Objective rationality foundations for (dynamic) α-MEU*

Mira Frick Ryota Iijima Yves Le Yaouanq

November 2, 2021

Abstract

We show how incorporating Gilboa et al.'s (2010) notion of objective rationality into the α-MEU model of choice under ambiguity can overcome several challenges faced by the baseline model without objective rationality. The decision-maker (DM) has a *subjectively rational* preference $\succsim^\wedge$, which captures the complete ranking over acts the DM expresses when forced to make a choice; in addition, we endow the DM with a (possibly incomplete) *objectively rational* preference $\succsim^*$, which captures the rankings the DM deems uncontroversial. Under the *objectively founded* α-MEU model, $\succsim^\wedge$ has an α-MEU representation and $\succsim^*$ has a unanimity representation à la Bewley (2002), where both representations feature the same utility index and set of beliefs. While the axiomatic foundations of the baseline α-MEU model are still not fully understood, we provide a simple characterization of its objectively founded counterpart. Moreover, in contrast with the baseline model, the model parameters are uniquely identified. Finally, we provide axiomatic foundations for prior-by-prior Bayesian updating of the objectively founded α-MEU model, while we show that, for the baseline model, standard updating rules can be ill-defined.

Keywords: ambiguity, α-MEU, objective rationality, updating.

1 Introduction

A widely used model of choice under ambiguity is the α-maxmin expected utility (α-MEU) criterion, dating back to Hurwicz (1951). This criterion represents a decision-maker's (DM's) preference $\succsim^\wedge$ over (Anscombe-Aumann) acts f by considering the weighted average of each act's worst-case and best-case expected utility,

$$\alpha \min_{\mu \in P} \mathbb{E}_\mu[u(f)] + (1-\alpha) \max_{\mu \in P} \mathbb{E}_\mu[u(f)], \tag{1}$$

*First version: July 2020. Frick: Yale University (mira.frick@yale.edu, Department of Economics, Yale University, P.O. Box 208281, New Haven, CT 06520-8281, United States of America); Iijima: Yale University (ryota.iijima@yale.edu, Department of Economics, Yale University, P.O. Box 208281, New Haven, CT 06520-8281, United States of America); Le Yaouanq: CREST, Ecole Polytechnique, IP Paris, LMU Munich and CEPR (corresponding author: yves.le-yaouanq@polytechnique.edu, CREST, 5 Avenue Henry Le Chatelier, TSA 96642, 91764 Palaiseau Cedex, FRANCE). This work was supported by the National Science Foundation [grant number SES-1824324] and the *Deutsche Forschungsgemeinschaft* [through CRC TRR 190].

according to some weight $\alpha \in [0,1]$, closed and convex set P of beliefs over states, and nonconstant and affine utility u over outcomes. Unlike Gilboa and Schmeidler's (1989) maxmin expected utility criterion (i.e., the special case when $\alpha = 1$), the general α-MEU model does not assume that the DM is uncertainty-averse (Schmeidler, 1989). Instead, in line with various experimental evidence (see the survey by Trautmann and van de Kuilen, 2015), (1) allows the DM to display a mix of ambiguity-averse and ambiguity-seeking tendencies, and the weight α and set of beliefs P are often interpreted as simple parameterizations of the DM's ambiguity attitude and perception of ambiguity, respectively. This has contributed to the model's popularity in applied work, which has employed α-MEU representations in both static and dynamic settings.[1]

Despite its popularity, the foundations of the α-MEU model are still not fully understood. In this paper, we point to several challenges that arise in the standard domain of preferences over acts, and show how incorporating the notion of objective rationality (Gilboa et al., 2010, henceforth, GMMS) into the model can overcome these challenges.

In Section 3, we begin by highlighting three main challenges in the standard domain. First, there is no known fully general axiomatic characterization of α-MEU in terms of the DM's preference $\succsim^\wedge$ over acts (Section 1.1 discusses existing work). The remaining two challenges are more fundamental. Second, as is well-known, the preference $\succsim^\wedge$ does not uniquely identify α and P, complicating the interpretation of these parameters as capturing the DM's ambiguity attitude and perception: In Proposition 1, we fully characterize the extent of multiplicity (building on Siniscalchi, 2006). Third, as we show in Example 1, the lack of identification of the model parameters creates the following problem for dynamic extensions of α-MEU: Common belief-updating rules, such as prior-by-prior Bayesian updating of all beliefs in P, are ill-defined at the level of preferences, as different representations of the same ex-ante preference $\succsim^\wedge$ may give rise to different updated preferences.

Motivated by these challenges, we consider the following *objectively founded α-MEU* model. We interpret $\succsim^\wedge$ as the DM's *subjectively rational* preference, which captures the complete ranking the DM expresses when forced to choose between any two acts. In addition, as in GMMS, we endow the DM with a (possibly incomplete) *objectively rational* preference $\succsim^*$, which models the rankings that appear uncontroversial to the DM. We then consider a joint representation of $\succsim^\wedge$ and $\succsim^*$, where for some utility u, set of beliefs P, and weight α:

1. The subjectively rational preference $\succsim^\wedge$ admits an α-MEU representation based on u, P, and α.

[1] In static settings, see, e.g., Cherbonnier and Gollier (2015); Chen et al. (2007); Bossaerts et al. (2010); Ahn et al. (2014); in dynamic settings, see, e.g., Saghafian (2018); Georgalos (2019); Beissner et al. (2020); Hedlund et al. (2020).

2. The objectively rational preference $\succsim^*$ is represented by u and P in the sense of Bewley (2002); that is, act f is deemed uncontroversially better than g if and only if the expected utility under u of f dominates the expected utility of g for every belief in P.

Thus, under this model, the DM employs the α-MEU criterion as a forced-choice completion of Bewley's (2002) unanimity criterion, where both criteria are based on the same set of beliefs P and the same utility u over outcomes.

In Section 4, we address the aforementioned challenges using the objectively founded α-MEU model. We first show that this model admits a simple axiomatic characterization (Theorem 1). We impose Bewley's (2002) axioms on the objectively rational preference; that is, $\succsim^*$ satisfies all subjective expected utility axioms, except that completeness is only assumed for the ranking over constant acts. The subjectively rational preference is required to be invariant biseparable (Ghirardato et al., 2004); that is, $\succsim^\wedge$ satisfies all subjective expected utility axioms, except that independence is only imposed for mixtures with constant acts. The final and key axiom, security-potential dominance, disciplines the completion rule from $\succsim^*$ to $\succsim^\wedge$: We require the DM to subjectively prefer act f to act g whenever f is both "more secure" than g and has "more potential" than g, where security and potential are defined in terms of the objective ranking against certain prospects.

Second, in contrast with the baseline model without objective rationality, the parameters α and P in Theorem 1 are uniquely identified. Thus, the interpretation of α and P as the DM's ambiguity attitude and perception is behaviorally founded, making it possible to conduct comparative statics of these parameters (Section 4.2).

Third, in contrast with Example 1, we show that prior-by-prior Bayesian updating of the objectively founded α-MEU model admits well-defined preference foundations. Suppose the DM's ex-ante subjective and objective preferences have an objectively founded α-MEU representation (u, P, α). Upon learning that the state of the world is contained in some event E, the DM forms a conditional subjective preference $\succsim_E^\wedge$. Theorem 2 characterizes when $\succsim_E^\wedge$ admits an α-MEU representation whose utility is u and whose set of beliefs P^E is derived from the unique ex-ante belief set P by prior-by-prior Bayesian updating. The key axiom imposes an intertemporal analog of security-potential dominance on the relationship between the ex-ante objective preference and the conditional subjective preference.

The contribution of Section 4 is not primarily technical (the results admit relatively simple proofs), but rather, to illustrate the methodological value of the objective rationality framework in shedding light on the α-MEU model. As we discuss in Section 5, our approach is not restricted to α-MEU. Indeed, we show that security-potential dominance characterizes linear completion rules for a broader class of incomplete preferences $\succsim^*$ beyond Bewley preferences. Just as for α-MEU, this makes it possible to provide foundations and characterize belief updating for several other representations that are difficult to analyze based on the subjectively rational preference $\succsim^{\wedge}$ alone.

1.1 Related literature

GMMS propose the objective and subjective rationality approach, and characterize when $\succsim^*$ and $\succsim^{\wedge}$ admit Bewley and maxmin expected utility representations with a common set of beliefs P and utility u. We impose the same axioms as GMMS on $\succsim^*$ and $\succsim^{\wedge}$ individually, but relax their main axiom, caution, that concerns the relationship between $\succsim^*$ and $\succsim^{\wedge}$ (see Section 4.1).[2] Several papers extend the results in GMMS in different directions. Kopylov (2009) characterizes when $\succsim^{\wedge}$ admits an ε-contamination representation. Cerreia-Vioglio (2016) allows $\succsim^{\wedge}$ to be a general uncertainty-averse preference (Cerreia-Vioglio et al., 2011). Faro and Lefort (2019) characterize prior-by-prior Bayesian updating under the Bewley-maxmin model in GMMS.[3] Grant et al. (2021) use a condition that is equivalent to security-potential dominance (along with weaker assumptions on $\succsim^{\wedge}$) to characterize a representation in which the subjectively rational model—ordinal Hurwicz expected utility—is more general than α-MEU; they do not characterize α-MEU and do not study belief updating.[4] We note that most aforementioned papers consider subjectively rational models that have well-understood foundations based on the primitive $\succsim^{\wedge}$ alone, and the focus is on understanding the consistency of the objective and subjective models. In contrast, in the present paper, the subjectively rational model—α-MEU—is not well-understood in isolation, and incorporating objective rationality plays a key role in enabling its axiomatic characterization, identification, and dynamic extension.

Several papers characterize α-MEU representations in terms of the subjectively rational preference $\succsim^{\wedge}$ alone, but impose specific assumptions on the structure of the belief set P. Kopylov (2003) considers the case in which P consists of beliefs that are derived from a particular class of subjectively risky acts. Ghirardato et al. (2004) require P to coincide with the Bewley set of the largest independent subrelation of $\succsim^{\wedge}$; for finite state spaces, Eichberger et al. (2011) show that this case reduces to maxmin or maxmax expected utility (see Remark 2). Chateauneuf et al. (2007) consider a neo-additive capacity model that evaluates each act according to a convex combination of the least favorable prize, most favorable

[2]GMMS also introduce a slight strengthening of caution, termed default to certainty, under which C-independence can be dropped from the assumptions on $\succsim^{\wedge}$.

[3]See also Bastianello et al. (2020) and Ceron and Vergopoulos (2020) for the connection with dynamic consistency.

[4]Relatedly, Nehring (2009) studies the compatibility of an incomplete preference over events and a complete preference over Savage acts. He considers the case where the latter preference is invariant biseparable, but does not characterize the special case of α-MEU.

prize, and the expected utility with respect to a fixed probability. Gul and Pesendorfer (2015) study the case in which P is the set of measures that are consistent with some benchmark belief μ over a given sigma-algebra of events. Klibanoff et al. (2021) consider a product state space $S = Y^\infty$ and assume that P consists of i.i.d. distributions.

In more recent work, Hartmann (2021) characterizes the α-MEU model with a general belief set P, based on axioms on $\succsim^\wedge$ that are indexed by an exogenously fixed $\alpha \neq \frac{1}{2}$. It is still unknown how to obtain a characterization that does not directly specify α.

Finally, Hill (2019) enriches the standard domain in a different direction from us, by considering a preference over acts f that map each state s to a *set* of objective lotteries $f(s)$. He characterizes an α-MEU representation $\alpha \min_{\mu \in P} \mathbb{E}_\mu[w(f)] + (1-\alpha) \max_{\mu \in P} \mathbb{E}_\mu[w(f)]$, where the utility $w(f(s)) = \alpha \min_{p \in f(s)} \mathbb{E}_p[u] + (1-\alpha) \max_{p \in f(s)} \mathbb{E}_p[u]$ over sets of lotteries also takes an α-MEU form. Relatedly, Jaffray (1994) and Olszewski (2007) directly consider preferences over sets of objective lotteries and characterize α-MEU representations for such preferences.

2 Model

2.1 Setup

Let Z be a set of prizes and let $\Delta(Z)$ denote the space of probability measures with finite support over Z. We refer to typical elements $p, q \in \Delta(Z)$ as lotteries. Let S be a finite set of states. An *(Anscombe-Aumann) act* is a mapping $f : S \to \Delta(Z)$. Let $\mathcal{F}$ be the space of all acts, with typical elements f, g, h. For any $f, g \in \mathcal{F}$ and $\alpha \in [0,1]$, define the mixture $\alpha f + (1-\alpha) g \in \mathcal{F}$ to be the act that in each state $s \in S$ yields lottery $\alpha f(s) + (1-\alpha) g(s) \in \Delta(Z)$. As usual, we identify each lottery $p \in \Delta(Z)$ with the constant act that yields lottery p in all states $s \in S$.

Let $\Delta(S)$ denote the set of all probability measures over S, which we embed in $\mathbb{R}^S$ and endow with the Euclidean topology. We refer to typical elements $\mu, \nu \in \Delta(S)$ as beliefs. Given any act $f \in \mathcal{F}$ and function $u : \Delta(Z) \to \mathbb{R}$, let $u(f)$ denote the element of $\mathbb{R}^S$ defined by $u(f)(s) = u(f(s))$ for all $s \in S$, and let $\mathbb{E}_\mu[u(f)] := \mu \cdot u(f)$. Given any functions $u, v : \Delta(Z) \to \mathbb{R}$, we write $u \approx v$ if u is a positive affine transformation of v.

We follow GMMS in endowing the DM with two binary relations, $\succsim^\wedge$ and $\succsim^*$, over $\mathcal{F}$. Relation $\succsim^\wedge$ is the DM's ***subjectively rational*** (for short, ***subjective***) preference, which models the rankings the DM expresses when forced to choose between any two acts and, as such, is complete. Relation $\succsim^*$ is the DM's ***objectively rational*** (for short, ***objective***) preference, which captures the rankings that appear uncontroversial to the DM and, as such, may be incomplete. As usual, we write $\succ$ (resp., $\sim$) for the asymmetric (resp., symmetric) part of a generic binary relation $\succsim$.

One possible interpretation of $\succsim^*$ is that it describes choices that are made with "confidence." For example, Kopylov (2009) interprets $\succsim^*$ as capturing choices that the DM would not want to revise at an interim stage (prior to the resolution of any uncertainty). As such, $\succsim^*$ might in principle be elicited by charging a small monetary cost for the option to revise choices, as in Danan and Ziegelmeyer (2006).[5]

2.2 Representation

We are interested in the following joint representation of $\succsim^*$ and $\succsim^\wedge$:

Definition 1. *An* ***objectively founded*** α***-MEU representation*** *of* $(\succsim^\wedge, \succsim^*)$ *consists of a nonconstant affine utility* $u : \Delta(Z) \to \mathbb{R}$*, a nonempty, closed and convex set of beliefs* $P \subseteq \Delta(S)$*, and a weight* $\alpha \in [0, 1]$ *such that*

(i.) (u, P, α) *is an* α***-MEU representation*** *of* $\succsim^\wedge$*; that is, for all* $f, g \in \mathcal{F}$*,*

$$f \succsim^\wedge g \Leftrightarrow \alpha \min_{\mu \in P} \mathbb{E}_\mu[u(f)] + (1-\alpha) \max_{\mu \in P} \mathbb{E}_\mu[u(f)] \geq \alpha \min_{\mu \in P} \mathbb{E}_\mu[u(g)] + (1-\alpha) \max_{\mu \in P} \mathbb{E}_\mu[u(g)]. \tag{2}$$

(ii.) (u, P) *is a* ***Bewley representation*** *of* $\succsim^*$*; that is, for all* $f, g \in \mathcal{F}$*,*

$$f \succsim^* g \iff \mathbb{E}_\mu[u(f)] \geq \mathbb{E}_\mu[u(g)] \; \forall \mu \in P. \tag{3}$$

The first condition says that when forced to choose between any two acts, the DM employs the α-MEU criterion based on utility u, set of beliefs P, and weight α. The second condition enriches the basic α-MEU model by requiring this choice procedure to be *objectively founded*, in the sense that the *same* set of beliefs P and utility u also represent the rankings the DM considers uncontroversial: Specifically, the DM deems act f uncontroversially better than act g if and only if the expected utility under u of f dominates the expected utility of g for every belief in P. Thus, the objectively founded α-MEU model captures a DM who uses the α-MEU criterion as a completion of an underlying unanimity criterion à la Bewley (2002).

[5] See also Sautua (2017) and Cettolin and Riedl (2019) for more recent experimental elicitations of incomplete preferences.

3 α-MEU without objective rationality

To motivate studying objectively founded α-MEU representations, we point to three challenges for the baseline α-MEU model without objective rationality. First, as discussed in the introduction, there is so far no fully general axiomatization of α-MEU representations in terms of the subjective preference $\succsim^\wedge$ alone. The remaining two challenges are more fundamental.

The second challenge is that the belief set P and weight α in representation (2) are not uniquely pinned down by $\succsim^\wedge$, complicating the common interpretation of these parameters as capturing the DM's ambiguity perception and attitude, respectively. The following result characterizes which pairs (P, α) give rise to the same preference, extending a result in Siniscalchi (2006).[6] Given any nonempty, closed and convex sets $P, Q \subseteq \Delta(S)$ and any $\gamma \geq 1$, we call Q the ***γ-expansion*** of P if

$$Q = \gamma P + (1-\gamma)P := \{\gamma\nu + (1-\gamma)\nu' : \nu, \nu' \in P\}. \tag{4}$$

Observe that (4) implies $Q \supseteq P$, with $Q = P$ if $\gamma = 1$.[7]

Proposition 1. *Suppose (u_1, P_1, α_1) and (u_2, P_2, α_2) are α-MEU representations of $\succsim_1^\wedge$ and $\succsim_2^\wedge$, respectively, such that $\alpha_i \neq 1/2$ and P_i is not a singleton for $i = 1, 2$.*[8] *Suppose $\alpha_1 \leq \alpha_2$.*

Then $\succsim_1^\wedge = \succsim_2^\wedge$ if and only if $u_1 \approx u_2$ and one of the following two statements holds:

(i). $\alpha_1, \alpha_2 > 1/2$ and P_1 is the γ-expansion of P_2 for $\gamma = \dfrac{\alpha_1 + \alpha_2 - 1}{2\alpha_1 - 1}$;

(ii). $\alpha_1, \alpha_2 < 1/2$ and P_2 is the γ-expansion of P_1 for $\gamma = \dfrac{1 - \alpha_1 - \alpha_2}{1 - 2\alpha_2}$.

Proposition 1 shows that while the DM's subjective preference pins down whether the weight α is greater or less than $1/2$, a range of different weights can be used to represent the same preference $\succsim^\wedge$. For each such weight α, the corresponding set of beliefs P is uniquely determined. In case 1, weight α_1 suggests a less extreme attitude towards ambiguity than α_2

[6] Siniscalchi (2006) (Proposition 6.1) considers the special case when $\succsim^\wedge$ admits a maxmin expected utility representation whose belief set P is bounded away from $\Delta(S)$ and shows that there is a continuum of α-MEU representations of $\succsim^\wedge$ with $\alpha < 1$ and belief sets $Q \supsetneq P$. His proof uses a different (but equivalent) definition of γ-expansion.

[7] Note that while the set $\gamma P + (1-\gamma)P \subseteq \mathbb{R}^S$ need not in general be a subset of the simplex $\Delta(S)$, condition (4) implicitly imposes this as $Q \subseteq \Delta(S)$.

[8] Rogers and Ryan (2012) cover the case where one belief set $P_i = \{\mu\}$ is a singleton ($\succsim_i$ is subjective expected utility): In this case, $\succsim_i^\wedge = \succsim_j^\wedge$ iff $u_i \approx u_j$ and either (i) $P_j = \{\mu\}$ or (ii) $\alpha_j = 1/2$ and P_j is centrally symmetric around μ. Appendix B considers the case where P_i, P_j are not singletons and $\alpha_i = 1/2$ (which implies $\alpha_j = 1/2$). As we show, this case admits a greater multiplicity of belief sets than in Proposition 1.

(as $\alpha_2 \geq \alpha_1$ is closer to $\{0,1\}$ than α_1), but the corresponding set of priors P_1 is larger than P_2, suggesting greater perceived ambiguity. In case 2, the opposite relationship obtains.

Third, we highlight that the non-uniqueness of the set of priors P poses a challenge for defining belief-updating under α-MEU. To illustrate, we focus on prior-by-prior Bayesian updating, which has been used in several applications.[9] Consider a DM whose ex-ante preference $\succsim^\wedge$ admits an α-MEU representation (u, P, α). Suppose the DM is informed that the true state of nature is contained in some event $E \subseteq S$ and based on this information forms a conditional preference $\succsim^\wedge_E$. Consider deriving $\succsim^\wedge_E$ from $\succsim^\wedge$ by prior-by-prior Bayesian updating of all beliefs in P. That is, assuming that $\mu(E) > 0$ for all $\mu \in P$, let $\succsim^\wedge_E$ be induced by the α-MEU representation (u, P^E, α) whose conditional set of beliefs is

$$P^E := \{\mu^E : \mu \in P\}, \text{ where } \mu^E(F) := \frac{\mu(E \cap F)}{\mu(E)} \quad \forall F \subseteq S. \tag{5}$$

The following example shows that this approach is not well-defined at the level of preferences. Indeed, if the ex-ante preference $\succsim^\wedge$ admits multiple α-MEU representations, prior-by-prior updating can induce a different conditional preference $\succsim^\wedge_E$ depending on which ex-ante representation is used:

Example 1. Suppose $S = \{1,2,3\}$. Fix any nonconstant affine utility u, and consider the two α-MEU representations (u, P_i, α_i), where

$$\alpha_1 = \frac{3}{4}, \qquad P_1 = \text{co}\left\{(\frac{5}{6}, \frac{1}{12}, \frac{1}{12}), (\frac{1}{6}, \frac{5}{12}, \frac{5}{12})\right\},$$

$$\alpha_2 = 1, \qquad P_2 = \text{co}\left\{(\frac{2}{3}, \frac{1}{6}, \frac{1}{6}), (\frac{1}{3}, \frac{1}{3}, \frac{1}{3})\right\}.$$

Let $\gamma = \dfrac{\alpha_1 + \alpha_2 - 1}{2\alpha_1 - 1} = 3/2$, and note that P_1 is the γ-expansion of P_2. Thus, by Proposition 1, the two representations represent the same ex-ante preference $\succsim^\wedge$. Now, consider the event $E = \{1,2\}$. The prior-by-prior Bayesian updates of P_1 and P_2 are

$$P_1^E = \text{co}\left\{(\frac{10}{11}, \frac{1}{11}, 0), (\frac{2}{7}, \frac{5}{7}, 0)\right\}, \quad P_2^E = \text{co}\left\{(\frac{4}{5}, \frac{1}{5}, 0), (\frac{1}{2}, \frac{1}{2}, 0)\right\}.$$

However, the γ-expansion of P_2^E is $\text{co}\left\{(\frac{19}{20}, \frac{1}{20}, 0), (\frac{7}{20}, \frac{13}{20}, 0)\right\} \neq P_1^E$. Hence, by Proposition 1, the conditional preferences represented by (u, P_1^E, α_1) and (u, P_2^E, α_2) are not the same. ▲

An implication of Example 1 is that Pires's (2002) coherency axiom, which characterizes

[9] See the references in Footnote 1. Analogous issues arise for other updating rules, such as maximum likelihood updating.

prior-by-prior updating under maxmin expected utility and some extensions, need not hold for prior-by-prior updating under α-MEU. Indeed, given that the conditional preference induced by prior-by-prior updating depends on the non-unique choice of ex-ante representation under α-MEU, this rule does not admit any axiomatic foundation in terms of the subjective preference alone.

4 Objectively founded α-MEU representations

We now show how incorporating objective rationality into the α-MEU model makes it possible to overcome the challenges discussed in the previous section.

4.1 Characterization and uniqueness

This section provides an axiomatic characterization of objectively founded α-MEU representations and shows that the pair $(\succsim^\wedge, \succsim^*)$ uniquely determines P and α. Our characterization imposes the same five axioms as GMMS on $\succsim^\wedge$ and $\succsim^*$ individually, but differs from GMMS in what we assume about the relationship between $\succsim^*$ and $\succsim^\wedge$.

First, we impose two basic rationality conditions, along with continuity and nondegeneracy, on both $\succsim^\wedge$ and $\succsim^*$. We state this axiom for a generic binary relation $\succsim$ on $\mathcal{F}$:

Axiom 1 (Basic conditions)**.**

1. *Transitivity: If $f, g, h \in \mathcal{F}$, $f \succsim g$, and $g \succsim h$, then $f \succsim h$.*
2. *Monotonicity: If $f, g \in \mathcal{F}$ and $f(s) \succsim g(s)$ for all $s \in S$, then $f \succsim g$.*
3. *Mixture continuity: If $f, g, h \in \mathcal{F}$, then the sets $\{\lambda \in [0,1] : \lambda f + (1-\lambda)g \succsim h\}$ and $\{\lambda \in [0,1] : h \succsim \lambda f + (1-\lambda)g\}$ are closed in $[0,1]$.*
4. *Non-degeneracy: $f \succ g$ for some $f, g \in \mathcal{F}$.*

The following two axioms are specific to the objective preference $\succsim^*$:

Axiom 2 (C-Completeness)**.** *If $p, q \in \Delta(Z)$, then either $p \succsim^* q$ or $q \succsim^* p$.*

Axiom 3 (Independence)**.** *If $f, g, h \in \mathcal{F}$ and $\lambda \in (0,1]$, then*

$$f \succsim^* g \Longleftrightarrow \lambda f + (1-\lambda)h \succsim^* \lambda g + (1-\lambda)h.$$

A binary relation on $\mathcal{F}$ satisfying Axioms 1–3 is called a ***Bewley preference***. Such preferences satisfy all subjective expected utility axioms, except that completeness is only imposed on the ranking over constant acts. C-completeness assumes that any difficulties the DM might have in determining an uncontroversial ranking are due to uncertainty, rather than incompleteness of tastes over certain outcomes. As is well-known (Bewley, 2002, GMMS), $\succsim^*$ is a Bewley preference if and only if it admits a Bewley representation (3).

The next two axioms are specific to the subjective preference $\succsim^\wedge$:

Axiom 4 (Completeness)**.** *If $f, g \in \mathcal{F}$, then either $f \succsim^\wedge g$ or $g \succsim^\wedge f$.*

Axiom 5 (C-Independence)**.** *If $f, g \in \mathcal{F}$, $p \in \Delta(Z)$, and $\alpha \in (0, 1]$, then*

$$f \succsim^\wedge g \Longleftrightarrow \alpha f + (1-\alpha)p \succsim^\wedge \alpha g + (1-\alpha)p.$$

A binary relation on $\mathcal{F}$ satisfying Axioms 1, 4, and 5 is called an ***invariant biseparable preference***. Unlike Bewley preferences, such preferences satisfy completeness, but differ from subjective expected utility in that independence is only assumed for mixtures with constant acts. We refer the reader to GMMS for a rationale for imposing C-independence on $\succsim^\wedge$, and to Ghirardato et al. (2004), Amarante (2009), and Chandrasekher et al. (2021) for representations of invariant biseparable preferences.

Our key axiom disciplines the completion rule from $\succsim^*$ to $\succsim^\wedge$. Consider any $f, g \in \mathcal{F}$. As in Kopylov (2009), we say that f ***is more secure*** than g if for all $p \in \Delta(Z)$,

$$g \succsim^* p \Longrightarrow f \succsim^* p.$$

We say that f ***has more potential*** than g if for all $p \in \Delta(Z)$,

$$p \not\succsim^* g \Longrightarrow p \not\succsim^* f.$$ [10]

Axiom 6 (Security-potential dominance)**.** *If $f, g \in \mathcal{F}$ and f is both more secure than g and has more potential than g, then $f \succsim^\wedge g$.*

Axiom 6 captures that in choosing between two uncertain acts f and g, the DM might compare how f and g rank objectively against prospects that are certain. Two dimensions might matter to the DM in comparing an uncertain act f with a constant act p. On the one hand, an ambiguity-averse DM might favor the "security" of certain prospects, and thus seek

[10] Kopylov (2009) introduces the notion of more security to define a strengthening of uncertainty aversion he calls cautious independence, and uses this to characterize the ε-contamination model. He uses the notion of more potential to characterize its uncertainty-seeking counterpart.

the assurance that f uncontroversially dominates p. On the other hand, an ambiguity-seeking DM might be drawn to the "potential" of uncertain prospects, and thus be content as long as p does not uncontroversially dominate f. If f is more secure (resp., has more potential) than g, then f performs at least as well as g along the first (resp., second) dimension. Security-potential dominance allows for the possibility that both dimensions are relevant to the DM, reflecting the idea that the α-MEU criterion accommodates a mix of ambiguity-averse and ambiguity-seeking tendencies. Thus, Axiom 6 only requires the DM to choose f over g if f is both more secure and has more potential than g.

Given transitivity of $\succsim^*$, note that if $f \succsim^* g$, then f is more secure than g and has more potential than g; thus, security-potential dominance implies the following consistency condition imposed by GMMS. This condition (together with Axiom 4) captures that the subjectively rational preference is a completion of the objectively rational preference:

Consistency. *If $f, g \in \mathcal{F}$ and $f \succsim^* g$, then $f \succsim^\wedge g$.*

By contrast, Axiom 6 does not entail the main substantive assumption in GMMS. This assumption requires the DM's completion rule to be cautious, in the sense that unless a general act f is uncontroversially superior to a constant act p, the DM prefers to choose the constant act:

Caution. *If $f \in \mathcal{F}$, $p \in \Delta(Z)$ and $f \not\succsim^* p$, then $p \succsim^\wedge f$.*

While GMMS show that caution and consistency characterize when the invariant biseparable preference $\succsim^\wedge$ is a maxmin expected utility completion of the Bewley preference $\succsim^*$, the following result shows that security-potential dominance characterizes α-MEU completions. Moreover, in contrast with Proposition 1, for the objectively founded α-MEU model, the parameters P and α are uniquely identified.

Theorem 1. *The following are equivalent:*

(i). $\succsim^*$ is a Bewley preference, $\succsim^\wedge$ is an invariant biseparable preference, and the pair $(\succsim^\wedge, \succsim^*)$ jointly satisfies security-potential dominance.

(ii). $(\succsim^\wedge, \succsim^*)$ admits an objectively founded α-MEU representation (u, P, α).

Moreover, in this case, u is unique up to positive affine transformation, P is unique, and α is unique if $\succsim^*$ is not complete.

To construct the representation, we first observe that, given a Bewley representation (u, P) of $\succsim^*$, security-potential dominance means that, for any f and g,

$$\left[\min_{\mu \in P} \mathbb{E}_\mu[u(f)] \geq \min_{\mu \in P} \mathbb{E}_\mu[u(g)] \text{ and } \max_{\mu \in P} \mathbb{E}_\mu[u(f)] \geq \max_{\mu \in P} \mathbb{E}_\mu[u(g)]\right] \Longrightarrow f \succsim^\wedge g. \quad (6)$$

The main step of the proof is to show that (6), together with the assumption that $\succsim^\wedge$ is invariant biseparable, guarantees that $\succsim^\wedge$ can be represented by a linear aggregation of the min and max functionals. To this end, Lemma A.3 (Appendix A.1) establishes a linear aggregation result for constant-linear functionals that also applies to the more general setting in Section 5.

Formally, a functional $I : \mathbb{R}^S \to \mathbb{R}$ is called ***constant-linear*** if $I(\phi + \underline{a}) = I(\phi) + a$ and $I(a\phi) = aI(\phi)$ for any $\phi \in \mathbb{R}^S, a \in \mathbb{R}$, where $\underline{a}$ denotes the constant vector $(a, \cdots, a) \in \mathbb{R}^S$. Lemma A.3 shows that, for any monotonic and constant-linear functionals I, I', I'' with $I' \leq I''$, functional I can be written as $I = \alpha I' + (1 - \alpha)I''$ for some $\alpha \in [0, 1]$, if and only if, for all $\phi, \psi \in \mathbb{R}^S$,

$$[I'(\phi) \geq I'(\psi) \text{ and } I''(\phi) \geq I''(\psi)] \Longrightarrow I(\phi) \geq I(\psi).$$

To apply Lemma A.3 to the current setting, we invoke the fact (Ghirardato et al., 2004) that preference $\succsim^\wedge$ is invariant biseparable if and only if it is represented by $I \circ u$ for some affine utility u and unique monotonic and constant-linear functional I. Given (6), Lemma A.3 then applies to I and the functionals I' and I'' defined by $I'(\phi) = \min_{\mu \in P} \mu \cdot \phi$ and $I''(\phi) = \max_{\mu \in P} \mu \cdot \phi$.

The uniqueness of u and P in Theorem 1 follows from the uniqueness of Bewley representations. Given that P is unique, $\succsim^\wedge$ pins down α, unless $P = \{\mu\}$ is a singleton (i.e., $\succsim^*$ is complete). In the latter case, $\succsim^* = \succsim^\wedge$ is the subjective expected utility preference corresponding to belief μ, and α can be chosen arbitrarily.

Remark 1. Identifying α and P does not require full observation of $\succsim^*$: Suppose that in addition to $\succsim^\wedge$, we only observe the restriction of $\succsim^*$ to binary bets, i.e., to acts that yield at most two different outcomes. This is enough to identify $\min_{\mu \in P} \mu(E)$ and $\max_{\mu \in P} \mu(E)$ for all events E, which in turn allows one to identify α from $\succsim^\wedge$ (unless $\succsim^\wedge$ is subjective expected utility). As long as $\alpha \neq \frac{1}{2}$, Proposition 1 then implies that P is identified.[11] ▲

Remark 2. For any invariant biseparable preference $\succsim^\wedge$, Ghirardato et al. (2004) define the ***unambiguous preference*** $\succsim^u$ as the largest independent subrelation of $\succsim^\wedge$; equivalently, $f \succsim^u g$ means that $\lambda f + (1 - \lambda)h \succsim^\wedge \lambda g + (1 - \lambda)h$ for all $\lambda \in (0, 1]$ and $h \in \mathcal{F}$. They show that $\succsim^u$ admits a Bewley representation (u, C) for some set of beliefs C.[12] Under the assumptions of Theorem 1, we have that $f \succeq^* g$ implies $f \succeq^u g$, or equivalently $C \subseteq P$.

[11] We thank an anonymous referee for this suggestion. Identifying P is not in general possible if $\alpha = \frac{1}{2}$.

[12] They also use the derived relation $\succsim^u$ to characterize the special case of α-MEU where the belief set P equals the induced C: Their Proposition 19 shows that $\succsim^\wedge$ admits such an α-MEU representation if and only if it is invariant biseparable and $C^u(f) = C^u(g)$ implies $f \sim^\wedge g$, where $C^u(f) := \{p \in \Delta(Z) : \forall q \in \Delta(Z), [q \succsim^u f \implies q \succsim^u p] \,\&\, [f \succsim^u q \implies p \succsim^u q]\}$ is the set of unambiguous certainty equivalents of f.

However, the opposite implication is typically not true. Thus, the unambiguous ranking $f \succsim^u g$ is necessary but not sufficient for the DM to deem f uncontroversially superior to g. As a result, Theorem 1 avoids the existence problem for finite-state α-MEU representations highlighted by Eichberger et al. (2011): While requiring P to be the Bewley set of $\succsim^u$ implies that either $\alpha = 1$ (maxmin), $\alpha = 0$ (maxmax), or P is a singleton (subjective expected utility), Theorem 1 imposes no such restrictions. ▲

Finally, strengthening security-potential dominance as follows characterizes the extreme cases of objectively founded maxmin ($\alpha = 1$) and maxmax ($\alpha = 0$) expected utility:

Axiom 7 (Security dominance)**.** *If $f, g \in \mathcal{F}$ and f is more secure than g, then $f \succsim^\wedge g$.*

Axiom 8 (Potential dominance)**.** *If $f, g \in \mathcal{F}$ and f has more potential than g, then $f \succsim^\wedge g$.*

Corollary 1. *The following are equivalent:*

(i). $\succsim^*$ is a Bewley preference, $\succsim^\wedge$ is an invariant biseparable preference, and the pair $(\succsim^\wedge, \succsim^*)$ jointly satisfies security (resp., potential) dominance.

(ii). $(\succsim^*, \succsim^\wedge)$ admits an objectively founded α-MEU representation (u, P, α) with $\alpha = 1$ (resp., $\alpha = 0$).

Corollary 1 provides an alternative to GMMS's foundation for maxmin expected utility completions. In particular, imposing security dominance on the completion rule is equivalent (given Axioms 1–5) to caution and consistency.

4.2 Comparative ambiguity attitudes

The unique identification of the parameters α and P in Theorem 1 behaviorally founds their interpretation as ambiguity attitude and perception and motivates conducting comparative statics. Consider two individuals $(\succsim_i^\wedge, \succsim_i^*)_{i=1,2}$ with objectively founded α-MEU representations $(u_i, P_i, \alpha_i)_{i=1,2}$. The belief sets (and utilities) are fully determined by the objective Bewley preferences $\succsim_i^*$, and the comparative statics of P_i are well-understood.[13] Moreover, when $u_1 \approx u_2$ and $P_1 = P_2$, standard arguments imply that $\alpha_1 \geq \alpha_2$ if and only if individual 1's subjective preference is *more ambiguity averse* than individual 2's (Ghirardato and Marinacci, 2002), in the sense that for all $p \in \Delta(Z)$ and $f \in \mathcal{F}$,

$$p \succsim_2^\wedge f \Longrightarrow p \succsim_1^\wedge f. \tag{7}$$

[13]In particular, if $u_1 \approx u_2$, then $P_1 \subseteq P_2$ if and only if $\succsim_2^* \subseteq \succsim_1^*$.

We now show how, by considering both subjective and objective preferences, one can compare ambiguity attitudes α_i across individuals whose perceived ambiguity P_i need not be the same.

Definition 2. *We call* $(\succsim_1^\wedge, \succsim_1^*)$ ***more security oriented*** *than* $(\succsim_2^\wedge, \succsim_2^*)$ *if the following condition holds: Whenever* $f, g \in \mathcal{F}$ *are such that for all* $q \in \Delta(Z)$, $f \succsim_1^* q \Leftrightarrow g \succsim_2^* q$ *and* $q \succsim_1^* f \Leftrightarrow q \succsim_2^* g$, *then for any* $p \in \Delta(Z)$,

$$p \succsim_2^\wedge g \Longrightarrow p \succsim_1^\wedge f.$$

Suppose that in terms of objective comparisons against constant acts, individual 1 ranks act f the same way as individual 2 ranks act g. Thus, objectively, f has the same level of security and potential for individual 1 as act g has for individual 2. If, subjectively, individual 1 is more inclined to prefer constant acts over f than individual 2 is to prefer constant acts over g, this suggests that individual 1's choices are more driven by security considerations than individual 2's. This is the content of Definition 2. Note that when $\succsim_1^* = \succsim_2^*$, more security orientation implies that $\succsim_1^\wedge$ is more ambiguity averse than $\succsim_2^\wedge$ in the sense of (7).

The following result shows that for a fixed utility u, a higher α corresponds to more security orientation, even across individuals with different belief sets:

Proposition 2. *Suppose* $(\succsim_i^\wedge, \succsim_i^*)_{i=1,2}$ *admit objectively founded* α*-MEU representations* $(u_i, P_i, \alpha_i)_{i=1,2}$, *where* $u_1 \approx u_2$ *and* P_i *is not a singleton for* $i = 1, 2$. *The following are equivalent:*

(i). $(\succsim_1^\wedge, \succsim_1^*)$ *is more security oriented than* $(\succsim_2^\wedge, \succsim_2^*)$.

(ii). $\alpha_1 \geq \alpha_2$.

Remark 3. Proposition 2 remains valid under the following alternative definition of "more security oriented:" Whenever $f, g \in \mathcal{F}$ are such that, for some $q, q' \in \Delta(Z)$, $q \succsim_1^* f \not\succsim_1^* q'$ and $q \not\succsim_2^* g \succsim_2^* q'$, then for any $p \in \Delta(Z)$, $p \succsim_2^\wedge g \Longrightarrow p \succsim_1^\wedge f$. In contrast with Definition 2, this condition is refutable, because verifying the antecedent does not require infinitely many observations of $\succsim_1^*$ and $\succsim_2^*$.

Proposition 2 can also be extended to allow for heterogeneous utilities. Specifically, assume instead of $u_1 \approx u_2$ that there exist some $p, q \in \Delta(Z)$ such that $p \succ_i^\wedge q$ for $i = 1, 2$. Then, if P_1 and P_2 are not singletons, one can show that imposing Definition 2 only on acts that have range in $\{\lambda p + (1 - \lambda)q : \lambda \in [0, 1]\}$ characterizes the condition $\alpha_1 \geq \alpha_2$.[14] ▲

[14]We thank an anonymous referee for this observation.

4.3 Belief updating

Finally, the fact that the objectively founded α-MEU model uniquely determines a set of priors makes it possible to provide preference foundations for prior-by-prior updating, avoiding the problem highlighted in Example 1. Fix ex-ante subjective and objective preferences $(\succsim^\wedge, \succsim^*)$ that admit an objectively founded α-MEU representation with belief set P. Call event $E \subseteq S$ ***non-null*** if $\mu(E) > 0$ for all $\mu \in P$. For any non-null event E, denote by $\succsim^\wedge_E$ the DM's subjective preference conditional on learning that the true state is in E. In this section, we characterize when $\succsim^\wedge_E$ admits an α-MEU representation whose set of beliefs P^E is the prior-by-prior update (5) of P.

To do so, we impose an intertemporal analog of security-potential dominance that relates the ex-ante objective preference $\succsim^*$ and conditional subjective preference $\succsim^\wedge_E$. For any $f, g \in \mathcal{F}$, let $f_E g$ denote the act such that $f_E g(s) = f(s)$ for all $s \in E$ and $f_E g(s) = g(s)$ for all $s \notin E$. Call f ***more secure than*** g ***at*** E[15] if for all $p \in \Delta(Z)$,

$$g_E p \succsim^* p \Longrightarrow f_E p \succsim^* p.$$

Likewise, f ***has more potential than*** g ***at*** E if for all $p \in \Delta(Z)$,

$$p \not\succsim^* g_E p \Longrightarrow p \not\succsim^* f_E p.$$

Axiom 9 (Intertemporal security-potential dominance)**.** *If $f, g \in \mathcal{F}$ and f is both more secure than g at E and has more potential than g at E, then $f \succsim^\wedge_E g$.*

Axiom 9 requires that if at the ex-ante stage, act f offers both more security and more potential than g when only considering their outcomes in event E, then ex post, upon learning that event E has realized, the DM will choose f over g.

The following result shows that Axiom 9 (along with the assumption that the conditional subjective preference $\succsim^\wedge_E$ remains invariant biseparable) characterizes when $\succsim^\wedge_E$ admits an α-MEU representation whose set of beliefs P^E is the prior-by-prior update of the ex-ante set P and whose utility u is the same as the ex-ante utility:

Theorem 2. *Suppose $(\succsim^\wedge, \succsim^*)$ admits an objectively founded α-MEU representation (u, P, α). Fix any non-null E and conditional subjective preference $\succsim^\wedge_E$. The following are equivalent:*

(i). $\succsim^\wedge_E$ is an invariant biseparable preference and the pair $(\succsim^\wedge_E, \succsim^*)$ jointly satisfies intertemporal security-potential dominance.

[15] This condition is analogous to the definition of "more secure on E" in Kopylov (2016), who studies the ε-contamination model in a setting with an exogenous set of priors.

(ii). There exists $\alpha_E \in [0,1]$ such that (u, P^E, α_E) is an α-MEU representation of $\succsim_E^{\wedge}$.

Moreover, in this case, α_E is unique if P^E is not a singleton.

Note that Theorem 2 does not restrict how the weight α_E in the conditional α-MEU representation relates to the ex-ante weight α. Indeed, for *any* $\alpha_E \in [0,1]$, the conditional preference $\succsim_E^{\wedge}$ represented by (u, P^E, α_E) satisfies intertemporal security-potential dominance with respect to the ex-ante objective preference $\succsim^*$ represented by (u, P). Thus, updating based on Axiom 9 allows for a flexible relationship between ex-ante and conditional ambiguity attitudes. This flexibility can capture that the DM's ambiguity attitude might be affected by the nature of the information he obtains—for example, ambiguity attitudes might differ following "surprising" (low ex-ante likelihood) vs. "unsurprising" events.[16]

At the same time, the case when $\alpha_E = \alpha$ can be characterized behaviorally by additionally requiring $(\succsim^*, \succsim_E^{\wedge})$ and $(\succsim^*, \succsim^{\wedge})$ to be "equally security-oriented," in a sense analogous to Definition 2.[17]

Remark 4. A prominent special case of α-MEU is the neo-additive capacity model, where the belief set takes the form

$$P = \delta\Delta(S) + (1-\delta)\{\nu\},$$

for some $\delta \in [0,1]$ and $\nu \in \Delta(S)$. Since this model is also a special case of Choquet expected utility, the literature has applied updating rules for capacities. In contrast with the flexible relationship between α_E and α in Theorem 2, these updating rules pin down a specific value of α_E from the ex-ante preference. For example, Eichberger et al. (2010) show that under the generalized Bayesian updating rule (Eichberger et al., 2007; Horie, 2013), the resulting conditional preference is represented by (u, P^E, α_E) with $\alpha_E = \alpha$. On the other hand, they show that under the Dempster-Shafer (resp. Optimistic) updating rule, the value of α_E always increases (resp. decreases) relative to α in a particular manner.[18] ▲

Remark 5. Theorem 2 characterizes prior-by-prior updating by relating the conditional subjective preference $\succsim_E^{\wedge}$ to the ex-ante objective preference $\succsim^*$. An alternative approach would be to introduce a conditional objective preference $\succsim_E^*$ as part of the primitives. In that case, Theorem 1 in Ghirardato et al. (2008) (see also Faro and Lefort, 2019) implies that $\succsim_E^*$

[16] Dillenberger and Rozen (2015) explore history-dependent *risk* attitudes, focusing on the effect of past payoff realizations, as opposed to realized information.

[17] Formally, the same argument as for Proposition 2 implies that $\alpha_E = \alpha$ is equivalent to the following condition: Suppose that for all q, $f \succsim^* q \iff g_E q \succsim^* q$ and $q \succsim^* f \iff q \succsim^* g_E q$. Then for all p, $p \succsim^{\wedge} f \iff p \succsim_E^{\wedge} g$.

[18] Eichberger et al. (2012) extend the results to the more general class of Jaffray-Philippe capacities, which is still a special case of α-MEU.

admits the Bewley representation (u, P^E) if and only if $(\succsim^*, \succsim^*_E)$ satisfies dynamic consistency (i.e., $f_E g \succsim^* g \Leftrightarrow f \succsim^*_E g$). Given this, our Theorem 1 implies that additionally imposing security-potential dominance (Axiom 6) on the conditional pair $(\succsim^*_E, \succsim^\wedge_E)$ also yields an α-MEU representation (u, P^E, α_E) of $\succsim^\wedge_E$. One advantage of the approach in Theorem 2 is that, as we show in the next section, it extends to more general objective preferences $\succsim^*$ that need not admit a dynamically consistent update $\succsim^*_E$.

We also note that, under prior-by-prior updating, the subjective preferences $(\succsim^\wedge, \succsim^\wedge_E)$ need not satisfy dynamic consistency, but the updating rule does satisfy consequentialism (i.e., $f_E g \sim^\wedge_E f_E h$ for all f, g, h).[19] ▲

5 Linear completion rules for other incomplete preferences

In the previous section, we provided foundations for α-MEU (as well as prior-by-prior updating of the model) by applying (intertemporal) security-potential dominance to a Bewley preference $\succsim^*$. We conclude the paper by showing that this approach can be extended to obtain linear completion rules for a broader class of incomplete preferences $\succsim^*$.

The following result, which again applies Lemma A.3, generalizes the static characterization in Theorem 1. Recall that every invariant biseparable preference $\succsim$ can be represented by $I \circ u$ for some affine utility u and unique monotonic and constant-linear functional I.

Proposition 3. *Suppose that $\succsim^*$ satisfies transitivity and C-completeness and that the associated more-secure and more-potential orders are invariant biseparable with respective representations $I' \circ u$ and $I'' \circ u$.*[20] *Then the following are equivalent:*

(i). $\succsim^\wedge$ is an invariant biseparable preference and the pair $(\succsim^\wedge, \succsim^*)$ jointly satisfies security-potential dominance.

(ii). There exists $\alpha \in [0,1]$ such that $\succsim^\wedge$ is represented by $I \circ u$ with $I = \alpha I' + (1-\alpha) I''$.

As an application of Proposition 3, suppose $\succsim^*$ admits a ***twofold conservatism representation***, as introduced by Echenique et al. (2020) and Miyashita and Nakamura (2020):

[19] Beissner et al. (2020) show that for a given α-MEU representation (u, P, α), imposing Epstein and Schneider's (2003) rectangularity condition on P is not in general sufficient to ensure that prior-by-prior updating is dynamically consistent, in contrast with maxmin expected utility. Siniscalchi (2011) provides a general analysis of dynamic choice without dynamic consistency.

[20] Transitivity and C-completeness of $\succsim^*$ ensures that $I' \leq I''$, so that Lemma A.3 applies.

There exist non-disjoint sets of beliefs P_1, P_2 and an affine u such that

$$f \succsim^* g \iff \min_{\mu \in P_1} \mathbb{E}_\mu[u(f)] \geq \max_{\mu \in P_2} \mathbb{E}_\mu[u(g)]. \tag{8}$$

Here, $\succsim^*$ is also C-complete and transitive, but unlike Bewley preferences, $\succsim^*$ does not satisfy full monotonicity and independence (unless $\succsim^*$ is complete, which is equivalent to $P_1 = P_2 = \{\mu\}$ for some belief μ, i.e., to $\succsim^*$ being a subjective expected utility preference). See the aforementioned papers for an axiomatization and interpretation of (8) as capturing difficulties with performing contingent reasoning.

The associated more-secure and more-potential orders are represented by $I'(\phi) = \min_{\mu \in P_1} \mu \cdot \phi$ and $I''(\phi) = \max_{\mu \in P_2} \mu \cdot \phi$. Thus, by Proposition 3, imposing security-potential dominance on the pair $(\succsim^\wedge, \succsim^*)$ characterizes the following ***asymmetric α-MEU representation*** of $\succsim^\wedge$:

$$f \succsim^\wedge g \iff \alpha \min_{\mu \in P_1} \mathbb{E}_\mu[u(f)] + (1-\alpha) \max_{\mu \in P_2} \mathbb{E}_\mu[u(f)] \geq \alpha \min_{\mu \in P_1} \mathbb{E}_\mu[u(g)] + (1-\alpha) \max_{\mu \in P_2} \mathbb{E}_\mu[u(g)]. \tag{9}$$

This model has been considered in the literature, because unlike symmetric α-MEU, it can accommodate source-dependent ambiguity attitudes (e.g., Chandrasekher et al., 2021). However, just as for symmetric α-MEU, there is no known characterization of (9) and the parameters in (9) are not identified based on $\succsim^\wedge$ alone. Incorporating the objective preference $\succsim^*$ addresses these issues. In particular, $\succsim^*$ uniquely determines P_1 and P_2, which in turn pins down α except when $\succsim^*$ is complete. A notable special case of (9) is when P_2 is a singleton, yielding the widely studied ***ε-contamination representation*** (e.g., Nishimura and Ozaki, 2006; Gajdos et al., 2008; Kopylov, 2009); for this model, it is again well-known that the parameters are not identified based on $\succsim^\wedge$ alone.[21]

More broadly, beyond (asymmetric) α-MEU, Proposition 3 can shed light on various other representations that may be difficult to analyze based on a subjective preference $\succsim^\wedge$ alone, by recasting these models as linear completion rules of suitable well-understood incomplete preferences $\succsim^*$. Appendix B.2 further illustrates this point using other examples of incomplete preferences from the recent literature.

Finally, our characterization of belief updating in Theorem 2 also generalizes to the current setting. For any monotonic and constant-linear functional I and event E, define the functional I_E by $I_E(\phi) = I(\phi_E \underline{I_E(\phi)})$ for all $\phi \in \mathbb{R}^S$. When $I(\phi) = \min_{\mu \in P} \mu \cdot \phi$ is maxmin expected utility, then $I_E(\phi) = \min_{\mu^E \in P^E} \mu^E \cdot \phi$ corresponds to prior-by-prior

[21] This model is characterized in our setting by requiring the more-potential order to be independent. Together with the axioms in Echenique et al. (2020) and Miyashita and Nakamura (2020), this ensures that $\succsim^*$ admits a twofold conservatism representation where P_2 is a singleton.

updating. More generally, if $\succsim$ is the invariant biseparable preference represented by $I \circ u$, then $I_E \circ u$ represents the unique conditional preference $\succsim_E$ obtained from $\succsim$ via Pires's (2002) coherency axiom (i.e., $f_E p \succsim p \iff f \succsim_E p$, for all $f \in \mathcal{F}, p \in \Delta(Z)$), and I_E is itself monotonic and constant-linear.[22]

The following result shows that intertemporal security-potential dominance corresponds to first updating the more-secure and more-potential functionals associated with $\succsim^*$ to I'_E and I''_E and then representing $\succsim^\wedge_E$ by a linear aggregation of these functionals:

Proposition 4. *Suppose that $\succsim^*$ satisfies transitivity and C-completeness and that the associated more-secure and more-potential orders are invariant biseparable with respective representations $I' \circ u$ and $I'' \circ u$. Fix any non-null E and conditional subjective preference $\succsim^\wedge_E$.*[23] *The following are equivalent:*

- *(i). $\succsim^\wedge_E$ is an invariant biseparable preference and the pair $(\succsim^\wedge_E, \succsim^*)$ jointly satisfies intertemporal security-potential dominance.*
- *(ii). There exists $\alpha_E \in [0,1]$ such that $\succsim^\wedge_E$ is represented by $I \circ u$ with $I = \alpha_E I'_E + (1-\alpha_E) I''_E$.*

When $\succsim^*$ admits a twofold conservatism representation, Proposition 4 characterizes prior-by-prior updating of the asymmetric α-MEU model:

$$f \succsim^\wedge_E g \iff \alpha_E \min_{\mu \in P_1^E} \mathbb{E}_\mu[u(f)] + (1-\alpha_E) \max_{\mu \in P_2^E} \mathbb{E}_\mu[u(f)] \geq \alpha_E \min_{\mu \in P_1^E} \mathbb{E}_\mu[u(g)] + (1-\alpha_E) \max_{\mu \in P_2^E} \mathbb{E}_\mu[u(g)].$$

Just as for symmetric α-MEU, this updating rule is not well-defined based on $\succsim^\wedge$ alone. This issue does not arise in the current setting due to the unique identification offered by $\succsim^*$.

A Proofs

A.1 Preliminaries

Throughout this appendix, for any non-empty, closed and convex $P \subseteq \Delta(S)$ and $\phi \in \mathbb{R}^S$, let

$$M_P(\phi) := \max_{\mu \in P} \phi \cdot \mu, \quad m_P(\phi) := \min_{\mu \in P} \phi \cdot \mu.$$

[22] Chandrasekher et al. (2021) show that for every invariant biseparable preference, I admits a *dual-self expected utility representation* of the form $I(\phi) = \max_{P \in \mathbb{P}} \min_{\mu \in P} \mu \cdot \phi$, where $\mathbb{P}$ is a set of belief sets P. They also show that the functional I_E is obtained by updating each belief set $P \in \mathbb{P}$ prior-by-prior, i.e., $I_E(\phi) = \max_{P \in \mathbb{P}} \min_{\mu^E \in P^E} \mu^E \cdot \phi$.

[23] Call E non-null if there exist $p, q \in \Delta(Z)$ such that $p_E q$ is both strictly more secure and has strictly more potential than q.

The following lemma shows that for any given $\alpha \neq \frac{1}{2}$, the sets of priors in the α-MEU functional are uniquely identified:

Lemma A.1. *Fix any $\alpha \in [0,1]$ with $\alpha \neq 1/2$ and any non-empty, closed and convex $P_1, P_2 \subseteq \Delta(S)$. For $i = 1,2$ and each $\phi \in \mathbb{R}^S$, let $I_i(\phi) := \alpha m_{P_i}(\phi) + (1-\alpha)M_{P_i}(\phi)$. If $I_1(\phi) = I_2(\phi)$ for all $\phi \in \mathbb{R}^S$, then $P_1 = P_2$.*

Proof. Take any $\phi \in \mathbb{R}^S$. Then for each $i = 1,2$,

$$I_i(-\phi) = -\alpha M_{P_i}(\phi) - (1-\alpha)m_{P_i}(\phi).$$

But $I_1(\phi) = I_2(\phi)$ and $I_1(-\phi) = I_2(-\phi)$ implies $M_{P_1}(\phi) = M_{P_2}(\phi)$, because

$$\begin{aligned}
& (1-\alpha)I_1(\phi) + \alpha I_1(-\phi) = (1-\alpha)I_2(\phi) + \alpha I_2(-\phi) \\
\iff & (1-2\alpha)M_{P_1}(\phi) = (1-2\alpha)M_{P_2}(\phi) \\
\iff & M_{P_1}(\phi) = M_{P_2}(\phi),
\end{aligned}$$

where the last equivalence uses $\alpha \neq 1/2$. Since this is true for any $\phi \in \mathbb{R}^S$, the support functions of P_1 and P_2 coincide, which implies $P_1 = P_2$. $\square$

The next lemma, which is used in the proof of Proposition 1, characterizes γ-expansions in terms of the relationship between the corresponding support functions.

Lemma A.2. *Consider two non-empty, closed and convex sets $P, Q \subseteq \Delta(S)$, and $\gamma \geq 1$. Then Q is the γ-expansion of P if and only if, for each $\phi \in \mathbb{R}^S$,*

$$M_P(\phi) = \frac{\gamma}{2\gamma-1}M_Q(\phi) + \frac{\gamma-1}{2\gamma-1}m_Q(\phi), \;\; m_P(\phi) = \frac{\gamma-1}{2\gamma-1}M_Q(\phi) + \frac{\gamma}{2\gamma-1}m_Q(\phi). \quad (10)$$

Proof. Suppose first that Q is the γ-expansion of P. Then $\mu \in Q$ if and only if $\mu = \gamma\nu + (1-\gamma)\nu'$ for some $\nu, \nu' \in P$. Since $\gamma \geq 1$, this implies that for any $\phi \in \mathbb{R}^S$,

$$M_Q(\phi) = \gamma M_P(\phi) + (1-\gamma)m_P(\phi), \;\; m_Q(\phi) = \gamma m_P(\phi) + (1-\gamma)M_P(\phi).$$

Solving this system yields (10).

Conversely, suppose (10) holds for all $\phi \in \mathbb{R}^S$. For any $s \in S$, define $\phi^s \in \mathbb{R}^S$ by $\phi^s(s) = 1$

and $\phi^s(s') = 0$ for each $s' \neq s$. By (10), we have

$$\begin{aligned}
\gamma \min_{\nu \in P} \nu(s) &= \gamma m_P(\phi^s) = \frac{\gamma^2}{2\gamma - 1} m_Q(\phi^s) + \frac{\gamma(\gamma-1)}{2\gamma-1} M_Q(\phi^s) \\
&\geq \frac{(\gamma-1)^2}{2\gamma-1} m_Q(\phi^s) + \frac{\gamma(\gamma-1)}{2\gamma-1} M_Q(\phi^s) \quad \text{since } \gamma^2 \geq (\gamma-1)^2 \text{ and } m_Q(\phi^s) \geq 0 \\
&= (\gamma - 1) M_P(\phi^s) \\
&= (\gamma - 1) \max_{\nu \in P} \nu(s).
\end{aligned}$$

This shows that for any $s \in S$ and $\nu, \nu' \in P$, we have $\gamma\nu(s) + (1-\gamma)\nu'(s) \geq 0$. Thus, $P^\gamma := \gamma P + (1-\gamma)P$ is a subset of $\Delta(S)$. Moreover, P^γ is non-empty (since it contains P), closed, and convex. Hence, for any $\phi \in \mathbb{R}^S$, we have

$$\frac{\gamma}{2\gamma-1} M_{P^\gamma}(\phi) + \frac{\gamma-1}{2\gamma-1} m_{P^\gamma}(\phi) = M_P(\phi) = \frac{\gamma}{2\gamma-1} M_Q(\phi) + \frac{\gamma-1}{2\gamma-1} m_Q(\phi),$$

where the first equality holds by the "only if" direction of the lemma and the second by (10). By Lemma A.1, this implies $Q = P^\gamma$, that is, Q is the γ-expansion of P. □

The final lemma is a general linear aggregation result for constant-linear functionals. This is central to the proof of Proposition 3 (and hence Theorem 1):

Lemma A.3. *Suppose functionals $I, I', I'' : \mathbb{R}^S \to \mathbb{R}$ are monotonic and constant-linear with $I' \leq I''$. Then the following are equivalent:*

(i). For all $\phi, \psi \in \mathbb{R}^S$,

$$[I'(\phi) \geq I'(\psi) \text{ and } I''(\phi) \geq I''(\psi)] \Longrightarrow I(\phi) \geq I(\psi).$$

(ii). There exists $\alpha \in [0,1]$ such that for all $\phi \in \mathbb{R}^S$, $I(\phi) = \alpha I'(\phi) + (1-\alpha) I''(\phi)$.

Proof. We show that (i) implies (ii); verifying the other direction is standard.[24] By (i), there exists a weakly increasing function $W : \{(I'(\phi), I''(\phi)) : \phi \in \mathbb{R}^S\} \to \mathbb{R}$ such that

$$I(\phi) = W(I'(\phi), I''(\phi)) \text{ for all } \phi \in \mathbb{R}^S. \tag{11}$$

Consider any $\phi \in \mathbb{R}^S$ such that $I'(\phi) = I''(\phi) =: c$. Since I', I'', I are constant-linear, they are normalized, i.e., $c = I(\underline{c}) = I'(\underline{c}) = I''(\underline{c})$. Thus, by (11), $I(\phi) = I(\underline{c}) = c$. Hence, $I(\phi) = \alpha I'(\phi) + (1-\alpha) I''(\phi)$ holds for any $\alpha \in \mathbb{R}$.

[24]Similar arguments were used in the proof of Lemma B.5 in Ghirardato et al. (2004), which considers the special case of Lemma A.3 when $I' = m_P$, $I'' = M_P$ for some P.

Next, consider any $\phi \in \mathbb{R}^S$ such that $I'(\phi) < I''(\phi)$; if there is no such ϕ, the previous paragraph establishes (ii). Then there exists a unique $\alpha(\phi) \in \mathbb{R}$ such that

$$I(\phi) = \alpha(\phi)I'(\phi) + (1 - \alpha(\phi))I''(\phi).$$

In particular,

$$\alpha(\phi) = \frac{I(\phi) - I''(\phi)}{I'(\phi) - I''(\phi)} = -\frac{I(\phi) - I''(\phi)}{I''(\phi) - I'(\phi)} = -I(\psi) = -W(I'(\psi), I''(\psi)),$$

where $\psi := \frac{\phi - I''(\phi)}{I''(\phi) - I'(\phi)}$ and the third equality holds since I is constant-linear. Note that $I'(\psi) = -1$ and $I''(\psi) = 0$. Thus, $\alpha(\phi) = -W(-1, 0) =: \alpha$, which does not depend on ϕ. Hence, for α thus defined, $I(\phi) = \alpha I'(\phi) + (1 - \alpha)I''(\phi)$ holds for all ϕ with $I'(\phi) < I''(\phi)$.

It remains to show that $\alpha \in [0, 1]$. Suppose that $\alpha < 0$. Then for any ϕ such that $I'(\phi) < I''(\phi)$, we have $I(\phi) > I''(\phi) = I(\underline{I''(\phi)})$, which contradicts (i), as $I'(\underline{I''(\phi)}) = I''(\phi) > I'(\phi)$ and $I''(\underline{I''(\phi)}) = I''(\phi)$. If $\alpha > 1$, we obtain a contradiction in an analogous manner. □

A.2 Proof of Proposition 1

"If" direction. For each $\phi \in \mathbb{R}^S$ and $i = 1, 2$, let $I_i(\phi) := \alpha_i m_{P_i}(\phi) + (1 - \alpha_i)M_{P_i}(\phi)$. Suppose case 1 in the proposition holds; the argument for case 2 is analogous. Note that since $\alpha_2 \geq \alpha_1 > \frac{1}{2}$, we have $\gamma := \frac{\alpha_1 + \alpha_2 - 1}{2\alpha_1 - 1} \geq 1$. Since P_1 is the γ-expansion of P_2, Lemma A.2 implies that for any $\phi \in \mathbb{R}^S$,

$$M_{P_2}(\phi) = \frac{\gamma}{2\gamma - 1}M_{P_1}(\phi) + \frac{\gamma - 1}{2\gamma - 1}m_{P_1}(\phi), \quad m_{P_2}(\phi) = \frac{\gamma}{2\gamma - 1}m_{P_1}(\phi) + \frac{\gamma - 1}{2\gamma - 1}M_{P_1}(\phi).$$

Then, for any $\phi \in \mathbb{R}^S$,

$$\begin{aligned}
&\alpha_2 m_{P_2}(\phi) + (1 - \alpha_2)M_{P_2}(\phi) \\
=&\alpha_2[\frac{\gamma}{2\gamma - 1}m_{P_1}(\phi) + \frac{\gamma - 1}{2\gamma - 1}M_{P_1}(\phi)] + (1 - \alpha_2)[\frac{\gamma}{2\gamma - 1}M_{P_1}(\phi) + \frac{\gamma - 1}{2\gamma - 1}m_{P_1}(\phi)] \\
=&\alpha_1 m_{P_1}(\phi) + (1 - \alpha_1)M_{P_1}(\phi),
\end{aligned}$$

as $\alpha_2\gamma/(2\gamma - 1) + (1 - \alpha_2)(\gamma - 1)/(2\gamma - 1) = \alpha_1$ by the definition of γ. Thus, given $u_1 \approx u_2$, (u_1, P_1, α_1) and (u_2, P_2, α_2) represent the same preference.

"Only if" direction. Assume that (u_1, P_1, α_1) and (u_2, P_2, α_2) with $\alpha_1 \leq \alpha_2$ represent the same preference, and let I_1 and I_2 denote the associated utility act functionals. Standard arguments imply that $u_1 \approx u_2$ and $I_1 = I_2$. Suppose that $\alpha_1 < \frac{1}{2}$ (the case $\alpha_1 > \frac{1}{2}$ is analogous). Note that for $i = 1, 2$ and any event $E \subseteq S$, we have

$$I_i(\underline{1}_E\underline{0}) + I_i(\underline{0}_E\underline{1}) = (2\alpha_i - 1)\left(\min_{\mu\in P_i}\mu(E) - \max_{\mu\in P_i}\mu(E)\right) + 1.$$

Since the left-hand side is the same for $i = 1, 2$ and each P_i is non-singleton, it follows that $\alpha_2 < \frac{1}{2}$.

For any $\phi \in \mathbb{R}^S$, the fact that $I_1(\phi) = I_2(\phi)$ and $I_1(-\phi) = I_2(-\phi)$ implies

$$\alpha_1 m_{P_1}(\phi) + (1-\alpha_1)M_{P_1}(\phi) = \alpha_2 m_{P_2}(\phi) + (1-\alpha_2)M_{P_2}(\phi)$$

and

$$(1-\alpha_1)m_{P_1}(\phi) + \alpha_1 M_{P_1}(\phi) = (1-\alpha_2)m_{P_2}(\phi) + \alpha_2 M_{P_2}(\phi).$$

Solving this system yields

$$M_{P_1}(\phi) = \frac{1-\alpha_1-\alpha_2}{1-2\alpha_1}M_{P_2}(\phi) + \frac{\alpha_2-\alpha_1}{1-2\alpha_1}m_{P_2}(\phi), \quad m_{P_1}(\phi) = \frac{1-\alpha_1-\alpha_2}{1-2\alpha_1}m_{P_2}(\phi) + \frac{\alpha_2-\alpha_1}{1-2\alpha_1}M_{P_2}(\phi)$$

which can be written as

$$M_{P_1}(\phi) = \frac{\gamma}{2\gamma-1}M_{P_2}(\phi) + \frac{\gamma-1}{2\gamma-1}m_{P_2}(\phi), \quad m_{P_1}(\phi) = \frac{\gamma}{2\gamma-1}m_{P_2}(\phi) + \frac{\gamma-1}{2\gamma-1}M_{P_2}(\phi),$$

where $\gamma := \frac{1-\alpha_1-\alpha_2}{1-2\alpha_2}$. By Lemma A.2, this implies that P_2 is the γ-expansion of P_1. □

A.3 Proof of Theorem 1

We show that (i) implies (ii); verifying that (ii) implies (i) is standard. Since $\succsim^*$ is a Bewley preference, it admits a Bewley representation (u, P) (see, e.g., Theorem 1 in GMMS).

Observe that f is more secure than g if and only if $\min_{\mu\in P}\mu \cdot u(f) \geq \min_{\mu\in P}\mu \cdot u(g)$. Likewise, f has more potential than g if and only if $\max_{\mu\in P}\mu\cdot u(f) \geq \max_{\mu\in P}\mu\cdot u(g)$. Thus, Proposition 3 yields some $\alpha \in [0, 1]$ such that (u, P, α) is an objectively founded α-MEU representation of $(\succsim^*, \succsim^\wedge)$.

For the moreover part, cardinal uniqueness of u and uniqueness of P follows from the uniqueness properties of Bewley representations (e.g., Theorem 1 in GMMS). Finally, whenever $\succsim^*$ is incomplete, then P is not a singleton. Thus, there exists $\phi \in \mathbb{R}^S$ with $\min_{\mu\in P}\mu\cdot\phi < \max_{\mu\in P}\mu\cdot\phi$. For any $\alpha' \neq \alpha$, this implies $\alpha'\min_{\mu\in P}\mu\cdot\phi + (1-\alpha')\max_{\mu\in P}\mu\cdot\phi \neq \alpha\min_{\mu\in P}\mu\cdot\phi + (1-\alpha)\max_{\mu\in P}\mu\cdot\phi$. Hence, α is unique by the uniqueness of the utility act functional I representing $\succsim^\wedge$. □

A.4 Proof of Corollary 1

We show that (i) implies (ii); verifying the other direction is standard. Consider the case in which security dominance holds (the argument when potential dominance holds is analogous). By Theorem 1, $(\succsim^*, \succsim^\wedge)$ admits some objectively founded α-MEU representation (u, P, α). If P is a singleton, the representation does not depend on the value of α, and we can set $\alpha = 1$. If P is not a singleton, then there exist $f \in \mathcal{F}$ and $p \in \Delta(Z)$ such that $\min_{\mu \in P} \mu \cdot u(f) = u(p) < \max_{\mu \in P} \mu \cdot u(f)$. Thus, p is more secure than f, and hence by security dominance $p \succsim^\wedge f$. By the representation, this means

$$u(p) \geq \alpha \min_{\mu \in P} \mu \cdot u(f) + (1 - \alpha) \max_{\mu \in P} \mu \cdot u(f),$$

which is only possible if $\alpha = 1$. □

A.5 Proof of Proposition 2

Observe first that there exist $\phi, \psi \in [-1, 1]^S$ such that $m_{P_1}(\phi) = m_{P_2}(\psi) < M_{P_1}(\phi) = M_{P_2}(\psi)$. Indeed, given that P_1 and P_2 are not singletons, there exist $\phi', \psi' \in \mathbb{R}^S$ such that $m_{P_1}(\phi') < M_{P_1}(\phi')$ and $m_{P_2}(\psi') < M_{P_2}(\psi')$. By setting $\phi := \varepsilon \frac{\phi' - m_{P_1}(\phi')}{M_{P_1}(\phi') - m_{P_1}(\phi')}$ and $\psi := \varepsilon \frac{\psi' - m_{P_2}(\psi')}{M_{P_2}(\psi') - m_{P_2}(\psi')}$ for a sufficiently small $\varepsilon > 0$, we obtain $\phi, \psi \in [-1, 1]^S$ and $m_{P_1}(\phi) = m_{P_2}(\psi) = 0 < \varepsilon = M_{P_1}(\phi) = M_{P_2}(\psi)$.

Since $u_1 \approx u_2$, we can assume without loss that $u_1 = u_2 =: u$ and that $[-1, 1] \subseteq u(Z)$ (up to performing a suitable positive affine transformation). Given this, consider any $f, g \in \mathcal{F}$, and observe that the equivalences $f \succsim_1^* p \Leftrightarrow g \succsim_2^* p$ and $p \succsim_1^* f \Leftrightarrow p \succsim_2^* g$ hold for all constant acts p, if and only if, $\min_{\mu \in P_1} \mu \cdot u(f) = \min_{\mu \in P_2} \mu \cdot u(g)$ and $\max_{\mu \in P_1} \mu \cdot u(f) = \max_{\mu \in P_2} \mu \cdot u(g)$. The equivalence of (i) and (ii) then follows from the fact that $\min_{\mu \in P_1} \mu \cdot u(f) = \min_{\mu \in P_2} \mu \cdot u(g) < \max_{\mu \in P_1} \mu \cdot u(f) = \max_{\mu \in P_2} \mu \cdot u(g)$ for some $f, g \in \mathcal{F}$ which satisfy $u(f) = \phi$ and $u(g) = \psi$ as in the previous paragraph. □

A.6 Proof of Theorem 2

Suppose $(\succsim^\wedge, \succsim^*)$ admits an objectively founded α-MEU representation (u, P, α). We will show that (i) implies (ii); verifying the other direction is standard.

Suppose $\succsim_E^\wedge$ is an invariant biseparable preference and the pair $(\succsim_E^\wedge, \succsim^*)$ jointly satisfies intertemporal security-potential dominance. Note that the more-secure and more-potential orders are represented by $I'(u(f)) = \min_{\mu \in P} \mu \cdot u(f)$ and $I''(u(f)) = \max_{\mu \in P} \mu \cdot u(f)$, respectively, and that $I'_E(\phi) = \min_{\mu^E \in P^E} \mu^E \cdot \phi$ and $I''_E(\phi) = \max_{\mu^E \in P^E} \mu^E \cdot \phi$. Thus, by

Proposition 4, there exists $\alpha_E \in [0,1]$ such that (u, P^E, α_E) represents $\succsim_E^\wedge$.

Moreover, if P^E is not a singleton, then α_E is unique by the same argument as in the proof of Theorem 1. □

A.7 Proof of Proposition 3

Lemma A.4. *Under the assumptions of Proposition 3, we have that $I' \leq I''$.*

Proof. It suffices to show that, for any $f \in \mathcal{F}$ and $p \in \Delta(Z)$, $I'(u(f)) \geq u(p)$ implies $I''(u(f)) \geq u(p)$. Suppose $I'(u(f)) \geq u(p)$. Then f is more secure than p. Since $p \succsim^* p$ by C-completeness, this implies $f \succsim^* p$. For any $q \in \Delta(Z)$ such that $q \succsim^* f$, we then have $q \succsim^* p$ by transitivity. Thus f has more potential than p, and hence $I''(u(f)) \geq u(p)$. □

Proof of Proposition 3. We show that (i) implies (ii); verifying that (ii) implies (i) is standard. Since $\succsim^\wedge$ is invariant biseparable, $\succsim^\wedge$ is represented by $I \circ v$ for some nonconstant and affine utility v and a unique constant-linear and monotonic functional I. Note that for any $p, q \in \Delta(Z)$, we have

$$u(p) \geq u(q) \Longrightarrow p \text{ is more secure and has more potential than } q \Longrightarrow v(p) \geq v(q),$$

where the last implication holds by security-potential dominance. Since u and v are nonconstant and affine, this implies $v \approx u$, and we can assume without loss that $v = u$.

Thus, $I \circ u$ represents $\succsim^\wedge$. Hence, security-potential dominance and the constant-linearity of I, I', I'' imply that for any $\phi, \psi \in \mathbb{R}^S$ with $I'(\phi) \geq I'(\psi)$ and $I''(\phi) \geq I''(\psi)$, we have $I(\phi) \geq I(\psi)$. Since $I' \leq I''$ (Lemma A.4), Lemma A.3 yields some $\alpha \in [0,1]$ such that $I(\phi) = \alpha I'(\phi) + (1-\alpha) I''(\phi)$ for all $\phi \in \mathbb{R}^S$. □

A.8 Proof of Proposition 4

Lemma A.5. *Under the assumptions of Proposition 4, there exist constant-linear functionals I'_E, I''_E such that for all $\phi \in \mathbb{R}^S$ and $\beta \in \mathbb{R}$, $I'_E(\phi) \geq \beta$ iff $I'(\phi_E \underline{\beta}) \geq \beta$ (resp. $I''_E(\phi) \geq \beta$ iff $I''(\phi_E \underline{\beta}) \geq \beta$). Moreover, f is more secure (resp. has more potential) than g at E iff $I'_E(u(f)) \geq I'_E(u(g))$ (resp. $I''_E(u(f)) \geq I''_E(u(g))$).*

Proof. The existence of such functionals I'_E and I''_E follows from Theorem 2 in Chandrasekher et al. (2021); see Footnote 22. Given this, the "moreover" part follows because

$$\begin{aligned} f \text{ is more secure than } g \text{ at } E \quad &\text{iff} \quad (\forall p \in \Delta(Z),\ [I'(u(g_E p)) \geq u(p)] \implies [I'(u(f_E p)) \geq u(p)]) \\ &\text{iff} \quad (\forall p \in \Delta(Z),\ [I'_E(u(g)) \geq u(p))] \implies [I'_E(u(f)) \geq u(p)]) \\ &\text{iff} \quad I'_E(u(f)) \geq I'_E(u(g)). \end{aligned}$$

The same argument applies to the more-potential order. □

Proof of Proposition 4. We show that (i) implies (ii); verifying that (ii) implies (i) is standard. We first observe that $I'_E \leq I''_E$. To see this, note that, for any $\phi \in \mathbb{R}^S$ and $\beta \in \mathbb{R}$,

$$I'_E(\phi) \geq \beta \implies I'(\phi_E \underline{\beta}) \geq \beta \implies I''(\phi_E \underline{\beta}) \geq \beta \implies I''_E(\phi) \geq \beta,$$

where the second implication uses $I' \leq I''$ (by Lemma A.4) and the other implications follow from Lemma A.5.

Let (I, v) be the representation of $\succsim_E^\wedge$. By intertemporal security-potential dominance, we can assume that $v = u$, by the same argument as in the proof of Proposition 3. Based on $I'_E \leq I''_E$ and Lemma A.3, intertemporal security-potential dominance then yields some $\alpha_E \in [0, 1]$ such that $I = \alpha_E I'_E + (1 - \alpha_E) I''_E$. □

B Additional results

B.1 Identification under $\alpha = 1/2$

The following result complements the identification result in Proposition 1 by covering the remaining case where $\alpha_i = 1/2$ for some i. Given two subsets A and B of $\Delta(S)$, we write $A - B := \{a - b : a \in A, b \in B\}$.

Proposition B.1. *Suppose $(u_1, P_1, 1/2)$ and (u_2, P_2, α_2) are α-MEU representations of $\succsim_1^\wedge$ and $\succsim_2^\wedge$, respectively, where P_i is not a singleton for $i = 1, 2$. Then $\succsim_1^\wedge = \succsim_2^\wedge$ if and only if $u_1 \approx u_2$, $\alpha_2 = 1/2$, and $P_1 - P_2 = P_2 - P_1$.*

Proof. The condition $u_1 \approx u_2$ is standard, and we can thus assume $u_1 = u_2 = u$ without loss of generality. Moreover, since $\alpha_1 = 1/2$ and the P_i are not singletons, the same argument as in the "only if" direction of Proposition 1 implies that if $\succsim_1^\wedge = \succsim_2^\wedge$, then $\alpha_2 = 1/2$.

Given that $\alpha_1 = \alpha_2 = 1/2$ and the uniqueness of the utility act functional, the condition $\succsim_1^\wedge = \succsim_2^\wedge$ is equivalent to

$$\frac{1}{2} \min_{\mu \in P_1} \mu \cdot \phi + \frac{1}{2} \max_{\mu \in P_1} \mu \cdot \phi = \frac{1}{2} \min_{\mu \in P_2} \mu \cdot \phi + \frac{1}{2} \max_{\mu \in P_2} \mu \cdot \phi$$

for all $\phi \in \mathbb{R}^S$. Re-arranging yields

$$\max_{(\mu_1,\mu_2)\in P_1\times P_2} \mu_1 \cdot \phi - \mu_2 \cdot \phi = \max_{(\mu_1,\mu_2)\in P_1\times P_2} \mu_1 \cdot \phi - \mu_2 \cdot \phi,$$

that is,

$$\max_{\mu\in P_1-P_2} \mu \cdot \phi = \max_{\mu\in P_2-P_1} \mu \cdot \phi.$$

Since $P_1 - P_2$ and $P_2 - P_1$ are both closed and convex subsets of $\mathbb{R}^S$, the above property is true for all $\phi \in \mathbb{R}^S$ if and only if $P_1 - P_2 = P_2 - P_1$. □

The multiplicity of belief sets allowed by Proposition B.1 is greater than in Proposition 1. Indeed, $P_1 - P_2 = P_2 - P_1$ is satisfied if P_2 is the γ-expansion of P_1 (or vice versa) for some $\gamma \geq 1$, irrespective of the value of γ. However, in contrast with Proposition 1, the opposite implication is not true, as $P_1 - P_2 = P_2 - P_1$ can hold even if P_1 and P_2 are not nested. The following example illustrates this: Consider $|S| = 3$, take ε with $0 < \varepsilon < 1/3$, and define

$$P_1 := \{\mu \in \Delta(S) : \mu_1 = \frac{1}{3}, |\mu_2 - \frac{1}{3}| \leq \varepsilon\} \text{ and } P_2 := \{\mu \in \Delta(S) : |\mu_1 - \frac{1}{3}| \leq \varepsilon, \mu_2 = \frac{1}{3}\}.$$

The sets P_1 and P_2 are not nested but satisfy

$$P_1 - P_2 = P_2 - P_1 = \{(\nu_1, \nu_2, -\nu_1 - \nu_2) : |\nu_1| \leq \varepsilon, |\nu_2| \leq \varepsilon\}.$$

B.2 Additional examples for Section 5

Beyond the cases where $\succsim^*$ admits a Bewley representation (as in Section 4) or a twofold conservatism representation (as in Section 5), Propositions 3–4 apply to several other classes of incomplete preferences.

For example, Valenzuela-Stookey (2020) considers a model of subjective complexity:

$$f \succsim^* g \iff \max_{h\in\mathcal{G}:f\geq h} \mathbb{E}_\mu[u(h)] \geq \min_{h\in\mathcal{G}:h\geq g} \mathbb{E}_\mu[u(h)],$$

where μ is a fixed belief, $\mathcal{G}$ is a set of acts that are "simple," and $\geq$ denotes the state-wise dominance relation. In this model, the more-secure and more-potential orders are represented by $I'(u(f)) = \max_{h\in\mathcal{G}:f\geq h} \mathbb{E}_\mu[u(h)]$ and $I''(u(f)) = \min_{h\in\mathcal{G}:h\geq f} \mathbb{E}_\mu[u(h)]$.

Another example is the multiple MEU model in Nascimento and Riella (2011) and Hara (2021), which considers a unanimity rule over MEU preferences:

$$f \succsim^* g \iff \min_{\mu\in P} \mathbb{E}_\mu[u(f)] \geq \min_{\mu\in P} \mathbb{E}_\mu[u(g)] \; \forall P \in \mathbb{P},$$

where $\mathbb{P} \subseteq 2^{\Delta}(S)$ is a (Hausdorff)-compact set of sets of beliefs. In this case, $I'(u(f)) = \min_{\mu \in \cup_{P \in \mathbb{P}} P} \mathbb{E}_\mu[u(f)]$ and $I''(u(f)) = \max_{P \in \mathbb{P}} \min_{\mu \in P} \mathbb{E}_\mu[u(f)]$.

Acknowledgments We are grateful to Matthias Lang, Marciano Siniscalchi, and two anonymous referees for valuable comments.

References

Ahn, D., S. Choi, D. Gale, and S. Kariv (2014). Estimating ambiguity aversion in a portfolio choice experiment. *Quantitative Economics 5*(2), 195–223.

Amarante, M. (2009). Foundations of neo-bayesian statistics. *Journal of Economic Theory 144*(5), 2146–2173.

Bastianello, L., J. H. Faro, and A. Santos (2020). Dynamically consistent objective and subjective rationality. Working paper.

Beissner, P., Q. Lin, and F. Riedel (2020). Dynamically consistent alpha-maxmin expected utility. *Mathematical Finance 30*(3), 1073–1102.

Bewley, T. F. (2002). Knightian decision theory. part i. *Decisions in Economics and Finance 25*(2), 79–110.

Bossaerts, P., P. Ghirardato, S. Guarnaschelli, and W. R. Zame (2010). Ambiguity in asset markets: Theory and experiment. *Review of Financial Studies 23*(4), 1325–1359.

Ceron, F. and V. Vergopoulos (2020). Recursive objective and subjective multiple priors. Working paper.

Cerreia-Vioglio, S. (2016). Objective rationality and uncertainty averse preferences. *Theoretical Economics 11*(2), 523–545.

Cerreia-Vioglio, S., F. Maccheroni, M. Marinacci, and L. Montrucchio (2011). Uncertainty averse preferences. *Journal of Economic Theory 146*(4), 1275–1330.

Cettolin, E. and A. Riedl (2019). Revealed preferences under uncertainty: Incomplete preferences and preferences for randomization. *Journal of Economic Theory 181*, 547–585.

Chandrasekher, M., M. Frick, R. Iijima, and Y. Le Yaouanq (2021). Dual-self representations of ambiguity preferences. *Econometrica*.

Chateauneuf, A., J. Eichberger, and S. Grant (2007). Choice under uncertainty with the best and worst in mind: Neo-additive capacities. *Journal of Economic Theory 137*(1), 538–567.

Chen, Y., P. Katuščák, and E. Ozdenoren (2007). Sealed bid auctions with ambiguity: Theory and experiments. *Journal of Economic Theory 136*(1), 513–535.

Cherbonnier, F. and C. Gollier (2015). Decreasing aversion under ambiguity. *Journal of Economic Theory 157*, 606–623.

Danan, E. and A. Ziegelmeyer (2006). Are preferences complete?: An experimental measurement of indecisiveness under risk. Working paper.

Dillenberger, D. and K. Rozen (2015). History-dependent risk attitude. *Journal of Economic Theory 157*, 445–477.

Echenique, F., L. Pomatto, and J. Vinson (2020). Twofold conservatism in choice under uncertainty. Working paper.

Eichberger, J., S. Grant, and D. Kelsey (2007). Updating choquet beliefs. *Journal of Mathematical Economics 43*(7-8), 888–899.

Eichberger, J., S. Grant, and D. Kelsey (2010). Comparing three ways to update choquet beliefs. *Economics Letters 107*(2), 91–94.

Eichberger, J., S. Grant, and D. Kelsey (2012). When is ambiguity–attitude constant? *Journal of Risk and Uncertainty 45*(3), 239–263.

Eichberger, J., S. Grant, D. Kelsey, and G. A. Koshevoy (2011). The α-meu model: A comment. *Journal of Economic Theory 146*(4), 1684–1698.

Epstein, L. G. and M. Schneider (2003). Recursive multiple-priors. *Journal of Economic Theory 113*(1), 1–31.

Faro, J. H. and J.-P. Lefort (2019). Dynamic objective and subjective rationality. *Theoretical Economics 14*(1), 1–14.

Gajdos, T., T. Hayashi, J.-M. Tallon, and J.-C. Vergnaud (2008). Attitude toward imprecise information. *Journal of Economic Theory 140*(1), 27–65.

Georgalos, K. (2019). An experimental test of the predictive power of dynamic ambiguity models. *Journal of Risk and Uncertainty 59*(1), 51–83.

Ghirardato, P., F. Maccheroni, and M. Marinacci (2004). Differentiating ambiguity and ambiguity attitude. *Journal of Economic Theory 118*(2), 133–173.

Ghirardato, P., F. Maccheroni, and M. Marinacci (2008). Revealed ambiguity and its consequences: updating. In *Advances in decision making Under Risk and Uncertainty*, pp. 3–18. Springer.

Ghirardato, P. and M. Marinacci (2002). Ambiguity made precise: A comparative foundation. *Journal of Economic Theory 102*(2), 251–289.

Gilboa, I., F. Maccheroni, M. Marinacci, and D. Schmeidler (2010). Objective and subjective rationality in a multiple prior model. *Econometrica 78*(2), 755–770.

Gilboa, I. and D. Schmeidler (1989). Maxmin expected utility with non-unique prior. *Journal of Mathematical Economics 18*(2), 141–153.

Grant, S., P. Rich, and J. Stecher (2021). Objective and subjective rationality and decisions with the best and worst case in mind. *Theory and Decision 90*(3), 309–320.

Gul, F. and W. Pesendorfer (2015). Hurwicz expected utility and subjective sources. *Journal of Economic Theory 159*, 465–488.

Hara, K. (2021). Incomplete preferences under uncertainty: Multiplicity in ambiguous beliefs. Working paper.

Hartmann, L. (2021). α-maxmin expected utility with non-unique prior. working paper.

Hedlund, J., T. F. Kauffeldt, and M. Lammert (2020). Persuasion under ambiguity. *Theory and Decision*, 1–28.

Hill, B. (2019). Beyond uncertainty aversion. Working paper.

Horie, M. (2013). Reexamination on updating choquet beliefs. *Journal of Mathematical Economics 49*(6), 467–470.

Hurwicz, L. (1951). Some specification problems and applications to econometric models. *Econometrica 19*(3), 343–344.

Jaffray, J. Y. (1994). Dynamic decision making with belief functions. In *Advances in the Dempster-Shafer Theory of Evidence*, pp. 331–352. Wiley.

Klibanoff, P., S. Mukerji, K. Seo, and L. Stanca (2021). Foundations of ambiguity models under symmetry: α-meu and smooth ambiguity. *Journal of Economic Theory*, 105202.

Kopylov, I. (2003). Alpha-maxmin expected utility. Working paper.

Kopylov, I. (2009). Choice deferral and ambiguity aversion. *Theoretical Economics 4*(2), 199–225.

Kopylov, I. (2016). Subjective probability, confidence, and bayesian updating. *Economic Theory 62*(4), 635–658.

Miyashita, M. and Y. Nakamura (2020). An axiomatic approach to failures in contingent reasoning. Working paper.

Nascimento, L. and G. Riella (2011). A class of incomplete and ambiguity averse preferences. *Journal of Economic Theory 146*(2), 728–750.

Nehring, K. (2009). Imprecise probabilistic beliefs as a context for decision-making under ambiguity. *Journal of Economic Theory 144*(3), 1054–1091.

Nishimura, K. G. and H. Ozaki (2006). An axiomatic approach to epsilon-contamination. *Economic Theory 27*(2), 333–340.

Olszewski, W. (2007). Preferences over sets of lotteries. *The Review of Economic Studies 74*(2), 567–595.

Pires, C. P. (2002). A rule for updating ambiguous beliefs. *Theory and Decision 53*(2), 137–152.

Rogers, A. and M. Ryan (2012). Additivity and uncertainty. *Economics Bulletin 32*(3), 1858–1864.

Saghafian, S. (2018). Ambiguous partially observable markov decision processes: Structural results and applications. *Journal of Economic Theory 178*, 1–35.

Sautua, S. I. (2017). Does uncertainty cause inertia in decision making? an experimental study of the role of regret aversion and indecisiveness. *Journal of Economic Behavior & Organization 136*, 1–14.

Schmeidler, D. (1989). Subjective probability and expected utility without additivity. *Econometrica*, 571–587.

Siniscalchi, M. (2006). A behavioral characterization of plausible priors. *Journal of Economic Theory 128*(1), 1–17.

Siniscalchi, M. (2011). Dynamic choice under ambiguity. *Theoretical Economics 6*(3), 379–421.

Trautmann, S. T. and G. van de Kuilen (2015). Ambiguity attitudes. *The Wiley Blackwell Handbook of Judgment and Decision Making 2*, 89–116.

Valenzuela-Stookey, Q. (2020). Subjective complexity under uncertainty. Working paper.