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Persistence and smooth dependence on parameters of periodic orbits in functional differential equations close to an ODE or an evolutionary PDE

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Abstract

We consider functional differential equations (FDEs) which are perturbations of smooth ordinary differential equations (ODEs). The FDE can involve multiple state-dependent delays, distributed delays, or implicitly defined delays (forward or backward). We show that, under some mild assumptions on the perturbation, if the ODE has a nondegenerate periodic orbit, then the FDE has a smooth periodic orbit. Moreover, when the perturbation depends on some parameters, we get smooth dependence of the periodic orbit and its frequency on the parameters with high regularity.

The method can also be applied to treat equations with small delays appearing in electrodynamics and FDEs which are perturbations of some evolutionary partial differential equations (PDEs).

The proof consists in solving functional equations satisfied by the parameterization of the periodic orbit and the frequency using a fixed-point approach. We do not need to consider the smoothness of the evolution or even the phase space of the FDEs.

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1. Introduction

In this paper, we first present a systematic approach to the study of periodic orbits of functional differential equations (FDEs) which are perturbations of smooth ordinary differential equations (ODEs) in $\mathbb{R}^n$. This is a singular perturbation problem since the phase space of an FDE is infinite dimensional, even if the perturbation looks small.

The approach we use bypasses completely the study of the evolution of FDEs and we do not even need to identify the phase space. In contrast with the standard procedure of constructing all the solutions and selecting the periodic ones, we start with the space of periodic functions and impose that they are solutions.

We formulate functional equations satisfied by parameterizations of the periodic orbits and their frequencies in appropriate spaces of smooth functions. We solve the functional equation using a fixed point approach, which gives existence of smooth solutions and dependence on parameters with high regularity.

One advantage of our approach is that the functional perturbations we cover can be rather general. For example, it may include terms with multiple state-dependent delays, distributed delays, or implicitly defined delays of the type appearing in electrodynamics (see Section 9). The delays can be either backward delays or forward delays (as in advanced equations).

Then, using a similar but more elaborate proof, we get results on periodic orbits for equations with small delays, which have applications in electrodynamics.

Finally, we extend the results to perturbations of partial differential equations (PDEs). We can consider PDEs which have good forward (but not backward) evolutions such as parabolic equations as well as some ill-posed equations (e.g. Boussinesq equation in water waves, which even if ill posed, admits many physically interesting solutions).

A philosophy similar to that of this paper has been used also in other papers. [55,39,38] develop functional equations for quasi-periodic solutions in several contexts and study them using KAM theory. In [91], one can find a theory of persistence of stable manifolds in some limited contexts. We hope that some of the previous studies can be extended to more dynamical objects. Notably we expect to get higher regularity of the center manifolds for state-dependent delay equations (SDDEs), which is essential for applications of the center manifold reduction to bifurcation theory [7]. Of course, removing the perturbative setting remains a long term goal, but this seems to pass through refining the theory of existence and regularity of [86].

1.1. Backgrounds on functional differential equations

In many applications, one needs to consider FDEs. They appear naturally as models in electrodynamics, control theory, biology, neuroscience, and economics, see [89,22,23,36,7,46,24,1,87,58] and references therein. In many cases where the delays depend on the states of the systems, one needs to consider SDDEs. For example, in the formulation of electrodynamics, the delays depend implicitly on the solutions. Sometimes several delays are involved in one equation, with different forms. Besides the interest in applications, the field of FDEs is a very rich mathematical subject worth of study because of its own depth.

The theory for delay equations with constant delays is well established [34,19]. However, many fundamental problems are not settled for SDDEs. For example, even identifying the correct phase space to formulate the equation is not clear. The paper [86] made a breakthrough considering a submanifold of C^1 space, the solution manifold, as the phase space for SDDEs on which the semiflow is C^1. A result on differentiability of solutions with respect to parameters for a class of SDDEs in the Sobolev sense (using quasi-Banach spaces) is in [37]. It seems that there is no result on higher regularity of the semiflow and dependence on parameters for a general solution. One can refer to [36] for a review of the applications and results in SDDEs. SDDEs display rich behaviors, see [7,41]. At the same time, some SDDEs, like the ones considered in this paper, have many solutions with regular behaviors, see [55,39,38,91]. See also [2,63] for results on low regularity via topological approach.

Periodic orbits are important landmarks in dynamical systems. There has been interest in studying periodic orbits of delay differential equations (DDEs) with constant delays, see [69,44, 59,43]. Some studies in the setting of SDDEs are in [61,62,60,78]. Some numerical works are in [83,80,81].

1.2. Related results in the literature

Results on persistence of non-degenerate periodic orbits and dependence on parameters for DDEs with constant delays were proven by studying the evolution operator, see [34,32], and [35]. This method is difficult to apply to SDDEs for regularities higher than C^1 since one would need to extend the regularity theory of the evolution [86] to higher regularities. The paper [66] also studied functional equations satisfied by periodic orbits, but treats them using topological methods, which do not allow to study regularity. See also the excellent surveys [67,70]. A modified Poincaré method for persistence of periodic orbits which is applicable to PDEs and neutral functional differential equations is in the paper [33].

1.3. Organization of the paper

A precise formulation of the problem is given in Section 2. Section 3 introduces the parameterization method for our problem. Section 4 states the main results of this paper. These results are formulated in terms of properties of the perturbation functional P. The detailed proofs of the main results are in Section 5. In Sections 6 and 7, we verify that several models which appear in the literature indeed satisfy the assumptions of the main result. These sections are the core of this paper.

In the other sections, we present extensions of the method and show that they lead to results for several models in the literature.

Section 8 is devoted to the analysis of equations with small delays, which requires an extension of the general result and indeed requires stronger regularity assumptions. Section 9 considers equations appearing in electrodynamics, which has been a very important motivation for the whole theory of FDEs. In particular, we give some justification to several procedures used in Physics such as the $1/c$ expansions.

Section 10 introduces a different method for the case that the unperturbed periodic orbits are hyperbolic. Even if this is a particular case of the previous results for perturbations of ODEs, it generalizes to perturbations of evolutionary PDEs. In Section 11, we present results for several evolutionary PDEs which have received attention in the literature. We note that, since our method dispenses with defining the evolution, the results apply even to ill-posed PDEs.

In Appendix A we have collected some results of analysis that we need to use.

2. Formulation of the problem

Consider an n-dimensional ODE

$$\dot{x}(t) = f(x(t)), \tag{1}$$

where, for the moment, $f: \mathbb{R}^n \to \mathbb{R}^n$ is a C^∞ vector field (later we will assume less regularity).

We assume that equation (1) has a periodic orbit with frequency $\omega_0 \neq 0$. The existence of periodic solutions for ODEs will not be discussed here. (We note however that the same methods discussed here can be used to produce periodic solutions of the ODEs perturbatively.)

We consider singular perturbations of equation (1) to FDEs with parameter γ:

$$\dot{x}(t) = f(x(t)) + \varepsilon P(x_t, \gamma), \tag{2}$$

where $\varepsilon \geqslant 0$ is the perturbation parameter, $P: \mathscr{R}[-h, h] \times O \to \mathbb{R}^n$, h is a positive constant. $\mathscr{R}[-h, h]$ is a space of regular functions from $[-h, h]$ to $\mathbb{R}^n$. The precise regularity of the functions in $\mathscr{R}[-h, h]$ will be specified later. The *"history segment"* $x_t \in \mathscr{R}[-h, h]$ is defined as $x_t(s) = x(t + s)$ for $s \in [-h, h]$. And $\gamma \in O$ is a parameter, where O is a bounded open set in $\mathbb{R}^m$. Note that we allow our history segments to involve also the future, so that the theory we will develop applies not just to delay equations but to equations that involve the future.

In many treatments of delay equations it is customary to think of $\mathscr{R}[-h, h]$ as the phase space in which one sets initial conditions and defines an evolution. For example, in the case of constant delay equations, it is customary to impose initial conditions in $C^0[-h, 0]$, with constant h being the delay. Nevertheless, in the case of SDDEs, this space includes many functions which cannot satisfy the equations and, therefore, have no physical meaning. As it will be clear later, our treatment bypasses the consideration of the evolution defined by the FDE, so that we will not think of $\mathscr{R}[-h, h]$ as the phase space of the evolution.

Under a nondegeneracy condition on the periodic orbit of equation (1) and some mild assumptions on P, see more details in the definition of $\mathscr{P}$ in (4) and assumptions (H2.1), (H3.1), (H2.2), and (H3.2), we show that for small enough ε, there exist periodic orbits for FDE (2). We also show that the periodic orbits for equation (2) depend on γ smoothly.

From now on, we will identify the periodic orbit for FDE (2) in a function space with a periodic function taking values in $\mathbb{R}^n$. Under this identification, we will see that the periodic orbit for FDE (2) is close to the periodic orbit for equation (1) for small ε.

3. Parameterization method

Let $K_0: \mathbb{T} \to \mathbb{R}^n$ be a parameterization of the periodic orbit of equation (1), where $\mathbb{T} = \mathbb{R}/\mathbb{Z}$. This means that for any fixed θ, $x(t) = K_0(\theta + \omega_0 t)$ solves equation (1). Equivalently, K_0 satisfies the functional equation (invariance equation):

$$\omega_0 D K_0(\theta) = f(K_0(\theta)). \tag{3}$$

Note that such K_0 is unique up to a phase shift. In this case, K_0 is C^∞ since f is C^∞.

We aim to find $K: \mathbb{T} \to \mathbb{R}^n$ and $\omega > 0$, such that for any θ, $x(t) = K(\theta + \omega t)$ solves equation (2). And we say such K parameterizes the periodic orbit of FDE (2).

The expression $x(t) = K(\theta + \omega t)$ solving equation (2) is equivalent to K satisfying the functional equation:

$$\omega DK(\theta) = f(K(\theta)) + \varepsilon \mathscr{P}(K, \omega, \gamma, \theta), \tag{4}$$

where $\mathscr{P}(K, \omega, \gamma, \theta)$ results from substituting $x(t) = K(\theta + \omega t)$ into $P(x_t, \gamma)$ in equation (2) and letting $t = 0$. See Sections 6, 7, and 8 for explicit formulations of $\mathscr{P}$ in some specific examples.

The equation (4) will be the centerpiece of our treatment. We will see that, using different methods of analysis, we can give results on existence of solutions of (4). Note that this analysis produces periodic solutions of (2) without discussing a general theory of existence and dependence on parameters of the solutions for FDEs.

4. Main results

4.1. Assumptions

For a given $\theta_0 \in \mathbb{T}$, let $\Phi(\theta; \theta_0)$ be the fundamental solution of a scaled variational equation of the ODE (1) along the periodic orbit parameterized by K_0, i.e.,

$$\omega_0 \frac{d}{d\theta} \Phi(\theta; \theta_0) = Df(K_0(\theta))\Phi(\theta; \theta_0), \quad \Phi(\theta_0; \theta_0) = Id. \tag{5}$$

We need to assume that the periodic orbit of (1) is nondegenerate, that is, we impose the following assumption on $\Phi(\theta_0 + 1; \theta_0)$:

(H1) $\Phi(\theta_0 + 1; \theta_0)$ has a simple eigenvalue 1 whose eigenspace is generated by $DK_0(\theta_0)$.

Note that, because of the existence and uniqueness of the solutions of (5), and the periodicity of K_0, we have that

$$\begin{aligned} \Phi(\theta_2; \theta_0) &= \Phi(\theta_2; \theta_1)\Phi(\theta_1; \theta_0); \\ \Phi(\theta_1 + 1; \theta_0 + 1) &= \Phi(\theta_1; \theta_0). \end{aligned}$$

As a consequence,

$$\Phi(\theta_0 + 1; \theta_0) = \Phi(\theta_0 + 1; 1)\Phi(1; 0)\Phi(0; \theta_0) = \Phi(0; \theta_0)^{-1}\Phi(1; 0)\Phi(0; \theta_0).$$

So that the spectrum of $\Phi(\theta_0 + 1; \theta_0)$, commonly called the Floquet multipliers, is independent of the starting point θ_0.

Under assumption (H1), there exists an $(n-1)$-dimensional linear space E_{θ_0} at $K_0(\theta_0)$, (the spectral complement of $\mathrm{Span}\{DK_0(\theta_0)\}$, corresponding to the eigenvalues of $\Phi(\theta_0 + 1; \theta_0)$ other than 1, $\mathbb{R}^n = E_{\theta_0} \oplus \mathrm{Span}\{DK_0(\theta_0)\}$), on which the matrix $[Id - \Phi(\theta_0 + 1; \theta_0)]$ is invertible. We denote the projections onto $\mathrm{Span}\{DK_0(\theta_0)\}$ and E_{θ_0} as $\Pi^{\top}_{\theta_0}$ and $\Pi^{\perp}_{\theta_0}$, respectively.

Remark 4.1. An equivalent formulation of (H1) in terms of functional analysis is (H1'). Define the operator $\mathscr{L}: C^1(\mathbb{T}, \mathbb{R}^n) \to C^0(\mathbb{T}, \mathbb{R}^n)$:

$$\mathscr{L}(v)(\theta) = \omega_0 Dv(\theta) - Df(K_0(\theta))v(\theta).$$

(H1') $Range(\mathscr{L})$ is of co-dimension 1, $Range(\mathscr{L}) \oplus \text{Span}\{DK_0\} = C^0(\mathbb{T}, \mathbb{R}^n)$.

The proofs of the Theorems in the next section imply the equivalence of (H1) and (H1').

To show the persistence of periodic orbit for a fixed $\gamma \in O$, the following assumptions on $\mathscr{P}$ are crucial. The assumption (H2.1) is about smoothness of $\mathscr{P}$ and expresses that $\mathscr{P}$ maps $C^{\ell+\text{Lip}}$ balls around zero into $C^{\ell-1+\text{Lip}}$ balls around zero (see Definition A.1 for $C^{\ell+\text{Lip}}$ spaces). (H3.1) is about Lipschitz property of $\mathscr{P}$ in C^0 for smooth K's. These properties are verified in the examples we study in Sections 6 and 7. For example, when the functional P is evaluation on $x(t - r(x(t)))$, the regularity is a consequence of the fact that we can control the C^ℓ norm of $f \circ g$ by the C^ℓ norm of f, g. (We can even loose a derivative.) The C^0 Lipschitz property results from the mean value theorem ($\|f \circ g_1 - f \circ g_2\|_{C^0} \leqslant \|f\|_{C^1}\|g_1 - g_2\|_{C^0}$).

In the following, ℓ is an arbitrarily fixed positive integer.

Let U_ρ be the ball of radius ρ in the space $C^{\ell+\text{Lip}}(\mathbb{T}, \mathbb{R}^n)$ centered at K_0, and let B_δ be the interval in $\mathbb{R}$ with radius δ centered at ω_0. Fix ρ and δ.

(H2.1) There exists constant $\phi_{\rho,\delta} > 0$, such that for all $K \in U_\rho$ and $\omega \in B_\delta$, $\mathscr{P}(K, \omega, \gamma, \cdot)$: $\mathbb{T} \to \mathbb{R}^n$ is $C^{\ell-1+\text{Lip}}$, with

$$\|\mathscr{P}(K, \omega, \gamma, \cdot)\|_{C^{\ell-1+\text{Lip}}} \leqslant \phi_{\rho,\delta}.$$

See (87) for definition of $C^{\ell+\text{Lip}}$ norm.

(H3.1) There exists constant $\alpha_{\rho,\delta} > 0$, such that for all $K,\ K' \in U_\rho$, and $\omega,\ \omega' \in B_\delta$,

$$|\mathscr{P}(K, \omega, \gamma, \theta) - \mathscr{P}(K', \omega', \gamma, \theta)| \leqslant \alpha_{\rho,\delta} \max\left\{|\omega - \omega'|, \|K - K'\|\right\}, \tag{6}$$

for all $\theta \in \mathbb{T}$, where $\|K - K'\|$ is the C^0 norm of $K - K'$ under the Euclidean distance on $\mathbb{R}^n$.

To show that the periodic orbits of the FDE (2) depend on the parameter γ smoothly, one needs to incorporate the dependence of the parameterization K and frequency ω on the parameter γ. Therefore, we abuse the notations K and ω and use the same letter when they are functions depending on the parameter γ. In this case, K is a function of θ and γ instead of a function of θ, and ω is a function of γ instead of a constant. (H2.2) and (H3.2) are similar to (H2.1) and (H3.1), respectively.

We let $\mathcal{U}_\rho$ be the ball of radius ρ in the space $C^{\ell+\text{Lip}}(\mathbb{T} \times O, \mathbb{R}^n)$ centered at γ independent function $K_0(\theta, \gamma) = K_0(\theta)$, and let $\mathcal{B}_\delta$ be the ball in $C^{\ell+\text{Lip}}(O, \mathbb{R})$ with radius δ centered at γ independent function $\omega_0(\gamma) = \omega_0$. Again, fix ρ and δ.

(H2.2) There exists constant $\phi_{\rho,\delta} > 0$, such that for all $K \in \mathcal{U}_\rho$ and $\omega \in \mathcal{B}_\delta$, $\mathscr{P}(K, \omega, \cdot, \cdot)$: $\mathbb{T} \times O \to \mathbb{R}^n$ is $C^{\ell+\text{Lip}}$ in γ, and $C^{\ell-1+\text{Lip}}$ in θ, with

$$\|\mathscr{P}(K, \omega, \cdot, \theta)\|_{C^{\ell+\text{Lip}}} \leqslant \phi_{\rho,\delta},$$

$$\|\mathscr{P}(K, \omega, \gamma, \cdot)\|_{C^{\ell-1+\text{Lip}}} \leqslant \phi_{\rho,\delta}.$$

(H3.2) There exists constant $\alpha_{\rho,\delta} > 0$, such that for all $K,\ K' \in \mathcal{U}_\rho$ and $\omega,\ \omega' \in \mathcal{B}_\delta$, and for all $\theta \in \mathbb{T}$ and $\gamma \in O$,

$$|\mathscr{P}(K, \omega, \gamma, \theta) - \mathscr{P}(K', \omega', \gamma, \theta)| \leqslant \alpha_{\rho,\delta} \max\left\{\|\omega - \omega'\|, \|K - K'\|\right\},$$

where $\|\cdot\|$ denotes the C^0 norm.

Remark 4.2. Note that our results work exactly the same if the perturbation depends on ε, i.e. we have $P(x_t, \gamma, \varepsilon)$ instead of $P(x_t, \gamma)$ in (2). We can get $\mathscr{P}(K, \omega, \gamma, \varepsilon, \theta)$ in this case. We need assumptions on $\mathscr{P}$ to hold uniformly in ε for all small ε.

More specifically, (H2.1), (H3.1) can be reformulated as:

(H2.1') There exists a positive function $\phi_{\rho,\delta}(\varepsilon)$, such that for all $K \in U_\rho$ and $\omega \in B_\delta$, $\mathscr{P}(K, \omega, \gamma, \varepsilon, \cdot) \colon \mathbb{T} \to \mathbb{R}^n$ is $C^{\ell-1+\mathrm{Lip}}$, with

$$\|\mathscr{P}(K, \omega, \gamma, \varepsilon, \cdot)\|_{C^{\ell-1+\mathrm{Lip}}} \leqslant \phi_{\rho,\delta}(\varepsilon).$$

Moreover, the function $\phi_{\rho,\delta}$ satisfies that $\varepsilon\phi_{\rho,\delta}(\varepsilon)$ converges to zero as $\varepsilon \to 0$.

(H3.1') There exists positive function $\alpha_{\rho,\delta}(\varepsilon)$, such that for all $K,\ K' \in U_\rho$, and $\omega,\ \omega' \in B_\delta$, and for all $\theta \in \mathbb{T}$,

$$|\mathscr{P}(K, \omega, \gamma, \varepsilon, \theta) - \mathscr{P}(K', \omega', \gamma, \varepsilon, \theta)| \leqslant \alpha_{\rho,\delta}(\varepsilon) \max\left\{|\omega - \omega'|, \|K - K'\|\right\}.$$

Moreover, the function $\alpha_{\rho,\delta}$ satisfies that $\varepsilon\alpha_{\rho,\delta}(\varepsilon)$ converges to zero as $\varepsilon \to 0$.

The assumptions similar to (H2.2), (H3.2) can be formulated similarly.

Remark 4.3. The assumptions we use are similar to assumptions in invariant manifold theory. For example in [53], the (H2.1) is called *propagated bounds*.

Remark 4.4. The most crucial part of assumptions (H2.1) and (H2.2) is that instead of requiring regularity of $\mathscr{P}$ as an operator on function spaces, we only require bounds of the derivatives of the function $\mathscr{P}(K)$ with respect to θ and γ assuming that K is smooth in θ and γ.

When the functional $\mathscr{P}$ is based on compositions (as in the case of SDDEs), the propagated bounds are easy to establish even if the regularity of $\mathscr{P}$ as an operator on function spaces is much more problematic.

This makes our assumptions much weaker than the ones in the literature. However, our results only concern periodic solutions. See also Remark 6.1.

Remark 4.5. We call attention to the fact that in Section 8 we will weaken substantially the assumption (H3.1) to be able to deal with equations with small delays. One possible way is to change the norm $\|\cdot\|$ of $K - K'$ in (6) to C^1 norm, thanks to the fact that we allow loss of a derivative.

4.2. Main theorems

Let $\mathbb{N}$ denote the set of positive integers.

Theorem 4.6 *(Persistence). For a given $\ell \in \mathbb{N}$, assume that f in (2) is $C^{\ell+\mathrm{Lip}}$, and that (H1), (H2.1), and (H3.1) are satisfied for a given $\gamma \in O$. Then, there exists $\varepsilon_0 > 0$, such that when $\varepsilon < \varepsilon_0$, the FDE (2) has a periodic orbit, which is parameterized by a $C^{\ell+\mathrm{Lip}}$ map $K: \mathbb{T} \to \mathbb{R}^n$. The smallness condition of ε_0 depends on ℓ, f, and P.*

The frequency ω for the periodic orbit of equation (2) is close to ω_0, the frequency of the periodic orbit of equation (1). $\|K - K_0\|_{C^\ell}$ is small under a suitable choice of the phases.

When the parameter γ varies, Theorem 4.7 shows that the unknowns K and ω depend smoothly on γ.

Theorem 4.7 *(Smooth dependence on parameter). For a given $\ell \in \mathbb{N}$, assume that f in (2) is $C^{\ell+\mathrm{Lip}}$, and that (H1), (H2.2), and (H3.2) are satisfied. Then, there is $\varepsilon_0 > 0$, such that if $\varepsilon < \varepsilon_0$, one can find $K_\gamma(\theta)$ which parameterizes the periodic orbit of FDE (2) persisted from the periodic orbit of (1). The smallness condition of ε_0 depends on ℓ, f, and P.*

K_γ has frequency ω_γ. $K_\gamma(\theta)$ is jointly $C^{\ell+\mathrm{Lip}}$ in θ and γ, ω_γ is $C^{\ell+\mathrm{Lip}}$ in γ.

4.3. Some comments on the Theorems 4.6 and 4.7

Remark 4.8. One physically important case where assumption (H1) fails is when there is a conserved quantity (for example, the energy in mechanical systems). We are not able to deal with this case by the method of this paper, but we hope to come back to this problem.

Remark 4.9. Note that K will not be unique. If $K(\theta)$ parameterizes the periodic orbit, then for any given θ_1, $K(\theta + \theta_1)$ also parameterizes the periodic orbit, with a shifted phase. Hence, in Theorem 4.6, the smallness of $\|K - K_0\|_{C^\ell}$ is interpreted under a suitable choice of the phases. Note that changing the phase of K_0 of the unperturbed periodic orbit changes the phase of the resulting K.

For the current case, one can also append a normalization (e.g., fixing the value of $\int_{\mathbb{T}} DK_0(s)^{\mathrm{T}} K(s)\, \mathrm{d}s$) to the equation of K (4). This idea is very convenient for fixing the phase in numerical studies, see [29].

Since the proofs of Theorems 4.6 and 4.7 are based on the contraction mapping argument, the parameterizations we found are locally unique up to possible choices of different phases.

Remark 4.10. The smallness of ε depends on ℓ, hence, the method cannot get a C^∞ result directly. Note, however, that in some cases, e.g. state-dependent delay perturbations in equation (28), one can bootstrap the regularity from C^1 to C^∞.

Remark 4.11. Our results apply to several types of FDEs, especially to many DDEs, see Sections 6 and 7. We only need that (H1), (H2.1), (H2.2), (H3.1), and (H3.2) are satisfied. Indeed, we allow several terms in the equation which may involve forward and backward delays. A comparison of assumptions in the present paper with the standard assumptions in the literature for SDDEs, see e.g., [36,50], is in Remark 6.1. In our case, we study special solutions rather than developing a general theory for all initial conditions of FDEs.

Remark 4.12. Our method allows to bypass the propagation of discontinuity in DDEs. Moreover, it has no restriction on the relation between the periods of the periodic orbits and the size of the delay.

Remark 4.13. The proofs we present are constructive, hence they can be implemented numerically. Indeed, we formulate the problem as a fixed point of a contractive operator, which concatenates several elementary operators. Implementations of these elementary operators for a 2D model are addressed in a numerical toolkit developed in [29].

The proofs, based on the fixed point approach, also lead to results in an a-posteriori format, which state that if there is an approximate solution (satisfying some mild assumptions), then there is a true solution which is close to the approximate one. See more details in Section 5.5.

Remark 4.14. A-posteriori theorems justify asymptotic expansions where solutions are written as formal expansions in terms of the small parameters, see [10,8]. Truncations of the formal power series provide approximate solutions. The a-posteriori theorem shows that there is one true solution close by.

A-posteriori theorems are also the base of computer-assisted proofs. Numerical methods produce approximate solutions. If one can estimate rigorously the error and the non-degeneracy conditions, then one has established the existence of the solution. The verification of the error in the approximation is a finite (but long) calculation which can be done using computers taking care of round-off and truncation. Some cases where computer-assisted proofs have been used in constant delay equations for periodic orbits and unstable manifolds are [45,30].

Remark 4.15. In a recent work [28], validated values of ε which make sure that the perturbed equation in this paper has a unique periodic orbit were obtained for some concrete examples with computer-assisted techniques.

5. Proofs

The proofs of Theorems 4.6 and 4.7 are based on the fixed point approach. We will provide the detailed proof of Theorem 4.6. The proof of Theorem 4.7 follows in the same manner by adding the parameters in the unknown functions, see Section 5.6.

The proof consists of several steps. First, we define an operator in an appropriate space of smooth functions. Then, we show that (i) the operator maps a ball in this space into itself (Section 5.3); (ii) the operator is a contraction in a C^0 type of distance (Section 5.4). The existence of fixed point in desired space is hence ensured using a generalization of contraction mapping [53], see also Lemma A.11.

5.1. Invariance equations

In this section, we reformulate the invariance equation (4). Since K and ω are expected to be close to K_0 and ω_0 respectively, it is natural to reformulate (4) as an equation for the corrections from the unperturbed ones (note that this includes making a choice among the K_0). In Section 5.2, we will manipulate the equation for the corrections into a fixed point problem.

Let

$$\begin{aligned} K(\theta) &:= K_0(\theta) + \widehat{K}(\theta), \\ \omega &:= \omega_0 + \widehat{\omega}, \end{aligned} \tag{7}$$

where $\widehat{K}: \mathbb{T} \to \mathbb{R}^n$ and $\widehat{\omega} \in \mathbb{R}$ are corrections to the parameterization and frequency of the periodic orbit of the unperturbed equation. Our goal is to find $\widehat{K}$ and $\widehat{\omega}$ so that K and ω satisfy the functional equation (4).

Using the notation in (7) and the invariance equation (3) for K_0 and ω_0, we are led to the following functional equation for $\widehat{K}$ and $\widehat{\omega}$,

$$\omega_0 D\widehat{K}(\theta) - Df(K_0(\theta))\widehat{K}(\theta) = B^\varepsilon(\widehat{K}, \widehat{\omega}, \gamma, \theta) - \widehat{\omega} DK_0(\theta), \tag{8}$$

where

$$\begin{aligned} B^\varepsilon(\widehat{K}, \widehat{\omega}, \gamma, \theta) &:= N(\theta, \widehat{K}) + \varepsilon \mathscr{P}(K, \omega, \gamma, \theta) - \widehat{\omega} D\widehat{K}(\theta), \\ N(\theta, \widehat{K}) &:= f(K_0(\theta) + \widehat{K}(\theta)) - f(K_0(\theta)) - Df(K_0(\theta))\widehat{K}(\theta). \end{aligned} \tag{9}$$

5.2. *The operator*

Recall $\Phi(\theta; \theta_0)$ introduced in (5) as the flow of the variational equations. Using the variation of parameters formula, equation (8) for $\widehat{K}$ and $\widehat{\omega}$ is equivalent to:

$$\widehat{K}(\theta) = \Phi(\theta; \theta_0) \left\{ u_0 + \frac{1}{\omega_0} \int_{\theta_0}^{\theta} \Phi(s; \theta_0)^{-1} \big(B^\varepsilon(\widehat{K}, \widehat{\omega}, \gamma, s) - \widehat{\omega} DK_0(s) \big) ds \right\}, \tag{10}$$

where the initial condition $\widehat{K}(\theta_0) = u_0$ is to be found imposing that the right-hand side of (10) is periodic. This will be discussed in the Section 5.2.1.

We can think of (10) as a fixed point equation. The right-hand side is an operator in $\widehat{K}$, see Section 5.2.3. We start with a given $\widehat{K}$, choose $\widehat{\omega}$ following Section 5.2.1 and we substitute them in the right-hand side of (10).

5.2.1. *Periodicity condition*

Since the right-hand side of equation (8) is periodic, $\widehat{K}$ is periodic if and only if $\widehat{K}(\theta_0) = \widehat{K}(\theta_0 + 1)$, i.e.,

$$\begin{aligned} [Id - \Phi(\theta_0 + 1; \theta_0)]u_0 = {} & \frac{1}{\omega_0} \Phi(\theta_0 + 1; \theta_0) \int_{\theta_0}^{\theta_0+1} \Phi(s; \theta_0)^{-1} B^\varepsilon(\widehat{K}, \widehat{\omega}, \gamma, s) ds \\ & - \frac{\widehat{\omega}}{\omega_0} \Phi(\theta_0 + 1; \theta_0) \int_{\theta_0}^{\theta_0+1} \Phi(s; \theta_0)^{-1} DK_0(s) ds. \end{aligned} \tag{11}$$

Since K_0 solves (3), and Φ satisfies (5), we have

$$\Phi(s;\theta_0)DK_0(\theta_0) = DK_0(s).$$

Then, the periodicity condition (11) becomes

$$[Id - \Phi(\theta_0+1;\theta_0)]u_0 = \frac{1}{\omega_0}\int_{\theta_0}^{\theta_0+1} \Phi(\theta_0+1;s)B^\varepsilon(\widehat{K},\widehat{\omega},\gamma,s)ds - \frac{\widehat{\omega}}{\omega_0}DK_0(\theta_0). \tag{12}$$

One is able to solve for u_0 if the right-hand side of equation (12) is in the range of $Id - \Phi(\theta_0+1;\theta_0)$. Thanks to assumption (H1), this can be achieved by choosing the correct $\widehat{\omega}$. The choice of $\widehat{\omega}$ is unique.

5.2.2. Spaces

Let $a > 0$ and define interval $I_a = [-a, a]$, let

$$\begin{aligned} \mathscr{B}_\beta = \Big\{ g\colon \mathbb{T} \to \mathbb{R}^n \,\big|\, g \text{ is } C^{\ell+\mathrm{Lip}},\ \Big\| \frac{d^i}{d\theta^i} g(\theta) \Big\| \leqslant \beta_i,\ i = 0, 1, \dots, \ell, \\ \mathrm{Lip}\Big(\frac{d^\ell}{d\theta^\ell} g(\theta)\Big) \leqslant \beta_\ell^{\mathrm{Lip}} \Big\}, \end{aligned} \tag{13}$$

where $\beta = (\beta_0, \beta_1, \dots, \beta_\ell, \beta_\ell^{\mathrm{Lip}})$. The constants a, β_i, $i = 0, 1, \dots, \ell$, and $\beta_\ell^{\mathrm{Lip}}$ will be chosen in the proof.

5.2.3. Definition of the operator

Define the operator Γ^ε on $I_a \times \mathscr{B}_\beta$,

$$\Gamma^\varepsilon(\widehat{\omega}, \widehat{K}) = \begin{pmatrix} \Gamma_1^\varepsilon(\widehat{\omega}, \widehat{K}) \\ \Gamma_2^\varepsilon(\widehat{\omega}, \widehat{K}) \end{pmatrix}. \tag{14}$$

Componentwise,

$$\Gamma_1^\varepsilon(\widehat{\omega}, \widehat{K}) = \frac{\langle \int_{\theta_0}^{\theta_0+1} \Pi_{\theta_0}^\top \Phi(\theta_0+1;s)B^\varepsilon(\widehat{K},\widehat{\omega},\gamma,s)ds,\, DK_0(\theta_0)\rangle}{|DK_0(\theta_0)|^2}, \tag{15}$$

where $\langle\cdot,\cdot\rangle$ is the standard inner product on $\mathbb{R}^n$.

$$\Gamma_2^\varepsilon(\widehat{\omega}, \widehat{K})(\theta) = \Phi(\theta;\theta_0)u_0 + \frac{1}{\omega_0}\int_{\theta_0}^{\theta} \Phi(\theta;s)\big(B^\varepsilon(\widehat{K},\widehat{\omega},\gamma,s) - \Gamma_1^\varepsilon(\widehat{\omega},\widehat{K})DK_0(s)\big)ds, \tag{16}$$

where $u_0 \in E_{\theta_0}$ (the spectral complement of the eigenspace of 1 for $\Phi(\theta_0+1;\theta_0)$, see Section 4.1) and it satisfies

$$
\begin{aligned}
[Id - \Phi(\theta_0+1;\theta_0)]u_0 &= \frac{1}{\omega_0}\int_{\theta_0}^{\theta_0+1} \Phi(\theta_0+1;s)B^{\varepsilon}(\widehat{K},\widehat{\omega},\gamma,s)ds - \frac{\Gamma_1^{\varepsilon}(\widehat{\omega},\widehat{K})}{\omega_0}DK_0(\theta_0) \\
&= \frac{1}{\omega_0}\int_{\theta_0}^{\theta_0+1} \Pi_{\theta_0}^{\perp}\Phi(\theta_0+1;s)B^{\varepsilon}(\widehat{K},\widehat{\omega},\gamma,s)ds. \qquad (17)
\end{aligned}
$$

Remark 5.1. The definition of $\Gamma_1^{\varepsilon}(\widehat{\omega},\widehat{K})$ ensures the right-hand side of (17) to be in the range of $Id - \Phi(\theta_0+1;\theta_0)$. Since the kernel of $Id - \Phi(\theta_0+1;\theta_0)$ is $\mathrm{Span}\{DK_0(\theta_0)\}$, equation (17) has infinitely many solutions, all of them are the same up to constant multiples of $DK_0(\theta_0)$. In the definition of the operator Γ^{ε}, we have chosen the solution for equation (17) which lies in the space E_{θ_0}. If we choose a different u_0 solving (17), we will get another parameterization of the periodic orbit corresponding to a different phase, see Remark 4.9.

Our goal is to find the fixed point $(\widehat{\omega}^*, \widehat{K}^*)$ of the operator Γ^{ε} in a ball $I_a \times \mathscr{B}_\beta$, which will solve the equation (8). Hence $\omega = \omega_0 + \widehat{\omega}^*$ and $K = K_0 + \widehat{K}^*$ satisfy (4), K parameterizes the periodic orbit of (2) with frequency ω.

To this end, under the assumptions (H1), (H2.1) and (H2.2), we show in Section 5.3 that for small ε, we can choose a and β so that Γ^{ε} maps $I_a \times \mathscr{B}_\beta$ back into itself.

In Section 5.4 we show that Γ^{ε} is a contraction in a C^0-like distance. The desired result of existence of a locally unique fixed point follows from a fixed point result in the literature that we have collected as Lemma A.8 and A.11.

5.3. *Propagated bounds for* Γ^{ε}

In this section, we will prove the following Lemma.

Lemma 5.2. *Assume ε is small enough, then a and β can be chosen such that $\Gamma^{\varepsilon}: I_a \times \mathscr{B}_\beta \to I_a \times \mathscr{B}_\beta$.*

Proof. Note that

$$
\|N(\theta,\widehat{K})\| \leqslant \frac{1}{2}\mathrm{Lip}(Df)\|\widehat{K}\|^2,
$$

where $\|\cdot\|$ means C^0 norm. Indeed, here and later in this proof we only need the Lipschitz constant of $Df(x)$ in a neighborhood of the periodic orbit of the unperturbed ODE, i.e. $K_0(\mathbb{T})$.

Using the integration by parts formula, for $\theta \in [\theta_0, \theta_0+1]$, we have

$$
\left|\int_{\theta_0}^{\theta} \Phi(\theta_0+1;s)\widehat{\omega}D\widehat{K}(s)ds\right| \leqslant \left(2\|\Phi(\theta_0+1;\theta)\| + \left\|\frac{d}{d\theta}\Phi(\theta_0+1;\theta)\right\|\right)|\widehat{\omega}|\,\|\widehat{K}\|,
$$

where

$$
\|\Phi(\theta_0+1;\theta)\| := \max_{\theta\in[\theta_0,\theta_0+1]} |\Phi(\theta_0+1;\theta)|,
$$

and

$$\left\|\frac{d}{d\theta}\Phi(\theta_0+1;\theta)\right\| := \max_{\theta\in[\theta_0,\theta_0+1]}\left|\frac{d}{d\theta}\Phi(\theta_0+1;\theta)\right|,$$

and $|\cdot|$ denotes the operator norm of the matrix. We will use similar conventions for norms from now on. Note that these standard notations are slightly imprecise, the left-hand sides of above definitions of norms do not depend on θ, which we keep to indicate the variable of the functions.

Since $(\widehat{\omega}, \widehat{K}) \in I_a \times \mathscr{B}_\beta$, we have

$$|\Gamma_1^\varepsilon(\widehat{\omega}, \widehat{K})| \leqslant \frac{\|\Pi_{\theta_0}^\top\|}{\left|DK_0(\theta_0)\right|}\Big[\|\Phi(\theta_0+1;s)\|\Big(\frac{1}{2}\operatorname{Lip}(Df)\beta_0^2+\varepsilon\|\mathscr{P}(K,\omega,\gamma,\theta)\|\Big) + \Big(2\|\Phi(\theta_0+1;\theta)\| + \left\|\frac{d}{d\theta}\Phi(\theta_0+1;\theta)\right\|\Big)a\beta_0\Big], \tag{18}$$

and,

$$\begin{aligned}\|\Gamma_2^\varepsilon(\widehat{\omega}, \widehat{K})\| \leqslant{}& \|\Phi(\theta;\theta_0)\|M\Big[\|\Phi(\theta_0+1;s)\|(\frac{1}{2}\operatorname{Lip}(Df)\beta_0^2+\varepsilon\|\mathscr{P}(K,\omega,\gamma,\theta)\|) \\ &+ \Big(2\|\Phi(\theta_0+1;\theta)\| + \left\|\frac{d}{d\theta}\Phi(\theta_0+1;\theta)\right\|\Big)a\beta_0\Big] \\ &+ \frac{1}{\omega_0}\Big[\|\Phi(\theta;s)\|(\frac{1}{2}\operatorname{Lip}(Df)\beta_0^2+\varepsilon\|\mathscr{P}(K,\omega,\gamma,\theta)\|) \\ &+ \Big(2\|\Phi(\theta;\theta_0)\| + \left\|\frac{d}{ds}\Phi(\theta;s)\right\|\Big)a\beta_0\Big] \\ &+ \frac{\|\Phi(\theta;s)\|}{\omega_0}\left\|DK_0(s)\right\||\Gamma_1^\varepsilon(\widehat{\omega}, \widehat{K})|,\end{aligned} \tag{19}$$

where

$$\|\Phi(\theta;s)\| := \max_{\theta\in[\theta_0,\theta_0+1]}\max_{s\in[\theta_0,\theta]}|\Phi(\theta;s)|,$$
$$\left\|\frac{d}{ds}\Phi(\theta;s)\right\| := \max_{\theta\in[\theta_0,\theta_0+1]}\max_{s\in[\theta_0,\theta]}\left|\frac{d}{ds}\Phi(\theta;s)\right|,$$

and

$$M := \frac{\|[Id-\Phi(\theta_0+1;\theta_0)]^{-1}\|\|\Pi_{\theta_0}^\perp\|}{\omega_0}. \tag{20}$$

We have used $[Id-\Phi(\theta_0+1;\theta_0)]^{-1}$ to denote the inverse of $[Id-\Phi(\theta_0+1;\theta_0)]$ in the $(n-1)$-dimensional space E_{θ_0} introduced in Section 4.1.

Note that for the right-hand sides of the inequalities (18) and (19) above, each term is either quadratic in a, β_0 or has a factor ε. Under smallness assumptions of a, β_0, and ε, we will have $|\Gamma_1^\varepsilon(\widehat{\omega}, \widehat{K})| \leqslant a$ and $\|\Gamma_2^\varepsilon(\widehat{\omega}, \widehat{K})\| \leqslant \beta_0$.

Now we consider the derivatives of $\Gamma_2^\varepsilon(\widehat{\omega}, \widehat{K})$.

The first derivative $\frac{d}{d\theta}\Gamma_2^\varepsilon(\widehat{\omega}, \widehat{K})(\theta)$ has the expression:

$$\left(\frac{d}{d\theta}\Phi(\theta;\theta_0)\right)\left\{u_0 + \frac{1}{\omega_0}\int_{\theta_0}^{\theta}\Phi(s;\theta_0)^{-1}\big(B^\varepsilon(\widehat{K},\widehat{\omega},\gamma,s) - \Gamma_1^\varepsilon(\widehat{\omega},\widehat{K})DK_0(s)\big)ds\right\}$$
$$+\frac{1}{\omega_0}\left\{B^\varepsilon(\widehat{K},\widehat{\omega},\gamma,\theta) - \Gamma_1^\varepsilon(\widehat{\omega},\widehat{K})DK_0(\theta)\right\}.$$

Recall that $\Phi(\theta;\theta_0)$ solves equation (5). Therefore,

$$\begin{aligned}\left\|\frac{d}{d\theta}\Gamma_2^\varepsilon(\widehat{\omega},\widehat{K})\right\| \leqslant & \frac{1}{\omega_0}\|Df(K_0(\theta))\|\|\Gamma_2^\varepsilon(\widehat{\omega},\widehat{K})\| + \frac{|\Gamma_1^\varepsilon(\widehat{\omega},\widehat{K})|}{\text{Lip}}(Df)\omega_0\left\|DK_0(\theta)\right\| \\ & + \frac{1}{\omega_0}\left(\frac{1}{2}\text{Lip}(Df)\|\widehat{K}\|^2 + \varepsilon\|\mathscr{P}(K,\omega,\gamma,\theta)\| + |\widehat{\omega}|\left\|D\widehat{K}(\theta)\right\|\right) \\ \leqslant & \frac{1}{\omega_0}\|Df(K_0(\theta))\|\beta_0 + \frac{a}{\omega_0}\left\|DK_0(\theta)\right\| \\ & + \frac{1}{\omega_0}\left(\frac{1}{2}\text{Lip}(Df)\beta_0^2 + \varepsilon\|\mathscr{P}(K,\omega,\gamma,\theta)\| + a\beta_1\right).\end{aligned}$$

If ε, a, and β_0 are small enough, we can choose β_1 to ensure that $\left\|\frac{d}{d\theta}\Gamma_2^\varepsilon(\widehat{\omega},\widehat{K})\right\| \leqslant \beta_1$.

Now we proceed inductively, for $n \geqslant 2$, $\frac{d^n}{d\theta^n}\Gamma_2^\varepsilon(\widehat{\omega},\widehat{K})$ is an expression involving Φ, K_0, and their derivatives up to order n, as well as B^ε and its derivatives up to order $n-1$. Within this expression, K_0 and its derivatives are always multiplied by the small factor $\Gamma_1^\varepsilon(\widehat{\omega},\widehat{K})$, which has absolute value bounded by constant a. It remains to consider B^ε and its derivatives.

Recall the definition of B^ε in (9), we now consider the three terms in B^ε separately:

- For derivatives of $N(\theta,\widehat{K})$, we use the Faa di Bruno formula. The j-th derivative of N is an expression which contains derivatives of f up to order $j+1$, derivatives of $\widehat{K}$ up to order j. All the terms in derivatives of N can be controlled taking advantage of the fact that N is of order at least 2 in $\widehat{K}$.
- Derivatives of $\mathscr{P}$ are bounded thanks to the assumption (H2.1). Moreover, note that in B^ε, $\mathscr{P}$ has the perturbation parameter ε as its coefficient. Hence, this term is less crucial.
- For the last term, $\widehat{\omega}D\widehat{K}(\theta)$, its j-th derivative is $\widehat{\omega}D^{j+1}\widehat{K}(\theta)$. All are under control since $|\widehat{\omega}| < a$. Notice that the $(n-1)$-th derivative of this term is $\widehat{\omega}D^n\widehat{K}(\theta)$, which is the only place that $D^n\widehat{K}(\theta)$ appears.

Taking all the terms above into consideration and using the triangle inequality, we obtain bounds

$$\left\|\frac{d^n}{d\theta^n}\Gamma_2^\varepsilon(\widehat{\omega},\widehat{K})\right\| \leqslant P_n(a,\beta_0,\dots,\beta_{n-1}) + \nu\beta_n, \tag{21}$$

where for each n, P_n is a polynomial expression with positive coefficients, and $0<\nu<1$. The coefficients of P_n are combinatorial numbers multiplied by derivatives of K_0, $\mathscr{P}$, f, and $\Phi(\theta;\theta_0)$.

Therefore, we can choose recursively the β_i's such that right-hand side of inequality (21) is bounded by β_n.

Similar estimation can be obtained for the Lipschitz constant of $\frac{d^\ell}{d\theta^\ell}\Gamma_2^\varepsilon(\widehat{\omega}, \widehat{K})$. Hence, we can choose a, β such that $\Gamma^\varepsilon : I_a \times \mathscr{B}_\beta \to I_a \times \mathscr{B}_\beta$. □

Remark 5.3. Note that for ε sufficiently small, we can choose constant a and each component of β to be as small as we want.

Remark 5.4. Note that $I_a \times \mathscr{B}_\beta \subset \mathbb{R} \times C(\mathbb{T}, \mathbb{R}^n)$ is compact and convex, and it is obvious that $\Gamma^\varepsilon : I_a \times \mathscr{B}_\beta \to I_a \times \mathscr{B}_\beta$ is continuous, so one could apply Schauder's fixed point Theorem to obtain existence of the fixed point. Indeed, weaker assumptions than assumption (H3.1) on $\mathscr{P}$ could also suffice to ensure continuity of Γ^ε.

We will later prove that Γ^ε is a contraction in C^0 topology, which will give local uniqueness of the fixed point and a-posteriori estimates on the difference between an initial guess and the fixed point.

In principle, the Banach contraction theorem provides estimates of the difference in C^0 norm, but, taking into account the propagated bounds, we can use interpolation inequalities (Lemma A.7) to obtain estimates in norms with higher regularity. See Section 5.5.

5.4. Contraction properties of Γ^ε

Define C^0-type distance on $I_a \times \mathscr{B}_\beta$:

$$d\left((\widehat{\omega}, \widehat{K}), (\widehat{\omega}', \widehat{K}')\right) := \max\left\{|\widehat{\omega} - \widehat{\omega}'|, \|\widehat{K} - \widehat{K}'\|\right\}. \tag{22}$$

Lemma 5.5. *For small enough ε, a, and β_0 (as in β), the operator in (14) is a contraction on $I_a \times \mathscr{B}_\beta$ with distance (22), i.e., there exists $0 < \mu < 1$, such that*

$$d\left(\Gamma^\varepsilon(\widehat{\omega}, \widehat{K}), \Gamma^\varepsilon(\widehat{\omega}', \widehat{K}')\right) < \mu \cdot d\left((\widehat{\omega}, \widehat{K}), (\widehat{\omega}', \widehat{K}')\right). \tag{23}$$

Proof. The proof of this lemma consists basically in adding and subtracting and estimating by the mean value theorem.

We first list some useful inequalities for proving this lemma:

$$\|N(\theta, \widehat{K}) - N(\theta, \widehat{K}')\| \leqslant \frac{1}{2}\mathrm{Lip}(Df)\left(\|\widehat{K}\| + \|\widehat{K}'\|\right)\|\widehat{K} - \widehat{K}'\|,$$

where $\mathrm{Lip}(Df)$ is still interpreted as the Lipschitz constant of $Df(x)$ in a neighborhood of the periodic orbit of the unperturbed ODE, as in the proof of Lemma 5.2.

For $\theta \in [\theta_0, \theta_0 + 1]$,

$$\left|\int_{\theta_0}^{\theta} \Phi(\theta_0 + 1; s)\widehat{\omega} D\widehat{K}(s)ds - \int_{\theta_0}^{\theta} \Phi(\theta_0 + 1; s)\widehat{\omega}' D\widehat{K}'(s)ds\right|$$
$$\leqslant \left(2\|\Phi(\theta_0 + 1; \theta)\| + \left\|\frac{d}{d\theta}\Phi(\theta_0 + 1; \theta)\right\|\right)\left(\|\widehat{K}\||\widehat{\omega} - \widehat{\omega}'| + |\widehat{\omega}'|\|\widehat{K} - \widehat{K}'\|\right).$$

Define

$$\omega' = \omega_0 + \widehat{\omega}', \quad K' = K_0 + \widehat{K}',$$

similar to (7).

By assumption (H3.1),

$$|\mathscr{P}(K,\omega,\gamma,\theta) - \mathscr{P}(K',\omega',\gamma,\theta)| \leqslant \alpha_{\rho,\delta} \max\left\{|\omega-\omega'|, \|K-K'\|\right\}.$$

Then,

$$\begin{aligned} &|\Gamma_1^\varepsilon(\widehat{\omega},\widehat{K}) - \Gamma_1^\varepsilon(\widehat{\omega}',\widehat{K}')| \\ &\leqslant \frac{\|\Pi_{\theta_0}^\top\|\Big(2\|\Phi(\theta_0+1;\theta)\| + \left\|\frac{d}{d\theta}\Phi(\theta_0+1;\theta)\right\|\Big)}{\left|DK_0(\theta_0)\right|}(\beta_0|\widehat{\omega}-\widehat{\omega}'| + a\|\widehat{K}-\widehat{K}'\|) \\ &\quad + \frac{\|\Pi_{\theta_0}^\top\|\|\Phi(\theta_0+1;\theta)\|}{\left|DK_0(\theta_0)\right|}\Big[\beta_0 \operatorname{Lip}(Df)\|\widehat{K}-\widehat{K}'\| + \varepsilon\alpha_{\rho,\delta} d\big((\widehat{\omega},\widehat{K}),(\widehat{\omega}',\widehat{K}')\big)\Big]. \end{aligned} \tag{24}$$

The initial conditions in both cases are:

$$u_0 = \frac{1}{\omega_0}[Id - \Phi(\theta_0+1;\theta_0)]^{-1}\int_{\theta_0}^{\theta_0+1} \Pi_{\theta_0}^\perp \Phi(\theta_0+1;s) B^\varepsilon(\widehat{K},\widehat{\omega},\gamma,s)ds;$$

$$u_0' = \frac{1}{\omega_0}[Id - \Phi(\theta_0+1;\theta_0)]^{-1}\int_{\theta_0}^{\theta_0+1} \Pi_{\theta_0}^\perp \Phi(\theta_0+1;s) B^\varepsilon(\widehat{K}',\widehat{\omega}',\gamma,s)ds.$$

As before, $[Id - \Phi(\theta_0+1;\theta_0)]^{-1}$ denotes the inverse of $[Id - \Phi(\theta_0+1;\theta_0)]$ in the $(n-1)$-dimensional space E_{θ_0} introduced in Section 4.1.

Therefore,

$$\begin{aligned} |u_0 - u_0'| \leqslant & M\Big(2\|\Phi(\theta_0+1;\theta)\| + \left\|\frac{d}{d\theta}\Phi(\theta_0+1;\theta)\right\|\Big)\big(\beta_0|\widehat{\omega}-\widehat{\omega}'| + a\|\widehat{K}-\widehat{K}'\|\big) \\ & + M\|\Phi(\theta_0+1;\theta)\|\Big[\beta_0 \operatorname{Lip}(Df)\|\widehat{K}-\widehat{K}'\| + \varepsilon\alpha_{\rho,\delta} d\big((\widehat{\omega},\widehat{K}),(\widehat{\omega}',\widehat{K}')\big)\Big], \end{aligned} \tag{25}$$

where M is defined as in (20). Therefore,

$$\begin{aligned} &\|\Gamma_2^\varepsilon(\widehat{\omega},\widehat{K}) - \Gamma_2^\varepsilon(\widehat{\omega}',\widehat{K}')\| \\ &\qquad \leqslant \|\Phi(\theta;\theta_0)\||u_0 - u_0'| + \frac{\|\Phi(\theta;s)\|}{\omega_0}\|DK_0(\theta)\||\Gamma_1^\varepsilon(\widehat{\omega},\widehat{K}) - \Gamma_1^\varepsilon(\widehat{\omega}',\widehat{K}')| \\ &\qquad\quad + \frac{\|\Phi(\theta;\theta_0)\| + \left\|\frac{d}{ds}\Phi(\theta;s)\right\| + 1}{\omega_0}\big(\beta_0|\widehat{\omega}-\widehat{\omega}'| + a\|\widehat{K}-\widehat{K}'\|\big) \end{aligned} \tag{26}$$

$$+\frac{\|\Phi(\theta;s)\|}{\omega_0}\Big[\beta_0\,\mathrm{Lip}(Df)\|\widehat{K}-\widehat{K}'\|+\varepsilon\alpha_{\rho,\delta}d\big((\widehat{\omega},\widehat{K}),(\widehat{\omega}',\widehat{K}')\big)\Big].$$

Combining (24), (25), and (26), if ε is sufficiently small, a and β_0 are chosen to be sufficiently small, we can find μ such that (23) is true, Γ^ε is a contraction. □

5.5. *Conclusion of the proofs of Theorem 4.6*

Since Γ^ε is a contraction, there exists a fixed point $(\widehat{\omega}^*,\widehat{K}^*)$. According to Lemma A.11, which is a corollary of Arzela-Ascoli Theorem (see Lemma A.8), $(\widehat{\omega}^*,\widehat{K}^*)\in I_a\times\mathscr{B}_\beta$. Hence, $(\widehat{\omega}^*,\widehat{K}^*)$ is a solution of the functional equation (8) with desired regularity. Then, $K=K_0+\widehat{K}^*$ gives a parameterization of the periodic orbit of (2).

The proof based on fixed point approach leads to a-posteriori type of results. Suppose we start with initial guess $(\widehat{\omega}^0,\widehat{K}^0)$ for $(\widehat{\omega},\widehat{K})$, since Γ^ε is contractive, see equation (23), we have

$$d\left((\widehat{\omega}^0,\widehat{K}^0),(\widehat{\omega}^*,\widehat{K}^*)\right)<\frac{1}{1-\mu}d\left((\widehat{\omega}^0,\widehat{K}^0),\Gamma^\varepsilon(\widehat{\omega}^0,\widehat{K}^0)\right). \tag{27}$$

Therefore, if we have a good choice of initial guess such that the error in the fixed point equation, $d\big((\widehat{\omega}^0,\widehat{K}^0),\Gamma^\varepsilon(\widehat{\omega}^0,\widehat{K}^0)\big)$, is small, then we know that the fixed point is close to the initial guess.

Using the interpolation inequalities in Lemma A.7, we also have

$$\|\widehat{K}^0-\widehat{K}^*\|_{C^m}\leqslant C\|\widehat{K}^0-\widehat{K}^*\|_{C^0}^{1-\frac{m}{\ell+1}},$$

for $0\leqslant m\leqslant\ell$, where the constant C depends on m, ℓ, and β. In particular, the distance between the initial guess $(\widehat{\omega}^0,\widehat{K}^0)=(0,0)$ and the fixed point $(\widehat{\omega}^*,\widehat{K}^*)$ is of order ε, therefore, $\|\widehat{K}^*\|_{C^m}$ is small for $0\leqslant m\leqslant\ell$. This finishes the proof of Theorem 4.6.

5.6. *Comments on proof of Theorem 4.7*

A very similar method proves Theorem 4.7. To discuss the dependence on parameters, we view $\widehat{\omega}$ as a function of γ, and $\widehat{K}$ as a function of θ and γ. Define operator $\widetilde{\Gamma}^\varepsilon$ using the same formulas as in (15) and (16) on the space $\mathcal{I}_\alpha\times\mathcal{F}_\beta$, where $\mathcal{I}_\alpha$ contains $C^{\ell+\mathrm{Lip}}$ functions from set O to $\mathbb{R}$ and $\mathcal{F}_\beta$ contains $C^{\ell+\mathrm{Lip}}$ functions from $\mathbb{T}\times O$ to $\mathbb{R}^n$, with bounded derivatives and Lipschitz constant of the ℓ-th derivative similar to (13). We then prove that for small enough ε, and suitable choices for constants in α and β (bounds on derivatives and Lipschitz constants for functions in spaces $\mathcal{I}_\alpha$ and $\mathcal{F}_\beta$), $\widetilde{\Gamma}^\varepsilon$ maps $\mathcal{I}_\alpha\times\mathcal{F}_\beta$ to itself using assumption (H2.2), and is a contraction in C^0 norm, taking advantage of assumption (H3.2). Therefore, there exists a fixed point for $\widetilde{\Gamma}^\varepsilon$ in the space $\mathcal{I}_\alpha\times\mathcal{F}_\beta$ solving equation (8). Same as above, Theorem 4.7 is proved.

6. Delay perturbation to autonomous ODE

In this section we show how several concrete examples fit into our general result, Theorem 4.6 and Theorem 4.7. In all the cases, we will show how to construct the operators $\mathscr{P}$ and to verify the properties in assumptions (H2) and (H3).

6.1. State-dependent delay perturbation

An important class of equations that one can consider is DDEs with state-dependent delays (backward or forward or mixed):

$$\dot{x}(t) = f(x(t)) + \varepsilon P\Big(x(t), x\big(t - r(x(t))\big), \gamma\Big), \tag{28}$$

where $f: \mathbb{R}^n \to \mathbb{R}^n$, $P: \mathbb{R}^n \times \mathbb{R}^n \times O \to \mathbb{R}^n$, and $r: \mathbb{R}^n \to [-h, h]$ are C^∞, and h is a positive constant.

Note that in this case, the operator $\mathscr{P}$ is,

$$\mathscr{P}(K, \omega, \gamma, \theta) = P(K(\theta), \widetilde{K}(\theta), \gamma), \tag{29}$$

where $\widetilde{K}(\theta) := K(\theta - \omega r(K(\theta)))$ comes from the delay.

Applying the composition Lemma A.3 repeatedly, we know that $\mathscr{P}$ above satisfies (H2.1) for all $\ell \in \mathbb{N}$. With the standard adding and subtracting terms method, one gets that $\mathscr{P}$ satisfies (H3.1). Similarly, $\mathscr{P}$ satisfies (H2.2) and (H3.2). Thus, Theorems 4.6 and 4.7 can be applied.

Note also that for the above equation (28), using Lemma A.6, we are able to prove that the operator Γ^ε is a contraction under $C^{\ell-1+\mathrm{Lip}}$ norm in the second component for $\varepsilon < \varepsilon_0$, where ε_0 depends on ℓ, f, P, and r.

We may improve the regularity conclusion of Theorem 4.6 for this case. Indeed, once we have that the parameterization K of the periodic orbit is C^1 in θ, we can use the standard bootstrapping argument to conclude higher regularity of K based on the smoothness of the equation, see Remark 6.5.

We can also consider more general state-dependent delays (the delay r is a functional depending on the whole trajectory, not a function on $\mathbb{R}^n$), which in particular include nested delays:

$$\dot{x}(t) = f(x(t)) + \varepsilon P\Big(x(t), x\big(t - r(x_t, \gamma)\big), \gamma\Big), \tag{30}$$

where $r: \mathscr{R}[-h, h] \times O \to \mathbb{R}$, positive constant h is an upper bound for $|r|$.

In this case,

$$\mathscr{P}(K, \omega, \gamma, \theta) = P\left(K(\theta), K\big(\theta - \omega r(K_{\theta,\omega}, \gamma)\big), \gamma\right), \tag{31}$$

where $K_{\theta,\omega}: [-h, h] \to \mathbb{R}^n$ is defined by

$$K_{\theta,\omega}(s) := K(\theta + \omega s). \tag{32}$$

If r is chosen such that (H2) and (H3) are verified, Theorems 4.6 and 4.7 can be applied.

Remark 6.1. Note that the operator $\mathscr{P}$ in (29) and (31) involves the composition operator, whose differentiability properties in function spaces are very complicated (See [17] for a systematic study). Hence, using the standard strategy of studying variational equations etc. to study regularity of the evolution will be rather complicated. Indeed, it will be hard to go beyond the first derivative.

On the other hand, the present strategy, only requires much simpler assumptions. We do not need regularity of the functional $\mathscr{P}$ on spaces of functions. In (31), we only need that if K is smooth, $\mathscr{P}(K)$ is also smooth, and that we have bounds on the derivatives of $\mathscr{P}(K)$ assuming bounds on the derivatives of K. We need smoothness of $\mathscr{P}(K)$ only in the dynamical variable θ for Theorem 4.6, whereas we need smoothness of $\mathscr{P}(K)$ in θ and γ for Theorem 4.7.

The above is the most important difference between our assumptions and the ones in the literature, e.g., [50,36,78]. Our goal is much more modest since we only consider certain types of solutions rather than providing a general theory for all initial conditions.

In (30), r needs to satisfy similar assumptions to the ones on $\mathscr{P}$, i.e., propagated bounds and Lipschitz in C^0. Indeed, we only need that (H2) and (H3) in Section 4.1 are satisfied. Besides the most essential difference mentioned above, our assumptions are also weaker in the following respects: (a) we only need assumptions for $C^{\ell+\mathrm{Lip}}$ periodic functions; (b) we can afford to lose one derivative in θ; (c) depending on the particular forms of r and P, less regularity assumptions on r may suffice to ensure (H2) and (H3).

An example for which our result does not apply is when the perturbation term P depends on the supremum norm of the solution $x(t)$, due to the fact that the supremum norm is not a differentiable functional even for C^∞ functions. Indeed, $\|x_t\| = \sup_{s\in[t-h,t+h]} |x(s)|$ may be only Lipschitz in t even if x is C^∞.

6.2. *Distributed delay perturbation*

Our results apply to models with distributed delays as well

$$\dot{x}(t) = f(x(t)) + \varepsilon P\Big(x(t), \int_{-r}^{0} x_t(s)d\mu(s), \gamma\Big), \tag{33}$$

where $P: \mathbb{R}^n \times \mathbb{R}^n \times O \to \mathbb{R}^n$ is a C^∞ map, r is a constant, and μ is a signed Borel measure. In this case,

$$\mathscr{P}(K, \omega, \gamma, \theta) = P\left(K(\theta), \int_{-r}^{0} K_{\theta,\omega}(s)d\mu(s), \gamma\right), \tag{34}$$

where $K_{\theta,\omega}$ is defined in (32).

Above $\mathscr{P}$ satisfies (H2), since we only care about derivatives with respect to θ. It is not hard to see that (H3) is also satisfied in this case. Therefore, Theorems 4.6 and 4.7 apply.

6.3. *Remarks on further applicability of Theorem 4.6*

Remark 6.2. It is straightforward to see that our results could be applied to systems similar to above systems with multiple forward or backward delays.

Remark 6.3. In some applications, the delays are defined by some implicit relations from the full trajectory.

Theorems 4.6 and 4.7 can be applied if we can justify (H2.1), (H3.1), (H2.2), and (H3.2). Notice that we only need to justify these hypotheses when $\widehat{K}$ lies in a ball in a space of differentiable functions. In such a case, we can often use the implicit function theorem.

Remark 6.4. The results so far do not include the models in which the perturbation is just adding a small delay. This small delay perturbation is more singular and seems to require extra assumptions and slightly different proofs. The extension of the results to the small delay case is done in Section 8.

Remark 6.5. In the case of state-dependent delay or distributed delay with smooth f, P, and r, it is automatic to show that if K is C^ℓ, then the right-hand side of (4) is C^ℓ, hence, by looking at the left-hand side of (4), we get that K is $C^{\ell+1}$. The bootstrap stops only when we do not have any more regularity of f, P, or r.

So, in case that f, P, and r are C^∞, we obtain that the K is C^∞.

One natural question that deserves more study is whether in the case that f, P, and r are analytic, then K is analytic. The remarkable paper [61] contains obstructions that equations with time dependent delays – heuristically better behaved than the ones considered here, may fail to have analytic solutions. In view of these results, it is natural to conjecture that the periodic solutions produced here, could fail to be analytic even if f, P, and r are analytic.

7. Delay perturbation to non-autonomous periodic ODE

Time periodic systems appear in many problems in physics, for example, see Section 9. And when there are conserved quantities in the ODE systems, periodic orbits cannot satisfy the assumption (H1). These are the motivations to consider a non-autonomous ODE:

$$\dot{x}(t) = f(x(t), t), \tag{35}$$

where $f\colon \mathbb{R}^n \times \frac{1}{\omega_0}\mathbb{T} \to \mathbb{R}^n$ (f is periodic in t with period $\frac{1}{\omega_0}$). Add the perturbation:

$$\dot{x}(t) = f(x(t), t) + \varepsilon P(x_t, \gamma). \tag{36}$$

Using the standard method of adding an extra variable to equation (35) to make it autonomous, we see that we can reduce the problem to the previous case. The autonomous equation corresponding to (35) is

$$\begin{pmatrix} \dot{x}(t) \\ \dot{\tau}(t) \end{pmatrix} = g(x, \tau) := \begin{pmatrix} f(x, \tau) \\ 1 \end{pmatrix}, \tag{37}$$

where $t \in \frac{1}{\omega_0}\mathbb{T}$.

Denote Ψ as the solution of the scaled variational equation for the periodic orbit of (37):

$$\omega_0 \frac{d}{d\theta}\Psi(\theta;\theta_0) = Dg\left(K_0(\theta), \frac{\theta}{\omega_0}\right)\Psi(\theta;\theta_0), \quad \Psi(\theta_0;\theta_0) = Id, \tag{38}$$

where K_0 is a parameterization of the periodic orbit of (35). Since

$$Dg\left(K_0(\theta), \frac{\theta}{\omega_0}\right) = \begin{pmatrix} D_1 f\left(K_0(\theta), \frac{\theta}{\omega_0}\right) & D_2 f\left(K_0(\theta), \frac{\theta}{\omega_0}\right) \\ 0 & 0 \end{pmatrix},$$

we have

$$\Psi(\theta;\theta_0) = \begin{pmatrix} \Phi(\theta;\theta_0) & * \\ 0 & 1 \end{pmatrix}, \tag{39}$$

where

$$\omega_0 \frac{d}{d\theta}\Phi(\theta;\theta_0) = D_1 f\left(K_0(\theta), \frac{\theta}{\omega_0}\right)\Phi(\theta;\theta_0). \tag{40}$$

If Ψ satisfies assumption (H1), then 1 is not an eigenvalue of $\Phi(\theta_0+1;\theta_0)$.

Equivalently, we could start the discussion in this section directly with the following assumption on Φ defined in (40):

(H1”) 1 is not an eigenvalue of $\Phi(\theta_0+1;\theta_0)$.

Under either assumption (H1) on Ψ or assumption (H1”) on Φ, we are able to solve the invariance equation (4) without adjusting the frequency. Indeed, a periodic solution of equation (36) must have frequency ω_0 when ε is small. More precisely, (4) applied to equation (37) is equivalent to:

$$\omega_0 DK(\theta) = f\left(K(\theta), \frac{\theta}{\omega_0}\right) + \varepsilon \mathscr{P}(K, \omega_0, \gamma, \theta). \tag{41}$$

Let $K = K_0 + \widehat{K}$ as in (7), we are led to

$$\omega_0 D\widehat{K}(\theta) - D_1 f\left(K_0(\theta), \frac{\theta}{\omega_0}\right)\widehat{K}(\theta) = B^\varepsilon(\theta, \omega_0, \widehat{K}, \gamma), \tag{42}$$

where

$$B^\varepsilon(\theta, \omega_0, \widehat{K}, \gamma) := N(\theta, \widehat{K}) + \varepsilon\mathscr{P}(K, \omega_0, \gamma, \theta),$$
$$N(\theta, \widehat{K}) := f\left(K(\theta), \frac{\theta}{\omega_0}\right) - f\left(K_0(\theta), \frac{\theta}{\omega_0}\right) - D_1 f\left(K_0(\theta), \frac{\theta}{\omega_0}\right)\widehat{K}(\theta).$$

Now we define an operator Υ^ε on the space $\mathscr{B}_\beta$ (see (13)) very similar to the second component of Γ^ε introduced in section 5.2.

$$\Upsilon^\varepsilon(\widehat{K})(\theta) := \Phi(\theta;\theta_0)u_0 + \frac{1}{\omega_0}\int_{\theta_0}^{\theta} \Phi(\theta;s)B^\varepsilon(s, \omega_0, \widehat{K}, \gamma)ds, \tag{43}$$

where

$$u_0 = \frac{1}{\omega_0}[Id - \Phi(\theta_0+1;\theta_0)]^{-1}\int_{\theta_0}^{\theta_0+1} \Phi(\theta_0+1;s)B^\varepsilon(s, \omega_0, \widehat{K}, \gamma)ds. \tag{44}$$

We have employed that, in the periodic case, the matrix $[Id - \Phi(\theta_0+1;\theta_0)]$ is invertible.

Under the assumption that $\mathscr{P}$ satisfies (H2.1), (H3.1), (H2.2), and (H3.2), we can prove that Υ^ε has a fixed point $\widehat{K}^* \in \mathscr{B}_\beta$ by proving $\Upsilon^\varepsilon : \mathscr{B}_\beta \to \mathscr{B}_\beta$ (similar to Lemma 5.2) and Υ^ε is a C^0 contraction (similar to Lemma 5.5). The periodic orbit of (36) is parameterized by $K = K_0 + \widehat{K}^*$. The analysis of the operator Υ^ε in (43) is actually simpler than the analysis presented for the operator Γ^ε in (14) because we do not need to adjust the frequency.

Remark 7.1. Similarly, we can also consider a non-autonomous perturbation $P(t, x_t, \gamma)$, we need that P to be periodic in t with the same period $\frac{1}{\omega_0}$.

8. The case of small delays

Many problems in the literature lead to equations of the form:

$$\begin{aligned} \dot{y}(t) &= g(y(t - \varepsilon r)), \\ \dot{y}(t) &= f(y(t - \varepsilon r), t), \end{aligned} \tag{45}$$

where r could be either a constant, an explicit function of t, a function of $y(t)$, or y_t, or an implicit function on the trajectory, and may depend on ε; and f is periodic of period 1 in t. Indeed, our results apply also to variants of (45) with perturbations involving several forward or backward delays.

In problems which present feedback loops, the feedback takes some time to start acting. The problems (45) correspond to the feedback taking a short time to start acting.

Equations of the form (45) play an important role in electrodynamics, where the small parameter $\varepsilon = \frac{1}{c}$ is the inverse of the speed of light and the delay r is a functional that depends on the trajectory. Given the physical importance of electrodynamics, we devote Section 9 to give more details and to show that it can be studied applying the main result of this section, Theorem 8.2.

Introducing a small delay to the ODE is a very singular perturbation, since the phase space becomes infinite dimensional. The limit is mathematically harder because the effect of a small delay is similar to adding an extra term containing the derivative $y(t - \varepsilon r) \approx y(t) + \varepsilon \dot{y}(t) r$. This shows that, heuristically, the perturbation is of the same order as the equation.

Remark 8.1. In the physical literature, one can find the use of higher order expansions to obtain heuristically even higher order equations, see [21]. As a general theory for all the solutions of the equations, these theories have severe paradoxes (e.g. *preacceleration*). The results of this paper show, however that the non-degenerate periodic solutions produced in many of these expansions, since they are very approximate solutions of the invariance equation, approximate true periodic solutions of the full system.

As a reflection of the extra difficulty of the small delay problem compared with the equations from previous sections, the main result of this section, Theorem 8.2, requires a more delicate proof than Theorem 4.6 and we need stronger regularity to obtain the C^0 contraction with our approach.

An important mathematical paper on the singular problem of small delay is [10]. We also point out that, there is a considerable literature in the formal study of $\frac{1}{c}$ limit in electrodynamics and in gravity [52,72,4,73]. Many famous consequences of relativity theory (e.g. the precession of the perihelion of Mercury) are only studied by formal perturbations.

Formal expansions of periodic and quasiperiodic solutions for small delays were considered in [8]. The results of this section establish that the formal expansions of periodic orbits obtained in [8] correspond to true periodic orbits and are asymptotic to the true periodic solutions in a very strong sense.

In this section, we establish results on persistence of periodic orbits for the models in (45), see Theorem 8.2. As we will see, when we perform the detailed discussion, we will not be able to reduce Theorem 8.2 to be a particular case of Theorem 4.6. The proof of Theorem 8.2 will be very similar to that of Theorem 4.6 and which is based on the study of operator Γ^ε very similar to those in (14). Nevertheless, the analysis of the operator Γ^ε in the current case will require to take advantage of an extra cancellation.

8.1. Formulation of the results

Our main result for the small delay problem (45) is as follows. Without specifying the delay functional r, we will use $r(\omega, K, \varepsilon)$ to denote the expression after substituting $K(\theta + \omega t)$ into r and letting $t = 0$.

Theorem 8.2. *For integer $\ell \geqslant 3$, assume that the function g (resp. f) in (45) is $C^{\ell+\mathrm{Lip}}$.*

Assume that for $\varepsilon = 0$, the ordinary differential equation $\dot{y} = g(y)$ has a periodic orbit satisfying (H1) (resp. $\dot{y} = f(y, t)$ has a periodic orbit satisfying (H1")). We denote by K_0 the parameterization of this periodic orbit with frequency ω_0.

Recall that U_ρ is the ball of radius ρ in $C^{\ell+\mathrm{Lip}}(\mathbb{T}, \mathbb{R}^n)$ centered at K_0, and B_δ is the interval with radius δ centered at ω_0.

Recall distance d defined in (22). Assume that the delay functional r satisfies:
(i) there exists $\phi_{\rho,\delta}(\varepsilon) > 0$, with $\varepsilon\phi_{\rho,\delta}(\varepsilon) \to 0$ as $\varepsilon \to 0$, such that

$$\|r(\omega, K, \varepsilon)\|_{C^{\ell-1+\mathrm{Lip}}(\mathbb{T},\mathbb{R}^n)} \leqslant \phi_{\rho,\delta}(\varepsilon) \quad \forall K \in U_\rho,\ \omega \in B_\delta. \tag{46}$$

(ii) there exists $\alpha_{\rho,\delta}(\varepsilon) > 0$, with $\varepsilon\alpha_{\rho,\delta}(\varepsilon) \to 0$ as $\varepsilon \to 0$, such that for any $K_1, K_2 \in U_\rho$, $\omega_1, \omega_2 \in B_\delta$,

$$\|r(\omega_1, K_1, \varepsilon) - r(\omega_2, K_2, \varepsilon)\|_{C^0} \leqslant \alpha_{\rho,\delta}(\varepsilon) d\big((\omega_1, K_1), (\omega_2, K_2)\big). \tag{47}$$

Then, there exists $\varepsilon_0 > 0$ such that for $\varepsilon < \varepsilon_0$, the problem (45) admits a periodic solution. There is a $C^{\ell+\mathrm{Lip}}$ parameterization K of the periodic orbit which is close to K_0 in the sense of C^ℓ.

Remark 8.3. As before, the requirements of smallness in ε for Theorem 8.2 depend on the regularity considered.

In many applied situations, the g and f considered are C^∞ or even analytic (for example in the electrodynamics applications considered in Section 9). In such a case, we can apply the result for any $\ell \geqslant 3$, for larger ℓ, we may need to take smaller ε. This allows us to obtain the a-posteriori estimates in more regular spaces as ε goes to zero.

Hence, the formal power series in [8] are asymptotic in the strong sense that the error in the truncation is bounded by a power of ε, where a stronger norm can be used for smaller ε.

We leave for the reader the formulation of a corresponding result for the smooth dependence on parameters similar to Theorem 4.7. The proof requires only small modifications from the discussion in Section 8.2, see comments in Section 5.6.

The proof of Theorem 8.2 will be given in Section 8.2. We first define the operator in this case. Then for this operator, we state and prove Lemma 5.2 in Section 8.2.1, and then state and prove Lemma 5.5 in Section 8.2.2. The existence of the fixed point of the operator is thus established. As it turns out, the analysis of the operator requires more care than in the case of Theorem 4.6.

8.2. Existence of fixed point

The equations (45) can be rearranged as

$$\begin{aligned}\dot{y}(t) &= g(y(t)) + [g(y(t-\varepsilon r)) - g(y(t))]\\ &= g(y(t)) - \varepsilon \int_0^1 \left[Dg\big(y(t-s\varepsilon r)\big)Dy(t-s\varepsilon r)r\right] ds,\\ \dot{y}(t) &= f(y(t),t) + [f(y(t-\varepsilon r),t) - f(y(t),t)]\\ &= f(y(t),t) - \varepsilon \int_0^1 \left[D_1 f\big(y(t-s\varepsilon r),t\big)Dy(t-s\varepsilon r)r\right] ds,\end{aligned} \tag{48}$$

For typographical convenience, we will discuss only the autonomous case, which is the most complicated. We refer the reader to Section 7 to see how the discussion simplifies in the non-autonomous case (the most relevant case for applications to electrodynamics).

Note that (48) is in the form of (2), with the operator P defined as

$$P(y_t) := -\int_0^1 \left[Dg\big(y(t-s\varepsilon r)\big)Dy(t-s\varepsilon r)r\right] ds. \tag{49}$$

Then,

$$\mathscr{P}(K,\omega,\gamma,\theta) := -\int_0^1 \left[Dg\big(K(\theta-\varepsilon s\omega r)\big)DK(\theta-\varepsilon s\omega r)\omega r\right] ds, \tag{50}$$

where the r's are $r(\omega, K, \varepsilon)$, the delay functional evaluated on the periodic orbit.

We define operator Γ^ε in the same way as in Section 5, substituting the expression of $\mathscr{P}$ in (50) into the general formula in (14).

In this section, we will proceed as before and show Lemmas 5.2 and 5.5 hold for the resulting operator Γ^ε with $\mathscr{P}$ defined in (50).

Lemma 5.2 is proven in this case, same as above, by noticing $\mathscr{P}$ satisfies assumption (H2.1). The proof for Lemma 5.5 is slightly different from Section 5. There we only needed to take advantage of the Lipschitz property of the operator $\mathscr{P}$ (assumption (H3.1)). In the present case, we will have to take into account that the operator Γ^ε involves not only $\mathscr{P}$, but also an integral, which has nice properties that compensate the bad properties of $\mathscr{P}$.

8.2.1. Propagated bounds

We observe that if $K \in U_\rho$ and $\omega \in B_\delta$, by the assumption (46), $r(\omega, K, \varepsilon)$ is in a $C^{\ell-1+\mathrm{Lip}}$ ball of size $\phi_{\rho,\delta}(\varepsilon)$ and, using the estimates on composition, Lemma A.3, $K(t - \varepsilon s\omega r)$ also lies in a $C^{\ell-1+\mathrm{Lip}}$ ball. If $g \in C^{\ell+\mathrm{Lip}}$, then $Dg \in C^{\ell-1+\mathrm{Lip}}$ and we conclude that $Dg \circ K(t - \varepsilon s\omega r)$ is contained in a $C^{\ell-1+\mathrm{Lip}}$ ball.

We also have that if K is in a $C^{\ell+\mathrm{Lip}}$ ball, $DK(t - \varepsilon s\omega r)\omega r$ is in a $C^{\ell-1+\mathrm{Lip}}$ ball whose size is a function of ρ, δ, and $\phi_{\rho,\delta}(\varepsilon)$.

Putting it all together we obtain that (H2.1) holds for $\mathscr{P}$ defined in (50). Therefore, Lemma 5.2 is proven in this case.

8.2.2. Contraction in C^0

Before estimating $\Gamma^\varepsilon(\widehat{\omega}, \widehat{K}) - \Gamma^\varepsilon(\widehat{\omega}', \widehat{K}')$, we estimate $\mathscr{P}(K, \omega, \gamma, \theta) - \mathscr{P}(K', \omega', \gamma, \theta)$ (we denote by r, r' the two delay terms corresponding to ω, K, and ω', K' respectively).

As before, adding and subtracting, we obtain that the difference in the integrands in $\mathscr{P}$,

$$Dg\big(K(\theta - \varepsilon s\omega r)\big)DK(\theta - \varepsilon s\omega r)\omega r - Dg\big(K'(\theta - \varepsilon s\omega' r')\big)DK'(\theta - \varepsilon s\omega' r')\omega' r'$$

can be written as a sum of 8 differences in which only one of the objects changes in each line, see (51) below. As it turns out, 7 of them will be straightforward to estimate and only one of them will require some effort. We give the details.

$$\begin{aligned}
&[Dg\big(K(\theta - \varepsilon s\omega r)\big) - Dg\big(K'(\theta - \varepsilon s\omega r)\big)]DK(\theta - \varepsilon s\omega r)\omega r \\
&+ [Dg\big(K'(\theta - \varepsilon s\omega r)\big) - Dg\big(K'(\theta - \varepsilon s\omega' r)\big)]DK(\theta - \varepsilon s\omega r)\omega r \\
&+ [Dg\big(K'(\theta - \varepsilon s\omega' r)\big) - Dg\big(K'(\theta - \varepsilon s\omega' r')\big)]DK(\theta - \varepsilon s\omega r)\omega r \\
&+ Dg\big(K'(\theta - \varepsilon s\omega' r')\big)[DK - DK'](\theta - \varepsilon s\omega r)\omega r \\
&+ Dg\big(K'(\theta - \varepsilon s\omega' r')\big)[DK'(\theta - \varepsilon s\omega r) - DK'(\theta - \varepsilon s\omega' r)]\omega r \\
&+ Dg\big(K'(\theta - \varepsilon s\omega' r')\big)[DK'(\theta - \varepsilon s\omega' r) - DK'(\theta - \varepsilon s\omega' r')]\omega r \\
&+ Dg\big(K'(\theta - \varepsilon s\omega' r')\big)DK'(\theta - \varepsilon s\omega' r')(\omega - \omega')r \\
&+ Dg\big(K'(\theta - \varepsilon s\omega' r')\big)DK'(\theta - \varepsilon s\omega' r')\omega'(r - r')
\end{aligned} \tag{51}$$

The C^0 norms of all the terms except for the 4th one are straightforward to estimate by some constant multiple of $d\big((\widehat{\omega}, \widehat{K}), (\widehat{\omega}', \widehat{K}')\big)$, keeping in mind bounds on the $C^{\ell+\mathrm{Lip}}$ norms of g, K, K', and assumptions (46) and (47) for r. We consider the first term for an example, the rest is similar.

$$\begin{aligned}
&\|[Dg\big(K(\theta - \varepsilon s\omega r)\big) - Dg\big(K'(\theta - \varepsilon s\omega r)\big)]DK(\theta - \varepsilon s\omega r)\omega r\|_{C^0} \\
&\qquad\qquad \leqslant \omega\,\mathrm{Lip}(Dg)\|DK\|\|r\|_{C^0}\|\widehat{K} - \widehat{K}'\|_{C^0}.
\end{aligned} \tag{52}$$

Observe the form of the operator Γ^ε in (14). Note that if we have a bound of

$$\int_{\theta_0}^{\theta} \Phi(\theta; s)\big(\mathscr{P}(K, \omega, \gamma, s) - \mathscr{P}(K', \omega', \gamma, s)\big)ds,$$

by a multiple of $d\big((\widehat{\omega}, \widehat{K}), (\widehat{\omega}', \widehat{K}')\big)$, we prove Lemma 5.5.

All terms except the 4th one in (51) are controlled using estimates similar to (52). Hence, to complete the proof, we just need to estimate the part coming from the 4th term in (51). We will take advantage of the integral which is an operator that improves the bounds.

We use integration by parts to get:

$$\begin{aligned}
&\int_{\theta_0}^{\theta} \Phi(\theta; s) \int_0^1 Dg\big(K'(s - \varepsilon\tau\omega' r')\big)[DK - DK'](s - \varepsilon\tau\omega r)\omega r d\tau ds \\
&= \int_0^1 \int_{\theta_0}^{\theta} \Phi(\theta; s) Dg(\cdot) \frac{\omega r}{1 - \varepsilon\tau\omega \frac{dr}{ds}} (1 - \varepsilon\tau\omega \frac{dr}{ds})[DK - DK'](s - \varepsilon\tau\omega r) ds d\tau \\
&= \int_0^1 \Big[\Phi(\theta; s) Dg(\cdot) \frac{\omega r}{1 - \varepsilon\tau\omega \frac{dr}{ds}} [K - K'](s - \varepsilon\tau\omega r)\Big|_{s=\theta_0}^{s=\theta} \\
&\qquad - \int_{\theta_0}^{\theta} \frac{d}{ds}\Big(\Phi(\theta; s) Dg(\cdot) \frac{\omega r}{1 - \varepsilon\tau\omega \frac{dr}{ds}}\Big)[K - K'](s - \varepsilon\tau\omega r) ds\Big] d\tau.
\end{aligned}$$

The C^0 norm of the above expression is bounded by a multiple of $\|K - K'\|_{C^0} \leqslant d\big((\widehat{\omega}, \widehat{K}), (\widehat{\omega}', \widehat{K}')\big)$, we have proved Lemma 5.5 in this case. The proof of Theorem 8.2 is finished.

Remark 8.4. Note that we need to differentiate r along the periodic orbit twice in the above expression, that is why we required $\ell \geqslant 3$ in Theorem 8.2, so that $r(\omega, K, \varepsilon)$ is more than C^2.

Remark 8.5. We thank an anonymous referee for pointing out an alternative approach to prove Theorem 8.2. In contrast to the distance d in (22), one could define a different distance

$$d_1\left((\widehat{\omega}, \widehat{K}), (\widehat{\omega}', \widehat{K}')\right) := \max\left\{|\widehat{\omega} - \widehat{\omega}'|, \|\widehat{K} - \widehat{K}'\|_{C^1}\right\}, \tag{53}$$

and take advantage of the integral format of the operator Γ^ε in (14). Notice that

$$\begin{aligned}
d\left(\Gamma^\varepsilon(\widehat{\omega}, \widehat{K}), \Gamma^\varepsilon(\widehat{\omega}', \widehat{K}')\right) &\leqslant \mu_0(\varepsilon, a, \beta_0) \cdot d_1\left((\widehat{\omega}, \widehat{K}), (\widehat{\omega}', \widehat{K}')\right) \\
\|D_\theta \Gamma_2^\varepsilon(\widehat{\omega}, \widehat{K}) - D_\theta \Gamma_2^\varepsilon(\widehat{\omega}', \widehat{K}')\|_{C^0} &\leqslant \mu_1(\varepsilon, a, \beta_0, \beta_1) \cdot d_1\left((\widehat{\omega}, \widehat{K}), (\widehat{\omega}', \widehat{K}')\right),
\end{aligned} \tag{54}$$

where constants $\mu_0(\varepsilon, a, \beta_0)$ and $\mu_1(\varepsilon, a, \beta_0, \beta_1)$ are less than 1 for small ε, a, β_0, and β_1 (see Remark 5.3 for smallness of a and β's). Then inequalities in (54) imply that Γ^ε is a contraction with respect to this new distance d_1. With this approach, we only need $\ell \in \mathbb{N}$ rather than $\ell \geqslant 3$.

9. Delays implicitly defined by the solution. Applications to electrodynamics

In this section, we show how to deal with delays that depend implicitly on the solution. The main motivation comes from electrodynamics, so we deal with this case in detail, but we formulate a more general mathematical result in Section 9.3.

We point out that implicitly defined delays appear naturally in other problems in which the delay of the effect is related to the whole trajectory. As we indicate later, the explicit state dependent delays appeared in (28) are often approximations of implicitly defined delays. One corollary of our treatment is a justification of the fact that the periodic solutions of these approximations approximate the true periodic solutions.

9.1. Motivation from electrodynamics

One of the original motivations for the whole field of delay equations was the study of forces in electrodynamics. The forces among charged particles, depend on the positions of the particles. Since the signals from a particle take time to reach another particle, this leads to a delay equation. Notice that the delay depends on the position (at a previous time) so that the delay is obtained by an implicit equation on the trajectory. This formulation was proposed very explicitly in [89], which we will follow.

Remark 9.1. An alternative description of electrodynamics uses the concept of fields. One problem of the concept of fields is to explain why particles do not interact with their own fields. We refer to [79] for a very lucid physical discussion of the paradoxes faced by a coherent formulation of classical electrodynamics.

Remark 9.2. Many Physicists object to [89] that it does not make clear what is the phase space and what are the initial conditions.

In this paper, we show that one does not need to answer these question to construct a theory of periodic solutions. We hope that similar results hold for other types of solutions. So that one can have a systematic theory of many solutions that resemble the classical ones.

Of course, it should also be possible to construct other solutions that are completely different from those of the systems without delays.

Remark 9.3. Even if one can have a rich theory of perturbative solutions, it is not clear that these solutions fit together in a smooth manifold. The paper [8] develops asymptotic expansions, which suggests that the resulting solutions may be difficult to fit together in a manifold.

We speculate that this may give a way to reconcile the successes of *predictive mechanics* [4] with the no-interaction theorems [15]. It could well happen that the results of predictive mechanics apply to the abundant solutions we construct, but, according to the no-interaction theorem, this set cannot be all the initial conditions. Of course, these speculations are far from being theorems.

9.2. Mathematical formulation

If we consider (time-dependent) external electric and magnetic fields as prescribed, the equations of a system of N particles in $\mathbb{R}^3$ are, denoting by $q_i(t)$ the position of the i-th particle,

$$\ddot{q}_i(t) = A_{i,\text{ext}}(t, q_i(t), \dot{q}_i(t)) + \sum_{j\neq i} A_{ij}\left(q_i(t), \dot{q}_i(t), q_j(t-\tau_{ij}), \dot{q}_j(t-\tau_{ij})\right), \tag{55}$$

where the time delay is defined implicitly by

$$\tau_{ij}(t) = \frac{1}{c}\left|q_i(t) - q_j(t-\tau_{ij}(t))\right|, \tag{56}$$

with c being the speed of light. For more explicit expressions, we refer to [89,74,23]. We just remark that (55) is the usual equation of acceleration equals force divided by the mass. The relativistic mass has some complicated expression depending on the velocity.

The term $A_{i,\text{ext}}$ denotes the external force. The terms A_{ij} correspond to the Coulomb and Lorenz forces of the fields obtained from Liénard–Wiechert potentials. This is a standard calculation which is classical in electrodynamics, see [52,42,92]. Roughly, they are the Coulomb and Ampere (electric and magnetic) forces at previous times but some derivative terms appear.

We observe that (55) is in the form imposed by the principle of relativity, and that any force which is relativistically invariant should have the form (55) with, of course, different expressions for the terms A_{ij}. Hence, the treatment discussed here should apply not only to electrodynamics but also to any forces subject to the rules of special relativity.

The exact form of the equations does not play an important role in this paper. We point out some properties that play a role:

(1) The expressions defining the forces are algebraic expressions. They have singularities when there are collisions ($q_i(t) = q_j(t)$ for some $i \neq j$) or when some particle reaches the speed of light ($|\dot{q}_i(t)| = c$ for some i).
(2) The delays τ_{ij} as in (56) are subtle. The expressions of τ_{ij} involve a small parameter $\varepsilon := 1/c$, and the delays can be approximated in first order as:

$$\tau_{ij}(t) = \varepsilon\left|q_i(t) - q_j(t)\right| + O(\varepsilon^2). \tag{57}$$

Keeping only the first order approximation in (57) makes (55) an SDDE, but with (56), the delay depends implicitly on the trajectory.
Note that it is not true that $\tau_{ij} = \tau_{ji}$ even if this symmetry is true in the first order approximation (57).
(3) In the case that $\tau_{ij} = 0$ and that the external forces are autonomous, the total energy is conserved. This has two consequences:
   - In the autonomous case, the periodic orbits do not satisfy the hypothesis (H1). Hence, we will only make precise statements in the case of time periodic external fields. In this case (very well studied in accelerator physics, plasma, etc.), there are many examples of periodic orbits satisfying assumption (H1”), so that the results presented here are not vacuous.

- If the external potential and external electric and magnetic fields are bounded, the periodic orbits of finite energy and away from collisions satisfy $|\dot{q}_i(t)| \leqslant \xi_1 c$ (where $\xi_1 \in (0, 1)$) and $|q_i(t) - q_j(t)| \geqslant \xi_2 > 0$, $i \neq j$. We will assume these two properties.

Denoting $y(t) := (q_1(t), \ldots, q_N(t), \dot{q}_1(t), \ldots \dot{q}_N(t))$, we can write the equation (55) in the form of (45) with the delays being implicitly defined. Note that there are $N(N-1)$ delays in total.

Remark 9.4. Even if we formulate the result for the retarded potentials, we point out that the mathematical treatment of Maxwell equations admits also advanced potentials.

It is customary to take only the retarded potentials because of "physical reasons" which are relegated to footnotes in most classical electrodynamics books. More detailed discussions appear in [74,79]. Note, that selecting only retarded potentials breaks, even at the classical level, the time-reversibility present in Maxwell's and Newton's equations. Mathematically any combination of advanced and retarded potentials would make sense from Maxwell equations. Indeed, [88] proposes a theory with half advanced and half retarded potentials.

We do not want to enter now into the physical arguments, which should be decided by experiment (we are not aware of explicit experimentation of these points). We just point out that the mathematical theory here and the asymptotic expansions [8] applies to retarded, advanced, or combination of advanced and retarded potentials.

9.3. Mathematical results for electrodynamics

In this section, we will collect the ideas we have been establishing and formulate our main result for the model (55). Note that we formulate the result only for periodic external fields, since when the external fields are time-independent, energy is conserved which prevents periodic orbits from satisfying assumption (H1).

We will assume that there exists $0 < \xi_1 < 1$, and $\xi_2 > 0$, such that for all t:

$$\begin{aligned} &|\dot{q}_j(t)| \leqslant \xi_1 c \\ &|q_i(t) - q_j(t)| \geqslant \xi_2. \end{aligned} \tag{58}$$

Note that (58) implies that the internal forces and the masses are analytic around the trajectory. Therefore, the regularity assumptions for the equation concern only the external fields.

Theorem 9.5. *Consider the model (55) with the delays defined in (56). Denote $1/c$ as ε, and treat it as a parameter.*

Assume that for $\varepsilon = 0$, the resulting time periodic ODE has a periodic solution satisfying hypothesis (H1") as well as (58). Assume that the external fields $\mathrm{A}_{i,\mathrm{ext}}$ are $C^{\ell+\mathrm{Lip}}$.

Then, for small enough ε, we can find a $C^{\ell+\mathrm{Lip}}$ periodic solution of (55).

Similarly, in the case that the external fields are jointly $C^{\ell+\mathrm{Lip}}$ in time, position, velocity, and in a parameter γ, the periodic solutions are jointly $C^{\ell+\mathrm{Lip}}$ as functions of the variable of the parameterization and the parameter γ.

The proof follows from Theorem 8.2 once we have the estimates on delays (46) and (47), which will be discussed in the next section.

Remark 9.6. Since the fully relativistic equations are cumbersome to handle, there are many approximations in the literature. [68,85,84,65,11] approximate the relativistic equations up to $O(c^{-m})$. Since the approximate models provide solutions that satisfy the invariance equation up to $O(c^{-m})$, using the a-posteriori format of Theorem 9.5, we conclude that the periodic orbits for the $O(c^{-m})$ approximate models are $O(c^{-m})$ close to the periodic orbits of the fully relativistic model when c is large enough.

Of course, the theory here is only for non-degenerate periodic solutions, and does not allow to conclude that other aperiodic solutions in the approximate model correspond to true solutions in the full electrodynamics.

9.4. Some preliminary results on the regularity of the delay

In this section, we study (56) as an equation for $\tau_{ij}(t)$ when we prescribe the trajectories q_i and q_j. This makes precise the notion that the delay is a functional of the whole trajectory.

In the following proposition, we collect the proofs of estimates that establish (46) and (47). Both follow rather straightforwardly from considering (56) as a contraction mapping.

Proposition 9.7. *Let q_i and q_j be continuously differentiable trajectories that satisfy* (58).

Then, for each $t \in \mathbb{R}$, we can find a unique $\tau_{ij}(t) > 0$ solving (56).

Moreover:

If the trajectories q_i and q_j are $C^{\ell+\mathrm{Lip}}$, then the τ_{ij} is $C^{\ell+\mathrm{Lip}}$, and there is an explicit algebraic expression ϕ such that

$$\|\tau_{ij}\|_{C^{\ell+\mathrm{Lip}}} \leqslant \phi(\|q_i\|_{C^{\ell+\mathrm{Lip}}}, \|q_j\|_{C^{\ell+\mathrm{Lip}}}, \xi_1, \xi_2). \tag{59}$$

Let $q_i, q_j, \bar{q}_i$, and $\bar{q}_j$ be trajectories satisfying (58)*. Denote by τ_{ij} and $\bar{\tau}_{ij}$ the solutions of* (56) *corresponding to q_i, q_j and to $\bar{q}_i, \bar{q}_j$, respectively. Then we can find a constant $C(\xi_1, \xi_2)$ (depend on ξ_1 and ξ_2), such that*

$$\|\tau_{ij} - \bar{\tau}_{ij}\|_{C^0} \leqslant C(\xi_1, \xi_2)\left(\|q_i - \bar{q}_i\|_{C^0} + \|q_j - \bar{q}_j\|_{C^0}\right). \tag{60}$$

Proof. Fix t and, hence, $q_i(t)$ and $q_j(t)$.

We treat (56) as a fixed point problem for the – long named – unknown $\tau_{ij}(t)$ with the functions q_i, q_j as well as the number t fixed.

The first part of the assumption (58) implies that the right-hand side of (56), as a function of $\tau_{ij}(t)$ has derivative with modulus bounded by $\xi_1 < 1$. Hence, we can apply the contraction mapping principle. This establishes existence and uniqueness.

Moreover, we can apply the implicit function theorem and obtain that $\tau_{ij}(t)$ is as differentiable with respect to t as the right-hand side of (56). Furthermore, we can get expressions for $\frac{d^k}{dt^k}\tau_{ij}(t)$ which are algebraic expressions involving derivatives with respect to t of $q_i(t)$, $q_j(t)$ up to order k, and derivatives of $\tau_{ij}(t)$ up to order $k-1$. The exact combinatorial formulas are very well known. By induction on the order of the derivative, we obtain (59).

To prove (60), we observe that since the contraction we used before is uniform in t, we can consider the right-hand side of (56) as a contraction in the C^0 topology.

We evaluate the right-hand side of (56) corresponding to $\bar{q}_i$ and $\bar{q}_j$ on τ_{ij}, note that

$$\bar{q}_i(\cdot) - \bar{q}_j(\cdot - \tau_{ij}(\cdot)) = (\bar{q}_i(\cdot) - q_i(\cdot)) + \left(q_j(\cdot - \tau_{ij}(\cdot)) - \bar{q}_j(\cdot - \tau_{ij}(\cdot))\right)$$

$$+\left(q_i(\cdot)-q_j(\cdot-\tau_{ij}(\cdot))\right).$$

Hence,

$$\left\|\frac{1}{c}\left|\bar{q}_i(\cdot)-\bar{q}_j(\cdot-\tau_{ij}(\cdot))\right|-\tau_{ij}(\cdot)\right\|_{C^0}\leqslant\frac{1}{c}\|q_i-\bar{q}_i\|_{C^0}+\frac{1}{c}\|q_j-\bar{q}_j\|_{C^0}.$$

From this, (60) follows from the Banach contraction mapping. □

Remark 9.8. Notice that the delays τ_{ij}'s contain the small factor $\frac{1}{c}$, so do the right-hand sides of the inequalities (59) and (60), as we can see in the proof above. We can view $\tau_{ij}:=\frac{1}{c}r_{ij}$ to fit in the case of small delays in Section 8.

10. The case of hyperbolic periodic orbits

Our main result Theorems 4.6 and 4.7 are based on the assumption (H1), which is automatically satisfied when the periodic orbit of the unperturbed equation is hyperbolic. Hence, the main results of this section can be viewed as corollaries of Theorems 4.6 and 4.7. In fact, we need slightly stronger regularity assumptions in this section.

In this section, we will introduce an operator, see (65), which is slightly different from the one introduced in Section 5.2.

Even if the operator considered in this section requires more regularity in the finite dimensional case, it generalizes our results to perturbations of PDEs, see Section 11, to perturbations of DDEs, and to other solutions that we will not discuss here (quasi-periodic, normally hyperbolic manifolds). We also note that the corrections needed in this section can be independent of the period. This makes it possible to develop a theory of aperiodic hyperbolic sets. We hope to come back to this problem.

10.1. Dynamical definition of hyperbolic periodic orbits

It is a standard notion that a periodic orbit of the ODE $\dot{x}=f(x)$ is hyperbolic when the following strengthening of (H1) holds.

With the same notation as in Section 3 and Section 4.1, we say that a periodic orbit is hyperbolic if:

(H1.1) $\Phi(\theta_0+1;\theta_0)$ has a simple eigenvalue 1 whose eigenspace is generated by $DK_0(\theta_0)$. Moreover, all the other eigenvalues of $\Phi(\theta_0+1;\theta_0)$ have modulus different from 1.

The assumption (H1.1) is equivalent to the following evolutionary formulation (H1.1') in terms of invariant decompositions. In the finite dimensional case, this formulation is easily obtained by taking the stable and unstable spaces of the monodromy matrix and propagating them by the variational equations. In the infinite dimensional cases, similar formulations are obtained using semi-group theory under appropriate spectral assumptions.

(H1.1') For every $\theta\in\mathbb{T}$, there is a decomposition

$$\mathbb{R}^n=E_\theta^s\oplus E_\theta^u\oplus E_\theta^c,\quad E_\theta^c=\mathrm{Span}\{DK_0(\theta)\},\tag{61}$$

depending continuously on θ such that E^s_θ is forward invariant, E^u_θ is backward invariant under the variational equation. Moreover, the forward semiflow (resp. backward semiflow) of the variational equation is contractive on E^s_θ (resp. E^u_θ).

More explicitly, we can find families of linear operators

$$\begin{aligned} &\{U^s_\theta(t)\}_{\theta\in\mathbb{T},t\in\mathbb{R}_+}, \quad U^s_\theta(t)\colon E^s_\theta \to E^s_{\theta+\omega_0 t} \quad t\in\mathbb{R}_+,\\ &\{U^u_\theta(t)\}_{\theta\in\mathbb{T},t\in\mathbb{R}_-}, \quad U^u_\theta(t)\colon E^u_\theta \to E^u_{\theta+\omega_0 t} \quad t\in\mathbb{R}_-,\end{aligned}$$

satisfying for all $\theta\in\mathbb{T}$

$$\begin{aligned} &\partial_t U^\sigma_\theta(t) = Df(K_0(\theta+\omega_0 t))U^\sigma_\theta(t) \quad \sigma\in\{s,u\}\\ &U^\sigma_\theta(0) = \mathrm{Id}\big|_{E^\sigma_\theta}, \end{aligned} \tag{62}$$

and

$$U^\sigma_\theta(t+\tau) = U^\sigma_{\theta+\omega_0 t}(\tau)\circ U^\sigma_\theta(t). \tag{63}$$

Moreover, there exist $C>0$, $\mu_s>0$, $\mu_u>0$ such that

$$\begin{aligned} &\|U^s_\theta(t)\| \leqslant Ce^{-\mu_s t} \qquad t\geqslant 0,\\ &\|U^u_\theta(t)\| \leqslant Ce^{-\mu_u |t|} \qquad t\leqslant 0. \end{aligned} \tag{64}$$

We can also define an evolution operator $U^c_\theta(t)$ in the E^c direction. Note that $U^c_\theta(\frac{1}{\omega_0}) = \mathrm{Id}|_{E^c_\theta}$.

10.2. Main result in hyperbolic case

The first result in this case is that Theorem 4.6 is true if assumption (H1) is changed to assumption (H1.1) or (H1.1’), and assumption (H2.1) is strengthened to (H2.1.1) as follows:

(H2.1.1) There is constant $\phi_{\rho,\delta}>0$, such that for all $K\in U_\rho$ and $\omega\in B_\delta$, $\mathscr{P}(K,\omega,\gamma,\cdot)\colon\mathbb{T}\to\mathbb{R}^n$ is $C^{\ell+\mathrm{Lip}}$, with

$$\|\mathscr{P}(K,\omega,\gamma,\cdot)\|_{C^{\ell+\mathrm{Lip}}} \leqslant \phi_{\rho,\delta}.$$

Recall that U_ρ is the ball of radius ρ in the space $C^{\ell+\mathrm{Lip}}(\mathbb{T},\mathbb{R}^n)$ centered at K_0, and B_δ is the interval in $\mathbb{R}$ with radius δ centered at ω_0.

The second result is that the results in Theorem 4.7 are true if assumption (H1) is substituted by assumption (H1.1) or (H1.1’), and assumption (H2.2) is strengthened to (H2.2.1) as follows:

(H2.2.1) There is constant $\phi_{\rho,\delta}>0$, such that for all $K\in\mathcal{U}_\rho$ and $\omega\in\mathcal{B}_\delta$, $\mathscr{P}(K,\omega,\cdot,\cdot)\colon\mathbb{T}\times O\to\mathbb{R}^n$ is $C^{\ell+\mathrm{Lip}}$, with

$$\|\mathscr{P}(K,\omega,\cdot,\cdot)\|_{C^{\ell+\mathrm{Lip}}} \leqslant \phi_{\rho,\delta}.$$

Recall that $\mathcal{U}_\rho$ is the ball of radius ρ in the space $C^{\ell+\mathrm{Lip}}(\mathbb{T}\times O,\mathbb{R}^n)$ centered at K_0, and $\mathcal{B}_\delta$ is the ball in $C^{\ell+\mathrm{Lip}}(O,\mathbb{R})$ with radius δ centered at the constant function ω_0.

Remark 10.1. We emphasize that the results in this section are weaker than Theorem 4.6 and Theorem 4.7, however, we want to introduce a different operator in the proof which has applications in ill-posed PDEs, see Section 11. Modification of the operator will be useful in the study of other dynamical objects.

10.3. Proof

We proceed as in Section 5.2 and manipulate (8) as a fixed point problem taking advantage of the geometric structures assumed in (H1.1').

Given the decomposition in (61), we define projections $\Pi_\theta^s,\Pi_\theta^u,\Pi_\theta^c$ over the spaces $E_\theta^s, E_\theta^u, E_\theta^c$. We also use the notation

$$\widehat{K}^\sigma(\theta) := \Pi_\theta^\sigma \widehat{K}(\theta), \qquad \sigma\in\{s,u\}.$$

Taking projections along the spaces of the decomposition, using the variation of parameters formula, and taking the initial conditions to infinity (this procedure is standard since [71]), we see that (8) implies

$$\begin{aligned}
\widehat{\omega} &= \omega_0 \frac{\langle \Pi_{\theta_0}^c \int_0^{\frac{1}{\omega_0}} U_{\theta_0+\omega_0 t}^c(\frac{1}{\omega_0}-t)B^\varepsilon(\widehat{K},\widehat{\omega},\gamma,\theta_0+\omega_0 t)dt, DK_0(\theta_0)\rangle}{|DK_0(\theta_0)|^2},\\
\widehat{K}^s(\theta) &= \int_{-\infty}^{0} U_{\theta+\omega_0 t}^s(-t)\Pi_{\theta+\omega_0 t}^s B^\varepsilon(\widehat{K},\widehat{\omega},\gamma,\theta+\omega_0 t)\,dt,\\
\widehat{K}^u(\theta) &= -\int_{0}^{\infty} U_{\theta+\omega_0 t}^u(-t)\Pi_{\theta+\omega_0 t}^u B^\varepsilon(\widehat{K},\widehat{\omega},\gamma,\theta+\omega_0 t)\,dt.
\end{aligned} \tag{65}$$

Define the right-hand side of (65) as an operator of $(\widehat{\omega},\widehat{K}^s,\widehat{K}^u)$, then one can get lemmas which are similar to Lemmas 5.2 and 5.5. Hence we can get a fixed point of the operator in this case.

When the solutions of (65) are smooth enough, and the integrands in the right-hand side of (65) decay fast enough that we can take derivatives inside of the integral sign (which will be the case of the fixed points that we produce), it is possible to show, taking derivatives of both sides of (65) and reversing the algebra that the well behaved fixed point of (65) indeed solves (8).

The remarkable aspect of (65) is that we only need U_θ^s for positive times, and U_θ^u for negative times. Hence, the assumed bounds (64) imply that the indefinite integrals in (65) converge uniformly in the $C^{\ell+\mathrm{Lip}}$ sense. At the same time, we pay the price of requiring one more derivative of $\mathscr{P}(K)$ with respect to θ while using this operator.

Another important feature of the operator (65) is that it does not require many assumptions on the evolution of the solutions beyond boundedness (in Section 5.2 we use heavily that the solutions we seek are periodic). This makes it possible to use analogues of (65) in several other problems. We hope to come back to these questions in the near future.

11. Evolutionary equations with delays

In this section we extend the results on ODEs in the previous sections to PDEs and other evolutionary equations (e.g. equations involving fractional operators or integral operators).

The key observation is that, the previous treatments of periodic solutions do not use much that the functions we are seeking take values in a finite dimensional space. For example, Lemma A.8 is valid for functions taking values in Banach spaces. Hence, we will show that the methods developed in the previous sections can be applied without much change to a wide class of PDEs.

Indeed, since one of the points of the previous theory was to avoid the discussion of the evolutions, the theory applies easily to PDEs using only very simple results on the evolutions of PDEs.

Remark 11.1. In this paper, we will not discuss the existence of periodic solutions of evolutionary equations before adding the delays. There is already a large literature in this area.

We point, however that in studying the periodic solutions of a PDE (which lie in an infinite dimensional space), it is natural to consider the periodic solutions of a finite dimensional truncation (e.g. a Galerkin approximation). The problem of going from the periodic solutions of a finite dimensional problem to the periodic solutions in an infinite dimensional space, has some similarity with the problems dealt with in the first parts of this paper.

A framework that systematizes the passing from periodic solutions of the Galerkin approximations to periodic solutions of the PDEs is in [26]. The methods of [26] have some points in common with the methods used in this paper. It bypasses the study of evolutionary equations and just studies the functional equations satisfied by a parametrization of a periodic orbit. The methods in [26] lead to computer-assisted proofs that have been implemented in [27,25]. Since the methods of [26] and this paper have points in common, one can hope to combine them and go from a periodic solution of a Galerkin truncation of the PDE to a periodic solution of the delay perturbation of the PDE.

11.1. Formulation of the problem and preliminary results

We use the standard set up of evolutionary equations (see [56,77]).

Consider problem of the form

$$\partial_t u(t) = \mathscr{F}(u(t)) + \varepsilon P(u(t), u_t; \gamma), \tag{66}$$

where $u(t)$, is the unknown and lies in a space X consisting of functions on a domain Ω. The points in Ω will be given the coordinate x, so that we can also consider $u(t, x)$ as a function on $\mathbb{R} \times \Omega$.

The function space X encodes regularity properties of the functions as well as boundary conditions. In particular, changing the boundary conditions, changes the space X and therefore, the functional analysis properties (e.g. spectra) of the operators acting on it.

The operator $\mathscr{F}$ is a (possibly nonlinear) differential (or fractional differential) operator.

As before (and contrary to the standard use in PDEs where u_t denotes partial derivative), we use u_t to denote a segment of the solution, which can be related with history or future. For $s \in [-h, h]$, $u_t(s) = u(t+s)$, so that $u_t \in \mathscr{R}([-h, h], X)$, a space of regular functions on $[-h, h]$ with values in X. To denote derivatives with respect to time we will always use $\partial_t u$.

We consider $P\colon X \times \mathscr{R}([-h, h], X) \times \mathbb{R}^m \to X$.

It is useful to think heuristically of

$$\partial_t u(t) = \mathscr{F}(u(t)) \tag{67}$$

as a differential equation in X and indeed, our results will be based on this heuristic principle. To make sense of this heuristic principle we have to overcome the problem that in the interesting applications (see e.g. Section 11.3), $\mathscr{F}$ is highly discontinuous (involving derivatives) and not defined everywhere so that the standard tools for smooth ODEs do not apply, but this is a well studied problem.

A research program which became specially prominent in the 60's shows that one can recover many of the results (existence, dependence on initial conditions, etc.) for equation (67) by assuming functional analysis properties of the operator $\mathscr{F}$, see [56,5,76,77,40,75,14]. Of course, the verification of the functional analysis assumptions in concrete examples, requires some hard analysis. One of the subtle points of this program is that the notion of solutions may be redefined to be weak or mild solutions.

Even if we will use the language and some material from the above program, we will take a different point of view.

- In this paper, we will not be interested in the theory of existence and well-posedness for **ALL** the possible initial conditions.
- Indeed, because we are not going to discuss the initial value problems, we can consider situations where the set of initial conditions for the delay problems are not clear. Nevertheless, we can get existence of smooth solutions.
- Since we are only aiming to produce some particular solutions, one gets stronger results by taking more reduced spaces so that the solutions are more regular and can be understood in the classical sense. In particular, in all the cases we will consider, the functions and their derivatives will be bounded. (This happens, e.g. if X is a Sobolev space of high enough order.)
  This is in contrast with the general theory of existence and uniqueness, where the figure of merit is considering a more general space of initial conditions.
- A more elaborate set-up for existence of evolutions that includes also FDEs is in [90]. In this paper, however, we will avoid discussing the evolution of the FDEs and need only some results on the evolution of the PDE.

11.2. Overview of the method

Roughly, we will formulate analogues of the operator Γ^ε in (15) and (16) as well as the operator in (65) and verify that similar contraction argument can be carried out.

The requirements of the above program on the theory of existence are very mild. The operator Γ^ε only requires the existence of solutions of the variational equation for finite time. The operators formulated in (65) only require the existence of partial evolutions (forward and backward evolutions in complementary spaces), which allows to consider ill-posed equations, see Section 11.5. Moreover, the smoothness requirements on the delay terms are very mild.

11.3. Examples

In this section, we will present some examples which are representative of the results we establish and which have appeared in applications.

Even if we hope that this section can serve as motivation, from the purely logical point of view, it can be skipped. Of course, our results apply to many more models and this section is not meant to be an exhaustive list but to provide some intuition.

11.3.1. Delay perturbations

One example of delay perturbation which considers long range interaction is

$$P(u(t), u_t; \gamma) = \int_{\mathbb{R}^d} W(x, y) \cdot u(t, x) \cdot u(t - \frac{1}{c}|x - y|, y) dy. \tag{68}$$

This models a situation in which the position x interacts with position y with a strength $W(x, y)$, with the interaction taking some time (proportional to the distance) to propagate. In (68) we have denoted by c the speed of propagation of the signal, which is assumed to be constant.

Note that the interaction term could be more general than quadratic, and may involve higher spacial derivatives thanks to the smoothing property of solutions. Meanwhile, the speed of propagation of the signal may not be constant (the propagation of signals may depend on their strength).

Another example

$$P(u(t), u_t; \gamma) = \int_0^\infty G(s, u(t - s, x))\, ds, \tag{69}$$

treating non-local interaction, is very typical in the modeling of materials with memory effects (for example *thixotropic* materials) where the properties of the materials depend on the history. The effect of the previous state at present time often decreases when the time delay grows. This is reflected that the function $G(s, u)$ decreases when s (the delay in the effect) increases.

Of course, the mathematical theory that will be developed accommodates more complicated effects such as G depending on spatial derivatives of u.

There are many other $P(u(t), u_t; \gamma)$ that we can consider. We only need P to satisfy some assumptions on regularity and Lipschitz property, see (H2.1*), (H3.1*), and (H2.1.1*), where we actually allow loss of regularity in the space variable.

In the coming sections, we see examples of unperturbed equations (67).

11.3.2. Parabolic equations

Consider the equation for $u: \mathbb{R} \times \mathbb{R}^d \to \mathbb{R}$:

$$\begin{aligned} \partial_t u &= \Delta u + N(x, u, \nabla u), \\ u(t, x) &= u(t, x + e) \quad \forall e \in \mathbb{Z}^d, \end{aligned} \tag{70}$$

with N vanishing to quadratic order. For simplicity, we have imposed periodic boundary conditions in space.

Notice that we have not imposed initial conditions at $t = 0$ in example (70). Indeed, the initial conditions needed require some thought.

As we will see, our treatment overcomes other possible complications not mentioned explicitly so far. We mention them because they are natural in modeling and eliminating them from the literature may be motivated by the need to have a more mathematically treatable problem.

Let us just mention briefly some small modifications.

- The unknown u could take values in $\mathbb{R}^d$. Note that considering systems rather than scalar equations makes a big difference in some PDE treatments (based on maximum principle), but it is not an issue in our case.
- The papers [48,49] consider damped wave equations with a delay. From the functional analysis point of view, the damped wave equations are similar to (70).

11.3.3. Kuramoto-Sivashinsky equations

The model below is called the Kuramoto-Sivashinsky equation

$$\begin{aligned} \partial_t u &= \Delta u + \Delta^2 u + \mu, \partial_x(u^2) \\ u(t,x) &= u(t, x+e) \quad \forall e \in \mathbb{Z}^d. \end{aligned} \tag{71}$$

The Kuramoto-Sivashinsky equations appear as *amplitude equations* for many problems arising in a variety of applications (water waves, chemical reactions, interactive populations, etc.).

From the mathematical point of view, when $d = 1$ (reduction of models with more variables), the equation is known to have an inertial manifold (all the solutions converge to a finite dimensional manifold), which can be analyzed by finite dimensional methods. The equation (71) is known to have many periodic solutions. A very large number was identified by non-rigorous, but reliable methods in [51]. Rigorous periodic solutions have been established in many papers, including bifurcations in [3,93]. From the point of view of this paper, it is interesting to note that [25,27] use computer-assisted proofs to establish the existence of periodic orbits.

The equations discussed in the above two sections are parabolic PDEs so that indeed, the evolution is well defined and the solutions gain smoothness. The linearized operator Φ that enters in (15) and (16) is also smoothing. Of course, for large solutions, there could be finite time blow ups, but we are in the regime of periodic solutions, which are well-behaved.

11.3.4. The Boussinesq equations in long wave approximation for water waves

In this section we present some physical equations that are ill-posed in the sense that it is impossible to define an evolution for every initial condition. On the other hand, these equations may possess many interesting and physically relevant solutions.

Since one of the main ideas of our treatment of FDEs is to bypass the evolution, we obtain results on delay perturbations of ill-posed equations. This indeed highlights the difference of the present method with the methods in evolutionary equations.

The material of this section is somewhat more sophisticated than the rest of the paper and does not affect any of the other results.

Consider the equation for $u\colon \mathbb{R} \times \mathbb{R} \to \mathbb{R}$, derived in [6], as a long wave approximation for water waves.

$$\partial_t^2 u = \mu \partial_x^4 u + \partial_x^2 u + \partial_x(u^2), \quad u(t, x+1) = u(t,x). \tag{72}$$

This equation (72) can be written as an evolutionary equation of the form (67) as follows:

$$\begin{aligned} &\partial_t u = v, \\ &\partial_t v = \mu \partial_x^4 u + \partial_x^2 u + \partial_x(u^2), \\ &u(t, x+1) = u(t, x); \quad v(t, x+1) = v(t, x). \end{aligned} \tag{73}$$

The linear part of the evolution is

$$\begin{aligned} &\partial_t u = v, \\ &\partial_t v = \mu \partial_x^4 u + \partial_x^2 u. \end{aligned} \tag{74}$$

Equations similar to (72) have also appeared in other contexts. In water wave theory, $\mu > 0$, which leads to (72) being ill-posed. Indeed, consider the linear part of the equation, the coefficient of the k-th Fourier mode $\hat{u}_k$ satisfies $\frac{d^2}{dt^2}\hat{u}_k = (\mu k^4 - k^2)\hat{u}_k$ (up to some constants related to 2π), which leads to exponentially growing solutions either in the future or in the past.

Nevertheless, it is well known that the Boussinesq equation contains many physically interesting solutions, including traveling waves and other periodic and quasi-periodic solutions that are not traveling waves. Notably, it contains a finite dimensional manifold (local center manifold) which is locally invariant and on which solutions can be defined till they leave the local center manifold [16,18,9]. In particular, the periodic and quasi-periodic solutions in the local center manifold are defined for all times.

For our purposes, the Boussinesq equation (72) is Hamiltonian, so that all the periodic solutions have monodromy with eigenvalues 1 – corresponding to the conservation of the energy – which make them unsuitable for the present version of our theory. Hence, we will consider, for $u: \mathbb{R} \times \mathbb{R}^d \to \mathbb{R}$, mainly time periodic perturbations of higher dimensional version of (72), which following the notation in [9], we write as:

$$\begin{cases} \partial_t \theta = \omega, \\ \partial_t^2 u = \mu \Delta^2 u + \Delta u + N_1(\theta, x) + N_2(\theta, x)u + N_3(\theta, x, u, \nabla u, \Delta u), \\ t \in \mathbb{R}, \quad \theta \in \mathbb{T}, \quad x \in \mathbb{T}^d. \end{cases} \tag{75}$$

The model (75) can be a long wave approximation of a water wave model perturbed periodically. These are physically sensible long wave approximations of a water wave subject to periodic forcing (e.g. waves in the ocean subject to tides or water waves in a vibrating table – Faraday experiment).

The result of [9] implies, under very mild regularity assumptions on N_1, N_2, N_3, that there is a finite dimensional local center manifold of (75) which is locally invariant.

This local center manifold is modeled on $\mathbb{T} \times \mathbb{R}^n$. The periodic solutions in the manifold are defined for all time. For specific forms of $N_{1,2,3}$, it is possible to prove the existence of periodic orbits of (75), which are non-degenerate in the center manifold.

A natural space to consider (73) is $(u, v) \in X := H^r \times H^{r-1}$ for sufficiently large r. Even if it is impossible to define an evolution of the linear part (74) in the full space X, it is easy to show using Fourier analysis that there are two complementary spaces in which one can define the evolution forwards and backwards. A remarkable result in [18,9] is that this splitting with partial evolution operators persists in the linearization near periodic orbits, provided that they stay close to the origin.

11.4. Result for well-posed PDE

The Theorem 11.2 will be our main result for well-posed PDEs. Essentially, the assumptions of the theorem are that we can formulate the operator in (15) and (16) and that the delay term possesses enough regularity so that the argument we used to prove Theorem 4.6 goes through unchanged.

Therefore, the proof of Theorem 11.2 is a trivial walk-through. On the other hand, the fact that the assumptions are satisfied in the cases (70), (71) for some choices of spaces X is not trivial and will be discussed in Section 11.4.5. Of course, similar verifications can be done in other models.

The only subtlety is that we will use the *two spaces approach* of [40]. (See also [82,13] for a more streamlined and refined version.) This allows to consider perturbations which are unbounded but of lower order than the evolution operator. For example in (70), the nonlinearity involves the first derivatives taking advantage of the fact that the main evolution operator is of second order. In the case of (71), since the linear term is a fourth order elliptic operator, the nonlinearity could involve terms of order up to three. As we will see, the two spaces approach also allows to lower the regularity requirements of the delay term. (See hypotheses in Theorem 11.2.)

11.4.1. The two spaces approach

The basic idea of the two spaces approach is that we study the evolution equation using two spaces X, Y consisting of functions with different regularity. In applications to PDEs, often $X = H^{r+k}$, $Y = H^r$ with H^r the standard Sobolev spaces or the product of these spaces. In our case, we will take r large enough so that the solutions are classical, and the space H^r enjoys properties that it is a Banach algebra and the composition operator is smooth.

Differential operators, which are unbounded from a space to itself become bounded from X to Y. Then, the main evolution operator, smooths things out, such that it maps Y to X in a bounded way. Of course, the bound of the evolution as an operator from the rough space Y to the smooth space X depends on the time that the evolution has been acting and becomes singular as the time goes to zero, but we assume that there are bounds for the exponents of the negative powers, which ensures integrability.

11.4.2. Setup of the result

Consider the evolutionary PDE (67). Let X, Y be Banach spaces consisting of smooth enough functions satisfying the boundary conditions imposed on (67). We will assume that Y consists of less smooth functions, such that $\mathscr{F}$ is a differentiable map from space X to space Y. One consequence is that X has a compact embedding into Y.

Let $K_0\colon \mathbb{T} \to X$ be a parameterization of the periodic orbit of (67). As in Section 5.1, we use the notation $K(\theta) = K_0(\theta) + \widehat{K}(\theta)$ with $\widehat{K}\colon \mathbb{T} \to X$, and we derive formally the equation (76).

$$\omega_0 D\widehat{K}(\theta) - D\mathscr{F}(K_0(\theta))\widehat{K}(\theta) = B^{\varepsilon}(K, \omega, \gamma, \theta) - \widehat{\omega} DK_0(\theta), \tag{76}$$

where

$$\begin{aligned} B^{\varepsilon}(K, \omega, \gamma, \theta) &:= N(\theta, \widehat{K}) + \varepsilon \mathscr{P}(K, \omega, \gamma, \theta) - \widehat{\omega} D\widehat{K}(\theta), \\ N(\theta, \widehat{K}) &:= \mathscr{F}(K_0(\theta) + \widehat{K}(\theta)) - \mathscr{F}(K_0(\theta)) - D\mathscr{F}(K_0(\theta))\widehat{K}(\theta). \end{aligned} \tag{77}$$

11.4.3. Statement of the result

We first formulate an abstract result, Theorem 11.2, whose proof is almost identical to the proof of Theorem 4.6. The deep result is to verify that the hypotheses of Theorem 11.2 hold in examples of interest. In Section 11.4.5, we show that the examples in Section 11.3 verify the hypotheses. We leave the verification in other models of interest to the readers.

Theorem 11.2. *Assume that when $\varepsilon = 0$, the equation (66) has a periodic orbit which satisfies:*

- *The linearized equation around the periodic orbit admits a solution. That is, for any $\theta_0 \in \mathbb{T}$ and $\theta_0 < \theta \in \mathbb{T}$, there is an operator $\Phi(\theta; \theta_0)$ mapping from Y to X solving*

$$\omega_0 \frac{d}{d\theta} \Phi(\theta; \theta_0) = D\mathscr{F}(K_0(\theta))\Phi(\theta; \theta_0); \tag{78}$$

- – *$1 \in Spec(\Phi(1; 0), X)$ is a simple eigenvalue.*
  – *The spectral projection on $Spec(\Phi(1; 0), X) \setminus \{1\}$ in X is bounded.*
- *The family of operators Φ is smoothing in the sense that it satisfies*

$$\|\Phi(t; \theta_0)\|_{Y,X} \leqslant C(t - \theta_0)^{-\alpha} \quad 0 < \alpha < 1, \tag{79}$$

where $\|\cdot\|_{Y,X}$ is the norm of an operator mapping from Y to X, C is a constant.

We also need the following two assumptions on the delay perturbation. Let $\ell > 0$ be an integer and fix the parameter γ. Denote the ball of radius ρ in the space $C^{\ell+\mathrm{Lip}}(\mathbb{T}, X)$ centered at K_0 as $\mathcal{U}_\rho$, and the interval in $\mathbb{R}$ centered at ω_0 with radius δ as B_δ.

(H2.1*) *There is constant $\phi_{\rho,\delta} > 0$, such that for all $K \in \mathcal{U}_\rho$ and $\omega \in B_\delta$, $\mathscr{P}(K, \omega, \gamma, \cdot) : \mathbb{T} \to Y$ is $C^{\ell-1+\mathrm{Lip}}$, with*

$$\|\mathscr{P}(K, \omega, \gamma, \cdot)\|_{C^{\ell-1+\mathrm{Lip}}(\mathbb{T},Y)} \leqslant \phi_{\rho,\delta}.$$

(H3.1*) *There is constant $\alpha_{\rho,\delta} > 0$, such that for all $K,\ K' \in \mathcal{U}_\rho$, and $\omega,\ \omega' \in B_\delta$, and for all $\theta \in \mathbb{T}$,*

$$\|\mathscr{P}(K, \omega, \gamma, \theta) - \mathscr{P}(K', \omega', \gamma, \theta)\|_Y \leqslant \alpha_{\rho,\delta} \max\left\{|\omega - \omega'|, \|K - K'\|_{C^0(\mathbb{T},X)}\right\}.$$

Then, for small enough ε, the equation (66) has a periodic orbit of frequency ω, which is parameterized by a $C^{\ell+\mathrm{Lip}}$ map $K : \mathbb{T} \to X$. Moreover, K is close to K_0 in the sense of $C^\ell(\mathbb{T}, X)$.

The proof of Theorem 11.2 is very easy. It suffices to observe that, thanks to the hypotheses of the theorem, the operator Γ^ε, defined in the same way as in (14), sends a ball in the space $\mathbb{R} \times C^{\ell+\mathrm{Lip}}(\mathbb{T}, X)$ to itself and that in this ball, Γ^ε is a contraction under the norm of $\mathbb{R} \times C^0(\mathbb{T}, X)$. Then, we apply Lemma A.8.

Similar to before, one can get smooth dependence on parameters result.

11.4.4. Some remarks

Remark 11.3. The assumption that equation (78) admits solutions with the bounds in (79) is rather nontrivial and its verification in concrete examples requires PDE techniques.

Remark 11.4. Thanks to (79), $\Phi(1;0)$ is bounded from Y to X and, hence compact from Y to Y. Therefore, the spectrum away from zero is characterized by the existence of finite dimensional eigenspaces.

However, for an operator A acting on two spaces $X \subset Y$, there is no relation of $Spec(A, X)$ and $Spec(A, Y)$ in general.

Remark 11.5. In our case, for the operator $\Phi(1;0)$, its point spectrum in space X agrees with its point spectrum in space Y. This is not hard to see from the eigenvector equation and the smoothing effect of the operator $\Phi(1;0)$.

11.4.5. Verification of the assumptions of Theorem 11.2 in some examples

For the parabolic equations (70) and (71), a very elegant formalism is developed in [40]. The case (71) will be simpler than (70) since the linearized operator being higher order leads to stronger smoothing properties of the evolution.

The space Y will be H^r, a Sobolev space of high enough order. We emphasize once again that for our purposes, the results are stronger if the space is more restrictive.

The semigroup theory tells us that we can solve the equation (78) and that the solution is smoothing in the sense that

$$\|\Phi(\theta;\theta_0)\|_{H^r,\ H^{r+a}} \leqslant C(a)(\theta-\theta_0)^{-a}. \tag{80}$$

11.5. Result for ill-posed PDE

In this section, we show how one can get existence of periodic solutions for delay perturbations of ill-posed PDEs.

We just need to assume that the linearized equation admits partial evolutions (one evolution forward in time and another one backward in time) defined in complementary spaces. If these evolutions are smoothing, the methods of Section 10 apply without change.

Again the deeper part is to show that the concrete examples satisfy the assumptions. In the case of the periodically forced Boussinesq equation (75) with a periodic solution which is hyperbolic, we will show that the periodic solution persists under delay perturbation. The assumption that (75) has a hyperbolic periodic orbit is a non-trivial – but easily verifiable in concrete models – assumption. We note that the time independent Boussinesq equation (72) does not have hyperbolic periodic orbits due to energy conservation. Our results require delicate regularity properties of the periodic orbits, which are verified for all the bounded small solutions in [9].

Since the partial evolutions involve smoothing properties, we still use the two spaces approach summarized in Section 11.4.1. We have used the same set up as [9] to help the reader check for the applications.

Remark 11.6. When the non-linear terms $N_{1,2,3}$ in (75) are analytic, the periodic orbits are analytic. As mentioned in Remark 6.5, we do not expect that the periodic orbits of the perturbed delay equations are analytic. So, we follow [9] and deduce the regularity of the periodic orbits from the C^ℓ regularity of the center manifold.

11.5.1. Abstract setup for the study of ill-posed equations

We will assume that there is a periodic solution of the evolution equation (75), which satisfies the following Definition 11.7. Definition 11.7 can be verified for the linear part of (75), and is shown to be stable under perturbations (which can be unbounded) in [18,9]. (Related notions of splittings and their stability using a different functional analysis set up appear also in [12,54]. We have found that the two spaces approach is more concrete and easier to adapt to the delay case.)

Definition 11.7 is motivated by an analogue of hyperbolicity for ill-posed equations. We do not assume that the linearized equations define an evolution such as Φ, but we assume that there are two evolutions (one in the future and one in the past) defined in complementary spaces. This is enough to follow the set up introduced in Section 10 and formulate a fixed point equation for the periodic orbit of the perturbed equation.

Let us make some remarks about some subtle technical points.

• We assume that when these evolutions are defined, they are *smoothing*. That is, they take functions of a certain degree of differentiability (in x) and map them into functions with more derivatives. As shown in [18,9], this allows to show that these structures are stable under perturbations, which can be unbounded but are of lower order. This generality is important in the treatment of examples such as (75) since it allows to show that the periodic solutions constructed in the above papers satisfy Definition 11.7.

• It is important to note that Definition 11.7 only needs to be applied to the periodic orbits of the problem without the delay. In this section the unperturbed problem will be a PDE, which is exactly the case discussed in [18,9]. As in Section 10, the invariant splitting will be used to set up a functional equation and it will remain fixed, so that once we verify the existence in the unperturbed case, it does not get updated.

• Both [18,9] consider situations more general than periodic orbits. The paper [18] considers quasi-periodic orbits and [9] considers bounded orbits. In the case of quasi-periodic (in particular periodic) orbits, it is natural in the examples considered to assume that the bundles are analytic. For orbits with a time-dependence more complicated than periodic, it is natural to assume only finite regularity. In this paper we have adopted the definition in [18], which includes analyticity, since it applies to the examples we have in mind. Notice, however that the solutions of the delay equation will only be shown to be finitely differentiable and depend regularly on parameters in finite differentiable topologies. Indeed, we do not expect that the solutions of the delay problem will be analytic. See Remark 6.5.

Definition 11.7. Let $X \subset Y$ be two Banach spaces. Let $\mathbb{T}_\rho := \{z \in \mathbb{C}/\mathbb{Z} : |Im\ z| < \rho\}$. We say that an embedding $K_0 \colon \mathbb{T}_\rho \to X$ is spectrally nondegenerate if for every θ in $\mathbb{T}_\rho$, we can find splittings:

$$\begin{aligned} X &= X_\theta^s \oplus X_\theta^c \oplus X_\theta^u, \\ Y &= Y_\theta^s \oplus Y_\theta^c \oplus Y_\theta^u, \end{aligned} \tag{81}$$

with associated bounded projections on X and Y. (We will abuse the notation and use $\Pi_\theta^{s,c,u}$ to denote the projections as maps in $L(X, X)$ or in $L(Y, Y)$.) The projections depend analytically on $\theta \in \mathbb{T}_\rho$, and have continuous extensions to the closure of $\mathbb{T}_\rho$. Spaces $X_\theta^{s,c,u}$ and $Y_\theta^{s,c,u}$ have the following properties.

- We can find families of operators

$$
\begin{aligned}
U_\theta^s(t) &\colon Y_\theta^s \to X_{\theta+\omega_0 t}^s, \quad t > 0, \\
U_\theta^u(t) &\colon Y_\theta^u \to X_{\theta+\omega_0 t}^u, \quad t < 0, \\
U_\theta^c(t) &\colon Y_\theta^c \to X_{\theta+\omega_0 t}^c, \quad t \in \mathbb{R}.
\end{aligned}
$$

- The operators $U_\theta^{s,c,u}(t)$ are cocycles over the rotation satisfying

$$
U_{\theta+\omega_0 t}^{s,c,u}(\tau) U_\theta^{s,c,u}(t) = U_\theta^{s,c,u}(\tau + t). \tag{82}
$$

- The operators $U_\theta^{s,c,u}(t)$ are smoothing in the time direction where they can be defined and they satisfy assumptions in the quantitative rates. There exist constants $\alpha_1, \alpha_2 \in [0, 1)$, $\beta_1, \beta_2, \beta_3^+, \beta_3^- > 0$ with $\beta_1 > \beta_3^-$, and $\beta_2 > \beta_3^+$, and $C > 1$, independent of θ, such that the evolution operators satisfy the following rate conditions:

$$
\|U_\theta^s(t)\|_{\rho,Y,X} \leqslant C e^{-\beta_1 t} t^{-\alpha_1}, \quad t > 0, \tag{83}
$$

$$
\|U_\theta^u(t)\|_{\rho,Y,X} \leqslant C e^{-\beta_2 |t|} |t|^{-\alpha_2}, \quad t < 0, \tag{84}
$$

  and

$$
\begin{aligned}
\|U_\theta^c(t)\|_{\rho,Y,X} &\leqslant C e^{\beta_3^+ t}, \quad t \geqslant 0, \\
\|U_\theta^c(t)\|_{\rho,Y,X} &\leqslant C e^{\beta_3^- |t|}, \quad t \leqslant 0.
\end{aligned} \tag{85}
$$

- The operators $U_\theta^{s,c,u}(t)$ are solutions of the variational equations in the sense that

$$
\begin{aligned}
U_\theta^s(t) &= Id + \int_0^t D\mathscr{F}^s(K_0(\theta + \omega_0 \tau)) U_\theta^s(\tau) d\tau, \quad t > 0, \\
U_\theta^u(t) &= Id + \int_0^t D\mathscr{F}^u(K_0(\theta + \omega_0 \tau)) U_\theta^u(\tau) d\tau, \quad t < 0, \\
U_\theta^c(t) &= Id + \int_0^t D\mathscr{F}^c(K_0(\theta + \omega_0 \tau)) U_\theta^c(\tau) d\tau, \quad t \in \mathbb{R}.
\end{aligned} \tag{86}
$$

In this paper, we will also need:

- The space X^c is unidimensional and it is spanned by the direction of the evolution along the periodic orbit.

Recall that $\mathcal{U}_\rho \subset C^{\ell+\mathrm{Lip}}(\mathbb{T}, X)$ is the ball of radius ρ centered at K_0, and $B_\delta \subset \mathbb{R}$ is the interval centered at ω_0 with radius δ. Compared with the hypothesis for well-posed equations in (H2.1*), we make similar but slightly stronger assumption on the delay term:

(H2.1.1*) There is constant $\phi_{\rho,\delta} > 0$, such that for all $K \in \mathcal{U}_\rho$ and $\omega \in B_\delta$, $\mathscr{P}(K, \omega, \gamma, \cdot): \mathbb{T} \to Y$ is $C^{\ell+\mathrm{Lip}}$, with

$$\|\mathscr{P}(K, \omega, \gamma, \cdot)\|_{C^{\ell+\mathrm{Lip}}(\mathbb{T}, Y)} \leqslant \phi_{\rho,\delta}.$$

11.5.2. Statement of the result

Theorem 11.8. *Assume that we have an evolution equation* (67) *that admits a periodic solution satisfying Definition 11.7, and that we perturb by delay terms satisfying assumptions (H2.1.1*) and (H3.1*).*

Then, for sufficiently small ε, the equation (66) *has a periodic solution of frequency ω, which is parameterized by a $C^{\ell+\mathrm{Lip}}$ map $K: \mathbb{T} \to X$. Moreover, K is close to K_0 in the sense of $C^\ell(\mathbb{T}, X)$.*

The proof of Theorem 11.8 follows the same line as in Section 10.3. We work with the fixed point equation (65). Using that we have evolution U_θ^c and partial evolutions U_θ^s and U_θ^u for the linearized equations satisfying Definition 11.7, we can find solution to equation (65), with $\widehat{\omega} \in \mathbb{R}$ and $\widehat{K}^s, \widehat{K}^u \in C^{\ell+\mathrm{Lip}}(\mathbb{T}, X)$.

As we have discussed, the regularity properties are verified for concrete examples in [9] for the time-perturbed Boussinesq equation (75). For this example, the frequency ω remains fixed and we only need to solve the last two equations in (65).

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Appendix A. Regularity properties

One of the sources of complication in the study of delay equations – especially state dependent delay equations – is that the equations involve compositions, which have many surprising properties. In this appendix we collect a few of them. A systematic study of the composition operator in C^r spaces which are the most natural for our problem is in [17].

A.1. Function spaces

Let ℓ be a positive integer, let X be a Banach space and $U \subset X$ be an open set. For functions on U taking values in another Banach space Y, we can define derivatives [20,57], and Lipschitz and Hölder regularity of the derivatives.

Let F be a function defined on U. Recall that the j derivative of F is a j-multilinear function from $X^{\otimes j}$ to Y and that there is a natural norm for multilinear functions (supremum of the norm of the values when the arguments have norm 1). The Lipschitz constant of F is

$$\mathrm{Lip}\,(F) = \sup_{x,y\in U, x\neq y} \|F(x) - F(y)\|_Y / \|x-y\|_X.$$

Definition A.1. We say that $K : U \to Y$ is in $C^{\ell+\mathrm{Lip}}(U, Y)$ when K has ℓ derivatives and the ℓ derivative is Lipschitz.

We endow $C^{\ell+\mathrm{Lip}}(U, Y)$ with the norm:

$$\|K\|_{C^{\ell+\mathrm{Lip}}} = \max\left\{ \max_{k=0,1,\dots,\ell} \left\{ \sup_x \|D^k K(x)\| \right\}, \mathrm{Lip}(D^\ell K) \right\}, \tag{87}$$

which makes $C^{\ell+\mathrm{Lip}}$ a Banach space.

A similar definition can be written when U is a Riemannian manifold. In this paper we will use the case that $U = \mathbb{T}$.

Remark A.2. We note that Definition A.1 assumes uniform bounds of the derivatives in the whole domain. There are other very standard definitions of differentiable sets that only assume continuity and bounds in compact subsets of U. Even when $U = \mathbb{R}^n$ these definitions (e.g. Whitney topology, very natural in differential geometry) do not lead to the space of $C^{\ell+\mathrm{Lip}}$ functions being a Banach space and we will not use them.

A.2. Simple estimates on composition

We will need the following property of the composition operator, one can refer to [17] for more details.

Lemma A.3. *Let X, Y, Z be Banach spaces. Let $E \subset X$, $F \subset Y$ be open subsets.*

Assume that: $g \in C^{\ell+\mathrm{Lip}}(E, Y)$, $f \in C^{\ell+\mathrm{Lip}}(F, Z)$ and that $g(E) \subset F$ so that $f \circ g$ can be defined. Then, $f \circ g \in C^{\ell+\mathrm{Lip}}(E, Z)$, and

$$\|f \circ g\|_{C^{\ell+\mathrm{Lip}}(E,Z)} \leqslant M_\ell \|f\|_{C^{\ell+\mathrm{Lip}}(F,Z)} \big(1 + \|g\|^{\ell+1}_{C^{\ell+\mathrm{Lip}}(E,Y)}\big). \tag{88}$$

The proof of Lemma A.3 just uses the Faà di Bruno's formula for the derivatives of the composition. To control the Lipschitz constant of the ℓ derivative, we use that the Lipschitz constant of product and composition satisfy the same formulas as those of the derivative with an inequality in place of equality.

In (88) we can take any set F that contains $g(E)$. The results are sharper when we take F as small as possible.

A.3. The mean value theorem

Definition A.4. We say that an open set $U \subset X$ is a compensated domain when it is connected, and there is $C > 0$ such that for any $x, y \in U$, there is a C^1 path $\gamma \subset U$ such that

$$\mathrm{length}(\gamma) \leqslant C\|x - y\|.$$

In particular, a convex domain is compensated with $C = 1$.

We also recall the fundamental theorem of calculus.

Theorem A.5. *Assume that $U \subset X$ is open connected, $F: U \to X$ is a C^1 function, $x, y \in U$ and that γ is a C^1 path joining x, y. Then*

$$F(x) - F(y) = \int_0^1 DF(\gamma(t))\, D\gamma(t)\, dt.$$

As a corollary of Theorem A.5 we have that

$$\|F(x) - F(y)\| \leqslant \|DF\|_{C^0} \cdot \mathrm{length}(\gamma) \leqslant \|F\|_{C^1} \cdot \mathrm{length}(\gamma).$$

If the domain U is compensated, we obtain that

$$\|F(x) - F(y)\| \leqslant C\|F\|_{C^1}\|x - y\|.$$

In particular, C^1 functions on compensated domains are Lipschitz.

The conclusion that C^1 implies Lipschitz, is not true if the domain is not compensated. It is not difficult to obtain examples of domains where C^1 functions are not continuous even when $X = \mathbb{R}^2$.

Lemma A.6. *Assume that for some $\ell \geqslant 1$, $\|f\|_{C^{\ell+\mathrm{Lip}}} \leqslant A$, $\|g_1\|_{C^{\ell-1+\mathrm{Lip}}}$, $\|g_2\|_{C^{\ell-1+\mathrm{Lip}}} \leqslant B$. Then:*

$$\|f \circ g_1 - f \circ g_2\|_{C^{\ell-1+\mathrm{Lip}}} \leqslant C(A, B)\|g_1 - g_2\|_{C^{\ell-1+\mathrm{Lip}}}. \tag{89}$$

Proof. By the fundamental theorem of calculus we have pointwise

$$f \circ g_1 - f \circ g_2 = \int_0^1 Df(g_2 + t(g_1 - g_2))(g_1 - g_2)\, dt.$$

If we interpret the above as identity among functions we have

$$\|f \circ g_1 - f \circ g_2\|_{C^{\ell-1+\mathrm{Lip}}} \leqslant \int_0^1 C\|Df(g_2 + t(g_1 - g_2))\|_{C^{\ell-1+\mathrm{Lip}}} \cdot \|(g_1 - g_2)\|_{C^{\ell-1+\mathrm{Lip}}}\, dt.$$

Using Lemma A.3, $\|Df(g_2 + t(g_1 - g_2))\|_{C^{\ell-1+\mathrm{Lip}}}$ is bounded by a function of A and B, we are done. □

A.4. Interpolation

We quote the following result from [31,47]. See [17] for a modern, very simple proof valid for functions on compensated domains in Banach spaces.

Lemma A.7. *Let U be a convex and bounded open subset of a Banach space E, F be a Banach space. Let r, s, t be positive numbers, $0 \leqslant r < s < t$, and $\mu = \frac{t-s}{t-r}$. There is a constant $M_{r,t}$, such that if $f \in C^t(U, F)$, then*

$$\|f\|_{C^s} \leqslant M_{r,t} \|f\|_{C^r}^{\mu} \|f\|_{C^t}^{1-\mu}.$$

A.5. Closure properties of $C^{\ell+\mathrm{Lip}}$ ball

We quote a very practical result which appears as Lemma 2.4 in [53]. (This paper is largely reproduced as a chapter in [64]. See Lemma (2.5) on p. 39.)

Lemma A.8. *Let $U \subset X$ be a compensated domain.*

Denote by $\mathbf{B}$ a closed ball in $C^{\ell+\mathrm{Lip}}(U, Y)$. Let $\{u_n\}_{n\in\mathbb{N}} \subset \mathbf{B}$ be such that u_n converges pointwise weakly to u. Then, $u \in \mathbf{B}$.

Furthermore, the derivatives of u_n of order up to ℓ converge weakly to the derivatives of u.

A similar result to Lemma A.8 is the following, which appears as Lemma 6.1.6 in [40, p. 151].

Lemma A.9. *Let $U \subset X$ be an open set. Denote by $\mathbf{B}$ a closed ball in $C^{\ell+\mathrm{Lip}}(U, Y)$. Let $\{u_n\}_{n\in\mathbb{N}} \subset \mathbf{B}$ be such that u_n converges uniformly to u. Then, $u \in \mathbf{B}$.*

Furthermore the derivatives of u_n of order up to ℓ converge uniformly to the derivatives of u away from the boundary of U.

Both Lemma A.8 and Lemma A.9 remain true when we replace the spaces of $C^{\ell+\mathrm{Lip}}$ functions by Hölder spaces.

Remark A.10. It is instructive to compare the proofs of Lemma A.8 and Lemma A.9 in their original references.

The proof of [53] is based on considering restrictions to lines. Then, one can apply Arzela-Ascoli theorem and extract converging subsequences. The assumption of a weak pointwise limit ensures that the limit is unique. The uniformity of the $C^{\ell+\mathrm{Lip}}$ norms of the functions ensures the existence of derivatives and the convergence.

The proof of [40] goes along different lines. It shows that there are bounds on the derivatives by the C^0 norms and the size of the ball. An alternative argument is to use interpolation inequalities in Lemma A.7, which provides uniform convergence of the derivatives on U (also near the boundary).

As a consequence of Lemma A.8, we have the following version of the contraction mapping.

Lemma A.11. *With the same notation of Lemma A.8.*

Assume $\mathscr{T} : \mathbf{B} \to \mathbf{B}$ satisfies that there exists $\kappa < 1$ such that

$$\|\mathscr{T}(u) - \mathscr{T}(v)\|_{C^0} \leqslant \kappa \|u - v\|_{C^0} \quad \forall u, v \in \mathbf{B}.$$

Then, $\mathscr{T}$ has a unique fixed point u^* in $\mathbf{B}$.

For any $u \in \mathbf{B}$, and $0 \leqslant j \leqslant \ell$

$$\|\mathscr{T}^n(u) - u^*\|_{C^j} \leqslant C\kappa^{n\frac{\ell+1-j}{\ell+1}} \|\mathscr{T}(u) - u\|_{C^0}^{\frac{\ell+1-j}{\ell+1}},$$

where C is a constant that depends on the radius of the ball $\mathbf{B}$ and j.

Furthermore,

$$\|u - u^*\|_{C^j} \leqslant C(1-\kappa)^{-\frac{\ell+1-j}{\ell+1}} \|\mathscr{T}(u) - u\|_{C^0}^{\frac{\ell+1-j}{\ell+1}}.$$

We note that the hypotheses of Lemma A.11 are easy to verify in operators that involve composition. The propagated bounds just amount to proving that the size of derivatives of composition of two functions can be estimated by the sizes of the derivatives of the original functions. The contraction properties are done under the assumption that the functions are smooth so that one can use the mean value theorem.

Proof. When X is finite dimensional (or just separable), Lemma A.8 is a corollary of Ascoli-Arzela theorem. For any subsequence of u_n we can extract a sub-subsequence that converges in C^ℓ sense. The limit of this sub-subsequence has to be u. It follows that the u_n converges to u in C^ℓ sense. It then follows that the ℓ-derivative is Lipschitz.

If X is infinite dimensional, one can repeat the above argument restricting to lines. The uniform regularity assumed on u_n translates to uniform regularity of u restricted to lines.

We refer to [53] for more details. Indeed, [53] only needs to assume that the sequence converges weakly pointwise. The convergence properties are only used to guarantee the uniqueness of the limit obtained through compactness (the paper [53] is written when the domain U is the whole space, but this is not used).

Once we have the closure property, the existence of the unique fixed point is as in Banach contraction. We observe that for any $u \in \mathbf{B}$,

$$\|\mathscr{T}^{n+1}(u) - \mathscr{T}^n(u)\|_{C^0} \leqslant \kappa^n \|\mathscr{T}(u) - u\|_{C^0}.$$

Using the interpolation inequalities Lemma A.7 and that the $C^{\ell+\mathrm{Lip}}$ norms of the iterates are bounded, we obtain

$$\|\mathscr{T}^{n+1}(u) - \mathscr{T}^n(u)\|_{C^j} \leqslant C\kappa^{n\frac{\ell+1-j}{\ell+1}} \|\mathscr{T}(u) - u\|_{C^0}^{\frac{\ell+1-j}{\ell+1}}. \tag{90}$$

From this one obtains that $\mathscr{T}^n(u) - u = \sum_{k=1}^n (\mathscr{T}^k(u) - \mathscr{T}^{k-1}(u))$ is an absolutely convergent series in the $C^{j+\mathrm{Lip}}$ sense. Let u^* be the fixed point. Using (90) to estimate the series, we obtain:

$$\|u - u^*\|_{C^j} \leqslant C(1 - \kappa^{\frac{\ell+1-j}{\ell+1}})^{-1} \|\mathscr{T}(u) - u\|_{C^0}^{\frac{\ell+1-j}{\ell+1}}.$$

On the other hand, from the standard Banach fixed point theory, we obtain that $\|u - u^*\|_{C^0} \leqslant (1-\kappa)^{-1}\|\mathcal{T}(u) - u\|_{C^0}$. By Lemma A.7 we obtain

$$\|u - u^*\|_{C^j} \leqslant C(1-\kappa)^{-\frac{\ell+1-j}{\ell+1}} \|\mathcal{T}(u) - u\|_{C^0}^{\frac{\ell+1-j}{\ell+1}}.$$

It is easy to see that this bound is better than the previously obtained one summing the series. □

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