for any compact set $S \subseteq \mathbb{R}^d$ (which is an appropriate time-independent Hamilton-Jacobi PDE on $\mathbb{R}^d \setminus S$) with a smooth enough boundary. These estimates improve at an exponential rate as the distance from S increases, and were then upgraded to similar estimates for S being any half-space.

Here we apply this strategy to reaction-diffusion equations, with the relevant estimates involving fluctuations of "arrival times" at any point $x \in \mathbb{R}^d$ for solutions initially approximating χ_S (we only need to consider $S = B_k(0)$ for any $k \in \mathbb{N}$). We still obtain an exponentially-in-$d(x, S)$ decaying estimate (see Proposition 3.8 below), albeit at a slower rate. However, we are also able to extend it to some reactions with infinite ranges of dependence (see Proposition 4.2) by carefully tracking the dependence of this rate on the range of dependence of f when the latter is finite, something that was described in [2] as completely open in the Hamilton-Jacobi setting (and appears to remain such at this time)!

After we upgrade this estimate from balls to half-spaces, we are able to prove existence of deterministic exclusive front speeds in all directions, and thus homogenization after using results from [16]. We note that while the effective equations in Hamilton-Jacobi homogenization are still Hamilton-Jacobi PDE (although some of their terms can disappear in the homogenization process), and the limiting functions are their continuous solutions, our limiting functions are *discontinuous* viscosity solutions to (1.4), which causes extra difficulties in the analysis. For a more thorough discussion of similarities and differences between Hamilton-Jacobi homogenization for non-convex Hamiltonians and reaction-diffusion homogenization, as well as for further references, we refer the reader to the introduction of [16].

1.1. **Hypotheses and Main Results.** Let us now turn to our main results. Our goal is to show that as $\varepsilon \to 0$, solutions to (1.3) with initial data approximating χ_A for any open set $A \subseteq \mathbb{R}^d$ converge to the unique (discontinuous viscosity) solution to (1.4) with initial data χ_A. Here, of course, $c^*(e)$ are the deterministic front speeds discussed above, and establishing their existence forms the bulk of our work.

One can show that if $c^* : \mathbb{S}^{d-1} \to (0,\infty)$ is Lipschitz (which will be our case), then for any open $A \subseteq \mathbb{R}^d$, there is an open set $\Theta^{A,c^*} \subseteq (0,\infty) \times \mathbb{R}^d$ such that the unique solution to (1.4) with initial data χ_A is $\bar{u} := \chi_{\Theta^{A,c^*}}$. In fact, this set can also be found from the formula

$$\Theta^{A,c^*} := \big\{(t,x) \in (0,\infty) \times \mathbb{R}^d \,\big|\, v(t,x) > 0\big\}, \tag{1.5}$$

where $v_0 : \mathbb{R}^d \to \mathbb{R}$ is any Lipschitz function satisfying $v_0 > 0$ on A and $v_0 < 0$ on $\mathbb{R}^d \setminus \overline{A}$, and v is the unique (continuous) viscosity solution to (1.4) with $v(0,\cdot) = v_0$. The open set Θ^{A,c^*} is then independent of the choice of v_0 as above, and $\partial\Theta^{A,c^*}$ has zero measure.

All these claims are contained in Theorem 5.3 in [16], which is a combination of results by Barles, Soner, and Souganidis [4], Crandall, Ishii, and Lions [7], Souganidis [21], and Soravia [20]. The reader can also consult Definition 5.1 in [16] for the definition of viscosity solutions to initial value problems for (1.4).

We also note that it was shown in the proof of Theorem 1.4(iii) in [16] that for any convex open $A \subseteq \mathbb{R}^d$ we have the explicit formula

$$\Theta^{A,c^*} = \bigcap_{e\in\mathbb{S}^{d-1}} \left\{ (t,x) \in (0,\infty) \times \mathbb{R}^d \,\middle|\, x \cdot e < \sup_{y\in\partial A} y \cdot e + c^*(e)t \right\}.$$

In particular, if $A = \{x \in \mathbb{R}^d \,|\, x \cdot e < 0\}$ is the half-space with outer normal e, then we obviously have $\Theta^{A,c^*} = \{(t,x) \in (0,\infty) \times \mathbb{R}^d \,|\, x \cdot e < c^*(e)t\}$. This also shows that if we let $(\Theta^{A,c^*})_t$ be the spatial slice of Θ^{A,c^*} at the time $t > 0$, then for any open bounded A we have

$$\lim_{t\to\infty} \frac{(\Theta^{A,c^*})_t}{t} = \bigcap_{e\in\mathbb{S}^{d-1}} \big\{ y \in \mathbb{R}^d \,\big|\, y \cdot e < c^*(e) \big\}$$

(e.g., in the sense of Hausdorff distances of boundaries of sets). Hence the set on the right-hand side is the *Wulff shape* for (1.4), and therefore also for (1.1) if homogenization holds.

We will consider here stationary ignition reactions that either have finite ranges of dependence, or can be uniformly approximated by such reactions (see Example 1.5 below for a simple example of the latter). These properties are summarized in the following definition and in hypothesis **(H1)** below.

Definition 1.1. Consider a probability space $(\Omega, \mathcal{F}, \mathbb{P})$ that is endowed with a group of measure-preserving bijections $\{\Upsilon_y : \Omega \to \Omega\}_{y\in\mathbb{R}^d}$ such that for all $y, z \in \mathbb{R}^d$ we have

$$\Upsilon_y \circ \Upsilon_z = \Upsilon_{y+z}.$$

A reaction function $f : \mathbb{R}^d \times [0,1] \times \Omega \to [0,\infty)$, uniformly continuous in the first two arguments and with the random variables $X_{x,u} := f(x,u,\cdot)$ being $\mathcal{F}$-measurable for all $(x,u) \in \mathbb{R}^d \times [0,1]$, is called *stationary* if for each $(x,y,u,\omega) \in \mathbb{R}^{2d} \times [0,1] \times \Omega$ we have

$$f(x,u,\Upsilon_y\omega) = f(x+y,u,\omega).$$

The *range of dependence* of such f is the infimum of all $r \in \mathbb{R}^+ \cup \{\infty\}$ such that

$$\mathcal{E}(U) \text{ and } \mathcal{E}(V) \text{ are } \mathbb{P}\text{-independent}$$

for any $U, V \subseteq \mathbb{R}^d$ with $d(U,V) \geq r$, where $\mathcal{E}(U)$ is the σ-algebra generated by the family of random variables $\{X_{x,u} \,|\, (x,u) \in U \times [0,1]\}$.

Remark. While stationary reactions with finite ranges of dependence are also stationary ergodic, we will not need to use this property here due to our quantitative approach. We note that although the main results in [16] apply to stationary ergodic reactions, that assumption is only needed to prove that all the deterministic (exclusive) front speeds for (1.1) exist and are strong (see Definition 6.1 below), which we instead prove in Sections 3–6.

We will consider here stationary reaction functions $f : \mathbb{R}^d \times [0,1] \times \Omega \to [0,\infty)$, and extend them to $\mathbb{R}^d \times \mathbb{R} \times \Omega$ by 0 whenever we need to evaluate them with $u \notin [0,1]$. Additionally, our reactions will be of the *ignition type*. That is, we will assume the following hypothesis.

(H1) The reaction f is stationary, Lipschitz in both x and u with constant $M \geq 1$, and there are $\theta_1 \in (0, \frac{1}{2})$, $m_1 > 1$, and $\alpha_1 > 0$ such that $f(\cdot, u, \cdot) \equiv 0$ for $u \in [0, \theta_1] \cup \{1\}$, $f(\cdot, u, \cdot) \geq \alpha_1 (1-u)^{m_1}$ for $u \in [1-\theta_1, 1)$, and f is non-increasing in $u \in [1-\theta_1, 1)$.

It is not difficult to see that one cannot hope for general reactions satisfying **(H1)** to lead to homogenization for (1.1), even if f is independent of (x, ω) (see, e.g., [28,29]). Indeed, if f is allowed to vanish at some intermediate value $\theta' \in (\theta_1, 1-\theta_1)$ and is also sufficiently large for some $u \in (\theta_1, \theta')$, solutions could easily form "plateaus" with values near θ' (or another intermediate value) whose widths grow linearly in time. And if that happens, the widths of these plateaus will not vanish even after the scaling from (1.2) is applied.

To avoid this scenario, one should assume that as the argument u grows from 0 to 1 (for any fixed (x, ω)), the reaction f cannot become arbitrarily small (except near $u = 1$) once it has become large enough. This is expressed in Definition 2.3 below, which was used in [29] to show that not only solutions to (1.1) do not develop such plateaus, but the transition from values $u \sim 0$ to values $u \sim 1$ in fact occurs over uniformly-in-time bounded distances in space (see, e.g., Lemma 2.4 below). Our most general results apply in this setting, as well as when one instead only assumes at most $O(t^\alpha)$ growth of the above transition distances, with $\alpha < 1$ (see hypothesis **(H2')** below).

However, for the sake of simplicity, in our first two results we will consider the case where the reaction does not become arbitrarily small (except near $u = 1$) after it has become just *positive*. That is, once u has exceeded the *ignition temperature*

$$\theta_{x,\omega} := \sup\{\theta \geq 0 \,|\, f(x, u, \omega) = 0 \text{ for all } u \in [0, \theta]\} \qquad (\in [\theta_1, 1-\theta_1)).$$

Of course, this is the case for any realistic model of combustion, where the reaction rate is positive at all temperatures above the ignition temperature (its vanishing at $u = 1$ is due to fuel exhaustion in systems of equations for temperature and concentration of the reactant, which in certain regimes simplify to (1.1) with $f(\cdot, 1, \cdot) \equiv 0$).

Definition 1.2. A reaction f satisfying **(H1)** is a stationary *pure ignition* reaction if for each $\eta > 0$ we have

$$\inf_{\substack{(x,\omega) \in \mathbb{R}^d \times \Omega \\ \theta_{x,\omega} + \eta < 1 - \theta_1}} f(x, \theta_{x,\omega} + \eta, \omega) > 0.$$

Remark. This definition (with the bound for $u \in [1-\theta_1, 1)$ being $\inf_{(x,\omega)} f(x, u, \omega) > 0$) is from [28]. Note that it is trivially satisfied, for instance, when $f(x, u, \omega) = g(x, \omega) F_0(u)$, with g bounded away from 0 and ∞, Lipschitz in x, and stationary in ω, and with Lipschitz

$F_0 : [0,1] \to [0,\infty)$ such that $F_0 = 0$ on $[0,\theta_0] \cup \{1\}$ and $F_0 > 0$ on $(\theta_0, 1)$ for some $\theta_0 \in (0,1)$, and F_0 is non-increasing and bounded below by $\alpha_1(1-u)^{m_1}$ near 1 (for some m_1, α_1).

We will therefore start by assuming the following hypothesis.

(H2) f is a stationary pure ignition reaction and $d \leq 3$.

The additional restriction $d \leq 3$ is necessitated by the above-mentioned surprising result from [29], where the second author showed that even for pure ignition reactions, transition from values $u = \eta$ to values $u = 1-\eta$ may only occur over linearly-in-time growing distances for solutions to (1.1) and all small $\eta > 0$ in dimensions $d \geq 4$ (while these distances remain bounded in dimensions $d \leq 3$).

We are now ready to state our first main homogenization result. In it and later we use the notation $B_r(A) := A + (B_r(0) \cup \{0\})$ and $A^0_r := A \backslash \overline{B_r(\partial A)}$ for $A \subseteq \mathbb{R}^d$ and $r \geq 0$. For the sake of generality, we also allow $O(1)$ shifts and $o(1)$ errors in initial data as $\varepsilon \to 0$ in (1.3).

Theorem 1.3. *If f satisfying **(H2)** has a finite range of dependence, then there is Lipschitz $c^* : \mathbb{S}^{d-1} \to (0,\infty)$ such that the following holds for any open $A \subseteq \mathbb{R}^d$ and Θ^{A,c^*} from* (1.5). *If $\Lambda > 0$, and for all $\omega \in \Omega$ and $\varepsilon > 0$, the function $u_\varepsilon(\cdot,\cdot,\omega)$ solves* (1.3) *and satisfies*

$$(1-\theta_1)\chi_{A^0_{\psi(\varepsilon)}} \leq u_\varepsilon(0,\cdot+y_\varepsilon,\omega) \leq \chi_{B_{\psi(\varepsilon)}(A)} + \psi(\varepsilon)\chi_{\mathbb{R}^d \backslash B_{\psi(\varepsilon)}(A)} \tag{1.6}$$

for some $y_\varepsilon \in B_\Lambda(0)$ and some ψ with $\lim_{\varepsilon\to 0}\psi(\varepsilon) = 0$ (when $y_\varepsilon = 0$ and $\psi(\varepsilon) = 0$, this becomes just $(1-\theta_1)\chi_A \leq u_\varepsilon(0,\cdot,\omega) \leq \chi_A$), then for almost all $\omega \in \Omega$ we have

$$\lim_{\varepsilon\to 0} u_\varepsilon(\cdot,\cdot+y_\varepsilon,\omega) = \chi_{\Theta^{A,c^*}}$$

locally uniformly on $([0,\infty)\times\mathbb{R}^d) \setminus \partial\Theta^{A,c^*}$.

Remark. Our proofs use results from [16] which in fact show that in all our main results, $1-\theta_1$ in (1.6) can be replaced by any θ satisfying $\inf_{(x,u,\omega)\in\mathbb{R}^d\times[\theta,1-\theta_1]\times\Omega} f(x,u,\omega) > 0$.

We next extend this to the case of reactions with infinite ranges of dependence that are uniform limits of reactions with finite ranges of dependence. Here we will also require some uniform decay of f near $u = 1$. This is the content of the next two hypotheses.

(H3) There are $m_3 \geq 1$ and $\alpha_3 > 0$ such that for all $\eta \in (0, \frac{1}{2}\theta_1]$ we have

$$\inf_{\substack{(x,\omega)\in\mathbb{R}^d\times\Omega \\ u\in[1-\theta_1/2,1]}} (f(x,u-\eta,\omega) - f(x,u,\omega)) \geq \alpha_3\eta^{m_3}.$$

(H4) There are $m_4, n_4, \alpha_4 > 0$ such that for each $n \geq n_4$, there exists a stationary reaction f_n with range of dependence $\leq n$ and $\|f_n - f\|_\infty \leq \alpha_4 n^{-m_4}$.

Theorem 1.4. *Theorem 1.3 holds for any f satisfying **(H2)–(H4)**.*

While this result does not cover all interesting random pure ignition reactions with correlations of $f(x,u,\cdot)$ and $f(y,v,\cdot)$ decreasing as $|x-y| \to \infty$ (for all $u, v \in [0,1]$), it does apply to many of them. Here is a simple such example.

Example 1.5. Let $d \le 3$, $F_0 : [0,1] \to [0,\infty)$ be Lipschitz with $F_0 = 0$ on $[0,\theta_0] \cup \{1\}$ and $F_0 > 0$ on $(\theta_0, 1)$ for some $\theta_0 \in (0,1)$, and $F_0'(u) \le -(1-u)^m$ near $u = 1$ for some m. Also pick some Lipschitz $g : \mathbb{R}^d \to [0,\infty)$ with $\sup_{x\in\mathbb{R}^d} |x|^{m'} g(x) < \infty$ for some $m' > 0$, and some Lebesgue measurable $a : [0,1] \to [0,1]$. Consider the product probability space $\Omega = [0,1]^{\mathbb{Z}^d}$, with the Lebesgue measure on each copy of $[0,1]$, and for any $k \in \mathbb{Z}^d$, denote by ω_k the k^{th} coordinate of $\omega \in \Omega$ (note that these are i.i.d. random variables). Let $\Upsilon_y : \Omega \to \Omega$ for $y \in \mathbb{Z}^d$ be given by $(\Upsilon_y \omega)_k := \omega_{y+k}$ for all $k \in \mathbb{Z}^d$. Then the random reaction strength model

$$f(x,u,\omega) := \left(1 + \sup_{k\in\mathbb{Z}^d} a(\omega_k) g(x-k)\right) F_0(u)$$

satisfies **(H2)**–**(H4)** (see next paragraph for stationarity), with f_n defined as f but with g replaced by $g_n(x) := g(x) \min\{1, d(x, \mathbb{R}^d \setminus B_{n/2}(0))\}$. Hence Theorem 1.4 applies. Note that f may have infinite range of dependence when g is not compactly supported.

Note also that while this f is stationary only with respect to integer shifts (i.e., $y \in \mathbb{Z}^d$ in Definition 1.1), such settings can be easily transformed to the case considered in the present paper by letting $\tilde\Omega := \Omega \times [0,1)^d$ with the product measure, $\tilde f(x,u,(\omega,z)) := f(x+z,u,\omega)$, and $\tilde\Upsilon_y(\omega,z) := (\Upsilon_{\lfloor y+z\rfloor}\omega, \{y+z\})$ for $y \in \mathbb{R}^d$. Since inclusion of y_ε in (1.6) shows that all our main results continue to hold if we replace the identified full-measure set $\tilde\Omega' \subseteq \tilde\Omega$ by $\bigcup_{y\in\mathbb{R}^d} \tilde\Upsilon_y \tilde\Omega'$, which is of the form $\Omega' \times [0,1)^d$, they then also apply in integer-shift settings.

In the above example and in Theorem 1.4, reactions f with infinite ranges of dependence are uniform limits of those with finite ranges of dependence. The next example is a natural situation when this need not be the case, a reaction-diffusion analog of the setting where sticks of random unbounded lengths are randomly positioned in $\mathbb{R}^d$. While Theorem 1.4 does not apply here, one can instead use its generalization, Theorem 1.7 below, which allows this.

Example 1.6. Consider the setting from Example 1.5, without the functions g and a. Instead pick some uniformly bounded and uniformly Lipschitz $g_j : \mathbb{R}^d \to [0,\infty)$ ($j \in \mathbb{N}$) that satisfy $\sup_{j\in\mathbb{N}} \sup_{|x|>j} |x|^{m'} g_j(x) < \infty$ for some $m' > 0$, and some Lebesgue measurable $a : [0,1] \to \mathbb{N}$ with $|a^{-1}(j)| \le j^{-\gamma}$ for some $\gamma > 3d+2$ and all $j \in \mathbb{N}$. Then

$$f(x,u,\omega) := \left(1 + \sup_{k\in\mathbb{Z}^d} g_{a(\omega_k)}(x-k)\right) F_0(u)$$

satisfies **(H3)** and **(H4')** below (see Example 1.5 for stationarity), with f_n defined as f but with g_j replaced by $g_{j,n}(x) := g_j(x) \min\{1, 2d(x, \mathbb{R}^d \setminus B_{n/2}(0))\}$. This uses the fact that

$$\sum_{j\ge n/2} (n^{1+m_4'} + j)^d |a^{-1}(j)| = o(n^{-(2d+1+m_4')})$$

for some $m_4' > 0$ (because $\gamma > 3d+2$), with the left-hand side being (up to a constant factor) an upper bound on the probability that $a(\omega_k) \ge \max\{\frac{n}{2}, |k| - n^{1+m_4'}\}$ for at least one $k \in \mathbb{Z}^d$. Hence Theorem 1.7 below applies. Note that f need not be a uniform limit of reactions with finite ranges of dependence when the functions g_j do not decay uniformly to 0 as $x \to \infty$.

1.2. **Generalizations.** As we indicated above, it is not clear whether the limitation on the dimension in **(H2)** is necessary to obtain a sufficiently general result. However, since both conditions in **(H2)** are only needed to guarantee certain estimates for some special solutions to (1.1) (see Lemma 2.4 below), including that the transition from values $u \sim 0$ to values $u \sim 1$ occurs over spatial distances that grow only sub-linearly in time, as we mentioned above (Lemma 2.4 shows that in the case of **(H2)** these distances are in fact uniformly bounded), we can extend our results to more general settings as long as these estimates still hold there. In particular, this might be the case for stationary ignition reactions in dimensions $d \geq 4$.

In order to state this alternative to hypothesis **(H2)**, let us define for any $0 < \eta < \theta < 1$ the *width of the transition zone* from η to θ for a solution $u : [0, \infty) \times \mathbb{R}^d \to [0, 1]$ to (1.1) at some time $t \geq 0$ to be (see [29])

$$L_{u,\eta,\theta}(t) := \inf \Big\{ L > 0 \,\Big|\, \{x \in \mathbb{R}^d \,|\, u(t,x) \geq \eta\} \subseteq B_L\left(\{x \in \mathbb{R}^d \,|\, u(t,x) \geq \theta\}\right) \Big\}. \tag{1.7}$$

The special solutions for which we need to assume certain bounds on these quantities will be essentially those evolving from characteristic functions of the balls $B_k(0) \subseteq \mathbb{R}^d$, with $k \in \mathbb{N}$.

It will however be more convenient to work with approximations $u_{0,k}$ of these characteristic functions that have two useful properties. First, they are close to 1 on $B_k(0)$ but are strictly below 1 (which will allow us to treat general initial data from (1.6)), and are supported on $B_{k+R_0}(0)$ for some fixed R_0. Specifically, we will require that

$$(1 - \theta^*)\chi_{B_k(0)} \leq u_{0,k} \leq (1 - \theta^*)\chi_{B_{k+R_0}(0)} \tag{1.8}$$

holds with $\theta^* > 0$ from (2.1) below. We note that we could in fact replace $1 - \theta^*$ in (1.8) by any $\theta < 1$ satisfying $\inf_{(x,u,\omega) \in \mathbb{R}^d \times [\theta, 1-\theta_1] \times \Omega} f(x, u, \omega) > 0$, but we make our choice for the sake of convenience (Lemma 2.1 shows that solutions $u : (0, \infty) \times \mathbb{R}^d \to [0, 1]$ with $u(t, x) \geq 1 - \theta^*$ for some $(t, x) \in [1, \infty) \times \mathbb{R}^d$ converge locally uniformly to 1). The second property is that the corresponding solutions to (1.1) satisfy $u_t > 0$. For this, it suffices to have

$$\Delta u_{0,k} + F(u_{0,k}) \geq 0 \tag{1.9}$$

with $F(u) := \inf_{(x,\omega) \in \mathbb{R}^d \times \Omega} f(x, u, \omega)$, which yields $\Delta u_{0,k} + f(\cdot, u_{0,k}, \omega) \geq 0$ for any $\omega \in \Omega$. Then $u_t > 0$ follows for the corresponding solution u at all positive times because $v := u_t$ solves the linear equation $v_t = \Delta v + f_u(x, u(t, x), \omega)v$ with $v(0, \cdot) \geq 0$ and $v(0, \cdot) \not\equiv 0$ (due to (1.8) and $F(1 - \theta^*) > 0$).

It is easy to construct radial functions satisfying (1.8) and (1.9), since then (1.9) becomes a simple ordinary differential inequality. (This is in fact possible for any set $S \subseteq \mathbb{R}^d$, without radial symmetry but still with a uniform R_0, and we do so in Lemma 2.2 below.) Let us now pick one such $u_{0,k}$ for each $k \in \mathbb{N}$ (any one can be chosen), and denote by $\mathcal{U}_f$ the set of all solutions u to (1.1) obtained by choosing any $\omega \in \Omega$ and initial data $u(0, \cdot) = u_{0,k}$ for any $k \in \mathbb{N}$. We can now replace **(H2)** by the following hypothesis.

(H2') f satisfies **(H1)** and there are $\alpha_2 < 1$ and $m_2 > 0$ such that

$$\begin{aligned} &\limsup_{t\to\infty} \sup_{u\in\mathcal{U}_f} \sup_{\eta>0} \frac{L_{u,\eta,1-\theta^*}(t)}{t^{\alpha_2}\eta^{-m_2}} < \infty, \\ &\liminf_{t\to\infty} \inf_{u\in\mathcal{U}_f} \inf_{u(t,x)\in[\theta^*,1-\theta^*]} u_t(t,x) > 0. \end{aligned} \tag{1.10}$$

Here $\mathcal{U}_f$ is as above, with some $u_{0,k}$ satisfying (1.8) and (1.9) for each $k \in \mathbb{N}$, and $\theta^* = \theta^*(M, \theta_1, m_1, \alpha_1)$ from (2.1) and $R_0 = R_0(M, \theta_1, m_1, \alpha_1)$ are independent of k.

Remarks. 1. The first statement in (1.10) allows $L_{u,\eta,1-\theta^*}(t)$ to grow algebraically in both $\eta \to 0$ and $t \to \infty$ (note that $\alpha_2 < 1$ is critical here because the scaling from (1.2) yields $L_{u_\varepsilon,\eta,1-\theta^*}(t) = \varepsilon L_{u,\eta,1-\theta^*}(\varepsilon^{-1}t)$, which will then vanish on any bounded time interval as we take $\varepsilon \to 0$). We note that Lemma 2.4 below shows that in the case of **(H2)**, the former growth is only logarithmic while the latter is non-existent.

2. Lemma 2.4 shows that the second statement in (1.10) holds as well if one assumes **(H2)** (recall also that all $u \in \mathcal{U}_f$ satisfy $u_t > 0$). Nevertheless, we will further weaken this hypothesis in Theorem 1.8 below.

3. We could also replace $B_k(0)$ and $B_{k+R_0}(0)$ in (1.8) by $B_{r_k}(0)$ and $B_{r_k+R_0}(0)$ for any sequence $r_k \to \infty$, without any change to our results.

After replacing **(H2)** by **(H2')**, we must also adjust **(H4)** in the extension of Theorem 1.4, in order to ensure that the reactions f_n will satisfy **(H2')** with uniform constants. Note that when **(H2)** holds, we will show in Corollary 2.7 that f_n from **(H4)** can be perturbed so that this is the case, but we do not know whether this remains true when we only assume **(H2')**.

In addition, we also state this new version of **(H4)** so that it applies to some f that are not uniform limits of reactions with finite ranges of dependence (see Example 1.6 above).

(H4') There are $m_4, m_4', n_4, \alpha_4 > 0$ such that for each $n \geq n_4$, there exists a stationary reaction f_n with range of dependence $\leq n$ and

$$\mathbb{P}\left(\sup_{|x|<n^{1+m_4'}} \sup_{u\in[0,1]} |f_n(x,u,\cdot) - f(x,u,\cdot)| > \alpha_4 n^{-m_4} \right) \leq n^{-(2d+1+m_4')}.$$

Moreover, **(H2')** holds uniformly in n (i.e., reactions f_n satisfy **(H1)** with the same $M, \theta_1, m_1, \alpha_1$, and (1.10) with $\mathcal{U}_f$ replaced by $\bigcup_{n\geq n_4} \mathcal{U}_{f_n}$).

We note that the initial data $u_{0,k}$ used in the definition of $\mathcal{U}_{f_n}$ are in principle allowed to be different for distinct n (but θ^* and R_0 are uniform in n; also r_k in Remark 3 above).

Theorem 1.7. *Theorem 1.3 holds for any f that either satisfies **(H2')** and has a finite range of dependence, or satisfies **(H3)** and **(H4')**.*

Finally, we show that one can also allow a power decay in time in the second statement in (1.10), at the expense of either having to extend this assumption to a slightly larger family of special solutions or obtaining the result for a smaller family of initial data.

For each $a \in [0, \frac{1}{2}\theta^*]$, let $\mathcal{U}_{f,a}$ be defined as $\mathcal{U}_f$ above, but with (1.8) replaced by

$$(1-a)(1-\theta^*)\chi_{B_k(0)} + a \leq u_{0,k,a} \leq (1-a)(1-\theta^*)\chi_{B_{k+R_0}(0)} + a \tag{1.11}$$

for initial data denoted $u_{0,k,a}$ instead of $u_{0,k}$ (so now $u_{0,k,a} - a$ is supported in $B_{k+R_0}(0)$). Obviously $\mathcal{U}_{f,0} = \mathcal{U}_f$, and one can find such initial data (for any $S \subseteq \mathbb{R}^d$ and with R_0 uniform in a) via Lemma 2.2 with α_1 replaced by $\alpha_1(1 - \frac{1}{8}\theta_1)^{m_1-1}$ (since $a \le \frac{1}{8}\theta_1$) and then applying the scaling $u_{0,S,a} := (1-a)u_{0,S} + a$.

We can now replace **(H2')** and **(H4')** by the following hypotheses.

(H2") f satisfies **(H1)** and there are $\alpha_2 < 1$, $m_2 > 0$, $a_2 \in [0, \frac{1}{2}\theta^*]$, and $\alpha_2' < \min\{\frac{1}{m_1-1}, \frac{1-\alpha_2}{m_2}\}$ such that

$$\begin{aligned} &\limsup_{t\to\infty} \sup_{a\in[0,a_2]} \sup_{u\in\mathcal{U}_{f,a}} \sup_{\eta>0} \frac{L_{u,\eta+a,1-\theta^*}(t)}{t^{\alpha_2}\eta^{-m_2}} < \infty, \\ &\liminf_{t\to\infty} \inf_{a\in[0,a_2]} \inf_{u\in\mathcal{U}_{f,a}} \inf_{u(t,x)\in[\theta^*,1-\theta^*]} u_t(t,x)\, t^{\alpha_2'} > 0. \end{aligned} \tag{1.12}$$

Here $\mathcal{U}_{f,a}$ is as above, with some $u_{0,k,a}$ satisfying (1.11) and (1.9) for each $k \in \mathbb{N}$, and $\theta^* = \theta^*(M, \theta_1, m_1, \alpha_1)$ from (2.1) and $R_0 = R_0(M, \theta_1, m_1, \alpha_1)$ are independent of k.

(H4") f satisfies **(H4')** with **(H2")** in place of **(H2')**, and also $\alpha_2' < \frac{m_4}{m_3}$.

Of course, these hypotheses coincide with **(H2')** and **(H4')** when $a_2 = 0 = \alpha_2'$. With them, we can now state our second generalization of Theorems 1.3 and 1.4.

Theorem 1.8. *Assume that f either satisfies **(H2")** and has a finite range of dependence, or satisfies **(H3)** and **(H4")**.*

(i) If $a_2 > 0$, then Theorem 1.3 holds for such f.

(ii) If $a_2 = 0$, then Theorem 1.3 holds for such f with A convex and (1.6) *replaced by*

$$(1-\theta_1)\chi_{A^0_{\psi(\varepsilon)}} \le u_\varepsilon(0, \cdot + y_\varepsilon, \omega) \le (1-\Lambda^{-1})\chi_{B_{\psi(\varepsilon)}(A)}\,.$$

1.3. **Organization of the Paper and Acknowledgements.** In Section 2 we collect most important notation and prove several preliminary results. These include Corollary 2.7, which shows that Theorems 1.3 and 1.4 follow from Theorem 1.7. It will therefore suffice to prove Theorems 1.7 and 1.8. We prove the first one in Section 6, after obtaining crucial quantitative estimates on long-time dynamics of solutions to (1.1) in Sections 3–5 (specifically, Propositions 3.8, 4.2, and 5.1, with the first two of these being essentially the same result but assuming **(H2')**+finite range in the first and **(H3)**+**(H4')** in the second). In Section 7 we then show how to extend all these results to the cases considered in Theorem 1.8.

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2. Preliminaries and Notation

In this section we collect some previous results and preliminary lemmas, all of which hold uniformly in ω and without needing to assume stationarity of the reaction. We will therefore use the following hypothesis.

(H1') f satisfies **(H1)** except possibly the stationarity hypothesis.

At the end of the section we also collect all the important notations in one place.

Let us start with a basic lower bound (see, e.g., [29, Lemma 3.1]), which shows that general solutions to (1.1) propagate with speed no less than some $c_0 = c_0(M, \theta_1, m_1, \alpha_1) > 0$. We will choose this to be the *unique front speed* for the homogeneous reaction $F_0 : [0,1] \to [0,\infty)$ defined to be the largest M-Lipschitz function with $F_0(u) \le \alpha_1(1-u)^{m_1}\chi_{[1-\theta_1,1]}(u)$ (so clearly $F_0 \le F$). Hence c_0 is the unique number such that the PDE $u_t = u_{xx} + F_0(u)$ in one space dimension has a *traveling front* solution $u(t,x) = U(x - c_0 t)$ with $U(-\infty) = 1$ and $U(\infty) = 0$.

Lemma 2.1. *There exists $\theta_2 = \theta_2(M, \theta_1, m_1, \alpha_1) < 1$ such that for each $c < c_0$ and $\theta < 1$, there is $\kappa_0 = \kappa_0(M, \theta_1, m_1, \alpha_1, c, \theta) \ge 1$ such that the following holds. If $u : (0,\infty) \times \mathbb{R}^d \to [0,1]$ is a solution to (1.1) with f satisfying **(H1')** and with some $\omega \in \Omega$, and if $u(t_0, y) \ge \theta_2$ for some $t_0 \ge 1$ and $y \in \mathbb{R}^d$, then for all $t \ge t_0 + \kappa_0$,*

$$\inf_{|x-y| \le c(t-t_0)} u(t,x) \ge \theta.$$

If also $u_t \ge 0$, then this clearly holds with any $t_0 \ge 0$ (and κ_0 increased by 1).

Let now

$$\theta^* := \frac{1}{4}\min\{1-\theta_2,\, \theta_1\}, \tag{2.1}$$

where $\theta_2 = \theta_2(M, \frac{1}{2}\theta_1, m_1, \alpha_1(1 - \frac{1}{8}\theta_1)^{m_1 - 1}) < 1$.

Remark. Addition of the factors $\frac{1}{2}$ and $(1-\frac{1}{8}\theta_1)^{m_1-1}$ here is due to the scaling $u \mapsto (1-a)u + a$ mentioned before **(H2")**, as we shall see in Section 7. All arguments before Proposition 7.4 will only require $\theta_2 = \theta_2(M, \theta_1, m_1, \alpha_1)$ here, and also only that $\theta^* \le \frac{1}{2}\min\{1-\theta_2, \theta_1\}$. So we could define θ^* this way in Theorem 1.7 and its proof.

In the rest of the paper we will primarily use Lemma 2.1 with $c = \frac{c_0}{2}$ and $\theta = 1 - \theta^*$, and we will therefore define

$$\kappa_0 := \kappa_0\left(M, \theta_1, m_1, \alpha_1, \frac{c_0}{2}, 1-\theta^*\right). \tag{2.2}$$

Having defined this θ^*, let us next construct the initial data $u_{0,S}$ from the introduction, which are perturbations of the functions χ_S that also satisfy (1.9).

Lemma 2.2. *There is $R_0 = R_0(M, \theta_1, m_1, \alpha_1) \ge 1$ such that for any f satisfying **(H1')** and $S \subseteq \mathbb{R}^d$, there is a smooth function $u_{0,S}$ satisfying (1.9) and*

$$(1-\theta^*)\chi_S \le u_{0,S} \le (1-\theta^*)\chi_{B_{R_0}(S)}. \tag{2.3}$$

Remark. Recall that (1.9) implies that the relevant solutions to (1.1) satisfy $u_t > 0$. Moreover, (2.3) yields the uniform bound $L_{u,\eta,1-\theta^*}(0) \le R_0$ for all $\eta \in (0, 1-\theta^*)$ and $S \subseteq \mathbb{R}^d$, which is relevant for the next result.

Lemma 2.2 is proved in Appendix A.

Let us now turn to the consequences of **(H2)** obtained in [29]. In fact, these results hold for the following more general classes of functions.

Definition 2.3. For $M, \theta_1, m_1, \alpha_1$ from **(H1)** (and F_0 defined above), and for any $\zeta, \xi > 0$, let $\mathcal{F}(F_0, M, \theta_1, \zeta, \xi)$ be the class of all f satisfying **(H1')** such that

$$\inf_{\substack{(x,\omega)\in\mathbb{R}^d\times\Omega \\ u\in[\gamma_f(x,\omega;\zeta),1-\theta_1]}} f(x,u,\omega) \geq \xi,$$

where (with the convention $\inf\emptyset = \infty$)

$$\gamma_f(x,\omega;\zeta) := \inf\{u \geq 0 \,|\, f(x,u,\omega) > \zeta u\}.$$

Remarks. 1. Although we could instead write $\mathcal{F}(M, \theta_1, m_1, \alpha_1, \zeta, \xi)$, we use notation from [29].

2. Note that pure ignition reactions belong to $\bigcup_{\xi>0} \mathcal{F}(F_0, M, \theta_1, \zeta, \xi)$ for any $\zeta > 0$.

It was shown in [29] that if $d \leq 3$ and f is from the class $\mathcal{F}(F_0, M, \theta_1, \zeta, \xi)$ for some F_0, M, θ_1, ξ and $\zeta < c_0^2/4$, then transitions from values $u \sim 0$ to values $u \sim 1$ for fairly general solutions to (1.1) occur over uniformly-in-time bounded distances. In view of our interest in solutions from $\mathcal{U}_f$, with initial data satisfying Lemma 2.2, the following result will be relevant.

Lemma 2.4. *Let $d \leq 3$, let F_0, M, θ_1 be as in Definition 2.3 and θ^* from (2.1), and consider any $\xi > 0$ and $\zeta < c_0^2/4$. There is $\Lambda > 0$ and for any $\eta \in (0, \frac{1}{2})$ there are $\mu_\eta, \kappa_\eta > 0$ such that if $f \in \mathcal{F}(F_0, M, \theta_1, \zeta, \xi)$ and u solves (1.1) with some $\omega \in \Omega$ and initial data satisfying Lemma 2.2 for some $S \subseteq \mathbb{R}^d$, then*

$$\sup_{t\geq 0\,\&\,\eta\in(0,1-\theta^*)} \frac{L_{u,\eta,1-\theta^*}(t)}{1+|\ln\eta|} \leq \Lambda \tag{2.4}$$

and for any $\eta > 0$ we have

$$\inf_{\substack{(t,x)\in(\kappa_\eta,\infty)\times\mathbb{R}^d \\ u(t,x)\in[\eta,1-\eta]}} u_t(t,x) \geq \mu_\eta. \tag{2.5}$$

Proof. Recall that we have $L_{u,\eta,1-\theta^*}(0) \leq R_0$ for all $\eta \in (0, 1-\theta^*)$. We then obtain (2.5) from [29, Theorem 2.5(i)] with $(\varepsilon', \varepsilon) = (1-\theta^*, \eta)$ (we can choose there $\varepsilon_0 = \theta^*$, and then $\varepsilon' = 1 - \varepsilon_0$), with independence of κ_η on u due to Remark 1 after the theorem.

To obtain (2.4), we instead use (4.14) in [29] with $(\varepsilon_0, h, t_0) = (\theta^*, 0, 0)$ (where ε_0 was used to define the function Z_y in (4.14)). Then the bound $Z_y(t_0) - Y_y^h(t_0) \leq R_0$ for all $y \in \mathbb{R}^d$ follows from $L_{u,\eta,1-\theta^*}(0) \leq R_0$ for all $\eta \in (0, 1-\theta^*)$, so (4.14) yields $Z_y(t) - Y_y^h(t) \leq \lambda$ for all $(t,y) \in [0,\infty)\times\mathbb{R}^d$ and some $\lambda = \lambda(F_0, M, \theta_1, \zeta, \xi)$. The definition of $Y_y^h(t)$ shows that if $u(t,y) \geq \eta$ for some (t,y), then $Y_y^h(t) \leq \psi^{-1}(\frac{1}{\eta})$, with $\psi''(r) + \frac{d-1}{r}\psi'(r) = \zeta\psi(r)$ on $(0,\infty)$ and $\psi(0) = 1$ (hence we have $\lim_{r\to\infty} r^{(d-1)/2} e^{-\sqrt{\zeta}r}\psi(r) \in (0,\infty)$). Therefore $\psi(r) \geq e^{\sqrt{\zeta}r/2}$ for all large enough r, so $Z_y(t) \leq \lambda + \frac{2}{\sqrt{\zeta}}|\ln\eta|$ for all small enough η. But since $Z_y(t)$ is the distance from y to the nearest point x with $u(t,x) \geq 1 - \varepsilon_0 = 1 - \theta^*$, we obtain $L_{u,\eta,1-\theta^*}(t) \leq \lambda + \frac{2}{\sqrt{\zeta}}|\ln\eta|$ for all $t \geq 0$ and all small enough $\eta > 0$. This yields (2.4) with some $\Lambda = \Lambda(F_0, M, \theta_1, \zeta, \xi)$. □

The next result is a counterpart of Lemma 2.1 (see [16, Lemma 2.2]). It shows that the speed of propagation of perturbations of solutions to (1.1) is bounded above by $2\sqrt{Md}$ (in fact, the bound $2\sqrt{M}$ works as well, but we will not need it here).

Lemma 2.5. *Let $r > 0$ and $y \in \mathbb{R}^d$, and let $u_1 : [0,\infty) \times B_r(y) \to (-\infty, 1]$ be a subsolution and $u_2 : [0,\infty) \times B_r(y) \to [0,\infty)$ a supersolution to (1.1) with some f satisfying **(H1')** and some $\omega \in \Omega$. If $u_1(0,\cdot) \le u_2(0,\cdot)$ on $B_r(y)$, then for all $(t,x) \in [0,\infty) \times B_r(y)$ we have*

$$u_1(t,x) \le u_2(t,x) + 2d\, e^{\sqrt{M/d}\,\left(|x-y|-r+2\sqrt{Md}\,t\right)}.$$

Remark. This was stated in [16] with u_1, u_2 having values in $[0,1]$ only, but the proof applies to our case without change.

This yields the following corollary (as above, c_1 in this result could be just $2\sqrt{M}$).

Corollary 2.6. *If $u : [0,\infty) \times \mathbb{R}^d \to [0,1]$ solves (1.1) with some f satisfying **(H1')** and some $\omega \in \Omega$, then for any $t \ge 0$ we have*

$$\{x \in \mathbb{R}^d \,|\, u(t,x) \ge 1-\theta_1\} \subseteq B_{c_1 t+\kappa_1}\big(\{x \in \mathbb{R}^d \,|\, u(0,x) \ge \theta_1\}\big),$$

where

$$c_1 := 2\sqrt{Md} > c_0 \qquad \text{and} \qquad \kappa_1 := 1 + \sqrt{d/M}\,\ln\frac{2d}{1-2\theta_1}.$$

Proof. The claim $c_1 > c_0$ follows from the well-known estimate $c_0 < 2\sqrt{M}$ for any M-Lipschitz ignition reaction F_0. If $t \ge 0$ and $y \notin B_{c_1 t+\kappa_1}(\{x \,|\, u(0,x) \ge \theta_1\})$, then Lemma 2.5 with $u_1 = u$, $u_2 \equiv \theta_1$, and $r = c_1 t + \kappa_1$ yields

$$u(t,y) \le \theta_1 + 2d\, e^{-\sqrt{M/d}\,\kappa_1} < 1-\theta_1,$$

finishing the proof. □

We can now use these results to show that Theorems 1.3 and 1.4 follow from Theorem 1.7, so it will suffice to prove the latter (and then Theorem 1.8). This also means that we will assume either **(H2')** or **(H4')** in the rest of Sections 2–6. Hence, then there will be $\mu_*, \kappa_* > 0$ such that for any $u \in \mathcal{U}_f$ or $u \in \bigcup_{n\ge n_4} \mathcal{U}_{f_n}$, respectively, we have

$$\begin{aligned} &\sup_{t\ge 0\,\&\,\eta>0} \frac{L_{u,\eta,1-\theta^*}(t)}{(1+t^{\alpha_2})\eta^{-m_2}} \le \mu_*^{-1}, \\ &\inf_{\substack{(t,x)\in[\kappa_*,\infty)\times\mathbb{R}^d \\ u(t,x)\in[\theta^*,1-\theta^*]}} u_t(t,x) \ge \mu_*. \end{aligned} \tag{2.6}$$

Notice that while (1.10) only allows us to state the first of these claims for $t \ge \kappa_*$, one can extend this to all $t \ge 0$ (with a different μ_*). This is because Lemma 2.5 with $x = y$ shows that if $u(t,y) \ge \eta$ for some $u \in \mathcal{U}_f$ (or $u \in \bigcup_{n\ge n_4} \mathcal{U}_{f_n}$), $(t,y) \in [0,\kappa^*] \times \mathbb{R}^d$, and $\eta > 0$, then

$$d(y, B_k(0)) \le R_0 + \sqrt{Md}\,\left(2\kappa_* + M^{-1}\ln\frac{2d}{\eta}\right)$$

(when $u(0,\cdot) = u_{0,k}$), so $L_{u,\eta,1-\theta^*}(t)$ satisfies the same upper bound because $u_t \ge 0$. We note that we could also include $t \in [0,\kappa_*)$ in the second claim of (2.6), at the expense of some extra work, but this would not be as useful.

Corollary 2.7. ***(H2)** implies **(H2')**. Also, **(H2)** and **(H4)** imply **(H4')** for some sequence of reactions g_n in place of f_n, with possibly different $M, \theta_1, n_4, \alpha_4$ and with $m_4' = \infty$.*

Proof. Remark 2 after Definition 2.3 and Lemma 2.4 show that **(H2)** implies **(H2')** with $\alpha_2 = 0$ and any $m_2 > 0$.

Let us now assume **(H2)** and **(H4)**, and with the convention that $[a, b] = \emptyset$ if $a > b$, let

$$\tilde{f}_n(x, u, \omega) := \min\{f_n(x, v, \omega) \,|\, v \in \{u\} \cup [1 - \theta_1, u]\}.$$

These reactions are non-increasing in $u \in [1 - \theta_1, 1]$, and still satisfy $\|\tilde{f}_n - f\|_\infty \le \alpha_4 n^{-m_4}$, because f is non-increasing in $u \in [1-\theta_1, 1]$. Also, each $\tilde{f}_n$ is obviously stationary with range of dependence no greater than that of f_n.

Next, let $\phi : \mathbb{R}^{d+1} \to [0, \infty)$ be a smooth function supported in $B_1(0)$ and with integral over $B_1(0)$ equal to 1. Then let $\phi_n(x, u) := n^{(d+1)m_4}\phi(n^{m_4}x, n^{m_4}u)$, and consider $\tilde{f}_n * \phi_n$ (with the convolution in (x, u); recall that all reactions are extended by 0 to $u \notin [0, 1]$). With ∇ being either ∇_x or ∂_u, we then have

$$\|\nabla \tilde{f}_n * \phi_n\|_\infty \le \|\nabla f * \phi_n\|_\infty + \|(\tilde{f}_n - f) * \nabla\phi_n\|_\infty \le M + \alpha_4 n^{-m_4}\|\nabla\phi_n\|_1 = M + \alpha_4\|\nabla\phi\|_1.$$

Finally, recall F from (1.9) and let $\bar{F}(u) := \sup_{(x,\omega)\in\mathbb{R}^d\times\Omega} f(x, u, \omega)$, and

$$\tilde{g}_n(x, u, \omega) := \min\{\max\{(\tilde{f}_n * \phi_n)(x, u, \omega), F(u)\}, \bar{F}(u)\}.$$

It is not hard to see that for all large enough n, these functions satisfy **(H1)** with (M, θ_1) replaced by $(M + \alpha_4\|\nabla_x\phi\|_1, \frac{1}{2}\theta_1)$. We also have $\|\tilde{g}_n - f\|_\infty \le (2M + \alpha_4)n^{-m_4}$ for all large enough n because $\|\tilde{f}_n - f\|_\infty \le \alpha_4 n^{-m_4}$ and $F \le f(x, \cdot, \omega) \le \bar{F}$ for all (x, ω), and the range of dependence of $\tilde{g}_n$ is at most $n + 2n^{-m_4} \le n + 2$.

Since $f \in \mathcal{F}(F_0, M, \theta_1, \zeta, \xi)$ for some $\zeta < c_0^2/4$ and $\xi > 0$, $\|\tilde{g}_n - f\|_\infty \le (2M + \alpha_4)n^{-m_4}$ yields $\tilde{g}_n \in \mathcal{F}(F_0, M + \alpha_4\|\nabla_x\phi\|_1, \frac{1}{2}\theta_1, \frac{1}{8}(4\zeta + c_0^2), \frac{1}{2}\xi)$ for all large enough n. Therefore, as at the start of this proof, we obtain that the $\tilde{g}_n$ satisfy **(H2')** uniformly in n, for all large enough n. Hence, the reactions $g_n := \tilde{g}_{n-2}$ satisfy **(H4')** with possibly different $M, \theta_1, n_4, \alpha_4$ and with $m_4' = \infty$. □

The next result uses Lemma 2.5 and the bound $f(\cdot, u, \cdot) \ge \alpha_1(1 - u)^{m_1}$ for u near 1 to essentially obtain an upper bound on $\kappa_0(M, \theta_1, m_1, \alpha_1, \frac{c_0}{4}, \theta)$ from Lemma 2.1 as $\theta \to 1$ (we could similarly do this for any $c < c_0$ in place of $\frac{c_0}{4}$).

Lemma 2.8. *Let $u : [0, \infty) \times \mathbb{R}^d \to [0, 1]$ solve (1.1) with f satisfying **(H1')** and some $\omega \in \Omega$. There is $D_1 = D_1(M, \theta_1, m_1, \alpha_1)$ such that if $u(t_0, y) \ge 1 - \theta^*$ for some $t_0 \ge 1$ and $y \in \mathbb{R}^d$, then for any $\theta \in [1 - \theta^*, 1)$ and $t \ge t_0 + D_1(1 - \theta)^{1-m_1}$ we have*

$$\inf_{|x-y|\le c_0(t-t_0)/4} u(t, x) \ge \theta.$$

Proof. Lemma 2.1 shows that with κ_0 from (2.2) we have

$$\inf_{|x-y|\le c_0(t-t_0)/2} u(t, x) \ge 1 - \theta^*$$

for any $t \ge t_0 + \kappa_0$. Since

$$U(s) = 1 - ((\theta^*)^{1-m_1} + (m_1 - 1)\alpha_1 s)^{-1/(m_1-1)}$$

solves the ODE $U' = \alpha_1(1-U)^{m_1}$ with $U(0) = 1-\theta^*$, it follows from Lemma 2.5 that

$$u(t+s,x) \geq U(s) - 2de^{\sqrt{M/d}\left(|x-y|-c_0(t-t_0)/2+2\sqrt{Md}\,s\right)}$$

for all $(s,x) \in [0,\infty)\times\mathbb{R}^d$. Picking $s := \frac{1}{(m_1-1)\alpha_1}[(\frac{1-\theta}{2})^{1-m_1} - (\theta^*)^{1-m_1}]$ makes $U(s) \geq \frac{1+\theta}{2}$. The second term on the right will be no more than $\frac{1-\theta}{2}$ when $|x-y| \leq \frac{c_0}{4}(t+s-t_0)$, provided

$$\frac{c_0(t+s-t_0)}{4} - \frac{c_0(t-t_0)}{2} + 2\sqrt{Md}\,s \leq \sqrt{d/M}\,\ln\frac{1-\theta}{4d},$$

which holds as long as $t \geq t_0 + D_1(1-\theta)^{1-m_1}$ for some $D_1 = D_1(M,\theta_1,m_1,\alpha_1) \geq \kappa_0$. Replacing D_1 by $D_1 + \frac{1}{(m_1-1)\alpha_1}2^{m_1-1}$ now yields the claim for $t+s$ in place of t whenever $t+s \geq t_0 + D_1(1-\theta)^{1-m_1}$. □

Finally, we will need two results providing estimates on how much solutions to (1.1) may change when the reaction f is perturbed. The first one concerns the case when the reaction can change only where the solution is initially close to 1. In it, we will also use the definition

$$T_u(x) := \inf\{t \geq 0 \,|\, u(t,x) \geq 1-\theta^*\}.$$

Lemma 2.9. *Let f_1 satisfy **(H2')** and f_2 satisfy **(H1')**, and let $M_* := \frac{1+M}{\mu_*}$, with μ_*, κ_* from (2.6) for all $u \in \mathcal{U}_{f_1}$. Fix some $\omega \in \Omega$ and let $u_1, u_2 : [0,\infty)\times\mathbb{R}^d \to [0,1]$ solve (1.1) with f_1, f_2 in place of f, respectively. If $u_1 \in \mathcal{U}_{f_1}$, $t_0 \geq 0$, and for some $\eta \in [0, \frac{1}{2}\min\{\theta^*, M_*^{-1}\}]$ we have*

$$f_1(x,u,\omega) = f_2(x,u,\omega) \qquad \text{whenever } u_1(t_0,x) < 1-\eta \text{ and } u \in [0,1], \tag{2.7}$$

then

$$u_+(t,x) := u_1((1+M_*\eta)t + t_0, x) + \eta$$

is a supersolution to (1.1) with f_2 in place of f on $(\kappa_*,\infty)\times\mathbb{R}^d$, and

$$u_-(t,x) := u_1((1-M_*\eta)t + t_0, x) - \eta$$

is a subsolution to (1.1) with f_2 in place of f on $(2\kappa_*,\infty)\times\{x \in \mathbb{R}^d \,|\, u_1(t_0,x) < 1-\eta\}$.

Moreover, there is $D_2 = D_2(M,\theta_1,m_1,\alpha_1) \geq 1$ such that if also

$$\sup_{x\in B_R(y)} (u_2(0,x) - u_1(t_0,x)) \leq \eta$$

for some $y \in \mathbb{R}^d$ and $R \geq D_2(1+T_{u_2}(y))$, then

$$T_{u_2}(y) \geq (1+M_*\eta)^{-1}(T_{u_1}(y) - t_0 - 2\kappa_* - \kappa_0). \tag{2.8}$$

The proof of Lemma 2.9 appears in Appendix B.

Our last preliminary lemma concerns the case when the reaction may be perturbed anywhere, although not by a lot.

Lemma 2.10. *Let f_1 satisfy **(H2')** and f_2 satisfy **(H1')**, with at least one of them satisfying **(H3)** with $\alpha_3 \leq 1$, and let M_*, D_2 be from Lemma 2.9. Fix some $\omega \in \Omega$ and let u_1, u_2 :*

$[0,\infty)\times\mathbb{R}^d\to[0,1]$ solve (1.1) with f_1, f_2 in place of f, respectively. If $u_1\in\mathcal{U}_{f_1}$, for some $y\in\mathbb{R}^d$, $R\ge D_2(1+T_{u_2}(y))$, and $\eta\in[0,\frac{1}{2}\min\{\theta^*,M_*^{-1}\}]$ we have

$$f_1(x,u,\omega)\ge f_2(x,u,\omega)-\alpha_3\eta^{m_3}\quad \text{for all } (x,u)\in B_R(y)\times[0,1], \tag{2.9}$$

and $u_2(0,x)\le u_1(t_0,x)$ for some $t_0\ge 0$ and all $x\in B_R(y)$, then (2.8) holds.

The proof of Lemma 2.10 appears in Appendix C.

2.1. **Notations.** Since Sections 3–6 are just the proof of Theorem 1.7, we will assume either **(H2')** or **(H3)**+**(H4')** in them. Any constants $C, C_0, C_\delta, C(\delta),\dots$ may depend on

$$M,\theta_1,m_1,\alpha_1,m_2,\alpha_2,m_3,\alpha_3,m_4,m_4',n_4,\alpha_4,\mu_*,\kappa_* \tag{2.10}$$

(except for $m_3,\alpha_3,m_4,m_4',n_4,\alpha_4$ when **(H2')** is assumed; dependence on d is implicit in the whole paper). Any other dependence will be explicitly declared, for instance, "for some C_δ" or "for some $C=C(\delta)$" will mean that this constant depends on δ as well as (2.10). These constants may also vary from line to line. We recall that M,θ_1,m_1,α_1 are from **(H1)**; m_2,α_2 from **(H2')**; m_3,α_3 from **(H3)**; m_4,m_4',n_4,α_4 from **(H4')**; and μ_*,κ_* from (2.6).

The constants

$$\theta_2,\theta^*,c_0,\kappa_0,c_1,\kappa_1,D_1,D_2,R_0,M_*$$

also only depend on subsets of (2.10), with θ_2,c_0,κ_0 from Lemma 2.1; θ^* from (2.1); R_0 from Lemma 2.2; c_1,κ_1 from Corollary 2.6; D_1 from Lemma 2.8; and M_*, D_2 from Lemma 2.9.

For $A\subseteq\mathbb{R}^d$ and $r\ge 0$ we let $B_r(A):=A+(B_r(0)\cup\{0\})$ and $A_r^0:=A\backslash\overline{B_r(\partial A)}$ (so A_0^0 is the interior of A). For sets $U,V\subseteq\mathbb{R}^d$, we let $d(U,V):=\inf_{x\in U\,\&\,y\in V}|x-y|$ and

$$d_H(U,V):=\max\left\{\sup_{x\in U}d(x,V),\sup_{y\in V}d(y,U)\right\}$$

be their standard and Hausdorff distances.

Widths $L_{u,\eta,\theta}(t)$ of transition zones of solutions are defined in (1.7), and the special sets $\mathcal{U}_f$ of solutions evolving from approximate characteristic functions of balls are defined in **(H2')**.

Finally, we recall that $\mathcal{E}(U)$ is the σ-algebra generated by the family of random variables $\{f(x,u,\cdot)\,|\,(x,u)\in U\times[0,1]\}$. Further important notation related to the dependence of reactions and solutions on ω appears below, particularly early in Sections 3 and 5.

3. Fluctuations for Reactions with Finite Ranges of Dependence

The proof of Theorem 1.7 will be based on the analysis of the dynamics of special solutions, starting from the approximate characteristic functions $u_{0,S}$ of sets $S\subseteq\mathbb{R}^d$ satisfying Lemma 2.2. For each S and $\omega\in\Omega$, we therefore let $u(\cdot,\cdot,\omega;S)$ be the solution to

$$\begin{aligned} &u_t=\Delta u+f(x,u,\omega) && \text{on } (0,\infty)\times\mathbb{R}^d,\\ &u(0,\cdot,\omega;S)=u_{0,S} && \text{on } \mathbb{R}^d. \end{aligned} \tag{3.1}$$

Recall that $u_t(\cdot,\cdot,\omega;S)\ge 0$ because $u_{0,S}$ satisfies (1.9). Let us also define for any $x\in\mathbb{R}^d$,

$$T(x,\omega;S):=\inf\{t\ge 0\,|\,u(t,x,\omega;S)\ge 1-\theta^*\}\ge 0, \tag{3.2}$$

with θ^* from (2.1), which can be thought of as the time when this solution reaches x.

Our goal is now to estimate the stochastic fluctuations of $T(x,\omega;S)$. In this section we will consider the first case in Theorem 1.7, when the reaction satisfies **(H2')** and has a finite range of dependence. We will only need to treat the cases when S is either a ball or a half-space. We start with S being a ball, when the solution in (3.1) will be precisely the one from $\mathcal{U}_f$ corresponding to (S,ω). Hence, below always $u(\cdot,\cdot,\omega;S)\in\mathcal{U}_f$ and (2.6) holds for it.

Remark. We note that if we enlarge $\mathcal{U}_f$ (or each $\mathcal{U}_{f_n}$) to include solutions with initial data $u_{0,S}$ from Lemma 2.2 for all S from some family $\mathcal{S}$ of bounded subsets of $\mathbb{R}^d$, and (2.6) still continues to hold with some $\mu_*,\kappa_*>0$, then everything in this section and the next holds without change (and with uniform constants) for either all $S\in\mathcal{S}$ (the results involving balls only) or for all local limits in Hausdorff distance of translations of sets from $\mathcal{S}$ (the results involving half-spaces). In particular, Remark 2 after Definition 2.3 and Lemma 2.4 show that if we assume **(H2)**, then we can let $\mathcal{S}$ be the family of all bounded subsets of $\mathbb{R}^d$.

Proposition 3.1. *Let f satisfy **(H2')** and let*

$$\beta_1:=\max\left\{\frac{m_1-1}{m_1},\,\frac{m_2+\alpha_2}{m_2+1}\right\}\qquad(\in(0,1)).\tag{3.3}$$

There is $C_0\geq 1$ such that if f has range of dependence at most $\rho\in[1,\infty)$ and $S=B_k(0)$ for some $k\in\mathbb{N}$, then for all $x\in\mathbb{R}^d$ and $\lambda>0$ we have

$$\mathbb{P}\left(\left|T(x,\cdot\,;S)-\mathbb{E}[T(x,\cdot\,;S)]\right|\geq\lambda\right)\leq 2\exp\left(\frac{-\lambda^2}{C_0(1+d(x,S))(\rho+d(x,S)^{\beta_1})}\right).\tag{3.4}$$

Remarks. 1. The point here is that by choosing $\lambda\sim d(x,S)^\gamma$ for some $\gamma\in(\frac{1+\beta_1}{2},1)$, one obtains a fast-decreasing bound on the probability of $O(d(x,S)^\gamma)$ fluctuations of $T(x,\cdot\,;S)$ from its mean (the latter is of course $\sim d(x,S)$ by Lemma 2.1 and Corollary 2.6).

2. Note that Remark 2 after Definition 2.3 and Lemma 2.4 show that **(H2)** implies **(H2')** with $\alpha_2=0$ and any $m_2>0$, so then $\beta_1=\frac{m_1-1}{m_1}$. If there is also $\theta<1$ such that $\inf_{(x,u,\omega)\in\mathbb{R}^d\times(\theta,1)\times\Omega}(1-u)^{-1}f(x,u,\omega)>0$, then this means that $\beta_1>0$ can be made arbitrarily small.

The rest of this section is devoted to proving Proposition 3.1 (and then extending it from balls to half-spaces in Proposition 3.8). We will therefore assume **(H2')** and the range of dependence of f being at most $\rho\in[1,\infty)$. We will also fix $S=B_k(0)$ and drop it from the notation (so the functions from (3.1) and (3.2) are $u(t,x,\omega)$ and $T(x,\omega)$, respectively) but all estimates will be independent of S (i.e., of k). Recall that all constants with C in them depend on (2.10), with any other dependence explicitly stated, and may vary from line to line.

3.1. **Construction of a martingale.** Let $\mathcal{K}$ be the set of all non-empty compact subsets of $\mathbb{R}^d$, and endow it with Hausdorff distance d_H. For each $a=(a_1,...,a_d)\in\mathbb{Z}^d$, let

$$\mathcal{C}_a:=[a_1,a_1+1)\times...\times[a_d,a_d+1),$$

and for each finite $\emptyset \neq A \subseteq \mathbb{Z}^d$, let

$$P_A := \left\{ K \in \mathcal{K} \,\middle|\, K \cap \frac{1}{\sqrt{d}}\mathcal{C}_a \neq \emptyset \text{ if and only if } a \in A \right\}.$$

Let us label all such A as $A_0, A_1, \ldots$, and denote $P_i := P_{A_i}$. Then $\{P_i\}_{i\in\mathbb{N}_0}$ is a (pairwise disjoint) partition of the metric space $(\mathcal{K}, d_H)$, and $\operatorname{diam}_H(P_i) = 1$ for each $i \in \mathbb{N}_0$.

For each i, let

$$K_i := \bigcup_{a\in A_i} \frac{1}{\sqrt{d}}\mathcal{C}_a = \bigcup_{K\in P_i} K$$

(note that K_i is not compact), so that for each $K \in P_i$ we have

$$K \subseteq K_i \subseteq B_1(K).$$

Notice also that for each $i, j \in \mathbb{N}_0$ we have

$$\text{if } P_i \ni K \subseteq K' \in P_j, \text{ then } K_i \subseteq K_j. \tag{3.5}$$

Next, for any $(t,\omega) \in [0,\infty)\times\Omega$ and $\theta \in (0,1)$, we let

$$\Gamma_{u,\theta}(t,\omega) := \{x \in \mathbb{R}^d \,|\, u(t,x,\omega) \geq \theta\}.$$

Since S is bounded, Lemma 2.5 shows that all these sets are compact. Then for all $i \in \mathbb{N}_0$ and $t \geq 0$ we let

$$E_i(t) := \{\omega \in \Omega \,|\, \Gamma_{u,\theta^*}(t,\omega) \in P_i\} \subseteq \Omega,$$

with θ^* from (2.1). Then $\{E_i(t)\}_{i\in\mathbb{N}_0}$ is a pairwise disjoint partition of Ω for each $t \geq 0$ because $\Gamma_{u,\theta^*}(t,\omega) \neq \emptyset$ (due to $\theta^* \leq 1-\theta^*$). Also note that $\alpha_2 < 1$ allows us to only track the evolution of one of the sets $\Gamma_{u,\theta}$ (see the remark after **(H2')**), and in this section we choose it to be Γ_{u,θ^*}. Finally, for $(t,\omega) \in [0,\infty)\times\Omega$, let

$$\iota(t,\omega) := i \qquad \text{when } \omega \in E_i(t),$$

and for each $(t,x) \in [0,\infty)\times\mathbb{R}^d$ let

$$F_{t,x} := \{\omega \in \Omega \,|\, x \in K_{\iota(t,\omega)}\}.$$

The latter is a slightly different version of the set of ω for which the solution u has reached x by time t (we have $F_{t,x} \supseteq \{\omega \in \Omega \,|\, T(x,\omega) \leq t\} \supseteq F_{t-C,x}$ for some $C > 0$ and all large enough t, due to (2.6)). Let us also pick A_0 so that $\iota(0,\omega) = 0$ for each $\omega \in \Omega$ (note that $\iota(0,\omega)$ does not depend on ω).

Since $u_t \geq 0$ and $f(\cdot,u,\cdot) \equiv 0$ for $u \in [0,\theta^*]$, the solution $u(\cdot,\cdot,\omega)$ only depends on the reaction inside $K_{\iota(t,\omega)}$ up to time t. Hence, we have the following lemma.

Lemma 3.2. *For any $(t,x) \in [0,\infty)\times\mathbb{R}^d$ and $s \in [0,t]$, the set $E_i(t)$ and the function $u(s,x,\cdot)\chi_{E_i(t)}(\cdot)$ are $\mathcal{E}(K_i)$-measurable for each $i \in \mathbb{N}_0$.*

Proof. Fix any $i \in \mathbb{N}_0$, and let g be any reaction satisfying **(H1')** and

$$f(x,u,\omega) = g(x,u,\omega) \qquad \text{for all } (x,u,\omega) \in K_i \times [0,1] \times \Omega.$$

Then for each $\omega \in \Omega$, let v be the solution to

$$v_t = \Delta v + g(x,v,\omega)$$

with the same initial data $v(0,\cdot,\omega) = u_{0,S}$ as u, and fix any $t \geq 0$. Since $u_t, v_t \geq 0$ and $f(\cdot,u,\cdot) \equiv 0$ for $u \in [0,\theta^*]$, it follows that $v(s,\cdot,\omega) = u(s,\cdot,\omega)$ whenever $\omega \in E_i(t)$ and $s \leq t$. Similarly, we have $v(s,\cdot,\omega) = u(s,\cdot,\omega)$ whenever $\Gamma_{v,\theta^*}(t,\omega) \in P_i$ and $s \leq t$ (with $\Gamma_{v,\theta}$ defined analogously to $\Gamma_{u,\theta}$). In particular, $\omega \in E_i(t)$ if and only if $\Gamma_{v,\theta^*}(t,\omega) \in P_i$.

Since this holds for each g as above, it follows that $E_i(t) \in \mathcal{E}(K_i)$, and then also that $u(s,x,\cdot)\chi_{E_i(t)}(\cdot)$ is $\mathcal{E}(K_i)$-measurable for any $(s,x) \in [0,t] \times \mathbb{R}^d$. □

For each $t \geq 0$, let $\mathcal{G}_t$ be the σ-algebra on Ω generated by

$$\bigcup_{i \in \mathbb{N}_0 \,\&\, s \in [0,t]} \mathcal{E}(K_i)|_{E_i(s)}.$$

That is, $\mathcal{G}_t$ is generated by all events $F \cap E_i(s)$ with $i \in \mathbb{N}_0, F \in \mathcal{E}(K_i), s \in [0,t]$. Then $\{\mathcal{G}_t\}_{t\geq 0}$ is a filtration on $(\Omega,\mathcal{F})$, and $\mathcal{G}_0 = \mathcal{E}(K_0)$ because $E_0(0) = \Omega$ and $E_i(0) = \emptyset$ for $i > 0$. Lemma 3.2 then shows that $u(s,x,\cdot)$ is $\mathcal{G}_t$-measurable for any $(t,x) \in [0,\infty) \times \mathbb{R}^d$ and $s \leq t$. Since $E_i(s) \in \mathcal{E}(K_i)$ for all $i \in \mathbb{N}_0$ and $s \geq 0$ (so also $F \cap E_i(s) \in \mathcal{E}(K_i)$ above), $\iota(s,\cdot)$ and $F_{s,x}$ are also $\mathcal{G}_t$-measurable for all $(t,x) \in [0,\infty) \times \mathbb{R}^d$ and $s \leq t$.

We note that $\mathcal{G}_t$ is actually simpler than its definition suggests, and for each $t \geq 0$ and $j \in \mathbb{N}_0$ we in fact have

$$\mathcal{G}_t|_{E_j(t)} = \mathcal{E}(K_j)|_{E_j(t)} \tag{3.6}$$

(recall also that $\{E_j(t)\}_{j\in\mathbb{N}_0}$ is a partition of Ω for each $t \geq 0$, and note that only finitely many of these sets are non-empty due to Lemma 2.5). Indeed, let us consider any $i \in \mathbb{N}_0$ and $s \in [0,t]$ such that $E_i(s) \cap E_j(t) \neq \emptyset$. Then there is $\omega \in \Omega$ such that

$$\iota(t,\omega) = j \qquad \text{and} \qquad \iota(s,\omega) = i.$$

From (3.5) and

$$\Gamma_{u,\theta^*}(r,\omega) \subseteq \Gamma_{u,\theta^*}(s,\omega)$$

for any such ω, we obtain $K_i \subseteq K_j$. But then $E_i(s) \in \mathcal{E}(K_i) \subseteq \mathcal{E}(K_j)$ and

$$\mathcal{E}(K_i)|_{E_i(s)\cap E_j(t)} \subseteq \mathcal{E}(K_j)|_{E_i(s)\cap E_j(t)} \subseteq \mathcal{E}(K_j)|_{E_j(t)}.$$

Since this clearly also holds when $E_i(s) \cap E_j(t) = \emptyset$, the definition of $\mathcal{G}_t$ proves (3.6).

Similarly to [2, 3], we will prove Proposition 3.1 by studying the $\mathcal{G}_t$-adapted martingale $\{X_t := \mathbb{E}[T(x,\cdot) \,|\, \mathcal{G}_t]\}_{t\geq 0}$ for any $x \in \mathbb{R}^d$ and estimating its increments, which will then allow us to apply Azuma's inequality to bound the fluctuations of $T(x,\cdot)$.

Lemma 3.3 (Azuma's inequality)**.** *Let $\{X_k\}_{k\in\mathbb{N}_0}$ be a martingale on $(\Omega,\mathcal{F},\mathbb{P})$. If for each $k \in \mathbb{N}$ there is $c_k \geq 0$ such that $|X_k - X_{k-1}| \leq c_k$ almost surely, then for all $\lambda > 0$ and $n \in \mathbb{N}$,*

$$\mathbb{P}[|X_n - X_0| \geq \lambda] \leq 2\exp\left(-\frac{\lambda^2}{2\sum_{k=1}^n c_k^2}\right).$$

We first show that for any $\omega \in F_{t,x}$ the difference $\mathbb{E}[T(x,\cdot) \,|\, \mathcal{G}_t](\omega) - T(x,\omega)$ is uniformly bounded, and 0 if $\omega \in F_{s,x}$ when $t - s$ is large enough (note that $F_{s,x} \subseteq F_{t,x}$ if $s \leq t$ because $u_t \geq 0$). This is due to $F_{t,x} \in \mathcal{G}_t$ and the above-mentioned relationship of $F_{t,x}$ and $\{\omega \in \Omega \,|\, T(x,\omega) \leq t\}$.

Lemma 3.4. *There is $C > 0$ such that for each $(t, x) \in [0, \infty) \times \mathbb{R}^d$ we have*

$$\left|\mathbb{E}[T(x,\cdot)\chi_{F_{t,x}} \,|\, \mathcal{G}_t] - T(x,\cdot)\chi_{F_{t,x}}\right| \le C \qquad \text{on } \Omega,$$

and if also $s \in [0, t - C]$, then

$$\mathbb{E}[T(x,\cdot)\chi_{F_{s,x}} \,|\, \mathcal{G}_t] = T(x,\cdot)\chi_{F_{s,x}} \qquad \text{on } \Omega. \tag{3.7}$$

Proof. For any $(x, \omega) \in \mathbb{R}^d \times \Omega$, let

$$\tau(x,\omega) := \inf\{t \ge 0 \,|\, x \in B_1(\Gamma_{u,\theta^*}(t,\omega))\} \le T(x,\omega).$$

If $\omega \in F_{t,x}$ for some $t \ge 0$, then $\tau(x, \omega) \le t$ due to $K_{\iota(t,\omega)} \subseteq B_1(\Gamma_{u,\theta^*}(t, \omega))$. And since $u(s, x, \cdot)$ is $\mathcal{G}_t$-measurable for all $s \le t$, we see that $\tau(x,\cdot)\chi_{F_{t,x}}(\cdot)$ is $\mathcal{G}_t$-measurable.

For each $(x, \omega) \in \mathbb{R}^d \times \Omega$, there is $y \in \overline{B_1(x)}$ with $u(\tau(x, \omega), y, \omega) \ge \theta^*$, so (2.6) shows that

$$u\left(\tau(x,\omega) + \kappa_* + \mu_*^{-1}, y, \omega\right) \ge 1 - \theta^*, \tag{3.8}$$

and then Lemma 2.1 yields

$$u(\tau(x,\omega) + \kappa_* + \mu_*^{-1} + \kappa_0 + 2c_0^{-1}, x, \omega) \ge 1 - \theta^*.$$

Therefore there is C such that $T(x, \omega) \le \tau(x, \omega) + C$, and hence $|T(x, \omega) - \tau(x, \omega)| \le C$. Hence for any $(t, x) \in [0, \infty) \times \mathbb{R}^d$ we obtain using $\mathcal{G}_t$-measurability of $\tau(x,\cdot)\chi_{F_{t,x}}$,

$$\left|\mathbb{E}[T(x,\cdot)\chi_{F_{t,x}} \,|\, \mathcal{G}_t] - T(x,\cdot)\chi_{F_{t,x}}\right| \le \left|\mathbb{E}[\tau(x,\cdot)\chi_{F_{t,x}} \,|\, \mathcal{G}_t] - \tau(x,\cdot)\chi_{F_{t,x}}\right| + 2C = 2C,$$

yielding the first claim. If also $s \le t - C$, then for all $\omega \in F_{s,x}$ we have

$$T(x,\omega) \le \tau(x,\omega) + C \le s + C \le t.$$

Since $u(t, x, \cdot)$ is $\mathcal{G}_t$-measurable, this shows that so is $T(x,\cdot)\chi_{F_{s,x}}$, and (3.7) follows. □

When $\omega \in \Omega \setminus F_{t,x}$ (that is, essentially, when the solution u has not yet reached x by the time t), we will obtain a different kind of estimate.

Let $\rho \ge 1$ be from Proposition 3.1, and for each $i \in \mathbb{N}_0$ let

$$g_i(x,u,\omega) := \psi_i(x)\mathbb{E}[f(x,u,\cdot)] + (1 - \psi_i(x))f(x,u,\omega),$$

where $0 \le \psi_i \le 1$ is Lipschitz with a uniform-in-i constant, with $\psi_i \equiv 1$ on $B_\rho(K_i)$ and $\psi_i \equiv 0$ on $\mathbb{R}^d \setminus B_{\rho+1}(K_i)$. Then g_i is Lipschitz in (x, u) (with a uniform M-dependent constant $\ge M$, which we will call M from now on), and $g_i(x, u, \cdot)$ is independent of $\mathcal{E}(K_i)$ for all $(x, u) \in \mathbb{R}^d \times [0, 1]$ because f has range of dependence at most ρ. Of course,

$$g_i \equiv f \quad \text{ on } (\mathbb{R}^d \setminus B_{\rho+1}(K_i)) \times [0,1] \times \Omega. \tag{3.9}$$

For each $\omega \in \Omega$, let now v_i be the solution to

$$\begin{aligned} (v_i)_t &= \Delta v_i + g_i(x, v_i, \omega) && \text{on } (0,\infty) \times \mathbb{R}^d, \\ v_i(0,\cdot,\omega) &= u_{0,K_i} && \text{on } \mathbb{R}^d, \end{aligned}$$

with u_{0,K_i} from Lemma 2.2. Then $v_i(t, x, \cdot)$ is independent of $\mathcal{E}(K_i)$ for each (t, x), and so is

$$T_i(x,\cdot) := \inf\{t \ge 0 \,|\, v_i(t,x,\cdot) \ge 1 - \theta^*\}.$$

Proposition 3.5. *There is $C \geq 1$ such that for each $(t,x) \in [0,\infty) \times \mathbb{R}^d$ and $\omega \in \Omega \setminus F_{t,x}$ we have*

$$\left|T(x,\omega) - t - T_{\iota(t,\omega)}(x,\omega)\right| \leq C(\rho + d(x,S)^{\beta_1}).$$

Remark. This shows that the difference of the time it takes to reach x from S and the sum of any smaller time t and the time it takes to reach x from $K_{\iota(t,\omega)}$ (which approximates $\Gamma_{u,\theta^*}(t,\omega)$) is sublinear in $d(x,S)$. Hence, this result yields a certain additive structure (up to lower order errors) for the arrival times of solutions with initial data from Lemma 2.2.

Proof. We will use (t_0,x_0) in place of (t,x) in the proof. Fix any $\omega \in \Omega \setminus F_{t_0,x_0}$ and let $i := \iota(t_0,\omega)$. Note that since $\omega \in E_i(t_0)$, we have

$$\Gamma_{u,\theta^*}(t_0,\omega) \subseteq K_i \subseteq B_1(\Gamma_{u,\theta^*}(t_0,\omega)). \tag{3.10}$$

Moreover, since $x_0 \notin K_i$, Lemma 2.1 shows that $t_0 \leq \kappa_0 + \frac{2}{c_0} d(x_0,S)$.

Let us first show that

$$T(x_0,\omega) - t_0 - T_i(x_0,\omega) \leq C(\rho + d(x_0,S)^{\beta_1}).$$

From (2.6) we know that with $L_1 := \mu_*^{-1}(1+t_0^{\alpha_2})(\theta^*)^{-m_2} + 1$ we have

$$\Gamma_{u,\theta^*}(t_0,\omega) \subseteq B_{L_1-1}(\Gamma_{u,1-\theta^*}(t_0,\omega)).$$

Hence we obtain

$$K_i \subseteq B_{L_1}(\Gamma_{u,1-\theta^*}(t_0,\omega)).$$

This and Lemma 2.1 now show

$$B_{R_0}(K_i) \subseteq \Gamma_{u,1-\theta^*}(t_0+t_1,\omega) \tag{3.11}$$

for

$$t_1 := \kappa_0 + 2c_0^{-1}(R_0 + L_1),$$

hence

$$u(t,\cdot,\omega) \geq u_{0,K_i} = v_i(0,\cdot,\omega) \tag{3.12}$$

for any $t \geq t_0 + t_1$. Since $t_0 \leq \kappa_0 + \frac{2}{c_0} d(x_0,S)$, there is $C > 0$ such that

$$t_1 \leq C(1 + d(x_0,S)^{\alpha_2}).$$

Now take

$$\eta := \min\left\{\frac{\theta^*}{2},\ \frac{1}{2M_*},\ d(x_0,S)^{-\gamma}\right\} > 0, \tag{3.13}$$

where

$$\gamma := \min\left\{\frac{1}{m_1}, \frac{1-\alpha_2}{m_2+1}\right\} \in (0,1).$$

Note that (3.3) shows that

$$\max\{\gamma(m_1-1), \alpha_2 + \gamma m_2, 1-\gamma\} = \beta_1 < 1. \tag{3.14}$$

It follows from (3.11) and Lemma 2.8 that for $t_2 := \frac{4}{c_0}(\rho+1) + D_1\eta^{1-m_1}$ we have

$$u(t_0+t_1+t_2,\cdot,\omega) \geq 1-\eta \qquad \text{on } B_{\rho+1}(K_i). \tag{3.15}$$

Moreover, from (3.13) and (3.14) we see that there is $C > 0$ such that

$$t_3 := t_1 + t_2 \le C(\rho + d(x_0, S)^{\alpha_2} + d(x_0, S)^{\gamma(m_1-1)}) \le C(\rho + d(x_0, S)^{\beta_1}). \tag{3.16}$$

We now apply Lemma 2.9 with $(f, g_i, u, v_i, t_0 + t_3, \infty)$ in place of $(f_1, f_2, u_1, u_2, t_0, R)$. Its hypotheses are satisfied due to (3.9), (3.12) and (3.15), and it yields

$$T(x_0, \omega) - t_0 - t_3 \le (1 + M_*\eta)T_i(x_0, \omega) + 2\kappa_* + \kappa_0.$$

This, (3.16), and $T_i(x_0, \omega) \le C(1 + d(x_0, K_i)) \le C(1 + d(x_0, S))$ (which follows from Lemma 2.1, with some $C > 0$) show that there is indeed some $C > 0$ such that

$$T(x_0, \omega) - t_0 - T_i(x_0, \omega) \le M_*\eta T_i(x_0, \omega) + 2\kappa_* + \kappa_0 + t_3 \le C(\rho + d(x_0, S)^{\beta_1}). \tag{3.17}$$

Let us now turn to

$$t_0 + T_i(x_0, \omega) - T(x_0, \omega) \le C(\rho + d(x_0, S)^{\beta_1}),$$

and let η be again from (3.13). We will now need to estimate u from above in terms of some time-shift of v_i. It follows from Lemma 2.9 that

$$u_-(t, x, \omega) := u((1 - M_*\eta)t + t_0, x, \omega) - \eta \qquad (\le 1 - \eta)$$

is a subsolution to (1.1) with reaction f on $(2\kappa_*, \infty) \times (\mathbb{R}^d \setminus \Gamma_{u,1-\eta}(t_0, \omega))$. We also know that v_i is a solution of the same equation on $(0, \infty) \times (\mathbb{R}^d \setminus B_{\rho+1}(K_i))$. In order to be able to compare them, we need some more estimates involving these sets.

Since $\theta^* \le \theta_1$, Corollary 2.6 and (3.10) yield

$$\Gamma_{u,1-\theta^*}(t_0 + 2\kappa_*, \omega) \subseteq B_{2c_1\kappa_*+\kappa_1}(\Gamma_{u,\theta^*}(t_0, \omega)) \subseteq B_{2c_1\kappa_*+\kappa_1}(K_i).$$

This and (2.6) show that with $L_2 := \mu_*^{-1}(1 + (t_0 + 2\kappa_*)^{\alpha_2})\eta^{-m_2}$ and

$$L_3 := \max\{L_2 + 2c_1\kappa_* + \kappa_1, \rho + 1\},$$

we have

$$\Gamma_{u_-,0}(2\kappa_*, \omega) \subseteq \Gamma_{u,\eta}(t_0 + 2\kappa_*, \omega) \subseteq B_{L_2}(\Gamma_{u,1-\theta^*}(t_0 + 2\kappa_*, \omega)) \subseteq B_{L_3}(K_i). \tag{3.18}$$

We note that this also implies

$$\Gamma_{u,1-\eta}(t_0, \omega) \subseteq \Gamma_{u,\eta}(t_0 + 2\kappa_*, \omega) \subseteq B_{L_3}(K_i). \tag{3.19}$$

Moreover, Lemma 2.8 shows that

$$B_{L_3}(K_i) \subseteq \Gamma_{v_i,1-\eta}(t_4, \omega), \tag{3.20}$$

with $t_4 := \frac{4}{c_0}L_3 + D_1\eta^{1-m_1}$. Then (3.13), (3.14), and $t_0 \le \kappa_0 + \frac{2}{c_0}d(x_0, S)$ show that there is $C > 0$ such that

$$t_4 \le C(\rho + t_0^{\alpha_2}\eta^{-m_2} + \eta^{1-m_1}) \le C(\rho + d(x_0, S)^{\beta_1}). \tag{3.21}$$

Using (3.20) and (3.18), we find that

$$\begin{aligned} v_i(t_4 + \cdot, \cdot, \omega) &\ge 1 - \eta \ge u_-(2\kappa_* + \cdot, \cdot, \omega) && \text{on } (0, \infty) \times B_{L_3}(K_i), \\ v_i(t_4, \cdot, \omega) &\ge 0 \ge u_-(2\kappa_*, \cdot, \omega) && \text{on } \mathbb{R}^d \setminus B_{L_3}(K_i). \end{aligned} \tag{3.22}$$

From (3.9), (3.19), and Lemma 2.9 we also see that v_i and u_- are, respectively, a solution and a subsolution to (1.1) with reaction f on $(2\kappa_*, \infty) \times (\mathbb{R}^d \setminus B_{L_3}(K_i))$. So the second claim in (3.22) and the comparison principle yield

$$u_-(2\kappa_* + \cdot, \cdot, \omega) \le v_i(t_4 + \cdot, \cdot, \omega) \qquad \text{on } [0, \infty) \times (\mathbb{R}^d \setminus B_{L_3}(K_i)).$$

If $x_0 \notin B_{L_3}(K_i)$, then this shows that

$$T(x_0, \omega) \ge (1 - M_*\eta)(T_i(x_0, \omega) - t_4 + 2\kappa_*) + t_0.$$

Using again $T_i(x_0, \omega) \le C(1 + d(x_0, S))$ (as we did above) and (3.21), we obtain

$$T_i(x_0, \omega) + t_0 - T(x_0, \omega) \le t_4 + M_*\eta T_i(x_0, \omega) \le C(\rho + d(x_0, S)^{\beta_1}) \tag{3.23}$$

for some $C > 0$. If instead $x_0 \in B_{L_3}(K_i)$, the first claim in (3.22) and (3.21) again yield

$$T_i(x_0, \omega) + t_0 - T(x_0, \omega) \le T_i(x_0, \omega) \le t_4 \le C(\rho + d(x_0, S)^{\beta_1})$$

because $x_0 \notin K_i$ (and hence $T(x_0, \omega) \ge t_0$). □

The last ingredient in the proof of Proposition 3.1 is an estimate on the difference of $T_{\iota(t,\omega)}(x, \omega)$ for two different times t.

Lemma 3.6. *There is $C > 0$ such that for all $(x, \omega) \in \mathbb{R}^d \times \Omega$ and $t_0, t_1 \ge 0$ we have*

$$\left|T_{\iota(t_1,\omega)}(x, \omega) - T_{\iota(t_0,\omega)}(x, \omega)\right| \le C\left(\rho + |t_1 - t_0| + d(x, S)^{\beta_1}\right).$$

Proof. Let $i_0 := \iota(t_0, \omega)$ and $i_1 := \iota(t_1, \omega)$, and then $\omega \in E_{i_0}(t_0) \cap E_{i_1}(t_1)$. Without loss of generality, let us assume $t_1 > t_0$. Then $u_t \ge 0$ and (3.5) show that $K_{i_0} \subseteq K_{i_1}$.

If $x \notin K_{i_1}$, Proposition 3.5 yields

$$|T_{i_1}(x, \omega) - T_{i_0}(x, \omega)| \le |t_1 - t_0| + C(\rho + d(x, S)^{\beta_1}).$$

If $x \in K_{i_0}$, then $T_{i_1}(x, \omega) = T_{i_0}(x, \omega) = 0$. The result follows in either case.

Let us now assume that $x \in K_{i_1} \setminus K_{i_0}$. Then $T_{i_1}(x, \omega) = 0$, while Lemma 2.1 shows that

$$T_{i_0}(x, \omega) \le \kappa_0 + 2c_0^{-1} d(x, K_{i_0}) \le C(1 + d_H(K_{i_0}, K_{i_1})) \tag{3.24}$$

for some $C > 0$. From $t_1 > t_0$ and (2.6) we also have

$$\Gamma_{u,\theta^*}(t_0, \omega) \subseteq \Gamma_{u,\theta^*}(t_1, \omega) \subseteq B_{\mu_*^{-1}(\theta^*)^{-m_2}(1+t_1^{\alpha_2})}(\Gamma_{u,1-\theta^*}(t_1, \omega)),$$

and Corollary 2.6 yields

$$\Gamma_{u,1-\theta^*}(t_1, \omega) \subseteq B_{c_1(t_1-t_0)+\kappa_1}(\Gamma_{u,\theta^*}(t_0, \omega)).$$

Hence there is $C > 0$ such that

$$d_H(\Gamma_{u,\theta^*}(t_0, \omega), \Gamma_{u,\theta^*}(t_1, \omega)) \le C(1 + t_1^{\alpha_2} + t_1 - t_0).$$

Since also $d_H(K_{\iota(t,\omega)}, \Gamma_{u,\theta^*}(t, \omega)) \le 1$ for all $t \ge 0$ (because $\Gamma_{u,\theta^*}(t, \omega) \in P_{\iota(t,\omega)}$), this implies

$$\begin{aligned} d_H(K_{i_0}, K_{i_1}) &\le d_H(K_{i_0}, \Gamma_{u,\theta^*}(t_0, \omega)) + d_H(\Gamma_{u,\theta^*}(t_0, \omega), \Gamma_{u,\theta^*}(t_1, \omega)) + d_H(\Gamma_{u,\theta^*}(t_1, \omega), K_{i_1}) \\ &\le C(1 + t_1^{\alpha_2} + t_1 - t_0). \end{aligned}$$

This and (3.24) yields the claim. □

3.2. **Proof of Proposition 3.1.** If $d(x,S) \le \rho$, then (3.4) holds for all $\lambda > 0$ as long as $C_0 \ge 2(\kappa_0 + \frac{2}{c_0})^2(\ln 2)^{-1}$. This is because $T(x,\cdot) \le \kappa_0 + \frac{2}{c_0} d(x,S)$ by Lemma 2.1, so one only needs to consider $\lambda \le (\kappa_0 + \frac{2}{c_0})(1 + d(x,S))$, for which the right-hand side of (3.4) with this C_0 is at least 1 due to $\rho \ge \frac{1}{2}(1 + d(x,S))$.

It therefore suffices to consider the case $d(x,S) > \rho$. In particular, we have $x \notin K_0$ due to $\rho \ge 1$. Let us fix any such x and consider the $\mathcal{G}_t$-adapted martingale $\{X_t\}_{t\ge 0}$ defined by

$$X_t = X_t(\omega) := \mathbb{E}[T(x,\cdot)\,|\,\mathcal{G}_t](\omega).$$

We want to apply Azuma's inequality to it, which means that we need to obtain a suitable ω-independent bound on $|X_t - X_s|$ for any $t > s > 0$ (which we fix). Using

$$\begin{aligned} X_t &= \mathbb{E}[T(x,\cdot)\chi_{F_{s,x}}\,|\,\mathcal{G}_t] + \mathbb{E}[T(x,\cdot)\chi_{F^c_{s,x}}\,|\,\mathcal{G}_t], \\ X_s &= \mathbb{E}[T(x,\cdot)\chi_{F_{s,x}}\,|\,\mathcal{G}_s] + \mathbb{E}[T(x,\cdot)\chi_{F^c_{s,x}}\,|\,\mathcal{G}_s], \end{aligned}$$

we find from Lemma 3.4 (recall that $F_{s,x} \subseteq F_{t,x}$) that there is $C > 0$ such that

$$\begin{aligned} |X_t - X_s| &\le |\mathbb{E}[T(x,\cdot)\chi_{F^c_{s,x}}\,|\,\mathcal{G}_t] - \mathbb{E}[T(x,\cdot)\chi_{F^c_{s,x}}\,|\,\mathcal{G}_s]| + C \\ &= \left|\Sigma_{x\notin K_i}\mathbb{E}[T(x,\cdot)\chi_{E_i(s)}\,|\,\mathcal{G}_t] - \Sigma_{x\notin K_i}\mathbb{E}[T(x,\cdot)\chi_{E_i(s)}\,|\,\mathcal{G}_s]\right| + C. \end{aligned} \tag{3.25}$$

Here we used that $\omega \notin F_{s,x}$ precisely when $x \notin K_{\iota(s,\omega)}$, and the sums are over all $i \in \mathbb{N}_0$ such that $x \notin K_i$. From Proposition 3.5 with s in place of t we have

$$\begin{aligned} &\left|\Sigma_{x\notin K_i}\mathbb{E}[T(x,\cdot)\chi_{E_i(s)}\,|\,\mathcal{G}_t] - \Sigma_{x\notin K_i}\mathbb{E}[T(x,\cdot)\chi_{E_i(s)}\,|\,\mathcal{G}_s]\right| \\ &\le \left|\Sigma_{x\notin K_i}\mathbb{E}[T_i(x,\cdot)\chi_{E_i(s)}\,|\,\mathcal{G}_t] - \Sigma_{x\notin K_i}\mathbb{E}[T_i(x,\cdot)\chi_{E_i(s)}\,|\,\mathcal{G}_s]\right| + C\left(\rho + d(x,S)^{\beta_1}\right) \\ &= \left|\Sigma_{i\in\mathbb{N}_0}\mathbb{E}[T_i(x,\cdot)\chi_{E_i(s)}\,|\,\mathcal{G}_t] - \Sigma_{i\in\mathbb{N}_0}\mathbb{E}[T_i(x,\cdot)\chi_{E_i(s)}\,|\,\mathcal{G}_s]\right| + C\left(\rho + d(x,S)^{\beta_1}\right). \end{aligned}$$

The last equality holds because $T_i(x,\omega) = 0$ when $x \in K_i$. Since

$$\mathbb{E}[T_i(x,\cdot)\chi_{E_i(s)}\,|\,\mathcal{G}_t] = \Sigma_{j\in\mathbb{N}_0}\mathbb{E}[T_i(x,\cdot)\chi_{E_i(s)\cap E_j(t)}\,|\,\mathcal{G}_t]$$

(recall that $E_i(s), E_j(t) \in \mathcal{G}_t$) and Lemma 3.6 yields

$$|\Sigma_{i,j}\mathbb{E}[T_i(x,\cdot)\chi_{E_i(s)\cap E_j(t)}\,|\,\mathcal{G}_t] - \Sigma_{i,j}\mathbb{E}[T_j(x,\cdot)\chi_{E_i(s)\cap E_j(t)}\,|\,\mathcal{G}_t]| \le C(\rho + |t-s| + d(x,S)^{\beta_1}),$$

it follows that with some $C > 0$ and $C^\rho_{t,s,x} := C(\rho + |t-s| + d(x,S)^{\beta_1})$, we have

$$\begin{aligned} |X_t - X_s| &\le \left|\Sigma_{i,j}\mathbb{E}[T_j(x,\cdot)\chi_{E_i(s)\cap E_j(t)}\,|\,\mathcal{G}_t] - \Sigma_i\mathbb{E}[T_i(x,\cdot)\chi_{E_i(s)}\,|\,\mathcal{G}_s]\right| + C^\rho_{t,s,x} \\ &= \left|\Sigma_j\mathbb{E}[T_j(x,\cdot)\chi_{E_j(t)}\,|\,\mathcal{G}_t] - \Sigma_i\mathbb{E}[T_i(x,\cdot)\chi_{E_i(s)}\,|\,\mathcal{G}_s]\right| + C^\rho_{t,s,x}. \end{aligned} \tag{3.26}$$

We now claim that for any $i \in \mathbb{N}_0$ we have

$$\mathbb{E}[T_i(x,\cdot)\chi_{E_i(s)}\,|\,\mathcal{G}_s] = \mathbb{E}[T_i(x,\cdot)]\chi_{E_i(s)}. \tag{3.27}$$

Since $E_i(s) \in \mathcal{G}_s$, to prove this, we only need to show that

$$\mathbb{E}[T_i(x,\cdot)\chi_A] = \mathbb{E}[T_i(x,\cdot)]\mathbb{P}(A) \tag{3.28}$$

for each $A \in \mathcal{G}_s$ such that $A \subseteq E_i(s)$. But then $A \in \mathcal{E}(K_i)$ by (3.6), so (3.28) follows from $T_i(x,\cdot)$ being independent of $\mathcal{E}(K_i)$.

Similarly to (3.27), we also have

$$\mathbb{E}[T_j(x,\cdot)\chi_{E_j(t)} \,|\, \mathcal{G}_t] = \mathbb{E}[T_j(x,\cdot)]\chi_{E_j(t)}$$

for any $j \in \mathbb{N}_0$. Then (3.26) becomes

$$\begin{aligned} |X_t - X_s| &\le \left|\Sigma_j \mathbb{E}[T_j(x,\cdot)]\chi_{E_j(t)} - \Sigma_i \mathbb{E}[T_i(x,\cdot)]\chi_{E_i(s)}\right| + C^\rho_{t,s,x} \\ &\le \Sigma_{i,j} \left|\mathbb{E}[T_j(x,\cdot)] - \mathbb{E}[T_i(x,\cdot)]\right| \chi_{E_j(t)\cap E_i(s)} + C^\rho_{t,s,x}. \end{aligned}$$

Lemma 3.6 now shows that there is $C > 0$ such that for all $\omega \in \Omega$ we have

$$|X_t(\omega) - X_s(\omega)| \le C(\rho + |t-s| + d(x,S)^{\beta_1}). \tag{3.29}$$

By Lemma 2.1, we have $F_{\tau_x,x} = \Omega$ when $\tau_x := \kappa_0 + \frac{2}{c_0} d(x,S)$. So with C from Lemma 3.4,

$$X_t = T(x,\cdot) \qquad \text{for all } t \ge \tau_x + C. \tag{3.30}$$

Let $\tau := \rho + d(x,S)^{\beta_1}$ and let N be the smallest integer such that $N\tau \ge \tau_x + C$. Then there is $C_1 > 0$ such that $N \le C_1 d(x,S)(\rho + d(x,S)^{\beta_1})^{-1}$ (recall that $d(x,S) > \rho \ge 1$). It follows from (3.29) that for $k = 0, \dots, N-1$ we have (uniformly in $\omega \in \Omega$)

$$|X_{(k+1)\tau} - X_{k\tau}| \le C(\rho + d(x,S)^{\beta_1}). \tag{3.31}$$

Now Azuma's inequality (Lemma 3.3), $X_0 = \mathbb{E}[T(x,\cdot)\,|\,\mathcal{G}_0]$, and (3.30) with $t = N\tau$ yield for any $\lambda \ge 0$ (with C changing from line to line),

$$\begin{aligned} \mathbb{P}\left[\left|T(x,\cdot) - \mathbb{E}[T(x,\cdot)\,|\,\mathcal{G}_0]\right| \ge \lambda\right] &\le 2\exp\left(\frac{-\lambda^2}{2CN(\rho + d(x,S)^{\beta_1})^2}\right) \\ &\le 2\exp\left(\frac{-\lambda^2}{Cd(x,S)(\rho + d(x,S)^{\beta_1})}\right). \end{aligned} \tag{3.32}$$

Since $F_{0,x} = \emptyset$ (because $x \notin K_0$), Proposition 3.5 with $t = 0$ (and $\iota(0,\cdot) \equiv 0$) yields

$$|T(x,\omega) - T_0(x,\omega)| \le C'(\rho + d(x_0,S)^{\beta_1})$$

for some $C' > 0$ and all $\omega \in \Omega$. This and $T_0(x,\cdot)$ being independent of $\mathcal{E}(K_0) = \mathcal{G}_0$ yield

$$\begin{aligned} |\mathbb{E}[T(x,\cdot)\,|\,\mathcal{G}_0] - \mathbb{E}[T(x,\cdot)]| &\le |\mathbb{E}[T_0(x,\cdot)\,|\,\mathcal{G}_0] - \mathbb{E}[T_0(x,\cdot)]| + C'(\rho + d(x,S)^{\beta_1}) \\ &= C'(\rho + d(x,S)^{\beta_1}). \end{aligned}$$

Hence, from (3.32) we obtain for any $\lambda > 0$,

$$\mathbb{P}\left[\left|T(x,\cdot) - \mathbb{E}[T(x,\cdot)]\right| \ge \lambda + C'(\rho + d(x,S)^{\beta_1})\right] \le 2\exp\left(\frac{-\lambda^2}{Cd(x,S)(\rho + d(x,S)^{\beta_1})}\right).$$

So for all $\lambda \ge C'(\rho + d(x,S)^{\beta_1})$ we have

$$\mathbb{P}\left[\left|T(x,\cdot) - \mathbb{E}[T(x,\cdot)]\right| \ge 2\lambda\right] \le 2\exp\left(\frac{-\lambda^2}{Cd(x,S)(\rho + d(x,S)^{\beta_1})}\right),$$

which yields (3.4) for all $\lambda \ge 2C'(\rho + d(x,S)^{\beta_1})$ as long as $C_0 \ge 4C$. But (3.4) also holds for all $\lambda \le 2C'(\rho + d(x,S)^{\beta_1})$ as long as $C_0 \ge 8(C')^2(\ln 2)^{-1}$ because $d(x,S) > \rho \ge 1 > \beta_1$ (and so the right-hand side of (3.4) is ≥ 1). This finishes the proof.

3.3. **Extension to half-spaces.** We now extend Proposition 3.1 to half-spaces, denoting

$$\mathcal{H}_e^- := \{x \in \mathbb{R}^d \,|\, x \cdot e \le 0\}$$

for $e \in \mathbb{S}^{d-1}$. This means that we need to enlarge $\mathcal{U}_f$ to include solutions initially approximating characteristic functions of half-spaces, with **(H2')** extending as well.

Lemma 3.7. ***(H2')*** *implies* ***(H2')*** *with $\mathcal{U}'_f$ in place of $\mathcal{U}_f$, with unchanged values of all the $\sup_{u\in\mathcal{U}_f}$ and $\inf_{u\in\mathcal{U}_f}$ in (1.10), and with $\mathcal{U}'_f$ defined as $\mathcal{U}_f$ but including the initial functions $u_{0,k}$ from $\mathcal{U}_f$ as well as all locally uniform limits of their translations. (These are then functions $u_{0,S}$ satisfying Lemma 2.2 for all balls $S = B_k(y)$ and all half-spaces $S = \mathcal{H}_e^- + le$, due to well-known elliptic regularity estimates.)*

In particular, Lemmas 2.9 and 2.10 hold with $\mathcal{U}_{f_1}$ replaced by $\mathcal{U}'_{f_1}$.

Proof. Stationarity of f again shows that adding translations of the $u_{0,k}$ to $\mathcal{U}_f$ does not change any of the sup or inf. Well known parabolic regularity estimates now show that the sup and inf also remain unchanged when we add locally uniform limits of these translations to $\mathcal{U}_f$. The proofs of Lemmas 2.9 and 2.10 then extend to $\mathcal{U}'_{f_1}$ in place of $\mathcal{U}_{f_1}$ without change.

We note that the elliptic and parabolic regularity (Krylov-Safonov and Schauder) estimates used here can be found in [11, Theorem 4.6], [15, Theorem 4.1], and [14, Theorem 8.6.1]. □

Proposition 3.8. *Proposition 3.1 holds for S being either any ball $B_k(y)$ with $(k, y) \in \mathbb{N}\times\mathbb{R}^d$ or any half-space $\mathcal{H}_e^- + le$ with $(e, l) \in \mathbb{S}^{d-1} \times \mathbb{R}$ (with the functions $u(\cdot,\cdot,\cdot; S)$ from $\mathcal{U}'_f$).*

Proof. The claim for balls is immediate from stationarity. For the same reason, in the half-space case we only need to consider $l = 0$. Hence let $S = \mathcal{H}_e^-$ for some $e \in \mathbb{S}^{d-1}$.

For each $k \in \mathbb{N}$, let $S_k := B_k(-ke)$. Then $\lim_{k\to\infty} d(x, S_k) = d(x, S)$ for each $x \in \mathbb{R}^d$, so by Proposition 3.1 (with C_0 independent of S), it suffices to show

$$\limsup_{k\to\infty} \sup_{\omega\in\Omega} |T(x, \omega; S) - T(x, \omega; S_k)| \le C$$

for some $C > 0$ (depending only on (2.10), as always) and any $x \in \mathbb{R}^d$.

Let $u_\omega := u(\cdot,\cdot,\omega; S)$ and $u_{k,\omega} := u(\cdot,\cdot,\omega; S_k)$ for each $(k, \omega) \in \mathbb{N} \times \Omega$, and $C := \kappa_0 + \frac{2R_0}{c_0}$. We then have $u_\omega(C,\cdot) \ge u_{k,\omega}(0,\cdot)$ by Lemma 2.1 and (2.3), so $T(\cdot,\omega; S) \le T(\cdot,\omega; S_k) + C$.

Similarly, we have $u_{k,\omega}(C + 1,\cdot) \ge u_\omega(0,\cdot)$ in $B_{k^{1/2}}(0)$. Then from the last claim in Lemma 2.9 with $(\eta, R) = (0, k^{1/2} - |x|)$ we obtain $T(x,\omega; S_k) \le T(x,\omega; S) + C + 1 + 2\kappa_* + \kappa_0$ whenever $k^{1/2} \ge |x| + D_2(1 + \kappa_0 + \frac{2}{c_0}(d(x, S) + 1))$, because then $d(x, S_k) \le d(x, S) + 1$ due to $|x| \le k^{1/2}$, so $k^{1/2} - |x| \ge D_2(1 + T(x,\omega; S_n))$ by Lemma 2.1. □

4. Fluctuations for General Reactions

First, we show that **(H4')** implies **(H2')**.

Lemma 4.1. ***(H4')*** *implies that f also satisfies* ***(H2')*** *(possibly after removing from Ω a measure-zero set that is invariant with respect to the group $\{\Upsilon_y\}_{y\in\mathbb{R}^d}$, which we then do).*

Proof. For each $k \in \mathbb{N}$, let $u_{0,k,n}$ be the initial datum for $B_k(0)$ that enters in the definition of $\mathcal{U}_{f_n}$ (these might in principle be different for distinct n, as we do not assume them to be those from the proof of Lemma 2.2). By (1.9), (1.8), and elliptic regularity estimates, there is a subsequence $\{n_j\}_{j\in\mathbb{N}}$ such that u_{0,k,n_j} converge uniformly to some $u_{0,k}$ satisfying (1.9) and (1.8) as $j \to \infty$. Since also **(H2')** holds uniformly for f_n, and the Borel-Cantelli Lemma shows that some subsequence of f_{n_j} converges to f locally uniformly on $\mathbb{R}^d$ for almost every $\omega \in \Omega$ (and if this holds for some ω, then it obviously also holds for $\Upsilon_y\omega$ with any $y \in \mathbb{R}^d$), it follows that **(H2')** also holds for f, with the above $u_{0,k}$ for each $k \in \mathbb{N}$ and after removal of a $\{\Upsilon_y\}$-invariant measure-zero set from Ω. □

We now extend Proposition 3.8 to reactions f satisfying **(H3)** and **(H4')**. Recall (3.2) and that constants with C in them only depend on (2.10) unless explicitly stated otherwise.

Proposition 4.2. *Let f satisfy **(H3)** and **(H4')**, and with β_1 from* (3.3) *let*

$$\beta_3 := \max\left\{\beta_1, \frac{m_3}{m_3+2m_4}, \frac{2d+2}{2d+2+m_4'}\right\} \qquad (\in (0,1)) \tag{4.1}$$

and

$$\nu(a) := \sup_{n\geq \max\{\lceil a\rceil, n_4\}} \mathbb{P}\left(\sup_{|z|<n^{1+m_4'}} \sup_{u\in[0,1]} |f_n(z,u,\cdot) - f(z,u,\cdot)| > \alpha_4 n^{-m_4}\right) \tag{4.2}$$

*for all $a \geq 0$ (then $\nu(a) \leq \max\{\lceil a\rceil, n_4\}^{-(2d+1+m_4')}$ by **(H4')**). There is $C_0' \geq 1$ such that for any S (and u) from Proposition 3.8, $x \in \mathbb{R}^d$, and $\lambda > 0$ we have*

$$\mathbb{P}\left(|T(x,\cdot\,;S) - \mathbb{E}[T(x,\cdot\,;S)]| \geq \lambda\right) \leq 2\exp\left(\frac{-\lambda^2}{C_0'(1+d(x,S)^{1+\beta_3})}\right) + \nu\left(d(x,S)^{\beta_3}\right).$$

Remark. Having this and Lemma 4.1, we will not need to use **(H3)** and **(H4')** again.

Proof. Let us assume without loss that $\alpha_3 \leq 1$ in **(H3)**. Lemma 4.1 shows that f also satisfies **(H2')**, and then Lemma 3.7 shows that **(H2')** holds with $\mathcal{U}_f' \cup \bigcup_{n\geq n_4} \mathcal{U}_{f_n}'$ in place of $\mathcal{U}_f$.

For each $\omega \in \Omega$ and S either a ball or a half-space, let $u(\cdot,\cdot,\omega,S)$ be the solution from $\mathcal{U}_f'$ corresponding to (ω, S), and for each $n \geq n_4$, let $u_n(\cdot,\cdot,\omega,S)$ be the analogous solution from $\mathcal{U}_{f_n}'$. (Note that we do not require the initial data for u and u_n to be the same, although they may be.) Also let

$$T_n(x,\omega;S) := \inf\{t \geq 0 \,|\, u_n(t,x,\omega;S) \geq 1-\theta^*\}.$$

Then **(H4')** and Proposition 3.8 show for all $n \geq n_4$,

$$\mathbb{P}\left(|T_n(x,\cdot\,;S) - \mathbb{E}[T_n(x,\cdot\,;S)]| \geq \lambda\right) \leq 2\exp\left(-\frac{\lambda^2}{C_0(1+d(x,S))(n+d(x,S)^{\beta_1})}\right). \tag{4.3}$$

Let $C_1 \geq n_4$ be such that $(\frac{\alpha_4}{\alpha_3})^{1/m_3} C_1^{-m_4/m_3} \leq \frac{1}{2}\min\{\theta^*, M_*^{-1}\}$.

Stationarity of f, f_n shows that the definition of $\nu(a)$ above is unchanged when in it we replace $|z|$ by $|z-x|$, for any $x \in \mathbb{R}^d$. So for each x and $n \geq n_4$, there is $\Omega_{n,x} \subseteq \Omega$ such that $\mathbb{P}(\Omega_{n,x}) \geq 1-\nu(n)$ and

$$|f(z,u,\omega) - f_n(z,u,\omega)| \leq \alpha_4 n^{-m_4} \qquad \text{for all } (z,u,\omega) \in B_{n^{1+m_4'}}(x) \times [0,1] \times \Omega_{n,x}.$$

Since f satisfies **(H3)**, and $u(\cdot,\cdot,\omega;S) \in \mathcal{U}_f$ and $u_n(\cdot,\cdot,\omega;S) \in \mathcal{U}_{f_n}$ for each $\omega \in \Omega_{n,x}$, Lemma 2.10 (see Lemma 3.7) applied twice with $\eta = (\frac{\alpha_4}{\alpha_3} n^{-m_4})^{1/m_3}$, $y = x$, $R = n^{1+m_4'}$, $t_0 = \kappa_0 + 2c_0^{-1}R_0$, and (f_1, f_2, u_1, u_2) being

$$(f, f_n, u(\cdot,\cdot,\omega;S), u_n(\cdot,\cdot,\omega;S)) \qquad \text{and} \qquad (f_n, f, u_n(\cdot,\cdot,\omega;S), u(\cdot,\cdot,\omega;S)),$$

respectively, yields for some $C > 0$ and all $n \geq \max\{C_1, [D_2(1 + \kappa_0 + \frac{2}{c_0} d(x,S))]^{1/(1+m_4')}\}$,

$$|T_n(x,\cdot;S) - T(x,\cdot;S)| \leq C n^{-\frac{m_4}{m_3}} (1 + d(x,S)) + C. \tag{4.4}$$

Here we also used Lemma 2.1 to show that $T_n(x,\cdot;S)$ and $T(x,\cdot;S)$ are at most $\kappa_0 + \frac{2}{c_0} d(x,S)$.

Let now n be the smallest integer such that

$$n \geq \max\left\{C_1,\, d(x,S)^{\beta_3}, [D_2(1 + \kappa_0 + 2c_0^{-1} d(x,S))]^{1/(1+m_4')}\right\}. \tag{4.5}$$

Since $\beta_3 \geq \frac{1}{1+m_4'}$, there is $C_2 > 0$ such that with C from (4.4) we have (uniformly in x, S)

$$\begin{aligned}
&(1 + d(x,S))(n + d(x,S)^{\beta_1}) \leq C_2(1 + d(x,S)^{1+\beta_3}),\\
&4Cn^{-\frac{m_4}{m_3}}(1 + d(x,S)) + 4C \leq C_2(1 + d(x,S)^{1-\beta_3\frac{m_4}{m_3}}) \leq C_2(1 + d(x,S)^{\frac{1+\beta_3}{2}}).
\end{aligned}$$

If now $\lambda \geq C_2(1 + d(x,S)^{(1+\beta_3)/2})$, then (4.3) and (4.4) imply

$$\begin{aligned}
\mathbb{P}\left(|T(x,\cdot;S) - \mathbb{E}[T(x,\cdot;S)]| \geq \lambda\right) &\leq \mathbb{P}\left(|T_n(x,\cdot;S) - \mathbb{E}[T_n(x,\cdot;S)]| \geq \frac{\lambda}{2}\right) + \mathbb{P}(\Omega \setminus \Omega_{n,x})\\
&\leq 2\exp\left(\frac{-\lambda^2}{4C_0C_2(1 + d(x,S)^{1+\beta_3})}\right) + \nu\left(d(x,S)^{\beta_3}\right).
\end{aligned}$$

Hence the result holds with $C_0' := \max\{4C_0C_2, 2C_2^2(\ln 2)^{-1}\}$, because then it obviously also holds for any $\lambda \leq C_2(1 + d(x,S)^{(1+\beta_3)/2})$. □

5. Convergence of the Mean Propagation Speeds

We now consider (3.1) with $S = \mathcal{H}_e^- + le$ for any $e \in \mathbb{S}^{d-1}$ and $l \in \mathbb{R}$, that is,

$$\begin{aligned}
&u_t = \Delta u + f(x,u,\omega) && \text{on } (0,\infty) \times \mathbb{R}^d,\\
&u(0,\cdot,\omega;\mathcal{H}_e^- + le) = u_{0,\mathcal{H}_e^- + le} && \text{on } \mathbb{R}^d.
\end{aligned} \tag{5.1}$$

Here $u_{0,\mathcal{H}_e^- + le}$ is the initial data used in the definition of $\mathcal{U}_f'$ in Lemma 3.7 (note that f satisfies **(H2')** in both cases under consideration, due to Lemma 4.1). Hence, $u(\cdot,\cdot,\omega;\mathcal{H}_e^- + le) \in \mathcal{U}_f'$ for all $(e,l,\omega) \in \mathbb{S}^{d-1} \times \mathbb{R} \times \Omega$ and (2.6) holds for it.

We will now prove that $\frac{1}{l}\mathbb{E}\left[T(le,\cdot;\mathcal{H}_e^-)\right]$ converges as $l \to \infty$, with $T(le,\cdot;\mathcal{H}_e^-)$ from (3.2) (stationarity shows that the expectation is the same if le is replaced by any y_l with $y_l \cdot e = l$). Note that the reciprocal of this limit can then be considered the asymptotic mean speed of propagation of the solutions u in direction e (this mean is technically harmonic).

We also note that all constants in this section will be uniform in e, and recall that all constants with C in them depend on (2.10), with any other dependence explicitly indicated, and may vary from line to line.

Proposition 5.1. *For each $e \in \mathbb{S}^{d-1}$ there is $\bar{T}(e) \in [\frac{1}{c_1}, \frac{1}{c_0}]$ (depending also on f) and for each $\delta > 0$ there is $C_\delta \geq 1$ such that the following hold. If f satisfies **(H2')** and has range of dependence at most $\rho \in [1, \infty)$, then with β_1 from (3.3) we have for all $l \geq 1$,*

$$\left| \frac{\mathbb{E}[T(le, \cdot\,; \mathcal{H}_e^-)]}{l} - \bar{T}(e) \right| \leq C_\delta \sqrt{\rho}\, l^{-1+\frac{1}{2}(1+\beta_1)+\delta}.$$

*If instead f satisfies **(H3)** and **(H4')**, then with $\beta_3 < 1$ from (4.1) we have for all $l \geq 1$,*

$$\left| \frac{\mathbb{E}[T(le, \cdot\,; \mathcal{H}_e^-)]}{l} - \bar{T}(e) \right| \leq C_\delta\, l^{-1+\frac{1}{2}(1+\beta_3)+\delta}.$$

We will fix e in the rest of this section and, for the sake of convenience, we will sometimes (but not always) drop $\mathcal{H}_e^-$ from the notation in (3.1) and (3.2) when $S = \mathcal{H}_e^-$. Hence we let

$$T(x, \omega) := T(x, \omega; \mathcal{H}_e^-) \qquad \text{and} \qquad u(t, x, \omega) := u(t, x, \omega; \mathcal{H}_e^-). \tag{5.2}$$

We will also prove both claims in Proposition 5.1 at the same time, with the notation

$$\beta := \frac{1+\beta_1}{2} \qquad \text{and} \qquad \bar{C}_\rho := C\sqrt{\rho} \qquad \text{and} \qquad \phi(l) := 0 \tag{5.3}$$

if f satisfies **(H2')** and has range of dependence at most $\rho \in [1, \infty)$, and

$$\beta := \frac{1+\beta_3}{2} \qquad \text{and} \qquad \bar{C}_\rho := C \qquad \text{and} \qquad \phi(l) := \nu(l^{\beta_3}) \tag{5.4}$$

if f satisfies **(H3)** and **(H4')**, with ν from (4.2). Here again, $C \geq 1$ will be a constant depending only on (2.10), which may vary from line to line. In particular, Propositions 3.8 and 4.2 show that in both cases we have for all $e \in \mathbb{S}^{d-1}$, $\lambda > 0$, $x \in \mathbb{R}^d$ with $x \cdot e \geq 1$ (and some $C > 0$ defining $\bar{C}_\rho$),

$$\mathbb{P}\left(\left| T(x, \cdot\,; \mathcal{H}_e^-) - \mathbb{E}[T(x, \cdot\,; \mathcal{H}_e^-)] \right| \geq \lambda \right) \leq 2 \exp\left(-\bar{C}_\rho^{-2}\, \lambda^2\, (x \cdot e)^{-2\beta} \right) + \phi(x \cdot e). \tag{5.5}$$

We start with the following simple result.

Lemma 5.2. *If $x \cdot e \leq l$ for some $(e, l, x) \in \mathbb{S}^{d-1} \times \mathbb{R} \times \mathbb{R}^d$, then*

$$\mathbb{E}\left[T(x, \cdot\,; \mathcal{H}_e^-) \right] \leq \mathbb{E}\left[T(le, \cdot\,; \mathcal{H}_e^-) \right] + \kappa_0 + 2c_0^{-1} R_0.$$

Proof. Since $u(0, \cdot, \omega; \mathcal{H}_e^-) \leq u(\tau_0, \cdot, \omega; \mathcal{H}_e^- + le - x)$ for $\tau_0 := \kappa_0 + \frac{2R_0}{c_0}$ by Lemma 2.1 and $x \cdot e \leq l$, we have

$$T(le, \cdot\,; \mathcal{H}_e^-) \geq T(le, \cdot\,; \mathcal{H}_e^- + le - x) - \tau_0.$$

Therefore,

$$\mathbb{E}[T(le, \cdot\,; \mathcal{H}_e^-)] \geq \mathbb{E}[T(le, \cdot\,; \mathcal{H}_e^- + le - x)] - \tau_0 = \mathbb{E}[T(x, \cdot\,; \mathcal{H}_e^-)] - \tau_0$$

because f is stationary (if we assume **(H4')**, this follows from Lemma 4.1) □

The next result is an immediate consequence of Lemma 2.1 and Corollary 2.6.

Lemma 5.3. *There is $t_0 \geq 1$, depending only on (2.10), such that for all $t \geq t_0$ and $\omega \in \Omega$,*

$$u(t,x,\omega;\mathcal{H}_e^-) \geq 1-\theta^* \qquad \text{when } x\cdot e \leq \frac{c_0 t}{2},$$

$$u(t,x,\omega;\mathcal{H}_e^-) < 1-\theta^* \qquad \text{when } x\cdot e \geq 2c_1 t.$$

In particular, for all $x \in \mathbb{R}^d$ with $x\cdot e \geq l_0 := 2c_1 t_0$ and $\omega \in \Omega$ we have

$$T(x,\omega;\mathcal{H}_e^-) \in \left[\frac{x\cdot e}{2c_1}, \frac{2\,x\cdot e}{c_0}\right]. \tag{5.6}$$

In order to prove Proposition 5.1, it will be necessary to simultaneously prove it for $T(x,\cdot\,;\mathcal{H}_e^-)$ with other points x satisfying $x\cdot e = l$. The Infinite Monkey "Theorem" shows that in dimensions $d \geq 2$, this cannot involve all the points in the unbounded set $\{x\cdot e = l\}$, but we will be able to include all such points with $|x| \leq O(l)$ (i.e., within a ball centered at le and with linearly-in-time growing radius due to (5.6)). This will be sufficient thanks to the speed of propagation of perturbations of solutions being finite (see Lemma 2.5).

This and Lemma 5.3 motivate the definitions of the cylinders

$$\mathcal{C}_e^-(R,l) := \left\{ x\in\mathbb{R}^d \,\middle|\, x\cdot e \in \left[\frac{c_0}{4c_1}l, l\right] \text{ and } |x-(x\cdot e)e| \leq R \right\},$$
$$\mathcal{C}_e^+(R,l) := \left\{ x\in\mathbb{R}^d \,\middle|\, x\cdot e \in \left[l, \frac{4c_1}{c_0}l\right] \text{ and } |x-(x\cdot e)e| \leq R \right\}$$

and of the corresponding times

$$\begin{aligned} T_e^-(R,l,\omega) :&= \inf\left\{t\geq 0 \,|\, u(t,\cdot,\omega;\mathcal{H}_e^-) \geq 1-\theta^* \text{ on } \mathcal{C}_e^-(R,l)\right\} \\ &= \sup\left\{T(x,\omega;\mathcal{H}_e^-) \,|\, x\in\mathcal{C}_e^-(R,l)\right\}, \\ T_e^+(R,l,\omega) :&= \sup\left\{t\geq 0 \,|\, u(t,\cdot,\omega;\mathcal{H}_e^-) < 1-\theta^* \text{ on } \mathcal{C}_e^+(R,l)\right\} \\ &= \inf\left\{T(x,\omega;\mathcal{H}_e^-) \,|\, x\in\mathcal{C}_e^+(R,l)\right\}. \end{aligned}$$

Obviously $T_e^+(R,l,\cdot) \leq T_e^-(R,l,\cdot)$ because $u_t(t,\cdot,\cdot;\mathcal{H}_e^-) > 0$ for all $t>0$.

Our next result shows that means of these times are sufficiently close to $\mathbb{E}\left[T(le,\cdot\,;\mathcal{H}_e^-)\right]$.

Lemma 5.4. *There is $C>0$, and for each $\delta>0$ there is $C_\delta \geq 1$, such that with $l_0 \geq 1$ from Lemma 5.3 we have for all $l \geq l_0$ and $R \geq 0$ (with $\bar{C}_\rho$ from (5.3) resp. (5.4)),*

$$\begin{aligned} &\mathbb{E}\left[T_e^-(R,l,\cdot)\right] \leq \mathbb{E}\left[T(le,\cdot\,;\mathcal{H}_e^-)\right] + \bar{C}_\rho(C_\delta + \max\{R,l\}^\delta l^\beta) + C\max\{R,l\}^{d-1} l^2\,\phi\left(\frac{c_0}{4c_1}l\right), \\ &\mathbb{E}\left[T_e^+(R,l,\cdot)\right] \geq \mathbb{E}\left[T(le,\cdot\,;\mathcal{H}_e^-)\right] - \bar{C}_\rho(C_\delta + \max\{R,l\}^\delta l^\beta) - C\max\{R,l\}^{d-1} l^2\,\phi(l). \end{aligned} \tag{5.7}$$

Proof. The definition of $T_e^\pm(R,l,\omega)$ shows that it suffices to consider the case $R \geq l \geq l_0$.

For any $\lambda > 0$ we have

$$\mathbb{P}\left[T_e^-(R,l,\omega) - \mathbb{E}\left[T(le,\cdot)\right] \geq \lambda\right] = \mathbb{P}\left[\sup\left\{T(x,\omega) - \mathbb{E}\left[T(le,\cdot)\right] \,|\, x\in\mathcal{C}_e^-(R,l)\right\} \geq \lambda\right].$$

Since Lemma 2.1 yields

$$|T(x,\omega) - T(y,\omega)| \le \kappa_0 + \frac{2|x-y|}{c_0},$$

for any $\lambda \ge 4\kappa_0 + \frac{4(\sqrt{d}+R_0)}{c_0} =: C_1$ we obtain

$$\begin{aligned}\mathbb{P}\left[T_e^-(R,l,\omega) - \mathbb{E}\left[T(le,\cdot)\right] \ge \lambda\right] &\le \mathbb{P}\left[\sup\left\{T(x,\omega) - \mathbb{E}\left[T(le,\cdot)\right] \mid x \in \mathcal{C}_e^-(R,l)\cap\mathbb{Z}^d\right\} \ge \frac{\lambda}{2}\right] \\ &\le \sum_{x\in\mathcal{C}_e^-(R,l)\cap\mathbb{Z}^d} \mathbb{P}\left[T(x,\omega) - \mathbb{E}\left[T(le,\cdot)\right] \ge \frac{\lambda}{2} + \kappa_0 + \frac{2R_0}{c_0}\right].\end{aligned}$$

Since $x \in \mathcal{C}_e^-(R,l)\cap\mathbb{Z}^d$ implies $x\cdot e \le l$ and f is stationary, Lemma 5.2 now yields

$$\mathbb{P}\left[T_e^-(R,l,\omega) - \mathbb{E}\left[T(le,\cdot)\right] \ge \lambda\right] \le \sum_{x\in\mathcal{C}_e^-(R,l)\cap\mathbb{Z}^d} \mathbb{P}\left[T(x,\omega) - \mathbb{E}\left[T(x,\cdot)\right] \ge \frac{\lambda}{2}\right].$$

The number of terms in this sum is bounded by $C_2R^{d-1}l$ for some $C_2 > 0$. Since each $x \in \mathcal{C}_e^-(R,l)$ satisfies $d(x,\mathcal{H}_e^-) \ge \frac{c_0}{4c_1}l$, by (5.5) we have for each $\lambda \ge C_1$,

$$\mathbb{P}\left[T_e^-(R,l,\omega) - \mathbb{E}\left[T(le,\cdot)\right] \ge \lambda\right] \le C_2R^{d-1}l\exp\left(-\bar{C}_\rho^{-2}\,\lambda^2\,l^{-2\beta}\right) + C_2R^{d-1}l\,\phi\left(\frac{c_0}{4c_1}l\right).$$

Moreover, this probability is clearly 0 when $\lambda > \kappa_0 + \frac{2}{c_0}l$. Thus, for each $\delta > 0$ we obtain

$$\begin{aligned}\mathbb{E}\left[T_e^-(R,l,\cdot) - \mathbb{E}\left[T(le,\cdot)\right]\right] &= \int_0^\infty \mathbb{P}\left[T_e^-(R,l,\omega) - \mathbb{E}\left[T(le,\cdot)\right] \ge \lambda\right] d\lambda \\ &\le \max\{C_1, \bar{C}_\rho R^\delta l^\beta\} + C_2R^{d-1}l\int_{\bar{C}_\rho R^\delta l^\beta}^\infty \exp\left(-\bar{C}_\rho^{-2}\lambda^2 l^{-2\beta}\right) d\lambda + C_3R^{d-1}l^2\,\phi\left(\frac{c_0}{4c_1}l\right) \\ &\le \max\{C_1, \bar{C}_\rho R^\delta l^\beta\} + C_2\bar{C}_\rho R^{d-1}l^{1+\beta}\int_{R^\delta}^\infty e^{-s^2}ds + C_3R^{d-1}l^2\,\phi\left(\frac{c_0}{4c_1}l\right),\end{aligned}$$

with some $C_3 > 0$. Since

$$\int_r^\infty e^{-s^2}ds = e^{-r^2}\int_0^\infty e^{-s^2-2sr}ds \le Ce^{-r^2},$$

it follows from $R \ge l$ that there is $C_\delta \ge 1$ such that

$$R^{d-1}l^{1+\beta}\int_{R^\delta}^\infty e^{-s^2}ds \le R^{d+\beta}e^{-R^{2\delta}} \le C_\delta.$$

This proves the first inequality in (5.7). The proof of the second is analogous. □

We can now show that $\mathbb{E}\left[T(le,\cdot;\mathcal{H}_e^-)\right]$ is close to being linear in l.

Proposition 5.5. *There is $C > 0$, and for each $\delta > 0$ there is $C_\delta \ge 1$, such that for all $l, m \ge 0$ we have (with $\bar{C}_\rho$ from (5.3) resp. (5.4))*

$$\left|\mathbb{E}\left[T((l+m)e,\cdot;\mathcal{H}_e^-)\right] - \mathbb{E}\left[T(le,\cdot;\mathcal{H}_e^-)\right] - \mathbb{E}\left[T(me,\cdot;\mathcal{H}_e^-)\right]\right| \le \bar{C}_\rho(C_\delta + (l+m)^{\beta+\delta}).$$

Remark. We will in fact only need the weaker upper bound $\bar{C}_\rho C_\delta (l+m)^{\beta+\delta}$.

Proof. Without loss of generality, we can assume that $l \geq m \geq 0$, and we also let l_0 be from Lemma 5.3. By Lemma 2.1, for all $m \leq \frac{4c_1}{c_0} l_0$ we have $T(me, \cdot; \mathcal{H}_e^-) \leq \kappa_0 + 8c_1 l_0$ and

$$u(\tau_0, x, \cdot\,; \mathcal{H}_e^- - me) \geq u(0, x, \cdot\,; \mathcal{H}_e^-),$$

with $\tau_0 := \kappa_0 + 8c_1 l_0 + 2c_0^{-1} R_0$. This and stationarity of f yield

$$\mathbb{E}\left[T((l+m)e, \cdot\,; \mathcal{H}_e^-)\right] = \mathbb{E}\left[T(le, \cdot\,; \mathcal{H}_e^- - me)\right] \leq \tau_0 + \mathbb{E}\left[T(le, \cdot\,; \mathcal{H}_e^-)\right].$$

All this and Lemma 5.2 with $(l+m, le)$ in place of (l, x) yield the claim for all $m \leq \frac{4c_1}{c_0} l_0$ and $l \geq m$, with any $C_\delta \geq 2\tau_0$.

Now assume that $l \geq m \geq \frac{4c_1}{c_0} l_0$, and let us first prove the direction

$$\mathbb{E}\left[T((l+m)e, \cdot\,)\right] - \mathbb{E}\left[T(le, \cdot\,)\right] - \mathbb{E}\left[T(me, \cdot\,)\right] \leq \bar{C}_\rho (C_\delta + l^{\beta+\delta}).$$

Pick $R_m := D_2(1 + \frac{2m}{c_0})$, with D_2 from Lemma 2.9, and denote $T_1(\omega) := T_e^-(R_m, l, \omega)$. Then Lemma 5.3 shows that for each $\omega \in \Omega$ we have $T_1(\omega) \in [\frac{l}{2c_1}, \frac{2l}{c_0}]$ and hence also

$$u_1(T_1(\omega), \cdot) \geq 1 - \theta^* \qquad \text{on } \mathcal{H}_e^- + (4c_1)^{-1} c_0 le$$

holds with $u_1 := u(\cdot, \cdot, \omega; \mathcal{H}_e^-)$. Therefore,

$$u_1(T_1(\omega), \cdot) \geq 1 - \theta^* \qquad \text{on } \mathcal{C}_e^-(R_m, l) \cup \left(\mathcal{H}_e^- + (4c_1)^{-1} c_0 le\right).$$

Hence Lemma 2.1 shows that with $u_2 := u(\cdot, \cdot, \omega; \mathcal{H}_e^- + le)$ and $\tau_0 := \kappa_0 + \frac{2R_0}{c_0}$, we have

$$u_1(T_1(\omega) + \tau_0, \cdot) \geq (1 - \theta^*) \chi_{\mathcal{H}_e^- + (l+R_0)e} \geq u_2(0, \cdot) \qquad \text{on } \left\{x \in \mathbb{R}^d \,\middle|\, |x - (x \cdot e)e| \leq R_m\right\}$$

Since the set above contains $B_{R_m}((l+m)e)$ and we also have $T((l+m)e, \omega; \mathcal{H}_e^- + le) \leq \frac{2m}{c_0}$ by Lemma 5.3, we can apply Lemma 2.9 (see Lemma 3.7) with $f_1 = f_2 = f$ and

$$(u_1, u_2, \eta, y, t_0, R) = (u_1, u_2, 0, (l+m)e, T_1(\omega) + \tau_0, R_m)$$

to obtain

$$T((l+m)e, \omega; \mathcal{H}_e^-) \leq T((l+m)e, \omega; \mathcal{H}_e^- + le) + T_1(\omega) + \tau_0 + 2\kappa_* + \kappa_0. \tag{5.8}$$

Taking expectations on both sides of this inequality and using stationarity of f yields

$$\begin{aligned}\mathbb{E}\left[T((l+m)e, \cdot\,; \mathcal{H}_e^-)\right] &\leq \mathbb{E}\left[T((l+m)e, \cdot\,; \mathcal{H}_e^- + le)\right] + \mathbb{E}\left[T_1(\cdot)\right] + \tau_0 + 2\kappa_* + \kappa_0 \\ &= \mathbb{E}\left[T(me, \cdot; \mathcal{H}_e^-)\right] + \mathbb{E}\left[T_1(\cdot)\right] + \tau_0 + 2\kappa_* + \kappa_0.\end{aligned}$$

Since $R_m \leq Cl$ and $l^{d+1}\phi(\frac{c_0}{4c_1} l) \leq C$ for some $C > 0$ due to **(H4')** and $\beta_3 \geq \frac{d+1}{2d+1+m_4'}$, it now follows from Lemma 5.4 that for any $\delta > 0$ we indeed have

$$\mathbb{E}\left[T((l+m)e, \cdot\,; \mathcal{H}_e^-)\right] \leq \mathbb{E}\left[T(me, \cdot\,; \mathcal{H}_e^-)\right] + \mathbb{E}\left[T(le, \cdot\,; \mathcal{H}_e^-)\right] + \bar{C}_\rho (C_\delta + l^{\beta+\delta}),$$

with some C, C_δ (and $\bar{C}_\rho$ from (5.3) resp. (5.4)).

Let us now turn to the other direction (again assuming $l \geq m \geq \frac{4c_1}{c_0} l_0$)

$$\mathbb{E}\left[T(le, \cdot\,; \mathcal{H}_e^-)\right] + \mathbb{E}\left[T(me, \cdot\,; \mathcal{H}_e^-) - \mathbb{E}\left[T((l+m)e, \cdot\,; \mathcal{H}_e^-)\right]\right] \leq \bar{C}_\rho (C_\delta + l^{\beta+\delta}),$$

the proof of which is a little more involved. With β_1 from (3.3), let

$$\eta := \min\left\{\frac{\theta^*}{2},\, \frac{1}{2M_*},\, l^{\beta_1-1}\right\} > 0, \tag{5.9}$$

and then (see Subsection 2.1 for the other constants)

$$R_l := \frac{1}{\mu_*\eta^{m_2}}\left(1+\left(\frac{2l}{c_0}\right)^{\alpha_2}\right) \qquad \text{and} \qquad R_m := D_2\left(1+\frac{2m}{c_0}\right).$$

For any $\omega\in\Omega$, denote $T_1'(\omega) := T_e^+(R_l+R_m, l, \omega)$ and

$$u_1'(t,x) := u(t+T_1'(\omega), x, \omega; \mathcal{H}_e^-),$$
$$u_2'(t,x) := u(t, x, \omega; \mathcal{H}_e^- + le).$$

Lemma 5.3 yields $T_1'(\omega)\in\left[\frac{l}{2c_1}, \frac{2l}{c_0}\right]$ and hence also

$$u_1'(0,\cdot) \le 1-\theta^* \qquad \text{on } \mathcal{H}_e^+ + 4c_1c_0^{-1}le,$$

where $\mathcal{H}_e^+ := \mathbb{R}^d\setminus\mathcal{H}_e^-$. Therefore,

$$u_1'(0,\cdot) \le 1-\theta^* \qquad \text{on } \mathcal{C}_e^+(R_l+R_m, l)\cup(\mathcal{H}_e^+ + 4c_1c_0^{-1}le).$$

From (2.6) and $T_1'(\omega)\le\frac{2l}{c_0}$ we have $R_l \ge L_{u,\eta,1-\theta^*}(T_1'(\omega))$, and so

$$u_1'(0,\cdot) \le \eta \qquad \text{on } \left\{x\in\mathbb{R}^d \,\middle|\, x\cdot e \ge l+R_l \text{ and } |x-(x\cdot e)e|\le R_m\right\}. \tag{5.10}$$

As for u_2', Lemma 2.1 shows that

$$u_2'(\kappa_0 + 2R_l{c_0}^{-1}, \cdot) \ge 1-\theta^* \qquad \text{on } \mathcal{H}_e^- + (l+R_l)e,$$

and Lemma 2.8 then shows that for $\tau_0^l := \kappa_0 + 2R_l{c_0}^{-1} + D_1\eta^{1-m_1}$ we have

$$u_2'(\tau_0^l, \cdot) \ge 1-\eta \qquad \text{on } \mathcal{H}_e^- + (l+R_l)e. \tag{5.11}$$

Note also that there is $C>0$ such that

$$\tau_0^l \le C(l^{\alpha_2}\eta^{-m_2} + \eta^{1-m_1}) \le Cl^{\beta_1} \tag{5.12}$$

because (3.3) shows that

$$\max\{(1-\beta_1)(m_1-1), \alpha_2 + (1-\beta_1)m_2\} = \beta_1.$$

From (5.10) and (5.11) we now have that

$$u_2'(\tau_0^l,\cdot) \ge u_1'(0,\cdot) - \eta \qquad \text{on } \left\{x\in\mathbb{R}^d \,\middle|\, |x-(x\cdot e)e|\le R_m\right\}.$$

Since the set above contains $B_{R_m}((l+m)e)$, we can apply Lemma 2.9 (see Lemma 3.7) with $f_1=f_2=f$ and

$$(u_1, u_2, \eta, y, t_0, R) = (u_2', u_1', \eta, (l+m)e, \tau_0^l, R_m)$$

to obtain

$$T((l+m)e, \omega; \mathcal{H}_e^- + le) \le (1+M_*\eta)\big[T((l+m)e, \omega; \mathcal{H}_e^-) - T_1'(\omega)\big] + \tau_0^l + 2\kappa_* + \kappa_0, \tag{5.13}$$

provided we also have

$$T((l+m)e, \omega; \mathcal{H}_e^-) - T_1'(\omega) \le 2m{c_0}^{-1} \tag{5.14}$$

(notice that $T((l+m)e,\omega;\mathcal{H}_e^-) \geq T_1'(\omega)$ because $l+m \leq 2l \leq \frac{4c_1}{c_0}l$). But since Lemma 2.1 yields $T((l+m)e,\omega;\mathcal{H}_e^- + le) \leq \kappa_0 + \frac{2m}{c_0}$, (5.13) obviously holds even if (5.14) fails. Since $T((m+l)e,\omega;\mathcal{H}_e^-) \leq \frac{2(m+l)}{c_0} \leq \frac{4l}{c_0}$ by Lemma 5.3, we get from (5.9) and (5.12),

$$\begin{aligned} T((l+m)e,\omega;\mathcal{H}_e^- + le) &\leq T((l+m)e,\omega;\mathcal{H}_e^-) - T_1'(\omega) + 4lc_0{}^{-1}M_*\eta + Cl^{\beta_1} \\ &\leq T((l+m)e,\omega;\mathcal{H}_e^-) - T_1'(\omega) + Cl^{\beta_1}, \end{aligned}$$

for some $C>0$. Taking expectations, using stationarity of f, $\beta_1 \leq \beta$, as well as $R_l + R_m \leq Cl$ and $l^{d+1}\phi(l) \leq C$ for some $C>0$ (due to **(H4')**, (3.3), and $\beta_3 \geq \frac{d+1}{2d+1+m_4'}$), and applying Lemma 5.4 shows that for any $\delta > 0$ we indeed have

$$\begin{aligned} \mathbb{E}\left[T((l+m)e,\cdot\,;\mathcal{H}_e^-)\right] &\geq \mathbb{E}\left[T((l+m)e,\cdot\,;\mathcal{H}_e^- + le)\right] + \mathbb{E}\left[T_1'(\cdot)\right] - Cl^{\beta_1} \\ &\geq \mathbb{E}\left[T(me,\cdot;\mathcal{H}_e^-)\right] + \mathbb{E}\left[T(le,\cdot\,;\mathcal{H}_e^-)\right] - \bar{C}_\rho(C_\delta + l^{\beta+\delta}), \end{aligned}$$

with some C, C_δ (and $\bar{C}_\rho$ from (5.3) resp. (5.4)). □

Now we are ready to prove the main result of this section.

5.1. **Proof of Proposition 5.1.** Let $G(l) := \frac{1}{l}\mathbb{E}[T(le,\cdot)] \geq 0$ and $\gamma := \beta + \delta$. It follows from Lemma 2.1 and Corollary 2.6 that

$$\frac{1}{c_1} \leq \liminf_{l\to\infty} G(l) \leq \limsup_{l\to\infty} G(l) \leq \frac{1}{c_0}.$$

It therefore suffices to show that there is $C_\delta > 0$ such that with either $\bar{C}_{\delta,\rho} := C_\delta\sqrt{\rho}$ (when f satisfies **(H2')**) or $\bar{C}_{\delta,\rho} := C_\delta$ (when f satisfies **(H3)** and **(H4')**), we have for all $l \geq m \geq 2$,

$$|G(l) - G(m)| \leq \bar{C}_{\delta,\rho} m^{\gamma-1}. \tag{5.15}$$

Since $\beta < 1$, we only need to consider $\delta > 0$ such that $\gamma < 1$.

By Lemma 2.1, $G(l)$ is no more than

$$\frac{1}{l}\left(\frac{2l}{c_0} + \kappa_0\right) \leq \frac{2}{c_0} + \kappa_0 \tag{5.16}$$

for all $l \geq 1$, and we also have $|T(le,\cdot) - T(me,\cdot)| \leq \frac{4}{c_0} + \kappa_0$ when $|l-m| \leq 2$. Therefore, there exists $C_0 > 0$ such that for all l, m satisfying $m+2 \geq l \geq m \geq 2$ we have

$$|G(l) - G(m)| \leq \left|\frac{m-l}{m}G(l)\right| + \frac{1}{m}\mathbb{E}[|T(le,\cdot) - T(me,\cdot)|] \leq C_0 m^{-1}. \tag{5.17}$$

Using Proposition 5.5, we also see that there is $C_\delta \geq 1$ such that for any $l \geq m \geq 2$ (and with $\bar{C}_{\delta,\rho} \geq 1$ given above),

$$\begin{aligned} |G(l) - G(m)| &= \frac{1}{l}\left|\mathbb{E}[T(le,\cdot)] - \mathbb{E}[T(me,\cdot)] - \frac{l-m}{m}\mathbb{E}[T(me,\cdot)]\right| \\ &\leq \frac{1}{l}\left|\mathbb{E}[T(le,\cdot)] - \mathbb{E}[T(me,\cdot)] - \mathbb{E}[T((l-m)e,\cdot)]\right| \\ &\quad + \frac{l-m}{l}\left|\frac{1}{m}\mathbb{E}[T(me,\cdot)] - \frac{1}{l-m}\mathbb{E}[T((l-m)e,\cdot)]\right| \\ &\leq \bar{C}_{\delta,\rho} l^{\gamma-1} + \frac{l-m}{l}|G(m) - G(l-m)|. \end{aligned} \tag{5.18}$$

Now assume that for some $p \in \{2,3,4,...\}$, there is $N_p \geq 2C_0$ such that for all $m \in [2, 2^p]$ and $l \geq m$ we have

$$|G(l) - G(m)| \leq N_p m^{\gamma-1}. \tag{5.19}$$

This is in fact true for $p = 2$, because (5.16) shows that (5.19) holds for all $m \in [2,4]$ and $l \geq m$ with $N_2 = \max\{4(\frac{2}{c_0} + \kappa_0), 2C_0\}$. We will then extend (5.19) to all $m \in [2, 2^{p+1}]$ and $l \geq m$, with a relevant new constant $N_{p+1} \geq N_p$.

First, note that for any $m \in [2, 2^{p+1}]$ and $l \in [m, \frac{3}{2}m]$ we have

$$|G(l) - G(m)| \leq (\bar{C}_{\delta,\rho} + 3^{-\gamma} N_p) l^{\gamma-1}. \tag{5.20}$$

Indeed, this holds for $l \in [m, m+2]$ due to (5.17) and $\frac{C_0}{m} \leq \frac{N_p}{m+2} \leq 3^{-\gamma} N_p (m+2)^{\gamma-1}$. And if instead $l \in [m+2, \frac{3}{2}m]$, then $l - m \in [2, \min\{2^p, \frac{l}{3}\}]$, so it follows from (5.18) and the induction hypothesis (5.19) that

$$|G(l) - G(m)| \leq \bar{C}_{\delta,\rho} l^{\gamma-1} + \frac{l-m}{l} N_p (l-m)^{\gamma-1} \leq (\bar{C}_{\delta,\rho} + 3^{-\gamma} N_p) l^{\gamma-1}.$$

Let us now consider $m \in (2^p, 2^{p+1}]$ and $l \geq \frac{3}{2}m$. Let $l_k := 2^{-k} l$ for $k = 0, 1, ..., j$, with j chosen so that $l_j \in (\frac{3}{4}m, \frac{3}{2}m]$. Since $l_k = l_{k-1} - l_k$, from (5.18) we obtain

$$|G(l) - G(l_j)| \leq \sum_{k=1}^{j} |G(l_{k-1}) - G(l_k)| \leq \sum_{k=1}^{j} \bar{C}_{\delta,\rho}\, l_{k-1}^{\gamma-1} \leq \bar{C}_{\delta,\rho}\, l_j^{\gamma-1} \sum_{k=1}^{j} \left(2^{j-k+1}\right)^{\gamma-1} \leq \bar{C}'_{\delta,\rho} m^{\gamma-1},$$

where $\bar{C}'_{\delta,\rho} := \bar{C}_{\delta,\rho} (\frac{4}{3})^{1-\gamma} \frac{2^{\gamma-1}}{1-2^{\gamma-1}} < \infty$ (recall that $\gamma < 1$).

If $l_j \in [m, \frac{3}{2}m]$, then (5.20) yields

$$|G(l_j) - G(m)| \leq (\bar{C}_{\delta,\rho} + 3^{-\gamma} N_p) l_j^{\gamma-1} \leq (\bar{C}_{\delta,\rho} + 3^{-\gamma} N_p) m^{\gamma-1}.$$

If instead $l_j \in \left(\frac{3}{4}m, m\right)$, then $l_j \in [2, 2^{p+1}]$ and $m \in [l_j, \frac{4}{3}l_j]$. Hence (5.20) again yields

$$|G(m) - G(l_j)| \leq (\bar{C}_{\delta,\rho} + 3^{-\gamma} N_p) m^{\gamma-1}.$$

In either case we obtain

$$|G(l) - G(m)| \leq |G(l) - G(l_j)| + |G(l_j) - G(m)| \leq \left(\bar{C}_{\delta,\rho} + \bar{C}'_{\delta,\rho} + 3^{-\gamma} N_p\right) m^{\gamma-1}.$$

This, (5.19), and (5.20) now prove (5.19) with $p+1$ in place of p (so it holds for all $m \in [2, 2^{p+1}]$ and $l \geq m$) and with

$$N_{p+1} = \bar{C}_{\delta,\rho} + \bar{C}'_{\delta,\rho} + 3^{-\gamma} N_p \leq \max\left\{ N_p, \frac{3^\gamma}{3^\gamma - 1}(\bar{C}_{\delta,\rho} + \bar{C}'_{\delta,\rho}) \right\}.$$

Since (5.19) holds for $p = 2$ with $N_2 = \max\{4(\frac{2}{c_0} + \kappa_0), 2C_0\}$, it follows that it holds for any $p = 2, 3, \ldots$ with $N_p = \max\left\{4(\frac{2}{c_0} + \kappa_0), 2C_0, \frac{3^\gamma}{3^\gamma - 1}(\bar{C}_{\delta,\rho} + \bar{C}'_{\delta,\rho})\right\} =: \bar{C}''_{\delta,\rho}$. This proves (5.15).

6. Deterministic Front Speeds and Proof of Theorem 1.7

We are now ready to prove Theorem 1.7. This is because it was shown in [16] that such homogenization results for reaction-diffusion equations and related models follow from appropriate estimates on the dynamics of the solutions to (5.1) for all vectors $e \in \mathbb{S}^{d-1}$. We will be able to obtain these estimates using the main results from Sections 3–5.

Specifically, we will use Proposition 5.1, and either Proposition 3.8 (when we assume **(H2')**) or Proposition 4.2 (when we assume **(H3)** and **(H4')**) in the proof. We will handle both cases at once, using that either of the latter two propositions yields (5.5) above, with the notation from either (5.3) in the first case or (5.4) in the second.

For us, the key result from [16] will be Theorem 5.4, which applies when for almost all $\omega \in \Omega$, the reaction $f(\cdot, \cdot, \omega)$ has deterministic strong exclusive front speeds in all directions $e \in \mathbb{S}^{d-1}$. We will first define these, following Definitions 1.3, 1.6, and Remark 3 after Hypothesis H' in [16], and then prove their existence.

Definition 6.1. Let f satisfy **(H1)** and let $e \in \mathbb{S}^{d-1}$. If there is $c^*(e) \in \mathbb{R}$ and $\Omega_e \subseteq \Omega$ with $\mathbb{P}(\Omega_e) = 1$ such that for each $\omega \in \Omega_e$ and compact $K \subseteq \mathcal{H}_e^+ = \{x \in \mathbb{R}^d \,|\, x \cdot e > 0\}$,

$$\lim_{t\to\infty} \inf_{x \in (c^*(e)e - K)t} u(t, x, \omega; \mathcal{H}_e^-) = 1,$$
$$\lim_{t\to\infty} \sup_{x \in (c^*(e)e + K)t} u(t, x, \omega; \mathcal{H}_e^-) = 0$$

holds for the solution to (5.1) with $l = 0$ and some $u_{0,\mathcal{H}_e^-}$ satisfying (1.9) and (2.3) with $S = \mathcal{H}_e^-$, then we say that $c^*(e)$ is a *deterministic front speed* in direction e for (1.1).

This speed is *strong* if for each such ω and K, and each $\Lambda \geq 0$, we have

$$\begin{aligned} &\lim_{t\to\infty} \inf_{|y| \leq \Lambda t} \inf_{x \in (c^*(e)e - K)t} u(t, x, \Upsilon_y\omega; \mathcal{H}_e^-) = 1, \\ &\lim_{t\to\infty} \sup_{|y| \leq \Lambda t} \sup_{x \in (c^*(e)e + K)t} u(t, x, \Upsilon_y\omega; \mathcal{H}_e^-) = 0. \end{aligned} \tag{6.1}$$

And if, in addition, for each such ω and K there is $\lambda_{K,\omega,e} : (0,1] \to (0,1]$ satisfying $\lim_{a\to 0} \lambda_{K,\omega,e}(a) = 0$ such that for each $\Lambda > 0$ and $a \in (0,1]$ we have

$$\limsup_{t\to\infty} \sup_{|y| \leq \Lambda t} \sup_{x \in (c^*(e)e + K)t} w_{e,a}(t, x, \Upsilon_y\omega) \leq \lambda_{K,\omega,e}(a), \tag{6.2}$$

where $w_{e,a}(\cdot, \cdot, \omega)$ solves (1.1) with initial data

$$w_{e,a}(0, \cdot, \omega) = \chi_{\mathcal{H}_e^-} + a\chi_{\mathcal{H}_e^+},$$

then $c^*(e)$ is a *deterministic strong exclusive front speed* in direction e for (1.1).

Remarks. 1. Lemma 2.1 and the comparison principle show that all these definitions are independent of the choice of $u_{0,\mathcal{H}_e^-}$ satisfying (1.9) and (2.3). We could equivalently choose $u_{0,\mathcal{H}_e^-} := (1-\theta^*)\chi_{\mathcal{H}_e^-}$ here, but having solutions with $u_t \geq 0$ will be more convenient.

2. We will show that in Theorem 1.7, $\lambda_{K,\omega,e}(a) = a$ for all (K,ω,e) as above and all small enough $a > 0$ (depending on $M, \theta_1, m_1, \alpha_1, \mu_*, K$).

Let us first show that the reactions we consider here have deterministic strong front speeds, and then we will show that all these speeds are also exclusive.

Proposition 6.2. *Assume that f either satisfies **(H2')** and has a finite range of dependence, or satisfies **(H3)** and **(H4')**. For each $e \in \mathbb{S}^{d-1}$, let $\bar{T}(e)$ be from Proposition 5.1. Then $c^*(e) := \bar{T}(e)^{-1} \in [c_0, c_1]$ is the deterministic strong front speed in direction e for* (1.1).

Proof. Fix any $e \in \mathbb{S}^{d-1}$, $\Lambda \geq 0$, and compact $K' \subseteq \mathcal{H}_e^+$. Let $K \subseteq \mathcal{H}_e^+$ be compact and such that $K' \subseteq K^0$, let $d_K := d(K, \mathcal{H}_e^-) > 0$, and let $A_K := 1 + \operatorname{diam}(K)^d < \infty$. Let us also use the notation (5.2), and for any $\eta \in (0, \theta^*]$ and $t \geq 0$, let

$$I_t^\eta(K,\Lambda) := \left\{ \omega \in \Omega \;\middle|\; \inf_{|y| \leq \Lambda(t+1)} \inf_{x \in (c^*(e)e - K)t} u(t,x,\Upsilon_y\omega) < 1-\eta \right\}.$$

Assume that $u(t,x,\Upsilon_y\omega) \geq 1-\theta^*$ for some $(t,x,y,\omega) \in [0,\infty) \times \mathbb{R}^{2d} \times \Omega$. Since for all $y' \in \mathbb{R}^d$ we have $u(\cdot,\cdot,\Upsilon_{y'}\omega) = u(\cdot,\cdot+y',\omega;\mathcal{H}_e^- + (y'\cdot e)e)$, Lemma 2.1 and comparison principle yield for all $y' \in B_{\sqrt{d}}(y)$,

$$u(\cdot+\tau_0, \cdot - y', \Upsilon_{y'}\omega) \geq u(\cdot, \cdot - y, \Upsilon_y\omega),$$

with $\tau_0 := \kappa_0 + 2c_0^{-1}(R_0 + \sqrt{d})$. Applying Lemma 2.1 again, we obtain for all $y' \in B_{\sqrt{d}}(y)$,

$$u(t+2\tau_0, x, \Upsilon_{y'}\omega) \geq u(t+\tau_0, x+y'-y, \Upsilon_y\omega) \geq 1-\theta^*.$$

Then Lemma 2.8 shows that if $\eta \in (0,\theta^*]$ and $\tau_\eta := 2\tau_0 + 4c_0^{-1}\sqrt{d} + D_1\eta^{1-m_1}$, then

$$u(t+\tau_\eta, \cdot, \Upsilon_{y'}\omega) \geq (1-\eta)\chi_{B_{\sqrt{d}}(x)} \tag{6.3}$$

for all $y' \in B_{\sqrt{d}}(y)$. This shows that we can only have $\omega \in I_t^\eta(K,\Lambda)$ for some $t \geq \tau_\eta$ if

$$\inf_{(x,y) \in Z_{K,\Lambda,t}} u(t-\tau_\eta, x, \Upsilon_y\omega) < 1-\theta^*,$$

where $Z_{K,\Lambda,t} := ((c^*(e)e - K)t \cap \mathbb{Z}^d) \times (B_{\Lambda(t+1)}(0) \cap \mathbb{Z}^d)$. From this we obtain

$$\begin{aligned} \mathbb{P}[I_t^\eta(K,\Lambda)] &\leq \sum_{(x,y) \in Z_{K,\Lambda,t}} \mathbb{P}\left[u(t-\tau_\eta, x, \Upsilon_y\omega) < 1-\theta^*\right] \\ &= \sum_{(x,y) \in Z_{K,\Lambda,t}} \mathbb{P}\left[T(x, \Upsilon_y\omega) > t - \tau_\eta\right]. \end{aligned}$$

If $t \geq 2(\tau_\eta + \kappa_0 + \frac{2}{c_0})$, then from Lemma 2.1 we have $T(x, \Upsilon_y\omega) \leq t - \tau_\eta$ whenever $x \cdot e \leq 1 + \frac{c_0 t}{4}$. Hence, with $Z'_{K,\Lambda,t} := \{(x,y) \in Z_{K,\Lambda,t} \,|\, x \cdot e > 1 + \frac{c_0 t}{4}\}$ we obtain

$$\mathbb{P}[I_t^\eta(K,\Lambda)] \leq \sum_{(x,y)\in Z'_{K,\Lambda,t}} \mathbb{P}\left[T(x,\Upsilon_y\omega) > t - \tau_\eta\right]. \tag{6.4}$$

Note also that there is $C > 0$ such that this sum has at most $C(1+\Lambda^d)A_K t^{2d}$ terms.

Consider any $(x,y) \in Z'_{K,\Lambda,t}$, where $t \geq 2(\tau_\eta + \kappa_0 + \frac{2}{c_0})$. With the notation from (5.3) resp. (5.4), and $\beta' := \frac{1+\beta}{2} \in (0,1)$, Proposition 5.1 and stationarity of f yield

$$\left| \frac{\mathbb{E}[T(x,\cdot)]}{x \cdot e} - \frac{1}{c^*(e)} \right| \leq \bar{C}_\rho (x \cdot e)^{\beta' - 1}.$$

Since $x \cdot e \leq (c^*(e) - d_K)t$ and $c^*(e) = \bar{T}(e)^{-1} \leq c_1$, we obtain

$$\mathbb{E}[T(x,\cdot)] \leq \frac{x \cdot e}{c^*(e)} + \bar{C}_\rho (x \cdot e)^{\beta'} \leq t - \frac{d_K t}{c^*(e)} + \bar{C}_\rho ((c^*(e) - d_K)t)^{\beta'} \leq t - \frac{d_K t}{2c_1}$$

whenever

$$t \geq \max\left\{ (2\bar{C}_\rho c_1^{1+\beta'} d_K^{-1})^{\frac{1}{1-\beta'}}, 2(\tau_\eta + \kappa_0 + 2{c_0}^{-1}), 4c_1 d_K^{-1} \tau_\eta \right\}.$$

Hence for such t, (5.5) yields $C > 0$ (defining $\bar{C}_\rho$ via (5.3) resp. (5.4)) such that

$$\begin{aligned}
\mathbb{P}\left[T(x,\Upsilon_y\omega) > t - \tau_\eta\right] &\leq \mathbb{P}\left[|T(x,\Upsilon_y\omega) - \mathbb{E}[T(x,\cdot)]| \geq \frac{d_K t}{2c_1} - \tau_\eta \right] \\
&\leq 2\exp\left(-\bar{C}_\rho^{-2} \left(\frac{d_K t}{2c_1} - \tau_\eta \right)^2 (x \cdot e)^{-2\beta} \right) + \phi(x \cdot e) \\
&\leq 2\exp\left(-\bar{C}_\rho^{-2} d_K^2 t^{2-2\beta} \right) + C t^{-\beta_3(2d+1+m'_4)}
\end{aligned}$$

when $(x,y) \in Z'_{K,\Lambda,t}$, where we also used that $\frac{c_0 t}{4} \leq x \cdot e \leq c_1 t$ and $\tau_\eta \leq \frac{d_K}{4c_1} t$ (recall that C can change from line to line). This and (6.4) show that for all large enough t we have

$$\mathbb{P}\left[I_t^\eta(K,\Lambda)\right] \leq C(1+\Lambda^d)A_K t^{2d} \left(\exp\left(-\bar{C}_\rho^{-2} d_K^2 t^{2-2\beta} \right) + t^{-\beta_3(2d+1+m'_4)} \right).$$

Then $\sum_{n\geq 1} \mathbb{P}[I_n^\eta(K,\Lambda)] < \infty$ since $\beta_3 > \frac{2d+1}{2d+1+m'_4}$, so the Borel-Cantelli Lemma shows that for a.e. $\omega \in \Omega$, there is N_ω such that $\omega \notin \bigcup_{n \geq N_\omega} I_n^\eta(K,\Lambda)$. But since $K' \subseteq K^0$ means there is τ such that

$$(c^*(e)e - K')t \subseteq (c^*(e)e - K)\lfloor t \rfloor$$

for all $t \geq \tau$, from $u_t \geq 0$ and the definition of $I_t^\eta(K,\Lambda)$ we obtain

$$\inf_{|y| \leq \Lambda t} \inf_{x \in (c^*(e)e - K')t} u(t,x,\Upsilon_y\omega) \geq 1 - \eta$$

for all such ω and all $t \geq \max\{N_\omega, \tau\}$. Applying this argument with $\eta = \frac{1}{n}$, $\Lambda = n$, and $K' = \{x \in \mathbb{R}^d \,|\, x \cdot e \in [\frac{1}{n}, n] \text{ and } |x - (x \cdot e)e| \leq n\}$ for each $n \in \mathbb{N}$ yields $\Omega_1 \subseteq \Omega$ with $\mathbb{P}(\Omega_1) = 1$ for which the first statement in (6.1) holds.

It remains to prove the the second statement for some Ω_2 with $\mathbb{P}(\Omega_2) = 1$, as we can then take $\Omega_e := \Omega_1 \cup \Omega_2$. The proof is similar to that of the first statement. With the setup from the start of its proof, let now

$$I_t^\eta(K,\Lambda) := \left\{ \omega \in \Omega \;\middle|\; \sup_{|y|\le \Lambda t} \sup_{x \in (c^*(e)e+K)t} u(t,x,\Upsilon_y\omega) > \eta \right\}.$$

Assume that $u(t,x,\Upsilon_y\omega) > \eta$ for some $(t,x,y,\omega) \in [0,\infty) \times \mathbb{R}^{2d} \times \Omega$. Then (2.6) shows that with $L_{t,\eta} := \mu_*^{-1}(1+t^{\alpha_2})\eta^{-m_2}$, there is $x' \in B_{L_{t,\eta}}(x)$ such that

$$u(t,x',\Upsilon_y\omega) \ge 1-\theta^*.$$

Lemma 2.1 now shows that

$$u(t+\kappa_0+2c_0^{-1}L_{t,\eta},x,\Upsilon_y\omega) \ge 1-\theta^*.$$

In the same way as we obtained (6.3) (but using Lemma 2.1 instead of Lemma 2.8), we now get for all $y' \in B_{\sqrt{d}}(y)$ and with $\tau_{t,\eta} := 3\kappa_0 + \frac{2}{c_0}(L_{t,\eta}+R_0+2\sqrt{d})$,

$$u(t+\tau_{t,\eta},\cdot,\Upsilon_{y'}\omega) \ge (1-\theta^*)\chi_{B_{\sqrt{d}}(x)}.$$

So similarly to (6.4), with $Z_{K,\Lambda,t} := ((c^*(e)e+K)t \cap \mathbb{Z}^d) \times (B_{\Lambda t}(0) \cap \mathbb{Z}^d)$ we get for all $t \ge 0$,

$$\mathbb{P}[I_t^\eta(K,\Lambda)] \le \sum_{(x,y)\in Z_{K,\Lambda,t}} \mathbb{P}\left[T(x,\Upsilon_y\omega) \le t+\tau_{t,\eta}\right]. \tag{6.5}$$

And again, there is $C>0$ such that this sum has at most $C(1+\Lambda^d)A_K t^{2d}$ terms.

Let $d'_K := d_H(K,\mathcal{H}_e^-)$ and consider any $t \ge \frac{1}{c_0}$ and $(x,y) \in Z_{K,\Lambda,t}$. Then $c^*(e)t \ge 1$ and

$$x\cdot e \in [(c^*(e)+d_K)t,\, (c^*(e)+d'_K)t]. \tag{6.6}$$

So Proposition 5.1 and $c^*(e) = \bar{T}(e)^{-1}$ imply as above (with $\beta' = \frac{1+\beta}{2}$),

$$\mathbb{E}[T(x,\cdot)] \ge \frac{x\cdot e}{c^*(e)} - \bar{C}_\rho (x\cdot e)^{\beta'} \ge t + \frac{d_K t}{c^*(e)} - \bar{C}_\rho((c^*(e)+d'_K)t)^{\beta'} \ge t + \frac{d_K t}{2c_1}$$

whenever

$$t \ge \max\left\{ (2\bar{C}_\rho(c_1+d'_K)^{1+\beta'}d_K^{-1})^{\frac{1}{1-\beta'}}, c_0^{-1}, C_{K,\eta} \right\},$$

where $C_{K,\eta}$ is such that $\tau_{t,\eta} \le \frac{d_K t}{4c_1}$ for all $t \ge C_{K,\eta}$ (this exists because $\alpha_2 < 1$, and will be used next). Hence for such t, (5.5) yields $C>0$ (defining $\bar{C}_\rho$ via (5.3) resp. (5.4)) such that

$$\begin{aligned}
\mathbb{P}\left[T(x,\Upsilon_y\omega) \le t+\tau_{t,\eta}\right] &\le \mathbb{P}\left[|T(x,\Upsilon_y\omega) - \mathbb{E}[T(x,\cdot)]| \ge \frac{d_K t}{2c_1} - \tau_{t,\eta} \right] \\
&\le 2\exp\left(-\bar{C}_\rho^{-2}\left(\frac{d_K t}{2c_1} - \tau_{t,\eta}\right)^2 (x\cdot e)^{-2\beta} \right) + \phi(x\cdot e) \\
&\le 2\exp\left(-\bar{C}_\rho^{-2} d_K^2 (1+d'_K)^{-2\beta} t^{2-2\beta}\right) + Ct^{-\beta_3(2d+1+m'_4)}
\end{aligned}$$

when $(x,y) \in Z_{K,\Lambda,t}$, where we also used (6.6) and $\tau_{t,\eta} \le \frac{d_K t}{4c_1}$ in the last inequality. This and (6.5) show that for all large enough t we have

$$\mathbb{P}[I_t^\eta(K,\Lambda)] \le C(1+\Lambda^d)A_K t^{2d}\left(\exp\left(-\bar{C}_\rho^{-2} d_K^2 (1+d_K')^{-2\beta} t^{2-2\beta}\right) + t^{-\beta_3(2d+1+m_4')}\right).$$

We can now conclude the proof of the second statement in (6.1) as we did the proof of the first statement, this time using that

$$(c^*(e)e + K')t \subseteq (c^*(e)e + K)\lceil t \rceil$$

for all large enough t. □

Remark. This proof shows that (6.1) holds with Λt replaced by $\exp(t^\gamma)$ for any $\gamma < 2 - 2\beta$.

Proposition 6.3. *Under the hypotheses of Proposition 6.2, for each $e \in \mathbb{S}^{d-1}$, the speed $c^*(e)$ is also a deterministic strong exclusive front speed in direction e for* (1.1).

Proof. Having Proposition 6.2, this proof is now similar to the one of [16, Theorem 1.7(i)].

For any $(e,a,\omega) \in \mathbb{S}^{d-1} \times (0,1) \times \Omega$, let $u(\cdot,\cdot,\omega;\mathcal{H}_e^-)$ and $w_{e,a}(\cdot,\cdot,\omega)$ be from Definition 6.1, and let $\tau_a := 1 + D_1 a^{1-m_1}$. Lemma 2.9 (see Lemma 3.7) shows that if $a \in (0, \frac{1}{2}\min\{\theta^*, M_*^{-1}\}]$, then

$$u_+(t,x) := u((1+M_*a)t + \tau_a, x, \omega; \mathcal{H}_e^-) + a$$

is a supersolution to (1.1) on $(\kappa_*, \infty) \times \mathbb{R}^d$. Moreover, $u_t \ge 0$ and Lemma 2.8 show that

$$u_+(\kappa_*, \cdot) \ge u(\tau_a, \cdot, \omega; \mathcal{H}_e^-) + a \ge w_{e,a}(0, \cdot, \omega).$$

The comparison principle now yields for all $t \ge 0$,

$$u_+(t + \kappa_* + \tau_a, \cdot) \ge w_{e,a}(t, \cdot, \omega).$$

It now follows from Proposition 6.2 that (6.2) holds with $\lambda_{K,\omega,e}(a) = a$ for all $\Lambda \ge 0$ and all compact $K \subseteq \mathcal{H}_e^+ + M_* a\, c^*(e)e$. This is true for all $a \in (0, \frac{1}{2}\min\{\theta^*, M_*^{-1}\}]$, so the result follows after letting $\lambda_{K,\omega,e}(a) := 1$ for all $a \in (\frac{1}{2}\min\{\theta^*, M_*^{-1}\}, 1]$. □

6.1. **Proof of Theorem 1.7.** For any $e \in \mathbb{S}^{d-1}$, let $\Omega_e \subseteq \Omega$ with $\mathbb{P}(\Omega_e) = 1$ be the set from Definition 6.1 and let $c^*(e)$ be the corresponding deterministic strong exclusive front speed for (1.1) from Proposition 6.3. Let $A \subseteq \mathbb{S}^{d-1}$ be a dense countable set and let $\Omega_0 := \bigcap_{e \in A} \Omega_e$. Then $\mathbb{P}(\Omega_0) = 1$, and for each $\omega \in \Omega_0$, (1.1) with this fixed ω has a strong exclusive front speed $c^*(e)$ in each direction $e \in A$ (i.e., (6.1) and (6.2) hold for this fixed ω and each $e \in A$, $\Lambda \ge 0$, and compact $K \subseteq \mathcal{H}_e^+$).

Then [16, Theorem 4.4(i)] shows that (1.1) with this fixed ω has a strong exclusive front speed $c_\omega^*(e)$ in each direction $e \in \mathbb{S}^{d-1}$, and c_ω^* is Lipschitz with Lipschitz constant only depending on M. But then $c_\omega^*(e)$ must be independent of $\omega \in \Omega_0$ for each $e \in \mathbb{S}^{d-1}$ (instead of just all $e \in A$), and hence equals $c^*(e)$ from Proposition 6.3 because $\mathbb{P}(\Omega_0) = 1$.

Theorem 1.7 now follows directly from [16, Theorem 5.4] applied separately to each $\omega \in \Omega_0$ (see also the remarks after Hypothesis H' in [16]).

7. Proof of Theorem 1.8

In this section we will show how to extend the above analysis to the cases considered in Theorem 1.8. We can obviously assume $\alpha_2' > 0$ without loss, and all constants with C in them may depend on (2.10) as well as on α_2'.

Since we now replace **(H2')** by **(H2")**, the estimates (2.6) instead become

$$\begin{aligned} &\sup_{t\ge 0\,\&\,\eta>0} \frac{L_{u,\eta+a,1-\theta^*}(t)}{(1+t^{\alpha_2})\eta^{-m_2}} \le \mu_*^{-1}, \\ &\inf_{\substack{(t,x)\in[\kappa_*,\infty)\times\mathbb{R}^d \\ u(t,x)\in[\theta^*,1-\theta^*]}} u_t(t,x)t^{\alpha_2'} \ge \mu_* \end{aligned} \tag{7.1}$$

for all $a \in [0, a_2]$ and either all $u \in \mathcal{U}_{f,a}$ (when assuming **(H2")**) or all $u \in \bigcup_{n\ge n_4} \mathcal{U}_{f_n,a}$ (when assuming **(H3)** and **(H4")**), again with some $\mu_*, \kappa_* > 0$.

We will first assume without loss that $a_2 = 0$. Then of course also $a = 0$ and $\mathcal{U}_{f,a} = \mathcal{U}_f$ above, so (7.1) is just (2.6) with the extra factor of $t^{\alpha_2'}$ in the second estimate. We will now show how the results in Sections 2–6 and their proofs change due to this.

Of the results in Section 2, clearly only Lemmas 2.9 and 2.10 are affected by this change. They will instead become the following two results.

Lemma 7.1. *Let f_1 satisfy **(H2")** and f_2 satisfy **(H1')**, and let $M_* := 2^{\alpha_2'}\frac{1+M}{\mu_*}$, with μ_*, κ_* from (7.1) for all $u \in \mathcal{U}_{f_1}$. Fix some $\omega \in \Omega$ and let $u_1, u_2 : [0,\infty)\times\mathbb{R}^d \to [0,1]$ solve (1.1) with f_1, f_2 in place of f, respectively. If $u_1 \in \mathcal{U}_{f_1}$, $t_0 \ge 0$, $T \ge 2\kappa_*$, and for some $\eta \in [0, \frac{1}{2}\min\{\theta^*, M_*^{-1}(T+t_0)^{-\alpha_2'}\}]$ we have*

$$f_1(x,u,\omega) = f_2(x,u,\omega) \qquad \text{whenever } u_1(t_0,x) < 1-\eta \text{ and } u \in [0,1],$$

then

$$u_+(t,x) := u_1((1+M_*(T+t_0)^{\alpha_2'}\eta)t + t_0, x) + \eta$$

is a supersolution to (1.1) with f_2 in place of f on $(\kappa_*, T)\times\mathbb{R}^d$, and

$$u_-(t,x) := u_1((1-M_*(T+t_0)^{\alpha_2'}\eta)t + t_0, x) - \eta$$

is a subsolution to (1.1) with f_2 in place of f on $(2\kappa_*, T)\times\{x\in\mathbb{R}^d \,|\, u_1(t_0,x) < 1-\eta\}$.

Moreover, there is $D_2 = D_2(M, \theta_1, m_1, \alpha_1) \ge 1$ such that if also $T_{u_2}(y) \le T$, and

$$\sup_{x\in B_R(y)} (u_2(0,x) - u_1(t_0,x)) \le \eta$$

for some $y \in \mathbb{R}^d$ and $R \ge D_2(1+T_{u_2}(y))$, then

$$T_{u_2}(y) \ge \left(1 + M_*(T+t_0)^{\alpha_2'}\eta\right)^{-1}(T_{u_1}(y) - t_0 - 2\kappa_* - \kappa_0).$$

Proof. The proof is the same as that of Lemma 2.9, replacing (2.6) by (7.1) and using $\tau_\pm(t) := (1 \pm M_*(T+t_0)^{\alpha_2'}\eta)t + t_0$. In particular, we use in it that for $t \le T$ we have

$$M_*(T+t_0)^{\alpha_2'}\eta\mu_*((1+M_*(T+t_0)^{\alpha_2'}\eta)t + t_0)^{-\alpha_2'} \ge M_*(T+t_0)^{\alpha_2'}\eta\mu_*(2T+t_0)^{-\alpha_2'} \ge (1+M)\eta.$$

We also have $D_2 := 2\sqrt{Md}\ln\frac{4d}{\theta^*}$ as before. □

Lemma 7.2. *Let f_1 satisfy **(H2")** and f_2 satisfy **(H1')**, with at least one satisfying **(H3)** with $\alpha_3 \le 1$, and let M_*, D_2 be from Lemma 7.1. Fix some $\omega \in \Omega$ and let $u_1, u_2 : [0,\infty)\times\mathbb{R}^d \to [0,1]$ solve (1.1) with f_1, f_2 in place of f, respectively. If $u_1 \in \mathcal{U}_{f_1}$, for some $y \in \mathbb{R}^d$, $R \ge D_2(1+T_{u_2}(y))$, and $\eta \in [0, \frac{1}{2}\min\{\theta^*, M_*^{-1}(\max\{T_{u_2}(y), 2\kappa_*\}+t_0)^{-\alpha_2'}\}]$ we have*

$$f_1(x,u,\omega) \ge f_2(x,u,\omega) - \alpha_3\eta^{m_3} \qquad \text{for all } (x,u) \in B_R(y)\times[0,1],$$

and $u_2(0,\cdot) \le u_1(t_0,\cdot)$ for some $t_0 \ge 0$ and all $x \in B_R(y)$, then

$$T_{u_2}(y) \ge \left(1 + M_*(\max\{T_{u_2}(y), 2\kappa_*\}+t_0)^{\alpha_2'}\eta\right)^{-1}\left(T_{u_1}(y) - t_0 - 2\kappa_* - \kappa_0\right).$$

Proof. The proof is the same as that of Lemma 2.10, replacing (2.6) and Lemma 2.9 by (7.1) and Lemma 7.1 with $T := \max\{T_{u_2}(y), 2\kappa_*\}$, and using $\tau_+(t) := (1+M_*(T+t_0)^{\alpha_2'}\eta)t + t_0$. □

We can now extend all of Sections 3–5 to **(H2")** in place of **(H2')**, and obtain the following analog of Proposition 6.2.

Proposition 7.3. *Assume that f either satisfies **(H2")** and has a finite range of dependence, or satisfies **(H3)** and **(H4")**. Then for each $e \in \mathbb{S}^{d-1}$, (1.1) has a deterministic strong front speed $c^*(e) \in [c_0, c_1]$ in direction e.*

Proof. In the whole proof, we assume without loss that $a_2 = 0$ (and so $a = 0$ as well). When extending results from Section 3, we assume that f satisfies **(H2")** and has range of dependence at most $\rho \in [1,\infty)$; in Section 4 we instead assume **(H3)** and **(H4")**; and in Sections 5 and 6 we assume either of these two cases, as before.

Most of Section 3 is unchanged, with (3.3) replaced by

$$\beta_1 := \max\left\{\frac{(1+\alpha_2')(m_1-1)}{m_1}, \frac{(1+\alpha_2')m_2+\alpha_2}{m_2+1}\right\}, \tag{7.2}$$

which is still in $(0,1)$ thanks to $\alpha_2' < \min\{\frac{1}{m_1-1}, \frac{1-\alpha_2}{m_2}\}$. The first adjustment is required in Lemma 3.4, where we used the second claim in (2.6) to obtain (3.8). We have here $u(\tau(x,\omega), y, \omega) \ge \theta^*$ for some $y \in \overline{B_1(x)}$, and can instead use Lemma 2.1 to get

$$u(\tau(x,\omega) + \kappa_0 + 2c_0^{-1}(L_{u,\theta^*,1-\theta^*}(\tau(x,\omega))+1), x, \omega) \ge 1-\theta^*.$$

Since the first claim in (7.1) (with $a = 0$ and $\eta = \theta^*$) yields $C > 0$ such that

$$\kappa_0 + 2c_0^{-1}(L_{u,\theta^*,1-\theta^*}(\tau(x,\omega))+1) \le C(1+\tau(x,\omega)^{\alpha_2})$$

(recall that $u \in \mathcal{U}_f$) and Lemma 2.1 also implies $\tau(x,\omega) \le C(1+t)$ whenever $\omega \in F_{t,x}$, we have $T(x,\cdot) \le \tau(x,\cdot) + C(1+t^{\alpha_2})$ on $F_{t,x}$. Hence the first claim of Lemma 3.4 becomes

$$\left|\mathbb{E}[T(x,\cdot)\chi_{F_{t,x}} \,|\, \mathcal{G}_t] - T(x,\cdot)\chi_{F_{t,x}}\right| \le C(1+t^{\alpha_2}) \qquad \text{on } \Omega,$$

while the second holds only for $s \in [0, t - C(1+t^{\alpha_2})]$.

As for Proposition 3.5, instead of (3.13) we let

$$\eta := C_1^{-1}(\rho + d(x_0, S))^{-\gamma},$$

where $C_1 > 0$ will be chosen shortly and

$$\gamma := \min\left\{\frac{1+\alpha_2'}{m_1}, \frac{1+\alpha_2'-\alpha_2}{m_2+1}\right\} > 0. \tag{7.3}$$

Note that then, similarly to (3.14), we have

$$\max\{\gamma(m_1-1), \alpha_2+\gamma m_2, 1+\alpha_2'-\gamma\} = \beta_1 < 1 \tag{7.4}$$

(in particular, $\gamma > \alpha_2'$), and (3.15) and (3.16) continue to hold. Now we pick C_1 so that with with $T := \max\{T(x_0,\omega), T_i(x_0,\omega), 2\kappa_*\}$ we have

$$\eta \le \min\left\{\frac{\theta^*}{2}, \frac{(T+t_0+t_3)^{-\gamma}}{2M_*}\right\},$$

which is possible due to $\max\{T(x_0,\omega), T_i(x_0,\omega), t_0\} \le C(1+d(x_0,S))$ and (3.16). Then we can use Lemma 7.1 with this T and η (instead of Lemma 2.9) to see that (3.17) becomes

$$T(x_0,\omega) - t_0 - T_i(x_0,\omega) \le M_*\eta\big(T+t_0+t_3\big)^{\alpha_2'} T_i(x_0,\omega) + 2\kappa_* + \kappa_0 + t_3 \le C(\rho + d(x_0,S)^{\beta_1})$$

because $\beta_1 \ge 1+\alpha_2'-\gamma$. This ends the first half of the proof. Using again Lemma 7.1 instead of Lemma 2.9 in the second half of it, with η as above, shows that

$$u_-(t,x) := u((1-M_*\eta(T+t_0)^{\alpha_2'})t + t_0, x) - \eta$$

is a subsolution to (1.1) on $(2\kappa_*, T) \times (\mathbb{R}^d \backslash \Gamma_{u,1-\eta}(t_0,\omega))$. The rest of the proof does not use the second claim in (2.6) and is unchanged (using this u_- and also (7.4)), with (3.23) becoming

$$T_i(x_0,\omega) + t_0 - T(x_0,\omega) \le t_4 + M_*\eta(T+t_0)^{\alpha_2'} T_i(x_0,\omega) \le C(\rho + d(x_0,S)^{\beta_1}).$$

This finishes the proof.

The proof of Lemma 3.6 remains the same. In the proof of Proposition 3.1, the change in Lemma 3.4 turns the C in (3.25) and (3.30) into $C(1+t^{\alpha_2})$ (recall that $s \le t$ in the argument), which is then added to the right-hand sides of (3.26) and (3.29). Then (3.30) shows that there is $C > 0$ such that $X_t = T(x,\cdot)$ for all $t \ge C(1+d(x,S))$, and we now pick N to be the smallest integer with $N\tau \ge C(1+d(x,S))$. The estimate $N \le C_1 d(x,S)(\rho + d(x,S)^{\beta_1})^{-1}$ now still holds (recall that $d(x,S) > \rho \ge 1$ here), and (3.31) remains unchanged because the term added to (3.29) is estimated by $C(1+(N\tau)^{\alpha_2}) \le C(1+d(x,S)^{\beta_1})$ because $\alpha_2 < \beta_1$. The rest of the proof of Proposition 3.1 remains the same.

Lemmas 3.7 and 4.1 are unchanged except for replacement of **(H2')**, **(H4')**, $\mathcal{U}_f$, $\mathcal{U}_f'$, Lemma 2.9, and Lemma 2.10 in their statements and proofs by **(H2")**, **(H4")**, $\mathcal{U}_{f,a}$, $\mathcal{U}_{f,a}'$, Lemma 7.1, and Lemma 7.2, respectively (here we can even allow any fixed $a_2 \in [0, \frac{1}{2}\theta^*]$ in **(H2")**, although $a_2 = 0$ is sufficient). The proof of Proposition 3.8 also remains the same, using Lemma 7.1 (with $T = \infty$ because $\eta = 0$) instead of Lemma 2.9.

In Proposition 4.2, we replace (4.1) by

$$\beta_3 := \max\left\{\beta_1, \frac{(1+2\alpha_2')m_3}{m_3+2m_4}, \frac{2d+2}{2d+2+m_4'}\right\}, \tag{7.5}$$

which is still in $(0,1)$ thanks to $\alpha_2' < \frac{m_4}{m_3}$. Then when we use Lemma 2.10 in the proof, we replace it by Lemma 7.2 with the same $\eta := (\frac{\alpha_4}{\alpha_3} n^{-m_4})^{1/m_3}$, but now we need to pick n so that

$$\alpha_4^{1/m_3} \alpha_3^{-1/m_3} n^{-\frac{m_4}{m_3}} \le \frac{1}{2} \min\{\theta^*,\, M_*^{-1}(T + \kappa_0 + 2c_0^{-1}R_0)^{-\alpha_2'}\},$$

with $T := \max\{T(x,\omega;S), T_n(x,\omega;S), 2\kappa_*\}$. Since $T \le C(1 + d(x,S))$ due to Lemma 2.1, and $\beta_3 \frac{m_4}{m_3} > \alpha_2'$ due to $\alpha_2' < \frac{m_4}{m_3}$ and (7.5), there is again $C_1 > 0$ such that it suffices to let n be the smallest integer for which (4.5) holds. Then a double application of Lemma 7.2 replaces (4.4) by

$$|T_n(x,\cdot\,;S) - T(x,\cdot\,;S)| \le C n^{-\frac{m_4}{m_3}} (1+T)^{1+\alpha_2'} + C \le C n^{-\frac{m_4}{m_3}} (1 + d(x,S))^{1+\alpha_2'} + C.$$

Since there is again $C_2 > 0$ such that

$$4C n^{-\frac{m_4}{m_3}} (1 + d(x,S))^{1+\alpha_2'} + 4C \le C_2 (1 + d(x,S)^{1+\alpha_2' - \beta_3 \frac{m_4}{m_3}}) \le C_2 (1 + d(x,S)^{\frac{1+\beta_3}{2}})$$

because $1 + \alpha_2' - \beta_3 \frac{m_4}{m_3} \le \frac{1+\beta_3}{2}$, the rest of the proof of Proposition 4.2 is unchanged.

Most of Section 5 is also unchanged, with the only two adjustments needed in the proof of Proposition 5.5. We used Lemma 2.9 when proving (5.8), and we can just replace it by Lemma 7.1 without any other change because there we had $\eta = 0$. We also used Lemma 2.9 when proving (5.13), and the change to Lemma 7.1 now requires us to replace (5.9) by

$$\eta := C_1^{-1} \min\left\{\theta^*, M_*^{-1} l^{-\gamma}\right\}$$

with γ from (7.3) (recall that (7.4) shows $\gamma \ge 1 + \alpha_2' - \beta_1 \ge 1 - \beta_1$, so $\eta \le l^{\beta_1 - 1}$ as well; in fact, we could have chosen this η in (5.9) as well) and $C_1 \ge 2$ such that with $T = \max\{\kappa_0 + \frac{2(l+m)}{c_0}, 2\kappa_*\}$ we have $\eta \le \frac{1}{2M_*}(T + \tau_0^l)^{-\alpha_2'}$. This is possible because of (5.12) and $\gamma \ge \alpha_2'$. Then replacing Lemma 2.9 by Lemma 7.1 with this T yields

$$T((l+m)e, \omega; \mathcal{H}_e^- + le) \le (1 + M_* \eta (T + \tau_0^l)^{\alpha_2'}) \big[T((l+m)e, \omega; \mathcal{H}_e^-) - T_1'(\omega)\big] + \tau_0^l + 2\kappa_* + \kappa_0$$

instead of (5.13). But the addition of $(T + \tau_0^l)^{\alpha_2'}$ here does not require further changes because

$$M_* \eta (T + \tau_0^l)^{\alpha_2'} T((l+m)e, \omega; \mathcal{H}_e^-) \le C l^{1 + \alpha_2' - \gamma} \le C l^{\beta_1}$$

by (7.4) (recall that we assume here $l \ge m \ge \frac{4c_1}{c_0} l_0$).

The proofs of Propositions 5.1 and 6.2 then remain unchanged, finishing the proof. □

We are only able to obtain an **(H2")**-version of Proposition 6.3 when $a_2 > 0$, and we do so below. But even without that, we can already prove Theorem 1.8(ii).

Proof of Theorem 1.8(ii). This is identical to the proof of Theorem 1.7 above, with the word "exclusive" and (6.2) dropped, and using Proposition 7.3 and [16, Theorem 1.4(iii)] instead of Proposition 6.3 and [16, Theorem 5.4], respectively. Note that f is also stationary ergodic in [16, Theorem 1.4(iii)], but this is only used in the first paragraph of its proof to show existence of deterministic strong front speeds for all $e \in \mathbb{S}^{d-1}$ (which we proved in Proposition 7.3), so that result extends to the case at hand. □

To prove Theorem 1.8(i), we need to show that the deterministic strong front speeds from Proposition 7.3 are exclusive. For this, we will need some uniform estimates on the reactions

$$f_a(x,u,\omega) := \frac{f(x,(1-a)u+a,\omega)}{1-a}$$

for $(x,u,\omega) \in \mathbb{R}^d \times [0,1] \times \Omega$. Note that the transformation $u \mapsto \frac{u-a}{1-a}$ turns solutions u to (1.1) for which $a \le u \le 1$ into solutions to (1.1) with f_a in place of f for which $0 \le u \le 1$.

Proposition 7.4. *Assume that f either satisfies **(H2")** with $a_2 > 0$ and has a finite range of dependence, or satisfies **(H3)** and **(H4")** with $a_2 > 0$. Then for each $(a,e) \in (0,a_2] \times \mathbb{S}^{d-1}$, (1.1) with f_a in place of f has a deterministic strong front speed $c_a^*(e) \in [c_0,c_1]$ in direction e, and $\lim_{a\to 0} c_a^*(e) = c^*(e)$ holds uniformly in $e \in \mathbb{S}^{d-1}$ (with $c^*(e)$ from Proposition 7.3).*

Proof. Clearly f_a satisfies **(H1)** with the same M and m_1, and θ_1 and α_1 replaced by $\frac{1}{2}\theta_1$ and $\alpha_1(1-\frac{1}{8}\theta_1)^{m_1-1}$, respectively (recall that $a \le a_2 \le \frac{1}{8}\theta_1$).

If we now assume **(H2")** and finite range of dependence of f, and let $u^a := \frac{u-a}{1-a}$ for some $u \in \mathcal{U}_{f,a}$ with initial datum $u_{0,k,a}$, then $u_t^a = \Delta u^a + f_a(x,u^a,\omega)$ on $(0,\infty)\times\mathbb{R}^d$ (with the same ω) and its initial datum $\frac{1}{1-a}(u_{0,k,a}-a)$ satisfies (1.8) and (1.9) (with F now defined via f_a). Moreover, (1.12) and $a_2 \le \frac{1}{2}$ show that we also have

$$\limsup_{t\to\infty} \sup_{a\in[0,a_2]} \sup_{u\in\mathcal{U}_{f,a}} \sup_{\eta>0} \frac{L_{u^a,\eta,1-2\theta^*}(t)}{t^{\alpha_2}\eta^{-m_2}} < \infty,$$

$$\liminf_{t\to\infty} \inf_{a\in[0,a_2]} \inf_{u\in\mathcal{U}_{f,a}} \inf_{u^a(t,x)\in[\theta^*,1-2\theta^*]} u_t^a(t,x)t^{\alpha_2'} > 0.$$

Hence for each such f_a we have **(H2")** with $a_2 = 0$, θ^* replaced by $2\theta^*$, the above constants in **(H1)**, and $\mathcal{U}_{f_a} := \{u^a \mid u \in \mathcal{U}_{f,a}\}$ (and the same m_2, α_2, α_2'). Moreover, there are $\mu_*, \kappa_* > 0$ such that (7.1) holds for all $a \in [0,a_2]$ and $u \in \mathcal{U}_{f_a}$, with θ^* replaced by $2\theta^*$. This and the remark after (2.1) (which shows that replacing θ^* by $2\theta^*$ in (7.1) does not change any of the above proofs) now show that Proposition 7.3 holds for all the f_a, and all constants in its proof are uniform in $a \in [0,a_2]$. In particular, (7.1) holds with the same $\mu_*, \kappa_* > 0$ (and θ^* replaced by $2\theta^*$) for all $a \in [0,a_2]$ and $u \in \mathcal{U}'_{f_a}$ (see Lemma 3.7), and the first claim in Proposition 5.1 holds for f_a with β_1 from (7.2) and C_δ uniform in $a \in [0,a_2]$.

The same argument applies when we assume **(H3)**+**(H4")**, where the passage to f_a and $f_{n,a}$ also replaces α_3 by $\alpha_3(1-\frac{\theta_1}{8})^{m_3-1}$ in **(H3)** and α_4 by $\alpha_4(1-\frac{\theta_1}{8})^{-1}$ in **(H4")**. Again we obtain Proposition 7.3 for all the f_a, as well as that (7.1) holds with some $\mu_*, \kappa_* > 0$ (and with θ^* replaced by $2\theta^*$) for all $a \in [0,a_2]$ and $u \in \mathcal{U}'_{f_a}$, and the second claim in Proposition 5.1 holds for f_a with β_3 from (7.5) and C_δ uniform in $a \in [0,a_2]$.

It therefore remains to prove the last claim, with the above deterministic strong front speeds denoted $c_a^*(e)$ (where clearly $c_0^*(e) = c^*(e)$). To achieve this, we will use uniformity of the estimates in Proposition 5.1 in $a \in [0,a_2]$. We therefore denote by $v_{e,a}(\cdot,\cdot,\omega)$ the solution to (5.1) with f replaced by f_a, some $(e,\omega) \in \mathbb{S}^{d-1}\times\Omega$, and $l = 0$ (then of course $v_{e,a}(\cdot,\cdot,\omega) \in \mathcal{U}'_{f_a}$). We also let

$$u_{e,a}(\cdot,\cdot,\omega) := (1-a)v_{e,a}(\cdot,\cdot,\omega) + a \in \mathcal{U}'_{f,a}, \tag{7.6}$$

with $\mathcal{U}'_{f,a}$ obtained from $\mathcal{U}_{f,a}$ as in Lemma 3.7, and for any $x \in \mathbb{R}^d$,

$$T_{e,a}(x,\omega) := \inf\{t \geq 0 \,|\, v_{e,a}(t,x,\omega) \geq 1-\theta^*\} = \inf\{t \geq 0 \,|\, u_{e,a}(t,x,\omega) \geq 1-(1-a)\theta^*\},$$
$$T'_{e,a}(x,\omega) := \inf\{t \geq 0 \,|\, u_{e,a}(t,x,\omega) \geq 1-\theta^*\}.$$

These definitions, Lemma 2.1, and $a_2 \leq \frac{1}{2}\theta^*$ show that there is C such that

$$T'_{e,a} \leq T_{e,a} \leq T'_{e,a} + C. \tag{7.7}$$

We will treat both cases **(H2")**+finite range and **(H3)**+**(H4")** at once, using the notation from either (5.3) in the first case or (5.4) in the second. Then the claims from Proposition 5.1, with $\delta := \frac{1-\beta}{2}$ and $\bar{C}'_\rho := C_{(1-\beta)/2}\bar{C}_\rho$ independent of (e,a,ω) become

$$\left|\frac{\mathbb{E}[T_{e,a}(le,\omega)]}{l} - \frac{1}{c^*_a(e)}\right| \leq \bar{C}'_\rho\, l^{-(1-\beta)/2} \tag{7.8}$$

for all $(e,a,\omega) \in \mathbb{S}^{d-1} \times [0,a_2] \times \Omega$ and all $l \geq 1$.

Lemma 2.1 shows that $u_{e,a}(\tau_0,\cdot,\cdot) \geq u_{e,0}(0,\cdot,\cdot)$ with $\tau_0 := \kappa_0 + \frac{2R_0}{c_0}$, hence the comparison principle yields $u_{e,a}(\tau_0+t,\cdot,\cdot) \geq u_{e,0}(t,\cdot,\cdot)$ for all $t \geq 0$. This and (7.6) immediately imply

$$T'_{e,a} \leq T_{e,0} + \tau_0,$$

so $c^*_a(e) \geq c^*(e)$ for all $(e,a) \in \mathbb{S}^{d-1} \times [0,a_2]$ by (7.7) (this also shows that $c^*_a(e) \geq c_0$).

Since $T'_{e,a}(le,\cdot) \leq Cl$ for all $l \geq 1$ by Lemma 2.1, Lemma 7.1 with $f_1 = f_2 = f$ and

$$(u_1,u_2,\eta,t_0,T,R) = (u_{e,0},u_{e,a},a,\tau_0,\max\{T'_{e,a}(le,\cdot),2\kappa_*\},\infty)$$

yields

$$T_{e,0}(le,\cdot) \leq T'_{e,a}(le,\cdot) + C(1+l^{1+\alpha'_2}a).$$

as long as $l \in [1,(C'a)^{-1/\alpha'_2}]$ (for some $C,C' > 0$). It follows by (7.7) that for such l we have

$$T_{e,0}(le,\cdot) \leq T_{e,a}(le,\cdot) + C(1+l^{1+\alpha'_2}a).$$

Picking $l := a^{-1/2\alpha'_2}$ and using (7.8) now yield for all small enough a (depending only on (2.10) and α_2),

$$c^*(e)^{-1} \leq c^*_a(e)^{-1} + Ca^{1/2} + \bar{C}'_\rho\, a^{(1-\beta)/4\alpha'_2}.$$

Since $\beta < 1$, $\alpha'_2 > 0$, and $c^*_a(e) \geq c^*(e)$, the uniform convergence claim follows. □

We can now extend Proposition 6.3 to the case $a_2 > 0$.

Proposition 7.5. *Under the hypotheses of Proposition 7.4, for each $e \in \mathbb{S}^{d-1}$, the speed $c^*(e)$ is a deterministic strong exclusive front speed in direction e for* (1.1).

Proof. Fix any $e \in \mathbb{S}^{d-1}$, and let $u_{e,a}$ be from (7.6) and $c^*_a(e)$ from Proposition 7.4. From that proposition and $f(\cdot,a,\cdot) \equiv 0$ for all $a \in [0,a_2]$ we know that for almost all $\omega \in \Omega$ we have

$$\lim_{t\to\infty} \sup_{|y|\leq \Lambda t} \sup_{x \in (c^*_a(e)e+K)t} u_{e,a}(t,x,\Upsilon_y\omega) = a \tag{7.9}$$

for each $a \in [0, a_2] \cap \mathbb{Q}$, $\Lambda > 0$, and compact $K \subseteq \mathcal{H}_e^+ = \{x \in \mathbb{R}^d \,|\, x \cdot e > 0\}$. Fix any such ω, and then any compact $K \subseteq \mathcal{H}_e^+$. Let $K' \subseteq \mathcal{H}_e^+$ be compact and such that $K \subseteq (K')^0$. Proposition 7.4 then yields $\delta \in (0, a_2]$ such that for all $a \in [0, \delta]$ we have

$$c^*(e)e + K \subseteq (c_a^*(e)e + K')(1 + 2\delta). \tag{7.10}$$

Since $\alpha_2' < \min\{\frac{1}{m_1 - 1}, \frac{1-\alpha_2}{m_2}\}$, there exists $T_0 \geq \kappa_*$ such that for all $T \geq T_0$ we have

$$\eta_T \leq \frac{\theta^*}{2} \qquad \text{and} \qquad \max\left\{M_*(T + \kappa_* + \tau_T)^{\alpha_2'}\eta_T, (\tau_T + 2\kappa_*)T^{-1}\right\} \leq \delta, \tag{7.11}$$

where $\eta_T := T^{-\gamma}$ with $\gamma := \frac{1}{2}(\alpha_2' + \frac{1}{m_1 - 1})$, and $\tau_T := 1 + D_1 \eta_T^{1-m_1}$ with D_1 from Lemma 2.8.

Now fix any $\Lambda > 0$. We see from (7.9) that for each $a \in [0, a_2] \cap \mathbb{Q}$, there is a function $\varphi_a : [0, \infty) \to [0, \infty)$ such that $\lim_{t \to \infty} \varphi_a(t) = 0$ and

$$\sup_{t \geq T} \sup_{|y| \leq \Lambda t} \sup_{x \in (c_a^*(e)e + K')t} u_{e,a}(t, x, \Upsilon_y \omega) \leq a + \varphi_a(T). \tag{7.12}$$

Pick any $T \geq T_0$ and $a \in [0, \delta] \cap \mathbb{Q}$, and let $w_{e,a}$ be from Definition 6.1. Then Lemma 2.8 yields $u_{e,a}(\tau_T, \cdot, \cdot) \geq 1 - \eta_T$ on $\mathcal{H}_e^-$, so from $u_{e,a} \geq a$ and $(u_{e,a})_t \geq 0$ we see that

$$u_{e,a}(t + \tau_T, \cdot, \cdot) + \eta_T \geq w_{e,a}(0, \cdot, \cdot) \tag{7.13}$$

for all $t \geq 0$. Since Lemma 7.1 and (7.11) show that

$$u_+(t, x, \cdot) := u_{e,a}((1 + M_*(T + \kappa_* + \tau_T)^{\alpha_2'}\eta_T)t + \tau_T, x, \cdot) + \eta_T$$

is a supersolution to (1.1) on $(\kappa_*, T + \kappa_*) \times \mathbb{R}^d$, the comparison principle and (7.13) yield

$$u_+(t + \kappa_*, x, \cdot) \geq w_{e,a}(t, x, \cdot)$$

for all $(t, x) \in [0, T] \times \mathbb{R}^d$. This, (7.11), (7.10), (7.12), and $(u_{e,a})_t \geq 0$ now show that

$$\begin{aligned}
&\sup_{|y| \leq \Lambda T} \sup_{x \in (c^*(e)e + K)T} w_{e,a}(T, x, \Upsilon_y \omega) \\
&\quad \leq \sup_{|y| \leq \Lambda T} \sup_{x \in (c^*(e)e + K)T} u_{e,a}((1 + M_*(T + \kappa_* + \tau_T)^{\alpha_2'}\eta_T)(T + \kappa_*) + \tau_T, x, \Upsilon_y \omega) + \eta_T \\
&\quad \leq \sup_{|y| \leq \Lambda T} \sup_{x \in (c^*(e)e + K)T} u_{e,a}((1 + \delta)T + \tau_T + 2\kappa_*, x, \Upsilon_y \omega) + \eta_T \\
&\quad \leq \sup_{|y| \leq \Lambda T} \sup_{x \in (c^*(e)e + K)T} u_{e,a}((1 + 2\delta)T, x, \Upsilon_y \omega) + \eta_T \\
&\quad \leq \sup_{|y| \leq \Lambda(1+2\delta)T} \sup_{x \in (c_a^*(e)e + K')(1+2\delta)T} u_{e,a}((1 + 2\delta)T, x, \Upsilon_y \omega) + \eta_T \\
&\quad \leq a + \varphi_a((1 + 2\delta)T) + T^{-\gamma}.
\end{aligned}$$

Hence

$$\lim_{t \to \infty} \sup_{|y| \leq \Lambda t} \sup_{x \in (c^*(e)e + K)t} w_{e,a}(t, x, \Upsilon_y \omega) \leq a$$

for all $a \in [0, \delta] \cap \mathbb{Q}$ (and the previously fixed (ω, K, Λ)). Since $w_{e,a}$ is non-decreasing in a, this extends to all $a \in [0, \delta]$. And since δ does not depend on Λ, we obtain (6.2) with $\lambda_{K,\omega,e}(a) = a + (1 - a)\chi_{(\delta,1]}(a)$, so the result follows. □

Proof of Theorem 1.8(i). This is now identical to the proof of Theorem 1.7 in Section 6, using Proposition 7.5 in place of Proposition 6.3. □

Appendix A. Proof of Lemma 2.2

If F_0 is the function defined before Lemma 2.1, then we have

$$\delta = \delta(M, \theta_1, m_1, \alpha_1) := \min_{u \in [1-2\theta_1/3, 1-\theta^*]} F_0(u) > 0. \tag{A.1}$$

We now claim that for each $L \geq 1$, there is $R_L := R_L(M, \theta_1, m_1, \alpha_1)$ and a smooth function $u_L : \mathbb{R}^d \to \mathbb{R}$ such that

$$(1-\theta^*)\chi_S \leq u_L \leq (1-\theta^*)\chi_{B_{R_L}(S)}, \tag{A.2}$$

$$|\Delta u_L| + |\nabla u_L|^2 \leq \frac{\delta}{L} \tag{A.3}$$

hold on $\mathbb{R}^d$, and for each $x \in \mathbb{R}^d$ with $u_L(x) < \frac{1-\theta^*}{3}$ we have

$$\Delta u_L(x) \geq 0. \tag{A.4}$$

Note that if we also had $1 - \frac{2}{3}\theta_1 \leq \frac{1-\theta^*}{3}$ (which is not the case), then (A.1) and (A.3) would show that for such u_L (with $L \geq 1$) we have

$$F_0\left(u_L(x)\right) \geq \delta \geq -\Delta u_L(x)$$

whenever $u_L(x) \in \left[\frac{1-\theta^*}{3}, 1-\theta^*\right]$, so this and (A.4) would yield

$$\Delta u_L + F_0(u_L) \geq 0$$

on $\mathbb{R}^d$. Hence the result would follow with $u_{0,S} := u_1$ and $R_0 := R_1$ because $F_0 \leq F$.

Let us now prove the claim. For any $a \in (0, \frac{1}{8})$, let $0 \not\equiv \xi_a : \mathbb{R}^d \to \mathbb{R}$ be a smooth, radially symmetric, non-negative function supported in $B_a(0)$, and define

$$\varphi_a := \frac{\zeta * \xi_a}{\|\zeta * \xi_a\|_{L^1}},$$

where

$$\zeta(x) := \begin{cases} \left(|x|^{2-d} - 2^{d-2}\right)_+ & \text{if } d \geq 3, \\ \ln_-(2|x|) & \text{if } d = 2. \end{cases}$$

Notice that ζ is sub-harmonic on $\mathbb{R}^d \backslash \{0\}$, and it is supported and integrable in $B_{1/2}(0)$. Therefore it is not hard to see that

$$\lim_{a \to \infty} \int_{B_a(0)} \varphi_a(x) dx = 0. \tag{A.5}$$

And since ξ_a is supported in $B_a(0)$, we also have

$$\Delta \varphi_a(x) = \int_{\mathbb{R}^d} \Delta\zeta(x-y)\xi_a(y) dy \geq 0$$

for all $x \in \mathbb{R}^d \setminus B_a(0)$. Thus, for any $R \geq 1$, the function

$$\varphi_{a,R}(x) := R^{-d}\varphi_a(R^{-1}x)$$

satisfies

$$\Delta\varphi_{a,R} \geq 0 \tag{A.6}$$

on $\mathbb{R}^d \setminus B_{aR}(0)$.

Next, for some $N \geq 1$ (to be determined later), take

$$u = u_{a,R,N,S} := (1-\theta^*)\chi_{B_{NR}(S)} * \varphi_{a,R}.$$

Direct computations then yield

$$|\nabla u(x)| \leq \int_{\mathbb{R}^d} |\nabla\varphi_{a,R}(y)|dy = R^{-1}\int_{B_1(0)} |\nabla\varphi_a(y)|dy,$$
$$|\Delta u(x)| \leq \int_{\mathbb{R}^d} |\Delta\varphi_{a,R}(y)|\,dy = R^{-2}\int_{B_1(0)} |\Delta\varphi_a(y)|\,dy$$

because φ_a is supported in $B_1(0)$. Hence (A.3) will hold with $u_L := u$ provided $R = R(a,\delta,L)$ is chosen large enough. And then $N \geq 1$ shows that (A.2) will also hold as long as we pick $R_L \geq (N+1)R$ (given this R, as well as some yet to be determined a and N).

It remains to show (A.4) when $u(x) < \frac{1-\theta^*}{3}$. If $d(x,S) \geq (N+a)R$, then (A.6) yields

$$\Delta u(x) = (1-\theta^*)\left(\chi_{B_{NR}(S)} * \Delta\varphi_{a,R}\right)(x) \geq 0,$$

so (A.4) holds. If $d(x,S) \leq (N+a)R$, let $z \in \overline{S} \cap \overline{B_{(N+a)R}(x)}$. Then

$$(\chi_{B_{NR}(S)} * \varphi_{a,R})(x) \geq \int_{B_{NR}(z)} \varphi_{a,R}(x-y)dy = \int_{B_N(z')} \varphi_a(y)dy,$$

with $z' := \frac{z-x}{R}$, so $|z'| \leq N+a$. From (A.5) and radial symmetry of φ_a, we get

$$\lim_{N\to\infty}\lim_{a\to 0}\int_{B_N((N+a)(1,0,\ldots,0))} \varphi_a(y)dy = \frac{1}{2},$$

so there are universal $a \in (0,\frac{1}{8})$ and $N \geq 1$ such that the last integral is at least $\frac{1}{3}$. Then

$$u(x) = (1-\theta^*)(\chi_{B_{NR}(S)} * \varphi_{a,R})(x) \geq \frac{1-\theta^*}{3}$$

holds when $d(x,S) \geq (N+a)R$, so (A.4) holds when $u(x) < \frac{1-\theta^*}{3}$ and the claim is proved.

Next, to prove the lemma, recall that $1 - \frac{2}{3}\theta_1 \in (\frac{1-\theta^*}{3}, 1-\theta^*)$ and take $u_{0,S} := \psi(u_L)$, for some $L \geq 1$ and some $\psi : [0, 1-\theta^*] \to [0, 1-\theta^*]$ satisfying the following:

(i) ψ is smooth and non-decreasing on $[0, 1-\theta^*]$;

(ii) $\psi(0) = 0$, $\psi(1-\theta^*) = 1-\theta^*$ and $\psi(\frac{1-\theta^*}{3}) = 1 - \frac{2}{3}\theta_1$,

(iii) $\psi'' = 0$ on $[0, \frac{1-\theta^*}{3}]$.

From (i,ii) and (A.2) we clearly have

$$(1-\theta^*)\chi_S \le u_{0,S} \le (1-\theta^*)\chi_{B_{R_L}(S)},$$

so it suffices to take $R_0 := R_L$ and verify (1.9).

When $u_L(x) < \frac{1-\theta^*}{3}$, (A.4) and (i,iii) yield

$$\Delta\psi(u_L(x)) + F_0(\psi(u_L(x))) \ge \psi'(u_L(x))\Delta u_L(x) + \psi''(u_L(x))|\nabla u_L(x)|^2 \ge 0.$$

When $u_L(x) \ge \frac{1-\theta^*}{3}$, (A.1) and (ii) yield $F_0(\psi(u_L(x))) \ge \delta$. Hence with

$$L = L(M,\theta_1,m_1,\alpha_1) := \max\{\|\psi'\|_\infty,\, \|\psi''\|_\infty\},$$

we get

$$\begin{aligned}\Delta\psi(u_L(x)) + F_0(\psi(u_L(x))) &\ge \psi'(u_L(x))\Delta u_L(x) + \psi''(u_L(x))|\nabla u_L(x)|^2 + \delta\\ &\ge \delta - L(|\Delta u_L(x)| + |\nabla u_L(x)|^2).\end{aligned}$$

So (1.9) follows from (A.3) and $F_0 \le F$, concluding the proof.

Appendix B. Proof of Lemma 2.9

Let us drop ω from the notation. Also recall that we extend the reactions by 0 to $u \notin [0,1]$.

Let us start with four estimates involving the reactions where $u_1(t,x) \notin (\theta^*, 1-\theta^*)$. From (2.1) and $\eta \le \frac{\theta^*}{2}$ we get $\theta^* + \eta \le \theta_1$. Hence **(H1)** shows that for $u \le \theta^*$ we have

$$f_1(\cdot,u) \equiv f_2(\cdot,u\pm\eta) \equiv 0 \tag{B.1}$$

on $\mathbb{R}^d$, while for $u \ge 1-\eta$ we have

$$f_1(\cdot,u) \ge 0 \equiv f_2(\cdot,u+\eta) \tag{B.2}$$

on $\mathbb{R}^d$. If $u_1(t,x) \in [1-\theta^*, 1-\eta)$ for some $(t,x) \in [t_0,\infty)\times\mathbb{R}^d$, then $(u_1)_t \ge 0$ shows that $u_1(t_0,x) < 1-\eta$, so **(H1)** and (2.7) yield

$$f_1(x,u_1(t,x)) = f_2(x,u_1(t,x)) \ge f_2(x,u_1(t,x)+\eta). \tag{B.3}$$

Finally, if $u_1(t,x) \in [1-\theta^*, 1]$ for some $(t,x) \in [t_0,\infty)\times\mathbb{R}^d$ and $u_1(t_0,x) < 1-\eta$, then **(H1)** and (2.7) again yield

$$f_1(x,u_1(t,x)) \le f_1(x,u_1(t,x)-\eta) = f_2(x,u_1(t,x)-\eta). \tag{B.4}$$

Denote $\tau_\pm(t) := (1 \pm M_*\eta)\,t + t_0$, so that $u_\pm(t,x) = u_1(\tau_\pm(t),x) \pm \eta$. If now $u_1(\tau_+(t),x) \notin (\theta^*, 1-\theta^*)$ for some $(t,x) \in (0,\infty)\times\mathbb{R}^d$, then (B.1), (B.2), and (B.3) yield

$$\begin{aligned}&[(u_+)_t - \Delta u_+ - f_2(\cdot,u_+)](t,x)\\ &\quad\ge (1+M_*\eta)(u_1)_t(\tau_+(t),x) - \Delta u_1(\tau_+(t),x) - f_1(x,u_1(\tau_+(t),x))\\ &\quad= M_*\eta(u_1)_t(\tau_+(t),x)\\ &\quad\ge 0.\end{aligned}$$

Similarly, if $u_1(\tau_-(t),x) \notin (\theta^*, 1-\theta^*)$ and $u_1(t_0,x) < 1-\eta$, then (B.1), (B.2), and (B.4) yield

$$[(u_-)_t - \Delta u_- - f_2(\cdot,u_-)](t,x) \le 0.$$

Let us now consider those $(t,x) \in (\kappa_*, \infty) \times \mathbb{R}^d$ for which $u_1(\tau_+(t),x) \in (\theta^*, 1-\theta^*)$. Then $(u_1)_t(\tau_+(t),x) \geq \mu_*$ by (2.6), so $|f_2(x,u_+(t,x)) - f_2(x,u_1(\tau_+(t),x))| \leq M\eta$ yields

$$\begin{aligned}
&[(u_+)_t - \Delta u_+ - f_2(\cdot,u_+)](t,x) \\
&\quad \geq (1+M_*\eta)(u_1)_t(\tau_+(t),x) - \Delta u_1(\tau_+(t),x) - f_2(x,u_1(\tau_+(t),x)) - M\eta \\
&\quad \geq (u_1)_t(\tau_+(t),x) - \Delta u_1(\tau_+(t),x) - f_1(x,u_1(\tau_+(t),x)) + M_*\eta\mu_* - M\eta \\
&\quad \geq 0,
\end{aligned}$$

where we again used (2.7) due to $u_1(t_0,x) < 1-\theta^* < 1-\eta$. Similarly if $u_1(\tau_-(t),x) \in (\theta^*, 1-\theta^*)$ for some $(t,x) \in (2\kappa_*, \infty) \times \mathbb{R}^d$ (so $\tau_-(t) > \kappa_*$ because $M_*\eta \leq \frac{1}{2}$), we obtain

$$[(u_-)_t - \Delta u_- - f_2(\cdot,u_-)](t,x) \leq 0.$$

This proves the claims about u_+ and u_-.

If now $u_2(0,\cdot) \leq u_1(t_0,\cdot) + \eta$ on $B_R(y)$, from $(u_1)_t \geq 0$ we also obtain $u_2(0,\cdot) \leq u_+(\kappa_*,\cdot)$ there. Since u_+ is a supersolution to (1.1) with f_2 in place of f on $(\kappa_*,\infty)\times B_R(y)$, Lemma 2.5 yields

$$u_2(t,y) \leq u_+(t+\kappa_*,y) + 2de^{2Mt - \sqrt{M/d}\,R}$$

for all $t \geq 0$. Hence,

$$u_2(t,y) \leq u_+(t+\kappa_*,y) + \frac{\theta^*}{2}$$

for all $t \in [0, T_{u_2}(y)]$ as long as

$$R \geq 2\sqrt{Md}\,T_{u_2}(y) + \sqrt{d/M}\ln\frac{4d}{\theta^*},$$

which will be guaranteed by taking $D_2 := 2\sqrt{Md}\ln\frac{4d}{\theta^*}$.

It follows from $\eta \leq \frac{\theta^*}{2}$ and the definition of $T_{u_2}(y)$ that,

$$u_1(\tau_+(T_{u_2}(y)+\kappa_*),y) \geq u_2(T_{u_2}(y),y) - \eta - \frac{\theta^*}{2} \geq 1 - 2\theta^*.$$

By Lemma 2.1, we have

$$u_1(\tau_+(T_{u_2}(y)+\kappa_*)+\kappa_0,y) \geq 1-\theta^*.$$

Therefore

$$T_{u_1}(y) \leq \tau_+(T_{u_2}(y)+\kappa_*) + \kappa_0 \leq (1+M_*\eta)T_{u_2}(y) + 2\kappa_* + \kappa_0 + t_0. \tag{B.5}$$

Appendix C. Proof of Lemma 2.10

We again have (B.1) and (B.2). For $(x,u) \in B_R(y) \times [1-\theta^*, 1-\eta]$, **(H3)** and (2.9) yield either

$$f_1(x,u) \geq f_1(x,u+\eta) + \alpha_3\eta^{m_3} \geq f_2(x,u+\eta)$$

(if f_1 satisfies **(H3)**) or

$$f_1(x,u) \geq f_2(x,u) - \alpha_3\eta^{m_3} \geq f_2(x,u+\eta)$$

(if f_2 does), replacing (B.3). Hence, as in the proof of Lemma 2.9 and with τ_+, u_+ from it, we have that if $u_1(\tau_+(t), x) \notin (\theta^*, 1-\theta^*)$ for some $(t,x) \in (0,\infty) \times B_R(y)$, then

$$[(u_+)_t - \Delta u_+ - f_2(\cdot, u_+)](t,x) \geq 0.$$

Let us now consider those $(t,x) \in (\kappa_*, \infty) \times B_R(y)$ for which $u_1(\tau_+(t), x) \in (\theta^*, 1-\theta^*)$. Then $(u_1)_t(\tau_+(t), x) \geq \mu_*$ by (2.6), so

$$f_2(x, u_+(t,x)) - f_1(x, u_1(\tau_+(t), x)) \leq \alpha_3 \eta^{m_3} + M\eta \leq (1+M)\eta$$

yields

$$\begin{aligned}
&[(u_+)_t - \Delta u_+ - f_2(\cdot, u_+)](t,x) \\
&\quad \geq (1 + M_*\eta)(u_1)_t(\tau_+(t), x) - (\Delta u_1)(\tau_+(t), x) - f_1(x, u_1(\tau_+(t), x)) - (1+M)\eta \\
&\quad \geq M_*\eta\mu_* - (1+M)\eta \\
&\quad = 0.
\end{aligned}$$

Hence u_+ is a supersolution to (1.1) with f_2 in place of f on $(\kappa_*, \infty) \times B_R(y)$. Since we also have $u_2(0,\cdot) \leq u_+(\kappa_*, \cdot)$ on $B_R(y)$ due to $(u_1)_t \geq 0$, (B.5) follows via Lemmas 2.5 and 2.1 as at the end of the proof of Lemma 2.9.

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