Sampling and Remote Estimation for the Ornstein-Uhlenbeck Process through Queues: Age of Information and Beyond

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***Abstract*—Recently, a connection between the age of information and remote estimation error was found in a sampling problem of Wiener processes: If the sampler has no knowledge of the signal being sampled, the optimal sampling strategy is to minimize the age of information; however, by exploiting causal knowledge of the signal values, it is possible to achieve a smaller estimation error. In this paper, we generalize the previous study by investigating a problem of sampling a stationary Gauss-Markov process named the Ornstein-Uhlenbeck (OU) process, where we aim to find useful insights for solving the problems of sampling more general signals. The optimal sampling problem is formulated as a constrained continuous-time Markov decision process (MDP) with an uncountable state space. We provide an exact solution to this MDP: The optimal sampling policy is a threshold policy on *instantaneous estimation error* and the threshold is found. Further, if the sampler has no knowledge of the OU process, the optimal sampling problem reduces to an MDP for minimizing a *nonlinear* age of information metric. The age-optimal sampling policy is a threshold policy on *expected estimation error* and the threshold is found. In both problems, the optimal sampling policies can be computed by low-complexity algorithms (e.g., bisection search and Newton's method), and the curse of dimensionality is circumvented. These results hold for (i) general service time distributions of the queueing server and (ii) sampling problems both with and without a sampling rate constraint. Numerical results are provided to compare different sampling policies.**

***Index Terms*—Age of information, Ornstein-Uhlenbeck process, sampling policy, threshold policy.**

I. INTRODUCTION

TIMELY updates of the system state are of significant importance for state estimation and decision making in networked control and cyber-physical systems, such as UAV navigation, robotics control, mobility tracking, and environment monitoring systems. To evaluate the freshness of state updates, the concept of *Age of Information*, or simply *age*, was introduced to measure the timeliness of state samples received from a remote transmitter [1]–[3]. Let U_t be the generation time of the freshest received state sample at time t. The age of information, as a function of t, is defined as $\Delta_t = t - U_t$, which is the time difference between the freshest samples available at the transmitter and receiver.

Recently, the age of information concept has received significant attention, because of the extensive applications of state updates among systems connected over communication networks. The states of many systems, such as UAV mobility trajectory and sensor measurements, are in the form of a signal X_t, that may change slowly at some time and vary more dynamically later. Hence, the time difference described by the age $\Delta_t = t - U_t$ only partially characterize the variation $X_t - X_{U_t}$ of the system state, and the state update policy that minimizes the age of information does not minimize the state estimation error. This result was first shown in [4], where a sampling problem of Wiener processes was solved and the optimal sampling policy was shown to have an intuitive structure. As the results therein hold only for signals that can be modeled as a Wiener process, one would wonder how to, and whether it is possible to, extend [4] for handling more general signal models.

In this paper, we generalize [4] by exploring a problem of sampling an Ornstein-Uhlenbeck (OU) process X_t. From the obtained results, we hope to find useful structural properties of the optimal sampler design that can be potentially applied to more general signal models. The OU process X_t is the continuous-time analogue of the well-known first-order autoregressive process, i.e., AR(1) process. The OU process is defined as the solution to the stochastic differential equation (SDE) [5], [6]

$$dX_t = \theta(\mu - X_t)dt + \sigma dW_t, \tag{1}$$

where μ, $\theta > 0$, and $\sigma > 0$ are parameters and W_t represents a Wiener process. It is the only nontrivial continuous-time process that is stationary, Gaussian, and Markovian [6]. Examples of first-order systems that can be described as the OU process include interest rates, currency exchange rates, and commodity prices (with modifications) [7], control systems such as node mobility in mobile ad-hoc networks, robotic swarms, and UAV systems [8], [9], and physical processes such as the transfer of liquids or gases in and out of a tank [10].

As shown in Fig. 1, samples of an OU process are forwarded to a remote estimator through a channel in a first-come, first-served (FCFS) fashion. The samples experience *i.i.d.* random transmission times over the channel, which is caused by random sample size, channel fading, interference, congestions, and etc. For example, UAVs flying close to WiFi access points may suffer from long communication delay and instability issues, because they receive strong interference from the WiFi access points [11]. We assume that at any time only one sample can be served by the channel. The samples that are waiting to be sent are stored in a queue at the transmitter. Hence,

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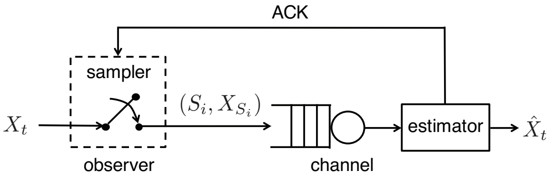

Fig. 1: System model.

the channel is modeled as an FCFS queue with *i.i.d.* service times. The service time distributions considered in this paper are quite general: they are only required to have a finite mean. This queueing model is helpful to analyze the robustness of remote estimation systems with occasionally long transmission times.

The estimator utilizes causally received samples to construct an estimate $\hat{X}_t$ of the real-time signal value X_t. The quality of remote estimation is measured by the time-average mean-squared estimation error, i.e.,

$$\mathsf{mse} = \limsup_{T \to \infty} \frac{1}{T} \mathbb{E}\left[\int_0^T (X_t - \hat{X}_t)^2 dt\right]. \tag{2}$$

Our goal is to find the optimal sampling policy that minimizes mse by causally choosing the sampling times subject to a maximum sampling rate constraint. In practice, the cost (e.g., energy, CPU cycle, storage) for state updates increases with the average sampling rate. Hence, we are striking to find the optimum tradeoff between estimation error and update cost. In addition, the unconstrained problem is also solved. The contributions of this paper are summarized as follows:

- The optimal sampling problem for minimize the mse under a sampling rate constraint is formulated as a constrained continuous-time Markov decision process (MDP) with an uncountable state space. Because of the curse of dimensionality, such problems are often lack of low-complexity solutions that are arbitrarily accurate. However, we were able to solve this MDP exactly: The optimal sampling policy is proven to be a threshold policy on *instantaneous* estimation error, where the threshold is a non-linear function $v(\beta)$ of a parameter β. The value of β is equal to the summation of the optimal objective value of the MDP and the optimal Lagrangian dual variable associated to the sampling rate constraint. If there is no sampling rate constraint, the Lagrangian dual variable is zero and hence β is exactly the optimal objective value. Among the technical tools developed to prove this result is a free boundary method [12], [13] for finding the optimal stopping time of the OU process.
- The optimal sampler design of Wiener process in [4] is a limiting case of the above result. By comparing the optimal sampling policies of OU process and Wiener process, we find that the threshold function $v(\beta)$ changes according to the signal model, where the parameter β is determined in the same way for both signal models.
- Further, we consider a class of signal-agnostic sampling policies, where the sampling times are determined without using knowledge of the signal value of the observed OU process; the parameters of the OU process are known. The optimal signal-agnostic sampling problem is equivalent to an MDP for minimizing the time-average of a nonlinear age function $p(\Delta_t)$, which has been solved recently in [14]. The age-optimal sampling policy is a threshold policy on *expected* estimation error, where the threshold function is simply $v(\beta) = \beta$ and the parameter β is determined in the same way as above.
- The above results hold for (i) general service time distributions with a finite mean and (ii) sampling problems both with and without a sampling rate constraint. Numerical results suggest that the optimal sampling policy is better than zero-wait sampling and the classic uniform sampling.

One interesting observation from these results is that the threshold function $v(\beta)$ varies with respect to the signal model and sampling problem, but the parameter β is determined in the same way.

A. Related Work

The results in this paper are tightly related to recent studies on the age of information Δ_t, e.g., [1], [14]–[35], which does not have a signal model. The average age and average peak age have been analyzed for various queueing systems in, e.g., [1], [18], [20], [21]. The optimality of the Last-Come, First-Served (LCFS) policy, or more generally the Last-Generated, First-Served (LGFS) policy, was established for various queueing system models in [24]–[26], [30]. Optimal sampling policies for minimizing non-linear age functions were developed in, e.g., [14], [15], [19], [35]. Age-optimal transmission scheduling of wireless networks were investigated in, e.g., [22], [23], [27]–[29], [31], [32].

On the other hand, this paper is also a contribution to the area of remote estimation, e.g., [10], [36]–[41]. In [37], [39], optimal sampling of Wiener processes was studied, where the transmission time from the sampler to the estimator is zero. Optimal sampling of OU processes was also considered in [37], which is solved by discretizing time and using dynamic programming to solve the discrete-time optimal stopping problems. However, our optimal sampler of OU processes is obtained analytically. Remote estimation over several different channel models was recently studied in, e.g., [40], [41]. In [10], [36]–[41], the optimal sampling policies were proven to be threshold policies. Because of the queueing model, our optimal sampling policy has a different structure from those in [10], [36]–[41]. Specifically, in our optimal sampling policy, sampling is suspended when the server is busy and is reactivated once the server becomes idle. In addition, we are able to characterize the threshold precisely. The optimal sampling policy for the Wiener process in [4] is a limiting case of ours. Remote estimation of the Wiener process with random two-way delay was recently considered in [42]. In [43], a jointly optimal sampler, quantizer, and estimator design was found for a class of continuous-time Markov processes under a bit-rate constraint. A recent survey on remote estimation systems was presented in [44].

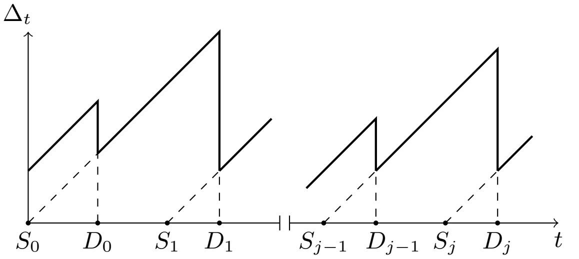

Fig. 2: Evolution of the age Δ_t over time.

II. MODEL AND FORMULATION

A. System Model

We consider the remote estimation system illustrated in Fig. 1, where an observer takes samples from an OU process X_t and forwards the samples to an estimator through a communication channel. The channel is modeled as a single-server FCFS queue with *i.i.d.* service times. The system starts to operate at time $t = 0$. The i-th sample is generated at time S_i and is delivered to the estimator at time D_i with a service time Y_i, which satisfy $S_i \leq S_{i+1}$, $S_i + Y_i \leq D_i$, $D_i + Y_{i+1} \leq D_{i+1}$, and $0 < \mathbb{E}[Y_i] < \infty$ for all i. Each sample packet (S_i, X_{S_i}) contains the sampling time S_i and the sample value X_{S_i}. Let $U_t = \max\{S_i : D_i \leq t\}$ be the sampling time of the latest received sample at time t. The *age of information*, or simply *age*, at time t is defined as [1], [2]

$$\Delta_t = t - U_t = t - \max\{S_i : D_i \leq t\}, \tag{3}$$

which is shown in Fig. 2. Because $D_i \leq D_{i+1}$, Δ_t can be also expressed as

$$\Delta_t = t - S_i, \text{ if } t \in [D_i, D_{i+1}),\ i = 0, 1, 2, \ldots \tag{4}$$

The initial state of the system is assumed to satisfy $S_0 = 0$, $D_0 = Y_0$, X_0 and Δ_0 are finite constants. The parameters μ, θ, and σ in (1) are known at both the sampler and estimator.

Let $I_t \in \{0, 1\}$ represent the idle/busy state of the server at time t. We assume that whenever a sample is delivered, an acknowledgement is sent back to the sampler with zero delay. By this, the idle/busy state I_t of the server is known at the sampler. Therefore, the information that is available at the sampler at time t can be expressed as $\{X_s, I_s : 0 \leq s \leq t\}$.

B. Sampling Policies

In causal sampling policies, each sampling time S_i is determined based on the up-to-date information that is available at the sampler, without using any future information. In probability theory, such sampling times are represented by *stopping times*.

To define *stopping time* precisely, the concepts of σ-field and filtration are needed. Let us define the σ-field

$$\mathcal{N}_t = \sigma(X_s, I_s : 0 \leq s \leq t),$$

which is the set of events whose occurrence are determined by the realization of the process $\{X_s, I_s, 0 \leq s \leq t\}$ up to time t. A filtration is a non-decreasing sequence of σ-fields. Our analysis requires a strong Markov property, which is satisfied when the filtration is right-continuous. Define

$$\mathcal{N}_t^+ = \cap_{s>t} \mathcal{N}_s, \tag{5}$$

then $\{\mathcal{N}_t^+, t \geq 0\}$ is a right-continuous filtration of the information process $\{X_s, I_s, t \geq 0\}$ [45]. In a causal sampling policy, each sampling time is a stopping time with respect to $\{\mathcal{N}_t^+, t \geq 0\}$, i.e.,

$$\{S_i \leq t\} \in \mathcal{N}_t^+,\ \forall t \geq 0. \tag{6}$$

In other words, whether sample i has been generated by time t (i.e., whether $\{S_i \leq t\}$ or $\{S_i > t\}$) is determined by the realization of the process $\{X_s, I_s, 0 \leq s \leq t\}$ up to time t.

Let $\pi = (S_1, S_2, ...)$ represent a sampling policy. We use Π to represent the set of *causal* sampling policies that satisfy two conditions: (i) Each sampling policy $\pi \in \Pi$ satisfies (6) for all i. (ii) The sequence of inter-sampling times $\{T_i = S_{i+1} - S_i, i = 0, 1, \ldots\}$ forms a *regenerative process* [46, Section 6.1]: There exists an increasing sequence $0 \leq k_1 < k_2 < \ldots$ of almost surely finite random integers such that the post-k_j process $\{T_{k_j+i}, i = 0, 1, \ldots\}$ has the same distribution as the post-k_0 process $\{T_{k_0+i}, i = 0, 1, \ldots\}$ and is independent of the pre-k_j process $\{T_i, i = 0, 1, \ldots, k_j - 1\}$; further, we assume that $\mathbb{E}[k_{j+1} - k_j] < \infty$, $\mathbb{E}[S_{k_1}] < \infty$, and $0 < \mathbb{E}[S_{k_{j+1}} - S_{k_j}] < \infty,\ j = 1, 2, \ldots$[1]

From this, we can obtain that S_i is finite almost surely for all i. We assume that the OU process $\{X_t, t \geq 0\}$ and the service times $\{Y_i, i = 1, 2, \ldots\}$ are mutually independent, and do not change according to the sampling policy.

A sampling policy $\pi \in \Pi$ is said to be *signal-agnostic* (*signal-aware*), if π is (not necessarily) independent of $\{X_t, t \geq 0\}$. Let $\Pi_{\text{signal-agnostic}} \subset \Pi$ denote the set of signal-agnostic sampling policies, defined as

$$\Pi_{\text{signal-agnostic}} = \{\pi \in \Pi : \pi \text{ is independent of } \{X_t, t \geq 0\}\}. \tag{7}$$

C. MMSE Estimator

According to (6), S_i is a finite stopping time. By using [52, Eq. (3)] and the strong Markov property of the OU process [12, Eq. (4.3.27)], X_t is expressed as

$$\begin{aligned} X_t =& X_{S_i} e^{-\theta(t-S_i)} + \mu\big[1 - e^{-\theta(t-S_i)}\big] \\ &+ \frac{\sigma}{\sqrt{2\theta}} e^{-\theta(t-S_i)} W_{e^{2\theta(t-S_i)}-1}, \text{ if } t \in [S_i, \infty). \end{aligned} \tag{8}$$

At any time $t \geq 0$, the estimator uses causally received samples to construct an estimate $\hat{X}_t$ of the real-time signal value X_t. The information available to the estimator consists of two parts: (i) $M_t = \{(S_i, X_{S_i}, D_i) : D_i \leq t\}$, which contains the sampling time S_i, sample value X_{S_i}, and delivery time D_i of the samples that have been delivered by time t

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[1] We will optimize $\limsup_{T\to\infty} \mathbb{E}[\int_0^T (X_t - \hat{X}_t)^2 dt]/T$, but operationally a nicer criterion is $\limsup_{i\to\infty} \mathbb{E}[\int_0^{D_i} (X_t - \hat{X}_t)^2 dt]/\mathbb{E}[D_i]$. These criteria are corresponding to two definitions of "average cost per unit time" that are widely used in the literature of semi-Markov decision processes. These two criteria are equivalent, if $\{T_1, T_2, \ldots\}$ is a regenerative process, or more generally, if $\{T_1, T_2, \ldots\}$ has only one ergodic class. If not condition is imposed, however, they are different. The interested readers are referred to [47]–[51] for more discussions.

and (ii) the fact that no sample has been received after the last delivery time $\max\{D_i : D_i \leq t\}$. Similar to [4], [37], [53], we assume that the estimator neglects the second part of information.[2] Then, as shown in Appendix C, the minimum mean square error (MMSE) estimator is determined by

$$\hat{X}_t = \mathbb{E}[X_t|M_t] = X_{S_i}e^{-\theta(t-S_i)} + \mu\big[1 - e^{-\theta(t-S_i)}\big], \\ \text{if } t \in [D_i, D_{i+1}),\ i = 0,1,2,\ldots \quad (9)$$

Hence, the estimation error of the MMSE estimator is

$$X_t - \hat{X}_t = \frac{\sigma}{\sqrt{2\theta}}e^{-\theta t}W_{e^{2\theta t}-1}, \\ \text{if } t \in [D_i, D_{i+1}), i = 0,1,2,\ldots \quad (10)$$

D. Problem Formulation

The goal of this paper is to find the optimal sampling policy that minimizes the mean-squared estimation error subject to an average sampling-rate constraint, which is formulated as the following problem:

$$\mathsf{mse}_{\text{opt}} = \inf_{\pi\in\Pi} \ \limsup_{T\to\infty} \frac{1}{T}\mathbb{E}\left[\int_0^T (X_t - \hat{X}_t)^2 dt\right] \quad (11)$$

$$\text{s.t.} \ \liminf_{n\to\infty} \frac{1}{n}\mathbb{E}\left[\sum_{i=1}^{n}(S_{i+1} - S_i)\right] \geq \frac{1}{f_{\max}}, \quad (12)$$

where $\mathsf{mse}_{\text{opt}}$ is the optimum value of (11) and $f_{\max}$ is the maximum allowed sampling rate. When $f_{\max} = \infty$, this problem becomes an unconstrained problem.

III. MAIN RESULTS

A. Signal-aware Sampling without Rate Constraint

Problem (11) is a constrained continuous-time MDP with a continuous state space. However, we found an exact solution to this problem. To present this solution, let us consider an OU process O_t with the initial state $O_t = 0$ and parameter $\mu = 0$. According to (8), O_t can be expressed as

$$O_t = \frac{\sigma}{\sqrt{2\theta}}e^{-\theta t}W_{e^{2\theta t}-1}. \quad (13)$$

Define

$$\mathsf{mse}_{Y_i} = \mathbb{E}[O_{Y_i}^2] = \frac{\sigma^2}{2\theta}\mathbb{E}[1 - e^{-2\theta Y_i}], \quad (14)$$

$$\mathsf{mse}_{\infty} = \mathbb{E}[O_{\infty}^2] = \frac{\sigma^2}{2\theta}. \quad (15)$$

In the sequel, we will see that mse_{Y_i} and mse_{∞} are the lower and upper bounds of $\mathsf{mse}_{\text{opt}}$, respectively. According to (10) and (13)-(15), mse_{Y_i} represents the estimation error when the estimation is made based on a sample that was generated Y_i seconds ago, and mse_{∞} represents the estimation error for the case that no sample has been delivered to the estimator before. We will also need to use the function[3]

$$G(x) = \frac{e^{x^2}}{x}\int_0^x e^{-t^2}dt = \frac{e^{x^2}}{x}\frac{\sqrt{\pi}}{2}\,\mathrm{erf}(x),\ x \in [0,\infty), \quad (16)$$

where $\mathrm{erf}(\cdot)$ is the error function [54], defined as

$$\mathrm{erf}(x) = \frac{2}{\sqrt{\pi}}\int_0^x e^{-t^2}dt. \quad (17)$$

We first consider the unconstrained optimal sampling problem, i.e., $f_{\max} = \infty$, such that the rate constraint (12) can be removed. In this scenario, the optimal sampler is provided in the following theorem.

Theorem 1. *(Sampling without Rate Constraint). If $f_{\max} = \infty$ and the Y_i's are i.i.d. with $0 < \mathbb{E}[Y_i] < \infty$, then $(S_1(\beta), S_2(\beta), \ldots)$ with a parameter β is an optimal solution to* (11)*, where*

$$S_{i+1}(\beta) = \inf\left\{t \geq D_i(\beta) : \big|X_t - \hat{X}_t\big| \geq v(\beta)\right\}, \quad (18)$$

$D_i(\beta) = S_i(\beta) + Y_i$, $v(\beta)$ is defined by

$$v(\beta) = \frac{\sigma}{\sqrt{\theta}}G^{-1}\left(\frac{\mathsf{mse}_{\infty} - \mathsf{mse}_{Y_i}}{\mathsf{mse}_{\infty} - \beta}\right), \quad (19)$$

$G^{-1}(\cdot)$ is the inverse function of $G(\cdot)$ in (16) *and β is the unique root of*

$$\mathbb{E}\left[\int_{D_i(\beta)}^{D_{i+1}(\beta)} (X_t - \hat{X}_t)^2 dt\right] - \beta\mathbb{E}[D_{i+1}(\beta) - D_i(\beta)] = 0. \quad (20)$$

The optimal objective value to (11) *is given by*

$$\mathsf{mse}_{\text{opt}} = \frac{\mathbb{E}\left[\int_{D_i(\beta)}^{D_{i+1}(\beta)}(X_t - \hat{X}_t)^2 dt\right]}{\mathbb{E}[D_{i+1}(\beta) - D_i(\beta)]}. \quad (21)$$

Furthermore, β is exactly the optimal value to (11)*, i.e., $\beta = \mathsf{mse}_{\text{opt}}$.*

The proof of Theorem 1 is explained in Section IV. The optimal sampling policy in Theorem 1 has a nice structure. Specifically, the $(i+1)$-th sample is taken at the earliest time t satisfying two conditions: (i) The i-th sample has already been delivered by time t, i.e., $t \geq D_i(\beta)$, and (ii) the estimation error $|X_t - \hat{X}_t|$ is no smaller than a pre-determined threshold $v(\beta)$, where $v(\cdot)$ is a non-linear function defined in (19). In Section IV, it is shown that $\mathsf{mse}_{Y_i} \leq \beta < \mathsf{mse}_{\infty}$. Further, it is not hard to show that $G(x)$ is strictly increasing on $[0,\infty)$ and $G(0) = 1$. Hence, its inverse function $G^{-1}(\cdot)$ and the threshold $v(\beta)$ are properly defined and $v(\beta) \geq 0$.

1) Three Algorithms for Solving (20)*:* We now present three algorithms for computing the root of (20). Because the $S_i(\beta)$'s are stopping times, numerically calculating the expectations in (20) appears to be a difficult task. Nonetheless, this challenge can be solved by resorting to the following lemma, which is obtained by using Dynkin's formula [13, Theorem 7.4.1] and the optional stopping theorem.

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[2] We note that this assumption can be removed by considering a joint sampler and estimator design problem. Specifically, it was shown in [10], [36], [38], [40], [41] that when the sampler and estimator are jointly optimized in discrete-time systems, the optimal estimator has the same expression no matter with or without the second part of information. As pointed out in [36, p. 619], such a structure property of the MMSE estimator can be also established for continuous-time systems. The goal of this paper is to find the closed-form expression of the optimal sampler under this assumption. The remaining task of finding the jointly optimal sampler and estimator design can be done by further using the majorization techniques developed in [10], [36], [38], [40], [41]; see [43] for a recent treatment on this task.

[3] If $x = 0$, $G(x)$ is defined as its right limit $G(0) = \lim_{x\to 0^+} G(x) = 1$.

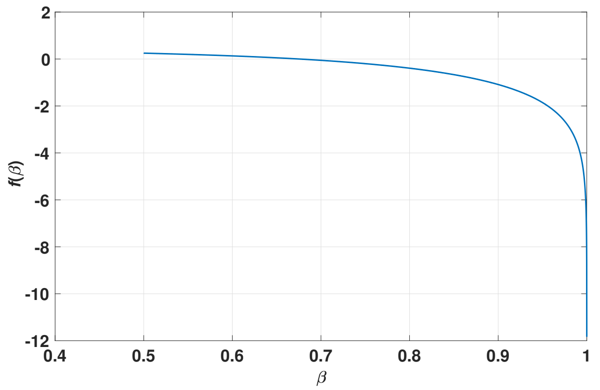

Fig. 3: $f(\beta)$ in (30) for *i.i.d.* exponential service time with $\mathbb{E}[Y_i]=1$, where the parameters of the OU process are $\sigma=1$ and $\theta=0.5$. For these parameters, $\mathsf{mse}_{Y_i}=0.5$ and $\mathsf{mse}_\infty=1$.

Lemma 1. *In Theorem 1, it holds that*

$$\begin{aligned}&\mathbb{E}[D_{i+1}(\beta)-D_i(\beta)]\\&=\mathbb{E}[\max\{R_1(v(\beta))-R_1(O_{Y_i}),0\}]+\mathbb{E}[Y_i],\end{aligned}\tag{22}$$

$$\begin{aligned}&\mathbb{E}\left[\int_{D_i(\beta)}^{D_{i+1}(\beta)}(X_t-\hat{X}_t)^2dt\right]\\&=\mathbb{E}[\max\{R_2(v(\beta))-R_2(O_{Y_i}),0\}]\\&\quad+\mathsf{mse}_\infty[\mathbb{E}(Y_i)-\gamma]+\mathbb{E}\left[\max\{v^2(\beta),O_{Y_i}^2\}\right]\gamma,\end{aligned}\tag{23}$$

where

$$\gamma=\frac{1}{2\theta}\mathbb{E}[1-e^{-2\theta Y_i}],\tag{24}$$

$$R_1(v)=\frac{v^2}{\sigma^2}\,{}_2F_2\left(1,1;\frac{3}{2},2;\frac{\theta}{\sigma^2}v^2\right),\tag{25}$$

$$R_2(v)=-\frac{v^2}{2\theta}+\frac{v^2}{2\theta}\,{}_2F_2\left(1,1;\frac{3}{2},2;\frac{\theta}{\sigma^2}v^2\right).\tag{26}$$

Proof. See Appendix I. ☐

In (25) and (26), we have used the generalized hypergeometric function, which is defined by [55, Eq. 16.2.1]

$$\begin{aligned}&{}_pF_q(a_1,a_2,\cdots,a_p;b_1,b_2,\cdots b_q;z)\\&=\sum_{n=0}^{\infty}\frac{(a_1)_n(a_2)_n\cdots(a_p)_n}{(b_1)_n(b_2)_n\cdots(b_p)_n}\frac{z^n}{n!},\end{aligned}\tag{27}$$

where

$$(a)_0=1,\tag{28}$$

$$(a)_n=a(a+1)(a+2)\cdots(a+n-1),\ n\geq 1.\tag{29}$$

Using Lemma 1, the expectations in (20) can be evaluated by Monte Carlo simulations of scalar random variables O_{Y_i} and Y_i, which is much simpler than directly simulating the entire random process $\{O_t,t\geq 0\}$.

For notational simplicity, we rewrite (20) as

$$f(\beta)=f_1(\beta)-\beta f_2(\beta)=0,\tag{30}$$

where $f_1(\beta)=\mathbb{E}\left[\int_{D_i(\beta)}^{D_{i+1}(\beta)}(X_t-\hat{X}_t)^2dt\right]$ and $f_2(\beta)=\mathbb{E}[D_{i+1}(\beta)-D_i(\beta)]$. The function $f(\beta)$ has several nice properties, which are asserted in the following lemma and illustrated in Fig. 3.

Algorithm 1 Bisection search method for solving (20)

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given $l = \mathsf{mse}_{Y_i}$, $u = \mathsf{mse}_\infty$, tolerance $\epsilon > 0$.
repeat
    $\beta := (l+u)/2$.
    $o := f_1(\beta) - \beta f_2(\beta)$.
    if $o \geq 0$, $l := \beta$; else, $u := \beta$.
until $u - l \leq \epsilon$.
return $\beta$.
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Algorithm 2 Newton's method for solving (20)

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given tolerance $\epsilon > 0$.
Pick initial value $\beta_0 \in [\mathsf{mse}_{\text{opt}}, \mathsf{mse}_\infty)$.
repeat
    $\beta_{k+1} := \beta_k - \frac{f(\beta_k)}{f'(\beta_k)}$.
until $|\frac{f(\beta_k)}{f'(\beta_k)}| \leq \epsilon$.
return $\beta_{k+1}$.
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Algorithm 3 Fixed-point iterations for solving (20)

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given tolerance $\epsilon > 0$.
Pick initial value $\beta_0 \in [\mathsf{mse}_{\text{opt}}, \mathsf{mse}_\infty)$.
repeat
    $\beta_{k+1} := \frac{f_1(\beta_k)}{f_2(\beta_k)}$.
until $|\beta_{k+1} - \frac{f_1(\beta_k)}{f_2(\beta_k)}| \leq \epsilon$.
return $\beta_{k+1}$.
```

Lemma 2. *The function $f(\beta)$ has the following properties:*

(i) $f(\beta)$ is concave, continuous, and strictly decreasing in β,

(ii) $f(\mathsf{mse}_{Y_i})>0$ and $\lim_{\beta\to\mathsf{mse}_\infty^-}f(\beta)=-\infty$.

Proof. See Appendix A. ☐

The uniqueness of the root of $f(\beta)$ follows immediately from Lemma 2.

Because $f(\beta)$ is decreasing and has a unique root, one can use a bisection search method to solve (20), which is illustrated in Algorithm 1. The bisection search method has a globally linear convergence speed.

To achieve an even faster convergence speed, we can use Newton's method [56]

$$\beta_{k+1}=\beta_k-\frac{f(\beta_k)}{f'(\beta_k)}\tag{31}$$

to solve (20), as shown in Algorithm 2. We suggest choosing the initial value β_0 of Newton's method from the set $[\mathsf{mse}_{\text{opt}},\mathsf{mse}_\infty)$, i.e., β_0 is larger than the root $\mathsf{mse}_{\text{opt}}$. Such an initial value β_0 can be found by taking a few bisection search iterations, or by using the mse of a sub-optimal sampling policy [57]. Because $f(\beta)$ is a concave function, the choice of initial value $\beta_0\in[\mathsf{mse}_{\text{opt}},\mathsf{mse}_\infty)$ ensures that β_k is a decreasing sequence converging to $\mathsf{mse}_{\text{opt}}$ [58]. Moreover, because $R_1(\cdot)$ and $R_2(\cdot)$ are twice continuously differentiable,

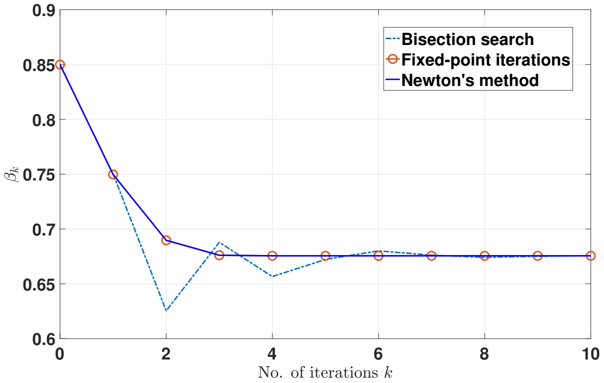

Fig. 4: Convergence of three algorithms for solving (20), where the service times are *i.i.d.* exponential with mean $\mathbb{E}[Y_i] = 1$, the parameters of the OU process are $\sigma = 1$ and $\theta = 0.5$.

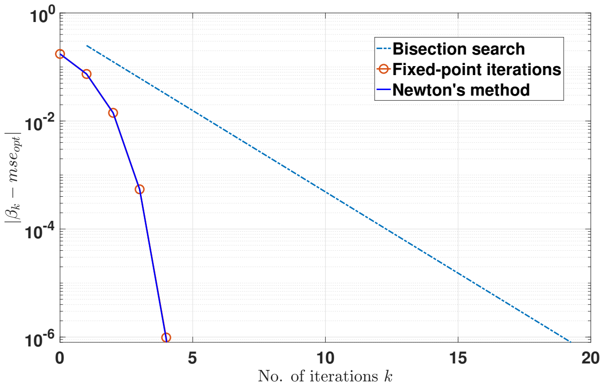

Fig. 5: Convergence of three algorithms for solving (20), where the service times are *i.i.d.* exponential with mean $\mathbb{E}[Y_i] = 1$, the parameters of the OU process are $\sigma = 1$ and $\theta = 0.5$. For bisection search, we plot the difference $|u - l|$ between the upper bound u and lower bound l, which is an upper bound of $|\beta_k - \mathsf{mse}_{\text{opt}}|$.

the function $f(\beta)$ is twice continuously differentiable. Therefore, Newton's method is known to have a locally quadratic convergence speed in the neighborhood of the root $\mathsf{mse}_{\text{opt}}$ [56, Chapter 2].

Newton's method requires to compute the gradient $f'(\beta_k)$, which can be solved by a finite-difference approximation, as in the secant method [56]. In the sequel, we introduce another approximation approach of Newton's method, which is of independent interest. In Theorem 1, we have shown that

$$\mathsf{mse}_{\text{opt}} = \underset{\beta \in [\mathsf{mse}_{Y_i}, \mathsf{mse}_\infty)}{\operatorname{argmax}} \frac{f_1(\beta)}{f_2(\beta)}. \tag{32}$$

Hence, the gradient of $f_1(\beta)/f_2(\beta)$ is equal to zero at the optimal solution $\beta = \mathsf{mse}_{\text{opt}}$, which leads to

$$f_1'(\mathsf{mse}_{\text{opt}}) f_2(\mathsf{mse}_{\text{opt}}) - f_1(\mathsf{mse}_{\text{opt}}) f_2'(\mathsf{mse}_{\text{opt}}) = 0. \tag{33}$$

Therefore,

$$\mathsf{mse}_{\text{opt}} = \frac{f_1(\mathsf{mse}_{\text{opt}})}{f_2(\mathsf{mse}_{\text{opt}})} = \frac{f_1'(\mathsf{mse}_{\text{opt}})}{f_2'(\mathsf{mse}_{\text{opt}})}. \tag{34}$$

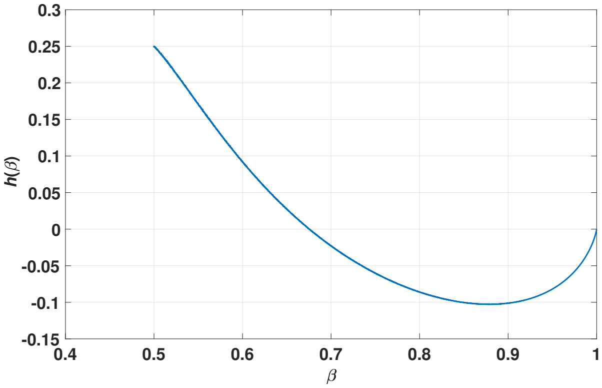

Fig. 6: The function $h(\beta)$ in (36) for *i.i.d.* exponential service time with $\mathbb{E}[Y_i] = 1$, where the parameters of the OU process are $\sigma = 1$ and $\theta = 0.5$. For these parameters, $\mathsf{mse}_{Y_i} = 0.5$ and $\mathsf{mse}_\infty = 1$.

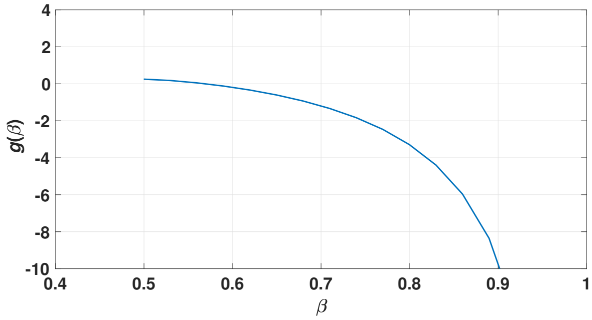

Fig. 7: The function $g(\beta)$ in (40) for *i.i.d.* exponential service time with $\mathbb{E}[Y_i] = 1$ and $f_{\max} = 0.8$, where the parameters of the OU process are $\sigma = 1$ and $\theta = 0.5$. For these parameters, $\mathsf{mse}_{Y_i} = 0.5$ and $\mathsf{mse}_\infty = 1$.

Because $f_1(\beta)$ and $f_2(\beta)$ are smooth functions, when β_k is in the neighborhood of $\mathsf{mse}_{\text{opt}}$, (34) implies that $f_1'(\beta_k) - \beta_k f_2'(\beta_k) \approx f_1'(\mathsf{mse}_{\text{opt}}) - \mathsf{mse}_{\text{opt}} f_2'(\mathsf{mse}_{\text{opt}}) = 0$. Substituting this into (31), yields

$$\begin{aligned} \beta_{k+1} =& \beta_k - \frac{f_1(\beta_k) - \beta_k f_2(\beta_k)}{f_1'(\beta_k) - f_2(\beta_k) - \beta_k f_2'(\beta_k)} \\ \approx& \beta_k - \frac{f_1(\beta_k) - \beta_k f_2(\beta_k)}{-f_2(\beta_k)} \\ =& \frac{f_1(\beta_k)}{f_2(\beta_k)}, \end{aligned} \tag{35}$$

which is a fixed-point iterative algorithm (see Algorithm 3) that was recently proposed in [57]. Similar to Newton's method, the fixed-point updates in (35) converge to $\mathsf{mse}_{\text{opt}}$ if the initial value $\beta_0 \in [\mathsf{mse}_{\text{opt}}, \mathsf{mse}_\infty)$. Moreover, (35) has a locally quadratic convergence speed, see [57] for a proof of this result. A numerical comparison of these three algorithms is shown in Fig. 4 and Fig. 5. One can observe that the fixed-point updates and Newton's method converge faster than bisection search.

We note that although (20), and equivalently (30), has a unique root $\mathsf{mse}_{\text{opt}}$, the fixed-point equation

$$h(\beta) = \frac{f_1(\beta)}{f_2(\beta)} - \beta = \frac{f_1(\beta) - \beta f_2(\beta)}{f_2(\beta)} = 0 \tag{36}$$

has two roots $\mathsf{mse}_{\text{opt}}$ and mse_{∞}. See Fig. 6 for an illustration of the two roots of $h(\beta)$. As shown in Appendix O, the correct root for computing the optimal threshold is $\mathsf{mse}_{\text{opt}}$. Interestingly, Algorithms 1-3 converge to the desired root $\mathsf{mse}_{\text{opt}}$, instead of mse_{∞}. Finally, we remark that these three algorithms can be used to find the optimal threshold in the age-optimal sampling problem studied in, e.g., [14], [15].

B. Signal-aware Sampling with Rate Constraint

When the sampling rate constraint (12) is taken into consideration, a solution to (11) is expressed in the following theorem:

Theorem 2. *(Sampling with Rate Constraint). If the Y_i's are i.i.d. with $0 < \mathbb{E}[Y_i] < \infty$, then* (18)-(20) *is an optimal solution to* (11). *The value of $\beta \geq 0$ is determined in two cases: β is the unique root of* (20) *if the root of* (20) *satisfies*

$$\mathbb{E}[D_{i+1}(\beta) - D_i(\beta)] > 1/f_{\max}; \tag{37}$$

otherwise, β is the unique root of

$$\mathbb{E}[D_{i+1}(\beta) - D_i(\beta)] = 1/f_{\max}. \tag{38}$$

The optimal objective value to (11) *is given by*

$$\mathsf{mse}_{\text{opt}} = \frac{\mathbb{E}\left[\int_{D_i(\beta)}^{D_{i+1}(\beta)} (X_t - \hat{X}_t)^2 dt\right]}{\mathbb{E}[D_{i+1}(\beta) - D_i(\beta)]}. \tag{39}$$

The proof of Theorem 2 is explained in Section IV. One can see that Theorem 1 is a special case of Theorem 2 when $f_{\max} = \infty$.

In Theorem 2, the calculation of β falls into two cases: In one case, β can be computed by solving (20) via Algorithms 1-3. For this case to occur, the sampling rate constraint (12) needs to be inactive at the root of (20). Because $D_i(\beta) = S_i(\beta) + Y_i$, we can obtain $\mathbb{E}[D_{i+1}(\beta) - D_i(\beta)] = \mathbb{E}[S_{i+1}(\beta) - S_i(\beta)]$ and hence (37) holds when the sampling rate constraint (12) is inactive.

In the other case, β is selected to satisfy the sampling rate constraint (12) with equality, as required in (38). Before we solve (38), let us first use $f_2(\beta)$ to express (38) as

$$g(\beta) = \frac{1}{f_{\max}} - f_2(\beta) = 0. \tag{40}$$

Lemma 3. *The function $g(\beta)$ has the following properties:*
(i) $g(\beta)$ is continuous and strictly decreasing in β,
(ii) $g(\mathsf{mse}_{Y_i}) \geq 0$ and $\lim_{\beta \to \mathsf{mse}_{\infty}^-} g(\beta) = -\infty$ if the root of (20) *does not satisfy* (37).

Proof. See Appendix B. □

According to Lemma 3, (38) has a unique root in $[\mathsf{mse}_{Y_i}, \mathsf{mse}_{\infty})$, which is denoted as β^*. In addition, the numerical results in Fig. 7 suggest that $g(\beta)$ should be concave, for which we do not have a proof.

Algorithm 4 Bisection search method for solving (38)

```
given $l = \mathsf{mse}_{Y_i}$, $u = \mathsf{mse}_{\infty}$, tolerance $\epsilon > 0$.
repeat
    $\beta := (l + u)/2$.
    $o := \mathbb{E}[D_{i+1}(\beta) - D_i(\beta)]$.
    if $o \geq 1/f_{\max}$, $u := \beta$; else, $l := \beta$.
until $u - l \leq \epsilon$.
return $\beta$.
```

Algorithm 5 Newton's method for solving (38)

```
given tolerance $\epsilon > 0$.
Pick initial value $\beta_0 \in [\beta^*, \mathsf{mse}_{\infty})$.
repeat
    $\beta_{k+1} := \beta_k - \frac{g(\beta_k)}{g'(\beta_k)}$.
until $|\frac{g(\beta_k)}{g'(\beta_k)}| \leq \epsilon$.
return $\beta_{k+1}$.
```

The root β^* can be solved by using bisection search and Newton's method, which are explained in Algorithms 4-5, respectively. Similar to the discussions in Section III-A1, the convergence of Algorithm 4 is ensured by Lemma 3. Moreover, if $g(\beta)$ is concave and $\beta_0 \in [\beta^*, \mathsf{mse}_{\infty})$, β_k in Algorithm 5 is a decreasing sequence converging to the root β^* of (38) [58].
ter

C. Special Case: Sampling of the Wiener Process

In the limiting case that $\sigma = 1$ and $\theta \to 0$, the OU process X_t in (1) becomes a Wiener process $X_t = W_t$. In this case, the MMSE estimator in (9) is given by

$$\hat{X}_t = W_{S_i}, \text{ if } t \in [D_i, D_{i+1}). \tag{41}$$

As shown in Appendix E, $v(\cdot)$ defined by (19) tends to

$$v(\beta) = \sqrt{3(\beta - \mathbb{E}[Y_i])}. \tag{42}$$

Theorem 3. *If $\sigma = 1$, $\theta \to 0$, and the Y_i's are i.i.d. with $0 < \mathbb{E}[Y_i] < \infty$, then $(S_1(\beta), S_2(\beta), \ldots)$ with a parameter β is an optimal solution to* (11), *where*

$$S_{i+1}(\beta) = \inf\left\{t \geq D_i(\beta) : \left|X_t - \hat{X}_t\right| \geq \sqrt{3(\beta - \mathbb{E}[Y_i])}\right\}, \tag{43}$$

$D_i(\beta) = S_i(\beta) + Y_i$. The value of $\beta \geq 0$ is determined in two cases: β is the unique root of

$$\mathbb{E}\left[\int_{D_i(\beta)}^{D_{i+1}(\beta)} (X_t - \hat{X}_t)^2 dt\right] - \beta \mathbb{E}[D_{i+1}(\beta) - D_i(\beta)] = 0, \tag{44}$$

if the root of (44) *satisfies $\mathbb{E}[D_{i+1}(\beta) - D_i(\beta)] > 1/f_{\max}$; otherwise, β is the unique root of $\mathbb{E}[D_{i+1}(\beta) - D_i(\beta)] = 1/f_{\max}$. The optimal objective value to* (11) *is given by*

$$\mathsf{mse}_{\text{opt}} = \frac{\mathbb{E}\left[\int_{D_i(\beta)}^{D_{i+1}(\beta)} (X_t - \hat{X}_t)^2 dt\right]}{\mathbb{E}[D_{i+1}(\beta) - D_i(\beta)]}. \tag{45}$$

Theorem 3 is an alternative form of Theorem 1 in [4] and hence its proof is omitted. The benefit of the new expression in Theorem 3 is that it allows to character β based on the optimal objective value $\mathsf{mse}_{\text{opt}}$ and the sampling rate constraint (12), in the same way as in Theorems 1-2. This appears to be more fundamental than the expression in [4]. The new form of optimal sampling policy of Wiener processes was also discovered in [42] without considering the constraint on (12).

D. Signal-agnostic Sampling

In signal-agnostic sampling policies, the sampling times S_i are determined based only on the service times Y_i, but not on the observed OU process $\{X_t, t \geq 0\}$.

Lemma 4. *If $\pi \in \Pi_{\textit{signal-agnostic}}$, then the mean-squared estimation error of the OU process X_t at time t is*

$$p(\Delta_t) = \mathbb{E}\left[(X_t - \hat{X}_t)^2 \big| \pi, Y_1, Y_2, \ldots\right] = \frac{\sigma^2}{2\theta}\left(1 - e^{-2\theta\Delta_t}\right), \tag{46}$$

which is a strictly increasing function of the age Δ_t.

Proof. See Appendix D. □

According to Lemma 4, for every policy $\pi \in \Pi_{\text{signal-agnostic}}$,

$$\mathbb{E}\left[\int_0^T (X_t - \hat{X}_t)^2 dt\right] = \mathbb{E}\left[\int_0^T p(\Delta_t) dt\right]. \tag{47}$$

Hence, minimizing the mean-squared estimation error among signal-agnostic sampling policies can be formulated as the following MDP for minimizing the expected time-average of the nonlinear age function $p(\Delta_t)$ in (46):

$$\mathsf{mse}_{\text{age-opt}} = \inf_{\pi \in \Pi_{\text{signal-agnostic}}} \limsup_{T \to \infty} \frac{1}{T} \mathbb{E}\left[\int_0^T p(\Delta_t) dt\right] \tag{48}$$

$$\text{s.t.} \quad \liminf_{n \to \infty} \frac{1}{n} \mathbb{E}\left[\sum_{i=1}^{n} (S_{i+1} - S_i)\right] \geq \frac{1}{f_{\max}}, \tag{49}$$

where $\mathsf{mse}_{\text{age-opt}}$ is the optimal value of (48). By (46), $p(\Delta_t)$ and $\mathsf{mse}_{\text{age-opt}}$ are bounded. Because $\Pi_{\text{signal-agnostic}} \subset \Pi$, it follows immediately that $\mathsf{mse}_{\text{opt}} \leq \mathsf{mse}_{\text{age-opt}}$.

Problem (48) is one instance of the problems recently solved in Corollary 3 of [14] for general strictly increasing functions $p(\cdot)$. From this, a solution to (48) for signal-agnostic sampling is given by

Theorem 4. *If the Y_i's are i.i.d. with $0 < \mathbb{E}[Y_i] < \infty$, then $(S_1(\beta), S_2(\beta), \ldots)$ with a parameter β is an optimal solution to* (48)*, where*

$$S_{i+1}(\beta) = \inf\left\{t \geq D_i(\beta) : \mathbb{E}[(X_{t+Y_{i+1}} - \hat{X}_{t+Y_{i+1}})^2] \geq \beta\right\}, \tag{50}$$

$D_i(\beta) = S_i(\beta) + Y_i$ and β is the unique root of

$$\mathbb{E}\left[\int_{D_i(\beta)}^{D_{i+1}(\beta)} (X_t - \hat{X}_t)^2 dt\right] - \beta \mathbb{E}[D_{i+1}(\beta) - D_i(\beta)] = 0, \tag{51}$$

if the root of (51) *satisfies $\mathbb{E}[D_{i+1}(\beta) - D_i(\beta)] > 1/f_{\max}$; otherwise, β is the unique root of*

$$\mathbb{E}[D_{i+1}(\beta) - D_i(\beta)] = 1/f_{\max}. \tag{52}$$

The optimal objective value to (48) *is given by*

$$\mathsf{mse}_{\text{age-opt}} = \frac{\mathbb{E}\left[\int_{D_i(\beta)}^{D_{i+1}(\beta)} (X_t - \hat{X}_t)^2 dt\right]}{\mathbb{E}[D_{i+1}(\beta) - D_i(\beta)]}. \tag{53}$$

Theorem 4 follows from Corollary 3 of [14] and Lemma 4. Similar to the case of signal-aware sampling, the roots of (51) and (52) can be solved by using Algorithms 1-5. In fact, Algorithms 1-5 can be used for minimizing general non-decreasing age penalty [14].

E. Discussions of the Results

The difference among Theorems 1-4 is only in the expressions (18), (43), (50) of threshold policies. In signal-aware sampling policies (18) and (43), the sampling time is determined by the *instantaneous* estimation error $|X_t - \hat{X}_t|$, and the threshold function $v(\cdot)$ is determined by the specific signal model. In the signal-agnostic sampling policy (50), the sampling time is determined by the *expected* estimation error $\mathbb{E}[(X_{t+Y_{i+1}} - \hat{X}_{t+Y_{i+1}})^2]$ at time $t + Y_{i+1}$. We note that if $t = S_{i+1}(\beta)$, then $t + Y_{i+1} = S_{i+1}(\beta) + Y_{i+1} = D_{i+1}(\beta)$ is the delivery time of the new sample. Hence, (50) requires that the expected estimation error upon the delivery of the new sample is no less than β. The parameter β in Theorems 1-4 is determined by the optimal objective value and the sampling rate constraint in the same manner. Later on in (69), we will further see that β is exactly equal to the summation of the optimal objective value of the MDP and the optimal Lagrangian dual variable associated to the sampling rate constraint. Finally, it is worth noting that Theorems 1-4 hold for all distributions of the service times Y_i satisfying $0 < \mathbb{E}[Y_i] < \infty$, and for both constrained and unconstrained sampling problems.

IV. PROOF OF THE MAIN RESULTS

We first provide the proof of Theorem 2. After that Theorem 1 follows immediately because it is a special case of Theorem 2. We prove Theorem 2 in four steps: (i) We first show that sampling should be suspended when the server is busy, which can be used to simplify (11). (ii) We use an extended Dinkelbach's method [59] and Lagrangian duality method to decompose the simplified problem into a series of mutually independent per-sample MDP. (iii) We utilize the free boundary method from optimal stopping theory [12] to solve the per-sample MDPs analytically. (iv) Finally, we use a geometric multiplier method [60] to show that the duality gap is zero. The above proof framework is an extension to that used in [4], [14], and the most challenging part is Step (iii).

A. Preliminaries

The OU process O_t in (13) with initial state $O_t = 0$ and parameter $\mu = 0$ is the solution to the SDE

$$dO_t = -\theta O_t dt + \sigma dW_t. \tag{54}$$

In addition, the infinitesimal generator of O_t is [61, Eq. A1.22]

$$\mathcal{G} = -\theta u \frac{\partial}{\partial u} + \frac{\sigma^2}{2} \frac{\partial^2}{\partial u^2}. \tag{55}$$

According to (8) and (9), the estimation error $(X_t - \hat{X}_t)$ is of the same distribution with O_{t-S_i}, if $t \in [D_i, D_{i+1})$. By using Dynkin's formula and the optional stopping theorem, we obtain the following lemma.

Lemma 5. *Let $\tau \geq 0$ be a stopping time of the OU process O_t with $\mathbb{E}[\tau] < \infty$, then*

$$\mathbb{E}\left[\int_0^\tau O_t^2 dt\right] = \mathbb{E}\left[\frac{\sigma^2}{2\theta}\tau - \frac{1}{2\theta}O_\tau^2\right]. \tag{56}$$

If, in addition, τ is the first exit time of a bounded set, then

$$\mathbb{E}[\tau] = \mathbb{E}[R_1(O_\tau)], \tag{57}$$

$$\mathbb{E}\left[\int_0^\tau O_t^2 dt\right] = \mathbb{E}[R_2(O_\tau)], \tag{58}$$

where $R_1(\cdot)$ and $R_2(\cdot)$ are defined in (25) *and* (26)*, respectively.*

Proof. See Appendix F. □

B. Suspend Sampling When the Server is Busy

By using the strong Markov property of the OU process X_t and the orthogonality principle of MMSE estimation, we obtain the following useful lemma:

Lemma 6. *Suppose that a feasible sampling policy for problem* (11) *is π, in which at least one sample is taken when the server is busy processing an earlier generated sample. Then, there exists another feasible policy π' for problem* (11) *which has a smaller estimation error than policy π. Therefore, in* (11)*, it is suboptimal to take a new sample before the previous sample is delivered.*

Proof. See Appendix G. □

A similar result was obtained in [4] for the sampling of Wiener processes. By Lemma 6, there is no loss to consider a sub-class of sampling policies $\Pi_1 \subset \Pi$ such that each sample is generated and sent out after all previous samples are delivered, i.e.,

$$\Pi_1 = \{\pi \in \Pi : S_i = G_i \geq D_{i-1} \text{ for all } i\}.$$

For any policy $\pi \in \Pi_1$, the *information* used for determining S_i includes: (i) the history of signal values $(X_t : t \in [0, S_i])$ and (ii) the service times $(Y_1, \ldots, Y_{i-1})$ of previous samples. Let us define the σ-fields $\mathcal{F}_t = \sigma(X_s : s \in [0, t])$ and $\mathcal{F}_t^+ = \cap_{r>t}\mathcal{F}_r$. Then, $\{\mathcal{F}_t^+, t \geq 0\}$ is the filtration (i.e., a non-decreasing and right-continuous family of σ-fields) of the OU process X_t. Given the service times $(Y_1, \ldots, Y_{i-1})$ of previous samples, S_i is a *stopping time* with respect to the filtration $\{\mathcal{F}_t^+, t \geq 0\}$ of the OU process X_t, that is

$$[\{S_i \leq t\}|Y_1, \ldots, Y_{i-1}] \in \mathcal{F}_t^+. \tag{59}$$

Hence, the policy space Π_1 can be expressed as

$$\begin{aligned}\Pi_1 = \{S_i : [\{S_i \leq t\}|Y_1, \ldots, Y_{i-1}] \in \mathcal{F}_t^+,\\ T_i \text{ is a regenerative process}\}.\end{aligned} \tag{60}$$

Let $Z_i = S_{i+1} - D_i \geq 0$ represent the *waiting time* between the delivery time D_i of the i-th sample and the generation time S_{i+1} of the $(i+1)$-th sample. Then, $S_i = \sum_{j=0}^{i-1}(Y_j + Z_j)$ and $D_i = \sum_{j=0}^{i-1}(Y_j + Z_j) + Y_i$ for each $i = 1, 2, \ldots$ Given $(Y_0, Y_1, \ldots)$, $(S_1, S_2, \ldots)$ is uniquely determined by $(Z_0, Z_1, \ldots)$. Hence, one can also use $\pi = (Z_0, Z_1, \ldots)$ to represent a sampling policy.

Because $\{X_t - \hat{X}_t, t \in [D_i, D_{i+1})\}$ and $\{O_{t-S_i}, t \in [D_i, D_{i+1})\}$ are of the same distribution, for each $i = 1, 2, \ldots$,

$$\begin{aligned}&\mathbb{E}\left[\int_{D_i}^{D_{i+1}} (X_t - \hat{X}_t)^2 dt\right]\\ =&\mathbb{E}\left[\int_{D_i}^{D_{i+1}} O_{t-S_i}^2 dt\right] = \mathbb{E}\left[\int_{Y_i}^{Y_i+Z_i+Y_{i+1}} O_s^2 ds\right].\end{aligned} \tag{61}$$

Because T_i is a regenerative process, the renewal theory [62] tells us that $\frac{1}{n}\mathbb{E}[S_n]$ is a convergent sequence and

$$\begin{aligned}&\limsup_{T\to\infty} \frac{1}{T}\mathbb{E}\left[\int_0^T (X_t - \hat{X}_t)^2 dt\right]\\ =& \lim_{n\to\infty} \frac{\mathbb{E}\left[\int_0^{D_n} (X_t - \hat{X}_t)^2 dt\right]}{\mathbb{E}[D_n]}\\ =& \lim_{n\to\infty} \frac{\sum_{i=1}^n \mathbb{E}\left[\int_{Y_i}^{Y_i+Z_i+Y_{i+1}} O_s^2 ds\right]}{\sum_{i=1}^n \mathbb{E}[Y_i + Z_i]}.\end{aligned} \tag{62}$$

Hence, (11) can be rewritten as the following MDP:

$$\begin{aligned}\mathsf{mse}_{\text{opt}} = \inf_{\pi\in\Pi_1} \lim_{n\to\infty} &\frac{\sum_{i=1}^n \mathbb{E}\left[\int_{Y_i}^{Y_i+Z_i+Y_{i+1}} O_s^2 ds\right]}{\sum_{i=1}^n \mathbb{E}[Y_i + Z_i]}\\ \text{s.t.} \lim_{n\to\infty} &\frac{1}{n}\sum_{i=1}^n \mathbb{E}[Y_i + Z_i] \geq \frac{1}{f_{\max}},\end{aligned} \tag{63}$$

where $\mathsf{mse}_{\text{opt}}$ is the optimal value of (63).

C. Reformulation of Problem (63)

In order to solve (63), let us consider the following MDP with a parameter $c \geq 0$:

$$h(c) = \inf_{\pi\in\Pi_1} \lim_{n\to\infty} \frac{1}{n}\sum_{i=1}^n \mathbb{E}\left[\int_{Y_i}^{Y_i+Z_i+Y_{i+1}} O_s^2 ds - c(Y_i + Z_i)\right] \tag{64}$$

$$\text{s.t.} \lim_{n\to\infty} \frac{1}{n}\sum_{i=1}^n \mathbb{E}[Y_i + Z_i] \geq \frac{1}{f_{\max}},$$

where $h(c)$ is the optimum value of (64). Similar with Dinkelbach's method [59] for nonlinear fractional programming, the following lemma holds for the MDP (63):

Lemma 7. *[4] The following assertions are true:*

(a). $\mathsf{mse}_{\text{opt}} \gtreqless c$ *if and only if* $h(c) \gtreqless 0$.

(b). *If $h(c) = 0$, the solutions to* (63) *and* (64) *are identical.*

Hence, the solution to (63) can be obtained by solving (64) and seeking $c = \mathsf{mse}_{\text{opt}} \geq 0$ such that

$$h(\mathsf{mse}_{\text{opt}}) = 0. \tag{65}$$

D. Lagrangian Dual Problem of (64)

Next, we use the Lagrangian dual approach to solve (64) with $c = \mathsf{mse}_{\text{opt}}$. We define the Lagrangian associated with (64) as

$$\begin{aligned} &L(\pi;\lambda)\\ &=\lim_{n\to\infty}\frac{1}{n}\sum_{i=1}^{n}\mathbb{E}\left[\int_{Y_i}^{Y_i+Z_i+Y_{i+1}}O_s^2ds-(\mathsf{mse}_{\text{opt}}+\lambda)(Y_i+Z_i)\right]\\ &\quad+\frac{\lambda}{f_{\max}}, \end{aligned}\tag{66}$$

where $\lambda \geq 0$ is the dual variable. Let

$$e(\lambda)=\inf_{\pi\in\Pi_1}L(\pi;\lambda).\tag{67}$$

Then, the dual problem of (64) is defined by

$$d=\max_{\lambda\geq 0}e(\lambda),\tag{68}$$

where d is the optimum value of (68). Weak duality [60] implies $d \leq h(\mathsf{mse}_{\text{opt}})$. In Section IV-F, we will establish strong duality, i.e., $d = h(\mathsf{mse}_{\text{opt}})$.

In the sequel, we decompose (67) into a sequence of mutually independent per-sample MDPs. Let us define

$$\beta=\mathsf{mse}_{\text{opt}}+\lambda.\tag{69}$$

As shown in Appendix H, by using Lemma 5, we can obtain

$$\begin{aligned} &\mathbb{E}\left[\int_{Y_i}^{Y_i+Z_i+Y_{i+1}}O_s^2ds-\beta(Y_i+Z_i)\right]\\ =&\mathbb{E}\left[\int_{Y_i}^{Y_i+Z_i}(O_s^2-\beta)ds+\gamma O_{Y_i+Z_i}^2\right]\\ &+\frac{\sigma^2}{2\theta}[\mathbb{E}(Y_{i+1})-\gamma]-\beta\mathbb{E}[Y_{i+1}], \end{aligned}\tag{70}$$

where γ is defined in (24). For any $s \geq 0$, define the σ-fields $\mathcal{F}_t^s = \sigma(O_{s+r} - O_s : r \in [0,t])$ and the right-continuous filtration $\mathcal{F}_t^{s+} = \cap_{r>t}\mathcal{F}_r^s$. Then, $\{\mathcal{F}_t^{s+}, t \geq 0\}$ is the filtration of the time-shifted OU process $\{O_{s+t} - O_s, t \in [0,\infty)\}$. Define $\mathfrak{M}_s$ as the set of integrable stopping times of $\{O_{s+t} - O_s, t \in [0,\infty)\}$, i.e.,

$$\mathfrak{M}_s=\{\tau\geq 0:\{\tau\leq t\}\in\mathcal{F}_t^{s+},\mathbb{E}[\tau]<\infty\}.\tag{71}$$

By using a sufficient statistic of (67), we can obtain

Lemma 8. *An optimal solution* $(Z_0, Z_1, \ldots)$ *to* (67) *satisfies*

$$\inf_{Z_i\in\mathfrak{M}_{Y_i}}\mathbb{E}\left[\int_{Y_i}^{Y_i+Z_i}(O_s^2-\beta)ds+\gamma O_{Y_i+Z_i}^2\middle|O_{Y_i},Y_i\right],\tag{72}$$

where $\beta \geq 0$ *and* $\gamma \geq 0$ *are defined in* (69) *and* (24)*, respectively.*

Proof. See Appendix J. □

By this, (67) is decomposed as a series of per-sample MDP (72).

E. Analytical Solution to Per-Sample MDP (72)

We solve (72) by using the free-boundary approach for optimal stopping problems [12].

Let us consider an OU process V_t with initial state $V_0 = v$ and parameter $\mu = 0$. Define the σ-fields $\mathcal{F}_t^V = \sigma(V_s : s \in [0,t])$, $\mathcal{F}_t^{V+} = \cap_{r>t}\mathcal{F}_r^V$, and the filtration $\{\mathcal{F}_t^{V+}, t \geq 0\}$ associated to $\{V_t, t \geq 0\}$. Define $\mathfrak{M}_V$ as the set of integrable stopping times of $\{V_t, t \in [0,\infty)\}$, i.e.,

$$\mathfrak{M}_V=\{\tau\geq 0:\{\tau\leq t\}\in\mathcal{F}_t^{V+},\mathbb{E}[\tau]<\infty\}.\tag{73}$$

Our goal is to solve the following optimal stopping problem for any given initial state $v \in \mathbb{R}$

$$\sup_{\tau\in\mathfrak{M}_V}\mathbb{E}_v\left[-\gamma V_\tau^2-\int_0^\tau(V_s^2-\beta)ds\right],\tag{74}$$

where $\mathbb{E}_v[\cdot]$ is the conditional expectation for given initial state $V_0 = v$, γ and β are given by (24) and (69), respectively. Hence, (72) is one instance of (74) with $v = O_{Y_i}$, where the supremum is taken over all stopping times τ of V_t. In this subsection, we focus on the case that β in (74) satisfies $\mathsf{mse}_{Y_i} \leq \beta < \mathsf{mse}_\infty$. Later on in Section IV-F, we will show that this condition is indeed satisfied by the optimal solution to (64).

In order to solve (74), we first find a candidate solution to (74) by solving a free boundary problem; then we prove that the free boundary solution is indeed the value function of (74):

1) A Candidate Solution to (74)*:* Now, we show how to solve (74). The general optimal stopping theory in Chapter I of [12] tells us that the following guess of the stopping time should be optimal for Problem (74):

$$\tau_*=\inf\{t\geq 0:|V_t|\geq v_*\},\tag{75}$$

where $v_* \geq 0$ is the optimal stopping threshold to be found. Observe that in this guess, the continuation region $(-v_*, v_*)$ is assumed symmetric around zero. This is because the OU process is symmetric, i.e., the process $\{-V_t, t \geq 0\}$ is also an OU process started at $-V_0 = -v$. Similarly, we can also argue that the value function of problem (74) should be even.

According to [12, Chapter 8], and [13, Chapter 10], the value function and the optimal stopping threshold v_* should satisfy the following free boundary problem:

$$\frac{\sigma^2}{2}H''(v)-\theta vH'(v)=v^2-\beta,\quad v\in(-v_*,v_*),\tag{76}$$

$$H(\pm v_*)=-\gamma v_*^2,\tag{77}$$

$$H'(\pm v_*)=\mp 2\gamma v_*.\tag{78}$$

In Appendix K, we use the integrating factor method [63, Sec. I.5] to find the general solution to (76), which is given by

$$\begin{aligned} H(v)=&-\frac{v^2}{2\theta}+\left(\frac{1}{2\theta}-\frac{\beta}{\sigma^2}\right){}_2F_2\left(1,1;\frac{3}{2},2;\frac{\theta}{\sigma^2}v^2\right)v^2\\ &+C_1\mathrm{erfi}\left(\frac{\sqrt{\theta}}{\sigma}v\right)+C_2,\qquad v\in(-v_*,v_*), \end{aligned}\tag{79}$$

where C_1 and C_2 are constants to be found for satisfying

(77)-(78), and $\operatorname{erfi}(x)$ is the imaginary error function, i.e.,

$$\operatorname{erfi}(x) = \frac{2}{\sqrt{\pi}} \int_0^x e^{t^2} dt. \tag{80}$$

Because $H(v)$ should be even but $\operatorname{erfi}(x)$ is odd, we should choose $C_1 = 0$. Further, in order to satisfy the boundary condition (77), C_2 is chosen as

$$C_2 = \frac{1}{2\theta} \mathbb{E}\left(e^{-2\theta Y_i}\right) v_*^2 - \left(\frac{1}{2\theta} - \frac{\beta}{\sigma^2}\right) {}_2F_2\big(1, 1; \frac{3}{2}, 2; \frac{\theta}{\sigma^2} v_*^2\big) v_*^2, \tag{81}$$

where we have used (24). With this, the expression of $H(v)$ is obtained in the continuation region $(-v_*, v_*)$. In the stopping region $|v| \geq v_*$, the stopping time in (75) is simply $\tau_* = 0$, because $|V_0| = |v| \geq v_*$. Hence, if $|v| \geq v_*$, the objective value achieved by the sampling time (75) is

$$\mathbb{E}_v \left[-\gamma v^2 - \int_0^0 (V_s^2 - \beta) ds \right] = -\gamma v^2. \tag{82}$$

Combining (79)-(82), we obtain a candidate of the value function for (74):

$$H(v) = \begin{cases} -\frac{v^2}{2\theta} + \left(\frac{1}{2\theta} - \frac{\beta}{\sigma^2}\right) {}_2F_2\big(1, 1; \frac{3}{2}, 2; \frac{\theta}{\sigma^2} v^2\big) v^2 + C_2, \\ \qquad\qquad\qquad\qquad\qquad\qquad \text{if } |v| < v_*, \\ -\gamma v^2, \qquad\qquad\qquad\qquad\qquad \text{if } |v| \geq v_*. \end{cases} \tag{83}$$

Next, we find a candidate value of the optimal stopping threshold v_*. By taking the gradient of $H(v)$, we get

$$H'(v) = -\frac{v}{\theta} + \left(\frac{\sigma}{\theta^{\frac{3}{2}}} - \frac{2\beta}{\sigma\sqrt{\theta}}\right) F\left(\frac{\sqrt{\theta}}{\sigma} v\right), \quad v \in (-v_*, v_*), \tag{84}$$

where

$$F(x) = e^{x^2} \int_0^x e^{-t^2} dt. \tag{85}$$

The boundary condition (78) implies that v_* is the root of

$$-\frac{v}{\theta} + \left(\frac{\sigma}{\theta^{\frac{3}{2}}} - \frac{2\beta}{\sigma\sqrt{\theta}}\right) F\left(\frac{\sqrt{\theta}}{\sigma} v\right) = -2\gamma v. \tag{86}$$

Substituting (14), (15), and (24) into (86), yields that v_* is the root of

$$(\mathsf{mse}_\infty - \beta)\, G\bigg(\frac{\sqrt{\theta}}{\sigma} v\bigg) = \mathsf{mse}_\infty - \mathsf{mse}_{Y_i}, \tag{87}$$

where $G(\cdot)$ is defined in (16). Because $\mathsf{mse}_{Y_i} \leq \beta < \mathsf{mse}_\infty$, $G(x)$ is strictly increasing on $[0, \infty)$, and $G(0) = 1$, we know that (87) has a unique non-negative root v_*. Further, the root v_* can be expressed as a function $v(\beta)$ of β, where $v(\beta)$ is defined in (19). By this, we obtain a candidate solution to (74).

2) Verification of the Optimality of the Candidate Solution: Next, we use Itô's formula to verify the above candidate solution is indeed optimal, as stated in the following theorem:

Theorem 5. *If $\mathsf{mse}_{Y_i} \leq \beta < \mathsf{mse}_\infty$, then for all $v \in \mathbb{R}$, $H(v)$ in* (83) *is the value function of the optimal stopping problem* (74)*. In addition, the optimal stopping time for solving* (74) *is τ_* in* (75)*, where $v_* = v(\beta)$ is given by* (19).

In order to prove Theorem 5, we need to establish the following properties of $H(v)$ in (83), for the case that $\mathsf{mse}_{Y_i} \leq \beta < \mathsf{mse}_\infty$ is satisfied in (74):

Lemma 9. $H(v) = \mathbb{E}_v\big[-\gamma V_{\tau_*}^2 - \int_0^{\tau_*} (V_s^2 - \beta) ds\big].$

Proof. See Appendix L. ☐

Lemma 10. *$H(v) \geq -\gamma v^2$ for all $v \in \mathbb{R}$.*

Proof. See Appendix M. ☐

A function $f(v)$ is said to be *excessive* for the process V_t if

$$\mathbb{E}_v f(V_t) \leq f(v), \forall t \geq 0, v \in \mathbb{R}. \tag{88}$$

By using Itô's formula in stochastic calculus, we can obtain

Lemma 11. *The function $H(v)$ is excessive for the process V_t.*

Proof. See Appendix N. ☐

Now, we are ready to prove Theorem 5.

Proof of Theorem 5. In Lemmas 9-11, we have shown that $H(v) = \mathbb{E}_v\big[-\gamma V_{\tau_*}^2 - \int_0^{\tau_*} (V_s^2 - \beta) ds\big]$, $H(v) \geq -\gamma v^2$, and $H(v)$ is an excessive function. Moreover, from the proof of Lemma 9, we know that $\mathbb{E}_v[\tau_*] < \infty$ holds for all $v \in \mathbb{R}$. Hence, $\mathbb{P}_v(\tau_* < \infty) = 1$ for all $v \in \mathbb{R}$. These conditions and Theorem 1.11 in [12, Section 1.2] imply that τ_* is an optimal stopping time of (74). This completes the proof. ☐

Because (72) is a special case of (74), we can get from Theorem 5 that

Corollary 1. *If $\mathsf{mse}_{Y_i} \leq \beta < \mathsf{mse}_\infty$, then a solution to* (72) *is $(Z_1(\beta), Z_2(\beta), \ldots)$, where*

$$Z_i(\beta) = \inf\{t \geq 0 : |O_{Y_i+t}| \geq v(\beta)\}, \tag{89}$$

and $v(\beta)$ is defined in (19).

F. Zero Duality Gap between (64) *and* (68)

Strong duality is established in the following thoerem:

Theorem 6. *If the service times Y_i are i.i.d. with $0 < \mathbb{E}[Y_i] < \infty$, then the duality gap between* (64) *and* (68) *is zero. Further, $(Z_0(\beta), Z_1(\beta), \ldots)$ is an optimal solution to both* (64) *and* (68)*, where $Z_i(\beta)$ is determined by*

$$Z_i(\beta) = \inf\{t \geq 0 : |O_{Y_i+t}| \geq v(\beta)\}, \tag{90}$$

$v(\beta)$ is defined in (19)*, $\beta \geq 0$ is the root of*

$$\mathbb{E}\left[\int_{Y_i}^{Y_i+Z_i(\beta)+Y_{i+1}} O_t^2 dt\right] - \beta \mathbb{E}[Y_i + Z_i(\beta)] = 0, \tag{91}$$

if $\mathbb{E}[Y_i + Z_i(\beta)] > 1/f_{\max}$; otherwise, β is the root of $\mathbb{E}[Y_i + Z_i(\beta)] = 1/f_{\max}$. In both cases, $\mathsf{mse}_{Y_i} \leq \beta < \mathsf{mse}_\infty$ is satisfied, and hence (19) *is well-defined. Further, the optimal objective value to* (63) *is given by*

$$\mathsf{mse}_{\text{opt}} = \frac{\mathbb{E}\left[\int_{Y_i}^{Y_i+Z_i(\beta)+Y_{i+1}} O_t^2 dt\right]}{\mathbb{E}[Y_i + Z_i(\beta)]}. \tag{92}$$

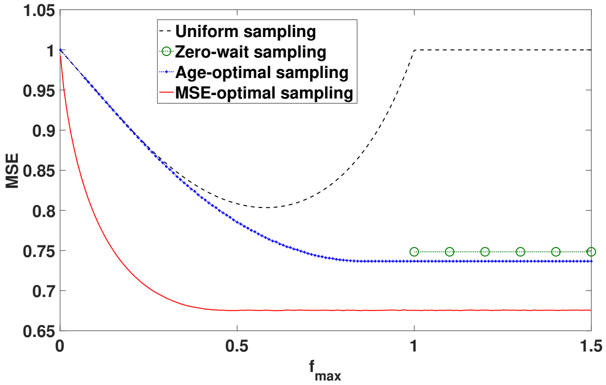

Fig. 8: MSE vs $f_{\max}$ tradeoff for *i.i.d.* exponential service time with $\mathbb{E}[Y_i] = 1$, where the parameters of the OU process are $\sigma = 1$ and $\theta = 0.5$.

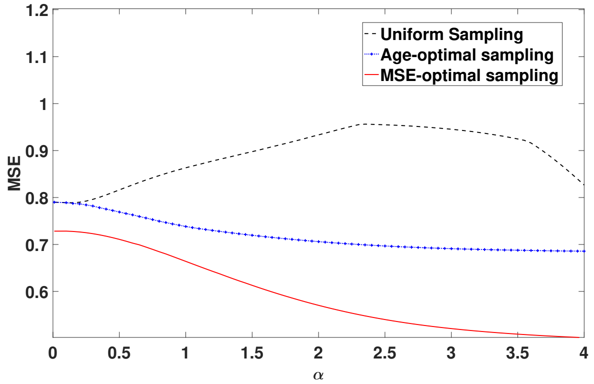

Fig. 9: MSE vs. the scale parameter α of *i.i.d.* normalized log-normal service time distribution with $\mathbb{E}[Y_i] = 1$ and $f_{\max} = 0.8$, where the parameters of the OU process are $\sigma = 1$ and $\theta = 0.5$. Zero-wait sampling is not feasible here as $f_{\max} < 1/\mathbb{E}[Y_i]$ and hence is not plotted.

Proof. We use [60, Prop. 6.2.5] to find a *geometric multiplier* for (64). This suggests that the duality gap between (64) and (68) must be zero, because otherwise there exists no geometric multiplier [60, Prop. 6.2.3(b)]. The details are provided in Appendix O. □

Hence, Theorem 2 follows from Theorem 6. Because Theorem 1 is a special case of Theorem 2, Theorem 1 is also proven.

V. NUMERICAL COMPARISONS

In this section, we evaluate the estimation error achieved by the following four sampling policies:

1. *Uniform sampling*: Periodic sampling with a period given by $S_{i+1} - S_i = 1/f_{\max}$.
2. *Zero-wait sampling* [1], [19]: The sampling policy given by

$$S_{i+1} = S_i + Y_i, \tag{93}$$

   which is infeasible when $f_{\max} < 1/\mathbb{E}[Y_i]$.
3. *Age-optimal sampling* [14]: The sampling policy given by Theorem 4.
4. *MSE-optimal sampling*: The sampling policy given by Theorem 1.

Let $\mathsf{mse}_{\text{uniform}}$, $\mathsf{mse}_{\text{zero-wait}}$, $\mathsf{mse}_{\text{age-opt}}$, and $\mathsf{mse}_{\text{opt}}$, be the MSEs of uniform sampling, zero-wait sampling, age-optimal sampling, MSE-optimal sampling, respectively. We can obtain

$$\begin{aligned}&\mathsf{mse}_{Y_i} \leq \mathsf{mse}_{\text{opt}} \leq \mathsf{mse}_{\text{age-opt}} \leq \mathsf{mse}_{\text{uniform}} \leq \mathsf{mse}_{\infty},\\ &\mathsf{mse}_{\text{age-opt}} \leq \mathsf{mse}_{\text{zero-wait}} \leq \mathsf{mse}_{\infty},\end{aligned} \tag{94}$$

whenever zero-wait sampling is feasible, which fit with our numerical results. The expectations in (25) and (26) are evaluated by taking the average over 1 million samples. The parameters of the OU process are given by $\sigma = 1$, $\theta = 0.5$, and μ can be chosen arbitrarily because it does not affect the estimation error.

Figure 8 illustrates the tradeoff between the MSE and $f_{\max}$ for *i.i.d.* exponential service times with mean $\mathbb{E}[Y_i] = 1$. Because $\mathbb{E}[Y_i] = 1$, the maximum throughput of the queue is 1. The lower bound mse_{Y_i} is 0.5 and the upper bound mse_{∞} is 1. In fact, as Y_i is an exponential random variable with mean 1, $\frac{\sigma^2}{2\theta}(1 - e^{-2\theta Y_i})$ has a uniform distribution on $[0, 1]$. Hence, $\mathsf{mse}_{Y_i} = 0.5$. For small values of $f_{\max}$, age-optimal sampling is similar to uniform sampling, and hence $\mathsf{mse}_{\text{age-opt}}$ and $\mathsf{mse}_{\text{uniform}}$ are close to each other in the regime. However, as $f_{\max}$ grows, $\mathsf{mse}_{\text{uniform}}$ reaches the upper bound mse_{∞} and remains constant for $f_{\max} \geq 1$. This is because the queue length of uniform sampling is large at high sampling frequencies. In particular, when $f_{\max} \geq 1$, the queue length of uniform sampling is infinite. On the other hand, $\mathsf{mse}_{\text{age-opt}}$ and $\mathsf{mse}_{\text{opt}}$ decrease with respect to $f_{\max}$. The reason behind this is that the set of feasible sampling policies satisfying the constraint in (11) and (48) becomes larger as $f_{\max}$ grows, and hence the optimal values of (11) and (48) are decreasing in $f_{\max}$. As we expected, $\mathsf{mse}_{\text{zero-wait}}$ is larger than $\mathsf{mse}_{\text{opt}}$ and $\mathsf{mse}_{\text{age-opt}}$. Moreover, all of them are between the lower bound mse_{Y_i} and upper bound mse_{∞}.

Figures 9 and 10 depict the MSE of *i.i.d.* normalized log-normal service time for $f_{\max} = 0.8$ and $f_{\max} = 1.2$, respectively, where $Y_i = e^{\alpha X_i}/\mathbb{E}[e^{\alpha X_i}], \alpha > 0$ is the scale parameter of log-normal distribution, and $(X_1, X_2, \ldots)$ are *i.i.d.* Gaussian random variables with zero mean and unit variance. Because $\mathbb{E}[Y_i] = 1$, the maximum throughput of the queue is 1. In Fig. 9, since $f_{\max} < 1$, zero-wait sampling is not feasible and hence is not plotted. As the scale parameter α grows, the tail of the log-normal distribution becomes heavier.

In both figures, $\mathsf{mse}_{\text{age-opt}}$ and $\mathsf{mse}_{\text{opt}}$ drop with α. This phenomenon may look surprising at first sight, because $\mathsf{mse}_{\text{age-opt}}$ and $\mathsf{mse}_{\text{opt}}$ grow quickly in α in the previous study [4] on the Wiener process. To understand this phenomenon, let us consider the age penalty function $p(\Delta_t)$ in (46) for the OU process. As the scale parameter α grows, the service time tends to become either shorter or much longer than the mean $\mathbb{E}[Y_i]$, rather than being close to $\mathbb{E}[Y_i]$. When Δ_t is small, $p(\Delta_t)$ reduces quickly in Δ_t, and hence the service time smaller than $\mathbb{E}[Y_i]$ leads to a fast decrease in the average age penalty;

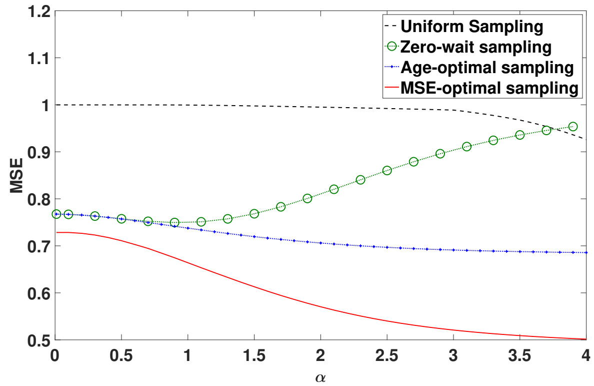

Fig. 10: MSE vs. the scale parameter α of *i.i.d.* normalized log-normal service time distribution $\mathbb{E}[Y_i] = 1$ and $f_{\max} = 1.2$, where the parameters of the OU process are $\sigma = 1$, $\theta = 0.5$.

when Δ_t is quite large, $p(\Delta_t)$ cannot increase much because it is upper bounded by mse_∞, hence the service time much longer than $\mathbb{E}[Y_i]$ would not increase the average age penalty by much. By combining these two trends, the average age penalty $\mathsf{mse}_{\text{age-opt}}$ decreases in α. The dropping of $\mathsf{mse}_{\text{opt}}$ in α can be understood in a similar fashion. On the other hand, the age penalty function of the Wiener process is $p(\Delta_t) = \Delta_t$, which is quite different from the case considered here. We also observe that in both figures, the gap between $\mathsf{mse}_{\text{opt}}$ and $\mathsf{mse}_{\text{age-opt}}$ increases as α grows.

VI. CONCLUSION

In this paper, we have studied the optimal sampler design for remote estimation of OU processes through queues. We have developed optimal causal sampling policies that minimize the estimation error of OU processes subject to a sampling rate constraint. These optimal sampling policies have nice structures and are easy to compute. A connection between remote estimation and nonlinear age metrics has been found. The structural properties of the optimal sampling policies shed lights on the possible structure of the optimal sampler designs for more general signal models, such as Feller processes, which is an important future research direction.

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